# THE MINKOWSKI EQUALITY OF FILTRATIONS

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ABSTRACT. Suppose that $R$ is an analytically irreducible or excellent local domain with maximal ideal $m_R$. We consider multiplicities and mixed multiplicities of $R$ by filtrations of $m_R$-primary ideals. We show that the theorem of Teissier, Rees and Sharp, and Katz, characterizing equality in the Minkowski inequality for multiplicities of ideals, is true for divisorial filtrations, and for the larger category of bounded filtrations. This theorem is not true for arbitrary filtrations of $m_R$-primary ideals.

## 1. Introduction

The study of mixed multiplicities of $m_R$-primary ideals in a local ring $R$ with maximal ideal $m_R$ was initiated by Bhattacharya [1], Rees [30] and Teissier and Risler [37]. In [14] the notion of mixed multiplicities is extended to arbitrary, not necessarily Noetherian, filtrations of $R$ by $m_R$-primary ideals ($m_R$-filtrations). It is shown in [14] that many basic theorems for mixed multiplicities of $m_R$-primary ideals are true for $m_R$-filtrations.

The development of the subject of mixed multiplicities and its connection to Teissier's work on equisingularity [37] can be found in [18]. A survey of the theory of mixed multiplicities of ideals can be found in [36, Chapter 17], including discussion of the results of the papers [31] of Rees and [35] of Swanson, and the theory of Minkowski inequalities of Teissier [37], [38], Rees and Sharp [34] and Katz [21]. Later, Katz and Verma [22], generalized mixed multiplicities to ideals that are not all $m_R$-primary. Trung and Verma [40] computed mixed multiplicities of monomial ideals from mixed volumes of suitable polytopes.

A filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of a ring $R$ is a descending chain

$$R = I_0 \supset I_1 \supset I_2 \supset \cdots$$

of ideals such that $I_iI_j \subset I_{i+j}$ for all $i, j \in \mathbb{N}$. An $m_R$-filtration $\mathcal{I} = \{I_n\}$ is a filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of $R$ such that $I_n$ is $m_R$-primary for $n \geq 1$.

A filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of a ring $R$ is said to be Noetherian if $\bigoplus_{n\geq 0} I_n$ is a finitely generated $R$-algebra.

The following theorem is the key result needed to define the multiplicity of an $m_R$-filtration. Let $\ell_R(M)$ denote the length of an $R$-module $M$.

**Theorem 1.1.** *([8, Theorem 1.1] and [9, Theorem 4.2]) Suppose that $R$ is a local ring of dimension $d$, and $N(\hat{R})$ is the nilradical of the $m_R$-adic completion $\hat{R}$ of $R$. Then the limit*

$$\lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d} \tag{1}$$

*exists for any $m_R$-filtration $\mathcal{I} = \{I_n\}$, if and only if $\dim N(\hat{R}) < d$.*

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The problem of existence of such limits (1) has been considered by Ein, Lazarsfeld and Smith [17] and Mustaţă [28]. When the ring $R$ is a domain and is essentially of finite type over an algebraically closed field $k$ with $R/m_R = k$, Lazarsfeld and Mustaţă [26] showed that the limit exists for all $m_R$-filtrations. Cutkosky [9] proved it in the complete generality stated above in Theorem 1.1. Lazarsfeld and Mustaţă use in [26] the method of counting asymptotic vector space dimensions of graded families using "Okounkov bodies". This method, which is reminiscent of the geometric methods used by Minkowski in number theory, was developed by Okounkov [29], Kaveh and Khovanskii [24] and Lazarsfeld and Mustaţă [26]. We also use this wonderful method. The fact that $\dim N(R) = d$ implies there exists a filtration without a limit was observed by Dao and Smirnov.

As can be seen from this theorem, one must impose the condition that the dimension of the nilradical of the completion $\hat{R}$ of $R$ is less than the dimension of $R$ to ensure the existence of limits. The nilradical $N(R)$ of a $d$-dimensional ring $R$ is

$$N(R) = \{x \in R \mid x^n = 0 \text{ for some positive integer } n\}.$$

We have that $\dim N(R) = d$ if and only if there exists a minimal prime $P$ of $R$ such that $\dim R/P = d$ and $R_P$ is not reduced. In particular, the condition $\dim N(\hat{R}) < d$ holds if $R$ is analytically unramified; that is, $\hat{R}$ is reduced. Thus it holds if $R$ is excellent and reduced. We define the multiplicity of $R$ with respect to the $m_R$-filtration $\mathcal{I} = \{I_n\}$ to be

$$e_R(\mathcal{I}) = e_R(\mathcal{I}; R) = \lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d/d!}.$$

The multiplicity of a ring with respect to a non Noetherian filtration can be an irrational number. A simple example on a regular local ring is given in [14].

Mixed multiplicities of filtrations are defined in [14]. Let $M$ be a finitely generated $R$-module where $R$ is a $d$-dimensional local ring with $\dim N(\hat{R}) < d$. Let

$$\mathcal{I}(1) = \{I(1)_n\}, \ldots, \mathcal{I}(r) = \{I(r)_n\}$$

be $m_R$-filtrations. In [14, Theorem 6.1] and [14, Theorem 6.6], it is shown that the function

$$P(n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\ell_R(M/I(1)_{mn_1} \cdots I(r)_{mn_r} M)}{m^d} \tag{2}$$

is a homogeneous polynomial of total degree $d$ with real coefficients for all $n_1, \ldots, n_r \in \mathbb{N}$. The mixed multiplicities of $M$ are defined from the coefficients of $P$, generalizing the definition of mixed multiplicities for $m_R$-primary ideals. Specifically, we write

$$P(n_1, \ldots, n_r) = \sum_{d_1+\cdots+d_r=d} \frac{1}{d_1! \cdots d_r!} e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; M) n_1^{d_1} \cdots n_r^{d_r}. \tag{3}$$

We say that $e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; M)$ is the mixed multiplicity of $M$ of type $(d_1, \ldots, d_r)$ with respect to the $m_R$-filtrations $\mathcal{I}(1), \ldots, \mathcal{I}(r)$. Here we are using the notation

$$e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; M) \tag{4}$$

to be consistent with the classical notation for mixed multiplicities of $M$ with respect to $m_R$-primary ideals from [37]. The mixed multiplicity of $M$ of type $(d_1, \ldots, d_r)$ with respect to $m_R$-primary ideals $I_1, \ldots, I_r$, denoted by $e_R(I_1^{[d_1]}, \ldots, I_r^{[d_r]}; M)$ ([37], [36, Definition 17.4.3]) is equal to the mixed multiplicity $e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; M)$, where the Noetherian $I$-adic filtrations $\mathcal{I}(1), \ldots, \mathcal{I}(r)$ are defined by $\mathcal{I}(1) = \{I_1^i\}_{i\in\mathbb{N}}, \ldots, \mathcal{I}(r) = \{I_r^i\}_{i\in\mathbb{N}}$.

We have that

$$e_R(\mathcal{I}(j); M) = e_R(\mathcal{I}(j)^{[d]}; M) \tag{5}$$

for all $j$.

The multiplicities and mixed multiplicities of powers of $m_R$-primary ideals are always positive ([37] or [36, Corollary 17.4.7]). The multiplicities and mixed multiplicities of $m_R$-filtrations are always nonnegative, as is clear for multiplicities, and is established for mixed multiplicities in [16, Proposition 1.3]. However, they can be zero. If $R$ is analytically irreducible, then all mixed multiplicities are positive if and only if the multiplicities $e_R(\mathcal{I}(j); R)$ are positive for $1 \le j \le r$. This is established in [16, Theorem 1.4].

When the module $M$ is $R$ and $R$ is understood, we will usually write $e(\mathcal{I}) = e_R(\mathcal{I})$ and $e(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}) = e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]})$.

Suppose that $R$ is a $d$-dimensional local domain, with quotient field $K$. A valuation $\mu$ of $K$ is called an $m_R$-valuation if $\mu$ dominates $R$ ($R \subset \mathcal{O}_\mu$ and $m_\mu \cap R = m_R$ where $\mathcal{O}_\mu$ is the valuation ring of $\mu$ with maximal ideal $m_\mu$) and $\mathrm{trdeg}_{R/m_R}\mathcal{O}_\mu/m_\mu = d - 1$.

Associated to an $m_R$-valuation $\mu$ are valuation ideals

$$I(\mu)_n = \{f \in R \mid \mu(f) \ge n\} \tag{6}$$

for $n \in \mathbb{N}$. In general, the $m_R$-filtration $\mathcal{I}(\mu) = \{I(\mu)_n\}$ is not Noetherian. In a two-dimensional normal local ring $R$, the condition that the filtration of valuation ideals of $R$ is Noetherian for all $m_R$-valuations dominating $R$ is the condition (N) of Muhly and Sakuma [27]. It is proven in [6] that a complete normal local ring of dimension two satisfies condition (N) if and only if its divisor class group is a torsion group. An example is given in [5] of an $m_R$-valuation $\mu$ of a 3-dimensional regular local ring $R$ such that the filtration $\mathcal{I}(\mu)$ is not Noetherian. The multiplicity $e(\mathcal{I}(\mu))$ is however a rational number. In Section 15, we give an example of an $m_R$-valuation $\mu$ dominating a normal excellent local domain $R$ of dimension three such that $e_R(\mathcal{I}(\mu)) = e_R(\mathcal{I}(\mu_{E_2}))$ is an irrational number. The filtration is necessarily non Noetherian.

Divisorial filtrations are defined in Section 5, and briefly discussed in SubSection 2.4. Divisorial filtrations are determined by prescribing multiplicities along a finite set of $m_R$-valuations.

Let $R$ be a local ring and $\mathcal{I} = \{I_m\}$ be an $m_R$-filtration. Then the integral closure $\overline{R[\mathcal{I}]}$ of $R[\mathcal{I}] = \sum_{m \ge 0} I_m u^m$ in $R[u]$ is $\overline{R[\mathcal{I}]} = \sum_{n \ge 0} J_m u^m$ where $\{J_m\}$ is the $m_R$-filtration defined by $J_m = \{f \in R \mid f^r \in \overline{I_{rm}}$ for some $r > 0\}$ (Lemma 5.6).

Bounded $m_R$-filtrations are defined in SubSection 5.6. A bounded $m_R$-filtration $\mathcal{I}$ is a filtration such that $\overline{R[\mathcal{I}]} = R[\mathcal{I}(D)]$ for some divisorial filtration $\mathcal{I}(D)$. Every adic filtration $\mathcal{I} = \{I^n\}$ of powers of a fixed $m_R$-primary ideal is bounded.

1.1. **Rees's Theorem.** Rees has shown in [30] that if $R$ is a formally equidimensional local ring and $I \subset I'$ are $m_R$-primary ideals then the following are equivalent:

1) $e(I') = e(I)$
2) $\overline{\sum_{n \ge 0} (I')^n t^n} = \overline{\sum_{n \ge 0} I^n t^n}$.
3) $\overline{I'} = \overline{I}$

The statement 3) $\Rightarrow$ 1) is true for an arbitrary local ring.

This raises the question of whether the conditions

1) $e(\mathcal{I}') = e(\mathcal{I})$
2) $\overline{\sum_{n \ge 0} I'_n t^n} = \overline{\sum_{n \ge 0} I_n t^n}$.

are equivalent for arbitrary $m_R$-filtrations $\mathcal{I}' \subset \mathcal{I}$.

The statement 2) $\Rightarrow$ 1) is true for arbitrary $m_R$-filtrations in a local ring which satisfies $\dim N(\hat{R}) < d$. This is shown in [14, Theorem 6.9] and Appendix [12]. However the statement 1) $\Rightarrow$ 2) is not true in general for $m_R$-filtrations (a simple example in a regular local ring is given in [14]).

Rees's theorem is true for bounded $m_R$-filtrations.

**Theorem 1.2.** *(Theorems 13.1 and 14.4) Suppose that $R$ is an excellent local domain or an analytically irreducible local domain, $\mathcal{I}(1)$ is a real bounded $m_R$-filtration and $\mathcal{I}(2)$ is an arbitrary $m_R$-filtration such that $\mathcal{I}(1) \subset \mathcal{I}(2)$. Then the following are equivalent*

1) $e(\mathcal{I}(1)) = e(\mathcal{I}(2))$.
2) *There is equality of integral closures*
$$\overline{\sum_{m\ge 0} I(1)_m t^m} = \overline{\sum_{m\ge 0} I(2)_m t^m}$$
*in $R[t]$.*

1.2. **The Minkowski inequalities and equality of mixed multiplicities.** The Minkowski inequalities were formulated and proven for $m_R$-primary ideals in reduced equicharacteristic zero local rings by Teissier [37], [38] and proven for $m_R$-primary ideals in full generality, for local rings, by Rees and Sharp [34]. The same inequalities hold for filtrations.

**Theorem 1.3.** *(Minkowski Inequalities for filtrations)(*[14, Theorem 6.3]*) Suppose that $R$ is a $d$-dimensional local ring with $\dim N(\hat{R}) < d$, $M$ is a finitely generated $R$-module and $\mathcal{I}(1) = \{I(1)_j\}$ and $\mathcal{I}(2) = \{I(2)_j\}$ are $m_R$-filtrations. Let $e_i = e_R(\mathcal{I}(1)^{[d-i]}, \mathcal{I}(2)^{[i]}; M)$ for $0 \le i \le d$. Then*

1) $e_i^2 \le e_{i-1}e_{i+1}$ *for* $1 \le i \le d-1$.
2) $e_i e_{d-i} \le e_0 e_d$ *for* $0 \le i \le d$.
3) $e_i^d \le e_0^{d-i} e_d^i$ *for* $0 \le i \le d$.
4) $e_R(\mathcal{I}(1)\mathcal{I}(2)); M)^{\frac{1}{d}} \le e_0^{\frac{1}{d}} + e_d^{\frac{1}{d}}$, *where* $\mathcal{I}(1)\mathcal{I}(2) = \{I(1)_j I(2)_j\}$.

We write out the last inequality without abbreviation as

$$e_R(\mathcal{I}(1)\mathcal{I}(2)); M)^{\frac{1}{d}} \le e_R(\mathcal{I}(1); M)^{\frac{1}{d}} + e_R(\mathcal{I}(2); M)^{\frac{1}{d}} \tag{7}$$

where $\mathcal{I}(1)\mathcal{I}(2) = \{I(1)_m I(2)_m\}$. This equation is called The Minkowski Inequality.

The fourth inequality 4) was proven for $m_R$-filtrations in a regular local ring with algebraically closed residue field by Mustaţă ([28, Corollary 1.9]) and more recently in this situation by Kaveh and Khovanskii ([23, Corollary 7.14]). The inequality 4) was proven with our assumption that $\dim N(\hat{R}) < d$ in [9, Theorem 3.1]. Inequalities 2) - 4) can be deduced directly from inequality 1), as explained in [37], [38], [34] and [36, Corollary 17.7.3].

There is a beautiful characterization of when equality holds in the Minkowski inequality (7) by Teissier [39] (for Cohen-Macaulay normal two-dimensional complex analytic $R$), Rees and Sharp [34] (in dimension 2) and Katz [21] (in complete generality).

They have shown that if $R$ is a formally equidimensional local ring and $I(1), I(2)$ are $m_R$-primary ideals then the following are equivalent:

1) The Minkowski inequality
$$e_R(I(1)I(2))^{\frac{1}{d}} = e(I(1))^{\frac{1}{d}} + e(I(2))^{\frac{1}{d}}$$
holds.

2) There exist positive integers $a$ and $b$ such that

$$\overline{\sum_{n\ge 0} I(1)^{an}t^n} = \overline{\sum_{n\ge 0} I(2)^{bn}t^n}.$$

3) There exist positive integers $a$ and $b$ such that $\overline{I(1)^a} = \overline{I(2)^b}$

The Teissier, Rees and Sharp, Katz theorem leads to the question of whether the following conditions are equivalent for $m_R$-filtrations $\mathcal{I}(1)$ and $\mathcal{I}(2)$.

1) The Minkowski equality

$$e_R(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}} = e(\mathcal{I}(1))^{\frac{1}{d}} + e(\mathcal{I}(2)^{\frac{1}{d}}$$

holds.

2) There exist positive integers $a$ and $b$ such that

$$\overline{\sum_{n\ge 0} I(1)^{an}t^n} = \overline{\sum_{n\ge 0} I(2)^{bn}t^n}.$$

We show in Theorem 8.4 that if $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are $m_R$-filtrations on a local ring $R$ such that $\dim N(\hat{R}) < d$ and condition 2) holds then the Minkowski equality 1) holds, but the converse statement, that the Minkowski equality 1) implies condition 2) is not true for filtrations, even in a regular local ring, as is shown in a simple example in [14].

In Theorems 13.2 and 14.5, we show that 1) and 2) are equivalent for bounded $m_R$-filtrations on an analytically irreducible or excellent local domain, giving a complete generalization of the Teissier, Rees and Sharp, Katz Theorem for bounded $m_R$-filtrations.

