Castelnuovo's bound and rigidity in almost complex geometry

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Abstract

This article is concerned with the question of whether an energy bound implies a genus bound for pseudo-holomorphic curves in almost complex manifolds. After reviewing what is known in dimensions other than six, we establish a new result in this direction in dimension six; in particular, for symplectic Calabi–Yau 3–folds. The proof relies on compactness and regularity theorems for pseudo-holomorphic currents.

1 Introduction

In 1889, Castelnuovo [Cas89] found a sharp upper bound for the genus of an irreducible, nondegenerate curve of a given degree in $\mathbf{C}P^n$; see [ACGH85, Chapter III Section 2] for a proof in modern language. A corollary of this result is that for every projective variety there is an upper bound for the genus of an irreducible curve representing a given homology class. Our starting point is the question:

Are there analogues of Castelnuovo's bound in almost complex geometry?

For curves in $\mathbf{C}P^2$ Castelnuovo's bound reduces to the degree-genus formula. The latter is a consequence of the adjunction formula, which generalizes to an inequality for almost complex 4–manifolds [MS12, Theorem 2.6.4]. The adjunction inequality directly implies the following well-known genus bound.

Proposition 1.1. *Suppose that (M, J) is an almost complex 4–manifold. If there exists a simple J–holomorphic map $u\colon \Sigma \to M$ representing $A \in H_2(M)$, then the genus $g(\Sigma)$ satisfies*

$$g(\Sigma) \leqslant \frac{1}{2}\left(A \cdot A - \langle c_1(M, J), A\rangle\right) + 1. \tag{1.2}$$

The following is a consequence of Gromov's h-principle for symplectic embeddings [Gro86, Section 3.4.2 Theorem (A)]. It shows that in higher dimensions there cannot be a genus bound which holds for all almost complex structures.

Proposition 1.3. *Let (M, ω) be a symplectic manifold of dimension $2n \geqslant 6$. For every $A \in H_2(M)$ with $\langle [\omega], A\rangle > 0$ and every $g \in \mathbf{N}$ there is an almost complex structure J compatible with ω and a J–holomorphic embedding $u\colon \Sigma \to M$ satisfying*

$$g(\Sigma) \geqslant g.$$

There are, however, genus bounds for *generic* almost complex structures. Here is a simple example, which follows easily from the index formula and transversality theorem for simple J–holomorphic maps [MS12, Chapter 3].

Proposition 1.4. *Let M be a manifold of dimension $2n$. Denote by $\mathcal{J}$ the space of smooth almost complex structures on M equipped with the C^∞_{loc}–topology. There is a comeager[1] subset $\mathcal{J}_\clubsuit \subset \mathcal{J}$ such that for every $J \in \mathcal{J}_\clubsuit$ the following holds: if there exists a simple J–holomorphic map $u\colon \Sigma \to M$ representing $A \in H_2(M)$, then*

$$\begin{cases} \langle c_1(M,J), A\rangle \geqslant 0 & \text{if } n = 3 \\ g(\Sigma) \leqslant \dfrac{\langle c_1(M,J), A\rangle}{n-3} + 1 & \text{if } n > 3. \end{cases} \tag{1.5}$$

Moreover, if M carries a symplectic form ω, then the same holds with $\mathcal{J}$ replaced by the space $\mathcal{J}(\omega)$ of smooth almost complex structures compatible with ω.

Proposition 1.3 and Proposition 1.4 are both well-known and we omit their proofs.

The preceding discussion leaves open the case of generic almost complex structures in dimension six and homology classes satisfying $\langle c_1(M,J), A\rangle \geqslant 0$. In the present article, we focus on the case

$$\langle c_1(M,J), A\rangle = 0,$$

that is: on classes for which the corresponding moduli space of J–holomorphic maps has expected dimension zero. This includes all homology classes in symplectic Calabi–Yau 3–folds, that is: symplectic manifolds (M,ω) such that $\dim M = 6$ and $c_1(M,J) = 0$ for some almost complex structure J compatible with ω. Our motivation for considering this case comes from our project to construct a symplectic analogue of the Pandharipande–Thomas invariants of projective Calabi–Yau 3–folds [DW19b, Section 7]. Another motivation comes from the Gopakumar–Vafa conjecture. Bryan and Pandharipande [BP01] defined the Gopakumar–Vafa BPS invariants $n_A^g(M,\omega)$ of a symplectic Calabi–Yau 3–fold (M,ω) in terms of its Gromov–Witten partition function. They conjectured that the BPS invariants $n_A^g(M,\omega)$ are integers and vanish for all but finitely many g [BP01, Conjecture 1.2]. The integrality conjecture has been proved by Ionel and Parker [IP18]. The finiteness conjecture remains open and is closely related to the question about the existence of genus bounds for symplectic Calabi–Yau 3–folds.

Motivated by Gromov–Witten theory, Bryan and Pandharipande [BP01] introduced the notion of *k–rigidity* for almost complex structures; see Definition 2.10. They conjectured that a generic almost complex structure is *∞–rigid* (or *super-rigid*), that is: k–rigid for every $k \in \mathbf{N}$. This has recently been proved by Wendl [Wen19b]; see Theorem 2.13. A concise exposition of Wendl's proof using the framework of equivariant Brill–Noether theory for elliptic operators can be found in [DW18].

The main result of this article shows that k–rigidity implies a Castelnuovo bound.

Theorem 1.6. *Let $k \in \mathbf{N} \cup \{\infty\}$. Let (M,J,g) be a compact almost Hermitian 6–manifold with a k–rigid almost complex structure J. Suppose $A \in H_2(M)$ satisfies $\langle c_1(M,J), A\rangle = 0$ and has*

[1] Let X be a topological space. A subset $A \subset X$ is called **comeager** (or **residual**) if it contains the intersection of countably many dense open subsets. A comeager subset of a complete metric space is dense.

divisibility at most k. Given any $\Lambda > 0$, there are only finitely many simple J–holomorphic maps representing A and with energy at most Λ.

Remark 1.7. Theorem 1.6 immediately implies a Castelnuovo bound for every *fixed* k–rigid almost complex structure J. Unlike in the $n > 3$ case of (1.5), however, this bound may depend on J. ♣

If J is tamed by a symplectic form ω, then imposing an upper bound for the energy is superfluous since the energy of any J–holomorphic map representing A is $\langle[\omega], A\rangle$.

Corollary 1.8. *Let (M, ω) be a compact symplectic Calabi–Yau 3–fold. Suppose J is a super-rigid almost complex structure compatible with ω. Then for every $A \in H_2(M)$ there are only finitely many simple J–holomorphic maps representing A.* ■

In the situation of Theorem 1.6, Gromov's compactness theorem [Gro85; PW93; Ye94; Hum97] shows that there are only finitely many J–holomorphic maps representing A from Riemann surfaces of *fixed genus*. It is thus of no use for proving Theorem 1.6. Instead, we use the following compactness result for *J–holomorphic cycles*, that is: formal sums of J–holomorphic curves, with respect to *geometric convergence*; see (1) and Definition 5.3.

