Strict Inequality for the Chemical Distance Exponent in Two-Dimensional Critical Percolation

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Abstract

We provide the first nontrivial upper bound for the chemical distance exponent in two-dimensional critical percolation. Specifically, we prove that the expected length of the shortest horizontal crossing path of a box of side length n in critical percolation on $\mathbb{Z}^2$ is bounded by $Cn^{2-\delta}\pi_3(n)$, for some $\delta > 0$, where $\pi_3(n)$ is the "three-arm probability to distance n." This implies that the ratio of this length to the length of the lowest crossing is bounded by an inverse power of n with high probability. In the case of site percolation on the triangular lattice, we obtain a strict upper bound for the exponent of $4/3$.

The proof builds on the strategy developed in our previous paper [9], but with a new iterative scheme, and a new large deviation inequality for events in annuli conditional on arm events, which may be of independent interest.

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1 Introduction

In this paper, we study the volume of crossing paths of a square $[-n,n]^2$ in two-dimensional critical Bernoulli bond percolation on $\mathbb{Z}^2$. Each edge of the lattice is declared to be open (with probability p) or closed (with probability $1-p$) independently, and in the critical case, we set $p = 1/2$. The minimal number of edges of any horizontal open crossing of a box is called the chemical distance between the left and right sides of the box. This terminology appears to have originated in the physics literature, where the intrinsic distance in the graph defined by large critical percolation clusters has been studied extensively [16, 18, 21, 22, 23, 24, 39]. An early reference is [22], where the authors credit the physicist S. Alexander for introducing them to the term "chemical distance." Let H_n be the event that there exists a horizontal open crossing of $[-n,n]^2$, and let S_n be the least number of edges in any horizontal open crossing of $[-n,n]^2$. A common assumption in this literature is the existence of a scaling exponent $d_{\min}$ for the chemical distance:

$$\mathbf{E}[S_n \mid H_n] \sim n^{d_{\min}}, \tag{1.1}$$

where the precise meaning of $\sim$ remains to be determined. Unlike for other critical exponents in percolation, there is not even a generally accepted prediction for the exact value of $d_{\min}$. The existence and determination of an exponent for the chemical distance in any two-dimensional short-range critical percolation model is thus far out of reach of current methods. In particular, as noted by O. Schramm in [37, Problem 3.3], the chemical distance is not likely to be accessible to SLE methods. One reason to doubt the relevance of SLE for this problem is that the chemical distance does not appear to be related in any simple way to conformally invariant quantities. Apart from its mathematical appeal, further progress on the chemical distance is a significant obstacle to analyzing random walks on low-dimensional critical percolation clusters (the last progress being by Kesten [27] in '86) and testing the validity of the celebrated Alexander-Orbach conjecture [4].

For long-range models and certain correlated fields, on the other hand, there has been much recent progress; see [6, 8, 14, 15], and also [13], where it is stated that "it is a major challenge to compute the exponent on the chemical distance ... for critical planar percolation." For a long-range, independent percolation model where sites x and y are connected with probability proportional to $|x-y|^{-s}$, $d < s < 2d$, Biskup [6] showed that the chemical distance satisfies

$$\mathrm{dist}_{\mathrm{chem}}(x,y) = (\log|x-y|)^{\frac{1}{\log(2d/s)}+o(1)} \tag{1.2}$$

as $|x-y| \to \infty$ through x,y in the infinite connected component. Biskup and Lin [7] sharpened (1.2) by showing that the left side can be bounded above and below by a constant factor times the quantity on the right, without the $o(1)$ error in the exponent.

In standard (nearest neighbor) Bernoulli percolation, it is known that the chemical distance in percolation clusters behaves linearly in the supercritical phase, when

$p > p_c$ [5, 20]. Antal and Pisztora [5] show that for $p > p_c$, there is a constant ρ depending on p and the dimension such that

$$\limsup_{|x|\to\infty} \frac{1}{|x|}\mathbf{1}[0 \leftrightarrow x]\mathrm{dist}_{\mathrm{chem}}(0,x) \le \rho(p,d)$$

almost surely. Thus, the chemical distance is at most linear in the supercritical phase. This is also true in the subcritical phase. Indeed, we have for $\lambda \ge 1$ large enough:

$$\mathbf{P}_p\left(\mathrm{dist}_{\mathrm{chem}}(x,y) \ge \lambda|x-y| \,\middle|\, x \leftrightarrow y\right) \le Ce^{-\lambda c_0|x-y|}, \quad p < p_c \tag{1.3}$$

where $\mathrm{dist}_{\mathrm{chem}}(x,y)$ is the chemical distance between the sites x and y in $\mathbb{Z}^d$ and $\mathbf{P}_p(\cdot \mid x \leftrightarrow y)$ denotes the Bernoulli percolation measure with density p, conditioned on the event that x and y are connected by an open path. This follows from exponential decay of the cluster volume, obtained by Aizenman-Newman [2, Proposition 5.1]: if $C(x)$ denotes the set of edges connected to x by open paths, then for $p < p_c$:

$$\mathbf{P}_p(\#C(x) > k) \le Ck^{-1/2}e^{-ck}$$

for $k \ge 1$ and constants $C, c > 0$ (where c is explicit). Since $\#C(x) \ge \mathrm{dist}_{\mathrm{chem}}(x,y)$ on $\{x \leftrightarrow y\}$, we obtain

$$\mathbf{P}_p\left(\mathrm{dist}_{\mathrm{chem}}(x,y) \ge \lambda|x-y| \,\middle|\, x \leftrightarrow y\right) \le C\frac{e^{-c\lambda|x-y|}}{\mathbf{P}_p(x \leftrightarrow y)}.$$

On the other hand, opening all edges along a deterministic path from x to y ensures the occurrence of $\{x \leftrightarrow y\}$, so

$$\mathbf{P}_p(x \leftrightarrow y) \ge e^{-c'(p)|x-y|}$$

for some $c'(p) > 0$. Choosing λ such that $c\lambda \ge 2c'(p)$, we obtain (1.3).

In critical percolation, connected paths are expected to be tortuous in the sense of [1, 30]; that is, they are asymptotically of dimension > 1. In dimension $d \ge 11$, precise estimates are known [17], and macroscopic connecting paths have dimension 2. Indeed, let $B(n) = [-n,n]^2$, $x \in B(n) \cap \mathbb{Z}^2$, and $\varepsilon > 0$. Denote by $\{0 \overset{\le \varepsilon n^2}{\leftrightarrow} x \text{ in } B(n)\}$ the event that the origin is connected to x by an open path inside $B(n)$ with no more than εn^2 edges. Then [38, Theorem 1.5, b)] states that

$$\limsup_{n\to\infty} \sum_{x\in\partial B(n)\cap\mathbb{Z}^2} \mathbf{P}_{p_c}\left(0 \overset{\le \varepsilon n^2}{\leftrightarrow} x \text{ in } B(n)\right) \le C\sqrt{\varepsilon},$$

which implies [38, Theorem 1.6] that with high probability, any open connection from 0 to x uses at least order $|x|^2$ many edges. Corresponding upper bounds appear in [31, 32] and [26, Theorem 2.8]. These estimates ultimately depend on results obtained using the lace expansion. See [25] for a good treatment of such high-dimensional results, as well as further references.

In the low-dimensional, critical case, the chemical distance is not well understood, even at the physics level of rigor. In [30], H. Kesten and Y. Zhang considered the number L_n of edges in the *lowest* horizontal open crossing of $B(n)$ and asked whether $S_n = o(L_n)$ with high probability. We answered this question affirmatively in [9], and provided a more quantitative estimate in the note [10]. The main result of the current paper (Theorem 1.1 below) is the first nontrivial upper bound on $d_{\min}$ which, combined with those of Aizenman-Burchard [1], implies that for some $\delta > 0$,

$$n^{1+\delta} \leq \mathbf{E}[S_n \mid H_n] \leq n^{-\delta} n^2 \pi_3(n). \tag{1.4}$$

Here, $\pi_3(n)$ is the "three-arm probability to distance n," defined in Section 3.2. (The term $n^2\pi_3(n)$ is the order of $\mathbb{E}L_n$, as shown in [34].) In site percolation on the triangular lattice, the right side of (1.4) is bounded by n^{1+s} for some $s < 1/3$, since for that model, $\mathbb{E}L_n$ is known to be equal to $n^{4/3+o(1)}$ [34, Theorem 1].

The possibility of the polynomial improvement over $n^2\pi_3(n)$ on the right side of (1.4) holding was mentioned in Kesten and Zhang [30]. It appears to have been expected by experts to be correct, but there is no simple, convincing heuristic for this expectation, and even no obvious reason to believe that there are open crossings of different dimensions. Indeed, for large d, the chemical distance exponent is 2, and this coincides with the exponent for the expected total number of points on all self-avoiding open paths between two vertices that are conditioned to be connected to each other [3].

1.1 Statement of main result

Our main result is that, conditioned on the existence of a horizontal crossing path of $B(n) = [-n,n]^2$, there exists with high probability a path whose volume is smaller than that of the lowest crossing by a factor of the form $n^{-\delta}$, for some $\delta > 0$. Recall that H_n is the event that there exists a horizontal open crossing of $[-n,n]^2$. Let l_n be the lowest open horizontal crossing. Finally, let $L_n = \#l_n$ and S_n be the least number of edges of any open horizontal crossing.

Theorem 1.1. *There is a $\delta > 0$ and a constant $C > 0$ such that*

$$\mathbf{E}[S_n \mid H_n] \leq C n^{-\delta} \mathbf{E}[L_n | H_n] \quad \textit{for all } n. \tag{1.5}$$

Our strategy builds on that in our previous paper [9]. The key idea introduced in that paper was to construct local modifications around an edge e which implied the existence of a shortcut path around e, conditionally on $e \in l_n$, rather than to attempt to construct modifications after conditioning on l_n itself. The latter point is essential; given the conditional independence of the region above the lowest crossing, a natural idea is to try to construct shortcuts around the lowest crossing in this "unexplored" region, conditionally on l_n. This type of approach is doomed to fail. The roughness of the lowest crossing prevents the use of the usual volume estimates based on arm exponents, making it difficult to control the size of potential shortcuts effectively.

To improve on the bounds from [9], one would hope to build shortcut paths on other shortcuts, saving length on those paths that are already shorter than portions of the lowest crossing, in an inductive manner. The main difficulty with this approach is that it is not clear how to manipulate the shortest crossing; we only have information on the lowest crossing. The idea at the heart of our proof is, instead of placing shortcuts on other shortcuts, to perform an iteration on the expected lengths of shortcuts. Roughly speaking, if one can produce paths on a certain scale which have a savings over the lowest crossing, then on larger scales, one can build paths using these shortcuts in places where the lowest crossing is abnormally long. This in turn gives a larger improvement on the higher scale. The main iterative result (for open paths in "U-shaped regions") appears in Section 7 as Proposition 7.1, and we quickly derive Theorem 1.1 from it in Section 8. A more detailed outline of the proof appears in the next section.

An important tool in our proof is Theorem 4.1, in Section 4, which is a new large deviation bound for sequences of events in disjoint annuli conditioned on arm events. See the discussion in Step 2 of the proof sketch in the next section. We believe this bound should be useful for other problems.

We present our argument in the case of Bernoulli percolation on $\mathbb{Z}^2$. However, given that our constructions are based on Russo-Seymour-Welsh estimates, the argument can be readily adapted to the triangular lattice for example.

2 Outline of the proof

We begin by outlining the proof. In this section and the rest of the paper, given an edge e and $L > 0$, $B(e,L)$ denotes the box of side length $2L$ centered at the lower-left endpoint of e; recall that $B(n) = [-n,n]^2$. (For further notation, we refer the reader to Section 3.) Theorem 1.1 is a consequence of an iterative bound given in Proposition 7.1, so we sketch the idea for the latter's proof.

This outline splits into two parts: Steps 1 – 3 summarize the construction of shortcut paths around portions of the lowest crossing l_n. These shortcuts are used to build a path σ which improves on l_n by a constant factor: it satisfies the bound in (2.4). Steps 4 – 5 describe the iterative procedure used to make improvements on open paths ℓ_k in U-shaped regions. If one can construct open paths on scale 2^k which improve on ℓ_k by a constant factor (see (2.6)), then for $m \geq k+C$, one can use these paths, with additional savings, to improve on ℓ_m by a smaller constant factor (see (2.7)). We now proceed to a more detailed outline.

Step 1. **Construction of shortcuts.** Given $\varepsilon > 0$, and an edge $e \in B(n)$, we define an event $E_k(e)$ depending on $B(e,2^K) \setminus B(e,2^k)$, with

$$K = k + \lfloor \log \frac{1}{\varepsilon} \rfloor, \tag{2.1}$$

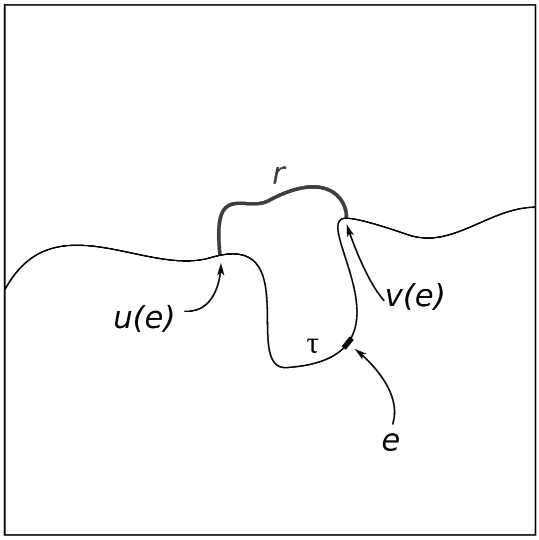

FIGURE 2.1. The topological plan of a shortcut. The outer box represents $B(n) = [-n,n]^2$. τ is a segment of the lowest crossing l_n, containing the edge e, with endpoints $u(e)$ and $v(e)$. The shortcut r, represented in grey, lies in the region above l_n, with endpoints $u(e)$ and $v(e)$. It bypasses the edge e.

such that

$$\mathbf{P}(E_k(e) \mid e \in l_n) \geq c\varepsilon^4, \tag{2.2}$$

for some $c > 0$ and such that the occurrence of $E_k(e)$ implies the existence of an open arc $r \subset B(e, 3 \cdot 2^k)$ with endpoints $u(e)$ and $v(e)$ on the lowest crossing l_n of $[-n,n]^2$ and otherwise not intersecting it. Moreover, letting $\tau = \tau(r)$ be the portion of l_n between $u(e)$ and $v(e)$, we have $e \in \tau$ and

$$\frac{\#r}{\#\tau} \leq \varepsilon. \tag{2.3}$$

See Figure 2.1. If $E_k(e)$ occurs and $e \in l_n$, there is a shortcut r on scale 2^k around edges of the lowest crossing which saves at least $(1/\varepsilon - 1) \cdot \#r$ edges. The open arc r is constructed in such a way that the shortcuts r, r' resulting from the occurrence of $E_k(e)$, $E_l(e')$, are either nested or disjoint.

The definition of $E_k(e)$ appears in Section 5.1.

Step 2. **Probability bound on shortcuts.** For each edge e, we define $\mathscr{S}(e)$ to be the collection of all shortcut paths around e arising from occurrence of an event $E_k(e')$ for some k, and some $e' \in l_n$. Using the lower bound (2.2), we show in Section 5.2 that if $\mathrm{dist}(e, \partial B(n)) \geq d$, then $\mathscr{S}(e) = \emptyset$ implies that none of the events $E_j(e)$ occur for $j = 1, \ldots, C \log d$. We will see that this implies

$$\mathbf{P}(\mathscr{S}(e) = \emptyset \mid e \in l_n) \leq Cd^{-\frac{\varepsilon^4 c}{\log \frac{1}{\varepsilon}}}.$$

The form of the right side follows from a large deviation bound conditional on a three-arm event from Section 4 (developed using tools from our recent study of arm events in invasion percolation [11]) that allows us to roughly decouple $E_k(e)$ and $E_j(e)$ on the event $e \in l_n$ so long as $|k-j| \geq C \log \frac{1}{\varepsilon}$. Note that in our previous work [9], we were only able to obtain a weaker probability bound of the form[1]

$$\mathbf{P}(\mathscr{S}(e) = \emptyset \mid e \in l_n) = o(1/\log d), \quad d \to \infty.$$

Step 3. **Construction of shorter crossing.** Forming an arc σ from a maximal collection of shortcuts and the remaining edges of l_n with no shortcuts around them, we find (a special case of equation (7.5)):

$$\mathbf{E}[\#\sigma \mid H_n] \leq (\varepsilon + Cn^{-\frac{c\varepsilon^4}{\log \frac{1}{\varepsilon}}}) \cdot \mathbf{E}[\#l_n \mid H_n]. \tag{2.4}$$

The term $\varepsilon \mathbf{E}[\#l_n \mid H_n]$ in (2.4) is the contribution from the shortcuts, and the term of the form $n^{-c}\mathbf{E}[\#l_n \mid H_n]$ comes from estimating the expected volume of the edges of the lowest crossing with no shortcut around them.

Step 4. **Iteration in U-shaped regions: initial step.** The shortcuts constructed in Steps 1-3 are contained in "U-shaped" regions of the form shown in Figures 2.2 and 6.1, attached to the lowest crossing.

In this iteration step, we consider candidate shortcuts contained in U-shaped regions, and shorten them using the procedure from Steps 1-3. To obtain even better gains, in Step 5 we will iterate this procedure, using shortcuts from one scale to reduce the length of possible shortcuts on higher scales. Although the iteration will happen at the level of expectation, the reader can think of this as "placing shortcuts on shortcuts on shortcuts ..."

To begin the iteration, define E_k' to be the event that there exist a closed and an open arc connecting two vertices contained in small boxes at opposite ends of the region (see Figure 6.1 again), which is of scale 2^k. (This E_k' is actually a version of the event $E_k(e)$ from step 1, adapted to U-shaped regions.) By construction, these vertices are five-arm points: they possess five connections, two dual closed arms, and three open arms, to macroscopic distance. See Figure 5.3, where the five-arm points are indicated in purple.

Denote the outermost open arc between the five-arm points by ℓ_k. The path ℓ_k is a potential shortcut to use on the lowest crossing. To estimate its length, one begins with an initial estimate in (7.16) (see [9, Lemma 5.2] for a similar bound):

$$\mathbf{E}[\#\ell_k \mid E_k'] \leq C 2^{2k} \pi_3(2^k).$$

[1] The estimate stated here does not appear in [9], but the method presented there can be quantified to obtain it. See the note [10].

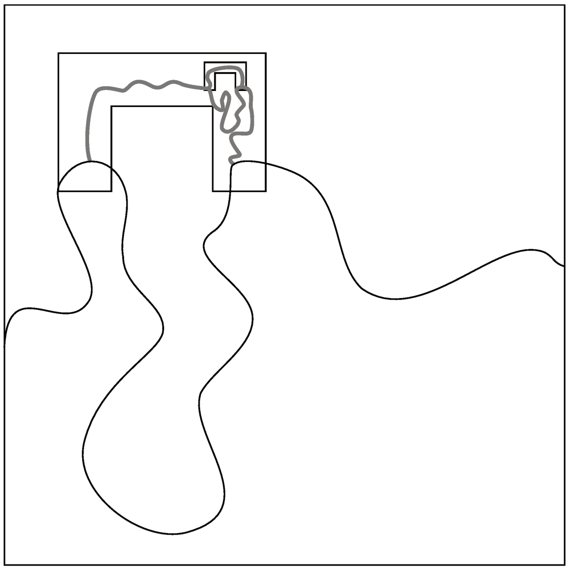

FIGURE 2.2. The shortcuts constructed in Step 1 lie inside "U-shaped" regions along the lowest crossing of the box. Our argument proceeds by successively improving the shortest paths joining points at opposite ends of these regions. An improvement on one scale allows more effective shortcuts to be constructed in much larger U-shaped regions.

The first step of the iteration uses the construction that led to the estimate (2.4). Inside the U-shaped region, we define a path σ joining the two five-arm points, consisting of portions of ℓ_k and shortcuts satisfying (2.3), and show in Section 7.3 that its length is bounded by

$$\begin{aligned}\mathbf{E}[\#\sigma \mid E_k'] &\leq (\varepsilon + 2^{-c\frac{\varepsilon^4 k}{\log\frac{1}{\varepsilon}}})2^{2k}\pi_3(2^k) \\ &\leq C\varepsilon^{1/2}2^{2k}\pi_3(2^k),\end{aligned} \tag{2.5}$$

whenever k is at least a constant s_1 depending on ε (see (7.25)). Comparing this estimate to the initial one, we see that as long as the U-shaped region is of scale at least 2^{s_1}, then we can modify a potential shortcut ℓ_k (by placing shortcuts on it) into a path σ which has smaller length, by a factor of $\varepsilon^{1/2}$.

Step 5. **Iteration in U-shaped regions: inductive step.** In this step, we iterate the construction from Step 4 on a large scale, shortening a candidate shortcut using shortcuts from lower scales. This procedure is one of the innovations in the current paper and is summarized in the central inequality (7.5) of Proposition 7.3. That inequality relates the savings in length on one scale to those on lower scales.

Let $\mathfrak{s}_k$ be an open path inside a U-shaped region of scale 2^k connecting the five-arm points at the ends of the "U" such that $\mathfrak{s}_k$ has minimal length.

(This is the "best possible" candidate shortcut on scale k.) By induction, we will assume in the i-th stage of the iteration that $\mathfrak{s}_k$ is already shorter than scale-k lowest crossings by a factor of $\delta_k(i)$. In other words, in Proposition 7.3, at stage i, we begin with initial assumptions (see (7.3))

$$\mathbf{E}[\#\mathfrak{s}_k \mid E_k'] \leq \delta_k(i)2^{2k}\pi_3(2^k) \tag{2.6}$$

for $k \geq 1$. The goal is to improve this inequality by replacing the right side of (2.6) with a better upper bound $\delta_k(i+1)2^{2k}\pi_3(2^k)$, so long as the scale k is large enough.

To make this improvement on (2.6), we use a version of the construction of Step 4 described in Section 7.1, modifying the outermost candidate shortcut ℓ_k to build a path σ out of shortcuts and portions of ℓ_k. This time, instead of saving a factor of ε (like in (2.3)), we use shortcuts that we have already constructed in previous stages, so they obey the length bound (2.6). Therefore their total savings over ℓ_k is a factor of $\kappa_k(i) := \varepsilon \cdot \delta_k(i)$ on scale k. In (7.5), we estimate the length of σ (and therefore the shortest candidate shortcut $\mathfrak{s}_k$) in terms of these factors $\kappa_k(i)$.

In Proposition 7.4 of Section 7.4, we bound this right side of (7.5) to eventually show

$$\mathbf{E}[\#\mathfrak{s}_k \mid E_k'] \leq \delta_k(i+1)2^{2k}\pi_3(2^k) \tag{2.7}$$

for $k \geq 1$, where the parameters $\delta_k(i+1)$ can roughly be taken as

$$\delta_k(i+1) \sim C'\varepsilon^{1/2}\delta_{k-C''}(i),$$

where C' is independent of ε and C'' has order $\varepsilon^{-4}(\log\frac{1}{\varepsilon})^2$. Equation (2.7), along with these values of $\delta_k(i+1)$, implies that if we move up $\varepsilon^{-4}\left(\log\frac{1}{\varepsilon}\right)^2$ scales, we accumulate an additional savings of $C'\varepsilon^{1/2}$. This is sufficient to conclude the induction for the general bound of Proposition 7.1: for k satisfying $2^k \geq (C\varepsilon^{-4}(\log\frac{1}{\varepsilon})^2)^L$ and $L \geq 1$,

$$\mathbf{E}[\#\mathfrak{s}_k \mid E_k'] \leq (C'\varepsilon^{1/2})^L 2^{2k}\pi_3(2^k).$$

The conditions on k and L now allow us to choose $L = c(\varepsilon)k$ for some (small) $c(\varepsilon) > 0$, which yields a bound of the form

$$\mathbf{E}[\#\mathfrak{s}_k \mid E_k'] \leq 2^{(2-\delta)k}\pi_3(2^k). \tag{2.8}$$

Step 6. The estimate (2.8) gives a bound for the size of optimal shortcuts that can be constructed around the lowest crossing on sufficiently large scales. In Section 8, we repeat Steps 1-3, but now using these improved shortcuts (along with portions of the lowest crossing) to improve the length of the lowest crossing of $B(n)$ and obtain the main result.