**Theorem 1.4.** *(Theorem 13.2 and Theorem 14.5) Suppose that $R$ is a $d$-dimensional analytically irreducible or excellent local domain and $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are bounded $m_R$-filtrations. Then the following are equivalent*

1) *The Minkowski equality*

$$e(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}} = e(\mathcal{I}(1))^{\frac{1}{d}} + e(\mathcal{I}(2))^{\frac{1}{d}}$$

*holds.*

2) *There exist positive integers $a, b$ such that there is equality of integral closures*

$$\overline{\sum_{n\ge 0} I(1)_{an}t^n} = \overline{\sum_{n\ge 0} I(2)_{bn}t^n}$$

*in $R[t]$.*

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## 2. An overview of the proof

In this section, we suppose that $R$ is a $d$-dimensional normal excellent local domain.

2.1. **Multiplicities of filtrations.** We summarize Sections 6 and 7 in this subsection. We use the method of counting asymptotic vector space dimensions of graded families by computing volumes of convex bodies associated to appropriate semigroups introduced in [29], [26] and [24]. Let $\nu$ be a valuation of the quotient field $K$ of $R$ which dominates $R$ and has value group isomorphic to $\mathbb{Z}^d$. Further suppose that $\nu(f) \in \mathbb{N}^d$ if $0 \ne f \in R$. Then we can associate to an $m_R$-filtration $\mathcal{I} = \{I_n\}$ a semigroup $\Gamma(\mathcal{I}) \subset \mathbb{N}^{d+1}$ defined by $\Gamma(\mathcal{I}) = \{(\nu(f), n) \mid f \in I_n\}$. Let $\Delta(\mathcal{I})$ be the intersection of the closure of the real cone generated by $\Gamma(\mathcal{I})$ with $\mathbb{R}^d\times\{1\}$. Similarly, we define $\Delta(R)$ to be the subset of $\mathbb{R}^d$ constructed from $\Gamma(R)$ by replacing $I_n$ with $R$ for all $n$.

For $c\in\mathbb{R}_{>0}$, let

$$H_c^-=\{(x_1,\ldots,x_d)\in\mathbb{R}^d\mid x_1+\cdots+x_d\le c\}.$$

Using some commutative algebra, we find a constant $c>0$ such that

$$\Delta(\mathcal{I})\setminus(\Delta(\mathcal{I})\cap H_c^-)=\Delta(R)\setminus(\Delta(R)\cap H_c^-). \tag{8}$$

Then $\Delta(\mathcal{I})\cap H_c^-$ and $\Delta(R)\cap H_c^-$ are compact convex sets and by (34),

$$\frac{e_R(\mathcal{I})}{d!}=\delta[\mathrm{Vol}(\Delta(R)\cap H_c^-)-\mathrm{Vol}(\Delta(\mathcal{I})\cap H_c^-)] \tag{9}$$

where $\delta=[\mathcal{O}_\nu/m_\nu:R/m_R]$.

2.2. **The Integral closure of a filtration $\mathcal{I}$ and the convex sets $\Delta(\mathcal{I})$.** Suppose that $\mathcal{I}'\subset\mathcal{I}$ are $m_R$-filtrations. Then we have $\Delta(\mathcal{I}')\subset\Delta(\mathcal{I})$, so we have $e_R(\mathcal{I})=e_R(\mathcal{I}')$ if and only if $\Delta(\mathcal{I}')=\Delta(\mathcal{I})$.

If $\mathcal{I}'$ is a Noetherian $m_R$-filtration, and $\mathcal{I}$ is an $m_R$-filtration such that $\mathcal{I}'\subset\mathcal{I}$, then we have that $e_R(\mathcal{I}')=e_R(\mathcal{I})$ if and only if $\Delta(\mathcal{I}')=\Delta(\mathcal{I})$ which holds if and only if $R[\mathcal{I}]=\sum_{m\ge0}I_mu^m\subset\overline{\sum_{m\ge0}I'_mu^m}=\overline{R[\mathcal{I}']}$. This can be proven as follows. By taking suitable Veronese subalgebras, we reduce to the case where $\mathcal{I}$ and $\mathcal{I}'$ are the filtrations of powers of fixed $m_R$-primary ideals $I$ and $I'$, so that the result then follows from Rees's Theorem [30] for normal excellent local domains. Rees's theorem was discussed at the beginning of Subsection 1.1.

For arbitrary $m_R$-filtrations $\mathcal{I}'\subset\mathcal{I}$ such that $R[\mathcal{I}]=\sum I_mt^m\subset\overline{\sum_{m\ge0}I'_mt^m}=\overline{R[\mathcal{I}']}$ we have that $e_R(\mathcal{I}')=e_R(\mathcal{I})$, as shown in [14, Theorem 6.9] and [12, Appendix]. However, as we mentioned in the beginning of Subsection 1.1, there exists a non-Noetherian $m_R$-filtration $\mathcal{I}'$ and a Noetherian $m_R$-filtration $\mathcal{I}$ such that $\mathcal{I}'\subset\mathcal{I}$, $e_R(\mathcal{I}')=e_R(\mathcal{I})$ and $R[\mathcal{I}]=\sum_{m\ge0}I_mt^m$ is not a subset of $\overline{R[\mathcal{I}']}=\overline{\sum_{m\ge0}I'_mt^m}$.

2.3. **The invariant $\gamma_\mu(\mathcal{I})$.** This subsection is a summary of Subsection 5.1. Let $\mu$ be an $m_R$-valuation and $\mathcal{I}$ be an $m_R$-filtration. Define $\tau_m=\min\{\mu(f)\mid f\in I_m\}$ and $\gamma_\mu(\mathcal{I})=\inf_m\{\frac{\tau_m}{m}\}$. The numbers $\tau_m\in\mathbb{Z}_{>0}$ for all $m$ but $\gamma_\mu(\mathcal{I})$ can be an irrational number, even when $\mathcal{I}$ is a divisorial $m_R$-filtration, as shown in Section 15) and explained in Subsection 2.7.

Theorem 7.3 shows that if $\mathcal{I}'\subset\mathcal{I}$ and $e_R(\mathcal{I}')=e_R(\mathcal{I})$ then $\gamma_\mu(\mathcal{I}')=\gamma_\mu(\mathcal{I})$ for all $m_R$-valuations $\mu$. This is proven by taking the valuation $\nu$ used to compute $\Delta$ to be composite with $\mu$, so $\nu(f)=(\mu(f),\cdots)\in\mathbb{N}^d$ for $f\in R$. The condition $e_R(\mathcal{I}')=e_R(\mathcal{I})$ implies $\Delta(\mathcal{I}')=\Delta(\mathcal{I})$ and $\gamma_\mu(\mathcal{I}'),\gamma_\mu(\mathcal{I})$ are the smallest points of the projections of $\Delta(\mathcal{I}')$, respectively $\Delta(\mathcal{I})$ onto the first coordinate of $\mathbb{R}^d$.

2.4. **Divisorial Filtrations.** In this subsection, we summarize material from Section 5. Let $\varphi:X\to\mathrm{Spec}(R)$ be a birational projective morphism such that $X$ is normal and is the blow up of an $m_R$-primary ideal. Let $E_1,\ldots,E_r$ be the prime exceptional divisors of $\varphi$, and for $1\le i\le r$, let $\mu_{E_i}$ be the $m_R$-valuation whose valuation ring is $\mathcal{O}_{X,E_i}$. Suppose that $D=\sum a_iE_i$ with $a_i\in\mathbb{N}$ is an effective Weil divisor on $X$ with exceptional support.

Define $\gamma_{E_i}(D)=\gamma_{\mu_{E_i}}(\mathcal{I}(D))$ for $1\le i\le r$. Then $\gamma_{E_i}(D)\ge a_i$ for all $i$. We have that $ma_i$ is the prescribed order of vanishing of elements of $I(mD)$ along $E_i$ but $m\gamma_{E_i}(D)$ is asymptotically the actual vanishing.

We remark that $\gamma_\mu(\mathcal{I}(D))$ can be an irrational number. By Theorem 15.2, the example $X$ of Section 15 has two prime exceptional divisors $E_1$ and $E_2$ such that

$$\gamma_{E_1}(E_2) = \frac{3}{9-\sqrt{3}}$$

is an irrational number. This example is surveyed in Subsection 2.7.

We have that

$$I(mD) = I(\lceil \sum m\gamma_{E_i}(D)E_i\rceil) \tag{10}$$

for all $m \in \mathbb{N}$, where $\lceil x \rceil$ is the round up of a real number $x$. In this way, we are led to extend our category of divisorial $m_R$-filtrations to real divisorial $m_R$-filtrations.

Now let $\mathcal{I} = \mathcal{I}(\sum_{i=1}^s a_i\mu_i)$ with $a_i \in \mathbb{N}$ be a divisorial $m_R$-filtration. A representation of $\mathcal{I}$ is a pair $\varphi : X \to \mathrm{Spec}(R)$ and a divisor $\sum_{i=1}^s a_iE_i$ such that $X$ is as in the above paragraph, and $\mu_{E_i} = \mu_i$ for $1 \le i \le s \le r$. We remark that it is not always possible to construct an $X$ for which $r = s$, even in dimension $d = 2$. An example of a two dimensional excellent normal local domain without a "one fibered ideal" is given in [6]. A one fibered ideal is an $m_R$-primary ideal $I$ such that the normalization of its blowup has only one prime exceptional divisor.

2.5. **Rees's theorem for divisorial $m_R$-filtrations.** It follows from Corollary 7.5 that if $\mathcal{I}(D_1) \subset \mathcal{I}(D_2)$ are divisorial $m_R$-filtrations such that $e(\mathcal{I}(D_2)) = e(\mathcal{I}(D_1))$, then $\mathcal{I}(D_2) = \mathcal{I}(D_1)$. This is proven in Section 7. Let $X \to \mathrm{Spec}(R)$ be a representation of $D_1$ and $D_2$, and write $D_1 = \sum a_iE_i$ and $D_2 = \sum b_iE_i$ as Weil divisors on $X$.

By Theorem 7.3, whose proof was discussed in Subsection 2.3, $\gamma_{E_i}(D_1) = \gamma_{E_i}(D_2)$ for $1 \le i \le r$. Thus $I(mD_1) = I(mD_2)$ for all $m \in \mathbb{N}$ by (10).

2.6. **The Teissier, Rees and Sharp, Katz Theorem for divisorial $m_R$-filtrations.** Suppose that we have equality in the Minkowski inequality (7) for the divisorial $m_R$-filtrations $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$. We will give an outline of our proof that there exist $a, b \in \mathbb{Z}_{>0}$ such that $I(maD_1) = I(bmD_2)$ for all $m \in \mathbb{N}$. Let

$$f(n_1,n_2) := \lim_{m\to\infty} \frac{\ell_R(R/I(mn_1D_1)I(mn_2D_2))}{m^d}.$$

Using the Minkowski inequalities $e_i^d \le e_0^{d-i}e_d^i$ of 3) of Theorem 1.3, we obtain in (67) of Section 9 that

$$f(n_1,n_2) = \frac{1}{d!}(e_0^{\frac{1}{d}}n_1 + e_d^{\frac{1}{d}}n_2)^d$$

where $e_0 = e_R(\mathcal{I}(D_1))$ and $e_d = e_R(\mathcal{I}(D_2))$.

We now survey Section 8. Define semigroups $\Gamma(n_1,n_2) = \Gamma(\{I(mn_1D_1)I(mn_2D_2)\})$ and associated closed convex sets $\Delta(n_1,n_2)$. We can find $\varphi \in \mathbb{R}_{>0}$ such that letting

$$H^-_{\Phi,n_1,n_2} = \{(x_1,\ldots,x_d) \in \mathbb{R}^d \mid x_1 + \cdots + x_d \le \varphi e_0^{\frac{1}{d}}n_1 + \varphi e_d^{\frac{1}{d}}n_2\},$$

$$\Delta_\Phi(n_1,n_2) = \Delta(n_1,n_2) \cap H^-_{\Phi,n_1,n_2}$$

and

$$\tilde{\Delta}_\Phi(n_1,n_2) = \Delta(R) \cap H^-_{\Phi,n_1,n_2},$$

we have (55) that

$$f(n_1,n_2) = \delta[\mathrm{Vol}(\tilde{\Delta}_\Phi(n_1,n_2) - \mathrm{Vol}(\Delta_\Phi(n_2,n_2))]$$

as in (9). Since $\Delta(R)$ is a closed cone with vertex at the origin, by (28) and (59)

$$\mathrm{Vol}(\tilde{\Delta}_\Phi(n_1,n_2)) = (n_1\alpha_1+n_2\alpha_2)^d\varphi^d\mathrm{Vol}(\Delta(R)\cap H_1^-).$$

We now survey Section 10. We define in (70)

$$h(n_1,n_2)=\mathrm{Vol}(\Delta_\Phi(n_1,n_2))=\mathrm{Vol}(\tilde{\Delta}_\Phi(n_1,n_2))-\frac{f(n_1,n_2)}{\delta}=\lambda(\alpha_1n_1+\alpha_2n_2)^d.$$

for some $\lambda\in\mathbb{R}_{>0}$.

Let

$$g(n_1,n_2):=\mathrm{Vol}(n_1\Delta_\Phi(1,0)+n_2\Delta_\Phi(0,1)),$$

which is a homogeneous real polynomial of degree $d$ (Theorem 4.2) Since

$$n_1\Delta_\Phi(1,0)+n_2\Delta_\Phi(0,1)\subset\Delta_\Phi(n_1,n_2),$$

we have that $g(n_1,n_2)\le h(n_1,n_2)$ for all $n_1,n_2\in\mathbb{N}$, $g(1,0)=h(1,0)$ and $g(0,1)=h(0,1)$. Thus for $0<t<1$,

$$\begin{array}{lll} h(1-t,t)^{\frac{1}{d}} & = & (1-t)h(1,0)^{\frac{1}{d}}+th(0,1)^{\frac{1}{d}}=(1-t)g(1,0)^{\frac{1}{d}}+tg(0,1)^{\frac{1}{d}}\\ & \le & g(1-t,t)^{\frac{1}{d}}\le h(1-t,t)^{\frac{1}{d}}. \end{array}$$

where the first inequality on the second line is the Brunn-Minkowski inequality of convex geometry (Theorem 4.3). We see from this equation that we have equality in the Brunn-Minkowski inequality. Thus by Theorem 4.3, we have that $\Delta_\Phi(1,0)$ and $\Delta_\Phi(0,1)$ are homothetic; that is, there is an affine transformation $T(\vec{x})=c\vec{x}+\gamma$ such that $T(\Delta_\Phi(1,0))=\Delta_\Phi(1,0)$. We then show in Theorem 10.1 that

$$e_d^{\frac{1}{d}}\Delta_\Phi(1,0)=e_0^{\frac{1}{d}}\Delta_\Phi(0,1),$$

and applying Theorem 10.3, which is proved like Theorem 7.3 discussed in Subsection 2.3, we get that

(11) $$\frac{\gamma_{E_j}(D_1)}{e_0^{\frac{1}{d}}}=\frac{\gamma_{E_j}(D_2)}{e_d^{\frac{1}{d}}}$$

for $1\le j\le r$.

It is shown in Theorem 11.4 that (assuming the Minkowski equality holds) the real number $\frac{e_d^{\frac{1}{d}}}{e_0^{\frac{1}{d}}}$ is actually a rational number $\frac{a}{b}$. This is in spite of the fact that the multiplicities $e_0$ and $e_d$ can be irrational numbers and the $\gamma_{E_j}(D_i)$ can be irrational numbers (as shown in the example of Section 15, which is surveyed in subsection 2.7).

Now combining this fact, (10) and (11) we obtain in Theorem 11.4 that

$$I(maD_1)=\Gamma(X,\mathcal{O}_X(-\lceil\sum_{i=1}^r ma\gamma_{E_i}(D_1)E_i\rceil)=\Gamma(X,\mathcal{O}_X(-\lceil\sum_{i=1}^r mb\gamma_{E_i}(D_2)E_i\rceil)=I(mbD_2)$$

for all $m\in\mathbb{N}$.

The proof of Theorem 11.4 uses the invariant

$$w_{\mathcal{I}}(f)=\max\{m\mid f\in I_m\}$$

for a filtration $\mathcal{I}=\{I_m\}$ and $f\in R$, which is either a natural number or $\infty$, and the fact that an integral divisorial $m_R$-filtration $\mathcal{I}(D)$ has the good property that for $f\in R$, there exists $d\in\mathbb{Z}_{>0}$ such that $w_{\mathcal{I}(D)}(f^{nd})=nw_{\mathcal{I}(D)}(f^d)$ for all $n\in\mathbb{N}$ (Lemma 11.3).

It is natural to define

$$\overline{w}_{\mathcal{I}}(f)=\limsup_{n\to\infty}\frac{w_{\mathcal{I}}(f^n)}{n}$$

which generalizes to filtrations the asymptotic Samuel function $\overline{\nu}_I(f)$ of an ideal in $R$ ([36, Definition 6.9.3]). We use a theorem of Rees in [33] about the asymptotic Samuel function (reduced order) $\overline{\nu}_{m_R}$ in our proof of Lemma 8.2.

