Restricted Percolation Critical Exponents in High Dimensions

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Abstract

Despite great progress in the study of critical percolation on $\mathbb{Z}^d$ for d large, properties of critical clusters in high-dimensional fractional spaces and boxes remain poorly understood, unlike the situation in two dimensions. Closely related models such as critical branching random walk give natural conjectures for the value of the relevant high-dimensional critical exponents; see in particular the conjecture by Kozma-Nachmias that the probability that 0 and $(n, n, n, \dots)$ are connected within $[-n, n]^d$ scales as n^{-2-2d}.

In this paper, we study the properties of critical clusters in high-dimensional half-spaces and boxes. In half-spaces, we show that the probability of an open connection ("arm") from 0 to the boundary of a sidelength n box scales as n^{-3}. We also find the scaling of the half-space two-point function (the probability of an open connection between two vertices) and the tail of the cluster size distribution. In boxes, we obtain the scaling of the two-point function between vertices which are any macroscopic distance away from the boundary. Our argument involves a new application of the "mass transport" principle which we expect will be useful to obtain quantitative estimates for a range of other problems.

1 Introduction

In this paper, we primarily consider the (*bond*) *percolation* model on the canonical d-dimensional *hypercubic lattice* $\mathbb{Z}^d$ and its subgraphs, the *half-space* with normal direction $\mathbf{e}_1$ and *boxes* or ℓ^∞ balls. When $d = 2$ or d is large, it is known that critical percolation on $\mathbb{Z}^d$ does not admit infinite open clusters. A great deal of research has been devoted to studying the finer properties of critical open clusters on $\mathbb{Z}^d$ for these values of d (and on the triangular lattice, where the model is closely related to the model on $\mathbb{Z}^2$).

There is also a fairly well-developed theory of critical percolation on half-planes and other sectors of $\mathbb{Z}^2$ and the triangular lattice. For instance, on the triangular lattice, the asymptotic behavior of the probability the open cluster of the right half

Communications on Pure and Applied Mathematics, 0001–0060 (PREPRINT)

plane containing 0 touches the line $\{\mathbf{z} : \mathbf{z} \cdot \mathbf{e}_1 = n\}$ (the "one-arm probability") obeys a power law, behaving as $n^{-1/3+o(1)}$. Importantly, this power law is related to several others via *scaling relations* similar to those proved in the entire triangular lattice. The proofs of these scaling relations are quite robust, even applying to subgraphs of $\mathbb{Z}^2$ (where exact computations of power laws are generally unavailable).

For $\mathbb{Z}^d$ with d large, by contrast, there has been little work on the behavior of critical percolation in half-spaces (or general sectors). In this paper, we build a foundation for such a study. We compute the asymptotic behavior of the one-arm probability in high-dimensional half-spaces along with several other power laws of interest; in this introduction, we also describe how these results imply certain scaling relations one would expect in high dimensions; see, for instance, (1.5). As part of our work, we build tools (e.g., Theorem 1.2) that may be of interest in a study of percolation in general high-dimensional sectors.

1.1 Definition of Model and Main Results

We will consider two graphs having vertex set $\mathbb{Z}^d$, as well as subgraphs of either of these. In the *hypercubic lattice*, we take as our edge set $\{\{\mathbf{x}, \mathbf{y}\}\colon \|\mathbf{x} - \mathbf{y}\|_1 = 1\}$. In the other, the *spread-out lattice*, we take as our edge set $\{\{\mathbf{x}, \mathbf{y}\}\colon \|\mathbf{x}-\mathbf{y}\|_\infty \leq \Lambda\}$, where Λ is a fixed positive integer. All definitions in this subsection apply equally well to either of these graphs.

The usual standard basis coordinates of a vertex $\mathbf{x} \in \mathbb{Z}^d$ will be denoted $x(i) = \mathbf{x} \cdot \mathbf{e}_i$, so $\mathbf{x} = (x(1), x(2), \ldots, x(d))$. The origin $0 = (0, 0, \ldots, 0)$. The *half-space* is the subgraph of either the hypercubic lattice or the spread-out lattice that is induced by the set of vertices $\mathbb{Z}_+^d$ that have a nonnegative first coordinate: $\mathbb{Z}_+^d = \{\mathbf{x} \in \mathbb{Z}^d\colon x(1) \geq 0\}$ (note that "half-space" for us always means one of these particular graphs or a translate thereof—we do not use the term in its more general sense). The boxes or ℓ^∞ balls in these graphs are the following vertex sets:

$$B(n) = [-n, n]^d \quad \text{and} \quad B_H(n) = B(n) \cap \mathbb{Z}_+^d, \quad \text{respectively.}$$

With some abuse of notation, we sometimes identify $\mathbb{Z}_+^d$, $B(n)$, and other vertex sets with the subgraphs of the hypercubic or spread-out lattice that they induce. In particular, when there is no ambiguity or when the choice of edge set is irrelevant, we write $\mathbb{Z}^d$ for either the hypercubic lattice or the spread-out lattice.

The main object of study will be the Bernoulli bond percolation model—*percolation* for brevity—on the above and other subgraphs of $\mathbb{Z}^d$. To define the model, fix $p \in [0, 1]$ and let $\omega = (\omega_e)_e$ be a collection of i.i.d. Bernoulli(p) random variables indexed by the edges e of the hypercubic or spread-out lattice. An edge e such that $\omega_e = 1$ (resp. $\omega_e = 0$) will be referred to as *open* (resp. *closed*). The model of percolation on $\mathbb{Z}^d$ consists of the study of the *open graph*, the random subgraph of the hypercubic or spread-out lattice whose vertex set is $\mathbb{Z}^d$ and whose edge set consists of the edges e that are open in ω.

The preceding definition induces a percolation model on subgraphs of the ambient hypercubic or spread-out lattice in the natural way. Given a set $G \subseteq \mathbb{Z}^d$ of vertices, consider the subgraph (which we, via abuse of notation, identify with G) of the ambient hypercubic or spread-out lattice which it induces. Using the same random variables $(\omega_e)_e$ to define the open/closed status of edges of G, one arrives at the definition of percolation on G: the open graph now has vertex set G and edge set consisting of those e of the ambient lattice having both endpoints in G and having $\omega_e = 1$. For an introduction to percolation on $\mathbb{Z}^d$ and its subgraphs, and for an expository treatment of fundamental results, we recommend [8]. See also [24, chap. 7] for the treatment of percolation on general graphs, including homogeneous trees.

For a particular realization of ω, the *open clusters* of G are the components of the open graph on G. For fixed ω and vertex $\mathbf{x}$ but different choices of G, the open cluster containing $\mathbf{x}$ may be different. To track this dependence on G, we write $\mathfrak{C}_G(\mathbf{x})$ for the open cluster containing $\mathbf{x}$ when considering percolation on G; we abbreviate $\mathfrak{C}(\mathbf{x}) = \mathfrak{C}_{\mathbb{Z}^d}(\mathbf{x})$ and $\mathfrak{C}_H(\mathbf{x}) = \mathfrak{C}_{\mathbb{Z}^d_+}(\mathbf{x})$. The symbol $\{\mathbf{x} \overset{G}{\longleftrightarrow} \mathbf{y}\}$ denotes the event that $\mathfrak{C}_G(\mathbf{x}) = \mathfrak{C}_G(\mathbf{y})$, and we again abbreviate

$$\{\mathbf{x} \overset{\mathbb{Z}^d}{\longleftrightarrow} \mathbf{y}\} \quad \text{to} \quad \{\mathbf{x} \leftrightarrow \mathbf{y}\}.$$

This notation extends naturally, replacing $\mathbf{x}$ and $\mathbf{y}$ by sets A, B of vertices: we write $A \overset{G}{\longleftrightarrow} B$ if there exist $\mathbf{x} \in A$ and $\mathbf{y} \in B$ such that $\mathbf{x} \overset{G}{\longleftrightarrow} \mathbf{y}$ (and omit the G superscript when $G = \mathbb{Z}^d$). We use the symbol $\not\leftrightarrow$ in the obvious way; for instance, $\mathbf{x} \not\leftrightarrow \mathbf{y}$ means that $\mathfrak{C}(\mathbf{x}) \neq \mathfrak{C}(\mathbf{y})$. When discussing a cluster $\mathfrak{C}_G$ or properties thereof in the case $G \neq \mathbb{Z}^d$, we sometimes use the term *restricted*; for instance, $\mathfrak{C}_{\mathbb{Z}^d_+}(\mathbf{x}) = \mathfrak{C}_H(\mathbf{x})$ is the cluster of $\mathbf{x}$ restricted to the half-space $\mathbb{Z}^d_+$.

The distribution of ω will be denoted by $\mathbb{P}_p$ to indicate its dependence on the parameter p (we soon will fix a particular value of p). We define the *critical probability* (of our ambient lattice, that is either the hypercubic or spread-out lattice) by

$$p_c := \inf\{p : \mathbb{P}_p(\#\mathfrak{C}_{\mathbb{Z}^d}(0) = \infty) > 0\} \tag{1.1}$$

(here and later # denotes cardinality). When $p < p_c$ (resp. $p = p_c$, $p > p_c$), the model is said to be *subcritical* (resp. *critical*, *supercritical*). This paper is exclusively concerned with critical percolation, and so in what follows we will always take $p = p_c$. In particular, we will often write $\mathbb{P}$ for $\mathbb{P}_{p_c}$, except when we wish to emphasize the fact that we are talking about the critical model. We stress that the value of p_c depends on the particular hypercubic or spread-out lattice being considered. We also note that p_c is taken relative to the ambient hypercubic or spread-out lattice, even when we are discussing percolation on a subgraph of this ambient lattice.

On the hypercubic or spread-out lattices with $d \geq 2$, it is widely conjectured that $\mathbb{P}_{p_c}$-almost surely there exists no infinite open cluster. Among others (see Section 1.2 below for background and references), this conjecture is proved in "high dimensions," which we define as follows.

DEFINITION 1. The phrases *high dimensions* and *high-dimensional* refer to both of the following settings:

- the hypercubic lattice, with $d \geq 11$;
- the spread-out lattice with $d > 6$ and Λ larger than some large d-dependent constant $\Lambda_0(d)$.

All new results of this paper are proved in the setting of high dimensions; after the introduction (see the "standing assumption" at the very end of the introduction) we will exclusively consider this setting. In fact, our arguments would apply to the hypercubic lattice for all $d > 6$, as well as other possible edge sets for $\mathbb{Z}^d$, if certain past work could be extended to this setting. We return to this issue in Section 1.2 after discussing more background, and we state this extension as a conditional theorem (see Theorem 1.3 below).

The main results of the paper, Theorems 1.1 and 1.2 in this section, relate to the behavior of the open clusters $\mathfrak{C}_{\mathbb{Z}^d_+}(x)$ and $\mathfrak{C}_{B(n)}(x)$ in high dimensions. To state them precisely, we now define several events and quantities (to allow us to discuss past results outside of the high-dimensional setting, we state them for general d).

DEFINITION 2. Consider critical percolation on either the hypercubic or spread-out lattice.

- The *two-point function* $\tau(\mathbf{x}, \mathbf{y})$ denotes the connectivity probability
$$\mathbb{P}(\mathbf{x} \leftrightarrow \mathbf{y}) = \mathbb{P}\big(\mathbf{x} \overset{\mathbb{Z}^d}{\longleftrightarrow} \mathbf{y}\big).$$
More generally, when $G \subseteq \mathbb{Z}^d$, the *two-point function restricted to* G is $\tau_G(\mathbf{x}, \mathbf{y}) = \mathbb{P}(\mathbf{x} \overset{G}{\longleftrightarrow} \mathbf{y})$. The particular case of the preceding when $G = \mathbb{Z}^d_+$ is the *half-space two-point function* and will be abbreviated to $\tau_H(\mathbf{x}, \mathbf{y}) = \tau_{\mathbb{Z}^d_+}(\mathbf{x}, \mathbf{y})$.
- The site $\mathbf{x}$ has an *arm* to distance n in G if $\sup\{\|\mathbf{y}-\mathbf{x}\|_\infty : \mathbf{y} \in \mathfrak{C}_G(\mathbf{x})\} \geq n$. In the case $G = \mathbb{Z}^d$, we often simply say that $\mathbf{x}$ has an arm to distance n without referring to G. Similarly, in the case that $G = \mathbb{Z}^d_+$, we say that $\mathbf{x}$ has a *half-space arm* to distance n—in other words, $\mathbf{x}$ has a half-space arm to distance n if $\sup\{\|\mathbf{y} - \mathbf{x}\|_\infty : \mathbf{y} \in \mathfrak{C}_{\mathbb{Z}^d_+}(\mathbf{x})\} \geq n$. The corresponding events are called *arm events* or *one-arm events*. The probability that the origin 0 has an arm (resp. half-space arm, arm in $G \ni 0$) to distance n will be denoted $\pi(n)$ (resp. $\pi_H(n)$, $\pi_G(n)$).

Note that there is another natural definition of $\pi_H(n)$, wherein we demand that the open cluster of 0 contain a vertex with $\mathbf{e}_1$-coordinate at least n (rather than any coordinate at least n). It will turn out that this probability has the same asymptotic behavior as the above-mentioned π_H; see Section 2.2 and (7.1) below.

We now state the first main theorem of this paper. In it, we use the usual asymptotic notation: given two functions $f, g : \{1, 2, \dots\} \to [0, \infty)$, we write $f(n) \asymp g(n)$ to mean that $\limsup_{n\to\infty} f(n)/g(n)$ and $\limsup_{n\to\infty} g(n)/f(n)$ are both finite. If f and g instead have domain $\mathbb{Z}^d \times \mathbb{Z}^d$ and if $A \subseteq \mathbb{Z}^d \times \mathbb{Z}^d$, we say that $f(\mathbf{x},\mathbf{y}) \asymp g(\mathbf{x},\mathbf{y})$ in A if both

$$\sup_{(\mathbf{x},\mathbf{y})\in A} \frac{g(\mathbf{x},\mathbf{y})}{f(\mathbf{x},\mathbf{y})} < \infty \quad \text{and} \quad \sup_{(\mathbf{x},\mathbf{y})\in A} \frac{f(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})} < \infty.$$

THEOREM 1.1. *In the setting of critical percolation in high dimensions, the following asymptotic power laws hold.*

(a) $\pi_H(n) \asymp n^{-3}$.

(b) *Fix a constant* $K > 0$. *Then*

$$\tau_H(\mathbf{x},\mathbf{y}) \asymp \begin{cases} \|\mathbf{x}-\mathbf{y}\|_\infty^{2-d} & \text{in } \{(\mathbf{x},\mathbf{y})\colon 0 < \|\mathbf{x}-\mathbf{y}\|_\infty < K\min\{x(1), y(1)\}\}, \\ \|\mathbf{x}-\mathbf{y}\|_\infty^{1-d} & \text{in } \{(\mathbf{x},\mathbf{y})\colon x(1) = 0 \text{ and } 0 < \|\mathbf{x}-\mathbf{y}\|_\infty < Ky(1)\}, \\ \|\mathbf{x}-\mathbf{y}\|_\infty^{-d} & \text{in } \{(\mathbf{x},\mathbf{y})\colon \mathbf{x} \neq \mathbf{y},\ x(1) = 0 \text{ and } y(1) = 0\}. \end{cases}$$

(c) $\mathbb{P}_{p_c}(\#\mathfrak{C}_H(0) > n) \asymp n^{-3/4}$.

In the high-dimensional settings of Definition 1, it is known that the "unrestricted" two-point function $\tau(\mathbf{x},\mathbf{y}) = \tau_{\mathbb{Z}^d}(\mathbf{x},\mathbf{y})$ is asymptotic to $\|\mathbf{x}-\mathbf{y}\|_\infty^{2-d}$ (see (1.7) below). In fact, this is a main input our proofs will require; we give an account of this and related high-dimensional results in Section 1.2. In this light, the first asymptotic of part (b) of Theorem 1.1 informally says that $\tau_H(\mathbf{x},\mathbf{y})$ behaves like $\tau(\mathbf{x},\mathbf{y})$ when both $\mathbf{x}$ and $\mathbf{y}$ are far from the boundary of $\mathbb{Z}^d_+$.

Our second main result, Theorem 1.2, is an analogous statement for the two-point function in boxes $B(n) \subseteq \mathbb{Z}^d$: roughly, $\tau_{B(n)}(\mathbf{x},\mathbf{y})$ scales as $\tau(\mathbf{x},\mathbf{y})$ for $\mathbf{x}$ and $\mathbf{y}$ far from the boundary of $B(n)$. This result is proved before Theorem 1.1 and is necessary for making key estimates in the proof of Theorem 1.1. We also believe it is interesting in its own right and is a potential tool for studying other properties of open clusters.

THEOREM 1.2. *Consider percolation in high dimensions, and fix any constant* $M > 1$. *There are constants* $C > c > 0$ (*depending on* M *and* d *only*) *such that for all* n *and for all* $\mathbf{x} \neq \mathbf{y} \in B(n)$,

$$c\,\|\mathbf{x}-\mathbf{y}\|_\infty^{2-d} \le \mathbb{P}_{p_c}\big(\mathbf{x} \overset{B(Mn)}{\longleftrightarrow} \mathbf{y}\big) \le C\,\|\mathbf{x}-\mathbf{y}\|_\infty^{2-d}.$$

As alluded to above (and stated formally at (1.7)), the two-point function $\tau(\mathbf{x},\mathbf{y})$ is known to scale as $\|\mathbf{x}-\mathbf{y}\|_\infty^{2-d}$ in both settings of Definition 1. Given that $\tau_{B(Mn)}(\mathbf{x},\mathbf{y}) \le \tau(\mathbf{x},\mathbf{y})$, the upper bound on $\tau_{B(Mn)}$ in Theorem 1.2 is trivial; the new result of the theorem is contained in the lower bound.

Changed "Since" to "Given that" to avoid a bad line break.

1.2 Background and Conditional Version of Results

Percolation has been the focus of a great deal of research in recent decades, with much of this effort dedicated to the case of percolation on the hypercubic lattice. The model has received attention on half-spaces, sectors, and other proper subgraphs of $\mathbb{Z}^d$ in part for the illuminating comparisons these settings provide with the full lattice. A key example relates to the well-known conjecture ("absence of percolation at criticality") that, at $p = p_c$, there is almost surely no infinite cluster on the hypercubic lattice for $d \geq 2$. While this conjecture is still open in general, the corresponding result is established for percolation on half-spaces [4].

Among hypercubic and spread-out lattices, absence of percolation at criticality has so far been proved on the $d = 2$ hypercubic lattice (by Kesten [18]) and in the high-dimensional setting of Definition 1. The latter result is due originally to Hara and Slade [10], though their work applied on the hypercubic lattice only for $d \geq 19$; this was improved by Fitzner and van der Hofstad [7] to the $d \geq 11$ setting considered in this paper. In such settings where absence of percolation is proved, one is led to ask for more quantitative results on just "how large" critical clusters may be.

Critical percolation on $\mathbb{Z}^d$ and in many other settings is believed to be characterized by the validity of *power laws* for various measures of open clusters. For instance, the upper tail of the cluster size is expected to obey $\mathbb{P}_{p_c}(\#\mathfrak{C}(0) > n) = n^{-1/\delta+o(1)}$, where δ is some *critical exponent*. According to a widely held *universality* conjecture, critical exponents should depend only on the "large scale" structure of the lattice, but not on the "microscopic details" thereof; for instance, the value of δ above should be the same on the d-dimensional hypercubic lattice and the d-dimensional spread-out lattice, but could change when d is changed.

In two dimensions and in high dimensions, many critical exponents are known, and others are explicitly conjectured. This leads naturally to the question of how or if these power laws change when the lattice is replaced by a half-space (or other sector). This question is the main focus of this paper. For definiteness, we restrict the discussion to the power laws studied in the above theorems—that is, on the behavior of the two-point function, the arm probability, and the upper tail of the *cluster size* $\#\mathfrak{C}(0)$. On the full lattice $\mathbb{Z}^d$, recall the traditional symbols for the relevant critical exponents (assuming they exist):

$$\pi(n) = n^{-1/\rho+o(1)}, \quad \tau(0,\mathbf{x}) = \|\mathbf{x}\|_\infty^{2-d+\eta+o(1)}, \quad \text{and} \quad \mathbb{P}(\#\mathfrak{C}(0) > n) = n^{-1/\delta+o(1)}. \tag{1.2}$$

While such power laws should hold for half-spaces, different exponents η should govern the behavior of $\tau_{\mathbb{Z}^d_+}(0,\mathbf{x})$ for $\mathbf{x}$ on and far away from the boundary of $\mathbb{Z}^d_+$, as in Theorem 1.1.

A comparison of critical exponents between the full lattice and half-spaces is most developed in two dimensions. Indeed, a main goal of the present paper is to begin to develop the comparison in high dimensions. We thus begin the discussion by recalling past work for percolation on the two-dimensional hypercubic ("square") lattice and *site percolation* on the two-dimensional triangular lattice, where sites are open or closed instead of bonds. In two dimensions, the precise power laws obeyed by the two-point function and other quantities considered in this paper are known only for critical site percolation on the triangular lattice, though the widely conjectured universality of these exponents suggests that the same behavior should hold on the square lattice.

Past Work in Two Dimensions

For the purpose of studying ρ, η, δ, there exist two broad families of techniques in two dimensions. One family, based on the Russo-Seymour-Welsh (RSW) technology and "gluing," is applicable in the settings of both the square and triangular lattice. The other family, based on conformal invariance and SLE methods, so far applies only in the setting of the triangular lattice. We begin by discussing what can be established using only gluing methods, or in other words what is known on $\mathbb{Z}^2$.

Gluing, RSW, and the square lattice. Gluing and RSW are not enough to establish the existence of the exponents in (1.2), let alone their values. However, it is relatively easy to establish inequalities for these exponents, in the sense that the relevant quantities are upper- and lower-bounded by constant multiples of particular power functions. See, for instance, [31] for a short and elegant argument, valid on both the square and triangular lattice, that $\pi(n) \geq cn^{-1/2}$. The values of certain critical exponents (though apparently not those of (1.2)) do follow from RSW-type arguments, including the exponent governing the "polychromatic five-arm event"; see the first exercise sheet of Werner's lecture notes on two-dimensional critical percolation [33].

Although gluing methods seem unable to establish the values of most critical exponents, they suffice to prove strong relationships between many of these exponents. Kesten [19] proved that if one of the exponents of (1.2) exists on the square or triangular lattice, so must the other two. Moreover, in this case, the values of the other two exponents are completely determined by the relationships

$$\eta = 2/\rho, \quad \delta = 2\rho - 1 = 4/\eta - 1. \tag{1.3}$$

Remarkably (and in contrast with the situation in high dimensions—see the discussion immediately following (1.7)), this result was proved long before the existence of any of these exponents was known on any two-dimensional lattice.

A straightforward application of the arguments leading to (1.3) gives the two-dimensional analogue of our Theorem 1.2 in two dimensions: if $M > 1$ is fixed, there is a constant $c = c(M) > 0$ such that

$$\tau_{B(Mn)}(\mathbf{x}, \mathbf{y}) \geq c\tau(\mathbf{x}, \mathbf{y}) \quad \text{uniformly in } n \text{ and } \mathbf{x}, \mathbf{y} \in B(n).$$

Moreover, such arguments would allow the proof of scaling relations for the quantities appearing in (1.2) in the two-dimensional half-plane (and other sectors). For instance, given the critical exponents governing π in $\mathbb{Z}^2$ and $\mathbb{Z}^2_+$, the critical exponent for $\tau_{\mathbb{Z}^2_+}(0, n\mathbf{e}_1)$ follows via the relationship

$$c\pi_{\mathbb{Z}^2}(n)\pi_{\mathbb{Z}^2_+}(n) \leq \tau_{\mathbb{Z}^2_+}(0, n\mathbf{e}_1) \leq C\pi_{\mathbb{Z}^2}(n)\pi_{\mathbb{Z}^2_+}(n). \tag{1.4}$$

As we will discuss further below, such scaling relations have taken longer to establish in high dimensions, and their half-space versions have been completely unexplored before now. A main goal of the present paper is to fill such gaps in high dimensions. One can consider our Theorem 1.1 as establishing an appropriate high-dimensional version of such scaling relations in the half-space. For instance, it implies the following analogue of (1.4):

$$cn^{6-d}\pi_{\mathbb{Z}^d}(n)\pi_{\mathbb{Z}^d_+}(n) \leq \tau_{\mathbb{Z}^d_+}(0, n\mathbf{e}_1) \leq Cn^{6-d}\pi_{\mathbb{Z}^d}(n)\pi_{\mathbb{Z}^d_+}(n). \tag{1.5}$$

The factor of n^{6-d} reflects the fact that in high-dimensional cubes having diameter of order n, there are order n^{d-6} distinct open clusters having diameter of order n, in contrast to the two-dimensional setting (where the number of such clusters is stochastically bounded).

Beyond the above, the RSW technology also suffices to prove a monotonicity property of the arm exponent ρ within sectors. Considering the sector $G_\varphi := \{(r\cos\theta, r\sin\theta) \in \mathbb{Z}^2 : r \geq 0,\ 0 \leq \theta \leq \varphi\}$, Kesten and Zhang [20] showed a version of the statement that the one-arm exponent ρ is strictly monotone in φ. Formally, given $\varphi < \psi \leq 2\pi$, there exist constants $C, \varepsilon > 0$ such that $\pi_{G_\varphi}(n) \leq Cn^{-\varepsilon}\pi_{G_\psi}(n)$ uniformly in n.

The triangular lattice and exact critical exponents. With the advent of SLE [28] and the proof of Cardy's formula [29], the existence of a critical exponent for π was shown for site percolation on the triangular lattice $\mathbb{T}$; in fact, $\rho = 48/5$ [23]. From this and Kesten's result (1.3), the values of δ and η follow. See Table 1.1 for a summary and comparison of exponents with the high-dimensional lattices and corresponding half-planes/spaces. The values of ρ, δ, and η should be the same on the triangular and square lattices, but this remains a challenging open problem.

SLE methods also give the value of the one-arm exponent ρ in the half-plane $\mathbb{T}_+ := \mathbb{T} \cap \big[[0,\infty) \times (-\infty,\infty)\big]$. Indeed, it has been shown (see [30, sec. 3]) that $\pi_{\mathbb{T}_+}(n) = n^{-1/3+o(1)}$. Using gluing methods, one can derive from this the scaling of the two-point function and the tail of the cluster size. See Table 1.1 for a summary. Using the value of ρ in $\mathbb{T}_+$ and the conformal invariance of the percolation scaling limit, one can also compute the value of ρ in the sectors G_φ, $0 \leq \varphi < 2\pi$.

Past Work and Conjectures in High Dimensions

In 1990, Hara and Slade [10] used the *lace expansion* to prove the absence of percolation at criticality on the square lattice for $d \geq 19$ and in the spread-out lattice setting of Definition 1. In fact, they proved the stronger *triangle condition* of Aizenman and Newman [2]. Combined with contemporaneous work of Barsky and Aizenman [3], this showed that the exponent δ of (1.2) exists and has the value 2 in these settings. In fact, the stronger bounded ratio asymptotic is known:

$$\mathbb{P}_{p_c}(\#\mathfrak{C}(0) > t) \asymp t^{-1/2}. \tag{1.6}$$

(Indeed, a more precise asymptotic has been shown for sufficiently high d—see [11]).

The fact that δ takes the same value for all large d (and differs from its two-dimensional value) is emblematic of the so-called *mean-field* behavior of high-dimensional percolation. Roughly speaking, when d is above the *upper critical dimension*—conjecturally, when $d > 6$—large critical clusters should exhibit a certain degree of independence. (See [16] for an extensive review of research on mean-field behavior in high-dimensional percolation, along with related results.) Many quantities of interest related to the critical model should exhibit the same behavior for all hypercubic and spread-out lattices with $d > 6$. As we will discuss below in Section 1.2, our results could be extended to any hypercubic or spread-out lattice with $d > 6$ if a few fundamental results—among them, (1.6)—were established in this generality.