Proposition 1.9. *Let M be a manifold and let $(J_n, g_n)_{n\in\mathbf{N}}$ be a sequence of almost Hermitian structures converging to an almost Hermitian structure (J, g) in the C^∞_{loc}–topology. Let $K \subset M$ be a compact subset and let $\Lambda > 0$. For each $n \in \mathbf{N}$ let C_n be a J_n–holomorphic cycle with support contained in K and of mass at most Λ. Then a subsequence of $(C_n)_{n\in\mathbf{N}}$ geometrically converges to a J–holomorphic cycle C.*

In dimension four, this result was proved by Taubes [Tau96a]. The proof in higher dimensions relies on results in geometric measure theory; in particular, the recent work of De Lellis, Spadaro, and Spolaor [DSS17b; DSS18; DSS17a; DSS20] on the regularity of semi-calibrated currents. The points of this theory most relevant to the present article are discussed in Section 4.

Remark 1.10. If (M, ω) is a symplectic manifold, $(J_n)_{n\in\mathbf{N}}$ is a sequence of ω–compatible almost complex structures, and $g_n = \omega(\cdot, J_n\cdot)$ is the corresponding sequence of Riemannian metrics, then Proposition 1.9 can be proved using earlier work of Rivière and Tian [RT09] on the regularity of *calibrated* currents. However, the proof of Theorem 1.6 leads to almost complex structures which are tamed by but (possibly) not compatible with a symplectic structure. Therefore, the work of De Lellis, Spadaro, and Spolaor is crucial even for Corollary 1.8. ♣

Remark 1.11. Since the first version of this article appeared, we used the $k = 1$ case of Theorem 1.6 to prove the Gopakumar–Vafa finitness conjecture for the BPS numbers $n^g_A(M, \omega)$ whenever A is a primitive homology class [DW19a]. The cited article also contains a version of Theorem 1.6 for homology classes satisfying $\langle c_1(M, \omega), A\rangle > 0$. ♣

Convention 1.12. Throughout this article, $f(x) \lesssim g(x)$ is an abbreviation for: $f(x) \leqslant cg(x)$ with a constant $c > 0$ independent of x.

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2 k–rigidity of J–holomorphic maps

Let us briefly recall the notion of k–rigidity as defined by Eftekhary. For a more detailed discussion we refer the reader to [Eft16, Section 2; Wen19b, Section 2.1] as well as [DW18, Section 2.1]. The notation and definitions in this article are consistent with those used in the last reference.

Henceforth, let (M, J, g) be an almost Hermitian $2n$–manifold; that is: J is an almost complex structure and g is a Riemannian metric such that $g(J\cdot, J\cdot) = g(\cdot,\cdot)$. In particular, we *do not* assume that the 2–form $g(J\cdot,\cdot)$ is closed or that M even admits a symplectic structure.

Definition 2.1. A **J–holomorphic map** $u\colon (\Sigma, j) \to (M, J)$ is a pair consisting of a closed, connected Riemann surface (Σ, j) and a smooth map $u\colon \Sigma \to M$ satisfying the non-linear Cauchy–Riemann equation

$$\bar\partial_J(u, j) \coloneqq \frac{1}{2}(\mathrm{d}u + J(u) \circ \mathrm{d}u \circ j) = 0. \tag{2.2}$$ •

Definition 2.3. Let $u\colon (\Sigma, j) \to (M, J)$ be a J–holomorphic map. Let $\phi \in \mathrm{Diff}(\Sigma)$ be a diffeomorphism. The **reparametrization** of u by ϕ is the J–holomorphic map $u \circ \phi^{-1}\colon (\Sigma, \phi_* j) \to (M, J)$. •

Definition 2.4. Let $u\colon (\Sigma, j) \to (M, J)$ be a J–holomorphic map and let $\pi\colon (\tilde\Sigma, \tilde j) \to (\Sigma, j)$ be a holomorphic map of degree $\deg(\pi) \geqslant 2$. The composition $u \circ \pi\colon (\tilde\Sigma, \tilde j) \to (M, J)$ is said to be a **multiple cover of** u. A J–holomorphic map is **simple** if it is not constant and not a multiple cover. •

Rigidity and k–rigidity are conditions on the infinitesimal deformation theory of J–holomorphic curves up to reparametrization. We will have to briefly review parts of this theory. The reader can find further details in [MS12, Chapter 3] and [Wen19a, Lectures 2 and 7], for example. The second reference, in particular, discusses varying the complex structure on higher genus Riemann surfaces.

The index of a J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$ is defined as

$$\mathrm{index}(u) \coloneqq 2\langle u^* c_1(M, J), [\Sigma]\rangle + (2n - 6)(1 - g(\Sigma)). \tag{2.5}$$

This is the Fredholm index of the linearization of (2.2) with respect to the map u and complex structure j (up to equivalence). The linearization with respect to u, with j fixed, is the operator

$$\xi \mapsto \frac{1}{2}\big(\nabla\xi + J \circ (\nabla\xi) \circ j + (\nabla_\xi J) \circ \mathrm{d}u \circ j\big). \tag{2.6}$$

Here ∇ denotes the Levi–Civita connection of g on TM and also the induced connection on u^*TM [MS12, Proposition 3.1.1].

Let $u\colon (\Sigma, j) \to (M, J)$ be a non-constant J–holomorphic map. There exists a unique complex subbundle

$$Tu \subset u^*TM$$

of rank one containing $\mathrm{d}u(T\Sigma)$ [IS99, Section 1.3]. The generalized normal bundle of u is defined as

$$Nu \coloneqq u^*TM/Tu.$$

If u is an immersion, then Nu is the usual normal bundle. If $\tilde{u} = u \circ \pi$ is a multiple cover of an immersion, then $N\tilde{u} = \pi^*Nu$. The operator (2.6) maps $\Gamma(Tu)$ to $\Omega^{0,1}(\Sigma, Tu)$. Thus, it induces an operator

$$\mathfrak{d}^N_{u,J}\colon \Gamma(Nu) \to \Omega^{0,1}(Nu) \tag{2.7}$$

called the **normal Cauchy–Riemann operator of** u [IS99, (1.5.1)]. The non-zero elements of the kernel of $\mathfrak{d}^N_{u,J}$ correspond to infinitesimal deformations of u which deform the image $u(\Sigma)$. The reader might find the summaries of Ivashkovich and Shevchishin's construction of Tu, Nu, and $\mathfrak{d}^N_{u,J}$ given in [Wen10, Section 3.3; DW18, Appendix 2A] helpful.

Definition 2.8. A non-constant J–holomorphic map u is **rigid** if $\ker \mathfrak{d}^N_{u,J} = 0$. •

A multiple cover $\tilde{u}$ of u may fail to be rigid, even if u itself is rigid.

Definition 2.9. Let $k \in \mathbf{N} \cup \{\infty\}$. A simple J–holomorphic map $u\colon (\Sigma, j) \to (M, J)$ is called k**–rigid** if it is rigid and all of its multiple covers of degree at most k are rigid. •

Rigidity and k–rigidity are mostly interesting for maps of index zero, as it follows from the index formula for the normal Cauchy–Riemann operator [IS99, Lemma 1.5.1] (see also [Wen10, Theorem 3; DW18, Proposition 2.7.1]) and standard transversality results that for a generic J there are no rigid simple J–holomorphic maps satisfying $\operatorname{index}(u) \neq 0$.

Definition 2.10. Suppose that $\dim M \geqslant 6$. Let $k \in \mathbf{N} \cup \{\infty\}$. An almost complex structure J is called k**–rigid** if the following hold:

(1) Every simple J–holomorphic map of index zero is k–rigid.

(2) Every simple J–holomorphic map has non-negative index.