2.7. **An Example.** The above concepts and results are analyzed in an example from [13] in Section 15. The example is of the blowup $\varphi: X\to \mathrm{Spec}(R)$ of an $m_R$ primary ideal in a normal and excellent three dimensional local ring $R$ which is a resolution of singularities. The map $\varphi$ has two prime exceptional divisors $E_1$ and $E_2$. The function

$$f(n_1,n_2)=\lim_{m\to\infty}\frac{\ell_R(R/I(mn_1E_1+mn_2E_2))}{m^3}$$

is computed in [13] and is reproduced here.

**Theorem 2.1.** *([13, Theorem 1.4]) For $n_1,n_2\in\mathbb{N}$,*

$$f(n_1,n_2)=\begin{cases}33n_1^3 & \text{if } n_2<n_1\\ 78n_1^3-81n_1^2n_2+27n_1n_2^2+9n_2^3 & \text{if } n_1\le n_2<n_1\left(3-\frac{\sqrt{3}}{3}\right)\\ \left(\frac{2007}{169}-\frac{9\sqrt{3}}{338}\right)n_2^3 & \text{if } n_1\left(3-\frac{\sqrt{3}}{3}\right)<n_2.\end{cases}$$

Thus $f(n_1,n_2)$ is not a polynomial, but it is "piecewise a polynomial"; that is, $\mathbb{R}^2_{\ge 0}$ consists of three triangular regions determined by lines through the origin such that $f(n_1,n_2)$ is a polynomial function within each of these three regions. The line separating the second and third regions has irrational slope, and the function $f(n_1,n_2)$ has an irrational coefficient in the third region. The middle region is the ample cone and is also the Nef cone.

We compute the functions $\gamma_{E_1}$ and $\gamma_{E_2}$ in [13, Theorem 4.1], as summarized in the following theorem. Observe that $\gamma_{E_1}$ is an irrational number in the third region.

**Theorem 2.2.** *([13, Theorem 4.1]) Let $D=n_1E_1+n_2E_2$ with $n_1,n_2\in\mathbb{N}$, an effective exceptional divisor on $X$.*

1) *Suppose that $n_2<n_1$. Then $\gamma_{E_1}(D)=n_1$ and $\gamma_{E_2}(D)=n_1$.*
2) *Suppose that $n_1\le n_2<n_1\left(3-\frac{\sqrt{3}}{3}\right)$. Then $\gamma_{E_1}(D)=n_1$ and $\gamma_{E_2}(D)=n_2$.*
3) *Suppose that $n_1\left(3-\frac{\sqrt{3}}{3}\right)<n_2$. Then $\gamma_{E_1}(D)=\frac{3}{9-\sqrt{3}}n_2$ and $\gamma_{E_2}(D)=n_2$.*

*In all three cases, $-\gamma_{E_1}(D)E_1-\gamma_{E_2}(D)E_2$ is nef on $X$.*

We determine the divisors for which Minkowski's inequality holds in the following Corollary, reproduced from Section 15.

**Corollary 2.3.** *(Corollary 15.3) Suppose that $D_1$ and $D_2$ are effective integral exceptional divisors on $X$. If $D_1$ and $D_2$ are in the first region of Theorem 15.1, then Minkowski's equality holds between them. If $D_1$ and $D_2$ are in the second region, then Minkowski's equality holds between them if and only if $D_2$ is a rational multiple of $D_1$. If $D_1$ and $D_2$ are in the third region, then Minkowski's equality holds between them. Minkowski's equality cannot hold between $D_1$ and $D_2$ in different regions.*

The above theorem allows us to compute the mixed multiplicities of any two divisors $D_1 = a_1E_1 + a_2E_2$ and $D_2 = b_1E_1 + b_2E_2$ by interpreting mixed multiplicities as the anti positive intersection multiplicities of (85).

In particular, we can compare $f(n_1, n_2)$ with the polynomial

$$P(n_1,n_2) = \lim_{m\to\infty} \frac{\ell_R(R/I(mn_1E_1)I(mn_2E_2))}{m^d}.$$

We calculate in (88) that

$$\begin{array}{rcl} P(n_1,n_2) &=& \frac{1}{3!}e(\mathcal{I}(E_1)^{[3]})n_1^3 + \frac{1}{2!}e(\mathcal{I}(E_1)^{[2]},\mathcal{I}(E_2)^{[1]})n_1^2n_2 \\ && +\frac{1}{2!}e(\mathcal{I}(E_1)^{[1]},\mathcal{I}(E_2)^{[2]})n_1n_2^2 + \frac{1}{3!}e(\mathcal{I}(E_2)^{[3]})n_2^3 \\ &=& 33n_1^3 + (\frac{891}{26} + \frac{99}{26}\sqrt{3})n_1^2n_2 + (\frac{12042}{338} - \frac{27}{338}\sqrt{3})n_1n_2^2 + \left(\frac{2007}{169} - \frac{9\sqrt{3}}{338}\right)n_2^3. \end{array}$$

## 3. Notation

We will denote the nonnegative integers by $\mathbb{N}$ and the positive integers by $\mathbb{Z}_{>0}$, the set of nonnegative rational numbers by $\mathbb{Q}_{\geq 0}$ and the positive rational numbers by $\mathbb{Q}_{>0}$. We will denote the set of nonnegative real numbers by $\mathbb{R}_{\geq 0}$ and the positive real numbers by $\mathbb{R}_{>0}$. For a real number $x$, $\lceil x \rceil$ will denote the smallest integer that is $\geq x$ and $\lfloor x \rfloor$ will denote the largest integer that is $\leq x$. If $E_1, \ldots, E_r$ are prime divisors on a normal scheme $X$ and $a_1, \ldots, a_r \in \mathbb{R}$, then $\lfloor \sum a_iE_i \rfloor$ denotes the integral divisor $\sum \lfloor a_i \rfloor E_i$ and $\lceil \sum a_iE_i \rceil$ denotes the integral divisor $\sum \lceil a_i \rceil E_i$.

A local ring is assumed to be Noetherian. The maximal ideal of a local ring $R$ will be denoted by $m_R$. The quotient field of a domain $R$ will be denoted by $\mathrm{QF}(R)$. We will denote the length of an $R$-module $M$ by $\ell_R(M)$. Excellent local rings have many excellent properties which are enumerated in [19, Scholie IV.7.8.3]. We will make use of some of these properties without further reference.

Divisorial $m_R$-filtrations $\mathcal{I}(D) = \{I(nD)\}$ will be defined in Section 5. If $R$ is an excellent local domain, $\mathcal{I}(D)$ is determined by an effective exceptional Weil divisor on the normalization of the blow up of an $m_R$-primary ideal.

## 4. Preliminaries

4.1. **Approximation of irrational numbers.** The following formula for approximation of real numbers appears in [20] (Remark on bottom of page 156).

**Lemma 4.1.** *Suppose that $\xi, \alpha \in \mathbb{R}_{>0}$. Then*

a) *There exist $p_0, q_0 \in \mathbb{Z}_{>0}$ such that*

$$0 \leq \xi - \frac{p_0}{q_0} < \frac{\alpha}{q_0}.$$

b) *There exist $p_0', q_0' \in \mathbb{Z}_{>0}$ such that*

$$-\frac{\alpha}{q_0'} < \xi - \frac{p_0'}{q_0'} \leq 0$$

*Proof.* If $\xi$ is a rational number we need only write $\xi = \frac{p_0}{q_0}$ with $p_0, q_0 \in \mathbb{Z}_{>0}$ (or $\xi = \frac{p_0'}{q_0'}$ with $p_0', q_0' \in \mathbb{Z}_{>0}$).

Suppose that $\xi$ is an irrational number. By [20, Theorem 170], we can express $\xi$ as an infinite simple continued fraction. Let $\frac{p_n}{q_n}$ be the convergents of this continued fraction for

$n \in \mathbb{Z}_{>0}$. By [20, Theorem 156], $q_n \geq n$, and by [20, Theorem 164] and [20, Theorem 171], we have that

$$\xi - \frac{p_n}{q_n} = \frac{(-1)^n \delta_n}{q_n q_{n+1}}$$

with $0 < \delta_n < 1$ for all $n$ from which the lemma follows. □

4.2. **The Brunn-Minkowski inequality in Convex Geometry.** Let $K$ and $L$ be compact convex subsets of $\mathbb{R}^d$. For $\lambda \in \mathbb{R}_{\geq 0}$, define

$$\lambda K = \{\lambda x \mid x \in K\}$$

and for $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$, define the Minkowski sum

$$\lambda_1 K + \lambda_2 L = \{\lambda_1 x + \lambda_2 y \mid x \in K, y \in L\}.$$

A proof of the following theorem can be found in [2, Section 29, page 42].

**Theorem 4.2.** *Suppose that $K_1, \ldots, K_r$ are compact convex subsets of $\mathbb{R}^d$. Then the volume function* $\mathrm{Vol}(\lambda_1 K_1 + \cdots + \lambda_r K_r)$ *is a homogeneous real polynomial of degree $d$ for $\lambda_1, \ldots, \lambda_r \in \mathbb{R}_{\geq 0}$.*

The coefficients of the polynomial of the theorem are called mixed volumes.

We now state the Brunn-Minkowski Theorem of convex geometry. A couple of proofs of this theorem are on [2, Page 94] and in [25].

**Theorem 4.3.** *Let $K$ and $L$ be compact convex subsets of $\mathbb{R}^d$. Then*

$$\mathrm{Vol}\left((1-t)K + tL\right)^{\frac{1}{d}} \geq (1-t)\mathrm{Vol}(K)^{\frac{1}{d}} + t\mathrm{Vol}(L)^{\frac{1}{d}} \tag{12}$$

*for $0 \leq t \leq 1$. Further, if $\mathrm{Vol}(K)$ and $\mathrm{Vol}(L)$ are positive, then equality holds in (12) for some $t$ with $0 < t < 1$ if and only if $K$ and $L$ are homothetic; that is, there exists $0 < c \in \mathbb{R}$ and $\vec{\gamma} \in \mathbb{R}^d$ such that $L = cK + \vec{\gamma}$.*

*If $K$ and $L$ are homothetic, then equality holds in (12) for all $t$ with $0 \leq t \leq 1$.*

## 5. $m_R$-valuations and divisorial $m_R$-filtrations on local domains $R$

5.1. **$m_R$-valuations and $m_R$-filtrations.** In this subsection, suppose that $R$ is a $d$-dimensional local domain, with quotient field $K$. A valuation $\mu$ of $K$ is called an $m_R$-valuation if $\mu$ dominates $R$ ($R \subset V_\mu$ and $m_\mu \cap R = m_R$ where $V_\mu$ is the valuation ring of $\mu$ with maximal ideal $m_\mu$) and $\mathrm{trdeg}_{R/m_R} V_\mu/m_\mu = d-1$.

Let $\mathcal{I} = \{I_i\}$ be an $m_R$-filtration. Let $\mu$ be an $m_R$-valuation. Let

$$I(\mu)_m = \{f \in R \mid \mu(f) \geq m\} = m_\mu^n \cap R,$$

and define

$$\tau_{\mu,m}(\mathcal{I}) = \mu(I_m) = \min\{\mu(f) \mid f \in I_m\}.$$

Since $\tau_{\mu,mn}(\mathcal{I}) \leq n\tau_{\mu,m}(\mathcal{I})$, we have that

$$\frac{\tau_{\mu,mn}(\mathcal{I})}{mn} \leq \min\{\frac{\tau_{\mu,m}(\mathcal{I})}{m}, \frac{\tau_{\mu,n}(\mathcal{I})}{n}\} \tag{13}$$

for $m, n \in \mathbb{N}$.

Define

$$\gamma_\mu(\mathcal{I}) = \inf_m \frac{\tau_{\mu,m}(\mathcal{I})}{m}. \tag{14}$$

**Lemma 5.1.** *Suppose that $R$ is an excellent local domain. Then a valuation $\mu$ of the quotient field $K$ of $R$ which dominates $R$ is an $m_R$-valuation of $R$ if and only if the valuation ring $\mathcal{O}_\mu$ is essentially of finite type over $R$.*

*Proof.* Since an excellent local domain is analytically unramified, the only if direction follows from [36, Theorem 9.3.2]. Now we establish the if direction. Since $\mathcal{O}_\mu$ is essentially of finite type over $R$, there exists a finite type $R$-algebra $S$ and a prime ideal $Q$ in $S$ such that $S$ is a sub $R$-algebra of $\mathcal{O}_\mu$ and $S_Q = \mathcal{O}_\mu$. In particular, $Q \cap R = m_R$. Since an excellent local domain is universally catenary, the dimension equality (c.f. [36, Theorem B.3.2.]) holds. Since a Noetherian valuation ring is a discrete valuation ring (c.f. [36, Corollary 6.4.5]) it has dimension 1, so that $\mathrm{ht}(Q) = 1$, from which it follows that $\mu$ is an $m_R$-valuation. □

5.2. **Divisors on blowups of normal local domains.** In this subsection suppose that $R$ is a normal excellent local domain. Let $\varphi : X \to \mathrm{Spec}(R)$ be a birational projective morphism such that $X$ is normal and $X$ is the blowup of an $m_R$-primary ideal. Let $E_1, \ldots, E_r$ be the prime divisors on $X$ with exceptional support. A real divisor $D$ on $X$ with exceptional support is a formal sum $D = \sum_{i=1}^r a_i E_i$ with $a_i \in \mathbb{R}$ for all $i$. $D$ is said to be effective if $a_i \geq 0$ for all $i$. $D$ is said to be a rational divisor if all $a_i \in \mathbb{Q}$ and $D$ is said to be an integral divisor if all $a_i \in \mathbb{Z}$.

Now suppose that $D$ is an effective integral divisor with exceptional support. In this case, $D$ is a Weil divisor on $X$. A rank one reflexive sheaf is associated to the Weil divisor $D$. Let $U$ be the open set of regular points of $X$ and let $i : U \to X$ be the inclusion. We have that $\dim(X \setminus U) \leq d - 2$ since $X$ is normal. Then $D|U$ is a Cartier divisor. The reflexive coherent sheaf $\mathcal{O}_X(-D)$ of $\mathcal{O}_X$-modules is defined by $\mathcal{O}_X(-D) = i_*\mathcal{O}_U(-D|U)$ The basic properties of this sheaf are developed for instance in [11, Section 13.2]. Since $R$ is normal, we have that $\Gamma(X, \mathcal{O}_X) = R$, and if $D$ is a nontrivial integral exceptional divisor with effective support, then $I(D) = \Gamma(X, \mathcal{O}_X(-D))$ is an $m_R$-primary ideal.

Now let $D = \sum_{i=1}^r a_i E_i$ be an effective real divisor with exceptional support. Let $\mathcal{I}(D)$ be the $m_R$-filtration $\mathcal{I}(D) = \{I(mD)\}$ where

$$I(mD) = \Gamma(X, \mathcal{O}_X(-\lceil \sum_{i=1}^r m a_i E_i \rceil)).$$

The round up $\lceil x \rceil$ of a real number $x$ is the smallest integer $a$ such that $x \leq a$. When $D$ is an integral divisor, we have that $I(mD) = \Gamma(X, \mathcal{O}_X(-mD))$ for all $m$.

Let $\mu_{E_i}$ be the $m_R$-valuation whose valuation ring is $\mathcal{O}_{X,E_i}$ for $1 \leq i \leq r$. Let $\tau_{m,i} = \tau_{E_i,m}(D) = \tau_{m,\mu_{E_i}}(\mathcal{I}(D))$. Now define

$$\gamma_{E_i}(D) = \gamma_{\mu_{E_i}}(\mathcal{I}(D)).$$

We have that

$$I(mD) = \Gamma(X, \mathcal{O}_X(-\lceil \sum_{i=1}^r a_i m E_i \rceil)) = \{f \in R \mid \mu_{E_i}(f) \geq \lceil m a_i \rceil \text{ for } 1 \leq i \leq r\}.$$

Thus $\tau_{E_i,m}(D) \geq m a_i$ for all $m \in \mathbb{N}$, and so

$$\gamma_{E_i}(D) \geq a_i \text{ for all } i. \tag{15}$$

**Lemma 5.2.** *(*[12, Lemma 3.1]*) We have that*

$$I(mD) = \Gamma(X, \mathcal{O}_X(-\lceil \sum_{i=1}^r a_i m E_i \rceil)) = \Gamma(X, \mathcal{O}_X(-\lceil \sum_{i=1}^r m \gamma_{E_i}(D) E_i \rceil))$$

*for all* $m \in \mathbb{N}$.

*Proof.* We have that

$$\Gamma(X, \mathcal{O}_X(-\lceil\sum_{i=1}^{r} m\gamma_{E_i}(D)E_i\rceil)) \subset \Gamma(X, \mathcal{O}_X(-\lceil\sum_{i=1}^{r} a_i m E_i\rceil))$$

by (15).