3 Notations

Throughout this paper, we consider the square lattice $\mathbb{Z}^2$, viewed as a graph with edges between nearest-neighbor vertices. We denote the set of edges by $\mathscr{E}^2$. The critical bond percolation measure $\mathbf{P}$ is the product measure

$$\mathbf{P} = \prod_{e \in \mathscr{E}^2} \frac{1}{2}(\delta_0 + \delta_1)$$

on $\Omega = \{0,1\}^{\mathscr{E}^2}$, with the product sigma-algebra. For an edge $e \in \mathscr{E}$, the translation of $e = \{v_1, v_2\}$ by a vertex $v \in \mathbb{Z}^2$ is

$$\tau_v e = \{v_1 + v, v_2 + v\}.$$

For $\omega \in \Omega$, the translation $\tau_v \omega$ is defined by

$$(\tau_v \omega)_e = \omega_{\{v_1+v, v_2+v\}}$$

for each edge $e = \{v_1, v_2\}$. For an event $E \subset \Omega$, we define $\tau_{-v}E$, the event translated by $-v$, by

$$\omega \in E \iff \tau_v \omega \in \tau_{-v} E.$$

A lattice path is a sequence of vertices and edges v_0, e_1, v_1, …, e_N, v_N such that $\|v_{k-1} - v_k\|_1 = 1$ and $e_k = \{v_{k-1}, v_k\}$. A path is called a circuit if $v_0 = v_N$. A path is called vertex self-avoiding if $v_i = v_j$ implies $i = j$. A path (or circuit) is said to be open if all its edges are open ($\omega(e_i) = 1$ for $i = 1, \ldots, N$). A circuit is said to be open with k defects if all but k edges on the circuit are open.

The coordinate vectors $\mathbf{e}_1$, $\mathbf{e}_2$ are

$$\mathbf{e}_1 = (1,0),\ \mathbf{e}_2 = (0,1).$$

The dual lattice $(\mathbb{Z}^2)^*$ is

$$(\mathbb{Z}^2)^* = \mathbb{Z}^2 + \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2).$$

To each edge $e \in \mathscr{E}^2$, we associate a dual edge e^*, the edge of $(\mathscr{E}^2)^*$ which shares a midpoint with e. For a configuration $\omega \in \Omega$, the dual configuration ω^* is defined by $\omega^*(e^*) = \omega(e)$. A dual path is a path made of dual vertices and edges. The definitions of circuit, openness, and circuit with defects extend to the case of dual paths in a straightforward way.

We will often refer to S^* for the set $S + (\frac{1}{2}, \frac{1}{2})$, when S is a subset of $\mathbb{R}^2$, a set of edges, or a set of vertices. In particular $B(n)^* = [-n + \frac{1}{2}, n + \frac{1}{2}]^2$.

3.1 Constants

Throughout the paper, the usual notation for the logarithm, log, is reserved for the logaritheorem in base 2; thus in our notation

$$2^{\log x} := 2^{\log_2 x} = x$$

for all $x > 0$. The nonnumbered constants C, C', c, c', and so on, will represent possibly different numbers from line to line.

3.2 Arm events

Events defined by the existence of connections from the center of a box to its boundary, or between the boundaries of annuli are referred to as arm events. Below, we denote by $\partial B(n)$ the topological boundary of $B(n) = [-n,n]^2$:

$$\partial B(n) = (\{n\} \times [-n,n]) \cup ([-n,n] \times \{n\}) \cup (\{-n\} \times [-n,n]) \cup ([-n,n] \times \{-n\}).$$

We make use of three main types of arm events: (polychromatic) three-arm, five-arm, and six-arm events.

Three-arm event. The three-arm event $A_3(n)$ is defined by the following connections:

(1) the edge $\{0, \mathbf{e}_1\}$ is connected to $\partial B(n)$ by two open vertex-disjoint paths,
(2) $(1/2)(\mathbf{e}_1 - \mathbf{e}_2)$ is connected to $\partial B(n)^*$ by a closed dual path.

For $v \in \mathbb{Z}^2$, $A_3(v,n)$ denotes the event $A_3(n)$ translated by v. If $A_3(v,n)$ occurs, we say that v is a three-arm point (to distance n).

Recall that $B(e,n)$ is the box of sidelength $2n$ centered at the lower-left endpoint of e. We also consider the three-arm event centered at an edge e, characterized by the conditions

(1) e is connected to $\partial B(e,n)$ by two vertex-disjoint open paths,
(2) the dual edge e^* is connected to $\partial B(e,n)^*$ by a closed dual path.

The probability of the three-arm event is denoted by

$$\pi_3(n) := \mathbf{P}(A_3(n)).$$

A fact concerning $\pi_3(n)$ that we will use several times is the existence of a $\beta = 1 - \gamma < 1$ such that

$$\frac{\pi_3(2^d)}{\pi_3(2^L)} \leq C_5 2^{\beta(L-d)}, \quad d \leq L, \tag{3.1}$$

for some $C_5 \geq 1$. See [9, Lemma 2.1].

For $n_1 < n_2$, we denote by $A_3(n_1,n_2)$ the probability that there are two vertex-disjoint open paths inside $B(n_2) \setminus B(n_1)$ from $\partial B(n_1)$ to $\partial B(n_2)$, and a closed dual connection inside $(B(n_2) \setminus B(n_1))^*$ from $\partial B(n_1)^*$ to $\partial B(n_2)^*$. We let

$$\pi_3(n_1,n_2) := \mathbf{P}(A_3(n_1,n_2))$$

be the corresponding probability. For convenience, if $n_1 \geq n_2$, we let $A_3(n_1,n_2)$ denote the entire sample space and correspondingly set $\pi_3(n_1,n_2) = 1$.

For a vertex v, the event $A_3(v,n_1,n_2)$ is the translation of $A_3(n_1,n_2)$ by v. For an edge e, we define $A_3(e,n_1,n_2) := A_3(w,n_1,n_2)$, where w is the lower-left endpoint of e.

Five-arm event. We say the origin is a five-arm point (to distance n) if:

(1) $(0,0)$ has three vertex-disjoint (except their initial vertex 0) open paths in $B(n)$ emanating from 0 and reaching $\partial B(n)$: one taking the edge $\{0,\mathbf{e}_1\}$ first, one taking the edge $\{0,-\mathbf{e}_1\}$ first, and one taking the edge $\{0,\mathbf{e}_2\}$ first,
(2) there two vertex-disjoint closed dual paths inside $B(n)^*$ emanating from dual neighbors of 0 and reaching $\partial B(n)^*$, one taking the dual edge $\{(-1/2)\mathbf{e}_1 + (1/2)\mathbf{e}_2, (-1/2)\mathbf{e}_1 + (3/2)\mathbf{e}_2\}$ first, and the other taking the dual edge $\{(1/2)\mathbf{e}_1 - (1/2)\mathbf{e}_2, (1/2)\mathbf{e}_1 - (3/2)\mathbf{e}_2\}$ first.
(3) the edge $\{0,-\mathbf{e}_2\}$ is closed.

We denote the event that the origin is a five-arm point to distance n by $A_5(n)$. We say that a vertex v is a five-arm point if $A_5(n)$ occurs in the configuration translated by $-v$.

For $4 \le n_1 < n_2$, we let $\pi_5(n_1,n_2)$ denote the probability that there are three vertex-disjoint open paths inside $B(n_2)\setminus B(n_1)$ from $\partial B(n_1)$ to $\partial B(n_2)$, and two vertex-disjoint closed dual paths inside $(B(n_2)\setminus B(n_1))^*$ between $\partial B(n_1)^*$ and $\partial B(n_2)^*$. The connections appear in the counterclockwise order closed, open, open, closed, open.

We denote by

$$\pi_5(n) := \mathbf{P}(A_5(n)).$$

Unlike the probabilities for other arm events, the exact scaling of $\pi_5(n)$ is known: there is a constant $C>0$, such that

$$(1/C)n^{-2} \le \pi_5(n) \le Cn^{-2}, \tag{3.2}$$

see [36, Theorem 24, 3.].

Six-arm event. Let $n_1 < n_2$. We say that the six-arm event $A_6(n_1,n_2)$ occurs if the following six connections occur:

(1) there are three vertex-disjoint open connections inside $B(n_2)\setminus B(n_1)$ from $\partial B(n_1)$ to $\partial B(n_2)$,
(2) there are three vertex-disjoint dual closed connections in $(B(n_2)\setminus B(n_1))^*$ from $\partial B(n_1)^*$ to $\partial B(n_2)^*$,
(3) the connections appear the in the counterclockwise order open, open, closed, open, open, closed.

Applying the van den Berg-Kesten-Reimer inequality [33], the RSW theorem (see Section 3.4 below), and using (3.2), we have

$$\mathbf{P}(A_6(n_1,n_2)) \le C\left(\frac{n_1}{n_2}\right)^{2+\delta_1},$$

for some $C,\delta_1>0$.

We also have the following lower bound for the six arm event:

$$\mathbf{P}(A_6(n_1,n_2)) \geq c\left(\frac{n_1}{n_2}\right)^4. \tag{3.3}$$

To see why, let $A_{3,HP}(n)$ be the event that there are two open paths from the origin to distance n and a dual vertex adjacent to the origin is connected to distance n by a closed dual path, with all paths lying in the upper half-plane $\mathbb{R}\times[0,\infty)$. We let $A_{3,HP}(n_1,n_2)$ be the event that there are two open paths connecting $\partial B(n_1)$ to $\partial B(n_2)$ and a closed dual path connecting $\partial B(n_1)^*$ to $\partial B(n_2)^*$, with all paths contained in $\mathbb{R}\times[0,\infty)$. The estimate (3.3) follows by quasi-multiplicativity [36, Proposition 12.2] of arm probabilities:

$$\mathbf{P}(A_{3,HP}(n_1,n_2)) \geq c\frac{\mathbf{P}(A_{3,HP}(n_2))}{\mathbf{P}(A_{3,HP}(n_1)))},$$

and the universal value of the half-plane exponent [36, Theorem 24.2]

$$(1/C)n^{-2} \leq \mathbf{P}(A_{3,HP}(n)) \leq Cn^{-2}.$$

If two half-plane three-arm events occur in disjoint half-planes, then $A_6(n_1,n_2)$ occurs, so:

$$\mathbf{P}(A_6(n_1,n_2)) \geq c\mathbf{P}(A_{3,HP}(n_1,n_2))^2 \geq c'(n_1/n_2)^4.$$

3.3 Characterizing the lowest crossing

Let $Q=[0,m]\times[0,n]$, where m and n are positive integers. It is well-known (see [19, p. 317]) that, on the event that there exists an open crossing of Q, there exists a lowest such open crossing. It is also well-known (see, for instance, [34, (1.1)] and surrounding discussion) that planar duality allows one to simply characterize this crossing in terms of arm events.

Consider the slightly modified rectangle $Q'=[1/2,m-1/2]\times[-1/2,n+1/2]$. The edge $e\in Q$ is in the lowest open horizontal crossing of Q if and only if it satisfies the following condition:

$$\begin{aligned}&e \text{ is open, the two endpoints of } e \text{ have disjoint open arms within } Q\\ &(\text{one to the left side } \{0\}\times[0,n] \text{ of } Q \text{ and one to the right side } \{m\}\times[0,n] \text{ of } Q),\\ &\text{and an endpoint of } e^* \text{ has a closed dual arm within } Q' \text{ to } \{-1/2\}\times[1/2,m-1/2]\,.\end{aligned} \tag{3.4}$$

One can use Kesten's arm separation method (See Section 3.4) to show that the probability that $\{0,\mathbf{e}_1\}$ is in the lowest crossing of $B(n)$ is bounded above and below by constant multiples of $\pi_3(n)$, uniformly in n.

3.4 FKG and gluing

We will repeatedly use "gluing" constructions based on combinations of the Russo-Seymour-Welsh (RSW) theorem, the generalized Fortuin-Kasteleyn-Ginibre (FKG) inequality, and arm events to obtain lower bounds on the probability of existence large-scale open or closed connections. These methods were pioneered

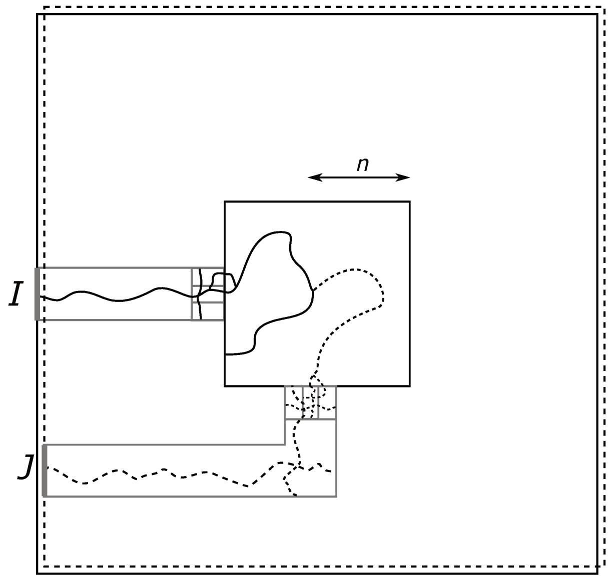

FIGURE 3.1. An example of a gluing construction. The inner central box is $B(n)$ and the outer box is $B(3n)$. The solid (open) arms, along with the connection to the vertical solid (open) crossing of the small box on the left make up the event A^+. The dotted (closed dual) arm, along with the connection to dotted (closed dual) horizontal crossing of the small box below make up the event A^-. B^+ is made up of the horizontal solid crossing of the thin box on the left, and B^- is made up of the horizontal dotted crossing of the thin box below, and the vertical dotted crossing connecting the previous crossing to the central box. The intersection of all these events implies A.

by H. Kesten (see, for example, [28, 29]), and have since become standard in the percolation literature. In this section, we describe this methodology and give an example. To limit the length of this paper and focus on the original aspects of the proofs, we will give less detail of such constructions, instead describing the requisite constructions.

We begin by recallling the Russo-Seymour-Welsh theorem, which ensures that crossings of rectangles exist with probability bounded away from 0 and 1 uniformly, with constants depending only on the aspect ratio of the rectangles. For $\rho > 0$, let $H(\rho n, n)$ be the event that there is a path of open edges inside $R(\rho n, n) = [-\rho n, \rho n] \times [-n, n]$ from the left to the right side of this rectangle.

Proposition 3.1 (Russo-Seymour-Welsh estimates)**.** *There exists $c = c(\rho) > 0$, such that for every $n \geq 1$,*

$$c \leq \mathbf{P}(H(\rho n, n)) \leq 1 - c. \tag{3.5}$$

See [36, Theorem 2].

The RSW theorem is typically used along with the Harris-Fortuin-Kasteleyn-Ginibre (Harris-FKG) inequality to "glue" paths together. For its statement, we recall that an event A is increasing (resp. decreasing) if whenever $\omega \in A$ and $\omega' \geq \omega$ coordinatewise (resp. $\omega' \leq \omega$ coordinatewise), then $\omega' \in A$. The Harris-FKG inequality states that if A and B are both increasing (or both decreasing), then $\mathbf{P}(A \cap B) \geq \mathbf{P}(A)\mathbf{P}(B)$. The event $H(\rho n, n)$ from Proposition 3.1 is increasing, so if we define $A = H(n,n)$ and B to be the event that there is a vertical crossing of $R(n,n)$, then by the RSW theorem and the Harris-FKG inequality, $\mathbf{P}(A \cap B)$ is bounded away from zero independently of n. By forcing A and B to occur simultaneously we have glued together two crossings.

In practice, we will not deal with only increasing events or only decreasing events, so we will need the so-called generalized Fortuin-Kasteleyn-Ginibre (FKG) inequality (see [36, Lemma 13]).

Proposition 3.2. *Consider A^+, B^+ two increasing events, and A^-, B^- two decreasing events. Assume there are three disjoint finite sets of edges $\tilde{A}$, $\tilde{A}^+$, $\tilde{A}^-$ such that A^+, A^-, B^+, B^- depend only on the edges in, respectively, $\tilde{A} \cup \tilde{A}^+$, $\tilde{A} \cup \tilde{A}^-$, $\tilde{A}^+$ and $\tilde{A}^-$. Then, we have*

$$\mathbf{P}(B^+ \cap B^- \mid A^+ \cap A^-) \geq \mathbf{P}(B^+)\mathbf{P}(B^-). \tag{3.6}$$

The proof of the next proposition illustrates a combination of the RSW theorem, the generalized FKG inequality, and Kesten's "arm separation" method to construct connections between arms and crossings.

Proposition 3.3. *Let*

$$I = \{-3n\} \times [-\frac{n}{8}, \frac{n}{8}] \subset \partial B(3n),$$
$$J = \left(\{-3n\} \times [-\frac{11n}{4}, -\frac{5n}{2}]\right)^*.$$

Let A be the event that

(1) there is an open connection inside $B(n)$ from the origin to $\partial B(n)$,
(2) there is an open connection, disjoint from the previous one, inside $[-3n,n] \times [-\frac{n}{4}, \frac{n}{4}]$ from 0 to I,
(3) there is a closed dual connection inside

$$\left([-3n, \frac{n}{4}] \times [-\frac{11n}{4}, -\frac{5n}{2}]\right)^* \cup \left([-\frac{n}{8}, \frac{n}{8}] \times [-\frac{11n}{4}, n]\right)^*,$$

from a dual neighbor of 0 to J.

Then, there is a constant c such that

$$\mathbf{P}(A) \geq c\pi_3(n).$$

Proof. Let A_+ be the event that

I there is a open connection $\mathfrak{c}_1$ inside $B(n)$ from $(0,0)$ to $\{-n\} \times [-\frac{n}{12}, \frac{n}{12}]$, II $\mathfrak{c}_1$ is connected inside $[-\frac{5n}{4}, -\frac{3n}{4}] \times [-\frac{n}{8}, \frac{n}{8}]$ to a vertical open crossing of $[-\frac{5n}{4}, -n] \times [-\frac{n}{8}, \frac{n}{8}]$.

III there is an open connection $\mathfrak{c}_1'$, edge-disjoint from $\mathfrak{c}_1$, inside $B(n)$ from 0 to $\partial B(n)$.

Similarly, let A_- be the event that

I there is a closed dual connection $\mathfrak{c}_2$ inside $B(0,n)^*$ from $(\frac{1}{2}, \frac{1}{2})$, to $([-\frac{n}{12}, \frac{n}{12}] \times \{-n\})^*$,

II $\mathfrak{c}_2$ is connected inside $\left([-\frac{n}{8}, \frac{n}{8}] \times [-\frac{5n}{4}, -\frac{3n}{4}]\right)^*$ to a horizontal closed dual crossing of $\left([-\frac{n}{8}, \frac{n}{8}] \times [-\frac{5n}{4}, -n]\right)^*$.

The event $A_+ \cap A_-$ is contained in $A_3(n)$, so clearly

$$\mathbf{P}(A_+ \cap A_-) \leq \mathbf{P}(A_3(n)).$$

Kesten's arms separation technique [29] implies that the probability of $A^+ \cap A^-$ is comparable to that of the three-arm event $A_3(n)$:

$$\mathbf{P}(A_+ \cap A_-) \geq c\mathbf{P}(A_3(n)) \tag{3.7}$$

(c does not depend on n). The results stated in [29] apply to the alternating four-arm event, but the reader can see [36, Theorem 11] for statements like inequality (3.7) which apply to general arm events, as well as to "near-critical" percolation. The essential point is that when $A_3(n)$ occurs, at the additional cost of a constant factor in the probability, we can specify certain subintervals ("landing zones") of $\partial B(n)$ for the endpoints of the arms in arm events, and we can extend the three arms through $\partial B(n)$ to connect to macroscopic connections outside of $B(n)$.

Next we attach more connections to the arms, so that they end at the intervals I and J. We will need to use the generalized FKG inequality (instead of the standard Harris-FKG inequality) because some not all of our events are of one type (increasing or decreasing). We let B_+ be the event that there is an open horizontal crossing of $[-3n, -n] \times [-\frac{n}{8}, \frac{n}{8}]$, and let B_- be the event that there is a closed dual horizontal crossing of $\left([-3n, \frac{n}{8}] \times [-\frac{11n}{4}, -\frac{5n}{2}]\right)^*$ and a closed dual vertical crossing of $\left([-\frac{n}{8}, \frac{n}{8}] \times [-\frac{11n}{4}, -n]\right)^*$.

By construction and planarity,

$$A_+ \cap A_- \cap B_+ \cap B_- \subset A,$$

where A is the event in the statement of the proposition. Successively applying the generalized FKG inequality (3.6), the lower bound (3.7), and the RSW theorem gives

$$\begin{aligned} \mathbf{P}(A) \geq \mathbf{P}(A_+ \cap A_- \cap B_+ \cap B_-) &\geq \mathbf{P}(A_+ \cap A_-)\mathbf{P}(B_+)\mathbf{P}(B_-) \\ &\geq c\mathbf{P}(A_3(n))\mathbf{P}(B_+)P(B_-) \\ &\geq c'\mathbf{P}(A_3(n)). \end{aligned}$$

To apply the generalized FKG inequality, one can choose the following regions to be $\tilde{A}$, $\tilde{A}^+$, and $\tilde{A}^-$:

$$\begin{aligned}\tilde{A} &= \text{ edges of } B(n),\\ \tilde{A}^+ &= \text{ edges of } [-3n,n]\times[-\frac{n}{8},\frac{n}{8}],\\ \tilde{A}^- &= \text{ edges of } [-3n,\frac{n}{8}]\times[-\frac{11n}{4},-n].\end{aligned}$$

□

4 Large deviation bound conditional on three arms

Our aim in this section is to give a bound on the conditional probability, given the three-arm event $A_3(2^n)$, that only a small number of events E_k, which satisfy the probability bound (4.1) below, occur. This result will be used later for a specific choice of E_k (see (5.21) for the definition) in Proposition 5.6.

On $A_3(2^n)$, we want to have closed dual circuits with defects around the origin to perform decoupling of various events. An open circuit with ℓ defects is a self-avoiding circuit all of whose edges are open, with the exception of ℓ edges which are closed. A closed dual circuit with defects is defined analogously. Fix an integer $N \geq 1$ and, given any k, let $\mathfrak{C}_k$ be the event that in $A(2^{kN},2^{(k+1)N}) = B(2^{(k+1)N}) \setminus B(2^{kN})$, there is a closed dual circuit with two defects around the origin. Let $\mathfrak{D}_k$ be the event that there is an open circuit with one defect in the same annulus, $A(2^{kN},2^{(k+1)N})$. We will need a large stack of these circuit events to decouple (seven in total), and so we define this compound circuit event for $k \geq 0$ as $\hat{\mathfrak{C}}_k$, the event that the following occur:

(1) for $i = 1,3,4,6,8,9$, the event $\mathfrak{C}_{10k+i}$ occurs and
(2) the event $\mathfrak{D}_{10k}$ occurs.

The exact definition of E_k will not be essential in this part of the paper. The assumptions on E_k we need are that

A. E_k depends on the state of edges in $A(2^{kN},2^{(k+1)N})$ and
B. for some constant $\tilde{c}_0 > 0$, one has for all $n \geq 0$ and integers k with $0 \leq k \leq \frac{n}{10N} - 1$,

$$\mathbf{P}\left(\mathfrak{B}_k \,\middle|\, A_3(2^n)\right) \geq \tilde{c}_0, \tag{4.1}$$

where

$$\mathfrak{B}_k = \hat{\mathfrak{C}}_k \cap E_{10k+5}. \tag{4.2}$$

In item B, we are requesting that E_{10k+5} occur, but also that it be surrounded on both sides by the total of seven defected circuits. These circuits will be needed for the "resetting" argument.

Define for $N \geq 1$ and $0 \leq n' \leq n$,

$$I_{n',n} = \left\{ j = \left\lceil \frac{n'}{10N} \right\rceil, \ldots, \left\lfloor \frac{n}{10N} \right\rfloor - 1 : \mathfrak{B}_j \text{ occurs} \right\}.$$

Note that if $n - n' \geq 40N$, then the range of j specified in $I_{n',n}$ is nonempty.

Theorem 4.1. *There exist universal $c_6 > 0$ and $N_0 > 0$ such that for any $N \geq N_0$, any $n', n \geq 0$ satisfying $n - n' \geq 40N$, and any events (E_k) satisfying conditions A and B,*

$$\mathbf{P}\left(\# I_{n',n} \leq c_6 \tilde{c}_0 \frac{n-n'}{N} \;\middle|\; A_3(2^n) \right) \leq \exp\left(-c_6 \tilde{c}_0 \frac{n-n'}{N} \right).$$

For the proof of Theorem 4.1, we first need to verify that conditional on $A_3(2^n)$, many of the events $\hat{\mathfrak{C}}_k$ occur. So for $0 \leq n' \leq n$, we set

$$J_{n',n} = \left\{ j = \left\lceil \frac{n'}{10N} \right\rceil, \ldots, \left\lfloor \frac{n}{10N} \right\rfloor - 1 : \hat{\mathfrak{C}}_j \text{ occurs} \right\}.$$

Proposition 4.2. *There exist $c_7 > 0$ and $N_0 \geq 1$ such that for all $N \geq N_0$ and $n, n' \geq 0$ with $n - n' \geq 40N$,*

$$\mathbf{P}\left(\# J_{n',n} \leq c_7 \frac{n-n'}{N} \;\middle|\; A_3(2^n) \right) \leq \exp\left(-c_7 (n-n') \right).$$

Proof. For $0 \leq n_1 \leq n_2$, let $A_3(2^{n_1}, 2^{n_2})$ be the event that there exist three arms from $B(2^{n_1})$ to $\partial B(2^{n_2})$: there are two open paths and one dual closed path, all disjoint, connecting $B(2^{n_1})$ to $\partial B(2^{n_2})$. First note that

$$\begin{aligned} \mathbf{P}\left(\# J_{n',n} \leq c_7 \frac{n-n'}{N}, A_3(2^n) \right) &\leq \mathbf{P}\left(A_3\left(2^{10N\left\lceil \frac{n'}{10N} \right\rceil} \right) \right) \mathbf{P}\left(A_3\left(2^{10N\left\lfloor \frac{n}{10N} \right\rfloor}, 2^n \right) \right) \\ &\quad \times \mathbf{P}\left(\bigcap_{m=10\left\lceil \frac{n'}{10N} \right\rceil}^{10\left\lfloor \frac{n}{10N} \right\rfloor - 1} A_3(2^{mN}, 2^{(m+1)N}),\ \# J_{n',n} \leq c_7 \frac{n-n'}{N} \right). \end{aligned} \tag{4.3}$$

Here we have used independence to decouple crossing events for disjoint annuli.