(3) Every simple J–holomorphic map of index zero is an embedding, and every two simple J–holomorphic maps of index zero either have disjoint images or are related by a reparametrization. •

Remark 2.11. In dimension four, one should weaken (3) and require only that every simple J–holomorphic map of index zero is an immersion with transverse self-intersections, and that two such maps are either transverse to one another or are related by reparametrization. However, we will only be concerned with dimension (at least) six. ♣

Definition 2.12. Denote by $\mathscr{J}(M)$ the Fréchet space of smooth almost complex structures on M equipped with the topology of C^∞ convergence over compact subsets. If ω is a symplectic form on M, denote by $\mathscr{J}(M,\omega)$ the subspace of almost complex structures in $\mathscr{J}(M)$ compatible with ω. For $k \in \mathbf{N} \cup \{\infty\}$ set

$$\mathscr{R}_k(M) := \{J \in \mathscr{J}(M) : J \text{ is } k\text{-rigid}\} \quad \text{and} \quad \mathscr{R}_k(M,\omega) := \mathscr{R}_k(M) \cap \mathscr{J}(M,\omega). \qquad \bullet$$

Theorem 2.13 (Wendl [Wen19b, Theorem A]). *Let M be a manifold of dimension at least six. The following hold:*

(1) *The subset $\mathscr{R}_\infty(M)$ is comeager in $\mathscr{J}(M)$.*

(2) *The subset $\mathscr{R}_\infty(M,\omega)$ is comeager in $\mathscr{J}(M,\omega)$.*

This result establishes the *super-rigidity conjecture* of Bryan and Pandharipande [BP01, p. 290]. Earlier progress on this conjecture was made by Eftekhary [Eft16, Theorem 1.2] who showed that $\mathscr{R}_4(M,\omega)$ is comeager in $\mathscr{J}(M,\omega)$ if $\dim M = 6$.

3 Real Cauchy–Riemann operators and almost complex structures

The purpose of this section is to explain that associated with every real Cauchy–Riemann operator defined on a Hermitian vector bundle there is a natural almost complex structure on the total space of that bundle. This construction is inspired by [Tau96b, p. 825–826]. In fact, the results below can be found in [Zin11, Section 3.2; Wen19b, Appendix B]. Nevertheless, we include them here for the reader's convenience.

Definition 3.1. Let (Σ, j) be a Riemann surface. Let $\pi\colon E \to \Sigma$ be a Hermitian vector bundle over Σ. A first order linear differential operator $\mathfrak{d}\colon \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ is called a **real Cauchy–Riemann operator** if

$$\mathfrak{d}(fs) = (\bar\partial f)s + i\mathfrak{d}j \tag{3.2}$$

for all $f \in C^\infty(M, \mathbf{R})$. The **anti-linear part** of $\mathfrak{d}$ is defined as

$$\mathfrak{n} = \mathfrak{n}_{\mathfrak{d}} := \frac{1}{2}(\mathfrak{d} + J\mathfrak{d}J) \in \Gamma(\mathrm{Hom}(E, \overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, E))). \qquad \bullet.$$

Every real Cauchy–Riemann operator can be written as

$$\mathfrak{d} = \bar\partial + \mathfrak{n}$$

where $\bar\partial_\nabla := \nabla^{0,1}$ is the Dolbeault operator associated with a Hermitian connection ∇ on E. Denote by $H_\nabla \subset TE$ the horizontal distribution of ∇. It induces an isomorphism

$$TE = H_\nabla \oplus \pi^*E \cong \pi^*T\Sigma \oplus \pi^*E. \tag{3.3}$$

Definition 3.4. The **complex structure** J_∇ on E associated with ∇ is defined by pulling back the standard complex structure $j \oplus i$ on $\pi^*T\Sigma \oplus \pi^*E$ by the isomorphism (3.3). $\bullet$

It is well-known that a section $s \in \Gamma(E)$ satisfies $\bar\partial_\nabla s = 0$ if and only if the map $s\colon \Sigma \to E$ is J_∇–holomorphic. The following proposition extends this to real Cauchy–Riemann operators.

Definition 3.5. Let $\mathfrak{d} = \bar\partial_\nabla + \mathfrak{n}$ be a real Cauchy–Riemann operator. Define $L_\mathfrak{n}\colon TE \to TE$ by

$$L_\mathfrak{n} = -2\mathfrak{n}(v)j\pi_*$$

at $v \in E$. The **almost complex structure** $J_\mathfrak{d}$ on E associated with $\mathfrak{d}$ is defined by

$$J_\mathfrak{d} := J_\nabla + L_\mathfrak{n}. \qquad \bullet$$

Lemma 3.6. *For every real Cauchy–Riemann operator* $\mathfrak{d}\colon \Gamma(E) \to \Omega^{0,1}(E)$ *the following hold:*

(1) $J_\mathfrak{d}$ *is an almost complex structure.*

(2) *The projection* $\pi\colon E \to \Sigma$ *is holomorphic with respect to* $J_\mathfrak{d}$.

(3) *For every* $x \in \Sigma$ *the fiber* $E_x = \pi^{-1}(x)$ *is a* $J_\mathfrak{d}$*–holomorphic submanifold of* E.

(4) *A section* $s \in \Gamma(E)$ *satisfies* $\mathfrak{d}s = 0$ *if and only if* $s\colon \Sigma \to E$ *is a* $J_\mathfrak{d}$*–holomorphic map.*

(5) *Denote by* $B_1(E) := \{e \in E : |e| < 1\}$ *the disc bundle of* E *with respect to the given Hermitian inner product on* E*. There exists a symplectic form* ω *on the total space of* $B_1(E)$ *which tames* $J_\mathfrak{d}$.

Proof. With respect to (3.3) we have

$$J_\mathfrak{d} = \begin{pmatrix} j & 0 \\ -2\mathfrak{n}(v)j & i \end{pmatrix} \tag{3.7}$$

at $v \in E$. Since $\mathfrak{n}(v)$ is anti-linear,

$$\mathfrak{n}(v)j^2 + i\mathfrak{n}(v)j = 0.$$

Therefore,

$$J_\mathfrak{d}^2 = -\mathrm{id};$$

that is, (1) holds.

Both (2) and (3) immediately follow from (3.7).

We prove (4). Let $s\colon \Sigma \to E$ be a section. The projection of $\mathrm{d}s$ to the first factor of (3.3) is $\pi_* \circ \mathrm{d}s = \mathrm{id}_{T\Sigma}$ and thus j–linear. The projection of $\mathrm{d}s\colon T\Sigma \to s^*TE$ to the second factor is its covariant derivative $\nabla s\colon T\Sigma \to s^*E$. It follows from (3.7) that the $J_\mathfrak{d}$–antilinear part of $\mathrm{d}s$ is

$$\begin{aligned} \frac{1}{2}(\mathrm{d}s + J_\mathfrak{d} \circ \mathrm{d}s \circ j) &= \frac{1}{2}(\nabla s + i \circ \nabla s \circ j) + \mathfrak{n}s \\ &= \bar\partial_\nabla s + \mathfrak{n}s = \mathfrak{d}s. \end{aligned}$$

Therefore, $\mathrm{d}s\colon T\Sigma \to TE$ is $J_\mathfrak{d}$–linear if and only if $\mathfrak{d}s = 0$.