The values of numerous other critical exponents have been rigorously established in high dimensions, through methods very different from those available in two dimensions. Over a decade after the establishment of the triangle condition, the stronger result $\eta = 0$ was shown in the spread-out [13] and $d \geq 19$ square lattice [9] settings. Here again an asymptotic result is known: there are lattice-dependent constants $0 < a_1 < A_1 < \infty$ such that

$$a_1 \|\mathbf{x} - \mathbf{y}\|^{2-d} \leq \tau(\mathbf{x}, \mathbf{y}) \leq A_1 \|\mathbf{x} - \mathbf{y}\|^{2-d} \quad \text{for all } \mathbf{x} \neq \mathbf{y} \in \mathbb{Z}^d. \tag{1.7}$$

We note that [9, 13] in fact show much more than (1.7) (namely, the precise leading-order behavior of τ, with error estimates); we direct the interested reader to the original articles for more information.

The time elapsed between determination of δ and η is in sharp contrast to the situation on the square lattice, where the early result (1.3) allows determination of one exponent from the other. This is one way in which two-dimensional techniques are more developed and robust than high-dimensional ones. Similarly, existing high-dimensional techniques seem less able to deal with settings (like half-spaces) lacking all the symmetries of $\mathbb{Z}^d$. It is hoped that the methods in this paper will provide a starting point for attacking other such problems in the future.

By developing an improvement of the lace expansion known as the *non-backtracking lace expansion*, Fitzner and van der Hofstad [7] established (more than) that (1.7) holds on the hypercubic lattice for all $d \geq 11$, i.e., the setting of Definition 1. By establishing the triangle condition, this work also allowed the extension of (1.6) to the $d \geq 11$ hypercubic lattice.

Obtaining the value of ρ required still more work over several years. A first attempt was due to Sakai [26], who gave an elegant scaling argument for $\rho = 1/2$ under unproven assumptions. In addition to assuming a form of existence of ρ, Sakai also assumed that

$$\mathbb{E}[\#[\mathfrak{C}(0) \cap B(n)] \,|\, 0 \nleftrightarrow \partial B(2n)] \asymp \mathbb{E}[\#[\mathfrak{C}(0) \cap B(n)]]. \tag{1.8}$$

In two dimensions, the above would again follow easily from the gluing methodology used to prove (1.3)—another instance where two-dimensional methods seem more robust in certain settings than high-dimensional ones. We note that Sakai's assumption (1.8) follows from Theorem 1.2 of this paper, whose proof is very different than its two-dimensional analogue. We direct the reader also to the work [6], where a statement of a similar flavor to Sakai's assumption (1.8) is shown for the *incipient infinite cluster* (IIC), an object which could be thought of as a critical percolation cluster conditioned to be infinite.

The asymptotic

$$a_2 n^{-2} \le \pi(n) \le A_2 n^{-2} \quad \text{for } n \ge 1 \text{ and constants } 0 < a_2 < A_2 < \infty \tag{1.9}$$

was shown in the setting of Definition 1, without any unproven assumptions, by Kozma and Nachmias [22]. The iterative method we use to upper-bound π_H as in Theorem 1.1 is related to that used for upper-bounding π in [22], though we must overcome several complications related to the fact that we work on $\mathbb{Z}_+^d$; see the discussion at the beginning of Section 6.

A conjecture made at [22, p. 378] was a major impetus for the present work. This conjecture suggested the correct asymptotic behavior of the two-point function $\tau_{B(n)}$ within a cube for vertices at the corner of the cube:

$$\mathbb{P}_{p_c}\big(0 \overset{B(n)}{\longleftrightarrow} (n, n, \dots, n)\big) \asymp n^{2-2d}. \tag{1.10}$$

One could hope to conjecture the correct asymptotic behavior of the two-point function in other subgraphs of $\mathbb{Z}^d$—for instance, quarter-spaces—based on a hypothesized connection between critical branching random walk (BRW) and critical percolation. It has been argued (see [16, sec. 2.2]) that BRW "can be viewed as the mean-field model for percolation," which would suggest that the probability appearing in (1.10) scales as the two-point function of a critical BRW started at 0 and killed at the boundary of $B(n)$, evaluated at $(n, n, \dots, n)$. This entry of this critical BRW two-point function indeed has the asymptotic n^{2-2d}. Establishing the scaling (1.10) appears difficult due to the comparative lack of symmetry of $B(n)$. We however believe the techniques developed in the present work will be useful for the study of critical exponents on general subgraphs of $\mathbb{Z}^d$, including the critical exponent appearing in (1.10).

There have been a number of other works studying properties of high-dimensional critical clusters, indeed far more than could be surveyed here; we will discuss several that are particularly relevant to our results. In [1], Aizenman showed that for $d > 6$, assuming (1.7), there are typically at least of order n^{d-6} *spanning clusters* of $B(n)$—that is, open clusters touching opposite sites of $B(n)$—and that the largest of these contains at most $n^{4+o(1)}$ vertices. A number of other authors have studied properties of large spanning clusters and the IIC, including the behavior of

	$\tau_G(0, n\mathbf{e}_1)$	$\tau_G(0, n\mathbf{e}_2)$	$\pi_G(n)$	$\mathbb{P}(\#\mathfrak{C}_G(0) > t)$
$G = \mathbb{T}$	$= n^{-5/24+o(1)}$	$= n^{-5/24+o(1)}$	$= n^{-5/48+o(1)}$	$= t^{-5/91+o(1)}$
$G = \mathbb{T}_+$	$= n^{-7/16+o(1)}$	$= n^{-2/3+o(1)}$	$= n^{-1/3+o(1)}$	$= t^{-16/91+o(1)}$
$G = \mathbb{Z}^d$, high dimensions	$= (c + o(1))n^{2-d}$	$= (c + o(1))n^{2-d}$	$\asymp n^{-2}$	$\asymp t^{-1/2}$
$G = \mathbb{Z}^d_+$, high dimensions	$\asymp n^{1-d}$	$\asymp n^{-d}$	$\asymp n^{-3}$	$\asymp t^{-3/4}$

TABLE 1.1. This table expresses the values of the exponents ρ, η, and δ described in (1.2) on the two-dimensional triangular lattice $\mathbb{T}$, the half-plane $\mathbb{T}_+$, the high-dimensional settings of Definition 1: both the case of the full lattice $\mathbb{Z}^d$ and the half-space subgraph $\mathbb{Z}^d_+$ of these high-dimensional graphs. The constant c, as well as the constants implied by "$\asymp$", depend on the particular lattice considered.

random walks on these clusters and closely related questions about resistances and intrinsic balls, and scaling limits of large open clusters [11, 12, 17, 21, 27].

Finally, we mention several papers that investigate the behavior of high-dimensional percolation not on subgraphs of $\mathbb{Z}^d$, but rather on large tori [14, 15, 32]. These works find, among other things, that percolation on a high-dimensional torus mimics the critical Erdős-Rényi random graph in several ways.

Conditional Version of Our Results

The results of Theorems 1.1 and 1.2 were stated unconditionally, under the "high-dimensional" assumption of Definition 1. As alluded to above, however, physicists believe that many critical exponents should take their mean-field values above the upper critical dimension $d = 6$ on a wide range of graphs. We will give a restatement of Theorems 1.1 and 1.2 here in a conditional form that makes clear that our proofs are valid on any hypercubic or spread-out lattice above the upper critical dimension. The missing ingredient, or in other words the reason why this version of the theorem is conditional, is the two-point function asymptotic provided by the lace expansion.

THEOREM 1.3. *Consider either the hypercubic or spread-out lattice* (*in the latter case,* $\Lambda \geq 1$ *is not required to be large*) *for* $d > 6$*. Suppose that the two-point function asymptotic of* (1.7) *holds. Under this assumption, all the results of Theorems* 1.1 *and* 1.2 *hold.*

The proofs of Theorems 1.1 and 1.2, verbatim, give the result of Theorem 1.3. In these proofs, the bounds (1.7) are used both directly and indirectly. The indirect usage of (1.7) occurs in three ways: through (1.6), through (1.9), and via the application of open cluster cardinality estimates appearing in Section 2.4. The arguments given in the original papers [3] and [22] to establish (1.6) and (1.9) are in fact valid under the assumptions of Theorem 1.3. The tail asymptotics appearing in Section 2.4 are either direct consequences of (1.7) (in the case of Lemma 2.2) or ultimately follow from (1.7) via the "tree graph inequalities" [2] of Aizenman and Newman.

Our arguments do not extend in their present form to $d \le 6$. Here they break down both on their own terms (various error terms are only small when $d > 6$) and because existing of proofs results on which they rely (e.g., (1.9)) explicitly require $d > 6$. It is easy to see that at least one of the asymptotics in Theorem 1.1 must be false for $d \le 5$. Indeed, for the vertices 0 and $n\mathbf{e}_2$ to be connected by an open path in $\mathbb{Z}_+^d$, each of these vertices must have an open arm to distance $\lfloor n/2 \rfloor$ in $\mathbb{Z}_+^d$. If Theorem 1.1 held for $\mathbb{Z}^d$, $d \le 5$, the probability of two such arms would be of order n^{-6}, but this would contradict the two-point function asymptotic of part (b) of that theorem. See [16, sec. 11.4] for more discussion of the upper critical dimension $d = 6$.

1.3 Summary of Some Main Arguments

We use this space to attempt to clarify the structure of certain parts of the proofs of Theorems 1.1 and 1.2. The most complicated and technically involved arguments are those used to establish Theorem 1.2 and the asymptotics on π_H from Theorem 1.1. In the interest of space, we give a detailed outline of the proof of these claims only (and in fact, only for the lower bound on $\pi_H(n)$).

Proof of $\pi_H(n) \ge cn^{-3}$. We first argue for the lower bound on π_H from Theorem 1.1; this argument does not depend on Theorem 1.2 or on any other new results of this paper. The main step of the proof involves establishing the result

$$\begin{aligned}(1.11)\quad &\mathbb{P}\big(\text{there are at least } cn^{4-d} \text{ vertices } \mathbf{x} \in \partial B(2n) \text{ satisfying } \mathbf{x} \overset{B(2n)}{\longleftrightarrow} \partial B(n)\big)\\ &\ge c > 0.\end{aligned}$$

To establish (1.11), we define a set $\mathscr{S}$ of open clusters $\mathcal{C}$ that touch both $\partial B(3n)$ and $\partial B(n)$ and that satisfy

$$\#\big\{\mathbf{x} \in \mathcal{C} \colon \mathbf{x} \overset{B(2n)}{\longleftrightarrow} \partial B(n)\big\} \ge cn^2. \tag{1.12}$$

The clusters in $\mathscr{S}$ also must contain order n^4 vertices of $B(3n)$, the "typical" number of vertices for a cluster of this diameter; see (3.2) for a precise definition of $\mathscr{S}$.

We argue that

$$\mathbb{E}[\#\mathscr{S}] \ge cn^{d-6} \quad \text{and} \quad \mathbb{E}[(\#\mathscr{S})^2] \le Cn^{2d-12}. \tag{1.13}$$

To show the bound on $\mathbb{E}[\#\mathscr{S}]$, we note that

$$\mathbb{E}[\#\{\mathbf{y} \in B(n) \colon \mathbf{y} \leftrightarrow \partial B(3n)\}] \ge cn^d \pi(n) \ge cn^{d-2}. \tag{1.14}$$

The cluster $\mathfrak{C}(\mathbf{y})$ of the typical $\mathbf{y}$ as in (1.14) should satisfy (1.12), with $\mathfrak{C}(\mathbf{y})$ playing the role of $\mathcal{C}$. Indeed, we show that, conditional on $\mathfrak{C}_{B(2n)}(\mathbf{y})$, the probability that $\mathfrak{C}(\mathbf{y})$ reaches $\partial B(3n)$ is at most $\pi(n) \times \#[\mathfrak{C}_{B(2n)}(\mathbf{y}) \cap \partial B(2n)]$ (see Lemma 3.2).

Since $\pi(2n) \geq c\pi(n)$, we must have $\#[\mathfrak{C}_{B(2n)}(\mathbf{y}) \cap \partial B(2n)]$ typically at least cn^2 on $\{\mathbf{y} \leftrightarrow \partial B(2n)\}$, so (1.14) holds. Since each cluster of $\mathscr{S}$ contains at most Cn^4 vertices of $B(n)$, (1.14) implies the bound on $\mathbb{E}[\#\mathscr{S}]$ from (1.13); this bound appears at (3.8) below.

The second moment bound of (1.13) follows by an argument of a similar flavor, but with additional complications; this argument may be found beginning at (3.9). Given (1.13), by the Paley-Zygmund inequality we see

$$\mathbb{P}(\#\mathscr{S} \geq cn^{d-6}) \geq c > 0,$$

so by (1.12), we have (1.11).

On the other hand, each vertex $\mathbf{x} \in \partial B(2n)$ having an open connection to $\partial B(n)$ as in (1.11) has an arm of length order n in an appropriate half-space, namely, a half-space whose boundary hyperplane contains a side of $\partial B(2n)$ in which $\mathbf{x}$ lies. The expected number of $\mathbf{x} \in \partial B(2n)$ having such half-space arms is at most $Cn^{d-1}\pi_H(n)$; comparing to (1.11), we find

$$n^{d-1}\pi_H(n) \geq cn^{4-d} \quad \text{so} \quad \pi_H(n) \geq cn^{-3}.$$

Proof of Theorem 1.2 The core of the proof of Theorem 1.2 is an iterative or inductive argument, where the induction is on a parameter M. The inductive hypothesis is that there exists a constant $c = c(M)$ such that

$$\text{for all } n \geq 1 \text{ and all } \mathbf{x} \in B(n) \quad \mathbb{P}\big(0 \overset{B(Mn)}{\longleftrightarrow} \mathbf{x}\big) \geq c|\mathbf{x}|^{2-d}. \tag{1.15}$$

The base case—i.e., the existence of some $M < \infty$ such that (1.15) holds—follows directly from existing results; see Proposition 4.1. In the inductive step, we show that (1.15) for a given $M > 1$ implies the existence of an $\alpha(M) < M$ such that the analogue of (1.15), with $\alpha(M)$ replacing M and a reduced value of the constant c, holds. (In fact, $\alpha(M)$ is essentially $(M + 1)/2$—see Claim 4.4—so the induction will eventually show (1.15) holds for any $M > 1$.)

For simplicity, let us describe the anatomy of the inductive step in a particular case: suppose we have shown (1.15) for $M = 3$, and we wish to show

$$\mathbb{P}\big(0 \overset{B(2n)}{\longleftrightarrow} n\mathbf{e}_1\big) \geq c' n^{2-d} \tag{1.16}$$

for some constant $c' > 0$. Define the random set

$$X = \big\{\mathbf{x} \in \partial[n\mathbf{e}_1 + B(2n/3)] \colon \mathbf{x} \overset{n\mathbf{e}_1 + B(2n/3)}{\longleftrightarrow} n\mathbf{e}_1\big\};$$

see Lemma 4.6, where the analogous variable is called X_D. By an argument similar to the one used to show (1.14) we establish

$$\begin{aligned}\mathbb{P}\big(&n\mathbf{e}_1 \leftrightarrow \partial[n\mathbf{e}_1 + B(2n/3)], cn^2 < \#[X \cap B(2n/3)] \leq \#X < Cn^2\big)\\ &\geq cn^{-2}\end{aligned} \tag{1.17}$$

for small enough $c > 0$. This is (4.4), where X_Q plays the role of $X \cap B(2n/3)$. The crucial point here is that the vertices in X are close enough to 0 to apply (1.15) to. We will show that, conditional on the event in (1.17), we can often find further open paths connecting some vertex of X to 0.

Let B be the event in (1.17) (compare B_η of (4.4)) and for each $\mathbf{x} \in \partial[n\mathbf{e}_1 + B(2n/3)]$, choose some neighbor $\mathbf{x}' \sim \mathbf{x}$ with $\mathbf{x}' \notin [n\mathbf{e}_1 + B(2n/3)]$. Let

$$Y = \big\{\text{edges } e = \{\mathbf{x}, \mathbf{x}'\} \text{ for some } \mathbf{x} \in X: \\ \mathbf{x}' \overset{B(2n)}{\longleftrightarrow} n\mathbf{e}_1 \text{ and } e \text{ is open and pivotal for } 0 \overset{B(2n)}{\longleftrightarrow} n\mathbf{e}_1\big\}.$$

Clearly $\#Y \le 1$, and if $\#Y = 1$, then we have $0 \overset{B(2n)}{\longleftrightarrow} n\mathbf{e}_1$. We devote the remainder of the argument to showing

$$\mathbb{P}(\#Y = 1 \mid B) \ge cn^{4-d}. \tag{1.18}$$

The result (1.18) establishes (1.16); the argument for (1.16) appears just below the statement of Lemma 4.6.

To show (1.18), we perform a second-moment argument conditional on B. To upper bound $\mathbb{E}[(\#Y)^2 \mid B]$, we note that the conditional probability a particular $\mathbf{x}'$ has a connection to 0 (necessarily avoiding $\mathfrak{C}_{n\mathbf{e}_1+B(n/2)}(n\mathbf{e}_1)$) is at most Cn^{2-d}. See (4.5) and the argument immediately following. Since the number of edges e as in the definition of Y is at most Cn^2 on B, and since $Y \le 1$ almost surely, the upper bound $\mathbb{E}[(\#Y)^2 \mid B] \le Cn^{4-d}$ follows.

We show $\mathbb{E}[\#Y \mid B] \ge cn^{4-d}$ by a similar but more delicate version of the above reasoning. Suppose $\mathbf{x} \in X \cap B(n/4)$ and $\mathbf{x}' \in B(2n/3)$ is a neighbor of $\mathbf{x}$ outside of $n\mathbf{e}_1 + B(2n/3)$ as above (note many such $\mathbf{x}$ exist on the event B). By the induction hypothesis, $\mathbf{x}'$ would have (unconditional) probability at least cn^{2-d} of being connected to 0 in $B(2n)$. To remove the effect of the conditioning and to guarantee the pivotality of the edge $e = \{\mathbf{x}, \mathbf{x}'\}$, we use cluster regularity estimates (see Theorem 2.3), which, combined with the two-point function asymptotic, allow us to establish "enough independence" between the cluster of $\mathbf{x}'$ and the cluster of $n\mathbf{e}_1$. See the proof of part (2B) of Lemma 4.6, where this entire argument is accomplished. This shows the conditional first moment of $\#Y$ is at least order $(n^2)(n^{2-d}) = n^{4-d}$, so

$$\mathbb{P}(\#Y = 1) \ge c\mathbb{P}(B)\frac{\mathbb{E}[\#Y \mid B]^2}{\mathbb{E}[(\#Y)^2 \mid B]} \ge cn^{2-d},$$

completing the proof.

1.4 Organization of the Paper, Constants, and a Standing Assumption

In Section 2, we provide further notation for graphs and subsets of $\mathbb{Z}^d$, along with some further notation related to percolation. We then (in Section 2.3) define the mass-transport method and prove an abstract mass-transport result, Lemma 2.1. Finally, we present useful results on the tail behavior of open cluster cardinalities.

In Section 3, we show the lower bound on the one-arm probability from (a) of Theorem 1.1: $\pi_H(n) \ge cn^{-3}$. In Section 4, we prove Theorem 1.2. Section

5 is devoted to results on cluster "extensibility" that will be crucial for proving the upper bound on the one-arm probability in Theorem 1.1. The proof of these extensibility results relies on Theorem 1.2.

In Section 6, we use the extensibility results of Section 5 to show $\pi_H(n) \le Cn^{-3}$, completing the proof of (a) from Theorem 1.1. This section breaks up into two parts: the choice and analysis of a particular mass-transport rule, and an iterative bound on π_H relying on our mass-transport results. Finally, in Section 7, we bound τ_H and the tail of the half-space cluster size distribution, proving (b) and (c) of Theorem 1.1.

A note about constants. The symbols C, c generally represent positive constants whose values may change from line to line (and even within lines); we sometimes number them to refer to them locally. Other symbols such as ε will sometimes refer to constants depending on context. When we wish to make clear the possible dependence of a constant on a parameter, we do it on a case-by-case basis, for instance, by writing $C = C(K)$. Numbered constants designed to be retained on a long-term or global basis will be denoted a_i, A_i; certain specially labeled constants, such as c_* from Theorem 5.1, will also be referred to several times throughout the paper.

Standing assumption. For the remainder of the paper, we consider critical percolation in one of the high-dimensional settings of Definition 1.

2 Further Definitions and Preliminary Results

In this section, we give some further definitions that will be useful in the course of our proofs (in Sections 2.1 and 2.2). We also state a version of the mass-transport principle (in Section 2.3) and several auxiliary results describing the tail behavior of percolation cluster sizes (in Section 2.4).

2.1 Graph Notation

Recall that we often abuse notation and write $\mathbb{Z}^d$ for the vertex set of the hypercubic or spread-out lattice, as well as for the lattice itself (with similar abuses common for subgraphs of $\mathbb{Z}^d$). We will write $\mathbf{x} \sim \mathbf{y}$ if $\mathbf{x}$ and $\mathbf{y}$ are neighbors in $\mathbb{Z}^d$—that is, if there is an edge e in the edge set of $\mathbb{Z}^d$ with $e = \{\mathbf{x}, \mathbf{y}\}$. The norm notation $\|\mathbf{x}\|$ refers to the ℓ^∞ norm $\|\mathbf{x}\|_\infty$ unless an alternate subscript is given. The symbol $\#A$ denotes the cardinality of a set A.

Define the shifted half-spaces

$$\mathbb{Z}^d_+(n) = \{\mathbf{x} \in \mathbb{Z}^d \colon x(1) \ge n\};$$

in this notation, we have $\mathbb{Z}^d_+ = \mathbb{Z}^d_+(0)$. The corresponding boundary "hyperplane" is

$$S(n) := \left\{\mathbf{x} \in \mathbb{Z}^d_+(n) \colon \mathbf{x} \sim \mathbf{y} \text{ for some } \mathbf{y} \notin \mathbb{Z}^d_+\right\}.$$

Note that $S(n)$ is a (discretized) hyperplane in the usual sense if the ambient graph is the hypercubic lattice; in the case of the spread-out lattice, it is a union of finitely

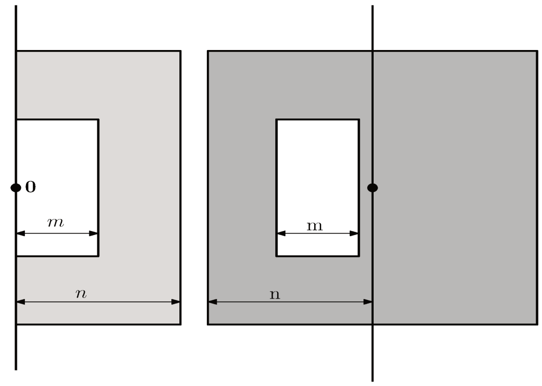

FIGURE 2.1. Left: $Ann_H(m,n)$. Right: $Ann'(m,n)$.

many such hyperplanes. Recalling the ℓ^∞-box is $B(n) := \{\mathbf{x}\colon \|\mathbf{x}\| \le n\}$, we define the shifted box centered at $\mathbf{x}$ by $\mathbf{x} + B(n)$. Note that the above definitions extend to noninteger values of n, so that for instance $B(3.5) = B(3)$.

Generally, for a set of vertices V, we let ∂V denote the interior vertex boundary relative to $\mathbb{Z}^d$:

$$\partial V = \{\mathbf{x} \in V\colon \exists \mathbf{y} \in \mathbb{Z}^d \setminus V \text{ such that } \mathbf{y} \sim \mathbf{x}\}.$$

The half-space analogue of a box will be denoted $B_H(n) := B(n) \cap \mathbb{Z}^d_+$. The boundary of $B_H(n)$, considered as a subgraph of $\mathbb{Z}^d_+$, is written

$$S'(n) := \{\mathbf{x} \in B_H(n)\colon \exists \mathbf{y} \in \mathbb{Z}^d_+ \setminus B_H(n) \text{ with } \mathbf{y} \sim \mathbf{x}\}.$$

This type of graph boundary with respect to a general subgraph of $\mathbb{Z}^d$ will often arise, so we introduce some notation to describe it. If $A_0 \subseteq A_1$ are subsets of (the vertices of) $\mathbb{Z}^d$, let

$$\partial_{A_1} A_0 = \{\mathbf{x} \in A_0\colon \text{there is some } \mathbf{y} \in A_1 \setminus A_0 \text{ with } \mathbf{y} \sim \mathbf{x}\}. \tag{2.1}$$

With the notation of (2.1), we can write $S'(n) = \partial_{\mathbb{Z}^d_+} B_H(n)$. We introduce one more standard box, $Rect(n) := [0,n] \times [-4n, 4n]^{d-1}$, as a fattened version of $B_H(n)$.

The annulus $Ann(m,n) := B(n) \setminus B(m)$. The corresponding half-space annuli are $Ann_H(m,n) := B_H(n) \setminus B_H(m)$. We will often refer to shifted annuli, where one side of the inner box lies along $S(0)$. Namely, we define $B_-(n) = -\mathbf{e}_1 - B_H(n)$, and (for $n \ge m$) $Ann'(m,n) = [B(n) \setminus B_-(m)]$ (see Figure 2.1). The outer boundaries of annuli are defined as the vertex boundaries of their outer boxes, relative to the ambient subgraph: $\partial^+ Ann(m,n) = \partial^+ Ann'(m,n) = \partial B(n)$, and $\partial^+ Ann_H(m,n) = S'(n)$. Similarly, the inner boundary $\partial^- Ann(m,n) = \partial(\mathbb{Z}^d \setminus B(m))$, with analogous definitions for the other annuli: $\partial^- Ann'(m,n) = \partial(\mathbb{Z}^d \setminus B_-(m))$, and $\partial^- Ann_H(m,n) = \partial_{\mathbb{Z}^d_+}(\mathbb{Z}^d_+ \setminus B_H(m))$.

2.2 Correlation inequalities and More about Percolation

Recall that a generic percolation configuration is written $\omega = (\omega_e)_e$, where the random variables ω_e are i.i.d. Bernoulli(p_c), and the edge e is open (resp. closed) if $\omega_e = 1$ (resp. $\omega_e = 0$). The open graph is the random subgraph of $\mathbb{Z}^d$ whose vertex set is $\mathbb{Z}^d$ and whose edge set is $\{e = \{\mathbf{x}, \mathbf{y}\} \colon \mathbf{x} \sim \mathbf{y},\ \omega_e = 1\}$. Recall that open clusters are subgraphs of this open graph; we sometimes identify an open cluster with its vertex set.

In many places where the two-point function asymptotics of (1.7) appear, we use the convention $\|0\|^{2-d} = 1$; this minor abuse allows us to avoid some cumbersome expressions when summing products of τ.

An edge e is said to be *pivotal* for an event A in a configuration ω if changing the status of ω_e while leaving the rest of ω fixed changes whether or not A occurs. In other words, letting $\omega^{(e,+)}$ (resp. $\omega^{(e,-)}$) denote the configuration agreeing with ω except possibly at ω_e, where it takes the value $\omega_e^{(e,+)} = 1$ (resp. 0), then e is pivotal for A in ω if $\mathbb{1}_A(\omega^{(e,+)}) \neq \mathbb{1}_A(\omega^{(e,-)})$. OK to change **1** to $\mathbb{1}$?

