Optimal Power Flow Models With Probabilistic Guarantees: A Boolean Approach

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***Abstract*—The legacy Optimal Power Flow (OPF) dispatch in electric power grids with high proliferation of renewables can be at risk due to the lack of awareness on major uncertainties, and sudden changes in renewable outputs. This may, in turn, result in conditions where transmission line power flows are significantly exceeded, and subsequent automatic protective actions take place. This letter presents a new generalized joint chance-constrained model for the OPF problem that effectively captures the stochasticity in renewable power generation in the system. In dealing with the complexity, and non-convexity of the proposed optimization model with probabilistic guarantees, we propose a novel tractable Boolean method to transform the model into an equivalent deterministic mixed-integer linear problem, which can be solved quickly, and efficiently by off-the-shelf solvers. Numerical results verify the effectiveness of the proposed model, and the suggested Boolean methodology.**

***Index Terms*—Boolean, joint chance constraints, optimal power flow (OPF), stochastic programming, uncertainty.**

NOMENCLATURE

A. Sets

- **G** Set of all system thermal generating units.
- **V** Set of all network buses.
- **W** Set of all wind farms in the network.
- **T** Set of all system transmission lines.
- **Ω** Set of all uncertainty scenarios.

B. Parameters

- h Constant cost of each power generating unit.
- e Vector of all 1's.
- μ Wind power forecast (MW).
- d Demanded load at each load point (MW).
- B Weighted-Laplacian susceptance matrix.
- $\hat{B}$ Modified susceptance matrix.
- β_{ij} Succeptance of transmission line ij.
- $f_{ij}^{\max}$ Maximum flow limit of transmission line ij.
- $y_g^{\max}$ Maximum power capacity of generator g.
- $y_g^{\min}$ Minimum power capacity of generator g.
- p Reliability level for joint chance constraints.
- ω Realization (scenarios) of random variables.
- m Mean change in wind power.

C. Random Variables

- ζ Multivariate random variable representing the stochastic changes in wind power.

D. Decision Variables

- y Generating unit output power (MW).
- α Affine control variable of each generator.
- $\bar{\theta}$ Updated phase angel at each node.

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I. INTRODUCTION

WIDESPREAD deployment of distributed energy resources (DERs) with high fluctuations and inherent stochasticity in power grids has brought about unique challenges to the legacy grid operation and control paradigms. Included among which is the standard Optimal Power Flow (OPF) mechanism, the solutions to which may be frequently challenged by sudden changes in renewables. A typical OPF engine receives estimates of the loads and renewable generation profiles for the upcoming time window and minimizes the system operation cost while adhering to operating limitations of the network's transmission lines and generating units. This is particularly concerning when the exogenous fluctuations in renewable resources are large, which may result in the transmission line flows significantly exceeded with potentially-devastating protection and stability consequences system-wide.

With the proliferation of uncertainties in generation portfolios and demand profiles in power grids, a wide range of analytical and simulation techniques for probabilistic OPF analysis has been presented and discussed in the literature [1]–[3]. In recent years, chance-constrained programming for the OPF problem under uncertainty has been also introduced [4]–[9]. In particular, reference [10] introduced a Chance-Constrained Optimal Power Flow (CC-OPF) model: assuming the availability of reliable renewable forecasts, it searches for the most probable realizations of the line overloads under renewable uncertainties, and satisfies all model constraints with a high probability while minimizing the system operation cost. Built on and different from the state-of-the-art CC-OPF models with a series of single chance constraints, we propose a CC-OPF model with *joint* *probabilistic constraints*, i.e., joint chance constraints. We further develop a novel Boolean modeling methodology to solve this class of stochastic optimization models considering discrete distributions for wind power uncertainty. A linear deterministic reformulation of the stochastic problem is next developed which elicits sets of minimally sufficient conditions to satisfy the proposed joint probabilistic constraint. The use of a joint chance constraint is motivated here to capture the actual reliability of the power grid as a whole. In contrast, the reliance upon a number of individual chance constraints enforcing a high reliability level for each single probabilistic constraint considered independently of the others could result in a low overall reliability of the entire power grid. The multivariate distribution of the vector of random variables is here represented by a large number of scenarios. The proposed model is computationally efficient in large-scale power grids with large number of binary variables. The proposed approach addresses a timely need for effectively managing prevailing uncertainties in power grids under stochastic risks.