The construction of a symplectic form ω in (5) is standard and goes back to Thurston [Thu76]; see also [Gom95, Lemma 2.2; MS98, Theorem 6.3; TMZ18, paragraph containing (2.9)]. Nevertheless, let us discuss the proof of (5). Let ω_Σ be an area form on Σ. Let ω_E be any closed 2–form on $B_1(E)$ which is positive when restricted to the fibers of E; that is, for all vertical tangent vectors v_E

$$\omega_E(v_E, J_\nabla v_E) \gtrsim |v_E|^2. \tag{3.8}$$

Such a form can be constructed by choosing local unitary trivializations $E|_{U_i} \cong U_i \times \mathbf{C}^r$, denoting by λ_i the corresponding Liouville 1–forms on $\mathbf{C}^r$ vanishing at zero, and setting

$$\omega_E = \mathrm{d}\left(\sum_i \chi_i \circ \pi \cdot \lambda_i\right)$$

for a partition of unity (χ_i). This form satisfies (3.8) on E. It remains to show that for $\tau \gg 1$ the closed 2–form $\omega = \tau\omega_\Sigma + \omega_E$ tames $J_\mathfrak{d}$ on $B_1(E)$. For a tangent vector w to E at a point $(x, v) \in B_1(E)$ denote by w_H and w_E its horizontal and vertical parts in the decomposition (3.3). We have

$$\begin{aligned}\omega(w, J_\mathfrak{d} w) &= (\tau\omega_\Sigma + \omega_E)(w, (J_\nabla + L_\mathfrak{n})w) \\ &= \tau\omega_\Sigma(w_H, jw_H) + \omega_E(w_E, J_\nabla w_E) + \omega_E(w_E, L_\mathfrak{n} w_H).\end{aligned}$$

From $|L_\mathfrak{n}(v)| \lesssim |v| < 1$ it follows that

$$|\omega_E(w_E, L_\mathfrak{n} w_H)| \lesssim |w_E||w_H|.$$

Since

$$\tau\omega_\Sigma(w_H, jw_H) + \omega_E(w_E, J_\nabla w_E) \gtrsim \tau|w_H|^2 + |v_E|,$$

it follows that ω tames $J_\mathfrak{d}$ provided $\tau \gg 1$. ∎

The next two propositions are concerned with the following situation. Let (M, J, g) be an almost Hermitian manifold and let $u\colon (\Sigma, j) \to (M, J)$ be a J–holomorphic embedding. Denote by $Nu \to \Sigma$ its normal bundle and by $\mathfrak{d}^N_{u,J}$ the normal Cauchy–Riemann operator introduced in (2.7). The almost complex structure J and Riemannian metric g on M induce a Hermitian structure on E. Write

$$J_u \coloneqq J_{\mathfrak{d}^N_{u,J}} \tag{3.9}$$

for the almost complex structure on the total space of Nu associated with $\mathfrak{d}^N_{u,J}$.

Lemma 3.10. *For every $\lambda > 0$ define $\sigma_\lambda\colon Nu \to Nu$ by*

$$\sigma_\lambda(v) \coloneqq \lambda v.$$

If $U \subset Nu$ is an open neighborhood of the zero section in Nu such that the exponential map $\exp\colon U \to M$ with respect to g is an embedding, then

$$\sigma_\lambda^* \exp^* J \to J_u \quad \text{as} \quad \lambda \to 0,$$

where the convergence is with respect to the C^∞_{loc}–topology.

Proof. Denote by ∇ the connection on $Nu \to \Sigma$ induced by the Levi–Civita connection of (M, g). Throughout this proof, we identify

$$TU = \pi^* T\Sigma \oplus \pi^* Nu$$

as in (3.3). The two almost complex structures J_∇ and $\exp^* J$ on $U \subset Nu$ agree along the zero section. The Taylor expansion of $\exp^* J$ is of the form

$$\exp^* J(x, v) = J_\nabla(x, 0) + \nabla_v J(x, 0) + O(|v|^2). \tag{3.11}$$

Set

$$L(x, v) := \nabla_n J(x, 0).$$

We write L as the matrix

$$L(x, v) = \begin{pmatrix} L_{11}(x, v) & L_{12}(x, v) \\ L_{21}(x, v) & L_{22}(x, v) \end{pmatrix}.$$

Here each L_{ij} is linear in v. The derivative $\mathrm{d}\sigma_\lambda$ is given by

$$\mathrm{d}\sigma_\lambda = \begin{pmatrix} \mathrm{id} & \\ & \lambda \end{pmatrix}.$$

Therefore,

$$\begin{aligned} (\sigma_\lambda)^* L(x, v) &= \begin{pmatrix} \mathrm{id} & \\ & \lambda^{-1} \end{pmatrix} \begin{pmatrix} L_{11}(x, \lambda v) & L_{12}(x, \lambda v) \\ L_{21}(x, \lambda v) & L_{22}(x, \lambda v) \end{pmatrix} \begin{pmatrix} \mathrm{id} & \\ & \lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda L_{11}(x, v) & \lambda^2 L_{12}(x, v) \\ L_{21}(x, v) & \lambda L_{22}(x, v) \end{pmatrix}. \end{aligned}$$

As λ tend to zero, all but the bottom left entry tend to zero.

By construction, $\sigma_\lambda^* J_\nabla = J_\nabla$. As λ tends to zero, the rescalings of terms of second order and higher in (3.11) tend to zero. It remains to identify the term L_{21}. By definition,

$$L_{21}(x, v) = \pi_{Nu} \circ \nabla_v J(x, 0) \circ \pi_*.$$

Comparing (2.6), Definition 3.1, and Definition 3.5, we see that $L_{21} = L_u$. This finishes the proof. ∎

Proposition 3.12. *If $\tilde{u} : (\tilde{\Sigma}, \tilde{j}) \to (Nu, J_u)$ is a simple J_u–holomorphic map whose image is not contained in the zero section, then the following hold:*

1. *The map $\varphi : (\tilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ given by $\varphi := \pi \circ \tilde{u}$ is non-constant and holomorphic.*
2. *The J–holomorphic map $u \circ \varphi : (\tilde{\Sigma}, \tilde{j}) \to (M, J)$ is not rigid; in particular, the J–holomorphic map $u : (\Sigma, j) \to (M, J)$ is not k–rigid for $k = \deg(\varphi)$.*

Proof. By Lemma 3.6 (2), $\pi\colon Nu \to \Sigma$ is J_u–holomorphic. Therefore, φ is holomorphic. The map φ is constant if and only if the image of $\tilde{u}$ is contained in a fiber of π. This is impossible, because then $\tilde{u}$ would be constant. This proves (1).

To prove (2), we use the fact that the normal bundle of the J–holomorphic map $u \circ \varphi$ is $Nu \circ \varphi = \varphi^* Nu$ and the corresponding normal Cauchy–Riemann operator is

$$\mathfrak{d}^N_{u\circ\varphi,J} = \varphi^* \mathfrak{d}^N_{u,J}. \tag{3.13}$$

Since $\tilde{u}$ takes values in Nu, for every $x \in \tilde{\Sigma}$ we have

$$\tilde{u}(x) \in Nu_{\pi(u(x))} = Nu_{\varphi(x)} = (\varphi^* Nu)_x.$$

This gives rise to the section $s \in \Gamma(\varphi^* Nu)$ defined by

$$s(x) \coloneqq \tilde{u}(x) \in (\varphi^* Nu)_x.$$

This section is not the zero section, because the image of $\tilde{u}$ is not contained in the zero section. By construction, s is holomorphic with respect to the almost complex structure on $\phi^* Nu$ induced from J_u. Lemma 3.6 (4) and (3.13) imply that

$$\mathfrak{d}^N_{u\circ\varphi,J} s = 0. \qquad \blacksquare$$

4 Regularity theory for 2–dimensional semicalibrated currents

The purpose of this section is to introduce a few notions of geometric measure theory and explain Theorem 4.13 due to De Lellis, Spadaro, and Spolaor. The standard reference for the foundations of geometric measure theory is Federer's voluminous monograph [Fed69]. The references [Sim83; De 16] are more accessible and easier to navigate, and are cited throughout this section.