We now recall Menger's Theorem from graph theory. An edge cutset for a pair of vertices x and y in a graph $G = (V, E)$ is a subset E' of E such that removing the edges in E' disconnects x and y. Menger's Theorem states that the minimal size of any edge cutset for x and y is equal to the maximum number of edge-disjoint paths from x and y.

By Menger's Theorem, for any m, the event $A_3(2^{mN}, 2^{(m+1)N}) \cap \mathfrak{C}_m^c$ implies $A_3(2^{mN}, 2^{(m+1)N}) \circ A_1(2^{mN}, 2^{(m+1)N})$, where $\circ$ indicates disjoint occurrence, and $A_1(2^{mN}, 2^{(m+1)N})$ is the event that there is one open path from $B(2^{mN})$ to $\partial B(2^{(m+1)N})$. By the RSW theorem and the van den Berg-Kesten-Reimer inequality, there is therefore $\alpha \in (0,1)$ such that

$$\mathbf{P}(A_3(2^{mN}, 2^{(m+1)N}) \cap \mathfrak{C}_m^c) \leq 2^{-\alpha N} \mathbf{P}(A_3(2^{mN}, 2^{(m+1)N})). \tag{4.4}$$

Similar reasoning shows that if $A_3(2^{mN}, 2^{(m+1)N}) \cap \mathfrak{D}_m^c$ occurs, then there are three arms as indicated by the A_3 event, but one additional closed dual arm crossing this annulus, and we obtain the same bound

$$\mathbf{P}(A_3(2^{mN}, 2^{(m+1)N} \cap \mathfrak{D}_m^c) \leq 2^{-\alpha N}\mathbf{P}(A_3(2^{mN}, 2^{(m+1)N})). \tag{4.5}$$

Using quasimultiplicativity of arm events [36, Proposition 12], independence, (4.4), and (4.5), there is a universal $C_8 \geq 1$ such that for all N and all $j \geq 0$,

$$\begin{aligned}
&\mathbf{P}\left(\cap_{l=0}^{9} A_3(2^{(10j+l)N}, 2^{(10j+l+1)N}) \cap \hat{\mathfrak{C}}_j^c\right)\\
&\leq \sum_{\substack{0\leq r\leq 9\\ r\neq 0,2,5,7}} \mathbf{P}\left(\cap_{l=0}^{9} A_3(2^{(10j+l)N}, 2^{(10j+l+1)N}) \cap \mathfrak{C}_{10j+r}^c\right)\\
&+ \mathbf{P}\left(\cap_{l=0}^{9} A_3(2^{(10j+l)N}, 2^{(10j+l+1)N}) \cap \mathfrak{D}_{10j}^c\right)\\
&= \sum_{\substack{0\leq r\leq 9\\ r\neq 0,2,5,7}} \left[\left(\prod_{\substack{0\leq l\leq 9\\ l\neq r}} \mathbf{P}(A_3(2^{(10j+l)N}, 2^{(10j+l+1)N}))\right)\mathbf{P}\left(A_3(2^{(10j+r)N}, 2^{(10j+r+1)N}), \mathfrak{C}_{10j+r}^c\right)\right]\\
&+ \left(\prod_{1\leq l\leq 9} \mathbf{P}\left(A_3(2^{(10j+l)N}, 2^{(10j+l+1)N}\right)\right)\mathbf{P}\left(A_3(2^{10jN}, 2^{(10j+1)N}), \mathfrak{D}_{10j}^c\right)\\
&\leq 7\cdot 2^{-\alpha N}\prod_{l=0}^{9}\mathbf{P}(A_3(2^{(10j+l)N}, 2^{(10j+l+1)N}))
\end{aligned}$$

(4.6)

$$\leq 7C_8^9 2^{-\alpha N}\mathbf{P}(A_3(2^{10jN}, 2^{10(j+1)N})).$$

These observations lead us to realizing the problem as one of concentration using independent variables. For any integer j with $\frac{n'}{10N} \leq j \leq \frac{n}{10N} - 1$, let X_j be the indicator of the event $\cap_{l=0}^{9} A_3(2^{(10j+l)N}, 2^{(10j+l+1)N}) \cap \hat{\mathfrak{C}}_j^c$. Then (4.3) implies

$$\begin{aligned}
\mathbf{P}\left(\#J_{n',n} \leq c_7\frac{n-n'}{N}, A_3(2^n)\right) &\leq \mathbf{P}\left(A_3\left(2^{10N\left\lceil \frac{n'}{10N}\right\rceil}\right)\right)\mathbf{P}\left(A_3\left(2^{10N\left\lfloor \frac{n}{10N}\right\rfloor}, 2^n\right)\right)\\
&\times \mathbf{P}\left(\sum_{j=\left\lceil \frac{n'}{10N}\right\rceil}^{\left\lfloor \frac{n}{10N}\right\rfloor - 1} X_j \geq \left\lfloor \frac{n}{10N}\right\rfloor - \left\lceil \frac{n'}{10N}\right\rceil - c_7\frac{n-n'}{N}\right).
\end{aligned} \tag{4.7}$$

Using (4.6) and the RSW theorem, the X_j's are independent Bernoulli random variables with parameters p_j that satisfy for some $\beta \geq 1$

$$2^{-\beta N} \leq p_j \leq 7C_8^9 2^{-\alpha N}\mathbf{P}(A_3(2^{10jN}, 2^{10(j+1)N})). \tag{4.8}$$

So we need an elementary lemma about concentration of independent Bernoulli random variables with suitable parameters.

Lemma 4.3. *Given $\varepsilon_1 \in (0,1)$ and $M \geq 1$, if $Y_1, \ldots, Y_M$ are any independent Bernoulli random variables with parameters $p_1, \ldots, p_M$ respectively satisfying $p_i \in [\varepsilon_1, 1]$ for all i, then for all $r \in (0,1)$,*

$$\mathbf{P}\left(\sum_{i=1}^{M} Y_i \geq rM\right) \leq (1/\varepsilon_1)^{M(1-r)} 2^M \prod_{i=1}^{M} p_i.$$

Proof. One has

$$\mathbf{P}(Y_1 + \cdots + Y_M \geq rM) = \sum_{\ell=\lceil rM \rceil}^{M} \mathbf{P}(Y_1 + \cdots + Y_M = \ell).$$

Also for ℓ with $\lceil rM \rceil \leq \ell \leq M$,

$$\begin{aligned}
\mathbf{P}(Y_1 + \cdots + Y_M = \ell) &= \sum_{\substack{y_1, \ldots, y_M \in \{0,1\} \\ y_1 + \cdots + y_M = \ell}} p_1^{y_1} \cdots p_M^{y_M} (1-p_1)^{1-y_1} \cdots (1-p_M)^{1-y_M} \\
&= \prod_{i=1}^{M} p_i \sum_{\substack{y_1, \ldots, y_M \in \{0,1\} \\ y_1 + \cdots + y_M = \ell}} \left(\frac{1-p_i}{p_i}\right)^{1-y_i} \\
&\leq \binom{M}{\ell} \left(\frac{1-\varepsilon_1}{\varepsilon_1}\right)^{M-\ell} \prod_{i=1}^{M} p_i \\
&\leq \binom{M}{\ell} (1/\varepsilon_1)^{M(1-r)} \prod_{i=1}^{M} p_i.
\end{aligned}$$

We sum over ℓ to obtain

$$\mathbf{P}(Y_1 + \cdots + Y_M \geq rM) \leq (1/\varepsilon_1)^{M(1-r)} \left(\sum_{\ell=\lceil rM \rceil}^{M} \binom{M}{\ell}\right) \prod_{i=1}^{M} p_i,$$

from which the lemma follows. $\square$

We now apply Lemma 4.3 to (4.7), using the bounds from (4.8), with $\varepsilon_1 = 2^{-\beta N}$. Note that if $c_7 < 1/20$ and $n - n' \geq 40N$, one has

$$\left\lfloor \frac{n}{10N} \right\rfloor - \left\lceil \frac{n'}{10N} \right\rceil - c_7 \frac{n-n'}{N} \geq \left(\left\lfloor \frac{n}{10N} \right\rfloor - \left\lceil \frac{n'}{10N} \right\rceil \right)(1 - 20c_7).$$

So if we put $r = 1-20c_7$ (noting that $r \in (0,1)$) and use Lemma 4.3, we continue from (4.7) to obtain

$$
\begin{aligned}
&\mathbf{P}\left(\#J_{n',n} \le c_7\frac{n-n'}{N}, A_3(2^n)\right)\\
&\le \mathbf{P}\left(A_3\left(2^{10N\lceil \frac{n'}{10N}\rceil}\right)\right)\mathbf{P}\left(A_3\left(2^{10N\lfloor \frac{n}{10N}\rfloor}, 2^n\right)\right)\\
&\times \mathbf{P}\left(\sum_{j=\lceil \frac{n'}{10N}\rceil}^{\lfloor \frac{n}{10N}\rfloor -1} X_j \ge \left(\left\lfloor \frac{n}{10N}\right\rfloor - \left\lceil \frac{n'}{10N}\right\rceil\right)(1-20c_7)\right)\\
&\le (2^{\beta N})^{\left(\lfloor \frac{n}{10N}\rfloor - \lceil \frac{n'}{10N}\rceil\right)\cdot 20c_7} 2^{\lfloor \frac{n}{10N}\rfloor - \lceil \frac{n'}{10N}\rceil}\\
&\times \mathbf{P}\left(A_3\left(2^{10N\lceil \frac{n'}{10N}\rceil}\right)\right)\mathbf{P}\left(A_3\left(2^{10N\lfloor \frac{n}{10N}\rfloor}, 2^n\right)\right)\prod_{j=\lceil \frac{n'}{10N}\rceil}^{\lfloor \frac{n}{10N}\rfloor -1} 7C_8^9 2^{-\alpha N}\mathbf{P}(A_3(2^{10jN}, 2^{10(j+1)N}))\\
&= \left(14C_8^9 2^{(20\beta c_7-\alpha)N}\right)^{\lfloor \frac{n}{10N}\rfloor - \lceil \frac{n'}{10N}\rceil}\\
&\times \mathbf{P}\left(A_3\left(2^{10N\lceil \frac{n'}{10N}\rceil}\right)\right)\mathbf{P}\left(A_3\left(2^{10N\lfloor \frac{n}{10N}\rfloor}, 2^n\right)\right)\prod_{j=\lceil \frac{n'}{10N}\rceil}^{\lfloor \frac{n}{10N}\rfloor -1} \mathbf{P}(A_3(2^{10jN}, 2^{10(j+1)N})). \qquad (4.9)
\end{aligned}
$$

Again by quasimultiplicativity of arm events,

$$
\begin{aligned}
&\mathbf{P}\left(A_3\left(2^{10N\lceil \frac{n'}{10N}\rceil}\right)\right)\mathbf{P}\left(A_3\left(2^{10N\lfloor \frac{n}{10N}\rfloor}, 2^n\right)\right)\prod_{j=\lceil \frac{n'}{10N}\rceil}^{\lfloor \frac{n}{10N}\rfloor -1} \mathbf{P}(A_3(2^{10jN}, 2^{10(j+1)N}))\\
&\qquad\qquad \le C_8^{\lfloor \frac{n}{10N}\rfloor - \lceil \frac{n'}{10N}\rceil + 1}\mathbf{P}(A_3(2^n)).
\end{aligned}
$$

Use this estimate in (4.9) to find for $c_7 < 1/20$, $n-n' \ge 40N$, and all $N \ge 1$,

$$
\begin{aligned}
\mathbf{P}\left(\#J_{n',n} \le c_7\frac{n-n'}{N} \;\middle|\; A_3(2^n)\right) &\le C_8\left(14C_8^{10}2^{(20\beta c_7-\alpha)N}\right)^{\lfloor \frac{n}{10N}\rfloor - \lceil \frac{n'}{10N}\rceil}\\
&\le \left(14C_8^{11}2^{(20\beta c_7-\alpha)N}\right)^{\lfloor \frac{n}{10N}\rfloor - \lceil \frac{n'}{10N}\rceil}.
\end{aligned}
$$

Lowering c_7 so that $c_7 < \alpha/(40\beta)$, one has $20\beta c_7 - \alpha \le -\alpha/2$. We also pick N_0 so large that for $N \ge N_0$, one has $14C_8^{11} \le 2^{\alpha N/4}$ and obtain

$$
\mathbf{P}\left(\#J_{n',n} \le c_7\frac{n-n'}{N} \;\middle|\; A_3(2^n)\right) \le 2^{-\alpha\frac{N}{4}\left(\lfloor \frac{n}{10N}\rfloor - \lceil \frac{n'}{10N}\rceil\right)}.
$$

If $n-n' \geq 40N$, then we obtain the upper bound $2^{-\alpha(n-n')/80}$, which completes the proof of Proposition 4.2. $\square$

Given the bound on the probability of existence of many decoupling circuits from Proposition 4.2, we move to the proof of Theorem 4.1.

Proof of Theorem 4.1. For $N \geq N_0$ and $n,n' \geq 0$ such that $n-n' \geq 40N$, we will estimate $\#I_{n',n}$ using the standard Chernoff bound along with a decoupling argument. So estimate using Proposition 4.2, for $c_9 > 0$ to be determined at the end of the proof,

$$\begin{aligned}
&\mathbf{P}\left(\#I_{n',n} \leq c_9 \frac{n-n'}{N} \,\middle|\, A_3(2^n)\right) \\
&\leq \exp\left(-c_7(n-n')\right) + \mathbf{P}\left(\#I_{n',n} \leq c_9 \frac{n-n'}{N}, \#J_{n',n} \geq c_7 \frac{n-n'}{N} \,\middle|\, A_3(2^n)\right) \\
&\leq \exp\left(-c_7(n-n')\right) + \exp\left(c_9 \frac{n-n'}{N}\right) \mathbf{E}\left[e^{-\#I_{n',n}} \mathbf{1}_{\{\#J_{n',n} \geq c_7 \frac{n-n'}{N}\}} \,\middle|\, A_3(2^n)\right].
\end{aligned} \tag{4.10}$$

The expectation we decompose over all possible sets $J_{n',n}$ as

$$\sum_{\#\mathscr{J} \geq c_7 \frac{n-n'}{N}} \mathbf{E}\left[e^{-\#I_{n',n}} \,\middle|\, J_{n',n} = \mathscr{J},\, A_3(2^n)\right] \mathbf{P}(J_{n',n} = \mathscr{J} \mid A_3(2^n)). \tag{4.11}$$

Last, we expand the expectation over a filtration. Enumerate the set $\mathscr{J} = \{j_1, \ldots, j_{r_0}\}$, where $r_0 \geq c_7 \frac{n-n'}{N}$. Then a.s. relative to the measure

$$\hat{\mathbf{P}} := \mathbf{P}\left(\cdot \mid J_{n',n} = \mathscr{J}, A_3(2^n)\right),$$

one has $\#I_{n',n} = \sum_{s=1}^{r_0} \mathbf{1}_{\{E_{10j_s+5}\}}$. For fixed $\mathscr{J}$, define the filtration $(\mathscr{F}_s)$ by

$$\mathscr{F}_s = \sigma\left\{E_{10j_1+5}, \ldots, E_{10j_{s-1}+5}\right\} \text{ for } s = 1, \ldots, r_0.$$

(Here, $\mathscr{F}_1$ is trivial.) Now the expectation in (4.11) can be written using the expectation $\hat{\mathbf{E}}$ relative to $\hat{\mathbf{P}}$ as

$$\hat{\mathbf{E}}\left[e^{-\mathbf{1}_{E_{10j_1+5}}} \cdots \hat{\mathbf{E}}\left[e^{-\mathbf{1}_{E_{10j_{s-1}+5}}} \hat{\mathbf{E}}\left[e^{-\mathbf{1}_{E_{10j_{r_0}+5}}} \,\middle|\, \mathscr{F}_{r_0}\right] \,\middle|\, \mathscr{F}_{s-1}\right] \cdots \,\middle|\, \mathscr{F}_1\right]. \tag{4.12}$$

For any $s = 1, \ldots, r_0$, one has $\hat{\mathbf{P}}$-a.s.,

$$\hat{\mathbf{E}}\left[e^{-\mathbf{1}_{E_{10j_s+5}}} \,\middle|\, \mathscr{F}_s\right] = 1 - \hat{\mathbf{P}}(E_{10j_s+5} \mid \mathscr{F}_s)(1-e^{-1}). \tag{4.13}$$

We bound this conditional probability uniformly over s and ω using the following decoupling estimate.

Lemma 4.4. *There exists a universal constant $c_1 > 0$ such that the following holds. For any $k, n \geq 0$ and $N \geq 1$ satisfying*

$$k \leq \left\lfloor \frac{n}{10N} \right\rfloor - 1,$$

and any events F and G depending on the status of edges in $B(2^{10kN})$ and $B(2^{10(k+1)N})^c$ respectively, one has

$$\mathbf{P}\left(E_{10k+5} \mid \hat{\mathfrak{C}}_k, A_3(2^n), F, G\right) \geq c_1 \mathbf{P}\left(E_{10k+5} \mid \hat{\mathfrak{C}}_k, A_3(2^n)\right). \tag{4.14}$$

Proof. We first prove a partial version of Lemma 4.4, where we remove the conditioning on F but not G: under the assumptions of Lemma 4.4, one has

$$\mathbf{P}\left(E_{10k+5} \mid \hat{\mathfrak{C}}_k, A_3(2^n), F, G\right) \geq c_2 \mathbf{P}\left(E_{10k+5} \mid \hat{\mathfrak{C}}_k, A_3(2^n), G\right). \tag{4.15}$$

The proof of (4.15) proceeds via decoupling using the block of circuits whose existence is guaranteed by $\hat{\mathfrak{C}}_k$. For $\ell = 1, 4, 6, 9$ and an outcome in $A_3(2^n)$, let $Circ_\ell(\mathscr{C})$ be the event that $\mathscr{C}$ is the innermost (vertex self-avoiding) closed dual circuit with exactly two defects in $A(2^{(10k+\ell)N}, 2^{(10k+\ell+1)N})$. If $A_3(2^n)$ does not occur, $Circ_\ell(\mathscr{C})$ is the event that $\mathscr{C}$ is a closed dual circuit with exactly two open defects in $A(2^{(10k+\ell)N}, 2^{(10k+\ell+1)N})$, such that no other such circuit in this annulus is contained in the union of $\mathscr{C}$ and its interior.

Conditioning on F can change the probabilities of the various $Circ_1(\mathscr{C})$ events. The role of the outer defected dual circuit (from $\mathfrak{C}_{10k+4}$) appearing before E_{10k+5} is to approximately remove this bias introduced by F. We make this decoupling explicit by breaking the intersection on the left-hand side of (4.15) into several pieces.

Any closed dual circuit $\mathscr{C}$ with exactly two defects has two disjoint closed arcs between these defects; order all defects and arcs arbitrarily and number the defects (resp. arcs) of $\mathscr{C}$ according to this ordering as $e_i(\mathscr{C})$ (resp. $\mathscr{A}_i(\mathscr{C})$) for $i = 1, 2$. For $\mathscr{C}$ a closed dual circuit with two defects in $A(2^{(10k+1)N}, 2^{(10k+2)N})$, let $X_-(\mathscr{C}, i)$ denote the event that

(1) $\mathfrak{D}_k \cap Circ_1(\mathscr{C})$ occurs;
(2) the edge $\{0, \mathbf{e}_1\}$ is connected to $e_1(\mathscr{C})$ and $e_2(\mathscr{C})$ in the interior of $\mathscr{C}$ via vertex-disjoint open paths;
(3) $\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)$ is connected to $\mathscr{A}_i(\mathscr{C})$ via a closed dual path.

We first make the following claim, which will be useful in decomposing the events appearing in (4.15):

$$\text{On } A_3(2^n) \cap \hat{\mathfrak{C}}_k, \text{ the event } X_-(\mathscr{C}, i) \text{ occurs for exactly one choice of } \mathscr{C} \text{ and } i. \tag{4.16}$$

We omit the proof of (4.16); the essential point is the presence of the open defected circuit in $A(2^{10kN}, 2^{(10k+1)N})$ having exactly one closed defect. This guarantees that exactly one $\mathscr{A}_i(\mathscr{C})$ can connect to $\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)$, since any closed path from the aforementioned defect will be confined by a pair of disjoint open paths leading to $e_1(\mathscr{C})$ and $e_2(\mathscr{C})$.

We will decompose $\hat{\mathfrak{C}}_k$ into inner, outer, and middle pieces; the above gives the "inner" piece. To build the outer piece, let $\hat{\mathfrak{C}}_k^+$ be the event that $\mathfrak{C}_{10k+\ell}$ occurs for $\ell = 6, 8, 9$. Similarly, to the above, let $\mathscr{D}$ be a dual circuit in $A(2^{(k+4)N}, 2^{(k+5)N})$ (it will eventually be taken closed with two defects) with two distinguished primal edges $\{e_i(\mathscr{D})\}_{i=1,2}$ crossing it and corresponding arcs $\mathscr{A}_i(\mathscr{D})$ between them. We define the event $X_+(\mathscr{D}, j)$ by the following conditions:

(1) $e_1(\mathscr{D})$ and $e_2(\mathscr{D})$ are connected to $\partial B(2^n)$ in the exterior of $\mathscr{D}$ via disjoint open paths;
(2) $\mathscr{A}_j(\mathscr{D})$ is connected in the exterior of $\mathscr{D}$ to $\partial B(2^n)$ via a closed dual path;
(3) $\hat{\mathfrak{C}}_k^+$ occurs.

We also need the probability of "transitions" between $\mathscr{C}$ and $\mathscr{D}$, and it is with these that we implement the decoupling from F. For $\mathscr{C}$ and $\mathscr{D}$ marked dual circuits in annuli as above, let $P(\mathscr{C}, \mathscr{D}, i, j)$ be the probability, conditional on the event that each $e_i(\mathscr{C})$ is open and all other edges of $\mathscr{C}$ are closed, that

(1) $Circ_4(\mathscr{D})$ occurs;
(2) there is a pair of disjoint open paths in the region between $\mathscr{C}$ and $\mathscr{D}$ connecting $e_1(\mathscr{C})$ to one of the marked edges $\{e_1(\mathscr{D}), e_2(\mathscr{D})\}$ and $e_2(\mathscr{C})$ to the other marked edge of $\mathscr{D}$;
(3) there is a closed dual path in the region between $\mathscr{C}$ and $\mathscr{D}$ connecting $\mathscr{A}_i(\mathscr{C})$ to $\mathscr{A}_j(\mathscr{D})$;
(4) $\mathfrak{C}_{10k+3}$ occurs.

Note that, conditioning on $X_-(\mathscr{C}, i)$ (and further conditioning on events depending on the status of edges in the interior of $\mathscr{C}$), the process outside $\mathscr{C}$ remains a free percolation. Conditioning also on $X_+(\mathscr{D}, j)$ and on any other events in the exterior of $\mathscr{D}$ leaves free percolation between $\mathscr{C}$ and $\mathscr{D}$. We last note that if $X_-(\mathscr{C}, i) \cap X_+(\mathscr{D}, j)$ occurs and if the defects of $\mathscr{C}$ and $\mathscr{D}$ are connected as in item 2 in the definition of $P(\cdot,\cdot,\cdot,\cdot)$, then $\mathscr{A}_i(\mathscr{C})$ is connected in the region between $\mathscr{C}$ and $\mathscr{D}$ to at most one of $\{\mathscr{A}_1(\mathscr{D}), \mathscr{A}_2(\mathscr{D})\}$. This follows by another trapping argument involving the open paths.