Suppose that $f \in \Gamma(X, \mathcal{O}_X(-\lceil\sum_{i=1}^{r} a_i m E_i\rceil))$. Then $\mu_{E_i}(f) \ge \tau_{E_i,m}(D) \ge m\gamma_{E_i}(D)$ for all $i$, so that $\mu_{E_i}(f) \ge \lceil m\gamma_{E_i}(D)\rceil$ for all $i$ since $\mu_{E_i}(f) \in \mathbb{N}$. ☐

5.3. **Divisors on blowups of local domains.** In this subsection, suppose that $R$ is an excellent $d$-dimensional local domain. Let $S$ be the normalization of $R$, which is a finitely generated $R$-module, and let $m_1, \ldots, m_t$ be the maximal ideals of $S$. Let $\varphi : X \to \mathrm{Spec}(R)$ be a birational projective morphism such that $X$ is the normalization of the blowup of an $m_R$-primary ideal. Since $X$ is normal, $\varphi$ factors through $\mathrm{Spec}(S)$. Let $\varphi_i : X_i \to \mathrm{Spec}(S_{m_i})$ be the induced projective morphisms where $X_i = X \times_{\mathrm{Spec}(S)} \mathrm{Spec}(S_{m_i})$. For $1 \le i \le t$, let $\{E_{i,j}\}$ be the prime exceptional divisors in $\varphi_i^{-1}(m_i)$.

A real divisor $D$ on $X$ with exceptional support is a formal sum $D = \sum a_{i,j}E_{i,j}$ with $a_{i,j} \in \mathbb{R}$ for all $i, j$. $D$ is said to be effective if all $a_{i,j} \ge 0$. $D$ is said to be a rational divisor if all $a_{i,j} \in \mathbb{Q}$ and $D$ is said to be an integral divisor if all $a_{i,j} \in \mathbb{Z}$.

Suppose that $D$ is an effective real divisor on $X$ with exceptional support. Write $D = \sum_{i,j} a_{i,j}E_{i,j}$ with $a_{i,j} \in \mathbb{R}_{\ge 0}$. Define $D_i = \sum_j a_{i,j}E_{i,j}$ for $1 \le i \le t$.

Let $D = \sum_{i,j} a_{i,j}E_{i,j}$ be an effective real divisor with exceptional support on $X$. Let $\mathcal{I}(D)$ be the $m_R$-filtration $\mathcal{I}(D) = \{I(mD)\}$ where

$$I(mD) = \Gamma(X, \mathcal{O}_X(-\lceil\sum_{i=1}^{r} m a_{i,j}E_{i,j}\rceil)) \cap R.$$

When $D$ is an integral divisor, we have that $I(mD) = \Gamma(X, \mathcal{O}_X(-mD)) \cap R$ for all $m$.

Now let $D$ be an effective integral divisor with exceptional support.

Let

$$\begin{aligned} J(D) &= \Gamma(X, \mathcal{O}_X(-D)),\\ J(D_i) &= \Gamma(X_i, \mathcal{O}_{X_i}(-D_i)),\\ I(D) &= J(D) \cap R,\\ I(D_i) &= J(D_i) \cap R. \end{aligned} \tag{16}$$

We have that

$$S/J(D) \cong \bigoplus_{i=1}^{t} S_{m_i}/\Gamma(X_i, \mathcal{O}_{X_i}(-D_i)) \cong \bigoplus_{i=1}^{t} S_{m_i}/J(D_i) \tag{17}$$

and so

$$\ell_R(S/J(D)) = \sum_{i=1}^{t} \ell_R(S_{m_i}/J(D_i)) = \sum_{i=1}^{t} [S/m_i : R/m_R]\ell_{S_{m_i}}(S_{m_i}/J(D_i)). \tag{18}$$

We have that $[S/m_i : R/m_R] < \infty$ for all $i$ since $S$ is a finitely generated $R$-module.

Let $D(1), \ldots, D(r)$ be effective integral divisors on $X$ with exceptional support.

**Lemma 5.3.** *(*[12, Lemma 2.2]*) For* $n_1, \ldots, n_r \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{\ell_R(R/I(nn_1D(1)) \cdots I(nn_rD(r)))}{n^d} = \lim_{n\to\infty} \frac{\ell_R(S/J(nn_1D(1)) \cdots J(nn_rD(r)))}{n^d}.$$

5.4. **Divisorial $m_R$-Filtrations.** In this subsection, let $R$ be a local domain.

Let $\mu_1,\ldots,\mu_s$ be $m_R$-valuations, and $a_1,\ldots,a_s\in\mathbb{N}$ with $a_1+\cdots+a_s>0$. Then we define a divisorial $m_R$-filtration

$$\mathcal{I}(a_1\mu_1+\cdots+a_s\mu_s)=\{I(a_1\mu_1+\cdots+a_s\mu_s)_n\}$$

by

$$I(a_1\mu_1+\cdots+a_s\mu_s)_n=I(\mu_1)_{na_1}\cap\cdots\cap I(\mu_s)_{na_s}.$$

We can also define real divisorial $m_R$-filtrations by taking $a_1,\ldots,a_s\in\mathbb{R}_{\geq 0}$ and defining an $m_R$-filtration $\mathcal{I}(a_1\mu_1+\cdots+a_s\mu_s)=\{I(a_1\mu_1+\cdots+a_s\mu_s)_n\}$ by

$$I(a_1\mu_1+\cdots+a_s\mu_s)_n=I(\mu_1)_{\lceil na_1\rceil}\cap\cdots\cap I(\mu_s)_{\lceil na_s\rceil}.$$

A real divisorial $m_R$-filtration will be called a rational divisorial $m_R$-filtration if $a_i\in\mathbb{Q}_{\geq 0}$ for all $i$ and will be called an integral divisorial $m_R$-filtration, or just a divisorial $m_R$-filtration if $a_i\in\mathbb{N}$ for all $i$.

The first statement of the following proposition is proven for the case when $\mathcal{I}=\mathcal{I}(a_1\mu_1+\cdots+a_t\mu_t)$ is an integral divisorial $m_R$-filtration in [12, Proposition 2.1]. However, the proof given there extends to the case when $\mathcal{I}$ is a real divisorial $m_R$-filtration. The second statement follows from [16, Theorem 1.4].

**Proposition 5.4.** ([12, Proposition 2.1], [16, Theorem 1.4]) *Suppose that $R$ is an excellent, analytically irreducible $d$-dimensional local domain.*

1) *Suppose that $\mathcal{I}=\mathcal{I}(a_1\mu_1+\cdots+a_t\mu_t)$ is a real divisorial $m_R$-filtration. Then*
$$e_R(\mathcal{I};R)>0.$$
2) *Suppose that $\mathcal{I}(1),\ldots,\mathcal{I}(r)$ are $m_R$-filtrations such that $e_R(\mathcal{I}(j))>0$ for all $j$. Then*
$$e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};R)>0$$
*for all $d_1,\ldots,d_r\in\mathbb{N}$ with $d_1+\cdots+d_r=d$.*

If $\mathcal{I}$ is a real divisorial $m_R$-filtration on an analytically irreducible excellent local ring $R$, then Rees's Izumi Theorem [33] shows that $\gamma_\mu(\mathcal{I})>0$ for all $m_R$-valuations $\mu$.

5.5. **Representations of divisorial $m_R$-filtrations on normal local rings.** In this subsection, suppose that $R$ is a normal excellent local domain. We now define a representation of a real divisorial $m_R$-filtration $\mathcal{I}(b_1\mu_1+\cdots+b_s\mu_s)$. Let $\varphi:X\to\mathrm{Spec}(R)$ be a birational projective morphism that is the blowup of an $m_R$-primary ideal such that $X$ is normal, and so that if $E_1,\ldots,E_r$ are the prime exceptional divisors of $\varphi$ and $\mu_{E_i}$ are the discrete valuations with valuation rings $\mathcal{O}_{X,E_i}$ for $1\leq i\leq r$, then $\mu_i=\mu_{E_i}$ for $1\leq i\leq s$ with $1\leq s\leq r$.

The pair of $X\to\mathrm{Spec}(R)$ and the real divisor $b_1E_1+\cdots+b_sE_s$ will be called a representation of the real divisorial $m_R$-filtration $\mathcal{I}(b_1\mu_1+\cdots+b_s\mu_s)$.

We remark that it may not be possible to construct an $X$ for which $r=s$, even in dimension $d=2$. This follows from the example of a two dimensional excellent normal local domain without a "one fibered ideal" given in [6].

We now tie this back in with our original real divisorial $m_R$-filtration $\mathcal{I}(b_1\mu_1+\cdots+b_s\mu_s)$, for which the pair of $X$ and $b_1E_1+\cdots+b_sE_s$ is a representation. Letting $D$ be the real divisor $D=b_1E_1+\cdots+b_sE_s$ on $X$, we have for all $m\in\mathbb{N}$ that

$$I(m(\gamma_{E_1}(D)E_1+\cdots+\gamma_{E_r}(D)E_r))=I(mD)=I(b_1\mu_1+\cdots+b_s\mu_s)_m$$

for all $m$. Thus we have equality of $m_R$-filtrations

$$\mathcal{I}(\gamma_{E_1}(D)E_1 + \cdots + \gamma_{E_r}(D)E_r) = \mathcal{I}(D) = \mathcal{I}(b_1\mu_1 + \cdots + b_s\mu_s).$$

In particular, every divisorial $m_R$-filtration has the form $\mathcal{I}(D)$ for some divisor $D = \sum a_iE_i$ with exceptional support on some $X$.

If the pair $X'$ and $D'$ is another representation of $\mathcal{I}(b_1\mu_1 + \cdots + b_s\mu_s)$, then there are prime exceptional divisors $E'_1, \ldots, E'_s$ on $X'$ such that we have equality of local rings $\mathcal{O}_{X,E_i} = \mathcal{O}_{X',E'_i} = \mathcal{O}_{\mu_i}$ for $1 \le i \le s$ and $D' = \sum_{i=1}^{s} b_iE'_i$.

We remark that even when $\mathcal{I}$ is an integral divisorial $m_R$-filtration, $\gamma_\mu(\mathcal{I})$ can be an irrational number for some $m_R$-valuation $\mu$. From 15.1, we find an example of $X$ with two prime exceptional divisors $E_1$ and $E_2$ such that

$$\gamma_{E_1}(E_2) = \frac{3}{9-\sqrt{3}}$$

is an irrational number. We will often abuse notation, denoting a real divisorial $m_R$-filtration by $\mathcal{I}(D)$.

5.6. **Bounded $m_R$-Filtrations.**

**Definition 5.5.** *Let $R$ be a local ring and $\mathcal{I}$ be an $m_R$-filtration. Let $R[\mathcal{I}]$ be the $R$-algebra*

$$R[\mathcal{I}] = \sum_{m \ge 0} I_m t^m$$

*and $\overline{R[\mathcal{I}]}$ be the integral closure of $R[\mathcal{I}]$ in the polynomial ring $R[t]$.*

If $I$ is an ideal in a local ring $R$, let $\overline{I}$ denote its integral closure.

**Lemma 5.6.** *Let $R$ be a local ring and $\mathcal{I}$ be an $m_R$-filtration. Then*

$$\overline{R[\mathcal{I}]} = \sum_{m \ge 0} J_m t^m$$

*where $\{J_m\}$ is the $m_R$-filtration*

$$J_m = \{f \in R \mid f^r \in \overline{I_{rm}} \text{ for some } r > 0\}.$$

**Remark 5.7.** *If $\mathcal{I} = \{I^i\}$ is the filtration of powers of a fixed $m_R$-primary ideal $I$ then $J_m = \overline{I_m}$ for all $m$.*

*Proof.* The ring $\overline{R[\mathcal{I}]}$ is graded by [36, Theorem 2.3.2]. Thus it suffices to show that for $f \in R$ and $n \in \mathbb{Z}_{>0}$ we have that $ft^n$ is integral over $R[\mathcal{I}]$ if and only if $f^r \in \overline{I_{rn}}$ for some $r \ge 1$. Now $ft^n$ is integral over $R[\mathcal{I}]$ if and only if there exists a homogeneous relation

$$(ft^n)^d + a_{d-1}t^n(ft^n)^{d-1} + \cdots + a_it^{n(d-i)}(ft^n)^i + \cdots + a_0t^{nd} = 0 \tag{19}$$

for some $d > 0$ with $a_i \in I_{n(d-i)}$ for all $i$.

We will show that $ft^n$ is integral over $R[\mathcal{I}]$ if and only if there exists $r > 0$ such that $f^r \in \overline{I_{rn}}$.

Suppose that $f^r \in \overline{I_{rn}}$. Then there exists a relation

$$(f^r)^d + a_{d-1}(f^r)^{d-1} + \cdots + a_i(f^r)^i + \cdots + a_0 = 0$$

with $a_i \in (I_{rn})^{d-i} \subset I_{rn(d-i)}$ for all $i$. Multiply this relation by $t^{rnd}$ to get a relation of type (19), showing that $(ft^n)^r$ is integral over $R[\mathcal{I}]$. Thus $ft^n$ is integral over $R[\mathcal{I}]$.

Now suppose that $ft^n$ is integral over $R[\mathcal{I}]$. We will break the proof up into two cases.

**Case 1.** Assume that $R[\mathcal{I}]$ is Noetherian. Then there exists $r > 0$ such that $I_{ri} = I_r^i$ for all $i \in \mathbb{Z}_{>0}$ by [4, Proposition 3, Section 1.3, Chapter III]. Since $f^r t^{rn}$ is integral over $R[\mathcal{I}]$, there exists a relation (19) with $f$ replaced with $f^r$ and $n$ with $rn$, so $a_i \in I_{rn(d-i)} = I_{rn}^{d-i}$ and thus $f^r \in \overline{I_{rn}}$.

**Case 2.**(General Case) Assume that $\mathcal{I}$ is an arbitrary $m_R$-filtration.

For $a \in \mathbb{Z}_{>0}$, let $\mathcal{I}_a = \{I_{a,n}\}$ where $I_{a,n} = I_n$ if $n \le a$ and if $n > a$ then $I_{a,n} = \sum I_{a,i} I_{a,j}$ where the sum is over $i, j > 0$ such that $i + j = n$.

Now $ft^n$ integral over $R[\mathcal{I}]$ implies there exits $a > 0$ such that $ft^n$ is integral over $R[\mathcal{I}_a]$. By Case 1, there exists $r > 0$ such that $f^r \in \overline{I_{a,rn}} \subset \overline{I_{rn}}$. ☐

**Lemma 5.8.** *Let $R$ be a local domain and $\mathcal{I}(D)$ be a divisorial $m_R$-filtration. Then $R[\mathcal{I}(D)]$ is integrally closed in $R[t]$.*

*Proof.* We have that $\mathcal{I}(D) = \mathcal{I}(\alpha_1\mu_1 + \cdots + \alpha_s\mu_s)$ where $\mu_1, \ldots, \mu_s$ are $m_R$-valuations and $\alpha_1, \ldots, \alpha_s \in \mathbb{R}_{>0}$. Since $\overline{R[\mathcal{I}]}$ is graded, we must show that if $f \in R$ and $n \in \mathbb{Z}_{>0}$ are such that $ft^n \in \overline{R[\mathcal{I}(D)]}$, then $f \in I(nD) = I(\mu_1)_{\lceil n\alpha_1 \rceil} \cap \cdots \cap I(\mu_s)_{\lceil n\alpha_s \rceil}$. Now $ft^n \in \overline{R[\mathcal{I}(D)]}$ implies there exists a relation

$$f^d + a_{d-1}f^{d-1} + \cdots + a_i f^i + \cdots + a_0 = 0$$

with $a_i \in I(n(d-i)D)$ for all $i$ by (19). Suppose that $f \notin I(nD)$. Then there exists $j$ such that $\mu_j(f) < \lceil n\alpha_j \rceil$. Thus $\mu_j(f) < n\alpha_j$ since $\mu_j(f) \in \mathbb{N}$ and so

$$(d-i)\mu_j(f) < n(d-i)\alpha_j \le \lceil n(d-i)\alpha_j \rceil$$

for all $i$ with $0 \le i < d$. Thus

$$d\mu_j(f) < \lceil n(d-i)\alpha_j \rceil + i\mu_j(f)$$

for all $i$ with $0 \le i < d$ so that

$$\mu_j(f^d + a_{d-1}f^{d-1} + \cdots + a_i f^i + \cdots + a_0) = d\mu_j(f) \in \mathbb{N}.$$

Thus $f^d + a_{d-1}f^{d-1} + \cdots + a_i f^i + \cdots + a_0 \ne 0$, a contradiction, and so $f \in I(nD)$. ☐