II. CC-OPF Model Formulation With Joint Probabilistic Constraints

The CC-OPF optimization model with joint probabilistic constraints has an objective function that minimizes the expected cost of the stochastic generation over a varying wind power output ζ, subject to a set of constraints [10]:

$$\mathbf{CC-OPF}:$$

$$\min \quad \mathbb{E}_{\zeta} \sum_{g\in\mathbf{G}} \left[h_g(y_g-(e^T\zeta)\alpha_g)\right] \tag{1}$$

$$\text{s.t.} \quad \sum_{g\in\mathbf{G}} \alpha_g = 1, \qquad \alpha \geq 0, \qquad \alpha_i = 0,\; i\in\mathbf{V}\setminus\mathbf{G} \tag{2}$$

$$\sum_{i\in\mathbf{V}} (y_i+\mu_i-d_i) = 0, \quad y\geq 0 \tag{3}$$

$$B\bar{\theta} = y+\mu-d \tag{4}$$

$$\bar{\theta} = \hat{B}(y+\mu-d) \tag{5}$$

$$-f_k^{\max} \leq \beta_k \Big[\bar{\theta}_i-\bar{\theta}_j+\hat{B}_i(m_i-(e^Tm)\alpha_i) - \hat{B}_j(m_j-(e^Tm)\alpha_j)\Big] \leq f_k^{\max} \quad k(i,j)\in\mathbf{T} \tag{6}$$

$$\mathbb{P}\begin{pmatrix} y_g-(e^T\zeta)\alpha_g \geq y_g^{\min}, & g\in\mathbf{G} \\ y_g-(e^T\zeta)\alpha_g \leq y_g^{\max}, & g\in\mathbf{G}\end{pmatrix} \geq p \tag{7}$$

The objective function, composed of a fixed term and a term that varies with wind, reflects an affine control strategy that allows the system generating units to respond to wind fluctuations between the current and the next—e.g., fifteen minute—time intervals. The decision variables in the proposed CC-OPF model are y_g, α_g, and $\bar{\theta}$. Constraint (2) sets the basic conditions needed by the affine control, i.e., to assume that all system generators' governors involved in controls always meet the generation-load balance in response to wind fluctuations proportionally. Constraint (3) represents the system-wide power balance equation, where $\mu_i = 0$ for $i \notin \mathbf{W}$. Constraint (4) states the stochastic power flow equations, in which $\bar{\theta}$ is formulated in (5). Note that $\hat{B}$ in (5) is an $(n-1)\times(n-1)$ matrix derived as follows

$$\hat{B} = \begin{pmatrix} \bar{B}^{-1} & 0 \\ 0 & 0\end{pmatrix}$$

where $\bar{B}$ is a sub-matrix obtained by eliminating the same column and row n of the susceptance matrix B, where n is neither a generator bus nor a wind farm bus, i.e., $n \notin (\mathbf{G}\cup\mathbf{W})$ [10]. Constraints (6) enforce the transmission line flows within the corresponding capacity limits for all lines k in the set $\mathbf{T}$ connecting bus i to bus j in the system. The proposed joint chance constraint (7) with random technology matrix enforces that the produced power from each thermal generating unit always stays within its minimum and maximum generation capacity limits with a pre-defined reliability level p.