Using the observations of the above paragraph and (4.16), we see that for events E_{10k+5}, F, G as in the statement of the proposition:

$$\begin{aligned} &\mathbf{P}\left(E_{10k+5}, \hat{\mathfrak{C}}_k, A_3(2^n), F, G\right) \\ &= \sum_{\mathscr{C}, \mathscr{D}, i, j} \mathbf{P}\left(F, X_-(\mathscr{C}, i)\right) P(\mathscr{C}, \mathscr{D}, i, j) \mathbf{P}\left(E_{10k+5}, X_+(\mathscr{D}, j), G\right). \end{aligned} \tag{4.17}$$

Similarly, we can decompose

$$\begin{aligned} &\mathbf{P}\left(\hat{\mathfrak{C}}_k, A_3(2^n), G\right) \\ &= \sum_{\mathscr{C}', \mathscr{D}', i', j'} \mathbf{P}\left(X_-(\mathscr{C}', i')\right) P(\mathscr{C}', \mathscr{D}', i', j') \mathbf{P}\left(X_+(\mathscr{D}', j'), G\right), \end{aligned} \tag{4.18}$$

and analogous decompositions hold for other quantities similar to $\mathbf{P}(\hat{\mathfrak{C}}_k, A_3(2^n), G)$.

To accomplish the decoupling, we use the following inequality which is adapted from, and whose proof is essentially the same as, [12, Lemma 6.1] (see also [9, Lemma 23]). It gives a form of comparability for the various circuit transition factors. There exists a uniform constant $C_{10} < \infty$ such that the following holds uniformly in k,N, as well as in choices of circuits $\mathscr{C}, \mathscr{C}', \mathscr{D}, \mathscr{D}'$ and arc indices i, j, i', j':

$$\frac{P(\mathscr{C}, \mathscr{D}, i, j)P(\mathscr{C}', \mathscr{D}', i', j')}{P(\mathscr{C}, \mathscr{D}', i, j')P(\mathscr{C}', \mathscr{D}, i', j)} < C_{10}. \tag{4.19}$$

To apply (4.19), multiply (4.17) and (4.18):

$$\begin{aligned}
&\mathbf{P}\left(E_{10k+5}, \hat{\mathfrak{C}}_k, A_3(2^n), F, G\right)\mathbf{P}\left(\hat{\mathfrak{C}}_k, A_3(2^n), G\right) \\
&= \sum_{\substack{\mathscr{C},\mathscr{D},i,j\\ \mathscr{C}',\mathscr{D}',i',j'}} \Big[\mathbf{P}\left(X_-(\mathscr{C}', i')\right)P(\mathscr{C}', \mathscr{D}', i', j')\mathbf{P}\left(X_+(\mathscr{D}', j'), G\right) \\
&\qquad \times \mathbf{P}\left(F, X_-(\mathscr{C}, i)\right)P(\mathscr{C}, \mathscr{D}, i, j)\mathbf{P}\left(E_{10k+5}, X_+(\mathscr{D}, j), G\right)\Big] \qquad (4.20)\\
&\geq C_{10}^{-1} \sum_{\substack{\mathscr{C},\mathscr{D},i,j\\ \mathscr{C}',\mathscr{D}',i',j'}} \Big[\mathbf{P}\left(X_-(\mathscr{C}', i')\right)P(\mathscr{C}', \mathscr{D}, i', j)\mathbf{P}\left(E_{10k+5}, X_+(\mathscr{D}, j), G\right) \\
&\qquad \times \mathbf{P}\left(F, X_-(\mathscr{C}, i)\right)P(\mathscr{C}, \mathscr{D}', i, j')\mathbf{P}\left(X_+(\mathscr{D}', j'), G\right)\Big] \\
&= C_{10}^{-1}\mathbf{P}\left(E_{10k+5}, A_3(2^n), \hat{\mathfrak{C}}_k, G\right)\mathbf{P}\left(A_3(2^n), \hat{\mathfrak{C}}_k, F, G\right).
\end{aligned}$$

Dividing both sides of the above by $\mathbf{P}(\hat{\mathfrak{C}}_k, A_3(2^n), G)$ and $\mathbf{P}\left(A_3(2^n), \hat{\mathfrak{C}}_k, F, G\right)$ gives

$$\mathbf{P}\left(E_{10k+5} \mid A_3(2^n), \hat{\mathfrak{C}}_k, F, G\right) \geq C_{10}^{-1}\mathbf{P}\left(E_{10k+5} \mid A_3(2^n), \hat{\mathfrak{C}}_k, G\right).$$

This is the claim of (4.15) with $c_2 = C_{10}^{-1}$.

Equation (4.15) allows us to first remove the conditioning on F, and using it, we see that to prove Lemma 4.4, it suffices to show the existence of a uniform $c_3 > 0$ such that

$$\mathbf{P}\left(E_{10k+5} \mid \hat{\mathfrak{C}}_k, A_3(2^n), G\right) \geq c_3\mathbf{P}\left(E_{10k+5} \mid \hat{\mathfrak{C}}_k, A_3(2^n)\right). \tag{4.21}$$

To show (4.21), we argue nearly identically to the proof of (4.15). The main difference is just the placement of the circuits and connections in the decoupling. We now have to condition on the values of innermost defected circuits in $A(2^{(k+6)N}, 2^{(k+7)N})$ and $A(2^{(k+9)N}, 2^{(k+10)N})$.

Just as before, the effect of conditioning on G is just to bias the distribution of circuits in $A(2^{(k+9)N}, 2^{(k+10)N})$, and (4.19) shows that the inner circuit approximately removes this bias. Expanding the product

$$\mathbf{P}\left(E_{10k+5}, \hat{\mathfrak{C}}_k, A_3(2^n), G\right)\mathbf{P}\left(\hat{\mathfrak{C}}_k, A_3(2^n)\right)$$

similarly to (4.20) and regrouping terms after applying (4.19), Lemma 4.4 follows. $\square$

Returning to the proof of Theorem 4.1, we apply Lemma 4.4 to prove the following statement. There exists a universal $c_{11} > 0$ such that for any $N \geq 1$, any $n', n \geq 0$ satisfying $n - n' \geq 40N$, any $j = \left\lceil \frac{n'}{10N} \right\rceil, \ldots, \left\lfloor \frac{n}{10N} \right\rfloor - 1$, any F depending on the state of edges in $B(2^{10jN})$, and any $\mathscr{J}$ containing j,

$$\mathbf{P}(\mathfrak{B}_j \mid F, J_{n',n} = \mathscr{J}, A_3(2^n)) \geq c_{11}\mathbf{P}(\mathfrak{B}_j \mid A_3(2^n)). \tag{4.22}$$

To show (4.22), write $\{J_{n',n} = \mathscr{J}\}$ as an intersection $\hat{F} \cap \hat{\mathfrak{C}}_j \cap G$, where $\hat{F}$ depends on the state of edges in $B(2^{10jN})$ and G depends on the state of edges in $B(2^{10(j+1)N})^c$. Applying Lemma 4.4 using $F \cap \hat{F}$ in place of F, we obtain

$$\begin{aligned}\mathbf{P}(\mathfrak{B}_j \mid F, J_{n',n} = \mathscr{J}, A_3(2^n)) &= \mathbf{P}(E_{10j+5} \mid F, \hat{F}, \hat{\mathfrak{C}}_j, G, A_3(2^n)) \\ &\geq c_1\mathbf{P}(E_{10j+5} \mid \hat{\mathfrak{C}}_j, A_3(2^n)) \\ &\geq c_1\mathbf{P}(\mathfrak{B}_j \mid A_3(2^n)),\end{aligned}$$

which is (4.22) with $c_{11} = c_1$.

We now apply (4.22) to the probability in (4.13). For a fixed $\mathscr{J} = \{j_1, \ldots, j_{r_0}\}$ with $r_0 \geq C_1\frac{n-n'}{N}$ and $s = 1, \ldots, r_0$, let $x_1, \ldots, x_{s-1} \in \{0,1\}$ and put

$$F = \{\mathbf{1}_{E_{10j_1+5}} = x_1, \ldots, \mathbf{1}_{E_{10j_{s-1}+5}} = x_{s-1}\}.$$

Then for $\omega \in F \cap \{J_{n',n} = \mathscr{J}\} \cap A_3(2^n)$, the event $\hat{\mathfrak{C}}_{j_s}$ occurs, and so

$$\begin{aligned}\hat{\mathbf{P}}(E_{10j_s+5} \mid \mathscr{F}_s)(\omega) &= \mathbf{P}(E_{10j_s+5} \mid F, J_{n',n} = \mathscr{J}, A_3(2^n)) \\ &= \mathbf{P}(\mathfrak{B}_{j_s} \mid F, J_{n',n} = \mathscr{J}, A_3(2^n)) \\ &\geq c_{11}\mathbf{P}(\mathfrak{B}_{j_s} \mid A_3(2^n)).\end{aligned}$$

Using assumption (4.1), we obtain $\hat{\mathbf{P}}$-a.s. for $\omega \in F \cap \{J_{n',n} = \mathscr{J}\} \cap A_3(2^n)$

$$\hat{\mathbf{P}}(E_{10j_s+5} \mid \mathscr{F}_s) \geq c_{11}\tilde{c}_0.$$

Because such events generate the sigma-algebra $\mathscr{F}_s$, the same inequality is valid $\hat{\mathbf{P}}$-a.s., and so replacing this in (4.13), we have

$$\hat{\mathbf{E}}\left[e^{-\mathbf{1}_{E_{10j_s+5}}} \;\middle|\; \mathscr{F}_s\right] \leq 1 - c_{11}\tilde{c}_0(1 - e^{-1}).$$

Starting with this bound for $s = r_0$, we place it in (4.12), and then repeat for $s = r_0 - 1$, and so on, until $s = 1$ to obtain the overall bound for $r_0 = \#\mathscr{J} \geq c_7\frac{n-n'}{N}$

$$\begin{aligned}\mathbf{E}\left[e^{-\#I_{n',n}} \;\middle|\; J_{n',n} = \mathscr{J}, A_3(2^n)\right] &\leq \left(1 - c_{11}\tilde{c}_0(1 - e^{-1})\right)^{r_0} \\ &\leq \left(1 - c_{11}\tilde{c}_0(1 - e^{-1})\right)^{c_7\frac{n-n'}{N}}.\end{aligned}$$

We sum this in (4.11) for

$$\begin{aligned}&\mathbf{E}\left[e^{-\#I_{n',n}}\mathbf{1}_{\left\{\#J_{n',n}\geq c_7\frac{n-n'}{N}\right\}}\,\middle|\,A_3(2^n)\right]\\&\leq\left(1-c_{11}\tilde{c}_0(1-e^{-1})\right)^{c_7\frac{n-n'}{N}}\mathbf{P}\left(J_{n',n}\geq c_7\frac{n-n'}{N}\,\middle|\,A_3(2^n)\right),\end{aligned}$$

and so, returning to (4.10), we conclude that

$$\mathbf{P}\left(\#I_{n',n}\leq c_9\frac{n-n'}{N}\,\middle|\,A_3(2^n)\right)\leq e^{-c_7(n-n')}+e^{c_9\frac{n-n'}{N}}(1-c_{11}\tilde{c}_0(1-e^{-1}))^{c_7\frac{n-n'}{N}}.$$

By the inequality $1-x\leq e^{-x}$, we get the upper bound

$$e^{-c_7(n-n')}+\exp\left(\frac{n-n'}{N}\left[c_9-c_{11}c_7\tilde{c}_0(1-e^{-1})\right]\right).$$

We therefore choose $c_9=c_{12}\tilde{c}_0$, where $c_{12}=\min\left\{1,c_{11}c_7(1-e^{-1})/2\right\}$ to obtain the bound

$$\mathbf{P}\left(\#I_{n',n}\leq c_{12}\tilde{c}_0\frac{n-n'}{N}\,\middle|\,A_3(2^n)\right)\leq e^{-c_7(n-n')}+\exp\left(-c_{12}\tilde{c}_0\frac{n-n'}{N}\right).$$

This implies for some universal $c_{13}>0$,

$$\mathbf{P}\left(\#I_{n',n}\leq c_{13}\tilde{c}_0\frac{n-n'}{N}\,\middle|\,A_3(2^n)\right)\leq\exp\left(-c_{13}\tilde{c}_0\frac{n-n'}{N}\right).$$

□

5 Definition of E_k

Suppose the event H_n that there exists a horizontal open crossing of $[-n,n]^2$ occurs. Any vertex self-avoiding open path connecting the vertical sides of $[-n,n]^2$ corresponds to a Jordan arc (a continuous injection of $[0,1]$ into $\mathbb{R}^2$) separating the top side $[-n,n]\times\{n\}$ from the bottom side $[-n,n]\times\{-n\}$. l_n is the vertex self-avoiding horizontal open crossing path such that the closed region $\mathscr{B}(l_n)$ of $[-n,n]^2$ below and including l_n is minimal.

A fact that we will use frequently, is that an edge e is in the lowest crossing l_n if and only if (a) it is open, (b) there are two vertex-disjoint open paths connecting e to the left and right sides of $B(n)$, and (c) there is a closed dual path connecting e^* to the bottom of $B(n)$. Using this, one can show that there are constants c,C such that if $e\in B(n)$ is an edge with $\mathrm{dist}(e,\partial B(n))=d$, then

$$c\pi_3(d)\pi_2(d,n)\leq\mathbf{P}(e\in l_n\mid H_n)\leq C\pi_3(d),\tag{5.1}$$

where $\pi_k(d,n)$ is the "k-arm" probability corresponding to crossings of an annulus $B(n)\setminus B(d)$. This estimate was already used extensively in our previous paper [9]. See for example Section 5.3 and Lemma 17 there.

Definition 5.1 (κ-shortcuts)**.** Let $\kappa \in (0,1)$. For an edge $e \in l_n$, the set $\mathscr{S}(e,\kappa)$ of κ-shortcuts around e is defined as the set of vertex self-avoiding open paths r with vertices $w_0,\dots,w_M$ such that

(1) for $i = 1,\dots M-1$, $w_i \in B(n) \setminus \mathscr{B}(l_n)$,
(2) the edges $\{w_0, w_0 + \mathbf{e}_1\}$, $\{w_0 - \mathbf{e}_1, w_0\}$, $\{w_M, w_M + \mathbf{e}_1\}$, and $\{w_M - \mathbf{e}_1, w_M\}$ are in l_n and $w_1 = w_0 + \mathbf{e}_2$, $w_{M-1} = w_M + \mathbf{e}_2$.
(3) writing τ for the subpath of l_n from w_0 to w_M, τ contains e, and the path $r \cup \tau$ is an open circuit in $[-n,n]^2$,
(4) The points $w_0 + (1/2)(-\mathbf{e}_1 + \mathbf{e}_2)$ and $w_M + (1/2)(\mathbf{e}_1 + \mathbf{e}_2)$ are connected by a dual closed vertex self-avoiding path $\mathfrak{c}$, whose first and last edges are vertical (translates of $\{0, \mathbf{e}_2\}$), and which lies in $[-n,n]^2 \setminus \mathscr{B}(l_n)$.
(5) $M = \#r \le \kappa \# \tau$.

For $0 < \varepsilon < 1$, we define the annulus

$$A(2^k, 2^K) := [-2^K, 2^K]^2 \setminus [-2^k, 2^k]^2,$$

where

$$K = k + \lfloor \log \frac{1}{\varepsilon} \rfloor. \tag{5.2}$$

For $\delta > 0$, we define an event $E_k = E_k(\varepsilon, \delta)$ depending only on the edges in the annulus $A(2^k, 2^K)$ which after translation by and edge e (see definition of $E_k(e)$ in the next paragraph) implies the existence of a $\delta\varepsilon$-shortcut around e when $e \in l_n$. The next subsection contains a precise description of E_k. It involves a large number of connections, and appears in equation (5.21), following Proposition 5.3. The event is illustrated in Figures 5.1 and 5.2. We encourage the reader to study these figures. The important features include the following:

- An open arc (shortcut), whose length is of order at most $\delta 2^{2k} \pi_3(2^k)$, connecting two arms emanating from the three-arm edge e. This arc lies inside a box of side length $3 \cdot 2^k$ centered at e, and is depicted as the top (solid) arc in Figure 5.2.
- A path with length of order at least $2^{2K} \pi_3(2^K)$, whose edges necessarily lie on the lowest crossing if e does. This path is depicted as the solid, "pendulous" arc intersecting the green box in Figure 5.1. This path will be "long" and cut short by the path in the preceding item.

We denote by $E_k(e) = E_k(e, \varepsilon, \delta)$ the event $\tau_{-e_x} E_k(\varepsilon, \delta)$, that is, the event that E_k occurs in the configuration $(\omega_{\tilde{e}+e_x})_{\tilde{e} \in \mathscr{E}^2}$ translated by the coordinates of the lower-left endpoint e_x of the edge e.

Two properties of $E_k(e)$ which will be crucial for the rest of the proof are the following:

(1) If $E_k(e)$ occurs for some k and e lies on l_n, then $\mathscr{S}(e, \delta\varepsilon) \neq \emptyset$. (See Proposition 5.4.)

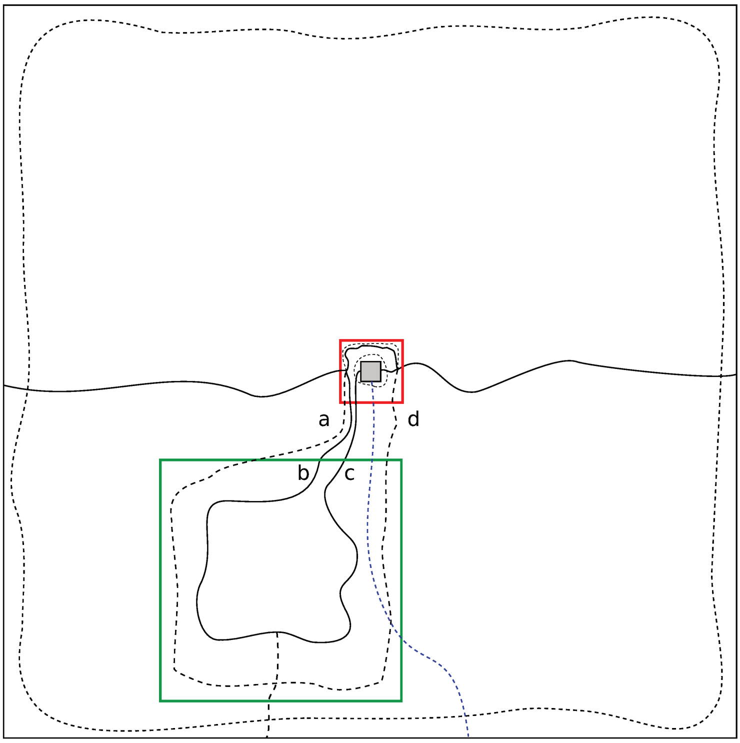

FIGURE 5.1. The event E_k. The outermost square represents the boundary of the box $[-2^K, 2^K]^2$. The box with red boundary is $[-3 \cdot 2^k, 3 \cdot 2^k]^2$ This box contains a shortcut which bypasses the lowest crossing. Details of the construction inside the red box appear in Figures 5.2 and 5.3. The blue path is not part of the definition of E_k. Rather, it is present if the grey box contains an edge of the lowest path of a larger box. In this case, all three-arm points in the green box with a closed dual arm to the bottom of that box also lie on the lowest path. The green box has size of order const. $\times 2^K$. The asymmetric placement of the green box serves to make it clear that both the shortcut and detoured paths in E_k (the blue paths in Figures 5.3 and 5.4, respectively) are contained in the box of side-length $2^K + 3 \cdot 2^k$ whose lower left corner coincides with that of $[-2^K, 2^K]^2$. This will be essential for the iteration scheme in Section 7.

(2) We have the following lower bound for the probability of E_k, assuming a bound of the form (5.18), expressing a length gain of δ: there is a constant $C > 0$ such that

$$\mathbf{P}(E_k \mid A_3(2^R)) \geq C\varepsilon^4.$$

for any $R \geq K$. (See Proposition 5.5 and (5.29).)

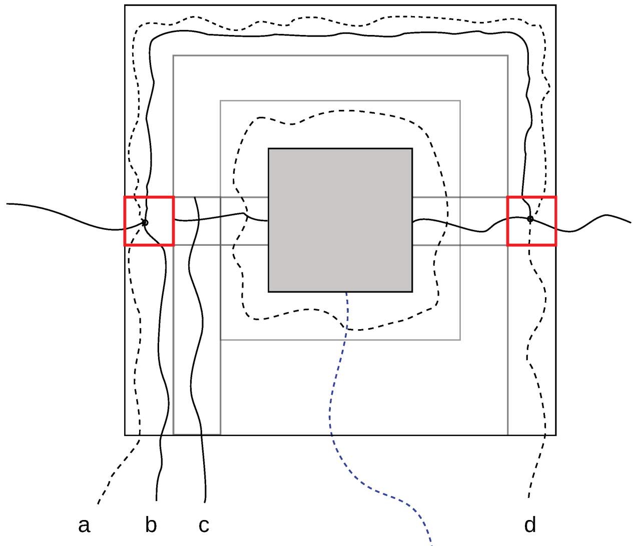

FIGURE 5.2. The inner box from Figure 5.1: the grey area represents the box $[-2^k,2^k]^2$. The outer square is the boundary of the box $[-3\cdot 2^k,3\cdot 2^k]^2$. The paths labeled a and d in the lower part of the figure are part of a dual closed path enclosing $[-2^k,2^k]^2$ (see Figure 5.1). The paths b and c are part of an arc containing on the order of $2^{2K}\pi_3(2^K)$ points, which all lie on the lowest crossing l_n if the box $[-2^k,2^k]^2$ contains an edge on the lowest crossing. The two red boxes (B_1 on the left, B_2 on the right) each contain a five-arm point (these points are far away from the boundaries of the red boxes, but are not necessarily directly in the center — their locations are random). These five-arm points $\star_1$ and $\star_2$ are connected by an open arc, the shortcut that bypasses the path between b and c. The latter is also depicted in blue in Figure 5.4.

5.1 Connections in E_k

In this section, we enumerate all the conditions for the occurrence of the event E_k. We first define an auxiliary event E_k', which will contain most of the conditions defining E_k.

Inside the box $[-3\cdot 2^k,3\cdot 2^k]^2$**.**

All connections described below remain in the annulus $[-2^K,2^K]^2\setminus[-2^k,2^k]^2$, so that the events are different for different values of k. First, we have a number of conditions depending on the status of edges inside $[-3\cdot 2^k,3\cdot 2^k]^2$ (see Figure 5.3).

(1) There is a horizontal open crossing of $[2^k,3\cdot 2^k]\times[-\frac{2^k}{3},\frac{2^k}{3}]$, and a horizontal open crossing of $[-\frac{7}{3}\cdot 2^k,-2^k]\times[-\frac{2^k}{3},\frac{2^k}{3}]$.
(2) There is a vertical open crossing of $[-\frac{7}{3}\cdot 2^k,-\frac{5}{3}\cdot 2^k]\times[-3\cdot 2^k,\frac{1}{3}\cdot 2^k]$.
(3) There is a five-arm point (represented by a purple dot in Figure 5.3) in the box dwith the following connections, in clockwise order:

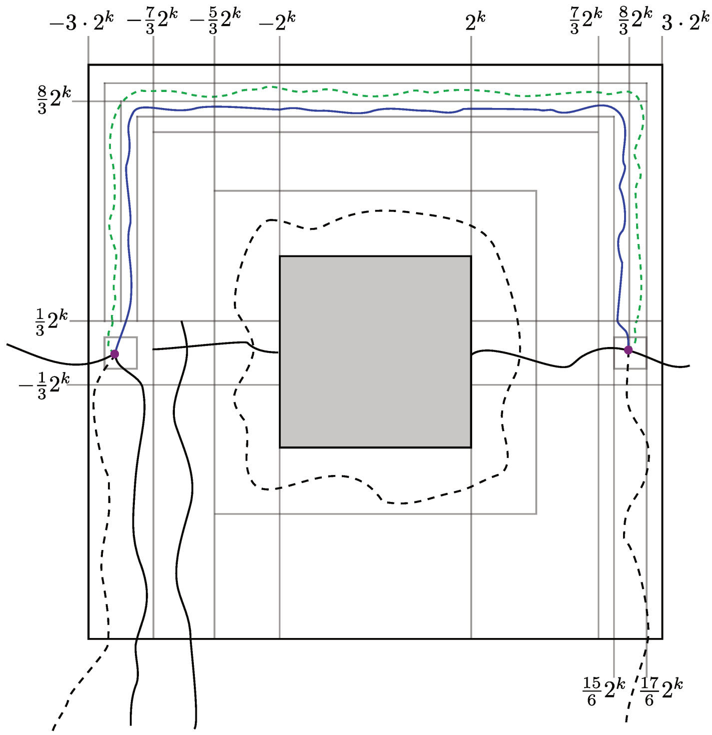

FIGURE 5.3. An illustration of the connections inside $[-3\cdot 2^k, 3\cdot 2^k]$ implied by the occurrence of E_k. The shortcut (item 8. in the definition of E'_k) appears in blue. The shielding dual closed arc (item 7. in the definition of E'_k) appears in green. The two five-arm points, $\star_1$ in B_1 and $\star_2$ in B_2 (items 3. and 4.), are depicted by purple dots.