**Definition 5.9.** *Suppose that $R$ is a local domain. An $m_R$-filtration $\mathcal{I} = \{I_n\}$ is said to be bounded if there exists an integral divisorial $m_R$-filtration $\mathcal{I}(D)$ such that*

$$\overline{R[\mathcal{I}]} = R[\mathcal{I}(D)].$$

*An $m_R$-filtration $\mathcal{I} = \{I_n\}$ is said to be real bounded if there exists a real divisorial $m_R$-filtration $\mathcal{I}(D)$ such that*

$$\overline{R[\mathcal{I}]} = R[\mathcal{I}(D)].$$

**Lemma 5.10.** *Suppose that $R$ is an excellent local domain and $\mathcal{I} = \{I^n\}$ is the $m_R$-filtration of powers of a fixed $m_R$-primary ideal $I$. Then $\mathcal{I}$ is bounded.*

*Proof.* We have that $\overline{R[\mathcal{I}]} = \oplus_{n \ge 0} \overline{I^n} u^n$ where $\overline{I^n}$ is the integral closure of $I^n$ in $R$. The algebra $\oplus_{n \ge 0} \overline{I^n} u^n$ is a finite $R[\mathcal{I}]$-module, so that $\{\overline{I^n}\}$ is a Noetherian filtration. Let $\varphi : X \to \mathrm{Spec}(R)$ be the normalization of the blowup of $I$ and $E_1, \ldots, E_t$ be the prime exceptional divisors of $\varphi$. Then $I\mathcal{O}_X = \mathcal{O}_X(-a_1E_1 - \cdots - a_tE_t)$ for some $a_1, \ldots, a_t \in \mathbb{Z}_{>0}$ is an ample Cartier divisor on $X$ and $I^n\mathcal{O}_X = \mathcal{O}_X(-na_1E_1 - \cdots - na_tE_t)$ for all $n \in \mathbb{N}$. Thus for $n \in \mathbb{N}$,

$$\overline{I^n} = \Gamma(X, \mathcal{O}_X(-na_1E_1 - \cdots - na_tE_t)) \cap R = I(a_1\mu_{E_1} + \cdots + a_t\mu_{E_t})_n$$

where $\mu_{E_i}$ is the $m_R$-valuation whose valuation ring is $\mathcal{O}_{X,E_i}$. Thus $\{\overline{I^n}\}$ is the divisorial filtration $\mathcal{I}(a_1\mu_{E_1} + \cdots + a_t\mu_{E_t}) = \mathcal{I}(D)$ where $D = a_1E_1 + \cdots + a_tE_t$ and $\overline{R[\mathcal{I}]} = R[\mathcal{I}(D)]$. ☐

**Proposition 5.11.** *Suppose that $R$ is a local ring with $\dim N(\hat{R}) < d$ and*

$$\mathcal{I}(1), \ldots, \mathcal{I}(r), \mathcal{I}'(1), \ldots, \mathcal{I}'(r)$$

*are $m_R$-filtrations such that $\overline{R[\mathcal{I}'(i)]} = \overline{R[\mathcal{I}(i)]}$ for $1 \le i \le r$. Then we have equality of all mixed multiplicities*

$$e(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}) = e(\mathcal{I}'(1)^{[d_1]}, \ldots, \mathcal{I}'(r)^{[d_r]}). \tag{20}$$

*Proof.* Write $\overline{R[\mathcal{I}(i)]} = \oplus_{n\ge 0} J(i)_n$ and let $\mathcal{J}(i) = \{J(i)_n\}$ for $1 \le i \le r$. We will show that for all mixed multiplicities,

$$e(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}) = e(\mathcal{J}(1)^{[d_1]}, \ldots, \mathcal{J}(r)^{[d_r]}). \tag{21}$$

The same argument applied to $\mathcal{I}'(1), \ldots, \mathcal{I}'(r)$ and $\mathcal{J}(1), \ldots, \mathcal{J}(r)$ will show show that equation (20) holds. Let

$$P(n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\ell_R(R/J(1)_{mn_1} \cdots J(r)_{mn_r})}{m^d}$$

and

$$Q(n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\ell_R(R/I(1)_{mn_1} \cdots I(r)_{mn_r})}{m^d}.$$

Since $\oplus_{m\ge 0} J(i)_m$ is integral over $\oplus_{m\ge 0} I(i)_m$ for all $i$, we have that the graded $R$-algebra $\oplus_{m_1,\ldots,m_r\ge 0} J(1)_{m_1} \cdots J(r)_{m_r}$ is integral over the graded $R$-algebra

$$\oplus_{m_1,\ldots,m_r\ge 0} I(1)_{m_1} \cdots I(r)_{m_r}.$$

Thus for fixed $n_1, \ldots, n_r \in \mathbb{N}$, we have that $\oplus_{m\ge 0} J(1)_{mn_1} \cdots J(r)_{mn_r}$ is integral over $\oplus_{m\ge 0} I(1)_{mn_1} \cdots I(r)_{mn_r}$. By [14, Theorem 6.9] or [12, Appendix] (summarized in Subsection 1.1) we have that

$$P(n_1, \ldots, n_r) = Q(n_1, \ldots, n_r)$$

for all $n_1, \ldots, n_r \in \mathbb{N}$. Since $P(n_1, \ldots, n_r)$ and $Q(n_1, \ldots, n_r)$ are homogeneous polynomials of the same degree $d$, we have that $P(n_1, \ldots, n_r)$ and $Q(n_1, \ldots, n_r)$ have the same values for all $n_1, \ldots, n_r$ in the infinite field $\mathbb{Q}$. Thus their coefficients are equal showing (21). ☐

## 6. A framework to compute multiplicities

In this section, we summarize a construction from [12, Section 3].

Let $R$ be an excellent local domain of dimension $d$ and let $\mu$ be an $m_R$-valuation. Since $R$ is excellent, there exists a birational projective morphism $\varphi : X \to \mathrm{Spec}(R)$ such that $X$ is the normalization of the blow up of an $m_R$-primary ideal, $X$ is normal and there exists a prime exceptional divisor $E$ on $X$ such that $\mu = \mu_E$.

Let $t$ be a generator of the maximal ideal of the valuation ring $\mathcal{O}_{X,E}$. Regarding $t^{-1}$ as an element of the quotient field $K$ of $R$, we compute its divisor $(t^{-1}) = -E + D$ on $X$, which is a Cartier divisor and where $D$ is a Weil divisor which does not contain $E$ in its support ($D$ will have non exceptional support). Write $D = D_1 - D_2$ where $D_1$ and $D_2$ are effective Weil divisors which do not contain $E$ in their supports.

Since $X \to \mathrm{Spec}(R)$ is projective, there exists an ample Cartier divisor $H$ on $X$.

For all $n$, there exist natural inclusions of reflexive rank 1 sheaves

$$\mathcal{O}_X(-D_2 - E + nH) \subset \mathcal{O}_X(-D_2 + nH) \subset \mathcal{O}_X(nH).$$

This can be seen by restricting to the nonsingular locus $U$ of $X$ (which has codimension $\geq 2$ in $X$) and then pushing the sequence forward to $X$. Taking global sections, we thus have inclusions

$$\Gamma(X, \mathcal{O}_X(-D_2 - E + nH)) \subset \Gamma(X, \mathcal{O}_X(-D_2 + nH)) \subset \Gamma(X, \mathcal{O}_X(nH)).$$

Since $H$ is an ample Cartier divisor, there exists a multiple $n$ of $H$ such that

$$\Gamma(X, \mathcal{O}_X(-D_2 - E + nH))$$

is a proper subset of $\Gamma(X, \mathcal{O}_X(-D_2 + nH))$. Thus there exists $\sigma \in \Gamma(X, \mathcal{O}_X(nH))$ such that the divisor $(\sigma)$ (considering $\sigma$ as a global section of $\mathcal{O}_X(nH)$) is an effective Cartier divisor which has the property that the Weil divisor $(\sigma) - D_2$ is effective and $E$ is not in the support of $(\sigma) - D_2$.

Thus $-E + D + (\sigma)$ is a Cartier divisor and

$$-E + D + (\sigma) = -E + D_1 - D_2 + (\sigma) = -E + F$$

where $F = D_1 - D_2 + (\sigma)$ is an effective Weil divisor which does not contain $E$ in its support.

The natural inclusions $\mathcal{O}_X(-nE) \to \mathcal{O}_X(-nE + nF)$ for $n \in \mathbb{N}$ induce inclusions

$$I(\mu)_n = \Gamma(X, \mathcal{O}_X(-nE)) \cap R \to \Gamma(X, \mathcal{O}_X(-nE)) \to \Gamma(X, \mathcal{O}_X(-nE + nF))$$

for all $n$.

Let $q \in E$ be a closed point that is nonsingular on both $X$ and $E$ and is not contained in the support of $F$. Let

$$X = X_0 \supset X_1 = E \supset \cdots \supset X_d = \{q\} \tag{22}$$

be a flag; that is, the $X_j$ are subvarieties of $X$ of dimension $d - j$ such that there is a regular system of parameters $b_1, \ldots, b_d$ in $\mathcal{O}_{X,q}$ such that $b_1 = \cdots = b_j = 0$ are local equations of $X_j$ for $1 \leq j \leq d$.

The flag determines a valuation $\nu$ on the quotient field $K$ of $R$ which dominates $R$ as follows. We have a sequence of natural surjections of regular local rings

$$\mathcal{O}_{X,q} = \mathcal{O}_{X_0,q} \overset{\sigma_1}{\to} \mathcal{O}_{X_1,q} = \mathcal{O}_{X_0,q}/(b_1) \overset{\sigma_2}{\to} \cdots \overset{\sigma_{d-1}}{\to} \mathcal{O}_{X_{d-1},q} = \mathcal{O}_{X_{d-2},q}/(b_{d-1}). \tag{23}$$

Define a rank-$d$ discrete valuation $\nu$ on $K$ (an Abhyankar valuation) by prescribing for $s \in \mathcal{O}_{X,q}$,

$$\nu(s) = (\mathrm{ord}_{X_1}(s), \mathrm{ord}_{X_2}(s_1), \cdots, \mathrm{ord}_{X_d}(s_{d-1})) \in (\mathbb{Z}^d)_{\mathrm{lex}}$$

where

$$s_1 = \sigma_1\left(\frac{s}{b_1^{\mathrm{ord}_{X_1}(s)}}\right), s_2 = \sigma_2\left(\frac{s_1}{b_2^{\mathrm{ord}_{X_2}(s_1)}}\right), \ldots, s_{d-1} = \sigma_{d-1}\left(\frac{s_{d-2}}{b_{d-1}^{\mathrm{ord}_{X_{d-1}}(s_{d-2})}}\right)$$

and $\mathrm{ord}_{X_{j+1}}(s_j)$ is the highest power of $b_{j+1}$ that divides $s_j$ in $\mathcal{O}_{X_j,q}$. We have that

$$\nu(s) = \left(\mu(s) = \mu_E(s), \omega\left(\frac{s}{b_1^{\mu_E(s)}}\right)\right)$$

where $\omega$ is the rank-$(d-1)$ Abhyankar valuation on the function field of $E$ determined by the flag

$$E = X_1 \supset \cdots \supset X_d = \{q\}$$

on the projective $k$-variety $E$, where $k = R/m_R$.

By our construction, $\mathcal{O}_X(-E+F)$ is an invertible sheaf on $X$ and so $\mathcal{O}_X(-E+F)\otimes\mathcal{O}_E$ is an invertible sheaf on $E$. Consider the graded linear series $L_n := \Gamma(E, \mathcal{O}_X(-nE+nF)\otimes_{\mathcal{O}_X}\mathcal{O}_E)$ on $E$. Recall that $b_1 = 0$ is a local equation of $E$ in $\mathcal{O}_{X,q}$. Let $g = b_1$. Thus, since $q$ is not in the support of $F$, for $n \in \mathbb{N}$, we have a natural commutative diagram

(24)

$$\begin{array}{ccccccc} I(\mu)_n & \subset \Gamma(X, \mathcal{O}_X(-nE)) & \to & \Gamma(X, \mathcal{O}_X(-nE+nF)) & \to & \Gamma(E, \mathcal{O}_X(-nE+nF)\otimes\mathcal{O}_E) \\ & \downarrow & & \downarrow & & \downarrow \\ & \mathcal{O}_X(-nE)_q & \stackrel{=}{\to} & \mathcal{O}_X(-nE+nF)_q & \to & \mathcal{O}_X(-nE+nF)_q \otimes_{\mathcal{O}_{X,q}} \mathcal{O}_{E,q} \\ & = \mathcal{O}_{X,q}g^n & & = \mathcal{O}_{X,q}g^n & & \cong \mathcal{O}_{E,q}\otimes_{\mathcal{O}_{X,q}} \mathcal{O}_{X,q}g^n \end{array}$$

where we denote the rightmost vertical arrow by $s \mapsto \varepsilon_n(s)\otimes g^n$ and the bottom horizontal arrow is

$$f \mapsto \left[\frac{f}{g^n}\right] \otimes g^n,$$

where $\left[\frac{f}{g^n}\right]$ is the class of $\frac{f}{g^n}$ in $\mathcal{O}_{E,q}$.

Let $\Xi$ be the semigroup defined by

(25) $$\Xi = \{(n, \omega(\varepsilon_n(s))) \mid n \in \mathbb{N} \text{ and } s \in \Gamma(E, \mathcal{O}_X(-nE+nF)\otimes_{\mathcal{O}_X}\mathcal{O}_E)\} \subset \mathbb{Z}^d,$$

and let

(26)

$\Delta(\Xi)$ be the intersection of the closed convex cone generated by $\Xi$ in $\mathbb{R}^d$ with $\{1\}\times\mathbb{R}^{d-1}$.

By the proof of Theorem 8.1 [8], $\Delta(\Xi)$ is compact and convex. Let

(27) $$\Xi_n := \{(n, \omega(\varepsilon_n(s))) \mid s \in \Gamma(E, \mathcal{O}_Z(-nE+nF)\otimes_{\mathcal{O}_X}\mathcal{O}_E)\}$$

be the elements of $\Xi$ at level $n$.

We will require the following important observation, which follows from the diagram (24).

(28) Suppose that $f \in R$ and $\nu(f) = (a_1, \ldots, a_d)$. Then $\nu(f) \in \Xi_{a_1}$.

## 7. Multiplicities of filtrations

Let notations be as in Section 6, so that $R$ is an excellent local domain. We further assume in this section that $R$ is analytically irreducible.

Let $\mathcal{I} = \{I_i\}$ be an $m_R$-filtration. For $m \in \mathbb{N}$, define

$$\Gamma(\mathcal{I})_m = \{(\nu(f), m) \mid f \in I_m\} \subset \mathbb{N}^{d+1}$$

which are the elements at level $m$ of the semigroup

$$\Gamma(\mathcal{I}) = \cup_{m\in\mathbb{N}}\{(\nu(f), m) \mid f \in I_m\}.$$

Define an associated closed convex set $\Delta(\mathcal{I}) \subset \mathbb{R}^d$ as follows. Let $\Sigma(\mathcal{I})$ be the closed convex cone with vertex at the origin generated by $\Gamma(\mathcal{I})$ and let $\Delta(\mathcal{I}) = \Sigma(\mathcal{I})\cap(\mathbb{R}^d\times\{1\})$. The set $\Delta(\mathcal{I})$ is the closure in the Euclidean topology of the set

$$\left\{\left(\frac{a_1}{i}, \cdots, \frac{a_d}{i}\right) \mid (a_1, \ldots, a_d, i) \in \Gamma(\mathcal{I}) \text{ and } i > 0\right\}.$$

For $m\in\mathbb{N}$, define

$$\Gamma(R)_m=\{(\nu(f),m)\mid f\in R\}\subset\mathbb{N}^{d+1}.$$

which are the elements at level $m$ of the semigroup

$$\Gamma(R)=\cup_{m\in\mathbb{N}}\{(\nu(f),m)\mid f\in R\}.$$

Define an associated closed convex set $\Delta(R)\subset\mathbb{R}^d$ as follows. Let $\Sigma(R)$ be the closed convex cone with vertex at the origin generated by $\Gamma(R)$ and let $\Delta(R)=\Sigma(R)\cap(\mathbb{R}^d\times\{1\})$. The set $\Delta(R)$ is the closure in the Euclidean topology of the set

$$\left\{\left(\frac{a_1}{i},\cdots,\frac{a_d}{i}\right)\mid (a_1,\ldots,a_d,i)\in\Gamma(R)\text{ and } i>0\right\}.$$

**Lemma 7.1.** *The closed convex set $\Delta(R)$ is a closed convex cone in $\mathbb{R}^d_{\ge 0}$ with vertex at the origin 0.*

*Proof.* We identify $\mathbb{R}^d\times\{1\}$ with $\mathbb{R}^d$. We have that $(\nu(1),1)=(0,\ldots,0,1)\in\Gamma(R)$. Thus $(0,\ldots,0)\in\Delta(R)\subset\mathbb{R}^d$.