Above, we defined the half-space one-arm probability by

$$\pi_H(n) := \mathbb{P}\big(0 \overset{\mathbb{Z}^d_+}{\longleftrightarrow} S'(n)\big).$$

There is a possible alternate definition of π_H: namely, $\mathbb{P}(0 \overset{\mathbb{Z}^d_+}{\longleftrightarrow} S(n))$, the probability that there is a half-space arm to distance n in the $\mathbf{e}_1$-direction. We note that the arguments in this paper in fact show that both of these probabilities are asymptotic to n^{-3}; see (7.1) and the surrounding discussion below.

Another minor issue arises when discussing $\pi_H(n)$ in the setting of the spread-out lattice. Since edges extend ℓ^∞ distance Λ here, each vertex $\mathbf{x}$ with $0 \le x(1) \le \Lambda - 1$ is on the "hyperplane" forming the boundary of $\mathbb{Z}^d_+$. But the probability that $\mathfrak{C}_{\mathbb{Z}^d_+}(\mathbf{x})$ has diameter at least n is not exactly $\pi_H(n)$. This issue is remedied by the following observation, whose proof is immediate:

(2.2) for each fixed $\mathbf{x}$ such that $\|\mathbf{x}\|_\infty \le \Lambda$,
$$\mathbb{P}(\exists \mathbf{y} \in \mathfrak{C}_{\mathbb{Z}^d_+}(\mathbf{x}) \colon \|\mathbf{y}\|_\infty \ge n) \asymp \pi_H(n).$$

We will make use of (2.2) sometimes without explicit reference.

We will make reference to the *Harris-FKG* (or "FKG") and *BK-Reimer* (or "BK") correlation inequalities. We direct the reader to [5, chap. 2] for statements of, and references to the literature on, these and related inequalities.

2.3 Mass Transport

Our proof of the upper bound $\pi_H(n) \le Cn^{-3}$ of Theorem 1.1 involves considering the point of view of a boundary vertex of a spanning cluster of a large box—that is, the configuration seen from a typical $\mathbf{x} \in \partial B(n)$ lying in such a spanning cluster. This is made precise by the following lemma, which is an application of the general mass-transport technique. See [24, chap. 8] for more information about mass transport.

LEMMA 2.1. *Let $h(\mathbf{x},\mathbf{y})$ be a function from $\mathbb{Z}^d \times \mathbb{Z}^d$ to $[0,\infty]$ that is translation-invariant in the following sense: $h(\mathbf{x}+\mathbf{z},\mathbf{y}+\mathbf{z}) = h(\mathbf{x},\mathbf{y})$ for all $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{Z}^d$. Then, for any $\mathbf{x} \in \mathbb{Z}^d$,*

$$\sum_{\mathbf{z}} h(0,\mathbf{z}) = \sum_{\mathbf{z}} h(\mathbf{x},\mathbf{z}) = \sum_{\mathbf{z}} h(\mathbf{z},\mathbf{x}) = \sum_{\mathbf{z}} h(\mathbf{z},0).$$

PROOF. Note that $h(0,\mathbf{z}) = h(0-\mathbf{z},\mathbf{z}-\mathbf{z}) = h(-\mathbf{z},0)$, so

$$\sum_{\mathbf{z}} h(0,\mathbf{z}) = \sum_{\mathbf{z}} h(-\mathbf{z},0) = \sum_{\mathbf{z}} h(\mathbf{z},0).$$

The fact that the value is unchanged when replacing 0 by $\mathbf{x}$ follows similarly, again using the translation invariance of h. □

Lemma 2.1 will be applied to particular *mass-transport rules*. A mass-transport rule is a function $\mathfrak{m}(\cdot,\cdot)$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ assigning to each pair $\mathbf{x},\mathbf{y}$ a nonnegative random variable $\mathfrak{m}(\mathbf{x},\mathbf{y}) = \mathfrak{m}[\omega](\mathbf{x},\mathbf{y})$ for each percolation configuration in a translation-covariant way. In other words, for almost every realization $\omega = (\omega_e)_e$ of the percolation process, we have

$$\mathfrak{m}[\omega](\mathbf{x}+\mathbf{z},\mathbf{y}+\mathbf{z}) = \mathfrak{m}[\Theta_{\mathbf{z}}\omega](\mathbf{x},\mathbf{y}),$$

where $(\Theta_{\mathbf{z}}\omega)_e = \omega_{e+\mathbf{z}}$ (and addition of a vertex and an edge is defined by $\{\mathbf{a},\mathbf{b}\} + \mathbf{z} = \{\mathbf{a}+\mathbf{z},\mathbf{b}+\mathbf{z}\}$).

Such an $\mathfrak{m}(\mathbf{x},\mathbf{y})$ is referred to as the "mass sent from $\mathbf{x}$ to $\mathbf{y}$." For a given choice of $\mathfrak{m}$, we apply Lemma 2.1 to $h(\mathbf{x},\mathbf{y}) = \mathbb{E}\mathfrak{m}(\mathbf{x},\mathbf{y})$ (translation invariance of h follows from the translation covariance of $\mathfrak{m}$). In this case, letting $\mathsf{send} = \sum_{\mathbf{z}} \mathfrak{m}(0,\mathbf{z})$ and $\mathsf{get} = \sum_{\mathbf{z}} \mathfrak{m}(\mathbf{z},0)$, the lemma states

$$\mathbb{E}\mathsf{send} = \mathbb{E}\mathsf{get}.$$

2.4 Open Cluster Cardinality Estimates

In this subsection, we provide several estimates on the cardinality of open clusters within boxes. We begin with a pair of well-known moment estimates in the following lemma. These will be useful for controlling the probability of events that imply that the cluster of a particular site is "large."

LEMMA 2.2. *We have*

$$\mathbb{E}[\#\mathfrak{C}(0) \cap B(n)] \asymp n^2, \quad \mathbb{E}\big[(\#\mathfrak{C}(0) \cap B(n))^2\big] \le Cn^6.$$

PROOF. The first moment is just $\sum_{\mathbf{x}\in B(n)} \tau(0,\mathbf{x})$, and the asymptotic follows by summing (1.7). The second moment bound follows using the "tree graph" method of Aizenman and Newman [2], decomposing $\mathbb{P}(\mathbf{x} \leftrightarrow 0, \mathbf{y} \leftrightarrow 0)$ based on the meeting point of the open paths from $\mathbf{x}$ to 0 and from $\mathbf{y}$ to 0. See, for instance, lemma 2.1 from [22]. □

The remainder of this subsection is devoted to the presentation of specialized cluster regularity results. Roughly speaking, these will allow us to argue that, even when the cluster of a particular site has large diameter, its intersection with a mesoscopic cube of sidelength s is very unlikely to contain more than $s^{4+\varepsilon}$ vertices. The fact that open clusters are so sparse will allow us to show that, when we explore the open cluster of a particular site, different stages of the exploration proceed approximately independently.

The statement of these regularity results requires further terminology. Consider a vertex $\mathbf{z}$ within some connected vertex set $D \subseteq \mathbb{Z}^d$ and some subset $Q \subseteq \partial D$ of its boundary. D is generally a box or annulus. We introduce the notation $X_Q(D, \mathbf{z})$ for the number of "boundary vertices" of $\mathfrak{C}_D(\mathbf{z})$ on Q:

$$\text{For } \mathbf{z} \in D \subseteq \mathbb{Z}^d \text{ and } Q \subseteq \partial D, \text{ let} \\ \#\{\mathbf{x} \in Q : \mathbf{x} \overset{D}{\longleftrightarrow} \mathbf{z}\} = \#[\mathfrak{C}_D(\mathbf{z}) \cap Q. \tag{2.3}$$

Our regularity result says roughly that, if X_Q is large, the clusters of most of the vertices contributing to X_Q are not larger than their typical size. For $s > 0$ and $\mathbf{x} \in \mathbb{Z}^d$ arbitrary, define the event

$$\mathcal{T}_s(\mathbf{x}) := \left\{\#(\mathfrak{C}(\mathbf{x}) \cap (\mathbf{x} + B(s))) \ < s^4 \log^7 s\right\}.$$

DEFINITION 3. Let $D \subseteq \mathbb{Z}^d$ and $\mathbf{x} \in \partial D$. For $s > 0$, we say that $\mathbf{x}$ is s-bad with respect to D if

$$\mathbb{P}(\mathcal{T}_s(\mathbf{x}) \mid \mathfrak{C}_D(\mathbf{x})) \le 1 - \exp(-\log^2 s).$$

We say that $\mathbf{x}$ is K-irregular with respect to D if $\mathbf{x} \in \partial D$ and there is some $s \ge K$ such that $\mathbf{x}$ is s-bad with respect to D. Otherwise, $\mathbf{x}$ is said to be K-regular with respect to D. We denote the set of K-regular vertices of D by $REG_D(K)$.

We define the "irregular version" of $X_Q(D, \mathbf{z})$, which counts the number of boundary vertices whose clusters are abnormally large (recall that $D \subseteq \mathbb{Z}^d$ and $Q \subseteq \partial D$):

$$X_Q^{K\text{-}irr}(D, \mathbf{z}) := \#\{\mathbf{x} \in \mathfrak{C}_D(z) \cap Q : \mathbf{x} \text{ is } K\text{-irregular with respect to } D\}. \tag{2.4}$$

The following lemma provides a tail bound for X_Q^{K-irr} when X_Q is large for a growing sequence of annuli or boxes D. Suppose that for each n, the set D is a dilation of the same box or annulus—that is, D is a translate of $\prod_{i=1}^d [\alpha_i n, \beta_i n]$, or the annuli $Ann(cn, n)$, $Ann'(cn, n)$, or $Ann_H(cn, n)$, where the α_i's, β_i's, or c are fixed. We say Q is a dilated subrectangle of ∂D for each n if Q is a $(d-1)$-dimensional rectangle in ∂D with nondegenerate sides and if, for each n, Q is dilated and translated as D is—i.e., as n increases, Q changes by the same dilations/translations as D.

LEMMA 2.3 (Cluster regularity). *Consider a sequence of growing (in n) domains D that are dilations/translations of the same box or annulus having sidelength order n as in the above paragraph. Suppose that Q is a dilated subrectangle of ∂D, also* *as above. There exist constants* $C > c > 0$ *and* $K_0 > 0$ *such that for any* n, M *and any* $K \geq K_0$*, the following holds*: *Uniformly in* $\mathbf{z} \in D$*, we have*

$$\mathbb{P}\left(X_Q(D,\mathbf{z}) \geq M \text{ and } X_Q^{K\text{-}irr}(D,\mathbf{z}) \geq \frac{1}{2} X_Q(D,\mathbf{z}) \right) \leq C n^d \exp(-c \log^2 M),$$

where X_Q *and* $X_Q^{K\text{-}irr}$ *are defined at* (2.3) *and* (2.4) *respectively.*

A version of Lemma 2.3 in the case that D is a cube $B(n)$ and $Q = \partial B(n)$ was proved as theorem 4 of [22]. Lemma 2.3 follows by an argument similar to the proof of that result; we omit the details. The main use of Lemma 2.3 will be in "extensibility" arguments allowing the enlargement of the cluster of a site $\mathbf{x}$, conditional on the value of $\mathfrak{C}_D(\mathbf{x})$.

3 Lower Bound on $\pi_H(n)$

Our main goal in this section is to prove the lower bound of part (a) of Theorem 1.1:

PROPOSITION 3.1. *There is a constant* $c = c(d)$ *such that* $\pi_H(n) \geq cn^{-3}$ *for all* $n \geq 1$.

Recall that $Ann(m,n) = B(n) \setminus B(m)$ is the annulus of in-radius m and out-radius n.

DEFINITION 4. For $r, s \in \mathbb{N}$ with $r < s$, let $[\![Ann(r,s)]\!]$ be the set of all open clusters of $\mathbb{Z}^d$ that intersect both $B(r)$ and $\partial B(s)$.

The clusters belonging to $[\![Ann(r,s)]\!]$ will be called $Ann(r,s)$-spanning clusters. Note that connectivity in the above definition is determined relative to $\mathbb{Z}^d$ and not the annulus; in particular, if $\mathcal{C} \in [\![Ann(r,s)]\!]$, then $\mathcal{C} \cap Ann(r,s)$ may be a disconnected set. We will mostly work with the annulus $Ann(n,3n)$. For $\mathcal{C} \in [\![Ann(n,3n)]\!]$, let $X_{\mathcal{C}}$ denote the number of vertices of $\partial B(2n) \cap \mathcal{C}$ that can access $\partial B(n)$ via open paths within $B(2n)$. More precisely,

$$X_{\mathcal{C}} := \#\{\mathbf{x} \in \partial B(2n) \cap \mathcal{C} : \mathbf{x} \overset{B(2n)}{\longleftrightarrow} B(n)\}. \tag{3.1}$$

Next we define a collection $\mathscr{S}$ of "regular" annulus spanning clusters with certain regularity properties. Roughly speaking, $\mathcal{C} \in \mathscr{S}$ if:

(1) $X_{\mathcal{C}}$ is large enough so that $\mathcal{C}$ is likely to extend to the boundary of a larger ball of radius $\Theta(n)$, say, $B(5n)$. That is, $X_{\mathcal{C}} \gtrsim n^2$.
(2) $\mathcal{C}$ contains $\approx n^4$ vertices in boxes of sidelength $\approx n$.

To be more precise, let $\eta > 0$ and

$$\begin{aligned} \mathscr{S}_\eta := \{\mathcal{C} \in [\![Ann(n,3n)]\!] \colon X_{\mathcal{C}} \geq \eta n^2,\ \#[\mathcal{C} \cap Ann(3n,5n)] \geq \eta n^4, \\ \#[\mathcal{C} \cap B(5n)] \leq \eta^{-1} n^4\}. \end{aligned} \tag{3.2}$$

Note that $\mathscr{S}_\eta$ depends on n.

The following lemma will be useful for showing that $X_{\mathcal{C}}$ is typically large (and thereby proving the existence of many points with half-space arms). We state it in a general form so that later in the paper it can also be applied to the case of, for instance, nested half-space boxes.

LEMMA 3.2. *Let $A_0 \subseteq A_1 \subseteq \mathbb{Z}^d$ be arbitrary finite vertex sets with $\mathbf{z} \in A_0$. Let $B \subseteq \partial A_1$ be a distinguished portion of the boundary of A_1, and suppose that the ℓ^∞ distance from A_0 to B is λ. Recall the definition of $\partial_{A_1} A_0$ from* (2.1). *Let $\mathcal{C}$ be a set of vertices of A_0 that is admissible in the sense that $\mathbb{P}(\mathfrak{C}_{A_0}(\mathbf{z}) = \mathcal{C}) > 0$, and suppose further that $\#[\mathcal{C} \cap \partial_{A_1} A_0] = M$. We then have*

$$\mathbb{P}\big(\mathbf{z} \xleftrightarrow{A_1} B \mid \mathfrak{C}_{A_0}(\mathbf{z}) = \mathcal{C}\big) \leq M\pi(\lambda).$$

PROOF. For a vertex set $\mathcal{C}$ of A_0, note that the event $\{\mathfrak{C}_{A_0}(\mathbf{z}) = \mathcal{C}\}$ depends only on the status of edges having either both endpoints in $\mathcal{C}$ or one endpoint in $\mathcal{C}$ and one endpoint in $A_0 \setminus \mathcal{C}$. Conditional on $\{\mathfrak{C}_{A_0}(\mathbf{z}) = \mathcal{C}\}$, if $\{\mathbf{z} \xleftrightarrow{A_1} B\}$ occurs, then there must be some $\mathbf{y} \in \mathcal{C} \cap \partial_{A_1} A_0$ (see Figure 3.1 for a sketch) such that $\mathbf{y} \leftrightarrow B$ off $\mathcal{C}$. That is, $\mathbf{y}$ has an open path (in $\mathbb{Z}^d$) to B which touches $\mathcal{C}$ only at $\mathbf{y}$. We thus have the inclusion

$$\begin{aligned}\big\{\mathbf{z} \xleftrightarrow{A_1} B,\ \mathfrak{C}_{A_0}(\mathbf{z}) = \mathcal{C}\big\} &\subseteq \{\mathfrak{C}_{A_0}(\mathbf{z}) = \mathcal{C}\} \\ &\quad \cap \{\exists \mathbf{y} \in \mathcal{C} \cap \partial_{A_1} A_0 \text{ with } \mathbf{y} \leftrightarrow B \text{ off } \mathcal{C}\}.\end{aligned} \tag{3.3}$$

For any fixed $\mathcal{C}$, the events on the right-hand side of (3.3) are independent, and the probability that any $\mathbf{y} \in \mathcal{C} \cap \partial_{A_1} A_0$ has such a connection is clearly bounded above by $\pi(\lambda)$. Thus, for any set $\mathcal{C} \subseteq A_0$ that is admissible as in the statement of the lemma, we have

$$\begin{aligned}&\mathbb{P}\big(\mathbf{z} \xleftrightarrow{A_1} B, \mathfrak{C}_{A_0}(\mathbf{z}) = \mathcal{C}\big) \\ &\quad = \mathbb{P}\big(\mathbf{z} \xleftrightarrow{A_1} B \mid \mathfrak{C}_{A_0}(\mathbf{z}) = \mathcal{C}\big)\mathbb{P}\big(\mathfrak{C}_{A_0}(\mathbf{z}) = \mathcal{C}\big) \bigcup_{y \in [\mathcal{C} \cap \partial_{A_1} A_0]} \\ &\quad \leq \mathbb{P}\big(\mathfrak{C}_{A_0}(\mathbf{z}) = \mathcal{C}\big)\mathbb{P}\Big(\bigcup_{y \in [\mathcal{C} \cap \partial_{A_1} A_0]} \{y \leftrightarrow B \text{ off } \mathcal{C}\}\Big) \\ &\quad \leq \mathbb{P}\big(\mathfrak{C}_{A_0}(\mathbf{z}) = \mathcal{C}\big) \sum_{y \in [\mathcal{C} \cap \partial_{A_1} A_0]} \mathbb{P}(y \leftrightarrow B \text{ off } \mathcal{C}) \\ &\quad \leq M\pi(\lambda)\mathbb{P}(\mathfrak{C}_{A_0}(\mathbf{z}) = \mathcal{C}).\end{aligned}$$

□

Our main technical work in the remainder of this section is to show the following.

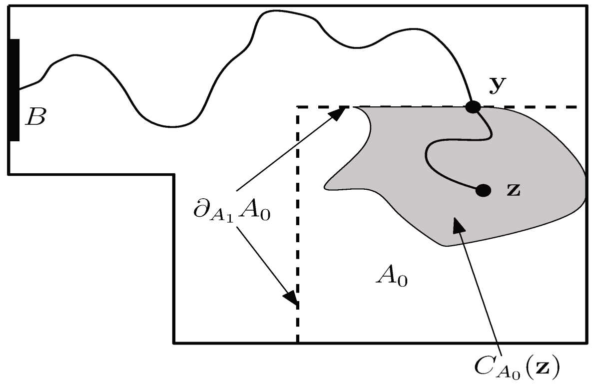

FIGURE 3.1. Depiction of a definition appearing in Lemma 3.2. This is an instance of the event $\{\mathbf{y} \leftrightarrow B\} \circ \{\mathbf{y} \xleftrightarrow{A_0} \mathbf{z}\}$ for $\mathbf{y} \in \partial_{A_1} A_0$ as illustrated.

LEMMA 3.3. *There exist $\eta_0 > 0$ and positive constants $c_1 = c_1(\eta_0, d)$ and $c_2 = c_2(\eta_0, d)$ such that, uniformly in n and $\eta \le \eta_0$,*

$$\mathbb{P}\big(\#\mathscr{S}_\eta \ge c_1 \eta n^{d-6}\big) \ge c_2.$$

We first assume the truth of Lemma 3.3 and use it to prove Proposition 3.1.

PROOF OF PROPOSITION 3.1. Let $\eta_0, c_1 = c_1(\eta_0, d)$ and $c_2 = c_2(\eta_0, d)$ be the constants from Lemma 3.3. First note that if $\mathcal{C}, \mathcal{C}' \in [\![Ann(n, 3n)]\!]$ are distinct, then $X_\mathcal{C}$ and $X_{\mathcal{C}'}$ count disjoint sets of vertices. So for any $\eta > 0$,

$$\#\big\{\mathbf{x} \in \partial B(2n) : \mathbf{x} \xleftrightarrow{B(2n)} B(n)\big\} \ge \sum_{\mathcal{C} \in \mathscr{S}_\eta} X_\mathcal{C}, \tag{3.4}$$

and hence $\#\{\mathbf{x} \in \partial B(2n) : \mathbf{x} \xleftrightarrow{B(2n)} B(n)\} \ge c_1 \eta^2 n^{d-4}$ on the event $\{\#\mathscr{S}_\eta \ge c_1 \eta n^{d-6}\}$. In view of Lemma 3.3, the above event has probability $\ge c_2$ for all $\eta \le \eta_0$. Therefore, for such an η,

$$\begin{aligned}
&\mathbb{E}\#\big\{\mathbf{x} \in \partial B(2n) : \mathbf{x} \xleftrightarrow{B(2n)} B(n)\big\} \\
&\quad \ge \mathbb{E}\big[\#\{\mathbf{x} \in \partial B(2n) : \mathbf{x} \xleftrightarrow{B(2n)} B(n)\}\mathbf{1}_{\{\#\mathscr{S}_\eta \ge c_1 \eta n^{d-6}\}}\big] \\
&\quad \ge c_1 \eta^2 n^{d-4} \mathbb{P}\big(\#\mathscr{S}_\eta \ge c_1 \eta n^{d-6}\big) \ge c_1 c_2 \eta^2 n^{d-4}.
\end{aligned} \tag{3.5}$$

On the other hand, if a vertex $\mathbf{x} \in \partial B(2n)$ satisfies $\mathbf{x} \xleftrightarrow{B(2n)} B(n)$, then $\mathbf{x}$ must have a half-space arm to distance n (in fact, a half-space arm in a "rectangle" similar to that of $Rect(n)$; we return to this point in the proof of Proposition 7.1).

Using (2.2), we obtain an upper bound for the expectation appearing in the last display:

$$\mathbb{E}\#\{\mathbf{x} \in \partial B(2n) : \mathbf{x} \overset{B(2n)}{\longleftrightarrow} B(n)\} \le C\pi_H(n)[\#\partial B(2n)] \le C_1 n^{d-1}\pi_H(n) \tag{3.6}$$

for some constant $C_1 = C_1(d)$. Comparing (3.4), (3.5), and (3.6) gives $\pi_H(n) \ge (c_1c_2\eta^2/C_1)n^{-3}$, which completes the proof of the proposition. □

To complete the proof of Proposition 3.1, it suffices to prove Lemma 3.3. The key fact that we need to prove Lemma 3.3 is the following. Recall that for $\mathbf{x} \in \mathbb{Z}^d$, $\mathfrak{C}(\mathbf{x})$ is the open cluster containing $\mathbf{x}$.

LEMMA 3.4. *There exists $\eta_0 > 0$ and $c = c(\eta_0, d) > 0$ such that for all $\eta \le \eta_0$ and $\mathbf{x} \in B(n/2)$, $\mathbb{P}(\mathfrak{C}(\mathbf{x}) \in \mathscr{S}_\eta) \ge cn^{-2}$.*

First, we show how to use Lemma 3.4 to prove Lemma 3.3; we then prove Lemma 3.4.

PROOF OF LEMMA 3.3. We apply Lemma 3.4 to obtain $\eta_0, c(\eta_0, d)$ such that

$$\mathbb{P}(\mathfrak{C}(\mathbf{x}) \in \mathscr{S}_\eta) \ge cn^{-2} \quad \text{uniformly in } n, \eta \le \eta_0 \text{ and } \mathbf{x} \in B(n/2). \tag{3.7}$$

Now we will use a second-moment argument for $\#\mathscr{S}_\eta$. First note that

$$\sum_{\mathbf{x} \in B(n/2)} \mathbb{1}_{\{\mathfrak{C}(\mathbf{x}) \in \mathscr{S}_\eta\}} = \sum_{\mathcal{C} \in \mathscr{S}_\eta} \#[\mathcal{C} \cap B(n/2)] \le \eta^{-1} n^4 \#\mathscr{S}_\eta.$$

The last inequality follows from the fact that $\#[\mathcal{C} \cap B(5n)] \le \eta^{-1}n^4$ for all $\mathcal{C} \in \mathscr{S}_\eta$. From the last display and (3.7),

$$\mathbb{E}\#\mathscr{S}_\eta \ge \frac{\eta}{n^4} \sum_{\mathbf{x} \in B(n/2)} \mathbb{P}(\mathfrak{C}(\mathbf{x}) \in \mathscr{S}_\eta) \ge \frac{\eta}{n^4} C_2 n^d cn^{-2} = C_2 c\eta n^{d-6} \tag{3.8}$$

for some constant $C_2 = C_2(d)$. Now we estimate the second moment of $\#\mathscr{S}_\eta$. Note that

$$\sum_{\mathbf{x} \in Ann(3n,5n)} \mathbb{1}_{\{\mathfrak{C}(\mathbf{x}) \in \mathscr{S}_\eta\}} = \sum_{\mathcal{C} \in \mathscr{S}_\eta} \#[\mathcal{C} \cap Ann(3n, 5n)] \ge \eta n^4 \#\mathscr{S}_\eta.$$

The last inequality follows from the fact that $\#[\mathcal{C} \cap Ann(3n, 5n)] \ge \eta n^4$ for all $\mathcal{C} \in \mathscr{S}_\eta$. Thus,

$$\#\mathscr{S}_\eta \le \frac{1}{\eta n^4} \sum_{\mathbf{x} \in Ann(3n,5n)} \mathbb{1}_{\{\mathfrak{C}(\mathbf{x}) \in \mathscr{S}_\eta\}} \tag{3.9}$$

$$\text{and so} \quad (\#\mathscr{S}_\eta)^2 \le \frac{1}{\eta^2 n^8} \sum_{\mathbf{x},\mathbf{y} \in Ann(3n,5n)} \mathbb{1}_{\{\mathfrak{C}(\mathbf{x}), \mathfrak{C}(\mathbf{y}) \in \mathscr{S}_\eta\}}.$$

For each of the above summands there are two possibilities based on whether $\mathfrak{C}(\mathbf{x})$ and $\mathfrak{C}(\mathbf{y})$ intersect or not. If $\mathfrak{C}(\mathbf{x}), \mathfrak{C}(\mathbf{y}) \in \mathscr{S}_\eta$ and $\mathfrak{C}(\mathbf{x}) \cap \mathfrak{C}(\mathbf{y}) = \varnothing$, then $\mathbf{x}$ and $\mathbf{y}$ are connected to $B(n)$ using disjoint paths, so using the BK inequality

$$\begin{aligned}\mathbb{P}(\mathfrak{C}(\mathbf{x}), \mathfrak{C}(\mathbf{y}) \in \mathscr{S}_\eta,\ \mathfrak{C}(\mathbf{x}) \cap \mathfrak{C}(\mathbf{y}) = \varnothing) &\le \mathbb{P}(\{\mathbf{x} \leftrightarrow B(n)\} \circ \{\mathbf{y} \leftrightarrow B(n)\}) \\ &\le \mathbb{P}(\mathbf{x} \leftrightarrow B(n))\mathbb{P}(\mathbf{y} \leftrightarrow B(n)).\end{aligned}$$

On the other hand, note that for any $\mathbf{x} \in Ann(3n, 5n)$,

$$\begin{aligned}&\sum_{\mathbf{y} \in Ann(3n,5n)} \mathbb{1}_{\{\mathfrak{C}(\mathbf{x}), \mathfrak{C}(\mathbf{y}) \in \mathscr{S}_\eta,\ \mathfrak{C}(\mathbf{x}) \cap \mathfrak{C}(\mathbf{y}) \neq \varnothing\}} \\ &\quad \le \sum_{\mathbf{y} \in \mathfrak{C}(\mathbf{x}) \cap B(5n)} \mathbb{1}_{\{\mathfrak{C}(\mathbf{x}) \in \mathscr{S}_\eta\}} = \#(\mathfrak{C}(\mathbf{x}) \cap B(5n)) \mathbb{1}_{\{\mathfrak{C}(\mathbf{x}) \in \mathscr{S}_\eta\}} \le \eta^{-1} n^4 \mathbb{1}_{\{\mathfrak{C}(\mathbf{x}) \in \mathscr{S}_\eta\}}\end{aligned}$$

by the definition of $\mathscr{S}_\eta$. Combining the last three displays,

$$\begin{aligned}\mathbb{E}\big[(\#\mathscr{S}_\eta)^2\big] \le \frac{1}{\eta^2 n^8}\Big[&\sum_{\mathbf{x} \in Ann(3n,5n)} \eta^{-1} n^4 \mathbb{P}(\mathfrak{C}(\mathbf{x}) \in \mathscr{S}_\eta) \\ &+ \sum_{\mathbf{x},\mathbf{y} \in Ann(3n,5n)} \mathbb{P}(\mathbf{x} \leftrightarrow B(n)) \mathbb{P}(\mathbf{y} \leftrightarrow B(n)) \Big].\end{aligned}$$

Using (1.9), we have $\mathbb{P}(\mathbf{x} \leftrightarrow B(n)) \le A_2 n^{-2}$ uniformly in $\mathbf{x} \in Ann(3n, 5n)$. Since $\#B(5n) = (5n + 1)^d$, the two terms in the right-hand side of the above display are at most $C\eta^{-3} n^{d-6}$ and $C\eta^{-2} n^{2d-12}$ respectively. Therefore, there is a constant $C_3 > 0$ such that

$$\mathbb{E}\big[(\#\mathscr{S}_\eta)^2\big] \le C_3 \eta^{-3} n^{2d-12}.$$

Using the estimates in the above display and (3.8), and applying the Paley-Zygmund inequality,

$$\begin{aligned}\mathbb{P}\left(\#\mathscr{S}_\eta \ge \frac{1}{2} C_2 \eta c n^{d-6}\right) &\ge \mathbb{P}\left(\#\mathscr{S}_\eta \ge \frac{1}{2}\mathbb{E}\#\mathscr{S}_\eta\right) \\ &\ge \frac{1}{4} \frac{(\mathbb{E}\#\mathscr{S}_\eta)^2}{\mathbb{E}[(\#\mathscr{S}_\eta)^2]} \\ &\ge \frac{1}{4} \frac{C_2^2 \eta^2 c^2 n^{2d-12}}{C_3 \eta^{-3} n^{2d-12}} = \frac{C_2^2 c^2}{4C_3} \eta^5.\end{aligned}$$

While the above bound depends on η, we can replace it by a constant for $\eta \le \eta_0$ since the probability appearing in the statement of Lemma 3.3 is decreasing in η. This completes the proof of the lemma. □

Lastly, we need to show Lemma 3.4. The lemma follows from moment estimates and Lemma 3.2, which says that clusters of boxes with a small number of boundary vertices are likely to die out.