(a) a closed dual arm connected to $[-\frac{17}{6}\cdot 2^k, -\frac{8}{3}\cdot 2^k]\times\{\frac{1}{3}\cdot 2^k\}$,
(b) an open arm connected to $[-\frac{8}{3}\cdot 2^k, -\frac{15}{6}\cdot 2^k]\times\{\frac{1}{3}\cdot 2^k\}$,
(c) an open arm connected to $[-\frac{8}{3}\cdot 2^k, -\frac{7}{3}\cdot 2^k]\times\{-\frac{1}{3}\cdot 2^k\}$,
(d) a closed dual arm connected to $[-3\cdot 2^k, -\frac{8}{3}\cdot 2^k]\times\{-\frac{1}{3}\cdot 2^k\}$,
(e) and an open arm connected to the "left side" of the box, $\{-3\cdot 2^k\}\times[-\frac{1}{3}\cdot 2^k, \frac{1}{3}\cdot 2^k]$.

We denote the unique such point in B_1 by $\star_1$. Note that $\star_1$ is at distance at least $(1/6)\cdot 2^k$ from the boundary of the box

$$B'_1 = [-3\cdot 2^k, -\frac{7}{3}2^k]\times[-\frac{1}{3}\cdot 2^k, \frac{1}{3}\cdot 2^k]\supset B_1.$$

(4) There is a five-arm point (represented by a purple dot in Figure 5.3) in the box

$$B_2 := [\frac{15}{6}\cdot 2^k, \frac{17}{6}\cdot 2^k]\times[-\frac{1}{6}\cdot 2^k, \frac{1}{6}\cdot 2^k],$$

with the following connections, in clockwise order:

(a) an open arm connected to $[\frac{15}{6}\cdot 2^k, \frac{8}{3}\cdot 2^k] \times \{\frac{1}{3}\cdot 2^k\}$,
(b) a closed dual arm connected to $[\frac{8}{3}\cdot 2^k, \frac{17}{6}\cdot 2^k] \times \{\frac{1}{3}\cdot 2^k\}$,
(c) an open arm connected to the "right side" of the box $\{3\cdot 2^k\} \times [-\frac{1}{3}\cdot 2^k, \frac{1}{3}\cdot 2^k]$,
(d) a closed dual arm connected to $[\frac{7}{3}\cdot 2^k, 3\cdot 2^k] \times \{-\frac{1}{3}\cdot 2^k\}$,
(e) and an open arm connected to $\{\frac{7}{3}\cdot 2^k\} \times [-\frac{1}{3}\cdot 2^k, \frac{1}{3}\cdot 2^k]$.

We denote the unique such point in B_2 by $\star_2$. The vertex $\star_2$ is at distance at least $(1/6)\cdot 2^k$ from the boundary of the box

$$B_2' := [\frac{7}{3}2^k, 3\cdot 2^k] \times [-\frac{1}{3}\cdot 2^k, \frac{1}{3}\cdot 2^k] \supset B_2.$$

(5) There is a closed dual circuit with two open defects around the origin inside the annulus $[-\frac{5}{3}2^k, \frac{5}{3}2^k]^2 \setminus [-2^k, 2^k]$. One of the defects is in the box $[-\frac{5}{3}2^k, -2^k] \times [-\frac{1}{3}2^k, \frac{1}{3}2^k]$, and the other is in $[2^k, \frac{5}{3}2^k] \times [-\frac{1}{3}2^k, \frac{1}{3}2^k]$.
(6) There is an open vertical crossing of $[-\frac{8}{3}\cdot 2^k, -\frac{7}{3}\cdot 2^k] \times [-3\cdot 2^k, -\frac{1}{3}\cdot 2^k]$, connected to the open arm that emanates from the five-arm point $\star_1$ in B_1 and lands in $[-\frac{8}{3}\cdot 2^k, -\frac{7}{3}\cdot 2^k] \times \{-\frac{1}{3}\cdot 2^k\}$. There is a dual closed vertical crossing of $[-3\cdot 2^k, -\frac{8}{3}\cdot 2^k] \times [-3\cdot 2^k, -\frac{1}{3}\cdot 2^k]$, connected to the closed dual arm that lands in $[-3\cdot 2^k, -\frac{8}{3}\cdot 2^k] \times \{-\frac{1}{3}\cdot 2^k\}$.
(7) There is a closed dual vertical crossing of $[\frac{7}{3}\cdot 2^k, 3\cdot 2^k] \times [-3\cdot 2^k, -\frac{1}{3}\cdot 2^k]$, connected to the dual arm that lands in $[\frac{7}{3}\cdot 2^k, 3\cdot 2^k] \times \{-\frac{1}{3}\cdot 2^k\}$.
(8) There is a closed dual arc (the shield, in green in Figure 5.3) in the half-annulus

$$\tilde{V}(k) := \left[[-\frac{17}{6}\cdot 2^k, \frac{17}{6}\cdot 2^k] \times [\frac{-1}{6}\cdot 2^k, \frac{17}{6}\cdot 2^k]\right] \setminus (-\frac{8}{3}\cdot 2^k, \frac{8}{3}\cdot 2^k)^2 \tag{5.3}$$

connecting the closed dual paths from the two five-arm points $\star_1$ and $\star_2$ in items 3 and 4.
(9) There is an open arc (the shortcut, in blue in Figure 5.3) in the region

$$\tilde{U}(k) := \left[[-\frac{8}{3}\cdot 2^k, \frac{8}{3}\cdot 2^k] \times [-\frac{1}{6}\cdot 2^k, \frac{8}{3}\cdot 2^k]\right] \setminus (-\frac{15}{6}\cdot 2^k, \frac{15}{6}\cdot 2^k)^2, \tag{5.4}$$

connecting the open paths from the two five-arm points in items 3 and 4 which land on the line $\{(x, \frac{1}{3}\cdot 2^k) : x \in \mathbb{Z}\}$.

5.2 The box $[-2^K, 2^K]^2$ and the large detoured path

The following connections occur in the box $[-2^K, 2^K]^2$. Refer to Figure 5.4 for an illustration and the relevant scales.

(10) There is a closed dual arc τ around D_2 in $D_1 \setminus D_2$, where

$$D_1 := [-\frac{3}{4}2^K + 3\cdot 2^k, 3\cdot 2^k] \times [-\frac{7}{8}2^K, -\frac{1}{8}2^K].$$

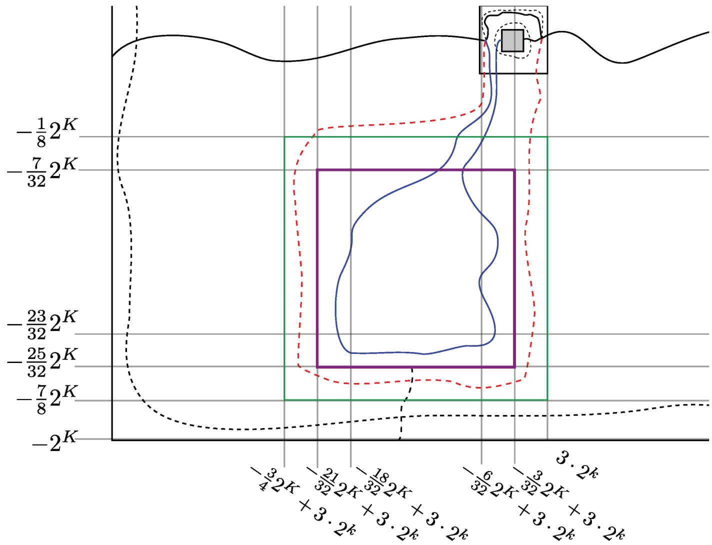

FIGURE 5.4. An illustration of the connections on scale 2^K implied by E_k. Note that this schematic is not to scale: in particular, the scale 2^k is much smaller than 2^K. For example, the right edge of the purple box is in reality far to the left of $[-3\cdot 2^k, 3\cdot 2^k]^2$.

is depicted in green in Figure 5.4, and

$$D_2 := [-\frac{21}{32}2^K + 3\cdot 2^k, -\frac{3}{32}2^K + 3\cdot 2^k] \times [-\frac{25}{32}2^K, -\frac{7}{32}2^K]$$

is in purple in Figure 5.4. The path τ joins $[-\frac{3}{4}2^K + 3\cdot 2^k, -\frac{21}{32}2^K + 3\cdot 2^k] \times \{-\frac{2^K}{8}\}$ to $[-\frac{3}{32}2^K + 3\cdot 2^k, 3\cdot 2^k] \times \{-\frac{2^K}{8}\}$. τ is the part of the path represented in red in Figure 5.4 that lies inside D_1.

(11) There are two disjoint closed paths inside $[-\frac{3}{4}2^K + 3\cdot 2^k, 3\cdot 2^k] \times ([-\frac{7}{8}2^K, 0])$: one joining the endpoint of the vertical closed crossing on $[-3\cdot 2^k, -\frac{8}{3}\cdot 2^k] \times \{-3\cdot 2^k\}$ to the endpoint of the closed dual arc in the previous item on $[-\frac{3}{4}2^K + 3\cdot 2^k, -\frac{21}{32}2^K + 3\cdot 2^k] \times \{-\frac{2^K}{8}\}$, the second, joining the endpoint of the vertical crossing on $[\frac{7}{3}\cdot 2^k, 3\cdot 2^k] \times \{-3\cdot 2^k\}$, to the endpoint of the closed dual arc in the previous item on $[-\frac{3}{32}2^K + 3\cdot 2^k, 3\cdot 2^k] \times \{-\frac{2^K}{8}\}$. The union of these two paths is represented in Figure 5.4 as the part of the red path outside of the box D_1.

(12) There is a horizontal open crossing of the rectangular box

$$R := [-\frac{18}{32}\cdot 2^K + 3\cdot 2^k, -\frac{6}{32}\cdot 2^K + 3\cdot 2^k] \times [-\frac{25}{32}\cdot 2^K, -\frac{23}{32}\cdot 2^K]. \tag{5.5}$$

This is the part of the path appearing in blue in Figure 5.4 which lies in R.

(13) There are two disjoint open paths contained in $[-\frac{21}{32}\cdot 2^K+3\cdot 2^k,-\frac{3}{32}\cdot 2^K+3\cdot 2^k]\times[-\frac{25}{32}\cdot 2^K,0]\setminus R$,
   (a) one joining the endpoint of the open vertical crossing of $[-\frac{8}{3}\cdot 2^k,-\frac{7}{3}\cdot 2^k]\times[-3\cdot 2^k,-\frac{1}{3}\cdot 2^k]$ to the left endpoint of the crossing in item 12. in the interval $\{-\frac{18}{32}\cdot 2^K+3\cdot 2^k\}\times\left[-\frac{49}{64}\cdot 2^K,-\frac{47}{64}\cdot 2^K\right]$ on the left side of R,
   (b) one joining the endpoint of the open vertical crossing of $[-\frac{7}{3}\cdot 2^k,-\frac{5}{3}\cdot 2^k]\times[-3\cdot 2^k,-\frac{1}{3}\cdot 2^k]$ to the right endpoint of the crossing in item 12. in the interval $\{-\frac{6}{32}\cdot 2^K+3\cdot 2^k\}\times\left[-\frac{49}{64}\cdot 2^K,-\frac{47}{64}\cdot 2^K\right]$ on the right side of R.

   The union of these two paths is the part of the path depicted in blue in Figure 5.4 lying outside of R.

(14) There is dual closed vertical crossing of $[-\frac{18}{32}\cdot 2^K+3\cdot 2^k,-\frac{6}{32}\cdot 2^K+3\cdot 2^k]\times[-2^K,-\frac{25}{32}\cdot 2^K]$.

Finally, we finish the description of the event by adding two more macroscopic conditions:

(15) There is a dual closed circuit with two open defects around the origin in $[-2^K,2^K]^2\setminus[-\frac{7}{8}\cdot 2^K,\frac{7}{8}\cdot 2^K]^2$. One of the defects is contained in $[-2^K,-\frac{7}{8}\cdot 2^K]\times[-\frac{2^K}{8},\frac{2^K}{8}]$, and the other in $[\frac{7}{8}\cdot 2^K,2^K]\times[-\frac{2^K}{8},\frac{2^K}{8}]$.

(16) There are two vertex-disjoint open arms: one from the left side $\{-3\cdot 2^k\}\times[-3\cdot 2^k,3\cdot 2^k]$ of the box $[-3\cdot 2^k,3\cdot 2^k]^2$ (touching the open arm from the five-arm point $\star_1$ that lands there) to the left side of $[-2^K,2^K]^2$, the other from the right side $\{3\cdot 2^k\}\times[-3\cdot 2^k,3\cdot 2^k]$ (touching the corresponding open arm from the five-arm point $\star_2$ there) to the right side of $[-2^K,2^K]^2$.

We denote by $E_k'=E_k'(\varepsilon)$ the intersection of the events listed in items 1-16 above. We also let $E_k'(e,\varepsilon)=\tau_{-e_x}E_k'(\varepsilon)$ be the event translated by e.

By considering three-arm points in the rectangle R (defined in (5.5)), we have the following proposition.

Proposition 5.2. *On E_k', let N_K be the number of open edges in R connected to the open paths from item 13. by two vertex-disjoint open paths inside R which moreover are connected inside R by a dual closed path to the dual path in item 14. There is a constant $c_0>0$ such that for $\varepsilon\in(0,1/4)$ and any $k\geq 1$, one has*

$$\mathbf{P}(\{N_K\geq c_0 2^{2K}\pi_3(2^K)\}\cap E_k')\geq c_0\mathbf{P}(E_k'). \tag{5.6}$$

Proof. This proof is similar to [9, Prop. 5.4]. We use the second moment method, in the form of the Paley-Zygmund inequality:

$$\mathbf{P}(X\geq\lambda\mathbf{E}[X])\geq(1-\lambda)^2\frac{\mathbf{E}[X]^2}{\mathbf{E}[X^2]}. \tag{5.7}$$

We apply this inequality to the conditional probability $\mathbf{P}(\cdot \mid E_k')$. We fix some $\lambda > 0$ (say $1/2$), and proceed to estimate the ratio

$$\frac{\mathbf{E}[N_K \mid E_k']^2}{\mathbf{E}[N_K^2 \mid E_k']}.$$

We show

$$\mathbf{E}[N_K \mid E_k'] \geq c2^{2k}\pi_3(2^k), \tag{5.8}$$

$$\mathbf{E}[N_K \mid E_k'] \leq C\big(2^{2k}\pi_3(2^k)\big)^2, \tag{5.9}$$

for some constants independent of k.

To show (5.8), we proceed by obtaining a lower bound for the number $\tilde{N}_K$ of points satisfying the conditions of the statement, which moreover lie in the rectangle $R' \subset R$, where

$$R' = \left[-\frac{15}{32}\cdot 2^K + 3\cdot 2^k, -\frac{9}{32}\cdot 2^K + 3\cdot 2^k\right] \times \left[-\frac{49}{64}\cdot 2^K, -\frac{47}{64}\cdot 2^K\right].$$

Clearly $N_K \geq \tilde{N}_K$. This rectangle has the same center, but half the side lengths of R. Let E_k'' be the event that all the connections in the description of E_k' occur *except* the horizontal crossing of R in point 13. Clearly $E_k' \subset E_k''$, and by RSW and gluing, we have

$$\mathbf{P}(E_k') \geq c\mathbf{P}(E_k''). \tag{5.10}$$

For an edge $e \in R'$, let $r = r(e) = \mathrm{dist}(e, \partial R)$. Note that $r \leq 2^K/16$. Define $I(e)$ to be event that

(1) one of the endpoints of e is connected inside $B(e,r)$ to the left side of $B(e,r)$, the other end point is connected inside $B(e,r)$ to the right side of $B(e,r)$, and the dual edge e^* has a dual closed connection in $B(e,r)$ to the bottom of $B(e,r)$;
(2) the open path from e to the left side of $B(e,r)$, denoted o_1, is connected inside R to the left side of R by a path of open edges;
(3) the open path from e to the right side of $B(e,r)$, denoted o_2, is connected inside R to the right side R by a path of open edges;
(4) the closed dual path from e^* to the bottom side of $B(e,r)$, c_1 is connected inside R^* to the bottom of R^*.

By the generalized FKG inequality (3.6) and a standard gluing construction

$$\begin{aligned}\mathbf{P}(I(e)) &\geq c\mathbf{P}(A_3(e,r))\\ &= c\pi_3(r)\\ &\geq c\pi_3(2^K).\end{aligned} \tag{5.11}$$

Let $J(e)$ be the event that the edge e satisfies the condition in the statement of the Proposition. Using gluing constructions to connect the open path o_1 to the open path in 13. (a) in the definition of E_k', the open path o_2 to the open path in 13. (b), and c_1 to the closed dual crossing in item 14. of the definition of E_k', we find:

$$\begin{aligned}
\mathbf{P}(J(e)\cap E_k') &\geq c''\mathbf{P}(I(e)\cap E_k'') && \text{(by generalized FKG)}\\
&= c''\mathbf{P}(I(e))\mathbf{P}(E_k'') && \text{(by independence)}\\
&\geq c''\mathbf{P}(I(e))\mathbf{P}(E_k'), && (5.10).
\end{aligned}$$

The first inequality follows by applications of FKG, first to construct connections between the open paths to sides of R guaranteed by E_k'' and the open arms emanating from e in the event $I(e)$. From this lower bound and (5.11), we have

$$\mathbf{P}(J(e)\cap E_k') \geq c\pi_3(2^K)P(E_k').$$

From this, we obtain

$$\mathbf{E}[\tilde N_K, E_k'] \geq c\mathbf{P}(E_k')\pi_3(2^K)\sum_{e\in R'} 1 \geq c\mathbf{P}(E_k')2^{2K}\pi_3(2^K).$$

It remains to estimate the second moment of

$$\tilde N_K \leq \sum_{e\in R'}\mathbf{1}[I(e)] \leq \sum_{e\in R'}\mathbf{1}[A_3(e,2^K/16)].$$

To do this, we use a method due to B. Nguyen [35, Section 2], writing:

$$\begin{aligned}
\mathbf{E}[\tilde N_K^2, E_k''] &\leq \mathbf{P}(E_k'')\sum_{e_1,e_2\in R'}\mathbf{P}(A_3(e_1,2^K/16),A_3(e_2,2^K/16))\\
&= \mathbf{P}(E_k'')\sum_{e_1\in R'}\sum_{d=1}^{\lceil\frac{3}{16}\cdot 2^K\rceil}\sum_{e_2:|e_1-e_2|_\infty=d}\mathbf{P}(A_3(e_1,2^K/16),A_3(e_2,2^K/16)). \qquad (5.12)
\end{aligned}$$

Let $m=\lfloor 2^K/16\rfloor$. Then, by independence of variables associated to disjoint regions, we have

$$\begin{aligned}
\mathbf{P}(A_3(e_1,m),A_3(e_2,m)) &\leq \mathbf{P}(A_3(e_1,d/2),A_3(e_1,3d/2,m),A_3(e_2,d/2))\\
&= \pi_3(d/2)\pi_3(3d/2,m)\pi_3(d/2).
\end{aligned}$$

The inner sum (5.12) is estimated, uniformly in e_1, by

(5.13)

$$\sum_{d=1}^{\lfloor 2m/3\rfloor} 8d\pi_3(d/2)\pi_3(3d/2,m)\pi_3(d/2)+\sum_{d=\lfloor 2m/3\rfloor+1}^{\lceil\frac{3}{16}\cdot 2^K\rceil} 8d\pi_3((d\wedge m)/2)\pi_3((d\wedge m)/2),$$

where $a\wedge b:=\min\{a,b\}$.

Using basic RSW theory [36, Proposition 11.1], we rescale the arguments of the probabilities at the cost of constant factors:

$$\begin{aligned}
\pi_3(d/2) &\leq C\pi_3(d),\\
\pi_3(d) &\leq C\pi_3(m), \quad d\geq 2m/3.
\end{aligned}$$

In addition, by quasi-multiplicativity [36, Proposition 12.2], we have

$$\pi_3(d/2)\pi_3(3d/2,m)\leq C\pi_3(m).$$

Using these estimates, we find that (5.13) is bounded up to a constant by

$$m\pi_3(m)\sum_{d=1}^{m}\pi_3(d) \le Cm\pi_3(m)^2\sum_{d=1}^{m}\pi_3(d,m)^{-1}, \tag{5.14}$$

where the inequality uses quasi-multiplicativity. To bound $\pi_3(d,m)^{-1}$ above, note that from (3.1), there is an $\eta > 0$ such

$$\pi_3(d,m)^{-1} \le C(m/d)^{\eta}. \tag{5.15}$$

Using this upper bound, we have

$$\sum_{d=1}^{m}(m/d)^{\eta} \le Cm. \tag{5.16}$$

Plugging (5.16) and (5.15) into (5.14), and using (5.10) and (5.12), we obtain the bound

$$\begin{aligned}\mathbf{E}[N_K^2, E_k'] &\le \mathbf{E}[\tilde{N}_K^2, E_k''] \\ &\le Cm^4(\pi_3(m))^2\mathbf{P}(E_k'') \\ &\le Cm^4(\pi_3(m))^2\mathbf{P}(E_k').\end{aligned}$$

Using (5.7) with the probability measure $\mathbf{P}(\cdot \mid E_k')$, we obtain (5.6). □

On E_k', let $\mathfrak{s}_k$ be the minimal length open path connecting the two five-arm points $\star_1$ and $\star_2$ in the U-shaped region

$$U(k) := \left[[-3\cdot 2^k, 3\cdot 2^k]\times[-\frac{1}{3}\cdot 2^k, 3\cdot 2^k]\right]\setminus(-\frac{7}{3}\cdot 2^k, \frac{7}{3}\cdot 2^k)^2. \tag{5.17}$$

Lemma 5.3. *Let c_0 be from Proposition 5.2. If for some $\varepsilon \in (0,1/4)$, $\delta > 0$ and $k \ge 1$ one has*

$$\mathbf{E}[\#\mathfrak{s}_k \mid E_k'] \le \delta 2^{2k}\pi_3(2^k), \tag{5.18}$$

then

$$\mathbf{P}(\#\mathfrak{s}_k \le 2(\delta/c_0)2^{2k}\pi_3(2^k) \mid N_K \ge c_0 2^{2K}\pi_3(2^K), E_k') \ge 1/2. \tag{5.19}$$

Proof. Let $\mathscr{N} = \{N_K \ge c_0 2^{2K}\pi_3(2^K)\}$. We have

$$\begin{aligned}\mathbf{E}[\#\mathfrak{s}_k \mid \mathscr{N}, E_k'] &\le \frac{\mathbf{E}[\#\mathfrak{s}_k \mathbf{1}_{E_k'}]}{\mathbf{P}(\mathscr{N}\cap E_k')} \\ &= \mathbf{E}[\#\mathfrak{s}_k \mid E_k']\cdot\frac{\mathbf{P}(E_k')}{\mathbf{P}(\mathscr{N}\cap E_k')}.\end{aligned} \tag{5.20}$$

By (5.6), the second factor is bounded above by $1/c_0$, so (5.20) is bounded by

$$\frac{\delta}{c_0}2^{2k}\pi_3(2^k).$$

The result then follows by Markov's inequality. □

We now define the event $E_k = E_k(\varepsilon,\delta)$ as

$$E_k(\varepsilon,\delta) := \{\#\mathfrak{s}_k \le 2(\delta/c_0)2^{2k}\pi_3(2^k)\} \cap \{N_K \ge c_0 2^{2K}\pi_3(2^K)\} \cap E_k', \tag{5.21}$$

as well as the translated event $E_k(e) = E_k(e,\varepsilon,\delta)$ as

$$E_k(e,\varepsilon,\delta) := \tau_{-e_x} E_k(\varepsilon,\delta). \tag{5.22}$$

For $e \in B(n)$, write $d = \mathrm{dist}(e,\partial B(n))$. The key property of E_k is the following:

Proposition 5.4. *There is an ε_0 such that if $\varepsilon < \varepsilon_0$, $\delta > 0$, and k satisfies $1 \le k \le \log d - \lfloor \log \frac{1}{\varepsilon} \rfloor$, the occurrence of*

$$\{e \in l_n\} \cap E_k(e,\varepsilon,\delta)$$

implies that there is an $\varepsilon\delta$-shortcut around e, i.e. $\mathscr{S}(e,\kappa) \neq \emptyset$ for $\kappa = \varepsilon \cdot \delta$.

Proof. We first claim:

$$\text{On } E_k(e,\varepsilon,\delta)\text{, there is a } \kappa\text{-shortcut around } e. \tag{5.23}$$

We have included a proof in Appendix A. See also [9, Sections 4.5 and 7] for a detailed proof of a similar claim. The event E_k there is defined differently, but the arguments remain essentially the same.