III. Boolean Reformulation Method

The main challenge with the standard scenario approach used to solve chance-constrained problems is that they typically require the introduction of one binary variable for each scenario and that their continuous relaxation is loose. The Boolean reformulation method permits to alleviate the above-mentioned issues as it allows for the derivation of a deterministic, compact, and exact mixed-integer linear programming reformulation of the chance-constrained problem where the number of binary variables does not depend on the number of scenarios used to represent uncertainty (this feature will be later highlighted in Theorem 1 and in the numerical experiments). The Boolean approach constructs the set of recombinations, binarizes the probability distribution, represents the feasibility of the chance constraint with a partially defined Boolean function, from which it extracts a system of mixed-integer inequalities providing an equivalent characterization of the feasible area of the chance constraint. Next, we describe succinctly the Boolean framework.

1) Binarization and Recombinations: To simplify the exposition, set $\xi = [\xi_1 = e^T\zeta \;\; \xi_2 = -e^T\zeta]$, $Tx_g = [T_1x_g = (y_g - y_g^{\min})/\alpha_g \;\; T_2x_g = (y_g^{\max}y_g)/\alpha_g]$, and $I = \{1,2\}$ which allows the chance constraint (7) to be rewritten in a more compact form:

$$\mathbb{P}(\xi \leq Tx) \geq p \;\Leftrightarrow\; \mathbb{P}(\xi_i \leq T_ix_g,\; g\in G, i\in I) \geq p. \tag{8}$$

The p-sufficiency concept [11] is used to derive sufficient conditions for (8) to hold. A realization ω^k is *p-sufficient* if and only if $\mathbb{P}(\xi \leq \omega^k) \geq p$ and is *p-insufficient* otherwise [11].

Let F_i denote the marginal probability distribution of ξ_i. The univariate-quantile inequalities $F_i(\omega_i^k) \geq p, i = 1,\ldots,|I|$ define the necessary conditions for (8) to hold. The set of recombinations $\bar{\Omega} = C_1 \times \ldots \times C_{|I|}$ [12] with

$$C_i = \left\{\omega_i^k : F_i(\omega_i^k) \geq p, k = 1,\ldots,|\Omega|\right\},\; i\in I \tag{9}$$

includes all points that can be p-sufficient.

The set $\bar{\Omega}$ is partitioned into the sets of p-sufficient $\bar{\Omega}^+ := \{\omega^k \in \bar{\Omega} : F(\omega^k) \geq p\}$ and p-insufficient $\bar{\Omega}^- := \{\omega^k \in \bar{\Omega} : F(\omega^k) < p\}$, recombinations. The two scenario index sets K and K^- are such that $\bar{\Omega} = \{\omega^k : k \in K\}$ and $\bar{\Omega}^- = \{\omega^k : k \in K^-\}$.

The next step is the binarization of the recombinations with a set of cut points. Let $n_i, i \in I$ be the number of cut points sorted in ascending order ($c_{i1} < c_{i2} < \ldots$) of each component i of the random vector ξ. The binarization of ω_i^k maps it to the vector $\beta_i^k = [\beta_{i1}^k, \ldots, \beta_{in_i}^k]$ with

$$\beta_{il}^k = 1 \text{ if } \omega_i^k \geq c_{il}, \ = 0 \text{ otherwise } \ l = 1, \ldots, n_i, i \in I. \quad (10)$$

The binarization process is carried out with the sufficient-equivalent set of cut points $C = (C_1, C_2, \ldots, C_{|I|})$ [11] with $C_i, i \in I$ defined by (9). The binary mapping of ω^k into β^k with the sufficient-equivalent set of cut points is injective over $\bar{\Omega}$, and ensures that $\bar{\Omega}_B^+ \bigcap \bar{\Omega}_B^- = \emptyset$, with $\bar{\Omega}_B^+$ and $\bar{\Omega}_B^-$ denoting the binary images of $\bar{\Omega}^+$ and $\bar{\Omega}^-$ obtained via (10), and that $\bar{\Omega}_B^+$ and $\bar{\Omega}_B^-$ are disjoint. The binary images of the recombinations allow for modeling of the feasible area of (8) with a partially defined Boolean function, from which we extract, in the next subsection, a system of mixed-integer inequalities representing exactly the feasible region of (8).