PROOF OF LEMMA 3.4. Since $\#\mathscr{S}_\eta$ is monotone in η, it suffices to show that there is a $\eta_0 > 0$ and $c = c(\eta_0, d) > 0$ such that $\mathbb{P}(\mathfrak{C}(\mathbf{x}) \in \mathscr{S}_{\eta_0}) \geq cn^{-2}$ for all $\mathbf{x} \in B(n/2)$. The proof consists of the following steps:

Step 1. There are positive constants C_1, C_2 depending on d such that for any $\mathbf{x} \in B(n/2)$,

$$\mathbb{P}\big(\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n, 5n)] > \eta n^4\big) \geq (C_1 - C_2\eta^2)n^{-2}. \tag{3.10}$$

Step 2. There are positive constants C_1, C_2, C_3 depending on d such that for any $\mathbf{x} \in B(n/2)$,

$$\begin{aligned} &\mathbb{P}\big(\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n, 5n)] > \eta n^4, \#\mathfrak{C}(\mathbf{x}) \cap B(5n) \leq \eta^{-1}n^4\big) \\ &\quad \geq \big(C_1 - C_2\eta^2 - C_3\eta\big)n^{-2}. \end{aligned} \tag{3.11}$$

Step 3. There are positive constants C_1, C_2, C_4 depending on d such that for any $\mathbf{x} \in B(n/2)$,

$$\mathbb{P}(\mathfrak{C}(\mathbf{x}) \in \mathscr{S}_\eta) \geq (C_1 - C_2\eta^2 - C_4\eta)n^{-2}. \tag{3.12}$$

The proof of the lemma follows from Step 3 by taking $c(\eta, d) := C_1 - C_2\eta^2 - C_4\eta$ and choosing $\eta_0 > 0$ small enough so that $c(\eta_0, d) > 0$. Now we give the proof of the three steps.

STEP 1. We will use a second moment argument for the distribution of $\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n, 5n)]$ given $\{\mathbf{x} \leftrightarrow \partial B(3n)\}$. First note that

$$\begin{aligned} &\mathbb{E}(\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n, 5n)] | \mathbf{x} \leftrightarrow \partial B(3n)) \\ &\quad = \sum_{\mathbf{y} \in Ann(3n,5n)} \mathbb{P}(\mathbf{x} \leftrightarrow \mathbf{y} | \mathbf{x} \leftrightarrow \partial B(3n)) \\ &\quad = \sum_{\mathbf{y} \in Ann(3n,5n)} \frac{\mathbb{P}(\mathbf{x} \leftrightarrow \mathbf{y})}{\mathbb{P}(\mathbf{x} \leftrightarrow \partial B(3n))}, \end{aligned}$$

as $\mathbf{x} \leftrightarrow \mathbf{y}$ implies $\mathbf{x} \leftrightarrow \partial B(3n)$ for all $\mathbf{y} \in Ann(3n, 5n)$. Equation (1.9) and the symmetries of the lattice give that $\mathbb{P}(\mathbf{x} \leftrightarrow \partial B(3n)) \asymp n^{-2}$. This, together with the two-point function estimate (1.7), gives

$$\begin{aligned} &\mathbb{E}(\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n, 5n)] | \mathbf{x} \leftrightarrow \partial B(3n)) \\ &\quad \geq c_1 n^2 \sum_{\mathbf{y} \in Ann(3n,5n)} ||\mathbf{x} - \mathbf{y}||^{2-d} \geq c_2 n^4 \end{aligned} \tag{3.13}$$

for some constants c_1, c_2 that depend only on d. Next note that

$$\begin{aligned}
&\mathbb{E}\big((\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n,5n)])^2 \mid \mathbf{x} \leftrightarrow \partial B(3n)\big) \\
&\quad = \sum_{\mathbf{y},\mathbf{z} \in Ann(3n,5n)} \mathbb{P}(\mathbf{x} \leftrightarrow \mathbf{y}, \mathbf{x} \leftrightarrow \mathbf{z} | \mathbf{x} \leftrightarrow \partial B(3n)) \\
&\quad = \sum_{\mathbf{y},\mathbf{z} \in Ann(3n,5n)} \frac{\mathbb{P}(\mathbf{x} \leftrightarrow \mathbf{y}, \mathbf{x} \leftrightarrow \mathbf{z})}{\mathbb{P}(\mathbf{x} \leftrightarrow \partial B(3n))},
\end{aligned}$$

as $\mathbf{x} \leftrightarrow \mathbf{y}, \mathbf{z}$ implies $\mathbf{x} \leftrightarrow \partial B(3n)$ for all $\mathbf{y}, \mathbf{z} \in Ann(3n,5n)$. Now,

$$\sum_{\mathbf{y},\mathbf{z} \in Ann(3n,5n)} \mathbb{P}(\mathbf{x} \leftrightarrow \mathbf{y},\ \mathbf{z})$$

is upper-bounded by $\mathbb{E}[(\#[\mathfrak{C}(\mathbf{x}) \cap [\mathbf{x} + B(6n)]])^2]$, which is at most $c_4 n^6$ for some constant $c_4 > 0$ by Lemma 2.2.

Combining this estimate with the fact that $\mathbb{P}(\mathbf{x} \leftrightarrow \partial B(3n)) \asymp n^{-2}$, we obtain

$$\mathbb{E}\big((\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n,5n)])^2 \,\big|\, \mathbf{x} \leftrightarrow \partial B(3n)\big) \le c_5 n^8 \tag{3.14}$$

for a constant c_5 that depends only on d. Using the inequalities in (3.13) and (3.14), and applying the Paley-Zygmund inequality, we find

$$\begin{aligned}
&\mathbb{P}\big(\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n,5n)] > \eta n^4 \big| \mathbf{x} \leftrightarrow \partial B(3n)\big) \\
&\quad \ge \mathbb{P}\big(\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n,5n)] > (\eta/c_2)\mathbb{E}\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n,5n)] \,\big|\, \mathbf{x} \leftrightarrow \partial B(3n)\big) \\
&\quad \ge (1 - \eta^2/c_2^2) \frac{\big(\mathbb{E}[\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n,5n)] | \mathbf{x} \leftrightarrow \partial B(3n)]\big)^2}{\mathbb{E}(\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n,5n) | \mathbf{x} \leftrightarrow \partial B(3n)]^2)} \\
&\quad \ge (1 - \eta^2/c_2^2) c_2^2 / c_5.
\end{aligned}$$

The above estimate together with the fact that $\mathbb{P}(\mathbf{x} \leftrightarrow \partial B(3n)) \asymp n^{-2}$ gives (3.10).

STEP 2. Combining the first moment bound of Lemma 2.2 with the Markov inequality gives

$$\mathbb{P}(\#[\mathfrak{C}(\mathbf{x}) \cap B(5n)] > \eta^{-1} n^4) \le c_9 \eta n^{-2}. \tag{3.15}$$

Using this with the estimate in (3.10), we get (3.11).

STEP 3. We now argue that the condition $X_{\mathfrak{C}(\mathbf{x})} > \eta n^2$ can be further imposed on the event in (3.15) without substantial probability cost. Write

$$\begin{aligned}
&\mathbb{P}(X_{\mathfrak{C}(\mathbf{x})} < \eta n^2, \mathbf{x} \leftrightarrow \partial B(3n)) \\
&= \mathbb{P}(0 < X_{\mathfrak{C}(\mathbf{x})} < \eta n^2, \mathbf{x} \leftrightarrow \partial B(3n)) \\
&= \sum_{M < \eta n^2} \sum_{\substack{\mathcal{C} \subseteq B(2n), \\ \#[\mathcal{C} \cap \partial B(2n)] = M}} \mathbb{P}\big(\mathfrak{C}_{B(2n)}(\mathbf{x}) = \mathcal{C},\, \mathbf{x} \leftrightarrow \partial B(3n)\big) \\
&\le \sum_{M < \eta n^2} \sum_{\substack{\mathcal{C} \subseteq B(2n), \\ \#[\mathcal{C} \cap \partial B(2n)] = M}} M \pi(n) \mathbb{P}\big(\mathfrak{C}_{B(2n)}(\mathbf{x}) = \mathcal{C}\big),
\end{aligned}$$

where in the final inequality we applied Lemma 3.2. Upper-bounding $M \le \eta n^2$ in the last display, using the asymptotics (1.9) for $\pi(n)$, and performing the sum over M and $\mathcal{C}$, we find

$$\begin{aligned}
\mathbb{P}(X_{\mathfrak{C}(\mathbf{x})} < \eta n^2 \mathbf{x} \leftrightarrow \partial B(3n)) &\le C\eta \mathbb{P}(0 < X_{\mathfrak{C}(\mathbf{x})} < \eta n^2) \\
&\le C\eta\, \mathbb{P}(\mathbf{x} \leftrightarrow \partial B(2n)) \le C_5 \eta n^{-2},
\end{aligned}$$

where $C_5 > 0$ is a constant, uniformly for $\mathbf{x} \in B(n/2)$.

Combining the above estimate with (3.11), we see

$$\begin{aligned}
&\mathbb{P}(\mathfrak{C}(\mathbf{x}) \in \mathscr{S}_\eta) \\
&\quad = \mathbb{P}\big(\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n, 5n)] > \eta n^4, \#[\mathfrak{C}(\mathbf{x}) \cap B(5n)] \le \eta^{-1} n^4\big) \\
&\qquad - \mathbb{P}\big(\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n, 5n)] > \eta n^4, \\
&\qquad\qquad \#[\mathfrak{C}(\mathbf{x}) \cap B(5n)] \le \eta^{-1} n^4, X_{\mathfrak{C}(\mathbf{x})} < \eta n^2\big) \\
&\quad \ge \mathbb{P}\big(\#[\mathfrak{C}(\mathbf{x}) \cap Ann(3n, 5n)] > \eta n^4, \#[\mathfrak{C}(\mathbf{x}) \cap B(5n)] \le \eta^{-1} n^4\big) \\
&\qquad - \mathbb{P}(X_{\mathfrak{C}(\mathbf{x})} < \eta n^2, \mathbf{x} \leftrightarrow \partial B(3n)) \\
&\quad \ge (C_1 - C_2 \eta^2 - C_3 \eta - C_5 \eta) n^{-2} =: c(\eta) n^{-2}.
\end{aligned}$$

This shows (3.12). □

4 Proof of Theorem 1.2

This section is entirely devoted to the proof of Theorem 1.2. Theorem 1.2 will be used in an essential way in the proofs of the remaining results of Theorem 1.1.

Note that the upper bound claimed in Theorem 1.2 follows from the unrestricted two-point function: $\tau_D(0, \mathbf{x}) \le \tau(0, \mathbf{x}) \le A_1 \|\mathbf{x}\|^{2-d}$ for any $D \subseteq \mathbb{Z}^d$. We will first give the matching lower bound in a more restrictive setting than claimed in the theorem. The restriction will be removed via an inductive argument that bootstraps a lower bound on the two-point function $\tau_{B(n)}$ far from the box boundary to one slightly closer to the box boundary.

We now state the "restrictive setting" version of Theorem 1.2 alluded to above.

PROPOSITION 4.1. *There exist constants $M_0 > 1$ and $c_1 > 0$ such that the following holds uniformly in n:*

$$\tau_{B(Mn)}(0,\mathbf{x}) \geq c_1\|\mathbf{x}\|^{2-d} \quad \textit{for all } \mathbf{x} \in B(n) \setminus \{0\} \textit{ and all } M > M_0.$$

PROOF. We say $\mathbf{x} \leftrightarrow \mathbf{y}$ through D if $\mathbf{x} \leftrightarrow \mathbf{y}$ but every open path from $\mathbf{x}$ to $\mathbf{y}$ uses a vertex of D. Suppose $\mathbf{x} \in B(n)$. Note that for any $M > 1$, the event $\{0 \leftrightarrow \mathbf{x}\}$ is a disjoint union of $\{0 \overset{B(Mn)}{\longleftrightarrow} \mathbf{x}\}$ and $\{0 \leftrightarrow \mathbf{x}$ through $B(Mn)^c\}$. Thus,

$$\tau_{B(Mn)}(0,\mathbf{x}) = \mathbb{P}(0 \leftrightarrow \mathbf{x}) - \mathbb{P}(0 \leftrightarrow \mathbf{x} \text{ through } B(Mn)^c).$$

The latter term of the right-hand side is bounded above by $C(Mn)^{2-d}$, uniformly in $\mathbf{x} \in B(n)$, by [32, (1.12)]. Using (1.7), the first term of the above is at least $a_1\|\mathbf{x}\|^{2-d}$. Choosing M large completes the proof. □

The result of Proposition 4.1 will serve as the base case for an induction argument, which will prove Theorem 1.2. In fact, our argument shows that the nested cubes of that theorem can be replaced by possibly oblong rectangles of arbitrary fixed aspect ratio. We state this strengthened version of the theorem for future reference:

THEOREM 4.2. *Fix $\alpha_i, \beta_i > 0$ for $1 \leq i \leq d$; fix also $M > 1$. For each n, let the rectangle*

$$R_n := [-\alpha_1 n, \beta_1 n] \times \cdots \times [-\alpha_d n, \beta_d n].$$

There is some $c = c(M, (\alpha_i), (\beta_i))$ such that, uniformly in n and in $\mathbf{x} \in R_n$,

$$\tau_{R_{Mn}}(0,\mathbf{x}) \geq c\|\mathbf{x}\|^{2-d}.$$

For use in the proof, we introduce some shorthand for the boundary vertices of cubes reachable from 0 within the cube. Recalling the definition of $X_Q(D,\mathbf{z})$ at (2.3), set

$$X^{box}(n) := X_{\partial B(n)}(B(n),0) = \#\{\mathbf{x} \in \partial B(n) : 0 \overset{B(n)}{\longleftrightarrow} \mathbf{x}\},$$

where in the first equality we use the notation of Section 5 with $D = B(n)$ and $Q = \partial B(n)$. We need a lemma bounding $\mathbb{E}X^{box}(n)$ for our proof of Theorem 1.2.

LEMMA 4.3 (Theorem 1.5(a) of [32]). *There is a constant $C_1 > 0$ such that $\mathbb{E}X^{box}(n) \leq C_1$ uniformly in $n \geq 1$.*

PROOFS OF THEOREM 1.2 AND THEOREM 4.2. We prove the notationally simpler case of a cube—that is, we prove Theorem 1.2—in detail, then describe the modifications necessary for other rectangular regions. Let

$$F^R(\cdot) := \|\cdot\|^{d-2}\tau_{B(R)}(0,\cdot).$$

For $M > 1$, say that τ is *M-good* if there are constants $c(M), n_0(M)$ so that $F^{Mn}|_{B(n)} \geq c$ for all $n \geq n_0$. The proof of Theorem 1.2 is inductive, and Proposition 4.1 initializes the induction. The inductive step is accomplished by the following claim. □

CLAIM 4.4. *If τ is M-good and $\alpha(M) := \min\{4/3, (M+1)/2\}$, then τ is $(M/\alpha(M))$-good.*

It is not hard to see that if τ is M_0-good for some $M_0 > 1$ (which is guaranteed by Proposition 4.1), then one can show that τ must be M-good for any $M \in (1, M_0)$ by applying Claim 4.4 finitely many times. This proves Theorem 1.2.

To prove Claim 4.4 it is enough to show that if $F^{Mn}|_{B(n)}$ is bounded away from 0, then so is $F^{Mn}|_{B(\alpha(M)n)}$. So, if $B_j(n) := \{\mathbf{x} \in \mathbb{Z}^d : |x(1)|, \ldots, |x(j)| \leq \alpha(M)n;\ |x(j+1)|, \ldots, |x(d)| \leq n\}$ obey

"obey" what? And what "obeys"? Please rewrite this and clarify.

CLAIM 4.5. *If $F^{Mn}|_{B_j(n)}$ (where $0 \leq j < d$) is bounded away from 0 for all n large enough, then so is $F^{Mn}|_{B_{j+1}(n)}$.*

then Claim 4.4 follows from Claim 4.5 by using induction on j. Note that the hypothesis of Claim 4.4 initializes the induction argument for Claim 4.5 at $j = 0$.

Please rework and clarify.

To show Claim 4.5 suppose $F^{Mn}|_{B_j(n)}$ is bounded away from 0 for some $0 \leq j < d$, so for some constant $c_M > 0$,

$$\tau_{B(Mn)}(0, \mathbf{x}) \geq c_M ||\mathbf{x}||^{2-d} \quad \text{for all } n \geq 1 \text{ and } \mathbf{x} \in B_j(n). \tag{4.1}$$

Fix an arbitrary $\mathbf{x} \in B_{j+1}(n) \setminus B_j(n)$. We will bound $\tau_{B(Mn)}(0, \mathbf{x})$ from below. Without loss of generality we can assume that $x(i) \geq 0$ for all i, as other cases are similar. Let

$$D = \mathbf{x} + B((\alpha(M) - 1)n), \quad \text{so} \quad D \subseteq B(Mn) \setminus B(n/3) \tag{4.2}$$

by our choice of $\alpha(\cdot)$. Also, ∂D contains the $(d-1)$-dimensional "quadrant"

$$Q := \big\{\mathbf{y} \in D : y(i) \leq x(i) \text{ for all } i \neq j+1, \text{ and } \mathbf{y} \sim \mathbf{y}' \text{ for some } \\ \mathbf{y}' \notin D \text{ with } y'(j+1) < x(j+1) - \lfloor(\alpha(M) - 1)n\rfloor\big\}.$$

Each vertex of Q has a lattice neighbor in $B_j(n)$ (as long as n is sufficiently large).

If $\mathbf{x}$ is on the i^{th} axis for some i, then all the vertices in an entire "side" of D (perpendicular to the i^{th} axis) containing Q are adjacent to vertices of $B_j(n)$. At the other extreme, when $\mathbf{x}$ is at the corner of $B_{j+1}(n)$ belonging to $\{\mathbf{y} \in \mathbb{Z}^d : y(i) \geq 0\}$, then no (or almost no) vertices of $\partial D \setminus Q$ are adjacent to vertices of $B_j(n)$. See Figure 4.1 for possible locations of D. Now note that if

$$F_{\mathbf{z}} := \big\{\mathbf{z} \overset{D}{\longleftrightarrow} \mathbf{x}, \mathbf{z} \overset{B(Mn)}{\longleftrightarrow} 0\big\},$$

then Claim 4.5 will follow if we show that there is a constant $c > 0$ (independent of $\mathbf{x}$ and n) such that

$$\mathbb{P}\Big(\bigcup_{\mathbf{z} \in Q} F_{\mathbf{z}}\Big) \geq cn^{2-d} \quad \text{for all } n \text{ large enough,} \tag{4.3}$$

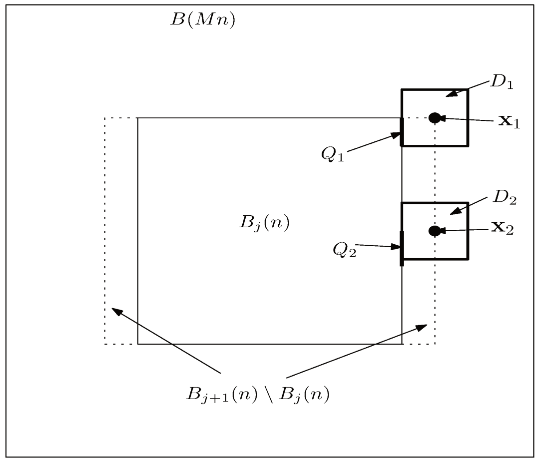

FIGURE 4.1. Referenced above (4.3). $(\mathbf{x}_i, D_i, Q_i), i = 1, 2$, are two possible locations of $(\mathbf{x}, D, Q)$.

because $\bigcup_{\mathbf{z}\in Q} F_{\mathbf{z}}$ implies $\{\mathbf{x} \overset{B(Mn)}{\longleftrightarrow} 0\}$. For each $\mathbf{z} \in Q$, fix a deterministic neighbor $\mathbf{z}' \sim \mathbf{z}$ with $\mathbf{z}' \in B_j(n) \setminus D$. To prove (4.3), let Y_Q^K be the number of $\mathbf{z}$ in $Q \cap \mathfrak{C}_D(\mathbf{x}) \cap REG_D(K)$ such that $\mathbf{z} \overset{B(Mn)}{\longleftrightarrow} 0$ and such that the edge $\{\mathbf{z}, \mathbf{z}'\}$ is pivotal for the event $\{\mathbf{x} \leftrightarrow 0\}$. The following lemma gives bounds for the (conditional) moments of Y_Q^K. As above, we introduce abbreviated notation for $X_Q(D, \mathbf{x})$ in order to make equations more readable.

LEMMA 4.6.

(1) *Let* $X_{\partial D} := \#\partial D \cap \mathfrak{C}_D(\mathbf{x}), X_Q := \#Q \cap \mathfrak{C}_D(\mathbf{x})$, *and* $X_Q^{K\text{-}reg} := \#Q \cap \mathfrak{C}_D(\mathbf{x}) \cap REG_D(K)$. *There are constants* $\eta, c_1(\eta) > 0$ *(independent of* $\mathbf{x}$ *and* n*) such that*

$$\text{(4.4)} \quad \text{if } B_\eta := \{\eta n^2 < X_Q^{K\text{-}reg} \le X_Q \le X_{\partial D} < \eta^{-1} n^2\}, \text{ then } \mathbb{P}(B_\eta) \ge c_1 n^{-2}.$$

(2) *Let* $\eta > 0$ *be such that* (4.4) *holds. There are constants* $K_0, C_2, c_2 > 0$ *such that for all* $K > K_0$ *and all* $\eta n^2 < N < \eta^{-1} n^2$,

$$\text{(2A)} \qquad \mathbb{E}\Big[\big(Y_Q^K\big)^2; X_Q^{K\text{-}reg} = N, B_\eta\Big] \le C_2 n^{4-d} \mathbb{P}\big(X_Q^{K\text{-}reg} = N; B_\eta\big),$$

$$\text{(2B)} \qquad \mathbb{E}\big[Y_Q^K; X_Q^{K\text{-}reg} = N, B_\eta\big] \ge c_2 n^{4-d} \mathbb{P}\big(X_Q^{K\text{-}reg} = N; B_\eta\big).$$

Note addition of "Proof" format added here.

PROOF. Using Lemma 4.6 and the second-moment method, if $K > K_0$ then

$$\mathbb{P}\big(Y_Q^K > 0 \mid X_Q^{K\text{-}reg} = N, B_\eta\big) \ge \frac{c_2^2}{C_2} n^{4-d} \quad \forall N \in (\eta n^2, \eta^{-1} n^2),$$

which implies

$$\begin{aligned}
&\mathbb{P}\big(Y_Q^K > 0\big) \\
&\geq \sum_{\eta n^2 < N < \eta^{-1} n^2} \mathbb{P}\big(Y_Q^K > 0 \mid X_Q^{K\text{-}reg} = N, B_\eta\big)\mathbb{P}\big(X_Q^{K\text{-}reg} = N, B_\eta\big) \\
&\geq \frac{c_2^2}{C_2} n^{4-d}\mathbb{P}(B_\eta) \geq \frac{c_1 c_2^2}{C_2} n^{2-d} \quad \text{using (4.4).}
\end{aligned}$$

This proves (4.3), as $\{Y_Q^K > 0\}$ implies $\bigcup_{\mathbf{z}\in Q} F_{\mathbf{z}}$, and thus completes the proof of Claim 4.5. □

We end the section by proving Lemma 4.6.

PROOF OF LEMMA 4.6. From the definition of Q and the symmetries of the lattice it is not hard to see that $\#[\mathfrak{C}_D(\mathbf{x}) \cap \partial D]$ is bounded above by a sum of $d2^d$ copies of X_Q that are identically distributed (but not independent). So, using a union bound and Lemma 3.4, there are constants $\eta_0(d) > 0$ and $c(\eta_0, d) > 0$ such that

$$\begin{aligned}
\mathbb{P}(X_Q > 2\eta n^2) &\geq \frac{1}{d2^d}\mathbb{P}(\#[\mathfrak{C}_D(\mathbf{x}) \cap \partial D] > d2^{d+1}\eta n^2) \\
&\geq \frac{c}{d2^d} n^{-2} \quad \text{for all } \eta \leq \eta_0.
\end{aligned}$$

Also, Lemma 2.3 implies

$$\mathbb{P}\big(X_Q > 2\eta n^2, X_Q^{K\text{-}reg} \leq \eta n^2\big) \leq C n^d \exp(-c \log^2(2\eta n^2))$$

for some constants $C, c > 0$. Finally, using Lemma 4.3 and the Markov inequality, $\mathbb{P}(X_{\partial D} \geq \eta^{-1} n^2) \leq C_1 \eta n^{-2}$. Combining this with the last two displays,

$$\mathbb{P}(B_\eta) \geq \frac{c}{d2^d} n^{-2} - C n^d \exp(-c \log^2(2\eta n^2)) - C_1 \eta n^{-2} \quad \text{for all } \eta \leq \eta_0.$$

So we get the desired result if we choose $\eta > 0$ small enough and n large enough.