For the path r, we choose a path in $\tau_{e_x}U(k)$ with length less than $2(\delta/c_0)2^{2k}\pi_3(2^k)$ between the two five-arm points in items 3. and 4. of the definition of E_k' above. On the other hand because the edges found in Proposition 5.2 are on the lowest crossing, the portion τ of l_n containing e between the two five-arm points has total volume greater than or equal to

$$N_K \ge c_0 2^{2K}\pi_3(2^K).$$

Thus,

$$\frac{\#r}{\#\tau} \le \delta \frac{(2/c_0)2^{2k}\pi_3(2^k)}{c_0 2^{2K}\pi_3(2^K)}. \tag{5.24}$$

Using (3.1), we have

$$\frac{2^{2k}\pi_3(2^k)}{2^{2K}\pi_3(2^K)} \le C_5 2^{(2-\beta)(k-K)} \le 2C_5 2^{2-\beta}\varepsilon^{2-\beta},$$

where $C_5 \ge 1$ is a constant, and $\beta = 1-\gamma$, for $\gamma > 0$. If

$$\varepsilon^\gamma < \min\left\{\frac{c_0^2}{8C_5 2^{2-\beta}}, 1/4^\gamma\right\}, \tag{5.25}$$

then we find

$$\#r < (\varepsilon\delta)\cdot\#\tau. \tag{5.26}$$

□

The following proposition gives a lower bound for the probability of $E_k(e,\varepsilon,\delta)$:

Proposition 5.5. *There is a constant $c_2 > 0$ such that for all $\varepsilon \in (0,1/4)$ and $k \geq 1$,*

$$\mathbf{P}(E_k') \geq c_2 \varepsilon^4. \tag{5.27}$$

In particular, by (5.6)*, if* (5.18) *holds for some $\varepsilon \in (0,1/2)$, $\delta > 0$, and $k \geq 1$, then*

$$\mathbf{P}(E_k(e,\varepsilon,\delta)) \geq \frac{c_0 c_2}{2} \varepsilon^4. \tag{5.28}$$

Proof. The second inequality is a combination of (5.6), (5.19) and (5.27).

For the first inequality, we apply the RSW and the generalized FKG inequalities and gluing constructions to build all the connections in the definition of E_k'. See Section 3.4 for a description of the methodology. Although the definition of E_k' is quite long and complicated, the probability that each of its 16 items occurs (individually) can be bounded below, and so the generalized FKG inequality allows us to glue all connections together (force them to occur simultaneously) with the lower bound given in (5.27).

We now list the individual lower bounds. The majority of the connections described in the definition of E_k' in Sections 5.1 and 5.2 are constructed from interlocking crossings or closed dual crossings of rectangles with fixed aspect ratio. By the RSW theorem (3.5), each of these particular items has probability bounded below independently of k and ε. The only connections in Sections 5.1, 5.2 whose probability cannot be bounded below simply by using RSW are the ones given below.

(1) The construction of the five-arm points in items 3. and 4. For these, we use the lower bound (3.2)

$$\mathbf{P}(A_5(n)) \geq cn^{-2},$$

where $A_5(n)$ is the event that there is a polychromatic five-arm sequence from 0 to distance n (see the definition at the beginning of Section 5.1). For item 3., say, we consider any vertex $v \in B_1$ and let $I(v)$ be the event that v has the properties listed in that item (which is to say that $v = \star_1$). By Kesten's arm separation method, the probability of $I(v)$ is comparable to the five-arm probability to distance 2^k; that is, $\mathbf{P}(I(v))$ is bounded above and below by constants times $\pi_5(2^k)$. Therefore one has

$$\mathbf{E} \sum_{v \in B_1} \mathbf{1}_{I(v)} \geq \pi_5(2^k) \cdot (\text{volume of } B_1),$$

which is bounded away from 0. Since there can be at most one point $\star_1$ in B_1, the sum takes value 0 or 1; therefore, the left side equals $\mathbf{P}(\sum_{v \in B_1} \mathbf{1}_{I(v)} = 1)$. We conclude that

$$\mathbf{P}(\text{events in items 3. and 4.}) \geq c > 0.$$

(2) The construction of circuits with defects from items 5. and 15. Note that the defected edges in these circuits are four-arm edges: they have two dual closed arms and two open arms ending on the boundary of boxes indicated

in these items (where the defects lie). To show that the events described in these items have probability bounded away from zero, by a gluing construction and the RSW theorem, it suffices to show that there exist four-arm edges in these boxes with probability bounded away from zero.

The probability that there exists such a four-arm edge in, say, the left box of item 5. (the box $[-\frac{5}{3}2^k, -2^k] \times [-\frac{1}{3}2^k, \frac{1}{3}2^k]$) can be bounded below as follows. By the RSW theorem, the probability that there is an open horizontal crossing of this box is bounded away from zero and one. By the van den Berg-Kesten inequality, one has

$$\mathbf{P}(\text{there are two such disjoint crossings}) \leq \mathbf{P}(\text{there is at least one such crossing})^2.$$

Therefore with probability bounded away from zero, there is one crossing, but one cannot find two disjoint crossings. On this event, there must be a "pivotal" edge: an edge which is open, but if it were made to be closed (without changing the status of any other edge), there would be no open crossing. By planar duality, such an edge is a four-arm edge. We conclude that

$$\mathbf{P}(\text{events in items 5. and 15.}) \geq c > 0.$$

(3) 6 arms (two dual closed and four open) from $\partial B(3 \cdot 2^k)$ and $\partial B(2^K/16)$, corresponding to the connections in items 11., 13. and 16. in Section 5.2. These are the following six paths shown in Figure 5.4: the solid (open) paths connecting the central box to the sides of the figure, the red dotted (closed dual) paths connecting the central box around the bottom box, and the blue solid (open) paths connecting the central box to the interior of the lower box. By Kesten's arm separation techniques, and a gluing argument, the probability of the existence of such arms is at least the probability that there are six arms in the counterclockwise order open, open, closed, open, open, closed which connect $B(3 \cdot 2^k)$ to $\partial B(\frac{1}{8}2^K)$. In other words,

$$\mathbf{P}(\text{events in items } 11, 13, 16) \geq c\mathbf{P}(A_6(3 \cdot 2^k, \frac{1}{8}2^K)).$$

Combining the above three cases (with the constant lower bound for the probabilities of all other items listed in the definition of E'_k) with a gluing argument as mentioned in the beginning of this proof, we use (3.3) to finish with

$$\mathbf{P}(E'_k) \geq c\mathbf{P}(A_6(3 \cdot 2^k, \frac{1}{8}2^K)) \geq c \left(\frac{3 \cdot 2^k}{\frac{1}{8}2^K} \right)^4 \geq c\varepsilon^4.$$

□

Since $E_k(e, \varepsilon, \delta)$ implies in particular the existence of 3 disjoint connections (2 open, one closed) between $\partial B(e, 2^k)$ and $\partial B(e, 2^K)$, by a straightforward gluing argument (see [9, Section 5.5]), we pass from the lower bound (5.28) to the following conditional bound. There is $c_4 > 0$ such that if (5.18) holds for some $\varepsilon \in (0,1/2)$, $\delta > 0$, and $k \geq 1$, then for all $L \geq 1$,

$$\mathbf{P}(E_k(e,\varepsilon,\delta) \mid A_3(e,2^L)) \geq c_4\varepsilon^4. \tag{5.29}$$

Proposition 5.6. *There is a constant $\hat{c}$ such that if $\delta_j > 0$, $j = 1,\dots,L$ is a sequence of parameters such that for some $\varepsilon \in (0,1/4)$,*

$$\mathbf{E}[\#\mathfrak{s}_j \mid E'_j] \leq \delta_j 2^{2j}\pi_3(2^j), \tag{5.30}$$

then for any, $L' < L$,

$$\mathbf{P}(\cap_{j=L'}^{L} E_j(e,\varepsilon,\delta_j)^c \mid A_3(e,2^L)) \leq 2^{-\hat{c}\frac{\varepsilon^4}{\log\frac{1}{\varepsilon}}(L-L')}. \tag{5.31}$$

Proof. Putting $E_j = E_j(e,\varepsilon,\delta_j)$, we have by (5.29),

$$\mathbf{P}(E_j \mid A_3(e,2^L)) \geq c_4\varepsilon^4, \quad j,L \geq 1.$$

Furthermore, using the notation of Theorem 4.1, straightforward gluing constructions can be used to show that, by possibly lowering c_4, one has

$$\mathbf{P}(E_{10j+5},\hat{\mathfrak{C}}_j \mid A_3(e,2^L)) \geq c_4\varepsilon^4, \text{ for } 0 \leq j \leq \frac{L}{10N} - 1,$$

where $N = \lfloor \log\frac{1}{\varepsilon} \rfloor$, and $\hat{\mathfrak{C}}_j$ is defined in the first paragraph of Section 4. We then use Theorem 4.1 with N as above, and $\tilde{c}_0 = c_4\varepsilon^4$ to find a constant $c_5 > 0$ such that for $L - L' \geq 40\lfloor \log\frac{1}{\varepsilon} \rfloor$, we have

$$\mathbf{P}(\cap_{j=L'}^{L} E_j^c \mid A_3(e,2^L)) \leq 2^{-c_5\varepsilon^4\frac{L-L'}{\log\frac{1}{\varepsilon}}}$$

By possibly decreasing c_5 to handle L' with $L' \geq L - 40\lfloor \log\frac{1}{\varepsilon} \rfloor$, this implies (5.31). □

6 U-shaped regions

Let $\varepsilon \in (0,1/2)$ and recall $\mathbf{e}_1 = (1,0)$. On the event $E_k(\{0,\mathbf{e}_1\},\varepsilon)$, in the box $[-3\cdot 2^k, 3\cdot 2^k]^2$ (see Figure 6.1), the U-shaped region

$$U(k) = \left[[-3\cdot 2^k, 3\cdot 2^k] \times [-\frac{1}{3}\cdot 2^k, 3\cdot 2^k]\right] \setminus (-\frac{7}{3}\cdot 2^k, \frac{7}{3}\cdot 2^k)^2,$$

contains an open arc on scale 2^k, joining two five-arm points $\star_1 \in B_1$ and $\star_2 \in B_2$. This arc is contained in the smaller region

$$\tilde{U}(k) \cup \tilde{V}(k) \subset U(k)$$

defined in (5.4).

Recall that we denote by $\mathfrak{s}_k$ an arc in $U(k)$ connecting the two five-arm points with the minimal number of edges.

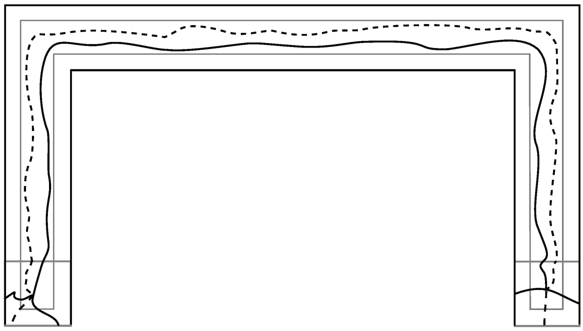

FIGURE 6.1. The U-shaped region. The region $\widetilde{U} \cup \tilde{V}$ appears in grey.

Definition 6.1. On the event E_k, the outermost open arc in $U(k)$ connecting $\star_1$ to $\star_2$ is defined to be the open arc ℓ_k in $U(k)$ whose initial and final edges are the vertical edges out of the five-arm points $\star_1$ and $\star_2$, and such that the compact region enclosed by the union of s and the dual closed arc $\mathfrak{c}$ between $\star_1$ and $\star_2$ (item 8. in the definition of E_k', in green in Figure 5.3) is minimal.

Note that since $r \subset \tilde{U}(k) \cup \tilde{V}(k)$, and E_k' implies the existence of an open path connecting $\star_1$ and $\star_2$ inside $\tilde{U}(k)$, we have

$$\ell_k \subset \tilde{V}(k) \cup \tilde{U}(k).$$

In particular,

$$\operatorname{dist}(\ell_k, \partial U(k)) \geq \frac{1}{6} \cdot 2^k. \tag{6.1}$$

The exact analogue of Proposition 5.4 holds in $U(k)$ with l_n replaced by ℓ_k, the key point being that belonging to the outermost arc ℓ_k is characterized locally by a three-arm event. By comparison with $\#\ell_k$ we have (see [9, Lemma 5.3] for a similar estimate) for all $k \geq 1$ and $\varepsilon \in (0, 1/2)$,

$$\mathbf{E}[\#\mathfrak{s}_k \mid E_k'] \leq C 2^{2k} \pi_3(2^k). \tag{6.2}$$

By Proposition 5.5, this implies, for $\varepsilon > 0$

$$\mathbf{P}(E_j(e, \varepsilon, C) \mid A_3(e, 2^j)) \geq c_3 \varepsilon^4. \tag{6.3}$$

To use (6.3) to construct a shorter path in the next section, we need the following:

Proposition 6.2. *There exists $C > 0$ with the following property. For any $x_1 \in B_1$, $x_2 \in B_2$, $e \in \tilde{U}(k)$, $\varepsilon \in (0, 1/4)$, $d = 2^j$, $j < k$ such that $B(e, d) \subset U(k)$, and $x_i \notin B(e, 4d)$, $i = 1, 2$, and any event E depending only on the status of edges in $B(e, d/100)$, we have*

$$\mathbf{P}(E \mid E_k', e \in \ell_k, \star_i = x_i, i = 1, 2) \leq C\mathbf{P}(E \mid A_3(e, d)) \tag{6.4}$$

Proof. We treat the case $e \in B_1$. The remaining cases are similar or simpler. See Figure 6.2 for an illustration.

For $x_1 \in B_1$, $x_2 \in B_2$, write

$$\mathbf{P}(E \mid E'_k, e \in \ell_k, \star_i = x_i, i = 1,2) = \frac{\mathbf{P}(E \cap E'_k, e \in \ell_k, \star_i = x_i, i = 1,2)}{\mathbf{P}(E'_k, e \in \ell_k, \star_i = x_i, i = 1,2)}. \tag{6.5}$$

We now decompose the events in the numerator and denominator into many smaller subevents representing connections that can be glued together. First, to decompose the event $\{e \in \ell_k, \star_1 = x_1\}$ we give the following events depending on edges in B'_1:

(1) the event $A_3(e,d)$ (the existence of three arms from e to distance d — this is because $e \in \ell_k$),
(2) the event $A_5(x_1,d)$ (the existence of five arms from x_1 to distance d — this is because $\star_1 = x_1$),
(3) five arms from $\partial B(x_1, 6d)$ (appearing in blue in Figure 6.2) to $\partial B'_1$, with the extremities of the arms lying in the intervals as prescribed in the definition of E'_k. This is again because $\star_1 = x_1$. We denote the event described in this item by $\tilde{A}_5(x_1,d,2^k)$.

To represent the portion of the event E'_k which does not involve $\star_1$, we let $\widetilde{E}_k$ be the event that the connections described in items 1., 2., and 4.-16. of E'_k occur. (Strictly speaking, in some of these items we need to eliminate reference to arms coming from $\star_1$. For example, in item 6., we do not require that the crossings connect to any arms.)

In total, we find by independence of the status of edges in disjoint regions,

$$\begin{aligned}
&\mathbf{P}(E \cap E'_k, e \in \ell_k, \star_i = x_i, i = 1,2) \\
&\le \mathbf{P}(E \cap A_3(e,d) \cap A_5(x_1,d) \cap \tilde{A}_5(x_1,d,2^k) \cap \widetilde{E}_k, \star_2 = x_2) \\
&= \mathbf{P}(E \cap A_3(e,d))\mathbf{P}(A_5(x_1,d))\mathbf{P}(\tilde{A}_5(x_1,d,2^k))\mathbf{P}(\widetilde{E}_k, \star_2 = x_2).
\end{aligned}$$

This is our upper bound for the numerator in (6.5).

For the denominator in (6.5), we use gluing and the generalized FKG inequality. On the event $A_3(e,d) \cap A_5(x_1,d) \cap \tilde{A}_5(x_1,d,2^k) \cap \widetilde{E}_k \cap \{\star_2 = x_2\}$, we can glue connections together with the generalized FKG inequality to force the event $E'_k \cap \{e \in \ell_k, \star_i = x_i, i = 1,2\}$ to occur. The reader may wish to consult Figure 6.2. For example, if $A_5(x_1,d) \cap A_3(e,d) \cap \tilde{A}_5(x_1,d,2^k)$ occurs, then we may glue the three arms from $A_3(e,d)$ together with the five arms from $A_5(x_1,d)$ and the five arms from $\tilde{A}_5(x_1,d,2^k)$ so that (a) the five arms from $A_5(x_1,d)$ connect directly to the five arms from $\tilde{A}_5(x_1,d,2^k)$, (b) the two open arms from $A_3(e,d)$ coincide with segments of an open arm from $A_5(x_1,d)$, and (c) the closed arm from $A_3(e,d)$ connects to a closed arm from the five continued arms. We then use Kesten's arm separation technique to ensure that the five arms land on the boundary of B'_1 in the areas described in item 3. of the definition of E'_k and two of the five arms reaching

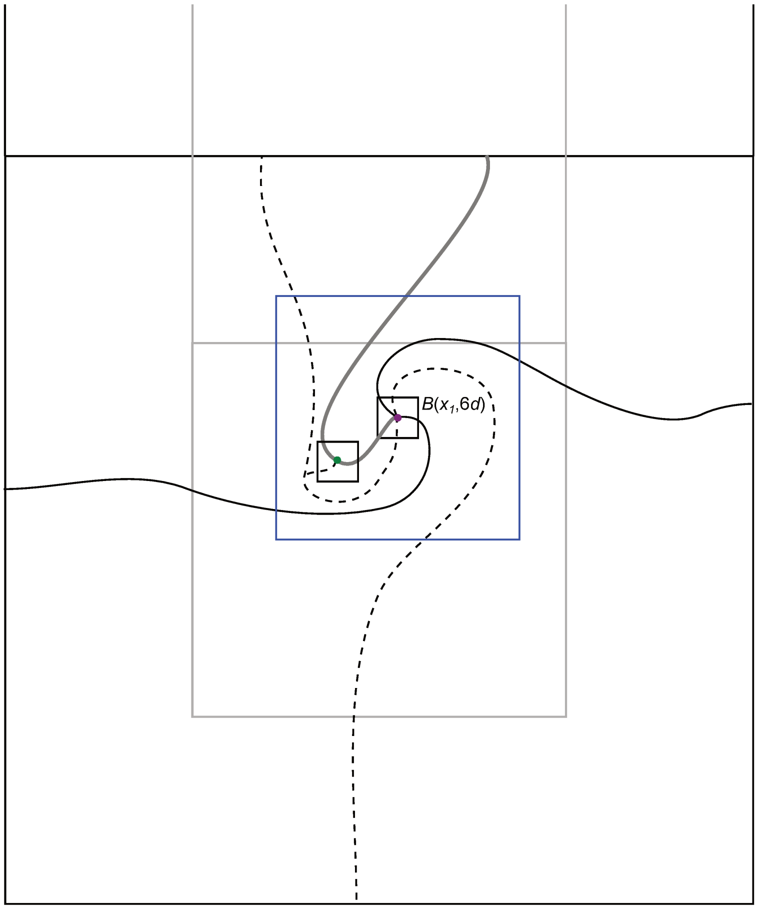

FIGURE 6.2. The gluing construction in B_1', the box with the same center as B_1 and twice the side length, used to obtain (6.4) in Proposition 6.2. The purple dot represents $\star_1$, and the green dot represents the edge e. The small boxes around $\star_1$ and e have side length d, and $\mathrm{dist}(e,\star_1) \geq 4d$. The blue box is centered around $\star_1$ and has side length $6d$. The portion of ℓ_k inside B_1 appears as a thicker grey curve.

the boundary of B_1' (open and closed) are connected to the open and closed paths depicted in the top of Figure 6.1 in the event $\widetilde{E}_k$. By these remarks, one then has

$$\mathbf{P}(E_k', e \in \ell_k, \star_i = x_i, i = 1,2) \geq c\mathbf{P}(A_3(e,d) \cap A_5(x_1,d) \cap \tilde{A}_5(x_1,d,2^k) \cap \widetilde{E}_k, \star_2 = x_2).$$

By independence applied to the right side,

$$\mathbf{P}(E_k', e \in \ell_k, \star_i = x_i, i = 1,2) \geq c\mathbf{P}(A_3(e,d))\mathbf{P}(A_5(x_1,d))\mathbf{P}(\tilde{A}_5(x_1,d,2^k))\mathbf{P}(\widetilde{E}_k, \star_2 = x_2).$$

This is our lower bound for the denominator in (6.5). Combining this with the upper bound for the numerator completes the proof.

□

7 Iteration

Our goal in this section is to derive the following proposition, which we use in Section 8 to prove the main result, Theorem 1.1:

Proposition 7.1. *There exist constants C, C' such that for any $\varepsilon > 0$ sufficiently small, $L \geq 1$, and $2^k \geq (C\varepsilon^{-4}(\log \frac{1}{\varepsilon})^2)^L$, we have*

$$\mathbf{E}[\#\mathfrak{s}_k \mid E_k'] \leq (C'\varepsilon^{1/2})^L 2^{2k} \pi_3(2^k). \tag{7.1}$$

The proof of Proposition 7.1 is split into four sections. In Section 7.1, we construct a family of candidate paths $(\sigma(i))_{i\geq 1} = (\sigma(i,k))_{i\geq 1}$ between the five-arm points in $U(k)$ using lower-scale optimal paths and give the central iterative bound on their lengths in Proposition 7.3. In the remaining sections, we estimate the right side of this inequality: in Section 7.2, we present basic inequalities and choices of parameters, in Section 7.3, we give the bound in the case $i = 0$, and in Section 7.4, we give the general case, $i \geq 1$.

7.1 Construction and estimation of shorter arcs

Proposition 7.1 follows from an iterative procedure wherein improvements on the outermost arc ℓ_k in $U(k)$ (which is actually in the smaller region $\tilde{U}(k) \cup \tilde{V}(k)$) are made on larger and larger scales. The best improvement so far on scale l is described by a sequence of parameters $\kappa_l(i)$, $l, i = 1, 2, \ldots$, nonincreasing in l, where i denotes the number of the current iteration in the argument. All definitions in this section will depend on the number of iterations so far, which we will call the generation i. The following is a key definition. It should be compared to Definition 5.1, where the shortcuts were constructed around the lowest crossing l_n of the box $B(n)$. Here the shortcuts are constructed around ℓ_k in the region $U(k)$ (see Section 6).

Definition 7.2. We say r is a <u>size l</u> shortcut in generation i if

(1) r is an $\kappa_l(i)$-shortcut in the sense of Definition 5.1. In particular, the "gain factor" $\#r/\#\tau$ is $\leq \kappa_l(i)$, where τ is the detoured part;
(2) the shortcut r is contained in a box of side length $3 \cdot 2^l$;
(3) the detoured part τ is contained in a box $B \subset U(k)$, with the same center as the box in the previous item, of side length $2^{\log \frac{1}{\varepsilon}} 2^l$, has ℓ^∞-diameter greater than $\frac{2}{3} 2^{\log \frac{1}{\varepsilon}} 2^l$, and

$$\mathrm{dist}(\tau, \{\star_1, \star_2\}) \geq \frac{1}{8} 2^{\log \frac{1}{\varepsilon}} 2^l. \tag{7.2}$$

Eventually, the gain factor will have the form $\kappa_l(i) = \varepsilon^{c \min\{i, c' l\}}$. We note that if ε is sufficiently small, the largest possible size of shortcut is no larger than $k + 1$. Furthermore, distinct shortcuts (regardless of their sizes) are either nested or disjoint. By nested, we mean that the region enclosed by the union of a shortcut and its detoured section of ℓ_k surrounds that of another shortcut. Both of these statements follow from the presence of "shielding" paths in item 4 of Definition 5.1. (See [9, Prop. 2.3].) Last, the definition of size l shortcuts is designed so that if $e \in \ell_k$ and if $E_l(e, \varepsilon, \kappa_l(i)/\varepsilon)$ occurs for an l such that (a) $B(e, 2^{l+\lfloor \log \frac{1}{\varepsilon} \rfloor}) \subset U(k)$ (which holds for $l \leq k - 3 - \log \frac{1}{\varepsilon}$ by (6.1)) and (b) $B(e, 2^{l+\lfloor \log \frac{1}{\varepsilon} \rfloor})$ does not contain the five-arm points $\star_i$, then there is a size l shortcut in generation i around e. This follows from the analogue of Proposition 5.4 for U-shaped regions (which gives item 1 above) and the construction of events E_k in the previous sections (the red box in Figure 5.1 for item 2 and the larger box from that figure and the existence of three-arm points in the rectangle R in (5.5) for item 3.)