Definition 4.1 (cf. [Sim83, Chapter 6 Definition 26.1 and Paragraphs 26.3, 26.10, 26.11; De 16, Definitions 2.1, 2.2]). Let M be a manifold. Let $\Omega^k_c(M)$ be the space of k–forms with compact support equipped with the strong C^∞ topology.[2]

(1) A k**–current** in M is a continuous linear map $T\colon\ \Omega^k_c(M) \to \mathbf{R}$.

(2) A sequence $(T_n)_{n\in\mathbf{N}}$ of k–currents **converges weakly** to a k–curent T if

$$\lim_{n\to\infty} T_n(\alpha) = T(\alpha) \quad \text{for every} \quad \alpha \in \Omega^k_c(M).$$

[2]For the definition, see, for example, [GG80, Chapter II Section 3]. A sequence $(\alpha_n)_{n\in\mathbf{N}}$ in $\Omega^k_c(M)$ converges in this topology if and only if there is a compact subset $K \subset M$ such that $\operatorname{supp}\alpha_n \subset K$ for all sufficiently large n and $(\alpha_n)_{n\in\mathbf{N}}$ converges uniformly with all derivatives over K. If M is compact, then the strong C^∞ topology agrees with the standard C^∞ topology making $\Omega^k(M)$ into a Fréchet space. If M is non-compact, then the strong C^∞ topology is not metrizable.

(3) The **boundary** of a k–current T is the $(k-1)$–current ∂T defined by

$$\partial T(\alpha) \coloneqq T(\mathrm{d}\alpha)$$

We say that T is **closed** if $\partial T = 0$.

(4) The **support** of a k–current T, denoted by $\operatorname{supp}(T)$, is the intersection of all closed subsets $A \subset M$ with the property that $T(\alpha) = 0$ for all $\alpha \in \Omega_c^k(M)$ with $\operatorname{supp}\alpha \cap A = \emptyset$.

(5) Given an open subset $U \subset M$ and a k–current T, the **restriction of** T **to** U is the k–current $T|_U \colon \Omega_c^k(U) \to \mathbf{R}$ defined by

$$T|_U(\alpha) \coloneqq T(\iota_*\alpha)$$

with $\iota_* \colon \Omega_c^k(U) \to \Omega_c^k(M)$ denoting the map extending a k–form with compact support in U by zero on $M \backslash U$. •

The archetypal example of a k–current is the following.

Example 4.2. Let

$$A = \sum_{i=1}^{I} m_i A_i$$

be a formal linear combination of oriented k–dimensional C^1 submanifolds $A_i \subset M$ possibly with non-empty boundary ∂A_i and with coefficients $m_1, \ldots, m_I \in \mathbf{N}$. The **Dirac delta associated with** A is the k–current $\delta_A \colon \Omega_c^k(M) \to \mathbf{R}$ defined by

$$\delta_A(\alpha) \coloneqq \sum_{i=1}^{I} m_i \int_{A_i} \alpha.$$

We have $\operatorname{supp}\delta_A = \bigcup_{i=1}^{I} A_i$. The boundary of A is the Dirac delta associated with the formal sum

$$\partial A = \sum_{i=1}^{I} m_i \partial A_i.$$

More generally, δ_A can be defined in the same way if each A_i is an oriented k–dimensional C^1 submanifolds away from a subset whose k–dimensional Hausdorff measure is zero. ♠

Definition 4.3 (cf. [De 16, Definition 3.2]). Let T be a k–current in M. A point $x \in \operatorname{supp}(T) \backslash \operatorname{supp}(\partial T)$ is **regular** if there exists an open neighborhood U of x such that $T|_U = \delta_{mA}$ for an oriented k–dimensional C^1 submanifold $A \subset U$ and $m \in \mathbf{N}$. Otherwise we say that x is **singular**. Denote by $\operatorname{reg}(T)$ and $\operatorname{sing}(T)$ the sets of regular and singular points in $\operatorname{supp}(T) \backslash \operatorname{supp}(\partial T)$. •

A generalization of the above example is the notion of an integral current, which we now describe. For the remainder of this section, let (M, g) be a Riemannian manifold and let $\mathscr{H}^k$ denote the k–dimensional Hausdorff measure induced by g.

Remark 4.4. While the definitions and results of geometric measure theory given below are typically stated for $M = \mathbf{R}^N$ equipped with the Euclidean metric, they immediately generalize to any Riemannian manifold (M, g) by embedding it isometrically into $\mathbf{R}^N$ for some N using the Nash embedding theorem, and considering currents in M as currents in $\mathbf{R}^N$. ♣

Definition 4.5 (cf. [Sim83, Chapter 6 Paragraphs 25.6, 26.4; De 16, Definitions 2.3, 3.4]). For $x \in M$ and $\alpha_x \in \Lambda^k T_x^* M$, set $|\alpha_x| := \sup\{|\alpha_x(\xi)|\}$ with the supremum taken over all simple k–vectors $\xi \in \Lambda^k T_x M$ with $|\xi| = 1$, with the norm induced by the Riemannian metric. The **comass** of $\alpha \in \Omega_c^k(M)$ is

$$\|\alpha\| := \sup_{x \in M} |\alpha_x|.$$

The **mass** of a k–current T in M is defined by

$$\mathbf{M}(T) := \sup\{T(\alpha) : \alpha \in \Omega_c^k(M) \text{ and } \|\alpha\| \leqslant 1\}.$$

•

Definition 4.6 (cf. [Sim83, Chapter 3 Section 11]). A subset $A \subset M$ is called k**–rectifiable** if there are subsets $A_0, A_1, A_2, \ldots \subset M$ such that $\mathscr{H}^k(A_0) = 0$, each A_i for $i \geqslant 1$ is a C^1–embedded k–dimensional submanifold, and

$$A \subset \bigcup_{i=0}^{\infty} A_i.$$

•

If $A \subset M$ is k–rectifiable, then for $\mathscr{H}^k$–almost every $x \in A$ there exists an **approximate tangent space** to A at x, which is a k–dimensional subspace of $T_x M$. We denote it by $\pi(A, x)$. For details, see [Sim83, Chapter 3 Section 11; De 16, Lemma 2.1.15],

Definition 4.7 (cf. [Sim83, Definition 27.1]). A k–current T in M is **integer rectifiable** if $\mathbf{M}(T) < \infty$ and there exist:

(1) a k–rectifiable subset $A \subset M$,

(2) an $\mathscr{H}^k$–measurable function $m\colon A \to \mathbf{N} \cup \{0\}$, and

(3) an $\mathscr{H}^k$–measurable section $\overrightarrow{T}$ of $\Lambda^k TM|_A$

such that:

(4) for $\mathscr{H}^k$–almost all $x \in A$, the k–vector $\overrightarrow{T}(x) \in \Lambda^k T_x M$ is given by $\overrightarrow{T}(x) = e_1 \wedge \ldots \wedge e_k$ for any orthonormal frame of $\pi(A, x)$, and

(5) T is given by

$$T(\alpha) = \int_A m(x) \langle \alpha(x), \overrightarrow{T}(x) \rangle \, \mathrm{d}\mathscr{H}^k \quad \text{for} \quad \alpha \in \Omega_c^k(M).$$

Here $\langle \cdot, \cdot \rangle$ is the pairing between k–forms and k–vectors and the integral is taken with respect to the k–dimensional Hausdorff measure.