Suppose that $(a_1,\ldots,a_d,i)\in\Gamma(R)$ with $i>0$. Let $x=(\frac{a_1}{i},\ldots,\frac{a_d}{i})\in\Delta(R)$. Let $\alpha\in\mathbb{Q}_{>0}$. Then $\alpha=\frac{m}{n}$ with $m,n\in\mathbb{Z}_{>0}$. There exists $f\in R$ such that $\nu(f)=(a_1,\ldots,a_d)$. Now $f^m\in R$ so $(\nu(f^m),in)=(ma_1,\ldots,ma_d,in)\in\Gamma(R)$. Thus $\alpha x\in\Delta(R)$.

Suppose that $x\in\Delta(R)$ is non zero. Let $U=\{tx\mid t\in\mathbb{R}_{\ge0}\}$. We must show that $U\subset\Delta(R)$. Let $y\in U$ be nonzero. Then $y=sx$ for some $s\in\mathbb{R}_{>0}$. Suppose that $\varepsilon\in\mathbb{R}_{>0}$. Choose $\delta\in\mathbb{R}_{>0}$ such that $\delta<\min\{1,\frac{1}{|s|},\frac{1}{|x|}\}\varepsilon$. There exists $(a_1,\ldots,a_d,i)\in\Gamma(R)$ with $i>0$ such that $|x-(\frac{a_1}{i},\ldots,\frac{a_d}{i})|<\delta$ and there exist $m,n\in\mathbb{Z}_{>0}$ such that $|s-\frac{m}{n}|<\delta$. Now $\frac{m}{n}(\frac{a_1}{i},\ldots,\frac{a_d}{i})\in\Delta(R)$ as we showed in the above paragraph. Let $\alpha=s-\frac{m}{n}$, $v=x-(\frac{a_1}{i},\ldots,\frac{a_d}{i})$. We compute

$$\begin{aligned}|y-\tfrac{m}{n}(\tfrac{a_1}{i},\ldots,\tfrac{a_d}{i})| &= |sx-(s-\alpha)(x-v)|=|sv+\alpha x-\alpha v|\\ &\le |s||v|+|\alpha||x|+|\alpha||v|\le|s|\delta+|x|\delta+\delta^2<3\varepsilon.\end{aligned}$$

Since we can make $\varepsilon$ arbitrarily small and $\Delta(R)$ is a closed set, we have that $y\in\Delta(R)$. ☐

For $c\in\mathbb{R}_{>0}$, let

$$H_c=\{(x_1,\ldots,x_d)\in\mathbb{R}^d\mid x_1+\cdots+x_d=c\},\tag{29}$$

$$H_c^-=\{(x_1,\ldots,x_d)\in\mathbb{R}^d\mid x_1+\cdots+x_d\le c\}\tag{30}$$

and

$$H_c^+=\{(x_1,\ldots,x_d)\in\mathbb{R}^d\mid x_1+\cdots+x_d\ge c\}.\tag{31}$$

Since $\Delta(R)$ is a closed cone in $\mathbb{R}^d$ with vertex 0 and $cH_1=H_c$, $cH_1^-=H_c^-$, we have

$$\Delta(R)\cap H_c=c(\Delta(R)\cap H_1)\text{ and }\Delta(R)\cap H_c^-=c(\Delta(R)\cap H_1^-).\tag{32}$$

The proof of the following lemma is a simplification of the proofs of Lemmas 8.2 and 8.3 in the following Section 8 (this is where the assumption that $R$ is analytically irreducible is needed).

**Lemma 7.2.** *There exists $\lambda\in\mathbb{Z}_{>0}$ such that $\Delta(\mathcal{I})\cap H_\lambda^+=\Delta(R)\cap H_\lambda^+$.*

For $c \in \mathbb{R}_{>0}$ define $\Delta_c(\mathcal{I}) = \Delta(\mathcal{I}) \cap H_c^-$ and $\Delta_c(R) = \Delta(R) \cap H_c^-$. These sets are compact convex subsets of $\mathbb{R}^d_{\geq 0}$.

Let $\lambda$ be the number defined in Lemma 7.2. If $\varphi \geq \lambda$, then

$$\Delta(\mathcal{I}) \setminus \Delta_c(\mathcal{I}) = \Delta(R) \setminus \Delta_c(R). \tag{33}$$

For $m \in \mathbb{N}$, let

$$\Gamma_c(\mathcal{I})_m = \{(\nu(f), m) \mid f \in I_m \text{ and } a_1 + \cdots + a_d \leq mc\}$$

and

$$\Gamma_c(R) = \{(\nu(f), i) \mid f \in R \text{ and } a_1 + \cdots + a_d \leq mc\}.$$

Define semigroups $\Gamma_c(\mathcal{I}) = \cup_{m \in \mathbb{N}} \Gamma_c(\mathcal{I})_m$ and $\Gamma_c(R) = \cup_{m \in \mathbb{N}} \Gamma_c(R)_m$. The semigroups $\Gamma_c(\mathcal{I})$ and $\Gamma_c(R)$ satisfy the condition (5) of [8, Theorem 3.2] since they are contained in $\mathbb{R}^{d+1}_{\geq 0} \cap H_c^-$.

We now verify that condition (6) of [8, Theorem 3.2] is satisfied; that is, that $\Gamma_c(\mathcal{I})$ generates $\mathbb{Z}^{d+1}$ as a group. Let $G(\Gamma_c(\mathcal{I}))$ be the subgroup of $\mathbb{Z}^{d+1}$ generated by $\Gamma_c(\mathcal{I})$. The value group of $\nu$ is $\mathbb{Z}^d$ and $e_j = \nu(b_j)$ for $1 \leq j \leq d$ is the natural basis of $\mathbb{Z}^d$. Write $b_j = \frac{f_j}{g_j}$ with $f_j, g_j \in R$ for $1 \leq j \leq d$. There exists $0 \neq h \in I_1$. Thus $hf_j, hg_j \in I_1$. Possibly replacing $\lambda$ with a larger value, we then have that $(\nu(hf_j), 1), (\nu(hg_j), 1) \in \Gamma_c(\mathcal{I})$ for $1 \leq j \leq d$. Thus $(e_j, 0) = (\nu(hf_j) - \nu(hg_j), 0) \in G(\Gamma_c(\mathcal{I}))$ for $1 \leq j \leq d$. Since $(\nu(hf_j), 1) \in \Gamma_c(\mathcal{I})$, we then have that $(0, 1) \in G(\Gamma_c(\mathcal{I}))$, and so condition (6) of [8, Theorem 3.2] is satisfied.

Thus the limits

$$\lim_{m \to \infty} \frac{\#\Gamma_c(\mathcal{I})_m}{m^d} = \mathrm{Vol}(\Delta_c(\mathcal{I}))$$

and

$$\lim_{m \to \infty} \frac{\#\Gamma_c(R)_m}{m^d} = \mathrm{Vol}(\Delta_c(R))$$

exist by [8, Theorem 3.2]. As in [9, Theorem 5.6], if $c \geq \lambda$, where $\lambda$ is chosen sufficiently large, then

$$\lim_{m \to \infty} \frac{\ell_R(R/I_m)}{m^d} = \delta[\mathrm{Vol}(\Delta_c(R)) - \mathrm{Vol}(\Delta_c(\mathcal{I}))] \tag{34}$$

where $\delta = [\mathcal{O}_{X,p}/m_p : R/m_R]$.

Thus the multiplicity

$$e_R(\mathcal{I}) := d! \lim_{m \to \infty} \frac{\ell_R(R/I_m)}{m^d} = d!\delta[\mathrm{Vol}(\Delta_c(R)) - \mathrm{Vol}(\Delta_c(\mathcal{I}))].$$

Define

$$\Delta_\lambda(\mathcal{I}) = \Delta(\mathcal{I}) \cap H_\lambda^- \text{ for an } m_R\text{- filtration } \mathcal{I} \text{ and } \lambda \in \mathbb{R}. \tag{35}$$

**Theorem 7.3.** *Suppose that $R$ is an analytically irreducible excellent local domain and that $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are $m_R$-filtrations such that $I(1)_i \subset I(2)_i$ for all $i$ and $e_R(\mathcal{I}(1)) = e_R(\mathcal{I}(2))$. Then*

$$\gamma_\mu(\mathcal{I}(1)) = \gamma_\mu(\mathcal{I}(2))$$

*for all $m_R$-valuations $\mu$ of $R$.*

The proof which we give below follows from the first part of the proof of [12, Theorem 3.4], applied to our filtrations $\mathcal{I}(1)$ and $\mathcal{I}(2)$ (instead of the divisorial $m_R$-filtrations $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ of Cartier divisors $D_1$ and $D_2$ of the statement of [12, Theorem 3.4]).

*Proof.* We apply the construction of Section 6 with $\mu_{E_1} = \mu$. Let $\pi_1 : \mathbb{R}^d \to \mathbb{R}$ be the projection onto the first factor. By the definition of $\gamma_\mu(\mathcal{I}(i))$ for $i = 1, 2$, and since for $c$ sufficiently large, $\gamma_\mu(\mathcal{I}(i))$ is in the compact set $\pi_1(\Delta_c(\mathcal{I}(i)),\ \pi_1^{-1}(\gamma_\mu(\mathcal{I}(i)) \cap \Delta_c(\mathcal{I}(i)) \neq \emptyset$ and $\pi_1^{-1}(a) \cap \Delta_c(\mathcal{I}(i)) = \emptyset$ if $a < \gamma_\mu(\mathcal{I}(i))$. Since $\mathcal{I}(1)_i \subset \mathcal{I}(2)_i$ for all $i$, we have that $\Delta_c(\mathcal{I}(1)) \subset \Delta_c(\mathcal{I}(2))$. Now $\mathrm{Vol}(\Delta_c(\mathcal{I}(1)) > 0$ for $c$ sufficiently large. Since we assume $e_R(\mathcal{I}(1)) = e_R(\mathcal{I}(2))$, we have that $\mathrm{Vol}(\Delta_c(\mathcal{I}(1)) = \mathrm{Vol}(\Delta_c(\mathcal{I}(2))$ by (34) and so $\Delta_c(\mathcal{I}(1)) = \Delta_c(\mathcal{I}(2))$ by [12, Lemma 3.2]. Thus $\gamma_\mu(\mathcal{I}(1)) = \gamma_\mu(\mathcal{I}(2))$. ☐

**Corollary 7.4.** *Let $R$ be a normal excellent local domain, $\mathcal{I} = \{I_m\}$ be an $m_R$-filtration and $\mathcal{I}(D)$ be a real divisorial $m_R$-filtration. Suppose that $I(mD) \subset I_m$ for all $m$ and $e_R(\mathcal{I}) = e_R(\mathcal{I}(D))$. Then $\mathcal{I} = \mathcal{I}(D)$.*

*Proof.* The ring $R$ is analytically irreducible since $R$ is normal and excellent. Let the pair $X \to \mathrm{Spec}(R)$ and $D = \sum_{i=1}^r a_i E_i$ be a representation of $\mathcal{I}(D)$. We have that $\gamma_{\mu_{E_i}}(\mathcal{I}) = \gamma_{E_i}(D)$ for $1 \le i \le r$ by Theorem 7.3. We have that $I(mD) = \cap_{i=1}^r I(\mu_{E_i})_{\lceil m\gamma_{E_i}(D)\rceil} \subset I_m$ for all $m$ by assumption. Suppose that $f \in I_m$. Then

$$\mu_{E_i}(f) \ge \tau_{\nu_{E_i},m}(\mathcal{I}) \ge m\gamma_{\mu_{E_i}}(\mathcal{I}) = m\gamma_{E_i}(D)$$

for $1 \le i \le r$. Thus $\mu_{E_i}(f) \ge \lceil m\gamma_{E_i}(D)\rceil$ for all $i$, and so $f \in \cap_{i=1}^r I(\mu_{E_i})_{\lceil m\gamma_{E_i}(D)\rceil} = I(mD)$. ☐

**Corollary 7.5.** *Let $R$ be an excellent local domain, $\mathcal{I}(D)$ be a real divisorial $m_R$-filtration and $\mathcal{I}$ be an arbitrary $m_R$-filtration. Suppose that $I(nD) \subset I_n$ for all $n$ and $e_R(\mathcal{I}(D)) = e_R(\mathcal{I})$. Then $\mathcal{I} = \mathcal{I}(D)$.*

*Proof.* If $R$ is normal, the corollary is immediate from Corollary 7.4, so we may assume that $R$ is not normal. We use the notation of Subsection 5.3. Let $S$ be the normalization of $R$ and let $m_1, \ldots, m_t$ be the maximal ideals of $S$. Let $X \to \mathrm{Spec}(R)$ and $D = \sum a_{i,j} E_{i,j}$ be a representation of $D$. Let $X_i = X \otimes_S S_{m_i}$ for $1 \le i \le t$. We have that $D = \sum_{i=1}^t D(i)$ where $D(i) = \sum_j a_{i,j} E_{i,j}$. Let $J(nD) = \Gamma(X, \mathcal{O}_X(-nD))$, so that $\mathcal{I}(D) = \{I(nD)\}$ where $I(nD) = J(nD) \cap R$. Further, we have real divisorial $m_i$-filtrations $\mathcal{J}(D(i)) = \{J(nD_i)\}$ on $S_{m_i}$ which are defined by $J(nD(i)) = \Gamma(X_i, \mathcal{O}_{X_i}(-nD(i))) = \Gamma(X, \mathcal{O}_X(-nD))S_{m_i}$.

Let $\mathcal{I}S_{m_i}$ be the $m_i$-filtration $\mathcal{I}S_{m_i} = \{I_n S_{m_i}\}_{n \ge 0}$. Then we have that

$$S/I_nS \cong \bigoplus_{i=1}^t (S_{m_i}/I_nS_{m_i})$$

for all $n$ and so

$$\ell_R(S/I_nS) = \sum_{i=1}^t [S/m_i : R/m_R]\ell_{S_{m_i}}(S_{m_i}/I_nS_{m_i}).$$

Now the proof of [12, Lemma 2.2] extends to this situation to show that

$$\lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d} = \lim_{n\to\infty} \frac{\ell_R(S/I_nS)}{n^d}$$

from which we deduce that

$$e_R(\mathcal{I}) = \sum_{i=1}^t [S/m_i : R/m_R] e_{S_{m_i}}(\mathcal{I}S_{m_i}). \tag{36}$$

Similarly,

$$e_R(\mathcal{I}(D)) = \sum_{i=1}^{t}[S/m_i : R/m_R]e_{S_{m_i}}(\mathcal{I}(D)S_{m_i}). \tag{37}$$

Let $0 \neq x$ be in the conductor of $S/R$. Then $xJ(nD) \subset I(nD)$ for all $n$. Let

$$A := R[\mathcal{I}(D)] = \sum_{n\geq 0} I(nD)t^n,$$

$$B := \sum_{n\geq 0} J(nD)t^n,$$

and for $a \in \mathbb{Z}_{>0}$, let

$${}^aA := R[\mathcal{I}(D)_a],$$

where $\mathcal{I}(D)_a$ is the $a$-th truncation of $\mathcal{I}(D)$, that is, ${}^aA$ is the sub $R$-algebra generated by $I(nD)$ such that $n \leq a$ and let ${}^aB$ be the sub $S$-algebra of $B$ generated by $J(nD)$ such that $n \leq a$.

We have $xB \subset A$ and $xB_a \subset A_a$. Suppose that $f \in J(mD)t^m$. Then $f \in ({}^mB)_m$ and $f^n \in ({}^mB)_{mn}$ for all $n$ so that $xf^n \in ({}^mA)_{mn}$ for all $n$ so that ${}^mA[f] \subset \frac{1}{x}({}^mA)$ which is a finitely generated ${}^mA$-module, so $f$ is integral over the Noetherian ring ${}^mA$, and therefore $f$ is integral over $A$. Thus $B$ is integral over $A$ and so $B$ is integral over $C := \sum_{n\geq 0} I(nD)St^n$, and thus $B_{m_i} = S_{m_i}[\mathcal{J}(D(i))]$ is integral over $C_{m_i} = S_{m_i}[\mathcal{I}(D)S_{m_i}]$ for $1 \leq i \leq t$. We then have that

$$e_{S_{m_i}}(\mathcal{I}(D)S_{m_i}) = e_{S_{m_i}}(\mathcal{J}(D(i)) \tag{38}$$

for $1 \leq i \leq t$ by [12, Theorem 1.4].

Let $G = \sum_{n\geq 0} L_nt^n$ be the integral closure of $F = \sum_{n\geq 0} I_nSt^n$ in $S[t]$. Then

$$e_{S_{m_i}}(\{L_nS_{m_i}\}) = e_{S_{m_i}}(\mathcal{I}S_{m_i}) \tag{39}$$

for $1 \leq i \leq t$ by [12, Theorem 1.4]. Now $I(nD)S_{m_i} \subset I_nS_{m_i}$ for all $i$, so that

$$J(nD(i)) \subset L_nS_{m_i} \tag{40}$$

for all $i$ and so

$$e_{S_{m_i}}(\mathcal{J}(D(i))) \geq e_{S_{m_i}}(\{L_nS_{m_i}\}) \tag{41}$$

for $1 \leq i \leq t$. We have that

$$\sum_{i=1}^{t}[S/m_i : R/m_R]e_{S_{m_i}}(\{L_nS_{m_i}\}) = \sum_{i=1}^{t}[S/m_i : R/m_R]e_{S_{m_i}}(\mathcal{J}(D(i)))$$

by equations (38), (39), (36) and (37). Thus, by (41), we have

$$e_{S_{m_i}}(\{L_nS_{m_i}\}) = e_{S_{m_i}}(\mathcal{J}(D(i))) \tag{42}$$

for $1 \leq i \leq t$. By Corollary 7.4, we then have that $L_nS_{m_i} = J(nD)S_{m_i}$ for $1 \leq i \leq t$, and so $L_n = J(nD)$ for all $n$. Thus

$$I(nD) = R \cap J(nD) \subset I_n \subset L_n \cap R = J(nD) \cap R = I(nD)$$

for all $n$.