Construction. Given the occurrence of E'_k, we define an arc $\sigma = \sigma(i)$ joining the two five-arm points in $U(k)$ as follows. For each $l = k+1, k, \ldots, 1$ in order, choose a maximal collection of (generation i) shortcuts of size l, in the following way. First, we select a collection of size $k+1$ shortcuts such that no two of their detoured paths share vertices and the total length of the detoured sections of ℓ_k is maximal. The remaining uncovered portion of ℓ_k splits into a union of disjoint segments. For each such segment, we select a collection of size k shortcuts such that no two of their detoured paths share vertices and the total length of the detoured sections of the segment is maximal. Continuing this way down to size 1 shortcuts, we obtain our maximal collection of shortcuts. Next we form the arc σ consisting of the union of these shortcuts, and all the segments of ℓ_k which are not covered by this collection. It can be argued similarly to [9, Lemma 2.4] that what results from the preceding construction is an open arc between the two five-arm points. Since the shortcuts are either nested or disjoint, this construction has the following essential property:

Claim 1. Given any edge e of the outermost arc ℓ_k of $U(k)$, if, after applying the above construction, the new arc σ does not include a shortcut around e of any size $l = k+1, k, \ldots, r-1$, then there is no shortcut of any size $k+1, k, \ldots, r-1$ around e at all.

Proof. Suppose σ does not include a shortcut around e of any size $l = k+1, k, \ldots, r-1$. Then for any such l, e must be on a segment π_l of ℓ_k that is uncovered after we place size l shortcuts of ℓ_k, and $\pi_l \subset \pi_{l+1}$ for all l, where we write $\pi_{k+2} = \ell_k$. If there is a shortcut r of size l' (not contained in σ) around e for some $l' = k+1, k, \ldots, r-1$, then note that r must have both of its endpoints on $\pi_{l'+1}$. This is trivial if $\pi_{l'+1} = \ell_k$; otherwise, the segment $\pi_{l'+1}$ has endpoints which are starting vertices of shortcuts r_1, r_2 of sizes $\geq l'+1$. (If one endpoint of $\pi_{l'+1}$ is one of the five-arm points $\star_i$, we only get one such shortcut r_1.) Because shortcuts are nested, if r has an endpoint on $\ell_k \setminus \pi_{l'+1}$, then the detoured path τ_i of some r_i would be contained in the detoured path τ of r. However, this is impossible by size considerations:

$$\frac{2}{3} 2^{\log \frac{1}{\varepsilon}} 2^{l'+1} \leq \operatorname{diam} \tau_i \leq \operatorname{diam} \tau \leq 2^{\log \frac{1}{\varepsilon}} 2^{l'}.$$

Therefore r has both endpoints on $\pi_{l'+1}$. Because $\pi_{l'+1}$ is uncovered when we add size l' shortcuts, and all such shortcuts are disjoint, maximality dictates that we must add r, or another shortcut of size l' that covers e, to σ. This is a contradiction. □

From Claim 1, we see that if the new arc σ contains a shortcut around e of size l, then there is no shortcut of any size $l+1,\ldots,k+1$ around e at all. Indeed, e must have been on an uncovered segment directly before we added shortcuts of size l, and is therefore not covered by a shortcut in σ of any size $l+1,\ldots,k+1$.

The following proposition is the main iterative bound of the paper.

Proposition 7.3. *Let $\varepsilon > 0$ and fix $i \in \mathbb{N}$. Suppose moreover that, for some nonincreasing sequence of parameters $\delta_l(i)$, $l \geq 1$, we have*

$$\mathbf{E}[\#\mathfrak{s}_l \mid E_l'] \leq \delta_l(i) 2^{2l} \pi_3(2^l). \tag{7.3}$$

Let

$$\kappa_l(i) := \begin{cases} \varepsilon \cdot \delta_l(i) & \text{if } l \geq 1 \\ 1 & \text{if } l \leq 0, \end{cases}$$

and $\sigma = \sigma(i)$ be defined as above, in terms of the sequence $\kappa_l(i)$, in the region U_k for some $k \geq 1$. For $d = 0,\ldots,k+1$, let $M > 0$ and $d_1 = d_1(d)$ be given as

$$d_1 = d - M\varepsilon^{-4} \left(\log \frac{1}{\varepsilon}\right)^2. \tag{7.4}$$

There are positive constants c_* and C_2 with $C_2 \geq 1$ such that for any ε sufficiently small, any $M > 0$ and $i \in \mathbb{N}$, any parameters $\delta_l(i)$ as above, and any $k \geq 1$,

$$\begin{aligned} \mathbf{E}[\#\mathfrak{s}_k \mid E_k'] &\leq \mathbf{E}[\#\sigma(i) \mid E_k'] \\ &\leq C_2 \sum_{d=0}^{k+1} 2^{2d} \pi_3(2^d) \cdot \left(2^{-\eta(\varepsilon)d} + \sum_{s=1}^{d_1} 2^{-\eta(\varepsilon)(d-s)} \kappa_s(i) + \kappa_{d_1}(i)\right), \end{aligned} \tag{7.5}$$

where

$$\eta(\varepsilon) := \frac{c_* \varepsilon^4}{\log \frac{1}{\varepsilon}}. \tag{7.6}$$

Proof. The first inequality follows because ℓ_k is in $\tilde{U}(k) \cup \tilde{V}(k) \subset U(k)$ and all its shortcuts are constructed in boxes in $U(k)$, so σ remains in $U(k)$. To estimate the length of σ, we begin by dividing the outermost arc ℓ_k, given σ, into a finite number of segments $\hat{\sigma}_\ell$, $\ell = 1, \ldots$, where each segment $\hat{\sigma}_\ell$ is either

(1) a single edge of the outermost arc also belonging to σ, or

(2) a segment of the outermost arc which is detoured by a connected subsegment of σ. That is, $\hat{\sigma}_\ell$ is the part of the outermost arc detoured by a shortcut σ_ℓ in σ.

To each shortcut σ_ℓ, we can associate a "gain factor" $\mathrm{gf}(\sigma_\ell)$, which is 1 if σ_ℓ is an edge of the outermost arc, and $\#\sigma_\ell/\#\hat{\sigma}_\ell$ otherwise.

By definition of σ, we have

$$\#\sigma = \sum_\ell \#\hat{\sigma}_\ell \times \mathrm{gf}(\sigma_\ell).$$

For a fixed generation i (initially $i=1$), we organize this sum according to the size of the shortcut σ_ℓ (we say the size is 0 if there is no shortcut, in which case the gain factor is 1):

$$\#\sigma = \sum_{s=0}^{k+1} \sum_{\ell:\mathrm{size}(\sigma_\ell)=s} \#\hat{\sigma}_\ell \times \mathrm{gf}(\sigma_\ell).$$

Note that for large values of s, many of the summands will be zero because there cannot exist shortcuts of such sizes. Nevertheless, the bound holds as stated.

The event E_k' is partitioned into the events:

$$F(x_1,x_2) := \{\star_1 = x_1, \star_2 = x_2\}, \quad x_1 \in B_1, x_2 \in B_2.$$

Note that $F(x,y) \cap F(x',y') = \emptyset$ on the event E_k' unless $x = x'$ and $y = y'$. Thus, we have

$$\#\sigma \le \sum_{x_1 \in B_1, x_2 \in B_2} \mathbf{1}_{F(x_1,x_2)} \sum_{s=0}^{k+1} \sum_{\ell:\mathrm{size}(\sigma_\ell)=s} \#\hat{\sigma}_\ell \cdot \kappa_s(i).$$

Next we divide the region $U(k)$ according to the distance d to the points x_1, x_2, obtaining, for

$$A_d = A_d(x_1,x_2) = \{e \in U(k) : 2^d \le \mathrm{dist}(e,x_1) \le 2^{d+1} \text{ or } 2^d \le \mathrm{dist}(e,x_2) \le 2^{d+1}\},$$

(and $A_0 = \{\mathrm{dist}(e,x_1) \le 1 \text{ or } \mathrm{dist}(e,x_2) \le 1\}$) the decomposition

$$\#\sigma \le \sum_{x_1 \in B_1, x_2 \in B_2} \mathbf{1}_{F(x_1,x_2)} \sum_{d=0}^{k+1} \sum_{s=0}^{d_0} \sum_{\ell:\mathrm{size}(\sigma_\ell)=s} \#(\hat{\sigma}_\ell \cap A_d) \cdot \kappa_s(i).$$

Here $d_0 = d_0(d) = \max(d + 4 - \log\frac{1}{\varepsilon}, 0)$. We do not need to consider larger sizes since they cannot occur at such distances by the condition (7.2).

By the remark following Claim 1, if a shortcut σ_ℓ surrounds an edge e and has size $s < k+1$, then there is no shortcut of any size $l = k+1, \ldots, s+1$ around e at all, so

$$\#\sigma \le \sum_{x_1 \in B_1, x_2 \in B_2} \mathbf{1}_{F(x_1,x_2)} \sum_{d=0}^{k+1} \left(\sum_{s=0}^{d_1} \#(B_s \cap A_d) \cdot \kappa_s(i) + \#(\ell_k \cap A_d) \cdot \kappa_{d_1}(i) \right), \tag{7.7}$$

where $B_s = B_s(\kappa_s(i))$ is the set of edges on ℓ_k with no generation i shortcuts of sizes $l = k+1, k, \ldots, s+1$. We have used monotonicity of $\delta_\ell(i)$ in ℓ. (Recall that $\kappa_{d_1} = 1$ for $d \le M\varepsilon^{-4}\left(\log\frac{1}{\varepsilon}\right)^2$).

From Propositions 5.6 (for which we use the assumed bounds (7.3)) and 6.2, and the fact that events $E_l(e)$ (see (5.22)) for l such that the box $B(e, 2^{l+\lfloor \log \frac{1}{\varepsilon} \rfloor}) \subset U(k)$ does not contain the five-arm points $\star_i$ guarantee the existence of size l shortcuts (see the discussion below Definition 7.2), we have

(7.8)

$$\begin{aligned}\mathbf{P}(e \in B_s \mid E_k', e \in \ell_k, F(x_1,x_2)) &\le \mathbf{P}(\cap_{l=s+1}^{d-\log \frac{1}{\varepsilon}-10} E_l(e,\varepsilon,\delta_l(i))^c \mid E_k', e \in \ell_k, F(x_1,x_2)) \\ &\le C\mathbf{P}(\cap_{l=s+1}^{d-\log \frac{1}{\varepsilon}-10} E_l(e,\varepsilon,\delta_l(i))^c \mid A_3(e,2^d)) \\ &\le C2^{-\frac{c_*\varepsilon^4}{\log \frac{1}{\varepsilon}}(d-s)}.\end{aligned}$$

whenever $e \in A_d$. From (7.8) and (7.7), we have the following estimate for the size of σ:

(7.9)

$$\begin{aligned}&\mathbf{E}[\#\sigma(i) \mid E_k'] \\ &\le \sum_{d=0}^{k+1} \sum_{x_1 \in B_1, x_2 \in B_2} \mathbf{P}(F(x_1,x_2) \mid E_k') \\ &\quad \times \Big[\sum_{s=0}^{d_1} \kappa_s(i) \sum_{e \in A_d} \mathbf{P}(e \in B_s \mid E_k', F(x_1,x_2), e \in \ell_k) \mathbf{P}(e \in \ell_k \mid E_k', F(x_1,x_2)) \\ &\quad + \kappa_{d_1}(i) \sum_{e \in A_d} \mathbf{P}(e \in \ell_k \mid E_k', F(x_1,x_2)) \Big] \\ &\le C_2 \sum_{d=0}^{k+1} 2^{2d} \pi_3(2^d)(2^{-\eta(\varepsilon)d} + \sum_{s=1}^{d_1} 2^{-\eta(\varepsilon)(d-s)} \kappa_s(i) + \kappa_{d_1}(i)).\end{aligned}$$

In passing to the final line of (7.9), we have used the estimate

$$\mathbf{P}(e \in \ell_k \mid E_k', F(x_1,x_2)) \le C\pi_3(2^d),$$

for $e \in A_d$, where C is some constant independent of the parameters (in particular, of the x_i's). This is the analogue (for ℓ_k instead of the lowest crossing l_n) of the upper bound in estimate (5.1). That the conditioning on E_k' and $F(x_1,x_2)$ results only in an additional constant factor is shown by a gluing construction very similar to the one illustrated in Figure 6.2. □

7.2 Some definitions

In estimating the volume of the new path $\sigma(i)$, $i \ge 1$, using (7.9), it is important to track the dependence on ε when performing the requisite summations. We begin by introducing some notations and simple bounds we will use repeatedly in Sections 7.3 and 7.4.

We first take $\varepsilon > 0$ sufficiently small that Proposition 7.3 holds. We will need ε to be possibly even smaller, and will state this at various points in what follows.

A key point is that the size of ε always depends on fixed parameters, and never on k nor on the generation i.

We define

$$m = M\varepsilon^{-4}(\log\frac{1}{\varepsilon})^2,$$

with M as in (7.4). To simplify notation, we will assume ε, M are taken so that m is an integer. With this notation we have $d_1 = d - m$.

Recall the definition of $\eta(\varepsilon)$ in (7.6) and the constant c_* appearing in the statement of Proposition 7.3. We have

$$\eta(\varepsilon)m = c_* M \log\frac{1}{\varepsilon}.$$

For $l \geq 1$, set

$$s_l = 3ml = 3Ml \cdot \varepsilon^{-4}\left(\log\frac{1}{\varepsilon}\right)^2, \quad l \geq 1, \tag{7.10}$$

with $s_0 = 0$.

We define

$$\theta(\varepsilon) = \frac{2^{\eta(\varepsilon)}}{2^{\eta(\varepsilon)}-1} \leq \frac{2}{2^{\eta(\varepsilon)}-1},$$

where the inequality holds if ε is sufficiently small. We choose M such that

$$M > \max(1, 7/c_*). \tag{7.11}$$

Since

$$2^{\frac{c_*}{\log\frac{1}{\varepsilon}}\varepsilon^4} - 1 \geq c_* \frac{\ln 2}{\log\frac{1}{\varepsilon}}\varepsilon^4, \tag{7.12}$$

we have

$$\frac{\varepsilon^{Mc_*}}{2^{\frac{c_*\varepsilon^4}{\log\frac{1}{\varepsilon}}} - 1} \leq \frac{\log\frac{1}{\varepsilon}}{c_* \ln 2}\varepsilon^3. \tag{7.13}$$

We will always choose $\varepsilon = \varepsilon(c_*)$ so small that the quantity in (7.13) is less than $2\varepsilon^2$:

$$\theta(\varepsilon)\varepsilon^{c_* M} \leq \varepsilon^2. \tag{7.14}$$

The constant $C_3 \geq 1$ is chosen such that for all $L \geq 1$ and any $\alpha \leq 1$,

$$\begin{aligned}
\sum_{d=0}^{L+1} 2^{2d}\pi_3(2^d)2^{-\alpha d} &= \pi_3(2^L)\sum_{d=0}^{L+1}\frac{\pi_3(2^d)}{\pi_3(2^L)}2^{(2-\alpha)d} \\
&\leq C_5\pi_3(2^L)2^{\beta L}\frac{2^{(L+2)(2-\beta-\alpha)}-1}{2^{2-\beta-\alpha}-1} \\
&\leq C_3 2^{(2-\alpha)L}\pi_3(2^L).
\end{aligned} \tag{7.15}$$

Here $0 < \beta < 1$ was introduced in (3.1).

7.3 Improvement by iteration

We use Propostion 7.3 inductively to obtain improvements on our estimates for $\#\mathfrak{s}_k$, starting from the initial estimate

$$\mathbf{E}[\#\mathfrak{s}_k \mid E_k'] \le C_1 2^{2k}\pi_3(2^k), \tag{7.16}$$

for some $C_1 \ge 1$. The inductive step is presented in Proposition 7.4 in Section 7.4. For the purposes of illustation, we carry out one step of the induction in this section.

We apply Proposition 7.3 with $\delta_s(0) = C_1$ (equivalently, $\kappa_s(0) = C_1\varepsilon$) for all $s \ge 1$. Defining the corresponding arc $\sigma(0)$, we obtain for $k \ge 1$,

$$\mathbf{E}[\#\sigma(0) \mid E_k'] \le C_2 \sum_{d=0}^{k+1} 2^{2d}\pi_3(2^d)(2^{-\eta(\varepsilon)d} + C_1\varepsilon \sum_{s=1}^{d_1} 2^{-\eta(\varepsilon)(d-s)} + \kappa_{d_1}(0)). \tag{7.17}$$

We use this last expression to obtain an improvement on (7.16) under the assumption

$$k > s_1 = 3m. \tag{7.18}$$

The quantity (7.17) is bounded by

$$C_2 \sum_{d=0}^{k+1} 2^{2d}\pi_3(2^d)2^{-\eta(\varepsilon)d} + C_2C_1\varepsilon \sum_{d=0}^{k+1} 2^{2d}\pi_3(2^d)2^{\eta(\varepsilon)}\frac{\varepsilon^{Mc_*}}{2^{\eta(\varepsilon)}-1} \tag{7.19}$$

$$+C_2 \sum_{d=0}^{k+1} 2^{2d}\pi_3(2^d)\kappa_{d_1}(0). \tag{7.20}$$

By definition of C_3 (see (7.15)), the first term in (7.19) is bounded by

$$C_2 \sum_{d=0}^{k+1} 2^{2d}\pi_3(2^d)2^{-\eta(\varepsilon)d} \le C_2C_3 2^{2k}2^{-\eta(\varepsilon)k}\pi_3(2^k). \tag{7.21}$$

Using (7.14) and (7.15), the second term in (7.19) is bounded by

$$2C_1C_2C_3\varepsilon^3 2^{2k}\pi_3(2^k). \tag{7.22}$$

Similarly, for (7.20) we have the upper bound

$$\begin{aligned} C_2 \sum_{d=0}^{m} 2^{2d}\pi_3(2^d) + C_2 \sum_{d=m+1}^{k+1} C_1\varepsilon 2^{2d}\pi_3(2^k) &\le C_2C_3 2^{2k}\pi_3(2^k)\left[2^{-(k-m)(2-\beta)} + C_1\varepsilon\right] \\ &\le 2C_1C_2C_3\varepsilon 2^{2k}\pi_3(2^k). \end{aligned} \tag{7.23}$$

In the second inequality we have assumed that $k \ge s_1 = 3m$ (see (7.18)) and taken ε sufficiently small (depending only on β).

Adding these three bounds, $\mathbf{E}[\#\sigma(0) \mid E_k']$ is bounded by

$$C_2C_3 2^{2k}\pi_3(2^k)(2^{-\eta(\varepsilon)k} + 2C_1\varepsilon^3 + 2C_1\varepsilon). \tag{7.24}$$

The quantity (7.24) is bounded by

$$2C_2C_3\varepsilon^{1/2}2^{2k}\pi_3(2^k),$$

if ε is small (depending on C_1) and (7.18) holds, since then $2^{-\eta(\varepsilon)k} \leq \varepsilon^{1/2}$. Thus,

$$\mathbf{E}[\#\sigma(0) \mid E_k'] \leq 4C_2C_3\varepsilon^{1/2} \cdot 2^{2k}\pi_3(2^k)$$

for k satisfying (7.18).

This completes our bounding of the right side of the main inequality in Proposition 7.3. In summary, we now have $\mathbf{E}[\#\mathfrak{s}_r \mid E_r'] \leq \delta_r(1)2^{2r}\pi_3(2^r)$, with

$$\delta_r(1) = \begin{cases} C_1, & r \leq s_1, \\ 4C_2C_3\varepsilon^{1/2}, & r > s_1. \end{cases} \tag{7.25}$$

We may now iterate Proposition 7.3 for further generations to obtain an improved bound. We formulate a general inductive result in the next section.

7.4 General case

The following proposition formalizes the inductive step, showing that our estimates on $\#\mathfrak{s}_k$ can be further improved.

Proposition 7.4. *Recall the definition of $s_l = 3ml$ from* (7.10). *Assume that*

$$\mathbf{E}[\#\mathfrak{s}_r \mid E_r'] \leq \delta_r(L)2^{2r}\pi_3(2^r), \tag{7.26}$$

holds for the choice of parameters

$$\delta_r(L) = \begin{cases} C_1 & \text{if } r \leq s_1 \\ (4C_2C_3)^l\varepsilon^{l/2} & \text{if } s_l < r \leq s_{l+1},\ l = 1,\ldots,L-1, \\ (4C_2C_3)^L\varepsilon^{L/2} & \text{if } r > s_L. \end{cases} \tag{7.27}$$

Then, (7.26) *also holds for $r \geq s_{L+1}$ and $\delta_r(L)$ replaced by*

$$\delta_r(L+1) = (4C_2C_3)^{L+1}\varepsilon^{(L+1)/2}. \tag{7.28}$$

Proof. By (7.25), we may assume $L \geq 1$ and $r = k \geq s_{L+1} = 3m(L+1)$. Start from an upper bound for the main inequality of Proposition 7.3:

$$\mathbf{E}[\#\sigma(L) \mid E_k'] \leq C_2 \sum_{d=0}^{k+1} 2^{2d}\pi_3(2^d)2^{-\eta(\varepsilon)d} \times \tag{7.29}$$

$$\Big(\sum_{s=0}^{3m} 2^{\eta(\varepsilon)s} + \varepsilon \sum_{l=1}^{L} \sum_{s=(s_{l-1}+1)\wedge d_1}^{s_l \wedge d_1} 2^{\eta(\varepsilon)s}(4C_2C_3\varepsilon^{1/2})^{l-1} + \varepsilon(4C_2C_3\varepsilon^{1/2})^L \sum_{s>s_L}^{d_1} 2^{\eta(\varepsilon)} \tag{7.30}$$

$$+C_2 \sum_{d \leq m} 2^{2d}\pi_3(2^d) + C_2 \sum_{d>m}^{k+1} 2^{2d}\pi_3(2^d) \cdot \kappa_{d_1}(L). \tag{7.31}$$

The final sum in (7.30) is zero if $s_L \geq d_1$. The term (7.31) corresponds to the $\kappa_{d_1(i)}$ term in (7.5). The term (7.30) corresponds to $2^{-\eta(\varepsilon)d}$ plus the term over sizes $1 \leq s < d_1$ in (7.5). Sizes $0 \leq s \leq s_1$ are bounded by the first term. Other sizes are split over ranges of $(s_{l-1}, s_l]$ up to d_1 in the second term of (7.30) and sizes $0 \leq s \leq s_1$ are double counted from the previous term.

The term (7.31)

The term (7.31) is bounded (since $C_1 \geq 1$) by

$$C_2 \sum_{d \leq m} 2^{2d} \pi_3(2^d) + C_1 C_2 \varepsilon \sum_{l=1}^{L} (4C_2C_3\varepsilon^{1/2})^{l-1} \sum_{d=3m(l-1)+m+1}^{3ml+m} 2^{2d} \pi_3(2^d) \tag{7.32}$$
$$+\varepsilon C_2 (4C_2C_3)^L \varepsilon^{L/2} \sum_{d: d_1 > s_L}^{k+1} 2^{2d} \pi_3(2^d).$$

By (7.15), the second sum in (7.32) is bounded by $C_1C_2\varepsilon$ times

$$\sum_{l=1}^{L} (4C_2C_3\varepsilon^{1/2})^{l-1} C_3 2^{2(3ml+m)} \pi_3(2^{3ml+m}). \tag{7.33}$$

Using (3.1), (7.33) is no greater than

$$C_5 C_3 (4C_2C_3\varepsilon^{1/2})^{-1} 2^{3\beta mL + 2m} \pi_3(2^{3mL+m}) \sum_{l=1}^{L} 2^{(2-\beta)3ml} 2^{l \log 4C_2C_3\varepsilon^{1/2}}. \tag{7.34}$$

The sum in (7.34) is bounded by $2^{(2-\beta)3mL}(4C_2C_3\varepsilon^{1/2})^L$ times

$$\frac{4C_2C_3\varepsilon^{1/2} \cdot 2^{(2-\beta)3m}}{4C_2C_3\varepsilon^{1/2} \cdot 2^{(2-\beta)3m} - 1} \leq 2,$$

if $m \geq \frac{1}{6} \frac{1}{2-\beta} \log \frac{1}{\varepsilon}$. This is true for ε small enough (depending on β). Thus using (3.1), (7.34) is bounded by

$$2 \cdot C_5^2 C_3 2^{2(\beta-2)m} 2^{2 \cdot 3m(L+1)} \pi_3(2^{3m(L+1)}) (4C_2C_3\varepsilon^{1/2})^{L-1}. \tag{7.35}$$

Recalling the extra factors C_1C_2 and ε, we find from (7.32) and (7.35) that (7.31) is bounded by

$$\begin{aligned} &C_2C_3 2^{2m} \pi_3(2^m) \\ &+ 2C_1C_5^2 (4C_2C_3\varepsilon^{1/2})^L \varepsilon^{1/2} 2^{2(\beta-2)m} 2^{2\cdot 3m(L+1)} \pi_3(2^{3m(L+1)}) \\ &+ \varepsilon C_2C_3 (4C_2C_3)^L \varepsilon^{L/2} 2^{2k} \pi_3(2^k). \end{aligned} \tag{7.36}$$

We compare the first two terms in (7.36) to the third using (3.1). We have

$$\begin{aligned} 2^{2k}\pi_3(2^k) 2^{2\cdot(m-k)} \frac{\pi_3(2^m)}{\pi_3(2^k)} &\leq C_5 2^{2k} \pi_3(2^k) 2^{(2-\beta)(m-k)} \\ &\leq C_5 2^{2k} \pi_3(2^k) 2^{m-k}. \end{aligned} \tag{7.37}$$

In the second step we have used $\beta < 1$ and $k \geq s_L + m$. Then if ε is small enough (depending on C_5), (7.37) is bounded by

$$C_5 2^{2k} \pi_3(2^k) 2^{-3mL} \leq C_5 2^{2k} \pi_3(2^k) \varepsilon^{10L} \leq 2^{2k} \pi_3(2^k) \varepsilon^{9L}. \tag{7.38}$$

For the second term in (7.36), we find (using $\beta < 1$ and $k \geq s_{L+1}$) for ε small (depending on β):

$$2^{2(\beta-2)m} 2^{2\cdot 3m(L+1)} \pi_3(2^{3m(L+1)}) \leq C_5 \varepsilon^{10} 2^{2k} \pi_3(2^k). \tag{7.39}$$

Putting (7.38) and (7.39) into (7.36), we find that (7.31) is bounded by

$$\varepsilon (4C_2C_3\varepsilon^{1/2})^L 2^{2k} \pi_3(2^k)(C_2C_3 + 2C_1C_5^3\varepsilon^{19/2}) + C_2C_3\varepsilon^{9L} 2^{2k} \pi_3(2^k), \tag{7.40}$$

when $k \geq s_{L+1} = s_L + 3m$.