(2A). First we argue that $Y_Q^K \leq 1$ a.s. via the method of contradiction. Suppose, if possible, $\mathbf{z}_1$ and $\mathbf{z}_2$ are two vertices counted in Y_Q^K. Then $\mathbf{x} \leftrightarrow 0$, so we can choose a self-avoiding open path γ joining $\mathbf{x}$ to 0. By pivotality, γ must contain the edges $\{\mathbf{z}_i, \mathbf{z}_i'\}$ for $i = 1, 2$. Suppose (without loss of generality) that γ passes through $\mathbf{z}_2$ first when traversed from 0 to $\mathbf{x}$. Then we can find a path $\gamma' \subseteq \gamma$ joining 0 and $\mathbf{z}_2$ such that the edge $\{\mathbf{z}_1, \mathbf{z}_1'\} \notin \gamma'$. On the other hand, since $\mathbf{z}_2 \in \mathfrak{C}_D(\mathbf{x})$, we also have a path γ'' that stays entirely within D and joins $\mathbf{x}$ and $\mathbf{z}_2$. This contradicts the fact that the edge $\{\mathbf{z}_1, \mathbf{z}_1'\}$ is pivotal for $\{\mathbf{x} \leftrightarrow 0\}$, as $\gamma' \cup \gamma''$ avoids the edge $\{\mathbf{z}_1, \mathbf{z}_1'\}$ and connects $\mathbf{x}$ and 0. Thus $Y_Q^K \leq 1$. In particular, $(Y_Q^K)^2 = \sum_{\mathbf{z}} \mathbb{1}_{\{\mathbf{z} \text{ counted in } Y_Q\}}$. Conditioning on the cluster

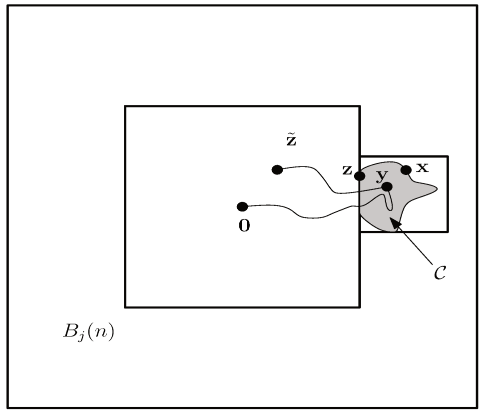

FIGURE 4.2. The event $\{\tilde{\mathbf{z}} \leftrightarrow \mathbf{y}\} \circ \{\mathbf{y} \leftrightarrow 0\}$ for $\mathbf{y} \in \mathcal{C}$ as in the proof of Claim 4.7. The shaded region represents $C_D(\mathbf{x})$.

of $\mathbf{x}$, $\mathbb{E}[(Y_Q^K)^2; X_Q^{K\text{-}reg} = N, B_\eta]$ is

$$\begin{aligned}
&\leq \sum_{\mathcal{C} \in B_\eta \cap \left\{X_Q^{K\text{-}reg} = N\right\}} \mathbb{E}\left[\left(Y_Q^K\right)^2; \mathfrak{C}_D(\mathbf{x}) = \mathcal{C}\right] \\
&\leq \sum_{\mathcal{C} \in B_\eta \cap \left\{X_Q^{K\text{-}reg} = N\right\}} \sum_{\substack{\mathbf{z} \in Q :\, \mathbf{z} \in \mathcal{C} \cap REG_D(K) \\ \text{when } \mathfrak{C}_D(\mathbf{x}) = \mathcal{C}}} \mathbb{P}(\mathfrak{C}_D(\mathbf{x}) = \mathcal{C}, \mathbf{z} \leftrightarrow 0 \text{ off } \mathcal{C})
\end{aligned} \tag{4.5}$$

(recall that "$\mathbf{z} \leftrightarrow 0$ off $\mathcal{C}$" means that there is an open path from $\mathbf{z}$ to 0 touching $\mathcal{C}$ only at $\mathbf{z}$). Using (1.7) and the fact that $Q \cap B(n/3) = \varnothing$, along with the independence of the above events, we see as in the proof of Lemma 3.2 that the above is bounded by

$$\begin{aligned}
&A_1(n/3)^{2-d} \sum_{\mathbf{z} \in Q} \mathbb{P}\left(\mathbf{z} \in \mathfrak{C}_D(\mathbf{x}) \cap REG_D(K), X_Q^{K\text{-}reg} = N; B_\eta\right) \\
&= A_1(n/3)^{2-d} \mathbb{E} X_Q^{K\text{-}reg} \mathbb{1}_{\left\{X_Q^{K\text{-}reg} = N\right\} \cap B_\eta} \\
&\leq A_1(n/3)^{2-d} \eta^{-1} n^2 \mathbb{P}\left(X_Q^{K\text{-}reg} = N; B_\eta\right).
\end{aligned}$$

This completes the proof of (2A) of Lemma 4.6.

(2B). We will define some events that force Y_Q^K to be nonzero. For $\mathbf{z} \in Q$, consider the box $\widetilde{D}_{\mathbf{z}} = \mathbf{z} - [K/2, K]^d$. Since $x(i) \geq 0$, $\widetilde{D}_{\mathbf{z}} \subseteq B_j(n) \setminus D$ as long as $n > K$ and also K is larger than some lattice-dependent constant. In fact, for K larger than a lattice-dependent constant, the ℓ^∞ distance of $\widetilde{D}_{\mathbf{z}}$ from D is at least $K/4$. In what follows, for a fixed $\mathbf{z} \in Q$, $\tilde{\mathbf{z}}$ will typically denote a vertex of $\widetilde{D}_{\mathbf{z}}$;

N will also always be a value between ηn^2 and $\eta^{-1}n^2$. Define

$$\mathcal{E}_1(\mathbf{z}, N) := B_\eta \cap \big\{\mathbf{x} \overset{D}{\leftrightarrow} \mathbf{z},\ \mathbf{z} \in REG_D(K),\ \text{and } X_Q^{K\text{-}reg} = N\big\}$$
$$\mathcal{E}_2(\mathbf{z}, \widetilde{\mathbf{z}}) := \big\{\widetilde{\mathbf{z}} \overset{B(Mn)}{\longleftrightarrow} 0 \text{ off } \mathfrak{C}_D(\mathbf{z})\big\},$$
$$\mathcal{E}_3(\mathbf{z}, \widetilde{\mathbf{z}}) := \{\mathfrak{C}(\mathbf{z}) \cap \mathfrak{C}(\widetilde{\mathbf{z}}) = \varnothing\}.$$

We successively bound probabilities of the intersections of the $\mathcal{E}_i$'s via a series of claims.

CLAIM 4.7. *Let c_M be the constant from* (4.1). *There is a constant $K_0 \geq 2$ (depending on c_M) such that $\mathbb{P}(\mathcal{E}_1(\mathbf{z}, N) \cap \mathcal{E}_2(\mathbf{z}, \widetilde{\mathbf{z}})) \geq (c_M/2)n^{2-d}\,\mathbb{P}(\mathcal{E}_1(\mathbf{z}, N))$ for all $\mathbf{x}$, $K > K_0$, $n \geq 10K$, $\mathbf{z} \in Q$, $\widetilde{\mathbf{z}} \in \widetilde{D}_{\mathbf{z}}$ and $N \geq 1$.*

Note that for any realization $\mathcal{C}$ of $\mathfrak{C}_D(\mathbf{z})$ satisfying $\mathcal{E}_1(\mathbf{z}, N)$,

$$\mathbb{P}(\mathcal{E}_2(\mathbf{z}, \widetilde{\mathbf{z}}) \mid \mathfrak{C}_D(\mathbf{z}) = \mathcal{C})$$

equals

$$\mathbb{P}\big(\widetilde{\mathbf{z}} \overset{B(Mn)}{\longleftrightarrow} 0 \text{ off } \mathcal{C}\big) \geq \mathbb{P}\big(\widetilde{\mathbf{z}} \overset{B(Mn)}{\longleftrightarrow} 0\big) - \mathbb{P}\Big(\bigcup_{\mathbf{y}\in\mathcal{C}} \{\widetilde{\mathbf{z}} \leftrightarrow \mathbf{y}\} \circ \{\mathbf{y} \leftrightarrow 0\}\Big). \tag{4.6}$$

See Figure 4.2 for a sketch. Using (4.1) and recalling that $\widetilde{\mathbf{z}} \in B_j(n)$, the first term in the RHS of (4.6) is $\geq c_M n^{2-d}$. Using a union bound and the BK inequality, (1.7), and the fact that $\mathcal{C} \subseteq (B(n/3))^c$ (see (4.2)), the second term in the RHS of (4.6) is $\leq A_1(n/3)^{2-d} \sum_{\mathbf{y}\in\mathcal{C}} \mathbb{P}(\widetilde{\mathbf{z}} \leftrightarrow \mathbf{y})$. From (4.6) and the last two observations, $\mathbb{P}(\mathcal{E}_2(\mathbf{z}, \widetilde{\mathbf{z}}) \mid \mathfrak{C}_D(\mathbf{z}) = \mathcal{C})$ is

$$\geq c_M n^{2-d} - A_1\left(\frac{n}{3}\right)^{2-d} \sum_{\mathbf{y}\in\mathcal{C}} \mathbb{P}(\widetilde{\mathbf{z}} \leftrightarrow \mathbf{y}). \tag{4.7}$$

In order to estimate the sum in (4.7), let $U_r := \widetilde{\mathbf{z}} + Ann(2^r, 2^{r+1})$ for $r \geq 0$. So $\mathbb{P}(\widetilde{\mathbf{z}} \leftrightarrow \mathbf{y}) \leq A_1 2^{r(2-d)}$ for all $\mathbf{y} \in U_r$, which gives

$$\sum_{\mathbf{y}\in\mathcal{C}} \mathbb{P}(\widetilde{\mathbf{z}} \leftrightarrow \mathbf{y}) \leq \sum_{r \geq \log_2(K/2)} A_1 2^{r(2-d)} (\#\mathcal{C} \cap U_r).$$

Since $\|\mathbf{z} - \widetilde{\mathbf{z}}\| \leq K$, we have $U_r \subseteq \mathbf{z} + B(2^{r+2})$ for all $r \geq \log_2(K/2)$. Hence, whenever $\mathcal{C}$ satisfies $\mathcal{E}_1(\mathbf{z}, N)$, we have

$$\begin{aligned}
\#\mathcal{C} \cap U_r &\leq \mathbb{E}[\#\mathfrak{C}(\mathbf{z}) \cap (\mathbf{z} + B(2^{r+2})) \mid \mathfrak{C}_D(\mathbf{z}) = \mathcal{C}] \\
&\leq 2^{4(r+2)} \log^7(2^{r+2}) + 2^{(r+4)d}\,\mathbb{P}(\mathcal{T}_{2^{r+2}}(\mathbf{z})^c \mid \mathfrak{C}_D(\mathbf{z}) = \mathcal{C}) \\
&\leq C 2^{4r} \log^7(2^r)
\end{aligned}$$

for all $r \geq \log_2(K/2)$, where C is independent of r and K (as long as K is large). In the above, we have used the definition of K-regularity and Lemma 2.3. This implies

$$\sum_{\mathbf{y}\in\mathcal{C}} \mathbb{P}(\widetilde{\mathbf{z}} \leftrightarrow \mathbf{y}) \leq c_1 \sum_{r \geq \log_2(K/2)} (r^7 2^{r(6-d)}) \leq c_2 K^{6-d} \log^7 K$$

for some constants c_1, c_2 (independent of K and n). Using this bound and (4.7), we see that if K is large enough then

$$\mathbb{P}(\mathcal{E}_2(\mathbf{z}, \widetilde{\mathbf{z}}) \mid \mathfrak{C}_D(\mathbf{z}) = \mathcal{C}) \geq (c_M/2) n^{2-d} \mathbb{1}_{\{\mathcal{C}\in\mathcal{E}_1(\mathbf{z},N)\}}.$$

Taking an expectation over $\mathfrak{C}_D(\mathbf{z})$ completes the proof of Claim 4.7.

Having proved Claim 4.7, we move on to the next subsidiary claim, which deals with $\mathcal{E}_3$.

CLAIM 4.8. *Let c_M be the constant from* (4.1). *There is a constant $K_1 > K_0$ (depending on c_M) such that for all* $\mathbf{x}$, *$K \geq K_1, n \geq 10K$ and $\mathbf{z} \in Q$, we can find a $\widetilde{\mathbf{z}} \in \widetilde{D}_{\mathbf{z}}$ satisfying*

$$\mathbb{P}(\mathcal{E}_1(\mathbf{z}, N) \cap \mathcal{E}_2(\mathbf{z}, \widetilde{\mathbf{z}}) \cap \mathcal{E}_3(\mathbf{z}, \widetilde{\mathbf{z}})) \geq (c_M/4) n^{2-d} \mathbb{P}(\mathcal{E}_1(\mathbf{z}, N)).$$

Claim 4.8 will follow if we show that there is a constant $K_1 > K_0$ such that, for any $z \in Q$, if ζ denotes a uniformly chosen random vertex in $\widetilde{D}_{\mathbf{z}}$ and if E_ζ denotes expectation over ζ, then

$$E_\zeta \mathbb{P}(\mathcal{E}_1(\mathbf{z}, N) \cap \mathcal{E}_2(\mathbf{z}, \zeta) \cap \mathcal{E}_3(\mathbf{z}, \zeta)) \geq (c_M/4) n^{2-d} \mathbb{P}(\mathcal{E}_1(\mathbf{z}, N)) \tag{4.8}$$

for all N and $K \geq K_1$.

Fix $\mathbf{z} \in Q$ and $\zeta \in \widetilde{D}_z$. Consider the event $(\mathcal{E}_1(\mathbf{z}, N) \cap \mathcal{E}_2(\mathbf{z}, \zeta)) \setminus \mathcal{E}_3(\mathbf{z}, \zeta)$. On this event, we can find a self-avoiding open path γ_1 joining ζ and 0 and avoiding $\mathfrak{C}_D(\mathbf{z})$, then subsequently find a path γ_2 starting at $\mathbf{z}$ and terminating at its first and only intersection point with γ_1. So if $\mathbf{v} \in \gamma_1 \cap \gamma_2$ is the unique such intersection point of γ_1 and γ_2, then the event $\{\mathbf{x} \leftrightarrow \mathbf{v}; \mathcal{E}_1(\mathbf{z}, N)\} \circ \{\zeta \leftrightarrow \mathbf{v}\} \circ \{\mathbf{v} \leftrightarrow 0\}$ occurs (see Figure 4.3 for a sketch). So, using the union bound, the BK inequality, (1.7), and the convention $0^{2-d} = 1$,

$$\begin{aligned}
&\mathbb{P}((\mathcal{E}_1(\mathbf{z}, N) \cap \mathcal{E}_2(\mathbf{z}, \zeta)) \setminus \mathcal{E}_3(\mathbf{z}, \zeta)) \\
&\quad \leq A_1^2 \sum_{\mathbf{v}\in B(n/100)} \mathbb{P}(\mathbf{x} \leftrightarrow \mathbf{v}; \mathcal{E}_1(\mathbf{z}, N)) \|\zeta - \mathbf{v}\|^{2-d} \|\mathbf{v}\|^{2-d} \\
&\qquad + A_1^2 \sum_{\mathbf{v}\notin B(n/100)} \mathbb{P}(\mathbf{x} \leftrightarrow \mathbf{v}; \mathcal{E}_1(\mathbf{z}, N)) \|\zeta - \mathbf{v}\|^{2-d} \|\mathbf{v}\|^{2-d} =: I_1 + I_2.
\end{aligned} \tag{4.9}$$

We bound $E_\zeta I_1$ and $E_\zeta I_2$ uniformly in K large, and in n large relative to K. First consider I_1. If $n \geq 10K$, then using the triangle inequality $\|\zeta - \mathbf{v}\| \geq \|\mathbf{z}\| - \|\mathbf{z} - \zeta\| - \|\mathbf{v}\| \geq n/2$ for each $\mathbf{v} \in B(n/100)$. Also,

$$\mathbb{P}(\mathbf{x} \leftrightarrow \mathbf{v}; \mathcal{E}_1(\mathbf{z}, N)) = \sum_{\mathcal{C}\in\mathcal{E}_1(\mathbf{z},N)} \mathbb{P}(\mathfrak{C}_D(\mathbf{x}) = \mathcal{C}) \mathbb{P}(\mathbf{x} \leftrightarrow \mathbf{v} \mid \mathfrak{C}_D(\mathbf{x}) = \mathcal{C}).$$

If $\{\mathbf{x} \leftrightarrow \mathbf{v}\}$ occurs, then there must be some $\mathbf{w} \in \mathfrak{C}_D(\mathbf{x}) \cap \partial D$ such that $\{\mathbf{w} \leftrightarrow \mathbf{v} \text{ off } \mathfrak{C}_D(\mathbf{x})\}$ occurs. In particular, using (1.7) and the fact that $\|\mathbf{v} - \mathbf{w}\| \geq n/2$ for all $\mathbf{w} \in \partial D$,

$$\begin{aligned}\mathbb{P}(\mathbf{x} \leftrightarrow \mathbf{v} \mid \mathfrak{C}_D(\mathbf{x}) = \mathcal{C}) \leq \sum_{\mathbf{w} \in \mathcal{C} \cap \partial D} \mathbb{P}(\mathbf{w} \leftrightarrow \mathbf{v}) &\leq A_1 (n/2)^{2-d} X_{\partial D} \\ &\leq A_1 \eta^{-1} n^2 (n/2)^{2-d}\end{aligned}$$

for all $\mathcal{C}$ satisfying $\mathcal{E}_1(\mathbf{z}, N)$. Pulling the above bounds together and summing over $\mathcal{C}$ and $\mathbf{v}$,

$$I_1 \leq c_1 \mathbb{P}(\mathcal{E}_1(\mathbf{z}, N)) n^{6-2d} \sum_{\mathbf{v} \in B(n/100)} \|\mathbf{v}\|^{2-d} \leq c_2 \mathbb{P}(\mathcal{E}_1(\mathbf{z}, N)) n^{8-2d} \tag{4.10}$$

uniformly in ζ, for some constants c_1, c_2 (independent of K and n).

To control I_2, we bound $\|\mathbf{v}\|^{2-d}$ uniformly by $(n/100)^{2-d}$. Define $\mathcal{C}_{\zeta,t} := \mathfrak{C}(\mathbf{x}) \cap [\zeta + Ann(2^{t-1}, 2^t)]$ for $t \geq 0$ and $t_K := \log_2(4K)$. Since $\|\zeta - \mathbf{v}\| \geq 2^{t-1}$ when $\mathbf{v} \in \mathcal{C}_{\zeta,t}$,

$$\begin{aligned} I_2 &\leq C(n/100)^{2-d} \sum_{t=0}^{\infty} \sum_{\mathbf{v} \in \zeta + Ann(2^{t-1}, 2^t)} \mathbb{P}(\mathbf{x} \leftrightarrow \mathbf{v}; \mathcal{E}_1(\mathbf{z}, N)) \|\zeta - \mathbf{v}\|^{2-d} \\ &\leq C\Big(\sum_{t \geq t_K} 2^{2t-dt} \mathbb{E}[\#\mathcal{C}_{\zeta,t};\, \mathcal{E}_1(\mathbf{z}, N)] \\ &\qquad + \sum_{\substack{t < t_K \\ \mathbf{v} \in \zeta + Ann(2^{t-1}, 2^t)}} \mathbb{P}(\mathbf{x} \leftrightarrow \mathbf{v}; \mathcal{E}_1(\mathbf{z}, N)) \|\zeta - \mathbf{v}\|^{2-d} \Big) n^{2-d} \\ &=: I_{21} + I_{22} \end{aligned} \tag{4.11}$$

for some constant $C > 0$. To bound I_{21} note that $\mathcal{C}_{\zeta,t} \subseteq \mathbf{z} + B(2^{t+1})$ for all $t \geq t_K$ and $\zeta \in \widetilde{D}_{\mathbf{z}}$, so using Lemma 2.3 and discarding a negligible contribution from the event $\mathcal{T}_{2^{t+1}}(\mathbf{z})^c$ as before, there is a constant C independent of n and (sufficiently large) K such that

$$\begin{aligned} &\mathbb{E}[\#\mathcal{C}_{\zeta,t}; \mathcal{E}_1(\mathbf{z}, N)] \\ &\quad = \sum_{\mathcal{C} \in \mathcal{E}_1(\mathbf{z}, N)} \mathbb{E}[\#[\mathfrak{C}(\mathbf{z}) \cap (\mathbf{z} + B(2^{t+1}))] \mid \mathfrak{C}_D(\mathbf{z}) = \mathcal{C}] \mathbb{P}(\mathfrak{C}_D(\mathbf{z}) = \mathcal{C}) \\ &\quad \leq C\, \mathbb{P}(\mathcal{E}_1(\mathbf{z}, N)) 2^{4t} \log^7(2^t), \end{aligned}$$

which implies

$$\begin{aligned} I_{21} &\leq C_3 \mathbb{P}(\mathcal{E}_1(\mathbf{z}, N)) n^{2-d} \sum_{t \geq t_K} t^7 2^{t(6-d)} \\ &\leq C_4 \mathbb{P}(\mathcal{E}_1(\mathbf{z}, N)) n^{2-d} K^{6-d} \log^7 K \end{aligned}$$

where the C_i's are constants independent of $\mathbf{x}$, of K sufficiently large, of $\mathbf{z}$ and ζ, and of n large relative to K.

We turn now to estimating $E_\zeta(I_{22})$. Consider the expectation E_ζ of the inner sum for a typical value of $t \le t_K$.

$$\begin{aligned}
E_\zeta & \sum_{\mathbf{v}\in\zeta+Ann(2^{t-1},2^t)} \mathbb{P}(\mathbf{x}\leftrightarrow\mathbf{v};\mathcal{E}_1(\mathbf{z},N))\|\zeta-\mathbf{v}\|^{2-d} \\
&= \big[\#\widetilde{D}_{\mathbf{z}}\big]^{-1} \sum_{\zeta\in\widetilde{D}_{\mathbf{z}}} \sum_{\mathbf{v}\in\zeta+Ann(2^{t-1},2^t)} \mathbb{P}(\mathbf{x}\leftrightarrow\mathbf{v};\mathcal{E}_1(\mathbf{z},N))\|\zeta-\mathbf{v}\|^{2-d} \\
&\le CK^{-d} \sum_{\mathbf{v}\in\bigcup_\zeta \zeta+Ann(2^{t-1},2^t)} \mathbb{P}(\mathbf{x}\leftrightarrow\mathbf{v};\mathcal{E}_1(\mathbf{z},N)) \\
&\qquad\cdot\Bigg[\sum_{\zeta\in\widetilde{D}_{\mathbf{z}}:|\zeta-\mathbf{v}|>K} K^{2-d} + \sum_{l=1}^{K}\sum_{\zeta\in\widetilde{D}_{\mathbf{z}}:\|\zeta-\mathbf{v}\|_\infty=l} l^{2-d}\Bigg] \\
&\le CK^{-d} \sum_{\mathbf{v}\in\bigcup_\zeta \zeta+Ann(2^{t-1},2^t)} \mathbb{P}(\mathbf{x}\leftrightarrow\mathbf{v};\mathcal{E}_1(\mathbf{z},N)) \\
&\qquad\cdot\Bigg[(K/2)^d K^{2-d} + \sum_{l=1}^{K} 2d\,l^{d-1}l^{2-d}\Bigg] \\
&\le C_5K^{2-d}\mathbb{E}\Big[\#\Big(\bigcup_{\zeta\in\widetilde{D}_{\mathbf{z}}}\mathcal{C}_{\zeta,t}\Big);\mathcal{E}_1(\mathbf{z},N)\Big] \quad \text{for some constant } C_5.
\end{aligned}$$

Note that $\mathcal{C}_{\zeta,t}\subseteq \mathbf{z}+B(5K)$ for all $t\le t_K$ and $\zeta\in\widetilde{D}_{\mathbf{z}}$, as $\|\zeta-\mathbf{z}\|\le K$. Therefore, the above is

$$\begin{aligned}
&\le C_5K^{2-d}\mathbb{E}[\#[\mathfrak{C}(\mathbf{z})\cap(\mathbf{z}+B(5K))];\mathcal{E}_1(\mathbf{z},N)] \\
&= C_5K^{2-d}\sum_{\mathcal{C}\in\mathcal{E}_1(\mathbf{z},N)}\mathbb{P}(\mathfrak{C}_D(\mathbf{z})=\mathcal{C})\mathbb{E}[\#[\mathfrak{C}(\mathbf{z})\cap(\mathbf{z}+B(5K))]\mid\mathfrak{C}_D(\mathbf{z})=\mathcal{C}] \qquad (4.12)\\
&\le C_5K^{2-d}(5K)^4\log^7(5K)\mathbb{P}(\mathcal{E}_1(\mathbf{z},N)) + C_6K^{2-d}K^de^{-t_K^2}\mathbb{P}(\mathcal{E}_1(\mathbf{z},N)),
\end{aligned}$$

again using K-regularity.

The second term of (4.12) is negligible, which implies

$$E_\zeta(I_{22}) \le C_7K^{6-d}\log^8(5K)n^{2-d}\mathbb{P}(\mathcal{E}_1(\mathbf{z},N))$$

for some constant C_7. Inserting our estimates for I_1, I_{21}, and $E_\zeta(I_{22})$ in (4.9), we bound $E_\zeta\mathbb{P}([\mathcal{E}_1(\mathbf{z},N)\cap\mathcal{E}_2(\mathbf{z},\zeta)]\setminus\mathcal{E}_3(\mathbf{z},\zeta))$. Using this bound, the LHS of (4.8) is at least

$$E_\zeta[\mathbb{P}(\mathcal{E}_1(\mathbf{z},N)\cap\mathcal{E}_2(\mathbf{z},\zeta))] - C_8n^{2-d}K^{6-d}\log^8(5K)\mathbb{P}(\mathcal{E}_1(\mathbf{z},N)) \qquad (4.13)$$

for some constant C_8. Choosing K large enough and applying Claim 4.7, (4.8) is established. This finishes the proof of Claim 4.8.

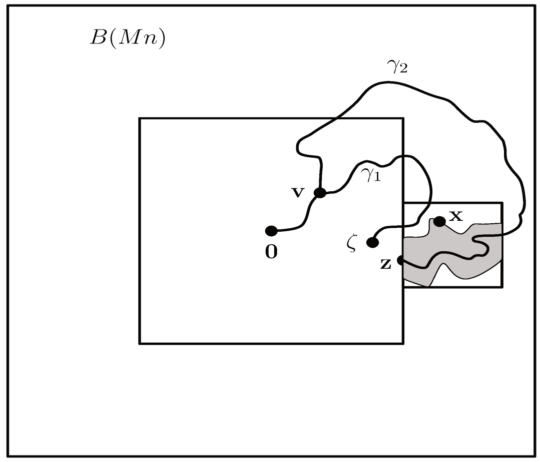

FIGURE 4.3. Bounding a cluster intersection event in the proof of Claim 4.8. Depicted is the event $\{\mathbf{x} \leftrightarrow \mathbf{v}; \mathcal{E}_1(\mathbf{z}, N)\} \circ \{\zeta \leftrightarrow \mathbf{v}\} \circ \{\mathbf{v} \leftrightarrow 0\}$.