2) Equivalent Reformulations: Lejeune [11] and Lejeune and Margot [12] have shown that it is possible to derive, from the partially defined Boolean function obtained from the above-defined binarization process, a set of mixed-integer linear inequalities representing the feasible area of (8).

Theorem 1: Let $\bar{\ell}^k(i) = \max\{1 \leq l \leq n_i \mid \beta_{il}^k = 1\}$, $i \in I, k \in K^-$ and $\gamma_{il} \in \{0,1\}, i \in I, 1 \leq l \leq n_i$ be binary variables, the number of which is equal to the number of cut points. Let $o_{il}, i \in I, 1 \leq l \leq n_i$ be parameters measuring the distance between two consecutive cut points c_{il} and c_{il+1} associated to i: $o_{il+1} = c_{il+1} - c_{il}, i \in I, l = 1, \ldots, n_i - 1$ and $o_{i1} = c_{i1}, i \in I$. The chance constraint **CC** can be equivalently rewritten as:

$$\sum_{i \in I} \gamma_{i\bar{\ell}^k(i)+1} \geq 1 \qquad k \in K^- \quad (11)$$

$$\gamma_{i,l-1} \geq \gamma_{il} \qquad i \in I, \ 2 \leq l \leq n_i \quad (12)$$

$$\gamma_{i1} = 1 \qquad i \in I \quad (13)$$

$$\gamma_{il} \in \{0,1\} \qquad i \in I, \ 1 \leq l \leq n_i \quad (14)$$

$$T_i x_g \geq \sum_{l=1}^{n_i} o_{il}\gamma_{il} \qquad i \in I, g \in \mathbf{G} \quad (15)$$

The equivalence between (8) and (11)-(14) was demonstrated in [12, p.945, Theorem 7]. As shown by the above formulation, the number of binary variables is equal to the number of cut points used in the binarization process (see (10)) and does not increase monotonically with the number of scenarios used to represent uncertainty, which is a key challenge in the traditional scenario reformulation method.

The equivalent deterministic reformulation of the **CC-OPF** optimization problem with joint probabilistic constraints and random technology matrix expressed with the original variables y_g, α_g, and $\bar{\theta}$ is the following mixed-integer linear problem:

$$\mathbf{REF-CC-OPF}:$$

$$\min \quad \mathbb{E}_\zeta \sum_{g \in \mathbf{G}} \left[h_g(y_g - (e^T\zeta)\alpha_g)\right]$$

$$\text{s.t.} \quad (2)\text{–}(6); \ (11)\text{–}(14)$$

$$(y_g - y_g^{\min})/\alpha_g \geq \sum_{l=1}^{n_1} o_{1l}\gamma_{1l} \ \ g \in \mathbf{G} \quad (16)$$

$$(y_g^{\max} - y_g)/\alpha_g \geq \sum_{l=1}^{n_2} o_{2l}\gamma_{2l} \ \ g \in \mathbf{G} \quad (17)$$

Note that the proposed model is generic enough to accommodate additional chance constraints, and the reformulation method would apply the same way.

IV. Numerical Results

The performance of the proposed analytics is tested on the IEEE 118-bus test system with $\mathbf{G} = 19$ conventional generators, $\mathbf{W} = 4$ wind farms, and $\mathbf{L} = 185$ transmission lines, the data on which are made available in [13]. We set the reliability parameter p to 95% and generate two sets of scenarios of size $|\Omega|$ =5,000 and 10,000. We compare two approaches, namely the standard scenario-based integer reformulation and the proposed Boolean reformulation. The problems are solved with the Gurobi 9.0.0 solver.