$\square$

Corollary 7.4 is proven when $R$ is an excellent local domain and $D_1$ and $D_2$ are Cartier divisors in addition to $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ being integral divisorial $m_R$-filtrations in [12, Theorem 3.5].

**Remark 7.6.** *Suppose that $R$ is an analytically irreducible excellent local ring, $\mathcal{I} = \{I_n\}$ is an $m_R$-filtration and $l \in \mathbb{Z}_+$. Let $\mathcal{J}_l$ be the $m_R$-filtration $\mathcal{J}_l = \{I_{ln}\}$. Then $\Delta(\mathcal{J}_l) = l\Delta(\mathcal{I})$.*

The following proposition will be used in our study of mixed multiplicities. Here, $\Delta(\mathcal{A})$, $\Delta(\mathcal{B})$ and $\nu$ are as defined in Section 6.

**Proposition 7.7.** *Suppose that $R$ is a normal excellent local domain and that $\varphi : X \to \mathrm{Spec}(R)$ is the normalization of the blow up of an $m_R$-primary ideal with prime exceptional divisors $E_1, \ldots, E_r$. Suppose that $\gamma_1, \ldots, \gamma_r, \xi \in \mathbb{R}_{\geq 0}$ are such that $\gamma_1 + \cdots + \gamma_r > 0$ and $\xi > 0$. Consider the $m_R$-filtrations $\mathcal{A} = \mathcal{I}(\sum_{i=1}^r \gamma_i E_i)$ and $\mathcal{B} = \mathcal{I}(\sum_{i=1}^r \xi\gamma_i E_i)$. Then*

$$\xi\Delta(\mathcal{A}) = \Delta(\mathcal{B}).$$

*Proof.* We have that $\mathcal{A} = \{A_n\}$ and $\mathcal{B} = \{B_n\}$ where

$$A_n = \Gamma(X, \mathcal{O}_X(-\lceil \sum_{i=1}^r n\gamma_i E_i \rceil)) \text{ and } B_n = \Gamma(X, \mathcal{O}_X(-\lceil \sum_{i=1}^r n\xi\gamma_i E_i \rceil)).$$

It suffices to show that for all $\tau \in \mathbb{R}$ sufficiently large, we have that

$$\xi\left(\Delta(\mathcal{A}) \cap H_\tau^-\right) = \Delta(\mathcal{B}) \cap H_{\tau\xi}^-.$$

The half space $H_c^-$ is defined in (30).

Let $\mathcal{C} = \{C_n\}$ be the $m_R$-filtration defined by

$$C_n = \Gamma(X, \mathcal{O}_X(-\lceil \sum_{i=1}^r (n\gamma_i + 1)E_i \rceil)).$$

Let $E = E_1 + \cdots + E_r$.

We now show that $\Delta(\mathcal{C}) = \Delta(\mathcal{A})$. Let $0 \neq f \in m_R$. Then $fA_n \subset C_n$ for all $n$. The elements of the form $\frac{\nu(g)}{m} = (\frac{a_1}{m}, \ldots, \frac{a_d}{m})$ with $g \in A_m$ and $m > 0$ are dense in $\Delta(\mathcal{A})$. Since $\Delta(\mathcal{C}) \subset \Delta(\mathcal{A})$ is a closed set, it suffices to show that given $\varepsilon > 0$, there exists $n \in \mathbb{Z}_{>0}$ and $h \in C_n$ such that $|\frac{\nu(h)}{n} - \frac{\nu(g)}{m}| < \varepsilon$. For all $t \in \mathbb{Z}_{>0}$, $(\nu(g^t), tm) \in A_{tm}$ and $(\nu(fg^t), tm) \in C_{tm}$. Thus $\frac{\nu(f)}{tm} + \frac{\nu(g)}{m} \in \Delta(\mathcal{C})$, with

$$|(\frac{\nu(f)}{tm} + \frac{\nu(g)}{m}) - \frac{\nu(g)}{m}| = \frac{1}{tm}|\nu(f)| < \varepsilon$$

for $t \gg 0$. Thus $\Delta(\mathcal{C}) = \Delta(\mathcal{A})$.

There exists $\tau_0 \in \mathbb{R}_{>0}$ such that

$$\Delta(R) \cap H_{\tau_0}^+ = \Delta(\mathcal{A}) \cap H_{\tau_0}^+ = \Delta(\mathcal{C}) \cap H_{\tau_0}^+ \tag{43}$$

and

$$\Delta(R) \cap H_{\xi\tau_0}^+ = \Delta(\mathcal{B}) \cap H_{\xi\tau_0}^+ \tag{44}$$

by Lemma 7.2.

Suppose that $\tau > \tau_0$. Choose $\delta \in \mathbb{R}_{>0}$ such that $\tau - \delta > \tau_0$. Let

$$\beta = \max\{|y| \mid y \in \Delta_\tau(\mathcal{A})\}.$$

The compact convex set $\Delta_\tau(\mathcal{A})$ is defined in (35). The numbers $\tau$, $\delta$ and $\beta$ will be fixed throughout the proof.

Given $\alpha \in \mathbb{R}_{>0}$, there exist $p_0, q_0 \in \mathbb{Z}_{>0}$ such that

$$-\frac{\alpha}{q_0} < \frac{p_0}{q_0} - \xi \leq 0 \tag{45}$$

by Lemma 4.1, so that

$$p_0 \leq \xi q_0 < p_0 + \alpha. \tag{46}$$

Let $m$ be a positive integer, and suppose that $\alpha$ is sufficiently small that

$$\alpha < \frac{1}{m \max\{\gamma_i\}}.$$

Set $p = mp_0$ and $q = mq_0$. Then

$$\gamma_i p \leq \xi \gamma_i q < \gamma_i p + \alpha \gamma_i m$$

for all $i$, so that

$$\lceil \gamma_i p \rceil \leq \lceil q \gamma_i \xi \rceil \leq \lceil p \gamma_i \rceil + 1$$

for all $i$ and so

$$-(\lceil p\gamma_i \rceil + 1) \leq -\lceil q\gamma_i \xi \rceil \leq -\lceil \gamma_i p \rceil$$

implying

$$-\lceil \sum_{i=1}^{r} (p\gamma_i + 1)E_i \rceil \leq -\lceil \sum_{i=1}^{r} q\gamma_i \xi E_i \rceil \leq -\lceil \sum_{i=1}^{r} \gamma_i p E_i \rceil$$

giving us that

$$C_p \subset B_q \subset A_p. \tag{47}$$

We will now show that $\Delta_{\xi\tau}(\mathcal{B}) = \xi \Delta_\tau(\mathcal{A})$.

First suppose that $v = (v_1, \ldots, v_d) \in \Delta_{\xi\tau}(\mathcal{B})$ and that $v_1 + \cdots + v_d \geq \xi\tau - \xi\delta$. Then $v \in \Delta(\mathcal{B}) \cap H^+_{\xi\tau - \xi\delta} = \Delta(R) \cap H^+_{\xi\tau-\xi\delta}$. Then since $\Delta(R)$ is a cone with vertex at the origin,

$$\frac{1}{\xi} v \in \left(\frac{1}{\xi}\Delta(R)\right) \cap H^+_{\tau-\delta} = \Delta(R) \cap H^+_{\tau-\delta} = \Delta(\mathcal{A}) \cap H^+_{\tau-\delta},$$

by (43) and so $v = \xi u$ for some $u \in \Delta_\tau(\mathcal{A})$.

Now suppose that $v = (v_1, \ldots, v_d) \in \Delta_{\xi\tau}(\mathcal{B})$ and $v_1 + \cdots + v_d < \xi\tau - \xi\delta$. Since the elements of $\Delta(\mathcal{B})$ of the form $\frac{\nu(f)}{m}$ with $m \in \mathbb{Z}_{>0}$ and $f \in B_m$ are dense in $\Delta(\mathcal{B})$, we may suppose that $v$ has this form. Let $\varepsilon > 0$. we will find $u \in \xi\Delta_\tau(\mathcal{A})$ such that $|v - u| < \varepsilon$. Since $\xi\Delta_\tau(\mathcal{A})$ is closed, this will show that $v \in \xi\Delta_\tau(\mathcal{A})$.

Choose $\alpha \in \mathbb{R}_{>0}$ such that

$$\alpha < \min\{\frac{1}{m \max \gamma_i}, \frac{\delta}{\tau}, \frac{\varepsilon}{\beta}\}.$$

Choose $p_0, q_0$ which satisfy (45). Thus

$$0 \leq \xi - \frac{p_0}{q_0} < \frac{\alpha}{q_0}.$$

Write $\xi = \frac{p_0}{q_0} + \lambda$ with $0 \leq \lambda < \frac{\alpha}{q_0}$.

Set $p = mp_0$ and $q = mq_0$, so that $v = \left(\frac{a_1}{q}, \ldots, \frac{a_d}{q}\right) = \frac{\nu(f^{q_0})}{q}$ where $f^{q_0} \in B_q$. Set

$$u = \xi\left(\frac{a_1}{p}, \ldots, \frac{a_d}{p}\right) \text{ and } w = -\lambda\left(\frac{a_1}{p}, \ldots, \frac{a_d}{p}\right)$$

so that $v = \frac{p}{q}\left(\frac{a_1}{p}, \ldots, \frac{a_d}{p}\right) = u + w$. Now $\left(\frac{a_1}{p}, \ldots, \frac{a_d}{p}\right) \in \Delta(\mathcal{A})$ by (47). Since by (46), $\frac{q}{p} < \frac{1}{\xi} + \frac{\alpha}{\xi p_0}$ and $\alpha < \frac{\delta}{\tau}$, we have that

$$\begin{array}{lll} \frac{a_1}{p} + \cdots + \frac{a_d}{p} & = & \frac{q}{p}\left(\frac{a_1}{q} + \cdots + \frac{a_d}{q}\right) < \frac{q}{p}(\xi\tau - \xi\delta) \\ & < & \left(\frac{1}{\xi} + \frac{\alpha}{\xi p_0}\right)(\xi\tau - \xi\delta) = \tau - \delta + \frac{\alpha\tau}{p_0} - \frac{\alpha\delta}{p_0} < \tau - \delta + \frac{\delta\tau}{\tau p_0} - \frac{\alpha\delta}{p_0} \\ & = & \tau - \delta\left(1 - \frac{1}{p_0}\right) - \frac{\alpha\delta}{p_0} < \tau. \end{array}$$

Thus $\left(\frac{a_1}{p}, \ldots, \frac{a_d}{p}\right) \in \Delta_\tau(\mathcal{A})$ and $|v - u| = |w| \le |\lambda|\beta < \frac{\alpha\beta}{q_0} < \varepsilon$. Since we can make $\varepsilon$ arbitrarily small, we have that $v \in \xi\Delta_\tau(\mathcal{A})$.

We will now show that $\xi\Delta_\tau(\mathcal{A}) \subset \Delta_{\xi\tau}(\mathcal{B})$.

First suppose that $u = (u_1, \ldots, u_d) \in \xi\Delta_\tau(\mathcal{A})$ and that $u_1 + \cdots + u_d \ge \xi\tau - \xi\delta$. Then $\frac{1}{\xi}u \in \Delta_\tau(\mathcal{A})$ with $\frac{1}{\xi}u_1 + \cdots + \frac{1}{\xi}u_d \ge \tau - \delta$, so that

$$\frac{1}{\xi}u \in \Delta(\mathcal{A}) \cap H^+_{\tau-\delta} = \Delta(R) \cap H^+_{\tau-\delta}$$

by (43) so

$$u \in \xi\left(\Delta(R) \cap H^+_{\tau-\delta}\right) = \Delta(R) \cap H^+_{\tau\xi-\delta\xi} = \Delta(\mathcal{B}) \cap H^+_{\tau\xi-\delta\xi}$$

by (44). Since $u_1 + \cdots + u_d < \tau\xi$ we have that $u \in \Delta_{\xi\tau}(\mathcal{B})$.

Now suppose that $u = (u_1, \ldots, u_d) \in \xi\Delta_\tau(\mathcal{A})$ and that $u_1 + \cdots + u_d < \xi\tau - \xi\delta$. Since the elements of $\Delta(\mathcal{C})$ of the form $\frac{\nu(f)}{m}$ with $m \in \mathbb{Z}_{>0}$ and $f \in C_m$ are dense in $\Delta(\mathcal{C}) = \Delta(\mathcal{A})$, we may suppose that $u$ is $\xi$ times an element of this form. Let $\varepsilon > 0$. we will find $v \in \Delta_{\tau\xi}(\mathcal{B})$ such that $|u - v| < \varepsilon$. Since $\Delta_{\tau\xi}(\mathcal{B})$ is closed, this will show that $u \in \Delta_{\tau\xi}(\mathcal{B})$.

Choose $\alpha \in \mathbb{R}_{>0}$ such that

$$\alpha < \max\{\frac{1}{m\max\{\gamma_i\}}, \frac{\varepsilon}{\beta}\}.$$

Choose $p_0, q_0$ which satisfy (45). Thus

$$0 \le \xi - \frac{p_0}{q_0} < \frac{\alpha}{q_0}.$$

Write $\xi = \frac{p_0}{q_0} + \lambda$ with $0 \le \lambda < \frac{\alpha}{q_0}$.

Set $p = mp_0$ and $q = mq_0$, so that $u = \xi\left(\frac{a_1}{p}, \ldots, \frac{a_d}{p}\right)$ with $\left(\frac{a_1}{p}, \ldots, \frac{a_d}{p}\right) = \frac{\nu(f^{p_0})}{p}$ where $f^{p_0} \in C_p$ and $\frac{a_1}{p} + \cdots + \frac{a_d}{p} < \tau - \delta$. Set $v = \frac{p}{q}\left(\frac{a_1}{p}, \ldots, \frac{a_d}{p}\right) = \left(\frac{a_1}{q}, \ldots, \frac{a_d}{q}\right)$ and $w = \lambda\left(\frac{a_1}{p}, \ldots, \frac{a_d}{p}\right)$ so that $u = v + w$.

We have that $v \in \Delta(\mathcal{B})$ by (47). We have that

$$\frac{a_1}{q} + \cdots + \frac{a_d}{q} < \frac{p}{q}(\tau - \delta) < \xi(\tau - \delta) < \xi\tau$$

so that $v \in \Delta_{\xi\tau}(\mathcal{B})$.

$|v - u| = |w| \le |\lambda|\beta < \frac{\alpha\beta}{q_0} < \varepsilon$. Since we can make $\varepsilon$ arbitrarily small, we have that $u \in \Delta_{\xi\tau}(\mathcal{B})$.

□

## 8. Computation of Mixed Multiplicities of filtrations

Let notation be as in Section 6, so that $R$ is a $d$-dimensional excellent local domain. We further assume that $R$ is analytically irreducible in this section.

Let $\mathcal{I}(1) = \{I(1)_i\}$, $\mathcal{I}(2) = \{I(2)_i\}$ be $m_R$-filtrations. We now define some sub semigroups of $\mathbb{N}^{d+1}$ which are associated to $\mathcal{I}(1)$ and $\mathcal{I}(2)$. For $n_1, n_2 \in \mathbb{N}$, define

$$\Gamma(n_1, n_2) = \{(\nu(f), i) \mid f \in I(1)_{in_1} I(2)_{in_2}\}.$$

We define an associated closed convex subset $\Delta(n_1, n_2)$ of $\mathbb{R}^d$ as follows. Let $\Sigma(n_1, n_2)$ be the closed convex cone with vertex at the origin generated by $\Gamma(n_1, n_2)$ and let $\Delta(n_1, n_2) = \Sigma(n_1, n_2) \cap (\mathbb{R}^d \times \{1\})$. The set $\Delta(n_1, n_2)$ is the closure of the set

$$\left\{\left(\frac{a_1}{i}, \ldots, \frac{a_d}{i}\right) \mid (a_1, \ldots, a_d, i) \in \Gamma(n_1, n_2) \text{ and } i > 0\right\}$$

in the Euclidean topology of $\mathbb{R}^d$. We have that $\Gamma(R) = \Gamma(0, 0)$ and $\Delta(R) = \Delta(0, 0)$ as defined in Section 7.