We now move to complete the proof of (2B) of Lemma 4.6. Suppose we have a pair $(\mathbf{z}, \widetilde{\mathbf{z}})$, where $\mathbf{z} \in Q$ and $\widetilde{\mathbf{z}} \in \widetilde{D}_{\mathbf{z}}$, as in Claim 4.8. We claim that there is a constant $c_9 = c_9(K) > 0$ such that

$$\mathbb{P}\Big(\mathbf{z} \text{ is counted in } Y_Q^K;\ X_Q^{K\text{-}reg} = N, B_\eta\Big) \geq c_9 \mathbb{P}\Big(\mathcal{E}_1(\mathbf{z}, N) \cap \bigcap_{i=2}^{3} \mathcal{E}_i(\mathbf{z}, \widetilde{\mathbf{z}})\Big). \tag{4.14}$$

The argument for (4.14) is a usual edge modification argument, which we now sketch. We define a function Υ mapping each edge configuration $\omega \in \mathcal{E}_1(\mathbf{z}, N) \cap \bigcap_{i=2}^3 \mathcal{E}_i(\mathbf{z}, \widetilde{\mathbf{z}})$ to a new edge configuration $\Upsilon(\omega)$ as follows. Consider such an outcome ω, with $\widetilde{\mathbf{z}}$ chosen as in Claim 4.8. We can choose according to some deterministic search algorithm a path π of open edges from $\widetilde{\mathbf{z}}$ to 0 lying entirely in $B(Mn)$. Since $\mathfrak{C}(\mathbf{z})$ and $\mathfrak{C}(\widetilde{\mathbf{z}})$ are disjoint, this path is guaranteed not to intersect $\mathfrak{C}(\mathbf{z})$. Now, we close all edges having an endpoint in the box $[\mathbf{z} + B(4K)] \setminus D$, except those edges belonging to π; we then open $\{\mathbf{z}, \mathbf{z}'\}$. Last, we open one-by-one the edges in a path from $\mathbf{z}'$ to π which lies entirely in

$$[\mathbf{z} + B(3K)] \setminus [\mathbf{x} + B((\alpha(M) - 1)n + 1)]$$

(i.e., the set D widened by one unit) except possibly for its initial vertex $\mathbf{z}'$.

It is easy to see that the above procedure connects $\mathbf{z}$ to 0 within $B(Mn)$ but that every open path from $\mathbf{z}$ to 0 must pass through $\mathbf{z}'$. Because, in the outcome ω, $\mathbf{z}$ was in $\mathfrak{C}_D(\mathbf{z}) \cap REG_D(K)$ and $B_\eta \cap \{X_Q^{K\text{-}reg} = N\}$ initially occurred, and since no edges of D were modified by Υ, these facts still hold true for $\Upsilon(\omega)$. Lastly, since the function is at most e^{CK^d}-to-one, the probability of the image $\Upsilon(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3)$ is at least $c(K)\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3)$.

Given (4.14), the conclusion of the proof is immediate. Summing (4.14) over $\mathbf{z}$, we find

$$\mathbb{E}\big[Y_Q^K; X_Q^{K\text{-}reg} = N; B_\eta\big] \geq c \sum_{\mathbf{z}} \mathbb{P}\left(\mathcal{E}_1(\mathbf{z}, N) \cap \bigcap_{i=2}^{3} \mathcal{E}_i(\mathbf{z}, \widetilde{\mathbf{z}})\right).$$

Using Claim 4.8, the probability appearing on the right-hand side is at least

$$cn^{2-d}\,\mathbb{P}(\mathcal{E}_1(\mathbf{z}, N))$$

when $\widetilde{\mathbf{z}}$ is chosen appropriately. Now, on $B_\eta \cap \{X_Q^{K\text{-}reg} = N\}$, there are N vertices $\mathbf{z}$ such that $\mathcal{E}_1(\mathbf{z}, N)$ occurs; since $N > \eta n^2$, this completes the proof. □

5 Extending Large Clusters

We now give a collection of results that could be said to relate to "extensibility"; these are collected in Theorem 5.1. We use this term in the sense of the two-dimensional percolation literature [25, prop. 12] to mean, roughly, that when $\mathfrak{C}_G(0)$ is conditioned to be large in some sense, it has nontrivial probability to be "still larger." Such extensibility arguments were also a key part of the argument showing (1.9) appearing in [22].

The setup we will use differs from previous high-dimensional extensibility results in a major way: namely, we typically want to extend clusters restricted to lie in the subgraph $G = \mathbb{Z}_+^d$. This poses a couple of serious obstacles. The first problem is that we cannot use the usual two-point function $\tau(\mathbf{x}, \mathbf{y})$ for lower bounds on the probability that long open connections exist, since $\tau(\mathbf{x}, \mathbf{y})$ includes contributions from the event where such connections leave $\mathbb{Z}_+^d$. More precisely, we need to compare $\tau_H(\mathbf{x}, \mathbf{y})$ to $\tau(\mathbf{x}, \mathbf{y})$. A main aim of Theorem 1.2 is to provide a comparison between these two connectivity probabilities when $\mathbf{x}$ and $\mathbf{y}$ are a macroscopic distance from $S(0)$.

The second problem relates to our inability to effectively localize the half-space arm from 0 on the event

$$\big\{0 \overset{\mathbb{Z}_+^d}{\longleftrightarrow} S'(n)\big\}.$$

Ideally, we would prove $\pi_H(2n) \geq c\pi_H(n)$ by conditioning on the existence of an arm to distance n and showing it is likely to be extended. This would require one to show that the distance-n arm does not typically terminate close to $S(0)$, since the two-point function in $\mathbb{Z}_+^d$ behaves very differently near $S(0)$ than far from $S(0)$. Proving that half-space arms can be localized away from the boundary appears to be difficult a priori; to solve this problem we work in an annulus Ann' and compare to the case of the half-space. As mentioned above, such a localization result does ultimately follow as a consequence of $\pi_H(n) \leq Cn^{-3}$ (see (7.1)); this will be important for our work on the two-point function in (b) of Theorem 1.1.

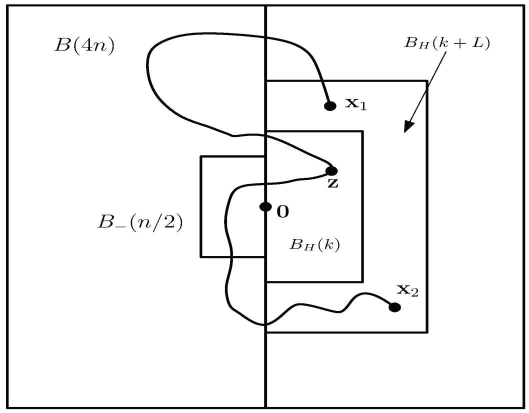

FIGURE 5.1. Depictions of a definition appearing in (5.1). All edges are closed except the two paths drawn above, and $\mathbf{x}_1 \in A^{out}_{\mathbf{z}}, \mathbf{x}_2 \notin A^{out}_{\mathbf{z}}$.

For simplicity, we introduce the following abbreviations for stating the extensibility result. If $\mathbf{z} \in B_H(k)$, where $n \le k \le 2n$, and if $0 < L \le 3n - k$ is an integer, we define (see Figure 5.1 for a sketch)

$$\mathsf{A}^{out}_{\mathbf{z}}(n,k,L) := \left[\mathfrak{C}_{Ann'(n/2,4n)}(\mathbf{z}) \cap Ann_H(k, k+L)\right]. \tag{5.1}$$

If $\mathbf{z} \in B_H(4n)$, we define

$$\mathsf{A}^{in}_{\mathbf{z}}(n) := [\mathfrak{C}(\mathbf{z}) \cap B_-(n/4)].$$

In this language, the main theorem on extensibility is as follows:

THEOREM 5.1. *There is some constant $c_* > 0$ such that the following hold uniformly in $n \ge c_*^{-1}$, in $n^{1/10} \le L \le 3n - k$, in $n \le k \le 2n$, and in M and $\mathbf{z}$ as specified.*

- *Let $D = B_H(k)$ and $Q = S'(k)$. Uniformly in $M \ge L^2/2$,*

$$\begin{aligned} &\mathbb{P}\left(\#\mathsf{A}^{out}_{0}(n,k,L) \le c_* M L^2, X_Q(D,0) = M\right) \\ &\quad \le (1-c_*)\mathbb{P}(X_Q(D,0) = M). \end{aligned} \tag{5.2}$$

- *Let $D = Rect(n)$ and $Q = \partial_{\mathbb{Z}^d_+} Rect(n)$ (the union of sides of $Rect(n)$ not lying along $S(0)$). Uniformly in $M \ge n^2/2$,*

$$\begin{aligned} &\mathbb{P}\left(\#\mathsf{A}^{out}_{0}(4n,4n,8n) \le c_* M n^2, X_Q(D,0) = M\right) \\ &\quad \le (1-c_*)\mathbb{P}(X_Q(D,0) = M). \end{aligned} \tag{5.3}$$

- *Let $D = Ann'(n/4, 5n)$ and $Q = \partial_- Ann'(n/4,5n)$. Uniformly in $\mathbf{z} \in B_H(4n)$ and in $M \ge n^2/2$,*

$$\mathbb{P}\left(\#\mathsf{A}^{in}_{\mathbf{z}}(n) \le c_* M n^2,\ X_Q(D,\mathbf{z}) = M\right) \le (1-c_*)\mathbb{P}(X_Q(D,\mathbf{z}) = M). \tag{5.4}$$

The remainder of this section is devoted to the proof of Theorem 5.1 (Section 5.1).

5.1 Proof of Theorem 5.1

We will prove only (5.2), since (5.3) has a very similar proof, and since both (5.2) and (5.3) are harder than (5.4) (involving, in particular, the restricted cluster appearing in A^{out}). For the purpose of abbreviation, for $X_{S'(m)}(B_H(m), 0)$ we write $X(m)$ throughout this section only, and similarly set

$$X^{K\text{-}irr}(m) = X^{K\text{-}irr}_{S'(m)}(B_H(m), 0) \quad \text{and} \quad REG(K, m) = REG_{B_H(m)}(K)$$

(recall Definition 3).

Although some parts of the arguments here are similar to that of Section 4, there are many differences in the details. We will need to build extensions of spanning clusters of large boxes, involving a number of parameters. The statements that follow will provide various bounds that are uniform in n sufficiently large with $n \le k \le 2n$, $n^{1/10} \le L \le 3n - k$, and $M \ge L^2/2$. The main restriction on n will come from it having to be very large relative to the regularity parameter K, which will be fixed relative to all other parameters but larger than some constant depending on d and the particular edge set of $\mathbb{Z}^d$ chosen.

We say a pair of vertices $(\mathbf{z}, \mathbf{y})$ is (k, L, K)-admissible if

(1) $\mathbf{z} \in S'(k)$ and $\mathbf{y} \in (\mathbf{z} + B_H(L)) \setminus B_H(k)$,

(2) $\mathbf{z} \in REG(K, k)$,

(3) $0 \overset{B_H(k)}{\longleftrightarrow} \mathbf{z}$,

(4) $\mathbf{z} \overset{Ann'(n/2,4n)}{\longleftrightarrow} \mathbf{y}$,

(5) The status of the edge $\{\mathbf{z}, \mathbf{z}'\}$ is pivotal for the event $0 \leftrightarrow \mathbf{y}$, where $\mathbf{z}'$ is a deterministically chosen neighbor of $\mathbf{z}$ in $[\mathbf{z} + B_H(K)] \setminus B_H(k)$.

Define the random number of admissible pairs

$$Y(k, L, K) = \#\{(\mathbf{z}, \mathbf{y}) \colon (\mathbf{z}, \mathbf{y}) \text{ is } (k, L, K)\text{-admissible}\}.$$

Let $X^{K\text{-}reg}(k) = X(k) - X^{K\text{-}irr}(k) = \#REG(K, k)$. The argument will follow from the second-moment method, using the bounds in the following pair of lemmas, followed by a local modification argument similar to that in the proof of Lemma 4.6.

LEMMA 5.2. *Let K be fixed larger than some dimension- and edge-set-dependent constant. There exists a constant $c = c(K) > 0$ such that*

$$\mathbb{E}Y(k, L, K)\mathbb{1}_{X^{K\text{-}reg}(k) = M} \ge cML^2\mathbb{P}(X^{K\text{-}reg}(k) = M), \tag{5.5}$$

uniformly in n large (relative to K), for $n \le k \le 2n$, $n^{1/10} \le L \le 3n - k$, and $M \ge L^2/2$.

LEMMA 5.3. *Let K be fixed larger than some dimension- and edge-set-dependent constant. There exists a constant $C = C(K)$ such that the following holds for all n large, for $n \leq k \leq 2n$, $n^{1/10} \leq L \leq 3n - k$, and $M \geq L^2/2$:*

$$\mathbb{E}Y(k, L, K)^2 \mathbb{1}_{X^{K\text{-}reg}(k)=M} \leq CM^2 L^4 \mathbb{P}(X^{K\text{-}reg}(k) = M). \tag{5.6}$$

PROOF OF LEMMA 5.2. As in the proof of (2B) from Lemma 4.6, we introduce three events that can be used to build connections from $\mathbf{z}$ to $\mathbf{y}$. In these definitions, we generally have $\mathbf{z} \in S'(k)$, $\mathbf{y} \in [\mathbf{z} + B_H(L)] \setminus B_H(k)$, and $\widetilde{\mathbf{z}} \in (\mathbf{z} + B_H(2K)) \setminus B_H(k + K)$. Let

$$\mathcal{E}_1(\mathbf{z}, K, M) := \{\mathbf{z} \overset{B_H(k)}{\longleftrightarrow} 0,\ \mathbf{z} \in REG(K, k), \text{ and } X^{K\text{-}reg}(k) = M\},$$

$$\mathcal{E}_2(\mathbf{z}, \widetilde{\mathbf{z}}, \mathbf{y}) := \{\widetilde{\mathbf{z}} \overset{Ann'(n/2,4n)}{\longleftrightarrow} \mathbf{y} \text{ off } \mathfrak{C}_{B_H(k)}(\mathbf{z})\},$$

$$\mathcal{E}_3(\mathbf{z}, \widetilde{\mathbf{z}}) := \{\mathfrak{C}(\mathbf{z}) \cap \mathfrak{C}(\widetilde{\mathbf{z}}) = \varnothing\}.$$

We continue by proving a pair of claims about the probabilities of these events.

CLAIM 5.4. *There exists a $c > 0$ depending only on d such that the following holds. Let K be larger than some fixed dimension- and edge-set-dependent constant, and n be large relative to K; let $n^{1/10} \leq L \leq 3n - k$ and $M \geq L^2/2$. For any $\mathbf{z} \in S'(k)$ and $\widetilde{\mathbf{z}} \in (\mathbf{z} + B_H(2K)) \setminus B_H(k + K)$, we have*

$$\sum_{\mathbf{y} \in [\mathbf{z}+B_H(L)] \setminus B_H(k)} \mathbb{P}(\mathcal{E}_1(\mathbf{z}, K, M) \cap \mathcal{E}_2(\mathbf{z}, \widetilde{\mathbf{z}}, \mathbf{y})) \geq cL^2 \mathbb{P}(\mathcal{E}_1(\mathbf{z}, K, M)).$$

PROOF. Note that the status of $\mathcal{E}_1$ can be determined by examining $\mathfrak{C}_{B_H(k)}(\mathbf{z})$. We can thus condition on $\mathfrak{C}_{B_H(k)}(\mathbf{z})$ and bound the conditional probability of $\mathcal{E}_2$, similarly to the beginning of the proof of Claim 4.7:

$$\begin{aligned}
&\sum_{\mathbf{y} \in [\mathbf{z}+B_H(L)] \setminus B_H(k)} \mathbb{P}(\mathcal{E}_1(\mathbf{z}, K, M) \cap \mathcal{E}_2(\mathbf{z}, \widetilde{\mathbf{z}}, y)) \\
&\geq \sum_{\mathcal{C} \in \mathcal{E}_1} \mathbb{P}(\mathfrak{C}_{B_H(k)}(\mathbf{z}) = \mathcal{C}) \\
&\qquad \cdot \sum_{\mathbf{y} \in [\mathbf{z}+B_H(L)] \setminus B_H(k)} \mathbb{P}(\widetilde{\mathbf{z}} \overset{Ann'(n/2,4n)}{\longleftrightarrow} \mathbf{y} \text{ off } \mathcal{C} \mid \mathfrak{C}_{B_H(k)}(\mathbf{z}) = \mathcal{C}) \\
&= \sum_{\mathcal{C} \in \mathcal{E}_1} \mathbb{P}(\mathfrak{C}_{B_H(k)}(\mathbf{z}) = \mathcal{C}) \sum_{\mathbf{y} \in [\mathbf{z}+B_H(L)] \setminus B_H(k)} \mathbb{P}(\widetilde{\mathbf{z}} \overset{Ann'(n/2,4n)}{\longleftrightarrow} \mathbf{y} \text{ off } \mathcal{C}),
\end{aligned}$$

where we have used the fact that the events in the last sum depend on disjoint sets of edges. We estimate the terms of the second sum using a union bound on vertices of $\mathcal{C}$:

$$\mathbb{P}\big(\widetilde{\mathbf{z}} \overset{Ann'(n/2,4n)}{\longleftrightarrow} \mathbf{y} \text{ off } \mathcal{C}\big) \geq \mathbb{P}\big(\widetilde{\mathbf{z}} \overset{Ann'(n/2,4n)}{\longleftrightarrow} \mathbf{y}) - \sum_{\zeta \in \mathcal{C}} \mathbb{P}(\{\zeta \leftrightarrow \widetilde{\mathbf{z}}\} \circ \{\zeta \leftrightarrow \mathbf{y}\}),$$

where we have used the fact that $\{\widetilde{\mathbf{z}} \leftrightarrow \zeta\} \supseteq \{\widetilde{\mathbf{z}} \overset{Ann'(n/2,4n)}{\longleftrightarrow} \zeta\}$ and similarly with $\{\zeta \leftrightarrow \mathbf{y}\}$. Applying the BK inequality gives the bound

$$\begin{aligned}
&\mathbb{P}\big(\widetilde{\mathbf{z}} \overset{Ann'(n/2,4n)}{\longleftrightarrow} \mathbf{y}\big) - \sum_{\zeta\in\mathcal{C}} \mathbb{P}(\{\zeta \leftrightarrow \widetilde{\mathbf{z}}\} \circ \{\zeta \leftrightarrow \mathbf{y}\}) \\
&\quad \geq \mathbb{P}\big(\widetilde{\mathbf{z}} \overset{Ann'(n/2,4n)}{\longleftrightarrow} \mathbf{y}\big) - \sum_{\zeta\in\mathcal{C}} \mathbb{P}(\zeta \leftrightarrow \widetilde{\mathbf{z}})\mathbb{P}(\zeta \leftrightarrow \mathbf{y}) \geq \\
&\quad \geq \mathbb{P}\big(\widetilde{\mathbf{z}} \overset{Ann'(n/2,4n)}{\longleftrightarrow} \mathbf{y}\big) - \sum_{t=\lfloor \log_2(K)\rfloor}^{\infty} \sum_{\substack{\zeta\in\mathcal{C},\\ \zeta\in[\widetilde{\mathbf{z}}+Ann(2^t,2^{t+1})]}} \mathbb{P}(\zeta \leftrightarrow \widetilde{\mathbf{z}})\mathbb{P}(\zeta \leftrightarrow \mathbf{y}).
\end{aligned}$$

Note we began the sum above not from $t = 0$ because $\widetilde{\mathbf{z}}$ is at least distance K away from $\mathcal{C}$.

We sum the above over $\mathbf{y}$ and use Theorem 1.2 on the first term on the right-hand side, finding a lower bound of cL^2 for a c uniform for parameter values as in the statement of Claim 5.4. (Our restrictions on the value of n and L force L to be large relative to K so that the distance between $\widetilde{\mathbf{z}}$ and the "typical" $\mathbf{y}$ is order L.) For the other term, we use (1.7) for an upper bound on the two-point function; the result is

$$\begin{aligned}
&\sum_{\mathbf{y}\in[\mathbf{z}+B_H(L)]\setminus B_H(k)} \mathbb{P}\big(\widetilde{\mathbf{z}} \overset{Ann'(n/2,4n)}{\longleftrightarrow} \mathbf{y} \text{ off } \mathcal{C}\big) \\
&\quad \geq cL^2 - CL^2 \sum_{t=\lfloor \log_2(K)\rfloor}^{\infty} \sum_{\substack{\zeta\in\mathcal{C},\\ \zeta\in[\widetilde{\mathbf{z}}+Ann(2^t,2^{t+1}])}} \mathbb{P}(\zeta \leftrightarrow \widetilde{\mathbf{z}}).
\end{aligned}$$

Furthermore, we have $\widetilde{\mathbf{z}} + B(2^s) \subseteq \mathbf{z} + B(2^{s+1})$ for $s \geq \log_2(2K)$, and note that for any $\mathcal{C}$ satisfying the requirements of $\mathcal{E}_1$ and any $m \geq K$, we necessarily have $\#(\mathcal{C} \cap \mathbf{z} + B(m)) \leq m^4 \log^7(m)$. Using these in the above gives a lower bound of

$$\begin{aligned}
&\geq cL^2 - CL^2 \sum_{t=\log_2(K)}^{\infty} (\#(\mathcal{C} \cap [\mathbf{z} + B(2^{t+2})]))2^{t(2-d)} \\
&\geq cL^2 - C'L^2 \sum_{t=\log_2(K)}^{\infty} t^7 2^{4t} 2^{t(2-d)} \\
&\geq cL^2 - C''L^2 K^{6-d} \log^7(K).
\end{aligned}$$

Again, the constant C'' is uniform for parameter values in the appropriate range. Therefore, whenever K is sufficiently large and fixed relative to the other parameters, the second term is negligible relative to the first. □

Our next claim gives the ability to add on $\mathcal{E}_3$ to the intersection in the last claim.

CLAIM 5.5. *For each $K > 0$ sufficiently large, there exists a $c = c(K) > 0$ such that the following holds uniformly in n, k, L, and M as in the statement of Theorem 5.2. For any $\mathbf{z} \in S'(k)$, there exists a $\widetilde{\mathbf{z}} \in [\mathbf{z} + B_H(2K)] \setminus B_H(k+K)$ such that*

$$\sum_{\mathbf{y}\in[\mathbf{z}+B_H(L)]\setminus B_H(k)} \mathbb{P}(\mathcal{E}_1(\mathbf{z}, K, M) \cap \mathcal{E}_2(\mathbf{z}, \widetilde{\mathbf{z}}, \mathbf{y}) \cap \mathcal{E}_3(\mathbf{z}, \widetilde{\mathbf{z}})) \geq cL^2 \mathbb{P}(\mathcal{E}_1(\mathbf{z}, K, M)).$$

PROOF. Let ζ be a uniformly chosen (independently of the percolation process) random vertex of $[\mathbf{z} + B_H(2K)] \setminus B_H(k+K)$, and let E_ζ denote expectation with respect to this random choice. We will prove that for K large,

$$E_\zeta \sum_{\mathbf{y}\in[\mathbf{z}+B_H(L)]\setminus B_H(k)} \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3) \geq cL^2 \mathbb{P}(\mathcal{E}_1), \tag{5.7}$$

where $\mathcal{E}_2 = \mathcal{E}_2(\mathbf{z}, \zeta, y)$ and $\mathcal{E}_3 = \mathcal{E}_3(\mathbf{z}, \zeta)$. This will suffice to show the claim. Indeed, for (5.7) to hold, there must be some $\widetilde{\mathbf{z}}$ such that, when $\zeta = \widetilde{\mathbf{z}}$, the quantity inside the expectation E_ζ is at least $cL^2\mathbb{P}(\mathcal{E}_1)$.

For any possible value of ζ, if $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3^c$ occurs, then there exists a vertex $\mathbf{v}$ such that

$$\mathcal{E}_1 \cap \{0 \leftrightarrow \mathbf{v}\} \circ \{\zeta \leftrightarrow \mathbf{v}\} \circ \{\mathbf{v} \leftrightarrow \mathbf{y}\}$$

occurs. (Compare to the reasoning above (4.9), where a similar vertex v is found.) In particular, by the BK inequality, for this value of ζ we have

$$\mathbb{P}\big(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3^c\big) \leq \sum_{\mathbf{v}\in\mathbb{Z}^d} \mathbb{P}(\mathcal{E}_1 \cap \{0 \leftrightarrow \mathbf{v}\})\mathbb{P}(\zeta \leftrightarrow \mathbf{v})\mathbb{P}(\mathbf{v} \leftrightarrow \mathbf{y}).$$

Summing the above over $\mathbf{y} \in [\mathbf{z} + B_H(L)] \setminus B_H(k)$ and using (1.7), we get a factor of at most a constant multiple of L^2, uniform in the value of ζ. Applying (1.7) again:

$$\begin{aligned} &\sum_{\mathbf{y}\in[\mathbf{z}+B_H(L)]\setminus B_H(k)} \mathbb{P}\big(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3^c\big) \\ &\quad \leq CL^2 \sum_{\mathbf{v}\in\mathbb{Z}^d} \mathbb{P}(\mathcal{E}_1 \cap \{0 \leftrightarrow \mathbf{v}\})\mathbb{P}(\zeta \leftrightarrow \mathbf{v}) \\ &\quad \leq C'L^2 \sum_{\mathbf{v}\in\mathbb{Z}^d} \mathbb{P}(\mathcal{E}_1 \cap \{0 \leftrightarrow \mathbf{v}\})\|\zeta - \mathbf{v}\|^{2-d}. \end{aligned} \tag{5.8}$$

The right-hand side of (5.8) is nearly identical to that of (4.11). The differences are that now 0 plays the role of $\mathbf{x}$, there is a different prefactor ($C'L^2$ instead of Cn^{2-d}), and the definition of $\mathcal{E}_1$ is somewhat modified. A proof very similar to the one used to treat (4.11) gives that (compare to the negative term in (4.13))

$$E_\zeta \sum_{\mathbf{v}\in\mathbb{Z}^d} \mathbb{P}(\mathcal{E}_1 \cap \{0 \leftrightarrow \mathbf{v}\})\|\zeta - \mathbf{v}\|^{2-d} \leq C''K^{6-d}\log^8(K)\mathbb{P}(\mathcal{E}_1),$$

uniformly over K sufficiently large and over n, k, L, M, and $\mathbf{z}$ as in the statement of Claim 5.5.

We can thus uniformly lower-bound $E_\zeta \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3)$:

$$\begin{aligned} &E_\zeta \sum_{\mathbf{y} \in [\mathbf{z}+B_H(L)] \setminus B_H(k)} \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3) \\ &= E_\zeta \sum_{\mathbf{y} \in [\mathbf{z}+B_H(L)] \setminus B_H(k)} [\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) - \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3^c)] \\ &\geq cL^2 \mathbb{P}(\mathcal{E}_1) - CL^2 K^{6-d} \log^8 K \, \mathbb{P}(\mathcal{E}_1), \end{aligned} \tag{5.9}$$

where we have used Claim 5.4 for the inequality. Taking K sufficiently large and using the uniformity of the constants c, C' establishes (5.7). □

We will now complete the proof of the first moment bound (5.5) from Theorem 5.2 using Claim 5.5. We claim that for any pair $(\mathbf{z}, \mathbf{y})$ with $\mathbf{z} \in S'(k)$ and $\mathbf{y} \in [\mathbf{z} + B_H(L)] \setminus B_H(k)$,

$$\begin{aligned} &\mathbb{P}\big((\mathbf{z}, \mathbf{y}) \text{ is } (k, L, K)\text{-admissible and } X^{K\text{-}reg}(k) = M\big) \\ &\quad \geq c(K) \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3) \end{aligned} \tag{5.10}$$

for a constant $c = c(K) > 0$, for all K larger than some constant (depending only on the dimension d and the edge set of $\mathbb{Z}^d$ being considered). The bound of (5.10) is uniform in n, k, L, and M as in the statement of Theorem 5.2, where $\widetilde{\mathbf{z}}$ is chosen for $\mathbf{z}$ according to Claim 5.5 (note $\mathbf{z}, \widetilde{\mathbf{z}}$ appear as arguments in the $\mathcal{E}_i$ events on the right-hand side). The proof of (5.10) is via an edge modification argument similar to the one used to prove (4.14), so we do not detail it here. Roughly speaking, one must open edges to connect $\mathbf{z}$ to $\widetilde{\mathbf{z}}$ in a way that guarantees the pivotality of $\{\mathbf{z}, \mathbf{z}'\}$ without, for instance, changing the condition $\mathbf{z} \in REG(K, k)$ guaranteed by $\mathcal{E}_1$.