Term (7.30)**: case** $s_L \leq d_1$

For (7.30), we distinguish the cases when $s_L \leq d_1$ and $s_L > d_1$. In the first case, the term in question is,

$$\sum_{s=0}^{3m} 2^{\eta(\varepsilon)s} + \varepsilon \sum_{l=1}^{L} \sum_{s=s_{l-1}+1}^{s_l} 2^{\eta(\varepsilon)s} (4C_2C_3\varepsilon^{1/2})^{l-1} + \varepsilon (4C_2C_3\varepsilon^{1/2})^L \sum_{s>s_L}^{d_1} 2^{\eta(\varepsilon)s}. \tag{7.41}$$

By a summation like the one leading to (7.35), the middle term in (7.41) is bounded by

$$4\varepsilon \cdot (4C_2C_3\varepsilon^{1/2})^{-1} \theta(\varepsilon) 2^{\eta(\varepsilon)d_1} (4C_2C_3\varepsilon^{1/2})^L \leq 2^{\eta(\varepsilon)d} \varepsilon^2 (4C_2C_3\varepsilon^{1/2})^L. \tag{7.42}$$

The first and third terms in (7.41) are bounded, respectively, by

$$\theta(\varepsilon) 2^{3m\eta(\varepsilon)}$$

and

$$(4C_2C_3\varepsilon^{1/2})^L \varepsilon \theta(\varepsilon) 2^{\eta(\varepsilon)d_1}.$$

Multiplying these bounds by $2^{-\eta(\varepsilon)d}$, and using (7.11) and (7.14) we find an estimate of

$$\varepsilon \cdot \varepsilon^L + \varepsilon^2 (4C_2C_3\varepsilon^{1/2})^L \tag{7.43}$$

if $L \geq 1$ and $d_1 \geq s_L$. Here we have taken ε small depending on C_2 and C_3.

Using (7.43), (7.42) and performing the sum over d, we find that the contribution to (7.30) from $d_1 \geq s_L$ is

$$C_2C_3\varepsilon^2 (4C_2C_3\varepsilon^{1/2})^L 2^{2k} \pi_3(2^k) + \varepsilon C_2C_3 \varepsilon^L 2^{2k} \pi_3(2^k). \tag{7.44}$$

Term (7.30)**: case** $s_L > d_1$**.**

We turn to the case $s_L > d_1$. We let

$$l_d = \max\{l : s_l \le d_1\}$$
$$= \lfloor \frac{d}{3m} - \frac{1}{3} \rfloor.$$

When $d_1 < s_L$, the inner sum in (7.30) is

(7.45)

$$\sum_{s=0}^{3m} 2^{\eta(\varepsilon)s} + \varepsilon \sum_{l=1}^{l_d} \sum_{s=s_{l-1}+1}^{s_l} 2^{\eta(\varepsilon)s}(4C_2C_3\varepsilon^{1/2})^{l-1} + \varepsilon \sum_{s=s_{l_d}+1}^{d_1} 2^{\eta(\varepsilon)s}(4C_2C_3\varepsilon^{1/2})^{l_d-1}.$$

(If $l_d \le 0$, the second and third terms are zero.) As in the case $s_L \le d_1$, the first summand in (7.45) is bounded by $\theta(\varepsilon)2^{3m}\eta(\varepsilon)$. Multiplying this by $C_2 2^{2d}\pi_3(2^d)2^{-\eta(\varepsilon)d}$ and summing over d from 0 to $k+1$, we find a bound of

$$C_2C_3 2^{2k}\pi_3(2^k)\varepsilon^{16L}, \tag{7.46}$$

for $k \ge s_{L+1}$.

Using $c_* 3M \log \frac{1}{\varepsilon} l_d \le \eta(\varepsilon)(d-m)$ and performing a dyadic summation similar to the one leading to (7.35), the second and third terms in (7.45) are seen to give a contribution bounded by

$$3\varepsilon\theta(\varepsilon)(4C_2C_3\varepsilon^{1/2})^{-1}2^{\eta(\varepsilon)(d-m)}2^{l_d \log 4C_2C_3\varepsilon^{1/2}}. \tag{7.47}$$

Multiplying (7.47) by $C_2 2^{2d}\pi_3(2^d)2^{-\eta(\varepsilon)d}$, and adding (7.46), we find that the contribution to (7.30) from $s_L > d_1$ is bounded by

(7.48)

$$C_2C_3 2^{2k}\pi_3(2^k)\varepsilon^{16L} + 3C_2(4C_2C_3\varepsilon^{1/2})^{-1}\theta(\varepsilon)\varepsilon^{c_*M}\varepsilon \sum_{d=m}^{s_L+m} 2^{2d}\pi_3(2^d)2^{l_d \log 4C_2C_3\varepsilon^{1/2}}.$$

Note that if

$$s_{l-1} < d \le s_l,$$

then $l-2 \le l_d \le l-1$. The sum in (7.48) is bounded by

$$\sum_{d=m}^{s_L+m} 2^{2d}\pi_3(2^d)2^{l_d \log 4C_2C_3\varepsilon^{1/2}}$$
$$\le 2C_5C_3 2^{2\cdot 3mL}\pi_3(2^{3mL})(4C_2C_3\varepsilon^{1/2})^{L-2} + C_3(4C_2C_3\varepsilon^{1/2})^{L-1}2^{2(s_L+m)}\pi_3(2^{s_L+m}).$$

Here we have performed a summation as in (7.34). By (7.14), the pre-factor in front of the sum in (7.48) is bounded by $4\varepsilon^2$ (if ε is small depending on C_2), so we find an estimate for the second term of (7.48) of

$$2C_5 2^{2\cdot 3mL}\varepsilon\pi_3(2^{3mL})(4C_2C_3\varepsilon^{1/2})^L + \varepsilon^{3/2}(4C_2C_3\varepsilon^{1/2})^L 2^{2(s_L+m)}\pi_3(2^{s_L+m}).$$

Returning to (7.48), we find that the contribution to (7.30) from d such that $d_1 \le s_L$ is bounded by

$$C_2C_3 2^{2k}\pi_3(2^k)\varepsilon^{16L} + 2^{2(3mL+m)}\pi_3(2^{3mL+m})(4C_2C_3\varepsilon^{1/2})^L(2C_5^2\varepsilon^2 + \varepsilon^{3/2})). \tag{7.49}$$

Using (3.1), we have

$$2^{2(3mL+m)}\pi_3(2^{3mL+m}) \le \varepsilon^{10}2^{2k}\pi_3(2^k),$$

when $k \ge s_L + 3m$ and ε is small. Putting this into (7.49), we find a bound of

$$\varepsilon(4C_2C_3\varepsilon^{1/2})^L 2^{2k}\pi_3(2^k), \tag{7.50}$$

$k \ge s_L + 3m$.

Reckoning

Combining (7.40), (7.44) and (7.50), we find for ε small enough,

$$\mathbf{E}[\#\sigma(L) \mid E_k'] \le \varepsilon(4C_2C_3)^{L+1}\varepsilon^{(L-1)/2}2^{2k}\pi_3(2^k), \tag{7.51}$$

for $k \ge s_L + 3m = s_{L+1}$, from which we obtain (7.28) for $k \ge s_{L+1}$. □

8 Proof of Theorem 1.1

The proof of the main theorem uses a similar but simpler construction to that which appeared in Section 7.1, and follows that of the main derivation of [9]. For this reason, we omit some details.

Using (3.1), we first choose $\delta > 0$ small enough so that for n large, one has

$$n^{1+2\delta} \le n^2\pi_3(n), \tag{8.1}$$

and define the truncated box

$$\hat{B}(n) = B(n - n^\delta).$$

This box is chosen so that the total number of edges contained in $B(n) \setminus \hat{B}(n)$ is at most $Cn^{-\delta}n^2\pi_3(n)$, and so this region does not significantly contribute to the volume of the lowest crossing l_n. Around each $e \in \hat{B}(n) \cap l_n$, we will search for shortcuts between scales $n^{\delta/8}$ and $n^{\delta/4}$ which give a savings compared to l_n of at least n^{-c} for some $c > 0$.

Precisely, from Proposition 7.1, we may choose $a < 1$ so that for ε sufficiently small,

$$\mathbf{E}[\#\mathfrak{s}_j \mid E_j'] \le a^j 2^{2j}\pi_3(2^j) \quad \text{for all large } n \text{ and } j \in \left(\frac{\delta}{8}\log n, \frac{\delta}{4}\log n\right).$$

From this we conclude that for $c = \frac{\delta}{8}\log\frac{1}{a}$, one has

$$\mathbf{E}[\#\mathfrak{s}_j \mid E_j'] \le n^{-c}2^{2j}\pi_3(2^j) \quad \text{for all large } n \text{ and } j \in \left(\frac{\delta}{8}\log n, \frac{\delta}{4}\log n\right). \tag{8.2}$$

We next place n^{-c}-shortcuts (as in Definition 5.1) on the lowest crossing in a maximal way, like before. That is, we select a collection of such shortcuts with the property that their detoured paths do not share any vertices, and the total length of their detoured paths is maximal. We then let σ be the open path consisting of the union of these shortcuts and the portions of l_n that are left undetoured. Just as in Claim 1, any edge on the lowest crossing that is not covered by such a shortcut in σ has no such shortcut around it at all. Because the events $E_k(e) \cap \{e \in l_n\}$ imply the existence of shortcuts (Proposition 5.4), one can again place (8.2) into Propositions 5.6 and 6.2 (just as in (7.8)) to find $\eta > 0$ such that for all large n, and uniformly in $e \subset \hat{B}(n)$, the probability that an edge e of the lowest crossing is not covered by a detour in σ is at most

$$\mathbf{P}(\text{there is no } n^{-c}\text{-shortcut around } e \mid e \in l_n) \leq \mathbf{P}\left(\cap_{j=\lceil \frac{\delta}{8}\log n\rceil}^{\lfloor \frac{\delta}{4}\log n\rfloor} E_j(e,\varepsilon,n^{-c})^c \mid e \in l_n\right)$$

$$\leq 2^{-\frac{\hat{c}\frac{\delta}{16}\log n}{\log\frac{1}{\varepsilon}}} \leq n^{-\eta}. \tag{8.3}$$

Last, we write (τ_ℓ) for the collection of detoured paths in l_n and use (8.1) and (8.3) to estimate the expected length of σ for n large as

$$\begin{aligned}
\mathbf{E}[\#\sigma \mid H_n] &\leq Cn^{1+\delta} + n^{-c}\sum_{\ell}\mathbf{E}[\#\tau_\ell \cap \hat{B}(n) \mid H_n] \\
&\quad + \mathbf{E}[\#\{e \in l_n \cap \hat{B}(n) : e \text{ has no } n^{-c}\text{-shortcut}\} \mid H_n] \\
&\leq Cn^{1+\delta} + n^{-c}\mathbf{E}[\#l_n \cap \hat{B}(n) \mid H_n] + n^{-\eta}\mathbf{E}[\#l_n \cap \hat{B}(n) \mid H_n] \\
&\leq C[n^{-\delta} + n^{-c} + n^{-\eta}]n^2\pi(n).
\end{aligned}$$

Because $S_n \leq \#\sigma$, this completes the proof of Theorem 1.1. □

Appendix: Topological considerations

In this section we prove that the event $E_k(e,\varepsilon,\delta)$ implies the existence of κ-shortcut around e, as stated in Proposition 5.4. The item numbers refer to the definition of E_k' in Section 5.1.

Proof of (5.23). We proceed by first establishing several topological facts. The ultimate topological goal, from which the proof of the proposition will follow, is to identify the vertices of N_K from Proposition 5.2 as vertices of the lowest crossing l_n and to show that the open arc of item 9. in the definition of E_k' is disjoint from l_n and connects to both five-arm points in the prescribed way.

Let $\mathfrak{C}$ denote (a choice of) the closed dual circuit with two defects from item 15. above (chosen according to some deterministic, measurable rule). We let $\mathfrak{C}_1$ denote the lower of the two closed arcs between the defects. The term “lower” can be defined precisely as follows using (deterministic choices of) the open arms from item 16. above. Consider the Jordan curve consisting of the concatenation of the open arm from the left side of $B(e,2^k)$ to the left side of $B(e,2^K)$, the arc of $\partial B(e,2^k)$ between the endpoints of the two open arms and containing the top of $B(e,2^k)$, the open arm from the right side of $B(e,2^k)$ to the right side of $B(e,2^K)$, and the arc of $\partial B(e,2^K)$ between the endpoints of the two open arms and containing the top of $B(e,2^K)$. $\mathfrak{C}$ crosses this Jordan curve twice (at the defects described in item 15.), and so we can unambiguously define $\mathfrak{C}_1$ as the arc lying in the exterior of the Jordan curve.

Recall the characterization of the lowest crossing from Section 3.3. Recall the notation Q' for a dual version, with slightly altered dimensions, of a rectangle Q — see the paragraph above (3.4).

Proposition A.1. *On $\{e \in l_n\} \cap E_k(e, \varepsilon, \delta)$, the arc $\mathfrak{C}_1$ has a closed dual connection to the bottom side of $B(n)'$. Moreover, any closed dual path from the interior of $\mathfrak{C}$ to the bottom side of $B(e, 2^K)^*$ which touches $\partial B(e, 2^K)^*$ at exactly one dual vertex must intersect $\mathfrak{C}_1$, and any closed dual path from the interior of $\mathfrak{C}$ to the bottom side of $B(n)'$ which touches $B(n)'$ at exactly one dual vertex must intersect $\mathfrak{C}_1$.*

Proof. We prove the first of the two claims of the proposition. By (3.4), since $e \in l_n$, there must exist a closed dual arm from an endpoint of e^* to the bottom side of $B(n)'$. The claim will be proved once it is shown that this arm can be chosen to touch $\mathfrak{C}_1$. Consider the Jordan curve $\mathfrak{J}$ consisting of the concatenation of the paths in the itemized list below (in all cases, choices of representatives of each open/closed crossing are made according to a deterministic and measurable rule):

- the closed dual path $\mathfrak{C}_1$ traversed from its endpoint at the "right" defect (the defect in $[\frac{7}{8} \cdot 2^K, 2^K] \times [-\frac{2^K}{8}, \frac{2^K}{8}]$) to the "left" defect (the defect in $[-2^K, -\frac{7}{8} \cdot 2^K] \times [-\frac{2^K}{8}, \frac{2^K}{8}]$);
- the diagonal straight line segment of length $1/\sqrt{2}$ beginning at the endpoint of $\mathfrak{C}_1$ at the left defect and ending at the endpoint of the open edge crossing the defect which lies in the interior of $\mathfrak{C}$;
- the concatenation of the two open arms from item 16. above with their extensions to $\star_1$ and $\star_2$ in items 3. (e) and 4. (c), further concatenated with the open arc from item 9. above;
- the diagonal straight line segment of length $1/\sqrt{2}$ beginning at the endpoint of the open edge crossing the right defect of $\mathfrak{C}$ which lies in the interior of $\mathfrak{C}$ and ending at the endpoint of $\mathfrak{C}_1$ at the right defect.

Since e is in the interior of $\mathfrak{J}$ and $\partial B(n)'$ is in the exterior of $\mathfrak{J}$, this closed dual arm must cross $\mathfrak{J}$. It cannot cross any of the open edges of $\mathfrak{J}$, nor can it cross the relative interior of either of the diagonal straight line segments. Therefore, it must intersect $\mathfrak{C}_1$. We have shown the first claim made in the statement of the proposition.

The claim in the statement of the proposition involving dual paths to the bottom of $B(n)'$ is proved in essentially the same way, replacing the role of e in the above argument with the initial vertex of the dual path. In order to run the argument as above, one need only show that the initial vertex of the dual path must be in the interior of $\mathfrak{J}$.

If this dual vertex did not lie in the interior of $\mathfrak{J}$, it would have to lie in the intersection of the interior of $\mathfrak{C}$ with the exterior of $\mathfrak{J}$. An easy Jordan curve argument shows that this is impossible. Indeed, recall that since $e \in l_n$, the two defects at the endpoints of $\mathfrak{C}_1$ have open arms to the left and right side of $B(n)$. Since $\mathfrak{C}_1$ has a closed arm to the bottom side of $B(n)'$, the open arm from the left defect must go to the left side of $B(n)$, and similarly with the right defect. Defining a Jordan curve using these open arms, the open portions of $\mathfrak{J}$, and the appropriate portions of the sides and top segment of $\partial B(n)$, one sees that any closed dual path starting in $(\operatorname{int}\mathfrak{C}) \cap \operatorname{ext}(\mathfrak{J})$ and touching the bottom of $B(n)'$ must cross this Jordan curve, an impossibility.

The proof of the remaining claim is quite similar; we describe only the essential changes to be made. Instead of $\mathfrak{J}$, we consider a Jordan curve $\mathfrak{J}'$ consisting of $\mathfrak{C}_1$, the portions of the open arms from item 16. above from the defects of $\mathfrak{C}$ to $\partial B(e,2^K)$ (along with appropriate diagonal line segments to connect with $\mathfrak{C}$), and the arc of $\partial B(e,2^K)$ connecting the endpoints of the open arms and containing the bottom side of $\partial B(e,2^K)$. Let γ be a closed dual arc from the bottom of $\partial B(e,2^K)^*$ (in particular, the exterior of $\mathfrak{J}'$) to the interior of $\mathfrak{C}$ and follow γ starting at $\partial B(e,2^K)^*$. Then γ must cross $\mathfrak{J}'$, and the first intersection happens no sooner than the far endpoint of its first dual edge, at which point γ has already entered the interior of $\mathfrak{J}'$. If this crossing happened via γ crossing a primal edge between vertices of $\partial B(e,2^K)$, then γ would intersect $\partial B(e,2^K)^*$ again, a contradiction. □

A direct consequence is that all the edges counted in N_K lie on l_n.

Corollary A.2. *On the event $\{e \in l_n\} \cap E_k(e,\varepsilon,\delta)$, consider the open edges in R connected to the open paths from item 13. by two vertex-disjoint open paths inside R which moreover are connected inside R by a dual closed path to the dual path in item 14. In other words, these are the edges counted in N_K from Proposition 3.3. Each such open edge is an edge of l_n, and the segment of l_n passing through any such edge does so between passing through the five-arm point $\star_1$ (taking the edge $\{\star_1, \star_1 + \mathbf{e}_1\}$) and the five-arm point $\star_2$ (taking the edge $\{\star_2, \star_2 - \mathbf{e}_1\}$).*

Proof. The dual crossing defined in item 14. touches the bottom of $B(e,2^K)$ by definition. If f is an edge as in the statement of the corollary, then this implies f^* has a closed dual arm to the bottom of $B(e,2^K)$. Moreover, f is in the interior of $\mathfrak{C}$ by construction. Thus, by Proposition A.1, f^* has a closed dual arm to $\mathfrak{C}_1$ and $\mathfrak{C}_1$ has a closed dual arm to the bottom of $B(n)'$; in particular, f^* has a closed dual arm to the bottom of $B(n)'$.

The fact that f is an edge of l_n will follow via (3.4) once we show that the endpoints of f have disjoint open arms to the sides of $\partial B(n)$. Note that the endpoints of f have disjoint open arms to the defects in $\mathfrak{C}$, obtained by following their disjoint connections to the open paths in item 13., then following these paths to $\star_1$ and $\star_2$, then following the open paths from $\star_1$ and $\star_2$ to the paths from item 16. above and then these paths to the defects in $\mathfrak{C}$.

The remaining portions of the open arms from f are furnished by segments of open arms from e. Since $e \in l_n$, it must have two disjoint open arms to the left and right side of $\partial B(n)$. Since e is in the interior of $\mathfrak{C}$, these arms must cross $\mathfrak{C}$; these crossings must occur at the two defects. Thus, we can use the portions of these arms after the defects on $\mathfrak{C}$ to complete the disjoint open arms from f to the left and right sides of $B(n)$.

We now show that any pair of disjoint open arms from f to $\partial B(n)$ must pass through $\star_1$ and $\star_2$ (one arm through each five-arm point). By construction, f is in the interior of another closed dual circuit with two defects (lying entirely in the interior of $\mathfrak{C}$). This dual circuit consists of the shield arc from item 8. above and the dual closed paths from items 10. and 11., corresponding to the union of the red dashed path from Figure 5.4 and the green dashed path from Figure 5.4. (To see that f is in the interior, note it is easy to construct a path from f to the boundary of D_1 from item 10. that passes through this circuit only once, then to extend this path to infinity without touching the circuit again.) Again, by planar duality, the crossings must occur at the defects — in other words, at the five-arm points $\star_1$ and $\star_2$.

We now explain why the lowest crossing uses the claimed edges at the two five-arm points; for brevity, we explain only the case of $\star_2$. By duality and the presence of the closed dual paths from $\star_2$, the segment of the lowest crossing from the right defect in the closed dual circuit of item 5. above must pass through either $\{\star_2 - \mathbf{e}_1, \star_2\}$ or $\{\star_2 - \mathbf{e}_2, \star_2\}$. Since the latter is closed (by the definition of a five-arm point), the lowest crossing must pass through the former. $\square$

It remains to use the above results to prove Proposition 5.4; we will show that $\mathfrak{s}_k \in \mathscr{S}(e, \varepsilon \cdot \delta)$. Since $\mathfrak{s}_k$ was defined to be the shortest open crossing between $\star_1$ and $\star_2$ disjoint (except for the vertices $\star_1$ and $\star_2$) from the lowest crossing of $B(n)$, by the characterization of the lowest crossing in Section 3.3 it must avoid the edges $\{\star_1, \star_1 + \mathbf{e}_1\}$, $\{\star_1, \star_1 - \mathbf{e}_1\}$, $\{\star_2, \star_2 + \mathbf{e}_1\}$, and $\{\star_2, \star_2 + \mathbf{e}_2\}$ (which lie on the lowest crossing). Since $\{\star_i, \star_i - \mathbf{e}_2\}$ is closed for $i = 1, 2$, the path $\mathfrak{s}_k$ must pass through $\{\star_i, \star_i + \mathbf{e}_2\}$. This establishes item 2. of the definition of κ-shortcuts (Definition 5.1). Item 3. of that definition follows from the assumed disjointness of l_n and $\mathfrak{s}_k$ and the fact that l_n passes through both $\star_1$ and $\star_2$ exactly once (and e is on the segment of l_n between these points).

To see that item 1. of Definition 5.1 holds, note that we can construct a curve starting at $\star_2 + \mathbf{e}_2$, passing through l_n once at the edge $\{\star_2, \star_2 + \mathbf{e}_2\}$, and (after taking a diagonal line segment to the appropriate dual neighbor of $\star_2$) following the closed dual arm from $\star_2$ to $\mathfrak{C}_1$, and then a dual path onward to the bottom of $B(n)'$ (by Proposition A.1). This crosses l_n exactly once, so $\star_2 + \mathbf{e}_2$ lies in $B(n) \setminus \mathscr{B}(l_n)$. The fact that $\mathfrak{s}_k$ is disjoint from l_n except at the five-arm points now gives that the midpoint of every edge of $\mathfrak{s}_k$ is in $B(n) \setminus \mathscr{B}(l_n)$, which is item 1. of the definition.

Item 4. of Definition 5.1 is immediate from the construction: the required dual path is furnished by using the closed dual paths from $\star_1$ and $\star_2$ as extended in items 10. and 11. above (see the red path from Figure 5.4). $\square$

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