We say that T is **integral** if both T and ∂T are integer rectifiable. In particular, a closed integer rectifiable current is integral. •

Definition 4.8 (cf. [De 16, Definition 5.5; DSS17a, Definition 0.1(b)]). A k–**semicalibration** on M is a k–form σ such that $\|\sigma\| \leqslant 1$. An integral k–current T in M is **semicalibrated** by σ if $\sigma_x(\overrightarrow{T}(x)) = 1$ for $\mathscr{H}^k$–almost every $x \in \operatorname{supp} T$. •

Remark 4.9. A semicalibration σ is called a **calibration** if $\mathrm{d}\sigma = 0$. This notion was introduced in a seminal article by Harvey and Lawson [HL82], who observed that a calibrated current is volume-minimizing. ♣

Theorem 4.10 (Federer–Fleming Compactness Theorem). *Let $(T_n)_{n\in\mathbf{N}}$ be a sequence of integral k–currents in M and let $K \subset M$ be a compact subset. If*

$$\sup_n\{\mathbf{M}(T_n) + \mathbf{M}(\partial T_n)\} < \infty \quad \textit{and} \quad \operatorname{supp}(T_n) \subset K \quad \textit{for every} \quad n \in \mathbf{N},$$

then after passing to a subsequence $(T_n)_{n\in\mathbf{N}}$ converges weakly to an integral current T. Moreover, if each T_n is closed, then so is T. If each T_n is semicalibrated by a semicalibration σ, then so is T.

The proof of the first two statements can be found, for example, in [Sim83, Chapter 6 Section 32]. The statement about semicalibrations follows immediately: an integral current T with $\operatorname{supp} T \subset K$ is semicalibrated by σ if and only if $T(\chi\sigma) = \operatorname{vol}(T)$, for any $\chi \in C_c^\infty(M)$ with $\chi = 1$ on K, and this condition is preserved by the weak convergence.

The upcoming discussion requires the following notation and definition.

Notation 4.11. Given $k \in \mathbf{N}$, set

$$\tilde{D}^k := \{(z, w) \in \mathbf{C}^2 : z = w^k \text{ and } |z| < 1\}.$$

$\tilde{D}^k\backslash\{0\}$ is an oriented smooth submanifold of $\mathbf{C}^2$ and the map $(z, w) \mapsto z$ is an orientation-preserving local diffeomorphism $\tilde{D}^k\backslash\{0\} \to D\backslash\{0\}$ where $D := \{z \in \mathbf{C} : |z| < 1\}$. We equip $\tilde{D}^k\backslash\{0\}$ with the pull-back of the flat metric on $D\backslash\{0\}$.[3]

Let $k \in \mathbf{N}$ and $\alpha \in (0, 1)$. Let $f\colon \tilde{D}^k \to \mathbf{R}^{2n-2}$ be a continuous injective map which is of class $C^{3,\alpha}$ on $\tilde{D}^k\backslash\{0\}$ and satisfies $f(0) = 0$ and $|\mathrm{d}f(z, w)| \lesssim |z|^\alpha$. Define $\underline{f}\colon \tilde{D}^k \to \mathbf{R}^{2n}$ by

$$\underline{f}(z, w) := (z, f(z, w)). \qquad \circ$$

Definition 4.12 (cf. [DSS17a, Definition 1.3]). Let $U \subset \mathbf{R}^{2n}$ be an open neighborhood of zero and let $\phi\colon U \to M$ be a smooth chart. Given f and ϕ as in Notation 4.11 the k**–branching associated with** (f, ϕ) is the integral 2–current $G_{f,\phi}$ on M given by

$$G_{f,\phi}\colon \Omega_c^2(M) \to \mathbf{R}$$

$$G_{f,\phi}(\alpha) := \int_{\tilde{D}^k\backslash\{0\}} \underline{f}^*\phi^*\alpha \quad \text{for} \quad \alpha \in \Omega_c^2(M). \qquad \bullet$$

[3] While $\tilde{D}^k$ is homeomorphic to D and thus has the structure of a smooth manifold, the pull-back metric on $\tilde{D}^k\backslash\{0\}$ does not extend through the origin unless $k = 1$.

We are ready to state the crucial regularity theorem for 2–dimensional semicalibrated currents used in this paper.

Theorem 4.13 (De Lellis, Spadaro, and Spolaor [DSS17a; DSS17b]). *Let σ be a 2–semicalibration on a Riemannian manifold (M, g) and let T be an integral 2–current in M semicalibrated by σ.*

(1) *For every $x \in \operatorname{supp}(T) \setminus \operatorname{supp}(\partial T)$ there exist a neighborhood U of x, a finite collection of maps $f_1, \dots, f_I$ and charts $\phi_1, \dots, \phi_I$ as in Notation 4.11 with $\phi_i(0) = x$, and weights $m_1, \dots, m_I \in \mathbf{N}$ such that*

$$T|_U = \sum_{i=1}^{I} m_i G_{f_i, \phi_i}. \tag{4.14}$$

(2) *The set $\operatorname{sing}(T)$ is discrete.*

Discussion of the proof. This result, for $M = \mathbf{R}^N$ with the Euclidean metric, is contained in [DSS17a, Theorem 0.2 and Section 1] and [DSS17b, Theorem 3.1, Section 3.2]. The result for an arbitrary M follows by the argument explained in Remark 4.4. Although only part (2) of the theorem is explicitly stated in the article [DSS17a], its authors actually prove (1) which is a slightly stronger statement. Indeed, (1) implies (2) because x is the only singular point of any current of the form (4.14). For the reader's convenience, we outline how to reconstruct the proof of (1) from the discussion in [DSS17a, Section 1, especially 1.4—1.5].

The first step in the proof, explained in [DSS17b, Step 4 in Section 3.2], is to show that without loss of generality we may assume that T is **irreducible** in the sense that it cannot be written in the form $T = T_1 + T_2$ for two integral currents T_1, T_2 with $\operatorname{supp} T_1 \cap \operatorname{supp} T_2 = \emptyset$. For $\lambda > 0$ denote by T_λ the rescaled current

$$T_\lambda(\alpha) := T(\lambda^* \alpha) \quad \text{for} \quad \alpha \in \Omega_c^2(\mathbf{R}^N),$$

where, by slight abuse of notation, $\lambda^*\alpha$ denotes the pull-back of α by the diffeomorphism $y \mapsto \lambda^{-1} y$. Denote by $B_r \subset \mathbf{R}^N$ the ball of radius $r > 0$ centered at zero. It is proved in [DSS17b, Theorem 3.1, Step 4 in Section 3.2] that in the situation at hand there exists an oriented 2–dimensional linear subspace $\pi \subset \mathbf{R}^N$ and $m \in \mathbf{N}$ such that for every sequence $(\lambda_n)_{n \in \mathbf{N}}$ converging to zero and $r > 0$, the sequence $(T_{\lambda_n}|_{B_r})_{n \in \mathbf{N}}$ converges weakly to the Dirac delta $\delta_{m\pi}|_{B_r}$; $\delta_{m\pi}$ is called the **tangent cone**. Note that this tangent cone is, in particular, a multiple of a 1–branching in the sense of Definition 4.12 for $f = 0$.