Given (5.10), the conclusion of the proof is immediate. Summing the bound over $\mathbf{y} \in [\mathbf{z} + B_H(L)] \setminus B_H(k)$ and using Claim 5.5 gives

$$\sum_{\mathbf{y} \in [\mathbf{z}+B_H(L)] \setminus B_H(k)} \mathbb{P}\big((\mathbf{z}, \mathbf{y}) \text{ is } (k, L, K)\text{-admissible and } X^{reg}(k) = M\big) \geq cL^2 \mathbb{P}(\mathcal{E}_1).$$

Summing now over $\mathbf{z}$ in the above gives a lower bound $cML^2 \mathbb{P}(X^{K\text{-}reg}(k) = M)$, since on $\mathcal{E}_1$ we have $X^{K\text{-}reg}(k) = M$ definitionally. □

PROOF OF LEMMA 5.3. We abbreviate $\mathbb{1}_M$ for $\mathbb{1}_{X^{K\text{-}reg}(k)=M}$ and $Y = Y(k, L, K)$ and write

$$\mathbb{E}\big[Y^2 \mathbb{1}_M\big] = \sum_{\substack{\mathbf{z}_1, \mathbf{y}_1; \\ \mathbf{z}_2, \mathbf{y}_2}} \mathbb{P}((\mathbf{z}_1, \mathbf{y}_1) \text{ and } (\mathbf{z}_2, \mathbf{y}_2) \text{ are } (k, L, K)\text{-admissible}). \tag{5.11}$$

A typical term of the above sum can be written as (using the abbreviation "$\mathbf{z}_i'$ pivotal" instead of "$\{\mathbf{z}_i, \mathbf{z}_i'\}$ pivotal")

$$\begin{aligned} &\mathbb{P}(\mathfrak{C}_{B_H(k)}(0) = \mathcal{C}) \\ &\quad \cdot \mathbb{P}\big(\mathbf{y}_i \overset{Ann'(n/2,4n)}{\longleftrightarrow} \mathbf{z}_i, \mathbf{z}_i' \text{ pivotal for } \{0 \leftrightarrow \mathbf{y}_i\}, \ i = 1, 2 \mid \mathfrak{C}_{B_H(k)}(0) = \mathcal{C}\big) \end{aligned} \tag{5.12}$$

where $\mathcal{C}$ is such that conditions 1, 2, and 3 of the definition of admissibility hold for the given $\mathbf{z}_1$ and $\mathbf{z}_2$ (note that these depend only on $\mathfrak{C}_{B_H(k)}(0)$). We consider first the case that $\mathbf{z}_1 \neq \mathbf{z}_2$ and $\mathbf{y}_1 \neq \mathbf{y}_2$.

On the event

$$\big\{\mathbf{y}_i \overset{Ann'(n/2,4n)}{\longleftrightarrow} \mathbf{z}_i,\ \mathbf{z}'_i \text{ pivotal for } \{0 \leftrightarrow \mathbf{y}_i\big\},\ i = 1,2\} \cap \{\mathfrak{C}_{B_H(k)}(0) = \mathcal{C}\}$$

we claim there exist disjoint open paths γ_1 (resp. γ_2) connecting $\mathbf{y}_1$ to $\mathbf{z}'_1$ (resp. $\mathbf{y}_2$ to $\mathbf{z}'_2$) and avoiding $\mathcal{C}$. To choose γ_1, consider any path σ from 0 to $\mathbf{y}_1$. Since $\{\mathbf{z}_1, \mathbf{z}'_1\}$ is pivotal for the connection, this path passes through $\mathbf{z}'_1$; the path must subsequently never intersect $\mathcal{C}$ (otherwise $\{\mathbf{z}_1, \mathbf{z}'_1\}$ could be bypassed, contradicting pivotality), and so the terminal segment of σ starting from $\mathbf{z}'_1$ may be used as γ_1. If one chooses γ_2 similarly, we see that necessarily $\gamma_1 \cap \gamma_2 = \varnothing$. Indeed, if γ_1 and γ_2 intersected at some $\mathbf{v}$, then following γ_2 from $\mathbf{y}_2$ to $\mathbf{v}$ and then following γ_1 from $\mathbf{v}$ to $\mathbf{z}'_1$ (or following γ_1 from $\mathbf{y}_1$ to $\mathbf{v}$ and then following γ_2), one sees that one of the edges $\{\mathbf{z}_i, \mathbf{z}'_i\}$ is not pivotal, a contradiction.

Having found such γ_1 and γ_2, one sees that when $\mathbf{z}_1 \neq \mathbf{z}_2$ and $\mathbf{y}_1 \neq \mathbf{y}_2$, the conditional probability in (5.12) is at most

$$\mathbb{P}\big(\mathbf{y}_1 \leftrightarrow \mathbf{z}'_1 \text{ off } \mathcal{C}\big)\mathbb{P}\big(\mathbf{y}_2 \leftrightarrow \mathbf{z}'_2 \text{ off } \mathcal{C}\big) \le A_1^2 \|\mathbf{z}'_1 - \mathbf{y}_1\|^{2-d} \|\mathbf{z}'_1 - \mathbf{y}_1\|^{2-d}.$$

Summing the above over $\mathbf{y}_1 \neq \mathbf{y}_2$ gives a uniform upper bound of CL^4. Putting this in (5.12) and performing the sum over $\mathcal{C}$, then doing an additional sum over $\mathbf{z}_1 \neq \mathbf{z}_2$ gives

$$\begin{aligned} &\sum_{\substack{\mathbf{z}_1 \neq \mathbf{z}_2,\\ \mathbf{y}_1 \neq \mathbf{y}_2}} \mathbb{P}((\mathbf{z}_1, \mathbf{y}_1) \text{ and } (\mathbf{z}_2, \mathbf{y}_2) \text{ are } (k, L, K)\text{-admissible}) \\ &\qquad \le CM^2 L^4 \mathbb{P}(X^{K\text{-}reg}(k) = M). \end{aligned} \tag{5.13}$$

When summing over terms in (5.11) where $\mathbf{z}_1 = \mathbf{z}_2$, one is essentially computing an upper bound of the second moment of the cluster size of $\mathbf{z}_1$; the resulting bound is $CML^6\mathbb{P}(X^{K\text{-}reg}(k) = M)$. Since $M \ge L^2/2$, this sum has an upper bound identical to that in (5.13), completing the proof. □

Given (5.5) and (5.6), Theorem 5.1 now follows by a second moment argument similar to the one immediately following Lemma 4.6 above. □

6 Upper Bound on $\pi_H(n)$

This section is devoted to the proof of the upper bound $\pi_H(n) \le Cn^{-3}$ from part (a) of Theorem 1.1, using the results of Theorem 5.1. This proof has two main ideas. The first main idea is an upper bound on the cardinality of $\mathfrak{C}_H(0) \cap Ann_H(n, 2n)$, which gives some information about scaling in large clusters and plays the role that knowledge of the cluster size exponent δ would otherwise play (recall we have not yet proved part (c) of Theorem 1.1). A key ingredient is a mass-transport inequality, which controls the number of large half-space clusters. The second main idea is an inductive argument that allows us to "bootstrap" control of $\pi_H(2n)$ from $\pi_H(n)$. This argument is based on a lemma that is similar in spirit to lemma 2.3 of [22], with some major differences. These reflect the different geometry of $\mathbb{Z}_+^d$ and the fact that we cannot use the two-point function or size exponents—which were used in [22]—having not yet proved parts (b) or (c) of Theorem 1.1.

Recall the definition of a mass-transport rule from Section 2.3. In proving the upper bound for $\pi_H(n)$, we fix a particular $\mathfrak{m}$ once and for all for each fixed value of n:

$$\mathfrak{m}(0, \mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \mathsf{A}_0^{\mathrm{out}}(n, n, 2n), \\ 0 & \text{otherwise.} \end{cases}$$

The set A_0^{out} was defined at (5.1).

The bound we will need for proving our main theorem comes from a comparison of asymptotics for $\mathbb{E}\mathsf{send}$ and $\mathbb{E}\mathsf{get}$. Let $\kappa > 0$ be arbitrary (in practice, typically small). We define the event

$$A(\kappa) := \{\mathsf{send} \geq \kappa n^4\}. \tag{6.1}$$

By the definition (6.1),

$$\mathbb{E}\mathsf{send} \geq \kappa n^4 \mathbb{P}(A(\kappa)). \tag{6.2}$$

An upper bound on $\mathbb{E}\mathsf{send}$ follows via Theorem 5.1; Lemma 2.1 and (6.2) then show a corresponding upper bound for $\mathbb{P}(A(\kappa))$. This is encapsulated in the following lemma.

LEMMA 6.1. *There exists a C such that, uniformly in n,*

$$\mathbb{E}\mathsf{get} \leq Cn. \tag{6.3}$$

In particular, we have the following bound uniformly in κ and n:

$$\mathbb{P}(A(\kappa)) \leq \frac{C}{\kappa n^3}. \tag{6.4}$$

PROOF. Note that 0 receives mass from $\mathbf{x}$ if and only if both

$$\text{(i) } 0 \in \mathbf{x} + Ann_H(n, 3n) \quad \text{and} \quad \text{(ii) } \Big\{0 \overset{\mathbf{x}+Ann'(n/2,4n)}{\longleftrightarrow} \mathbf{x}\Big\}$$

(recall $Ann_H(\ell_1, \ell_2) = B_H(\ell_2) \setminus B_H(\ell_1)$). The set o f$\mathbf{x}$ which satisfies the non-random condition (i) is exactly $-Ann_H(n, 3n)$. We break get into a sum of contributions over "slices" depending on $\mathbf{e}_1$-distance, setting $T(j) = [-Ann_H(n, 3n) \cap S(-j)]$ and

$$Y(j) = \Big\{\mathbf{x} \in T(j), \mathbf{x} \overset{\mathbf{x}+Ann'(n/2,4n)}{\longleftrightarrow} 0\Big\} \quad 0 \leq j \leq 3n. \tag{6.5}$$

See Figure 6.1 for a sketch. In particular, $\mathsf{get} = \sum_j \#Y(j)$. We will use (5.4) of Theorem 5.1 to argue that if $Y(j)$ is too large, then $\mathfrak{C}(0) \cap [\mathbf{z} + B_-(n/2)]$ is abnormally large for some choice of $\mathbf{z} \in T(j)$.

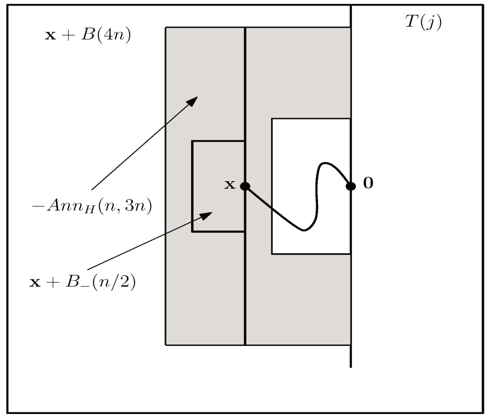

FIGURE 6.1. Referenced below (6.5). This is an instance of the event when $\mathbf{x} \in T(j)$ has an open connection to 0 staying within $\mathbf{x} + B(4n)$ and avoiding $\mathbf{x} + B_-(n/2)$.

To that end, for $\mathbf{x} \in T(j)$ we set

$$X_{\mathbf{x}}(j) := \#\{\mathbf{y} \in Y(j) \colon \|\mathbf{y} - \mathbf{x}\| \le n/4\}.$$

There exists a deterministic set $\mathcal{S}_j \subseteq T(j)$ of no more than 5^{d-1} vertices such that, for any $\mathbf{y} \in T(j)$, there is an $\mathbf{x} \in \mathcal{S}_j$ such that $\|\mathbf{y} - \mathbf{x}\| \le n/4$. It follows that $\#Y(j) \le \sum_{\mathbf{x} \in \mathcal{S}_j} X_{\mathbf{x}}(j)$. If we can show that

$$\mathbb{E} X_{\mathbf{z}}(j) \le C \quad \text{uniformly in } n,\ 0 \le j \le 3n,\ \mathbf{z} \in \mathcal{S}_j, \tag{6.6}$$

we can immediately conclude that $\sum_j \mathbb{E}\#Y(j) \le Cn$ and the lemma is proved.

We now prove (6.6). We will apply the third part of Theorem 5.1, but in shifted form. Define $D = \mathbf{x} + Ann'(n/4, 5n)$ and $Q = \partial_-[\mathbf{x} + Ann'(n/4, 5n)]$. Each $\mathbf{y} \in Y(j)$ having $\|\mathbf{y} - \mathbf{x}\| \le n/4$ also satisfies $\mathbf{y} \in Q$. The vertex $\mathbf{y}$ is connected to 0 by a path lying entirely in $\mathbf{y} + Ann'(n/2, 4n)$; in particular, this path lies in D. We therefore have the upper bound

$$\begin{aligned} &\mathbb{P}(X_{\mathbf{x}}(j) \ge M) \le \mathbb{P}(X_Q(D, 0) \ge M) \quad \text{for all } M, \\ &\qquad \text{and so} \quad \mathbb{E}[X_{\mathbf{x}}(j)] \le \mathbb{E}[X_Q(D, 0)]. \end{aligned} \tag{6.7}$$

We now bound the right-hand side of (6.7) by

$$\begin{aligned} &\mathbb{P}\big(\#\mathfrak{C}(0) \cap [\mathbf{x} + B(5n)] \ge c_* n^2 M\big) \\ &\quad + \mathbb{P}\big(X_Q(D, 0) \ge M,\ \#\mathfrak{C}(0) \cap [\mathbf{x} + B(5n)] \le c_* n^2 M\big), \end{aligned}$$

where c_* is from Theorem 5.1. We note that the shifted analogue of $\#\mathsf{A}_{\mathbf{0}}^{in}$ (shifted so $\mathbf{x}$ plays the role of 0) is a lower bound for $\#[\mathfrak{C}(0) \cap [\mathbf{x} + B(5n)]]$. Applying

Theorem 5.1 to the second term in the case when $M \geq n^2$ and rearranging, we see

$$\mathbb{P}(X_Q(D,0) \geq M) \leq c_*^{-1}\mathbb{P}\big(\#[\mathfrak{C}(0) \cap [\mathbf{x} + B(5n)]] \geq c_* n^2 M\big) \quad \text{for all } M \geq n^2. \tag{6.8}$$

Thus, beginning with (6.7), we see

$$\begin{aligned}
&\mathbb{E}X_{\mathbf{x}}(j)\\
&\leq \mathbb{E}[X_Q(D,0);\, 0 < X_Q(D,0) \leq n^2] + \mathbb{E}[X_Q(D,0);\, X_Q(D,0) > n^2]\\
&\leq n^2\mathbb{P}(X_Q(D,0) > 0) + \sum_{M=n^2}^{\infty} \mathbb{P}(X_Q(D,0) \geq M)\\
&\leq n^2\mathbb{P}(X_Q(D,0) > 0) + c_*^{-1} \sum_{M=n^2}^{\infty} \mathbb{P}\big(\#[\mathfrak{C}(0) \cap [\mathbf{x} + B(5n)]] \geq c_* n^2 M\big)\\
&\leq n^2\pi(n) + c_*^{-2}n^{-2}\mathbb{E}\big[\#[\mathfrak{C}(0) \cap [\mathbf{x} + B(5n)]];\#[\mathfrak{C}(0) \cap [\mathbf{x} + B(5n)]] \geq c_* n^4\big],
\end{aligned} \tag{6.9}$$

where in the second-to-last line we used (6.8). Using (1.9), the first term of (6.9) is uniformly bounded by a constant. Using Lemma 2.2, the second term of (6.9) is also bounded by a constant, giving $\mathbb{E}X_{\mathbf{x}}(j) \leq C$ and completing the proof of Lemma 6.1. □

We continue with the proof of the upper bound from part (a) of Theorem 1.1, namely

$$\pi_H(n) \leq Cn^{-3}. \tag{6.10}$$

The main remaining ingredient is the following lemma, which relates $\pi_H(n(1+\lambda))$ to $\pi_H(n)$, where $\lambda > 0$ is small but fixed relative to n.

LEMMA 6.2. *There exist positive constants C_1, C_2, c_1 such that the following hold: For each $\lambda \in (0,1]$, there exists a constant $\varepsilon_0 = \varepsilon_0(\lambda) \in (0,1)$ such that, for all $0 < \varepsilon \leq \varepsilon_0$,*

$$\pi_H(n(1+\lambda)) \leq \frac{C_1}{\varepsilon n^3} + C_2\varepsilon^{3/5}\lambda^{-2}\pi_H(n) + (1-c_1)\pi_H(n) \tag{6.11}$$

uniformly in n large relative to λ.

We first prove (6.10) assuming the veracity of Lemma 6.2 and then establish the lemma.

PROOF OF (6.10). We begin by choosing λ small enough to make the third term of (6.11) negligible. Namely, fix $0 < \lambda < 1/2$ such that

$$(1+\lambda)^3(1-c_1) \leq (1-c_1/2). \tag{6.12}$$

We will bootstrap a bound for $\pi_H(n)$ assuming it holds for $\pi_H(m)$, $m < n$. To this end, set $n_0 := \lceil 8\lambda^{-1} + c_*^{-1}\rceil$ and let $K > 0$ be a large constant such that

$$\pi_H(m) \leq K/m^3 \quad \text{for all } 1 \leq m \leq n_0.$$

We will also enlarge K if necessary so that

$$\max\left\{\frac{C_1(36C_2)^{5/3}}{c_1^{5/3}\lambda^{10/3}},\ \frac{C_1}{\varepsilon_0}\right\} \le c_1K/64. \tag{6.13}$$

We show inductively that, for each $m \ge 0$,

$$\begin{gathered}\pi_H(n_0(1+\lambda)^{m+1}) \le K/(n_0(1+\lambda)^{m+1})^3 \\ \text{assuming} \quad \pi_H(n_0(1+\lambda)^m) \le K/(n_0(1+\lambda)^m)^3.\end{gathered} \tag{6.14}$$

Setting $n = n_0(1+\lambda)^m$, we apply (6.11) with the choice

$$\varepsilon = \min\left\{\varepsilon_0,\ \frac{c_1^{5/3}\lambda^{10/3}}{(36C_2)^{5/3}}\right\}.$$

Note that $(1+\lambda)^3 \le 8$, so by the bound (6.13) we have

$$\text{First term of (6.11)} \le \frac{c_1K}{8[(1+\lambda)n]^3}.$$

A direct calculation similarly gives

$$\text{Second term of (6.11)} \le \frac{2c_1K}{9[(1+\lambda)n]^3},$$

$$\text{Third term of (6.11)} \le \frac{K}{[(1+\lambda)n]^3}[1-c_1/2].$$

Pulling the three bounds above together completes the proof of (6.14).

To finish the argument for (6.10), let $n > n_0$ be arbitrary and fix m to be the largest integer such that $N := (1+\lambda)^m n_0 \le n$. Note that, since $(1+\lambda) \le 2$, we have $N \ge n/2$. Using (6.14), (6.13), and the monotonicity of π_H, we see

$$\pi_H(n) \le \pi_H(N) \le KN^{-3} \le 8Kn^{-3},$$

establishing (6.10) with $C = 8K$. □

We now prove Lemma 6.2.

PROOF OF LEMMA 6.2. Fix λ as in the statement of the lemma. If $\varepsilon \le n^{-1}$, then the above bound is simple using the one-arm exponent. Indeed, using (1.9) we see

$$\pi_H((1+\lambda)n) \le \pi_H(n) \le \pi(n) \le \frac{A_2}{n^2} \le \frac{A_2}{\varepsilon n^3},$$

and we are done. Otherwise, we will prove the bound by breaking up the connection event to $S'((1+\lambda)n)$, depending on the structure of the cluster of 0.

Recall the definition of the event $A(\kappa)$ in (6.1) and the definition of the mass-transport rule $\mathfrak{m}$ above it. We write $X(k)$ for $X_Q(D,0)$ with $D = B_H(k)$ and $Q = S'(k)$, where k is an integer satisfying $(1+\lambda/4)n \le k \le (1+\lambda/2)n$. The reason for emphasizing the k-dependence is that we will wish, in our definition of $\mathcal{D}_1$ below, to consider the first such integer value of k for which $X(k)$ is small. The idea here is similar to that of the proof of Lemma 6.1, but from the perspective of vertices receiving mass instead of sending it.

Given a value of $\varepsilon \in (n^{-1}, 1)$, we define $L = \varepsilon^{3/10} n$ and the events

$$\mathcal{D}_1(\varepsilon) := \{\exists k \in [n + \lambda n/4, n + \lambda n/2]\colon X(k) \le L^2 \text{ and } 0 \overset{\mathbb{Z}^d_+}{\longleftrightarrow} S'((1+\lambda)n)\};$$
$$\mathcal{D}_2(\varepsilon) := \{\forall k \in [n + \lambda n/4, n + \lambda n/2]\colon X(k) \ge L^2 \text{ and } \mathsf{send} < \varepsilon n^4\}.$$

The union bound and (6.4) give

$$\begin{aligned}\pi_H((1+\lambda)n) &\le \mathbb{P}(A(\varepsilon)) + \mathbb{P}(\mathcal{D}_1(\varepsilon)) + \mathbb{P}(\mathcal{D}_2(\varepsilon)) \\ &\le \frac{C}{\varepsilon n^3} + \mathbb{P}(\mathcal{D}_1(\varepsilon)) + \mathbb{P}(\mathcal{D}_2(\varepsilon)).\end{aligned}$$

It suffices to show that the two $\mathbb{P}(\mathcal{D}_i)$ terms above have upper bounds of the form of the second and third terms of (6.11).

To bound the second term, let I denote the (random) smallest integer value of k as in the definition of $\mathcal{D}_1$ such that $0 < X(k) \le L^2$. Note that on $\mathcal{D}_1$ we never have $X(k) = 0$, so we set $I = 0$ whenever some $X(k)$ is equal to 0. We explore the cluster of the origin in successive half-space boxes $B_H(k)$ until reaching $k = I$. At this point, the probability of further connection to $S'(n(1+\lambda))$ is, by Lemma 3.2:

$$\begin{aligned}&\mathbb{P}(\mathcal{D}_1(\varepsilon)) \\ &\quad = \sum_{k \in [n(1+\lambda/4), n(1+\lambda/2)]} \sum_{\mathcal{C} \in \{I = k\}} \mathbb{P}\big(0 \overset{\mathbb{Z}^d_+}{\longleftrightarrow} S'(n(1+\lambda)), \mathfrak{C}_{B_H(k)}(0) = \mathcal{C}\big) \\ &\quad \le L^2 \pi(\lambda n/4) \sum_{k \ge n(1+\lambda/4)} \mathbb{P}(I = k) \le \frac{C\varepsilon^{3/5}}{\lambda^2} \pi_H(n),\end{aligned}$$

where the second sum is over $\mathcal{C}$ giving $I = k$ and where we have used (1.9) to bound π.

The bound on $\mathcal{D}_2$ is where we use Theorem 5.1, namely (5.2). Since $\varepsilon > n^{-1}$, we have $L \ge n^{7/10}$, and so our choice of L from above is a valid choice of L in the statement of the theorem.

To set up our application of Theorem 5.1, we consider a sequence of values of k and corresponding annular regions in which extensions can be made. For each integer $i \in [0, \frac{\lambda}{4}\varepsilon^{-3/10}]$ set $k_i = (1 + \lambda/4)n + iL$ and note that $(1+\lambda/4)n \le k_i \le (1+\lambda/2)n$. Recall that c_* is the constant from Theorem 5.1 and set

$$\mathfrak{I} = \{i\colon X(k_i) > L^2, \#\mathsf{A}_0^{out}(n, k_i, L) < c_* L^4\}.$$

Here we recall the notation

$$\mathsf{A}_0^{out}(n, k, L) := \mathfrak{C}_{Ann'(n/2, 4n)}(0) \cap Ann_H(k, k+L).$$

Note that (by disjointedness of the annuli $Ann_H(k_i, k_i + L)$)

$$c_* L^4 \#\{i\colon \#\mathsf{A}_0^{out}(n, k_i, L) \geq c_* L^4\} \leq \mathsf{send},$$

and so on $\mathcal{D}_2(\varepsilon)$ we have

$$\#\{i\colon \#\mathsf{A}_0^{out}(n, k_i, L) \geq c_* L^4\} \leq \frac{1}{c_* \varepsilon^{1/5}}. \tag{6.15}$$

In particular, on $\mathcal{D}_2$, the cardinality of $\mathfrak{I}$ must be large; namely,

$$\text{on } \mathcal{D}_2(\varepsilon), \quad \#\mathfrak{I} \geq \lfloor (\lambda/4)\varepsilon^{-3/10} \rfloor - c_*^{-1}\varepsilon^{-1/5}. \tag{6.16}$$

On the other hand, using Theorem 5.1 on each value of i and summing, we have

$$\mathbb{E}\#\mathfrak{I} \leq (1 - c_*)\pi_H(n)[(\lambda/4)\varepsilon^{-3/10} + 1]. \tag{6.17}$$

z We may now apply Markov's inequality with the bound (6.17) and compare to the lower bound for $\mathbb{E}\#\mathfrak{I}$ in terms of $\mathcal{D}_2$ which follows from (6.16). This yields

$$\begin{aligned} &\big[(\lambda/4)\varepsilon^{-3/10} - c_*^{-1}\varepsilon^{-1/5} - 1\big]\mathbb{P}(D_2(\varepsilon)) \\ &\quad \leq (1 - c_*)\pi_H(n)[(\lambda/4)\varepsilon^{-3/10} + 1]. \end{aligned} \tag{6.18}$$

If ε is sufficiently small (relative only to λ and c_*), the left-hand side of (6.18) is at least

$$\frac{1 - c_*}{1 - c_*/2}[(\lambda/4)\varepsilon^{-3/10} + 1]\mathbb{P}(\mathcal{D}_2(\varepsilon)).$$

Comparing the above to (6.18) gives $\mathbb{P}(\mathcal{D}_2(\varepsilon)) \leq (1 - c_*/2)\pi_H(n)$ and completes the proof. □

7 Half-Space Two-Point Function and Cluster Sizes

In this section, we prove the remaining parts of Theorem 1.1 involving the two-point function and the tail of $\#\mathfrak{C}_{\mathbb{Z}_+^d}(0)$. The arguments use the asymptotics for $\pi_H(n)$ that have already been proved (in Sections 3 and 6). As a first step, we prove the promised alternate formulation of the half-space one-arm probability.

7.1 Alternate Version of $\pi_H(n)$

In this section, we show that the probability that 0 has an arm in $\mathbb{Z}_+^d$ to $S(n)$ is of order n^{-3}; in other words, it has the same asymptotic behavior as the probability that 0 has an arm in $\mathbb{Z}_+^d$ to $S'(n)$. Recall that $Rect(n) = [0, n] \times [-4n, 4n]^{d-1}$.