The basic scenario-based approach for reformulating the chance constraint (8) is to introduce one binary variable for each scenario. The notation Ω refers to the set of all realizations (scenarios) $\omega^k \in \mathbb{R}_+^{|I|}, k \in K^S$ that characterize the joint probability distribution function F of ξ. Let q^k be the probability of ω^k, θ^k be a binary variable taking value 1 if the conditions imposed by the scenario ω_k are violated for at least one $g \in I$, and equal to 0 otherwise, while M^k is a sufficiently large positive number. The chance constraint can be reformulated as follows:

$$y_g - (e^T\omega^k)\alpha_g + M^k\Omega^k \geq y_g^{\min}, \ g \in \mathbf{G} \quad (18)$$

$$y_g - (e^T\omega^k)\alpha_g - M^k\Omega^k \leq y_g^{\max}, \ g \in \mathbf{G} \quad (19)$$

$$\sum_{k \in K^S} q^k\theta^k \leq 1 - p \quad (20)$$

$$\theta^k \in \{0,1\}, \ k \in K^S \quad (21)$$

The knapsack constraint (20) prevents the sum of the probabilities of the violated scenarios to exceed the complement $(1-p)$ of the enforced reliability level.

We solve two problem instances in which respectively 5,000 and 10,000 scenarios are considered to account for the uncertainty of the wind power and report the results in Table I. We denote by t the CPU time in seconds, by n_B the number of binary decision variables, by n_C the number of constraints, and by G the optimality gap of the best solution found in 1 hour and half of CPU time.

TABLE I
COMPARISON OF THE BOOLEAN- AND SCENARIO-BASED REFORMULATION METHODS

$\|\Omega\|$	**Reformulation Method**							
	Boolean				Scenario			
	n_B	n_C	t	G	n_B	n_C	t	G
5,000	20	780	0.3	0	5,000	140,279	5,400	0.105%
10,000	19	780	0.3	0	10,000	280,404	5,400	0.103%

TABLE II
OUT-OF-SAMPLE VALIDATION

$\|\Omega\|$	p	p_V
5,000	95%	3.28%
10,000	95%	2.29%

The Boolean reformulation method solves the largest (i.e., $|\Omega|=$ 10,000 scenarios) instance to optimality in less than 0.3 seconds, while the scenario-based model could not be solved in 5,400 seconds. This can be easily explained by the size and complexity of the reformulations obtained with the two approaches. First, the Boolean reformulation contains more than 50 times less binary variables than the scenario reformulation does (i.e., 19 vs. 10,000). Second, the number of constraints in the Boolean reformulation is 359 times smaller than that in the scenario reformulation (780 vs. 280,404). This illustrates the promising advantage of the proposed Boolean method which provides a computationally efficient reformulation that permits to take into account many more scenarios, thereby providing a much finer representation of uncertainty, than the standard scenario method. The scalability of the Boolean reformulation is highlighted in Table I: the solution time, the number of constraints and variables are each invariant with the number of scenarios. In particular, the number of binary variables is not an increasing function of the number of scenarios (20 for 5,000 scenarios vs. 19 for 10,000 scenarios).

In order to test the robustness of the proposed model, we have carried out the following validation test. We have generated 1000 additional scenarios, thereby forming the scenario test set. Since these scenarios were not used to construct the model and determine the optimal policy, they can be used to validate the robustness of the model and perform an out-of-sample validation study. In particular, we have checked how the optimal solution obtained for each considered problem instance fares on the testing scenario set and the application of the optimal policy would result or not in the violation of the model constraints and would permit or not to attain the prescribed probability level p. In Table II, we report for each problem instance the proportion p_V of scenarios in the testing for which applying the optimal policy leads to the violation of at least one constraint.

We can see that in each instance the value of p_V is below the complement $(1-p)$ of the enforced reliability level p, which confirms the out-of-sample robustness of the model.

V. CONCLUSION

This letter introduced a new class of power grid chance constrained optimization problems with joint probabilistic constraints. Applied to the legacy OPF models with chance constraints (i.e., the CC-OPF), a novel Boolean modeling method is proposed which captures the interactions between the components of a multidimensional random vector to reach a predefined system-wide reliability level. As numerically demonstrated and outperforming the scenario-based method, the proposed methodology allows for a very fast solution of stochastic optimization models in which the random variables are represented by a large number of scenarios.

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