The crucial step in the proof is [DSS17a, Theorem 1.8] which asserts the following. Suppose that for some $r > 0$ the restriction $T|_{B_r}$ is **approximated by a k–branching**: the precise definition is rather technical and is stated in [DSS17a, Assumption 1.7]. If this is the case, then

(1) either $T|_{B_\rho} = \ell G_{f,\phi}$ for some $\rho > 0$, $\ell \in \mathbf{N}$, and a k–branching $G_{f,\phi}$,

(2) or there are $k' > k$ and $\lambda > 0$ such that $T_\lambda|_{B_r}$ is approximated by a k'–branching.

Now the theorem can be proved by induction. It follows from [DSS17b, Theorem 3.1] that for sufficiently small $\lambda > 0$ and $r > 0$, $T_\lambda|_{B_r}$ is approximated by a 1–branching, namely the tangent cone. Thus, either $T_\lambda|_{B_\rho}$ is a multiple of a 1–branching for some $\rho > 0$, or after further rescaling is approximated by a k–branching with $k > 1$. If the latter is true, we can apply the theorem again and the process continues, so that T is after rescaling approximated by a k–branching for larger values of k. The process must, however, terminate at some point, because the sequence of rescaling converges to the tangent cone $\delta_{m\pi}$ from which it follows that $k \leqslant m$ [DSS17a, third paragraph of Section 2.1]. Thus, after finitely many steps, we obtain that $\lambda^* T|_{B_\rho} = \ell G_{f,\phi}$ for some $\lambda > 0$, $\rho > 0$, $\ell \in \mathbf{N}$ and a k–branching $G_{f,\phi}$. We conclude that $T|_{B_{\lambda\rho}} = \ell G_{f,\tilde{\phi}}$ for $\tilde{\phi} := \lambda \cdot \phi$. ■

5 J–holomorphic cycles and geometric convergence

In this section we introduce the notions of a J–holomorphic cycle and geometric convergence. We then compare these with the notions of an integral currents and weak convergence. This comparison, combined with the results discussed in Section 4, implies Proposition 1.9.

Throughout, let (M, J, g) be an almost Hermitian manifold. Denote by

$$\sigma := g(J\cdot, \cdot)$$

the corresponding Hermitian form. It follows from Wirtinger's inequality that σ is a semicalibration on (M, g) and that a 2–dimensional submanifold is semicalibrated by σ if and only if it is J–holomorphic [Fed69, Section 5.4.19; HL82, Section II.3 Example 1].

Remark 5.1. If σ is closed, that is σ is a symplectic form and J is compatible with σ, then σ is a calibration and Remark 4.9 recovers the well-known fact that J–holomorphic curves in symplectic manifolds minimize volume. ♣

Definition 5.2.

(1) A **J–holomorphic curve** is a subset of M which is the image of a simple J–holomorphic map to M. A **J–holomorphic cycle** C is a formal linear combination

$$C = \sum_{i=1}^{I} m_i C_i$$

of J–holomorphic curves $C_1, \dots, C_I$ with coefficients $m_1, \dots, m_I \in \mathbf{N}$.

(2) The **homology class** of a J–holomorphic curve is the homology class of the corresponding simple map and the homology class of a J–holomorphic cycle C is

$$[C] := \sum_{i=1}^{I} m_i [C_i].$$

(3) We say that C is **smooth** if $C_1, \dots, C_I$ are embedded and pairwise disjoint.

(4) For a J–holomorphic cycle C denote by $\delta_C\colon\ \Omega^2_c(M) \to \mathbf{R}$ the associated Dirac delta 2–current defined in Example 4.2. The support $\mathrm{supp}(C)$ and mass $\mathbf{M}(C)$ of C are defined as the support and mass of δ_C respectively. Explicitly,

$$\mathrm{supp}(C) \coloneqq \mathrm{supp}(\delta_C) = \bigcup_{i=1}^{I} C_i \quad \text{and} \quad \mathbf{M}(C) \coloneqq \mathbf{M}(\delta_c) = \sum_{i=1}^{I} m_i \,\mathrm{area}(C_i). \qquad \bullet$$

Definition 5.3 (Taubes [Tau98, Definition 3.1]). Let M be a manifold and let $(J_n, g_n)_{n\in\mathbf{N}}$ be a sequence of almost Hermitian structures converging to an almost Hermitian structure (J, g) in the C^∞ topology. For every $n \in \mathbf{N}$ let C_n be a J_n–holomorphic cycle. We say that $(C_n)_{n\in\mathbf{N}}$ **converges geometrically** to a J–holomorphic cycle C if:

(1) $(\delta_{C_n})_{n\in\mathbf{N}}$ converges weakly to δ_C and

(2) $(\mathrm{supp}(C_n))_{n\in\mathbf{N}}$ converges to $\mathrm{supp}(C)$ in the Hausdorff distance; that is:

$$\lim_{n\to\infty} d_H(\mathrm{supp}(C), \mathrm{supp}(C_n)) \to 0. \tag{5.4}$$

Recall that the Hausdorff distance between two closed sets X and Y is defined by

$$d_H(X,Y) \coloneqq \max\left\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\right\},$$

with d denoting the distance with respect to the Riemannian metric g. $\bullet$

The following results compare J–holomorphic cycles and geometric convergence with closed σ–semicalibrated currents and weak convergence.

Lemma 5.5. *If T is a closed integer rectifiable current with compact support which is semicalibrated by σ, then there exists a J–holomorphic cycle C such that*

$$T = \delta_C.$$

For symplectic 4–manifolds this result was proved by Taubes [Tau96a, Proposition 6.1]. Taubes' argument and the work of Rivière and Tian [RT09] establish the result for symplectic manifolds, that is: when σ is a calibration.

Proof. Let $\mathring{\Sigma} = \mathrm{reg}(T)$ be the set of regular points of T, so that $\mathring{\Sigma}$ is an oriented C^1 submanifold. Since T is semicalibrated by σ, the tangent spaces to $\mathring{\Sigma}$ are J–invariant. It follows from elliptic regularity that $\mathring{\Sigma}$ is a smooth submanifold and has a canonical structure of a Riemann surface. By Theorem 4.13, the singular locus $\mathrm{sing}(T)$ is discrete and so finite since $\mathrm{supp}(T)$ is compact. Moreover, every $x \in \mathrm{sing}(T)$ has a neighborhood U such that

$$\mathring{\Sigma} \cap U \cong D\backslash\{0\} \sqcup \cdots \sqcup D\backslash\{0\} \quad \text{and} \quad \mathrm{sing}(T) \cap U = \{x\}.$$

Here $D = \{z \in \mathbf{C} : |z| < 1\}$. Thus, $\mathring{\Sigma}$ can be compactified to a Riemann surface Σ by adding finitely many points. The compact Riemann surface Σ comes with a continuous map $u\colon\ \Sigma \to M$.