PROPOSITION 7.1. *There exists a uniform $c > 0$ such that*

$$\begin{aligned} &\widetilde{\pi}_H(n, c) := \textit{if}\ \mathbb{P}\big(0 \overset{Rect(n)}{\longleftrightarrow} S(n),\ \#[\mathfrak{C}_{Rect(n)}(0) \cap S(n)] > cn^2\big), \\ &\qquad \textit{then}\ \widetilde{\pi}_H(n, c) \geq cn^{-3}. \end{aligned} \tag{7.1}$$

Proposition 7.1 shows that the probability of 0 having a half-space arm directed in the $\mathbf{e}_1$-direction has the same order as the probability of an undirected half-space arm. We will make use of the strengthened result (7.1) later in this section (see (7.9)).

PROOF OF PROPOSITION 7.1. The proof is very similar to that of Proposition 3.1 in Section 3, and we will need to use results from that section in the argument. The main new input in the present argument is the upper bound $\pi_H(n) \leq Cn^{-3}$ proved at (6.10) above.

Suppose $\mathbf{x} \in \partial B(2n)$ is such that

$$\mathbf{x} \overset{B(2n)}{\longleftrightarrow} B(n)$$

as in (3.6); let us for simplicity consider the case that $\mathbf{x} \sim \mathbf{y}$ for some $\mathbf{y}$ with $y(1) < -2n$. Note that any open path from $\mathbf{x}$ to $B(n/2)$ in $B(2n)$ must first reach $S(n)$ without exiting $\mathbf{x} + Rect(n)$, then continue to $B(n/2)$ without exiting $\mathbf{x} + B(4n)$. Thus, applying Lemma 3.2 with $B = B(n/2)$, $A_0 = \mathbf{x} + Rect(n)$, and $A_1 = x + B(4n)$, we see that

$$\begin{aligned} &\mathbb{P}\big(\mathbf{x} \overset{\mathbf{x}+B(4n)}{\longleftrightarrow} B(n/2) \bigm| \#[\mathfrak{C}_{\mathbf{x}+Rect(n)}(\mathbf{x}) \cap S(n)] = M\big) \\ &\quad \leq M\pi(n/2) \leq CMn^{-2}. \end{aligned} \tag{7.2}$$

If instead $\mathbf{x} \sim \mathbf{y}$ for $\mathbf{y}$ having $y(1) > 2n$, or $|y(i)| > 2n$ for some $i \neq 1$, the situation is similar, with a shifted and rotated version of $Rect(n)$ used instead. Applying (3.5) with these observations and using (2.2) along with a variant for the event $\{\#\big[\mathfrak{C}_{\mathbf{x}+Rect(n)}(\mathbf{x}) \cap S(n)\big] \geq \lambda n^2\}$, we see that there are some uniform $C, c > 0$ such that, for each $\lambda > 0$,

$$\begin{aligned} cn^{d-4} &\leq \mathbb{E}\#\big\{\mathbf{x} \in \partial B(2n)\colon \mathbf{x} \overset{B(2n)}{\longleftrightarrow} B(n/2)\big\} \\ &\leq Cn^{d-1}\big[\mathbb{P}\big(\#[\mathfrak{C}_{Rect(n)}(0) \cap S(n)] \geq \lambda n^2\big) + \lambda \pi_H(n)\big] \\ &= C\lambda n^{d-4} + Cn^{d-1}\mathbb{P}(\#[\mathfrak{C}_{Rect(n)}(0) \cap S(n)] \geq \lambda n^2). \end{aligned}$$

In the last line of the above, we applied (6.10) to upper-bound the $\pi_H(n)$ term. Choosing $\lambda > 0$ small and fixed relative to n, we see that the left side of the above is at least twice the first term on the right for all large n. In particular,

$$c' n^{d-4} \leq Cn^{d-1}\mathbb{P}\big(\#[\mathfrak{C}_{\mathbf{x}+Rect(n)}(\mathbf{x}) \cap S(n)] \geq \lambda n^2\big),$$

uniformly in n. Dividing both sides of the above by n^{d-1} completes the proof. □

7.2 Two-Point Function

To better separate the proofs of the individual pieces, we restate the contents of part (b) of Theorem 1.1, consisting of bounds on the two-point function in $\mathbb{Z}_+^d$.

THEOREM 7.2. *There exists a constant* $C_1 > 0$ *such that*

$$\tau_H(0, \mathbf{x}) \leq C_1 \|\mathbf{x}\|^{1-d} \quad \textit{uniformly in } \mathbf{x} \in \mathbb{Z}_+^d \setminus \{0\}. \tag{7.3}$$

Fix $\varepsilon > 0$. Then there exists a constant $c_1 = c_1(\varepsilon)$ such that a matching lower bound holds for all points macroscopically far from $S(0)$, relative to ε:

$$\tau_H(0,\mathbf{x}) \geq c_1\|\mathbf{x}\|^{1-d} \quad \text{uniformly in } \mathbf{x} \in \mathbb{Z}^d_+ \text{ with } x(1) \geq \varepsilon\|\mathbf{x}\|. \tag{7.4}$$

There exist constants $c_2, C_2 > 0$ such that the following holds uniformly in $\mathbf{y} \in \mathbb{Z}^d_+ \setminus \{0\}$ with $y(1) = 0$:

$$c_2\|\mathbf{y}\|^{-d} \leq \mathbb{P}\big(0 \overset{\mathbb{Z}^d_+}{\longleftrightarrow} \mathbf{y}\big) \leq C_2\|\mathbf{y}\|^{-d}. \tag{7.5}$$

Our proof of Theorem 7.2 relies crucially on the result of Theorem 1.2 as input. We first prove a lemma that is in some respects a half-space analogue of Lemma 3.4 and Lemma 4.3. For the statement, recall the definition $Rect(n) = [0,n] \times [-4n,4n]^{d-1}$.

LEMMA 7.3. *Let $D = Rect(n)$, and let $Q_1 = \partial_{\mathbb{Z}^d_+} Rect(n)$ and $Q_2 = S(n) \cap Rect(n)$ (the "top" of $Rect(n)$). Define $X_Q(D,0)$ as usual for $Q = Q_1, Q_2$. There exists $C_2 > 0$ such that, uniformly in n,*

$$\mathbb{E}X_{Q_1}(D,0) \leq C_2 n^{-1}. \tag{7.6}$$

Recall the definition of K_0: the constant from Theorem 2.3, chosen for the growing sequence $(Rect(n))_n$. There exist $\eta, c_2 > 0$ and such that the following holds uniformly in n and in $K > K_0$:

$$\mathbb{P}\big(\eta n^2 < X^{K\text{-}reg}_{Q_2}(D,0) \leq X_{Q_1}(D,0) < \eta^{-1}n^2\big) \geq c_2 n^{-3}. \tag{7.7}$$

PROOF. We first show the bound on the expectation. By Lemma 6.1 (recall the notation of A^{out} defined before Theorem 5.1), we have

$$\mathbb{E}\mathsf{A}^{out}_0(4n,4n,8n) \leq Cn. \tag{7.8}$$

By (5.3) from Theorem 5.1, we have

$$\mathbb{E}[\mathsf{A}^{out}_0(4n,4n,8n) \mid X_{Q_1}(D,0)] \geq c_*^2 X_{Q_1}(D,0)n^2 \quad \text{on } \{X_{Q_1}(D,0) \geq n^2/2\}.$$

Combined with (7.8), the above gives

$$\mathbb{E}[X_{Q_1}(D,0); X_{Q_1}(D,0) \geq n^2/2] \leq Cn^{-1}.$$

On the other hand,

$$\mathbb{E}[X_{Q_1}(D,0); X_{Q_1} < n^2/2] \leq (n^2/2)\pi_H(n) \leq Cn^{-1},$$

where we have used part (a) of Theorem 1.1. This completes the proof of (7.6).

To show (7.7), we note that by Theorem 2.3 it suffices to show

$$\mathbb{P}\big(\eta n^2 < X_{Q_2}(D,0) < X_{Q_1}(D,0) < \eta^{-1}n^2\big) \geq cn^{-3} \quad \text{for all } n \tag{7.9}$$

for some $c, \eta > 0$. By (7.1), we have $\mathbb{P}(X_{Q_2} > \eta n^2) \geq cn^{-3}$ for some fixed small c (independent of η and n) for η sufficiently small. By (7.6) and the Markov inequality, $\mathbb{P}(X_{Q_1}(D,0) > \eta^{-1}n^2)$ is at most $C\eta n^{-3}$.

Bounding the probability in (7.9) by $\mathbb{P}(X_{Q_2}(D,0) > \eta n^2) - \mathbb{P}(X_{Q_1}(D,0) > \eta^{-1}n^2)$ and taking η sufficiently small completes the proof. □

PROOF OF (7.3) AND THE UPPER BOUND OF (7.5). To prove (7.3). Let $8n = \|\mathbf{x}\|$. Note that if $0 \overset{\mathbb{Z}_+^d}{\longleftrightarrow} \mathbf{x}$, there exists a $\mathbf{z} \in \partial_{\mathbb{Z}_+^d} Rect(n)$ such that $\{0 \overset{Rect(n)}{\longleftrightarrow} \mathbf{z}\} \circ \{\mathbf{z} \leftrightarrow \mathbf{x}\}$ occurs. Taking a union bound and using the BK inequality gives

$$\mathbb{P}\big(0 \overset{\mathbb{Z}_+^d}{\longleftrightarrow} \mathbf{x}\big) \leq \sum_{\mathbf{z} \in \partial_{\mathbb{Z}_+^d} Rect(n)} \mathbb{P}\big(0 \overset{Rect(n)}{\longleftrightarrow} \mathbf{z}\big)\mathbb{P}(\mathbf{z} \leftrightarrow \mathbf{x}) \tag{7.10}$$

$$\leq Cn^{2-d} \sum_{\mathbf{z} \in \partial_{\mathbb{Z}_+^d} Rect(n)} \mathbb{P}\big(0 \overset{Rect(n)}{\longleftrightarrow} \mathbf{z}\big) = Cn^{2-d}\mathbb{E}X_{Q_1}(D, 0) \tag{7.11}$$

with $D = Rect(n)$ and $Q_1 = \partial_{\mathbb{Z}_+^d} Rect(n)$, and where we have used (1.7). Applying (7.6) completes the proof.

The upper bound of (7.5) follows from a decoupling argument similar to the one used for (7.3), this time using (7.3) as input. As before, letting $8n = \|\mathbf{y}\|$, for $0 \overset{\mathbb{Z}_+^d}{\longleftrightarrow} \mathbf{y}$ to hold, there must be a $\mathbf{z} \in \partial_{\mathbb{Z}_+^d} Rect(n)$ such that $\{0 \overset{Rect(n)}{\longleftrightarrow} \mathbf{z}\} \circ \{\mathbf{z} \overset{\mathbb{Z}_+^d}{\longleftrightarrow} \mathbf{y}\}$ holds. This gives (7.10) with $\mathbf{x}$ replaced by $\mathbf{y}$ and the connection from $\mathbf{z}$ to $\mathbf{y}$ restricted to $\mathbb{Z}_+^d$. Now the same reasoning used to produce (7.11), but now using the upper bound from (7.3) to estimate $\mathbb{P}(\mathbf{z} \overset{\mathbb{Z}_+^d}{\longleftrightarrow} \mathbf{y})$, gives the analogue of (7.11), with Cn^{2-d} replaced by Cn^{1-d}. Using (7.6) as before completes the proof. □

PROOF OF (7.4) AND THE LOWER BOUND OF (7.5). We first prove (7.4). The argument is a modification of the proof of Theorem 1.2: roughly, we condition on 0 having an arm to distance of order $n \approx \|\mathbf{x}\|$, and then show an open connection from $\mathbf{x}$ to this arm can be made. There are three major modifications. First, if the arm from 0 terminated too close to $S(0)$ (more carefully speaking: if $\mathfrak{C}_{B_H(n)}(0) \cap S'(n)$ had too few vertices at macroscopic distance from $S(0)$), this connection would not be possible; because of the lack of symmetry in the half-space, we must resort to the second part of Lemma 7.3 to direct this arm. Second, there is no inductive improvement needed in the argument. Third, we must rely on the result of Theorem 1.2 as input to insure the further connection to $\mathbf{x}$ does not cross the half-space boundary (the earlier argument required only information about the unrestricted τ as input in the base case).

Fix $\varepsilon > 0$ and suppose $\mathbf{x} \in Ann_H(8n, 16n)$ with $x(1) \geq \varepsilon n$. Let $D = Rect(n)$, and let $X_{Q_2} = X_{Q_2}(D, 0)$ be as in the statement of Lemma 7.3. Let $K > K_0$ be fixed, to be chosen. Define $Y_{Q_2}^K = Y_{Q_2}^K(\mathbf{x})$ to be the number of $\mathbf{z} \in Q_2$ such that (a) $\mathbf{z} \overset{D}{\longleftrightarrow} 0$, (b) $\mathbf{z} \in REG_D(K)$ (recall Definition 3), and (c) the edge $\{\mathbf{z}, \mathbf{z} + \mathbf{e}_1\}$

is pivotal for $\{0 \overset{\mathbb{Z}^d_+}{\longleftrightarrow} \mathbf{x}\}$. As in the proof of Lemma 4.6, we have $Y^K_{Q_2} \leq 1$ a.s., since no pair of vertices $\mathbf{z}_1 \neq \mathbf{z}_2$ can simultaneously satisfy parts (a) and (c) of the definition.

Defining B_η to be the event in (7.7), we argue that for $K > K_0$ fixed sufficiently large,

$$\mathbb{E}\big[Y^K_{Q_2}; X^{K\text{-}reg}_{Q_2} = N, B_\eta\big] \geq cn^{4-d}\,\mathbb{P}\big(X^{K\text{-}reg}_{Q_2} = N; B_\eta\big) \quad \text{for } c = c(K), \tag{7.12}$$

uniformly in $\mathbf{x} \in Ann_H(8n, 16n)$ with $x(1) \geq \varepsilon n$ and in $\eta n^2 \leq N \leq \eta^{-1} n^2$. Set $\widetilde{D}_\mathbf{z} := \mathbf{z} + [K/2, K]^d$ and let $\widetilde{\mathbf{z}}$ range over vertices of $\widetilde{D}_\mathbf{z}$; define $R_n = B_H(0, 20n)$. We show (7.12) by defining events

$$\mathcal{E}_1(\mathbf{z}, N) = B_\eta \cap \big\{0 \overset{D}{\longleftrightarrow} \mathbf{z}, \mathbf{z} \in REG_D(K) \text{ and } X^{K\text{-}reg}_{Q_2} = N\big\},$$
$$\mathcal{E}_2(\mathbf{x}, \widetilde{\mathbf{z}}, \mathbf{z}) = \big\{\widetilde{\mathbf{z}} \overset{R_n}{\longleftrightarrow} \mathbf{x} \text{ off } \mathfrak{C}_D(\mathbf{z})\big\}, \quad \mathcal{E}_3(\mathbf{z}, \widetilde{\mathbf{z}}) = \{\mathfrak{C}(\mathbf{z}) \cap \mathfrak{C}(\widetilde{\mathbf{z}}) = \varnothing\}.$$

Similar arguments to those of Claim 4.8 show that we can choose $K > K_0$ and find a constant $c > 0$ such that the following holds: for each n, each $\mathbf{x} \in Ann_H(8n, 16n)$ with $x(1) \geq \varepsilon n$, and each $\mathbf{z} \in Q_2$, there is a $\widetilde{\mathbf{z}} \in \widetilde{D}_\mathbf{z}$ such that

$$\mathbb{P}(\mathcal{E}_1(\mathbf{z}, N) \cap \mathcal{E}_2(\mathbf{x}, \widetilde{\mathbf{z}}, \mathbf{z}) \cap \mathcal{E}_3(\mathbf{z}, \widetilde{\mathbf{z}})) \geq cn^{2-d}\,\mathbb{P}(\mathcal{E}_1(\mathbf{z}, N)). \tag{7.13}$$

A main complication in proving (7.13), compared with the proof of Claim 4.8, comes in the bound on $\mathbb{P}(\mathcal{E}_1 \setminus \mathcal{E}_2)$. Namely: for the analogue of (4.6), we bound, on the event $\mathfrak{C}_D(0) = \mathcal{C}$,

$$\mathbb{P}\big(\widetilde{\mathbf{z}} \overset{R_n}{\longleftrightarrow} \mathbf{x} \text{ off } \mathcal{C}\big) \geq \mathbb{P}\big(\widetilde{\mathbf{z}} \overset{R_n}{\longleftrightarrow} \mathbf{x}\big) - \sum_{\mathbf{y} \in \mathcal{C}} \mathbb{P}(\{\widetilde{\mathbf{z}} \leftrightarrow \mathbf{y}\} \circ \{\mathbf{y} \leftrightarrow \mathbf{x}\}). \tag{7.14}$$

To show the first term of the above is at least $c(\varepsilon)n^{2-d}$ using Theorem 1.2, we use crucially the fact that $\mathbf{z}$ is macroscopically distant from $S(0)$. This necessitates the condition $\mathbf{z} \in Q_2$, and this ultimately requires our arm-directedness statement (7.1). The second term of (7.14) can be bounded similarly to before: the probability that $\mathbf{y} \leftrightarrow \mathbf{x}$ is of order n^{2-d}, and the sum of probabilities $\mathbb{P}(\widetilde{\mathbf{z}} \leftrightarrow \mathbf{y})$ is small for K large using the regularity in $\mathcal{E}_1$.

Having established (7.13), we note that an edge-modification argument again gives the existence of a constant $c_1 = c_1(K)$ such that

$$\begin{aligned} &\mathbb{P}\big(\mathbf{z} \text{ is counted in } Y^K_{Q_2}; B_\eta \cap \big\{X^{K\text{-}reg}_{Q_2} = N\big\}\big) \\ &\quad \geq c_1 \mathbb{P}\big(\mathcal{E}_1(\mathbf{z}, N) \cap \mathcal{E}_2(\mathbf{x}, \widetilde{\mathbf{z}}, \mathbf{z}) \cap \mathcal{E}_3(\widetilde{\mathbf{z}}, \mathbf{z})\big) \\ &\quad \geq cn^{2-d}\,\mathbb{P}(\mathcal{E}_1(\mathbf{z}, N)), \end{aligned}$$

where $\widetilde{\mathbf{z}}$ is chosen so that (7.13) holds. Summing over $\mathbf{z} \in Q_1$, we get

$$\begin{aligned}\mathbb{E}\big[Y_{Q_2}^K;\ B_\eta \cap \{X_{Q_2}^{K\text{-}reg} = N\}\big] &\geq cn^{2-d} \sum_{\mathbf{z}\in Q_2} \mathbb{P}(\mathcal{E}_1(\mathbf{z}, N))\\ &\geq cn^{4-d}\,\mathbb{P}\big(X_{Q_2}^{K\text{-}reg} = N, B_\eta\big),\end{aligned}$$

which is (7.12).

Having established (7.12), we move to complete the proof of (7.4). Note that $\mathbb{P}(0 \overset{\mathbb{Z}_+^d}{\longleftrightarrow} \mathbf{x}) \geq \mathbb{P}(Y_{Q_2}^K > 0)$. We use a conditional second-moment argument to bound the latter probability. The fact that $Y_{Q_2}^K \leq 1$ a.s., and an argument similar to the one used to show (2A) of Lemma 4.6, give

$$\mathbb{E}\big[\big(Y_{Q_2}^K\big)^2 \mid X_{Q_2}^{K\text{-}reg} = N, B_\eta\big] \leq Cn^{4-d}. \tag{7.15}$$

Combining (7.15) with (7.12), we find

$$\begin{aligned}&\mathbb{P}\big(Y_{Q_2}^K > 0 \mid X_{Q_2}^{K\text{-}reg} = N, B_\eta\big)\\ &\quad\geq \frac{\mathbb{E}\big[Y_{Q_2}^K \mid X_{Q_2}^{K\text{-}reg} = N, B_\eta\big]^2}{\mathbb{E}\big[\big(Y_{Q_2}^K\big)^2 \mid X_{Q_2}^{K\text{-}reg} = N, B_\eta\big]}\\ &\quad\geq cn^{4-d} \quad \text{uniformly in } n \text{ and } \mathbf{x} \in Ann_H(8n, 16n) \text{ with } x(1) \geq \varepsilon n.\end{aligned}$$

Recalling that B_η was the event in (7.7) and applying the probability bound there, we see

$$\mathbb{P}\big(0 \overset{\mathbb{Z}_+^d}{\longleftrightarrow} \mathbf{x}\big) \geq \mathbb{P}\big(Y_{Q_2}^K > 0\big) \geq cn^{4-d}\,\mathbb{P}(B_\eta) \geq cn^{1-d} \geq c\|\mathbf{x}\|^{1-d},$$

completing the proof of (7.4).

We now outline the proof of the lower bound of (7.5); the proof is similar to the above, so we describe only the major differences. Suppose $\mathbf{y}$ has $y(1) = 0$ and $\mathbf{y} \in Ann_H(8n, 16n)$. As before, we set $D = Rect(n)$ and let $X_{Q_2}(D, 0)$ be as in Lemma 7.3, and we define $Y_{Q_2}^K$ exactly as before (with references to $\mathbf{x}$ replaced by $\mathbf{y}$).

The events $\mathcal{E}_i$ are defined as previously. except in $\mathcal{E}_2$ we ask instead that $\widetilde{\mathbf{z}} \overset{\mathbb{Z}_+^d}{\longleftrightarrow} \mathbf{x}$ off $\mathfrak{C}_D(\mathbf{z})$. Estimates involving the probability of this connection are made using (7.4) instead of the bound on the box-restricted two-point function; upper bounds on the probability of appropriate portions of large-loop connections are made using the upper bound of (7.3). For instance, the right-hand side of (7.13) is replaced by $cn^{1-d}\mathbb{P}(\mathcal{E}_1(\mathbf{z}, N))$. This reflects the fact that (7.14) is replaced by

$$\mathbb{P}\big(\tilde{\mathbf{z}} \overset{\mathbb{Z}^d_+}{\longleftrightarrow} \mathbf{y} \text{ off } \mathcal{C}\big) \geq \mathbb{P}\big(\tilde{\mathbf{z}} \overset{\mathbb{Z}^d_+}{\longleftrightarrow} \mathbf{y}\big) - \sum_{\zeta \in \mathcal{C}} \mathbb{P}\big(\{\tilde{\mathbf{z}} \leftrightarrow \zeta\} \circ \{\zeta \overset{\mathbb{Z}^d_+}{\longleftrightarrow} \mathbf{y}\}\big). \tag{7.16}$$

The first term of (7.16) is uniformly at least cn^{1-d} by (7.4). $\mathbb{P}(\zeta \overset{\mathbb{Z}^d_+}{\longleftrightarrow} \mathbf{y})$ is at most Cn^{1-d} by the upper bound of (7.3), and again the sum of probabilities $\mathbb{P}(\tilde{\mathbf{z}} \leftrightarrow \zeta)$ is small for K large.

Making similar adaptations to the remaining estimates, we find that the conditional (on B_η) first and second moments of $Y^K_{Q_2}$ are both of order n^{3-d}. A conditional second-moment argument as before gives

$$\mathbb{P}\big(0 \overset{\mathbb{Z}^d_+}{\longleftrightarrow} \mathbf{y}\big) \geq \mathbb{P}\big(Y^K_{Q_2} > 0\big) \geq cn^{3-d}\mathbb{P}(B_\eta) \geq c\|\mathbf{y}\|^{-d}. \qquad \square$$

7.3 Cluster Sizes

We now prove part (c) of Theorem 1.1. For clarity, we restate the claim here as Theorem 7.4.

THEOREM 7.4. *There exist constants $c, C > 0$ such that*

$$ct^{-3/4} \leq \mathbb{P}(\#\mathfrak{C}_H(0) > t) \leq Ct^{-3/4}. \tag{7.17}$$

PROOF. We begin by proving the first inequality. We will compute cluster size moments conditional on $H_n := \{0 \overset{\mathbb{Z}^d_+}{\longleftrightarrow} S'(n)\}$. Abbreviate

$$Y_n = \#[\mathfrak{C}_H(0) \cap Ann_H(n, 2n)].$$

We can lower bound the (conditional) first moment of Y_n by considering only those $\mathbf{x}$ having $x(1) \geq n$:

$$\begin{aligned} \mathbb{E}[Y_n \mid H_n] &\geq cn^3 \sum_{\mathbf{x} \in Ann_H(n,2n)} \tau_H(0, \mathbf{x}) \\ &\geq cn^3 \sum_{\substack{\mathbf{x} \in Ann_H(n,2n), \\ x(1) \geq n}} \tau_H(0, \mathbf{x}) \geq \sum_{\substack{\mathbf{x} \in Ann_H(n,2n), \\ x(1) \geq n}} cn^{4-d} \geq cn^4, \end{aligned} \tag{7.18}$$

where we have used (7.4) and the asymptotics of π_H.

We can upper-bound Y_n^2 by $(\#[\mathfrak{C}_H(0) \cap B_H(2n)])^2$. Writing the latter quantity as a sum and using (7.3) gives

$$\begin{aligned}
&\mathbb{E}((\#[\mathfrak{C}_H(0) \cap B_H(2n)])^2 \mid H_n) \\
&\le Cn^3 \sum_{\mathbf{x},\mathbf{y} \in B_H(2n)} \mathbb{P}\big(0 \overset{\mathbb{Z}_+^d}{\longleftrightarrow} \mathbf{x},\ 0 \overset{\mathbb{Z}_+^d}{\longleftrightarrow} \mathbf{y}\big) \\
&\le Cn^3 \sum_{\substack{\mathbf{x},\mathbf{y} \in B_H(2n), \\ \mathbf{z} \in \mathbb{Z}_+^d}} \mathbb{P}\big(\{0 \overset{\mathbb{Z}_+^d}{\longleftrightarrow} \mathbf{z}\} \circ \{\mathbf{z} \leftrightarrow \mathbf{x}\} \circ \{\mathbf{z} \leftrightarrow \mathbf{y}\}\big) \qquad (7.19) \\
&\le Cn^3 \sum_{\substack{\mathbf{x},\mathbf{y} \in B_H(2n), \\ \mathbf{z} \in \mathbb{Z}_+^d}} \|\mathbf{z}\|^{1-d} \|\mathbf{z}-\mathbf{x}\|^{2-d} \|\mathbf{z}-\mathbf{y}\|^{2-d} \le Cn^8.
\end{aligned}$$

Using the Paley-Zymund inequality with (7.18) and (7.19), we find that there is a constant $c > 0$ such that, uniformly in n,

$$\mathbb{P}\big(Y_n > cn^4 \bigm| H_n\big) \ge c.$$

Using the fact that $\mathbb{P}(H_n) = \pi_H(n) \ge cn^{-3}$ gives $\mathbb{P}(Y_n \ge cn^4) \ge cn^{-3}$. Since $\#\mathfrak{C}_H(0) \ge Y_n$, setting $n = Ct^{1/4}$ for C sufficiently large completes the proof of the first inequality of (7.17).

To prove the second inequality of (7.17), first note that a calculation similar to that in (7.19) shows $\mathbb{E}((\#[\mathfrak{C}_H(0) \cap B_H(2n)])^2) \le Cn^5$. Using this fact and Chebyshev's inequality, we see that for each $m > 0$,

$$\begin{aligned}
\mathbb{P}(\#\mathfrak{C}_H(0) > t) &\le \pi_H(m) + \mathbb{P}(\#[\mathfrak{C}_H(0) \cap B_H(m)] > t) \\
&\le Cm^{-3} + Cm^5/t^2.
\end{aligned}$$

Setting $m = t^{1/4}$ completes the proof. □

Acknowledgment. The authors would like to thank L.-P. Arguin for helpful discussions. They also acknowledge the careful reading and extensive comments provided by the reviewer of the first version of this paper. These greatly improved the presentation and articulation of the manuscript.

S. Chatterjee acknowledges support from a grant from the Simons Foundation (#430073). J. Hanson acknowledges support from National Science Foundation Grant DMS-1612921.

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Please note change in article title and DOI of [31] to match bibliographical information provided.

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Received October 2018.