Its restriction to $\mathring{\Sigma}$ is a smooth J–holomorphic embedding. Since the image of u is the compact set $\operatorname{supp}(T)$ and the energy of u is $\mathbf{M}(T) < \infty$, it follows from the removable singularity theorem [MS12, Theorem 4.2.1] that u is, in fact, smooth and J–holomorphic on all of Σ. Moreover, the above discussion shows that

$$T(\alpha) = \int_\Sigma m \cdot u^*\alpha \quad \text{for all } \alpha \in \Omega_c^2(M)$$

for some locally constant function $m\colon \Sigma \to \mathbf{N}$. Denoting by $C_1, \dots, C_I$ the images of the connected components of Σ and by $m_1, \dots, m_I \in \mathbf{N}$ the corresponding values of m yields

$$T = \delta_C \quad \text{with } C := \sum_{i=1}^{I} m_i C_i. \qquad \blacksquare$$

Lemma 5.6. *In the situation of Definition 5.3, if condition (1) holds and there exists a compact subset containing* $\operatorname{supp}(C_n)$ *for every* $n \in \mathbf{N}$*, then condition (2) holds as well.*

This result, which is an immediate consequence of the monotonicity formula for J–holomorphic maps, is contained in the proof of [Tau98, Proposition 3.3] and also a well-known fact in geometric measure theory, so we omit the proof.

Proof of Proposition 1.9. Since $\sup_n \mathbf{M}(C_n) < \infty$ and $\operatorname{supp}(C_n) \subset K$ for every $n \in \mathbf{N}$, by Theorem 4.10 there exists a subsequence which converges weakly to a closed integer rectifiable current semicalibrated by σ. By Lemma 5.5, this current is of the form δ_C for a J–holomorphic cycle C. By Lemma 5.6, the sequence of pseudo-holomorphic cycles (C_n) geometrically converges to C. $\blacksquare$

6 Proof of Theorem 1.6

Suppose that J is k–rigid and that $A \in H_2(M)$ satisfies $\langle c_1(M, J), A\rangle = 0$ and its divisibility is at most k. If the conclusion of the theorem fails, then there are are infinitely many *distinct* J–holomorphic curves $C_n \subset M$ representing A and of energy at most Λ. By Proposition 1.9, after passing to a subsequence, the sequence (C_n) converges geometrically to a J–holomorphic cycle

$$C_\infty = \sum_{i=1}^{I} m_i C_\infty^i.$$

Proposition 6.1. *C_∞ is connected, smooth, and its multiplicity is at most the divisibility of A.*

Proof. By Definition 5.3 (1), $[C_\infty] = [A]$. Let $u_i\colon \Sigma_i \to M$ be a simple J–holomorphic map whose image is C_∞^i. The index formula (2.5) yields

$$\sum_{i=1}^{I} m_i \operatorname{index}(u_i) = \sum_{i=1}^{I} 2m_i\langle c_1(M,J), [C_\infty^i]\rangle = 2\langle c_1(M,J), [C_\infty]\rangle = 0.$$

Since J is k-rigid, by Definition 2.10 (2), there are no J–holomorphic curves of negative index. Thus, we have $\operatorname{index}(u_i) \geqslant 0$ for every $i \in \{1, \dots, I\}$ and the above computation shows that

$$\operatorname{index}(u_1) = \cdots = \operatorname{index}(u_I) = 0.$$

Therefore, by Definition 2.10 (3), the J–holomorphic curves $C_\infty^1, \dots, C_\infty^I$ are embedded and pairwise disjoint. This proves that C_∞ is smooth.

To see that C_∞ is connected, observe that if C_∞ were disconnected, then Definition 5.3 (2) would imply that C_n is disconnected for $n \gg 1$. However, C_n is a J–holomorphic curve and thus connected by definition.

Since $A = m_1[C_\infty^1]$, it follows that m_1 is at most the divisibility of A. ∎

In the following, we rescale the sequence (C_n) and extract a further limit $\tilde{C}_\infty$. The properties of $\tilde{C}_\infty$ will give a contradiction to J being k–rigid.

Henceforth, we denote by C_∞^1 the J–holomorphic curve underlying the J–holomorphic cycle C_∞. Since the curves C_n are all distinct, we can assume that they are all distinct from C_∞^1. We can also assume that every C_n is contained in a sufficiently small tubular neighborhood of C_∞^1. By slight abuse of notation, we regard C_n as an $\exp^* J$–holomorphic curve in the normal bundle NC_∞^1 and C_∞^1 as the zero section in NC_∞^1.

For every $\lambda > 0$ let σ_λ be as in Lemma 3.10. Choose (λ_n) such that such that the sets

$$\tilde{C}_n \coloneqq \sigma_{\lambda_n}^{-1}(C_n)$$

satisfy

$$d_H(\tilde{C}_n, C_\infty^1) = 1/2. \tag{6.2}$$

Set

$$J_n \coloneqq \sigma_{\lambda_n}^* \exp^* J.$$

By construction, the $\tilde{C}_n$ are J_n–holomorphic. By Lemma 3.10, the sequence (J_n) converges to the almost complex structure J_u associated with the J–holomorphic map $u\colon C_\infty^1 \hookrightarrow M$. The sequence $(\tilde{C}_n)$ is contained in the compact disc bundle $\bar{B}_{1/2}(NC_\infty^1) \subset NC_\infty^1$. By Lemma 3.6 (5), J_u is tamed by a symplectic form ω on $B_1(NC_\infty^1)$. Consequently, for $n \gg 1$ the almost complex structure J_n is tamed by ω as well. Define a Riemannian metric g on $B_1(NC_\infty^1)$ by

$$g \coloneqq \frac{1}{2}(-\omega(J_u\cdot, \cdot) + \omega(\cdot, J_u\cdot)).$$

The analogously defined metrics g_n are Hermitian with respect to J_n and converge to g. By the energy identity [MS12, Lemma 2.2.1],

$$\lim_{n\to\infty} \mathbf{M}(\tilde{C}_n) = \lim_{n\to\infty} \delta_{\tilde{C}_n}(\omega) = \delta_{C_\infty^1}(\omega) < \infty.$$

Therefore, the mass of $\tilde{C}_n$ with respect to g_n (and thus also g) can be bounded independent of n.

By Proposition 1.9, a subsequence of $(\tilde{C}_n)_{n\in\mathbf{N}}$ geometrically converges to a J–holomorphic cycle

$$\tilde{C}_\infty = \sum_{i=1}^{I} \tilde{m}_i \tilde{C}^i_\infty.$$

Let $d_i \in \mathbf{N}$ be such that $[\tilde{C}^i_\infty] = d_i[C^1_\infty]$. Condition (6.2) guarantees that $\operatorname{supp}(\tilde{C}_\infty) \neq C^1_\infty$. Therefore, without loss of generality, $\tilde{C}^1_\infty$ is not contained in the zero section. Since

$$m_1[C^1_\infty] = A = [\tilde{C}_\infty] = \sum_{i=1}^{I} \tilde{m}_i d_i [C^1_\infty],$$

we have $d_1 \leqslant \tilde{m}_1 d_1 \leqslant m_1 \leqslant k$. Proposition 3.12 applies and the map $\varphi\colon \tilde{C}^1_\infty \to C^1_\infty$ defined there has degree d_1. This contradicts J being k–rigid. ■

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