**Lemma 8.1.** *For all $m_1, m_2, n_1, n_2 \in \mathbb{N}$, we have that*

$$\Delta(m_1, m_2) + \Delta(n_2, n_2) \subset \Delta(m_1 + n_1, m_2 + n_2).$$

*In particular,*

$$n_1\Delta(1, 0) + n_2\Delta(0, 1) \subset \Delta(n_1, n_2).$$

*Proof.* The set of points

$$\left\{\left(\frac{a_1}{i}, \ldots, \frac{a_d}{i}\right) \mid (a_1, \ldots, a_d, i) \in \Gamma(m_1, m_2) \text{ and } i > 0\right\}$$

is dense in the closed set $\Delta(m_1, m_2)$. Thus it suffices to show that if $(a_1, \ldots, a_d, i) \in \Gamma(m_1, m_2)$ and $(b_1, \ldots, b_d, j) \in \Gamma(n_1, n_2)$ with $i, j > 0$, then

$$\left(\frac{a_1}{i} + \frac{b_1}{j}, \ldots, \frac{a_d}{i} + \frac{b_d}{j}\right) \in \Delta(m_1 + n_1, m_2 + n_2).$$

With this assumption, there exists $f \in I(1)_{im_1} I(2)_{im_2}$ such that $\nu(f) = (a_1, \ldots, a_d)$ and there exists $g \in I(1)_{jn_1} I(2)_{jn2}$ such that $\nu(g) = (b_1, \ldots, b_d)$. We have that $f^j g^i \in I(1)_{ij(m_1+n_1)} I(2)_{ij(m_2+n_2)}$ so

$$(\nu(f^j g^i), ij) = (ja_1 + ib_1, \ldots, ja_d + ib_d, ij) \in \Gamma(m_1 + n_1, m_2 + n_2).$$

Thus

$$\left(\frac{ja_1 + ib_1}{ij}, \ldots, \frac{ja_d + ib_d}{ij}\right) = \left(\frac{a_1}{i} + \frac{b_1}{j}, \ldots, \frac{a_d}{i} + \frac{b_d}{j}\right) \in \Delta(m_1 + n_1, m_2 + n_2).$$

□

**Lemma 8.2.** *There exists $\overline{\lambda} \in \mathbb{Z}_{>0}$ such that for all $n_1, n_2 \in \mathbb{N}$,*

$$\begin{aligned}&\Delta(n_1, n_2) \cap \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 \geq (n_1 + n_2)\overline{\lambda}\}\\ &= \Delta(R) \cap \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 \geq (n_1 + n_2)\overline{\lambda}\}.\end{aligned}$$

*Proof.* Let $\nu_1, \ldots, \nu_t$ be the Rees valuations of $m_R$. Since $R$ is analytically irreducible, the topologies of the $\nu_j$ on $R$ are linearly equivalent to the topology of $\mu$ on $R$ by Rees's Izumi Theorem [33]. Let $\overline{\nu}_{m_R}$ be the reduced order. By the Rees valuation theorem (recalled in [33]), for $x \in R$,

$$\overline{\nu}_{m_R}(x) = \min_j \left\{\frac{\nu_j(x)}{\nu_j(m_R)}\right\}$$

so the topology of $\overline{\nu}_{m_R}$ is linearly equivalent to the topology induced by each $\nu_j$. Further, $\overline{\nu}_{m_R}$ is linearly equivalent to the $m_R$-topology by [32] since $R$ is analytically irreducible. Thus there exists $\alpha \in \mathbb{Z}_{>0}$ such that $I(\mu)_{m\alpha} \subset m_R^m$ for all $m \in \mathbb{N}$. Since $I(1)_1$ and $I(2)_1$ are $m_R$-primary, there exists $c \in \mathbb{Z}_{>0}$ such that $m_R^c \subset I(1)_1$ and $m_R^c \subset I(2)_1$, so that $m_R^{c(n_1+n_2)} \subset I(1)_{n_1} I(2)_{n_2}$ for all $n_1, n_2 \in \mathbb{Z}_{\geq 0}$. Let $\overline{\lambda} = c\alpha$. Then

$$I(\mu)_{(n_1+n_2)\overline{\lambda}} \subset m_R^{c(n_1+n_2)} \subset I(1)_{n_1} I(2)_{n_2} \tag{48}$$

for all $n_1, n_2 \in \mathbb{Z}_{\geq 0}$.

Suppose $(a_1, \ldots, a_d, m) \in \Gamma(R)$ is such that $m > 0$ and $\frac{a_1}{m} \geq (n_1+n_2)\overline{\lambda}$. Then there exists $f \in R$ such that $\nu(f) = (a_1, \ldots, a_d)$. In particular, $\mu(f) = a_1$. Thus $\mu(f) \geq m(n_1+n_2)\overline{\lambda}$ so that $f \in I(a)_{mn_1} I(2)_{mn_2}$ by (48). Thus $(a_1, \ldots, a_d, m) \in \Gamma(n_1, n_2)$ and so

$$\left(\frac{a_1}{m}, \ldots, \frac{a_d}{m}\right) \in \Delta(n_1, n_2).$$

□

**Lemma 8.3.** *There exists $\lambda \in \mathbb{Z}_{>0}$ such that for all $n_1, n_2 \in \mathbb{N}$,*

$$\Delta(n_1, n_2) \cap H^+_{(n_1+n_2)\lambda} = \Delta(R) \cap H^+_{(n_1+n_2)\lambda}.$$

*Proof.* Recall the definitions of $\Xi$, $\Xi_n$ and $\Delta(\Xi)$ in equations (25), (26) and (27). The set $\Delta(\Xi) \subset \{1\} \times \mathbb{R}^{d-1}$ is compact and convex as explained after (26). Thus there exists $b \in \mathbb{Z}_{>0}$ such that $\Delta(\Xi) \subset \{1\} \times [0, b]^{d-1}$. Suppose that $f \in R$ and $\mu(f) \leq \delta$ for some $\delta$. Let $\nu(f) = (a_1 = \mu(f), a_2, \ldots, a_d)$. Then $\nu(f) \in \Xi_{a_1}$ by (28) which implies

$$\left(1, \frac{a_2}{a_1}, \ldots, \frac{a_d}{a_1}\right) \in \Delta(\Xi)$$

so

$$a_i \leq \delta b \text{ for all } i. \tag{49}$$

Choose $\lambda > \overline{\lambda} bd$ where $\overline{\lambda}$ is the constant of Lemma 8.2. Suppose $(a_1, \ldots, a_d, m) \in \Gamma(R)$ is such that $m > 0$ and

$$\frac{a_1}{m} + \cdots + \frac{a_d}{m} \geq (n_1+n_2)\lambda. \tag{50}$$

If

$$\frac{a_1}{m} > (n_1+n_2)\overline{\lambda} \text{ then } \left(\frac{a_1}{m}, \ldots, \frac{a_d}{m}\right) \in \Delta(n_1, n_2)$$

by Lemma 8.2. Suppose $\frac{a_1}{m} \leq (n_1+n_2)\overline{\lambda}$. Then $a_1 \leq m(n_1+n_2)\overline{\lambda}$ so that $a_i \leq m(n_1+n_2)\overline{\lambda} b$ by (49). Thus

$$\frac{a_1}{m} + \cdots + \frac{a_d}{m} \leq (n_1+n_2) bd\overline{\lambda} < (n_1+n_2)\lambda,$$

a contradiction to our assumption (50). Thus the conclusions of our lemma hold. □

Given $\Phi = (\alpha_1, \alpha_2, \varphi) \in \mathbb{R}^3_{>0}$, define

$$H^+_{\Phi, n_1, n_2} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 + \cdots + x_d > (\alpha_1 n_1 + \alpha_2 n_2)\varphi\}.$$

Let $\lambda$ be the number defined in Lemma 8.3. If

$$\varphi \geq \frac{\lambda}{\min\{\alpha_1, \alpha_2\}}, \tag{51}$$

then for $n_1, n_2 \in \mathbb{Z}_{\geq 0}$ with $n_1 + n_2 > 0$, we have

$$(\alpha_1 n_1 + \alpha_2 n_2)\varphi \geq (n_1+n_2)\lambda$$

so

$$(52)\qquad \Delta(R)\cap H^+_{\Phi,n_1,n_2}=\Delta(n_1,n_2)\cap H^+_{\Phi,n_1,n_2}=(n_1\Delta(1,0)+n_2\Delta(0,1))\cap H^+_{\Phi,n_1,n_2}.$$

The second equality in (52) is obtained as follows. Lemma 8.3 implies that

$$\Delta(1,0)\cap H^+_{\Phi,1,0}=\Delta(R)\cap H^+_{\Phi,1,0}\text{ and }\Delta(0,1)\cap H^+_{\Phi,0,1}=\Delta(R)\cap H^+_{\Phi,0,1}.$$

Taking the Minkowski sum, we thus have

$$(n_1\Delta(1,0)+n_2\Delta(0,1))\cap H^+_{\Phi,n_1,n_2}=\Delta(R)\cap H^+_{\Phi,n_1,n_2}.$$

Set

$$(53)\qquad \Delta_\Phi(n_1,n_2)=\Delta(n_1,n_2)\setminus\Delta(n_1,n_2)\cap H^+_{\Phi,n_1,n_2},$$

$$\tilde\Delta_\Phi(n_1,n_2)=\Delta(R)\setminus\Delta(R)\cap H^+_{\Phi,n_1,n_2}.$$

These are compact, convex subsets of $\mathbb{R}^d$. For all $n_1,n_2\in\mathbb{N}$, we have that the Minkowski sum

$$(54)\qquad n_1\Delta_\Phi(1,0)+n_2\Delta_\Phi(0,1)\subset\Delta_\Phi(n_1,n_2)\subset\tilde\Delta_\Phi(n_1,n_2).$$

We now fix $n_1,n_2\in\mathbb{N}$. For $m\in\mathbb{N}$, let

$$\begin{aligned}&\Gamma_\Phi(n_1,n_2)_m\\&=\{(\nu(f),m)=(a_1,\ldots,a_d,m)\mid f\in I(1)_{mn_1}I(2)_{mn_2}\text{ and }a_1+\cdots+a_d\le m(\alpha_1n_1+\alpha_2n_2)\varphi\},\end{aligned}$$

$$\begin{aligned}&\tilde\Gamma_\Phi(n_1,n_2)_m\\&=\{(\nu(f),m)=(a_1,\ldots,a_d,m)\mid f\in R\text{ and }a_1+\cdots+a_d\le m(\alpha_1n_1+\alpha_2n_2)\varphi\}.\end{aligned}$$

The semigroups $\Gamma_\Phi(n_1,n_2)=\cup_{m\in\mathbb{N}}\Gamma_\Phi(n_1,n_2)_m$ and $\tilde\Gamma_\Phi(n_1,n_2)=\cup_{m\in\mathbb{N}}\tilde\Gamma_\Phi(n_1,n_2)_m$ satisfy conditions (5) and (6) of [8, Theorem 3.2] by the argument after Lemma 7.2. Thus the limits

$$\lim_{m\to\infty}\frac{\#\tilde\Gamma_\Phi(n_1,n_2)_m}{m^d}=\mathrm{Vol}(\tilde\Delta_\Phi(n_1,n_2))$$

and

$$\lim_{m\to\infty}\frac{\#\Gamma_\Phi(n_1,n_2)_m}{m^d}=\mathrm{Vol}(\Delta_\Phi(n_1,n_2))$$

exist by [8, Theorem 3.2]. As in [9, Theorem 5.6], if $\varphi\ge\frac{\lambda}{\min\{\alpha_1,\alpha_2\}}$, then

$$(55)\qquad \lim_{m\to\infty}\frac{\ell_R(R/I(1)_{mn_1}I(2)_{mn_2})}{m^d}=\delta[\mathrm{Vol}(\tilde\Delta_\Phi(n_1,n_2))-\mathrm{Vol}(\Delta_\Phi(n_1,n_2))]$$

where

$$(56)\qquad \delta=[\mathcal{O}_\nu/m_\nu:R/m_R].$$

The function

$$(57)\qquad f(n_1,n_2):=\lim_{m\to\infty}\frac{\ell_R(R/I(1)_{mn_1}I(2)_{mn_2})}{m^d}$$

for $n_1,n_2\in\mathbb{N}$ of (2) and (3) is a homogeneous polynomial in $\mathbb{R}[x]$ of degree $d$.

Since $\Delta_\Phi(1,0)$ and $\Delta_\Phi(0,1)$ are compact convex subsets of $\mathbb{R}^d$, the function

$$(58)\qquad g(n_1,n_2)=\mathrm{Vol}(n_1\Delta_\Phi(1,0)+n_2\Delta_\Phi(0,1))$$

is a homogeneous polynomial over $\mathbb{R}$ of degree $d$ for all $n_1, n_2 \in \mathbb{R}_{\geq 0}$ by Theorem 4.2. We have that

$$
\begin{aligned}
\text{Vol}(\tilde{\Delta}_\Phi(n_1,n_2)) &= \text{Vol}(\Delta(R) \cap H^-_{(n_1\alpha_1+n_2\alpha_2)\varphi}) \\
&= \text{Vol}\left((n_1\alpha_1+n_2\alpha_2)\varphi(\Delta(R)\cap H_1^-)\right) \\
&= (n_1\alpha_1+n_2\alpha_2)^d\varphi^d\text{Vol}(\Delta(R)\cap H_1^-)
\end{aligned} \tag{59}
$$

by (32). Thus by (55) and (59),

$$
\text{Vol}(\Delta_\Phi(n_1,n_2)) = (n_1\alpha_1+n_2\alpha_2)^d\varphi^d\text{Vol}(\Delta(R)\cap H_1^-) - \frac{1}{\delta}f(n_1,n_2) \tag{60}
$$

where $f(n_1,n_2)$ is the function of (57). We have that

$$
\text{Vol}(\Delta_\Phi(n_1,n_2)) \geq g(n_1,n_2) \tag{61}
$$

for all $n_1, n_2 \in \mathbb{N}$ by (54).

**Theorem 8.4.** *Suppose that $R$ is a local ring of dimension $d$ with $\dim N(\hat{R}) < d$ and $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are $m_R$-filtrations of $R$. Suppose that there exist $a, b \in \mathbb{Z}_{>0}$ such that*

$$
\overline{\sum_{n\geq 0} I(1)_{an}t^n} = \overline{\sum_{n\geq 0} I(2)_{bn}t^n}. \tag{62}
$$

*Then $\mathcal{I}(1)$ and $\mathcal{I}(2)$ satisfy the Minkowski equality*

$$
e_R(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}} = e_R(\mathcal{I}(1))^{\frac{1}{d}} + e_R(\mathcal{I}(2))^{\frac{1}{d}}.
$$

*Proof.* Let

$$
P(n_1,n_2) = \lim_{m\to\infty} \frac{\ell_R(R/I(1)_{mn_1}I(2)_{mn_2})}{m^d}.
$$

Let $P_1, \ldots, P_s$ be the minimal primes of $\hat{R}$ such that $\dim \hat{R}/P_i = d$. Let $R_i = \hat{R}/P_i$ for $1 \leq i \leq s$. The $R_i$ are analytically irreducible excellent local domains.

By the proof of Theorem 4.7 of [8] we have that

$$
P(n_1,n_2) = \sum_{k=1}^{s} \lim_{m\to\infty} \frac{\ell_{R_i}(R_i/I(1)_{mn_1}I(2)_{mn_2}R_i)}{m^d}.
$$

We first suppose that $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are such that $R[\mathcal{I}(1)] = \overline{R[\mathcal{I}(1)]}$ and $R[\mathcal{I}(2)] = \overline{R[\mathcal{I}(2)]}$ are integrally closed. Then $I(1)_{ma} = I(2)_{mb}$ for all $m \in \mathbb{Z}_{>0}$.

Let $J_m = I(1)_{ma} = I(2)_{mb}$ for $m \in \mathbb{N}$ and $\mathcal{J} = \{J_m\}$. Let

$$
Q(n_1,n_2) = \lim_{m\to\infty} \frac{\ell_R(R/J_{mn_1}J_{mn_2})}{m^d}.
$$

For $1 \leq i \leq s$, let

$$
Q_i(n_1,n_2) = \lim_{m\to\infty} \frac{\ell_{R_i}(R_i/J(i)_{mn_1}J(i)_{mn_2})}{m^d}
$$

where $J(i)_m = J_m R_i$. We have that

$$
Q(n_1,n_2) = \sum_{i=1}^{s} Q_i(n_1,n_2)
$$

by the proof of Theorem 4.7 of [8].

For each $i$, we apply the construction of Section 7 to the $m_{R_i}$-filtration $\{J(i)_m\}$ on $R_i$ and we apply the construction of this section (Section 8) to the $m_{R_i}$-filtrations $\{J(i)_m\}$ and $\{J(i)_m\}$ on $R_i$. We use the notation of these sections.

We have by Remark 7.6 that for all $n_1, n_2 \in \mathbb{N}$ with $n_1 + n_2 > 0$ that

$$
\begin{aligned}
&(n_1+n_2)\Delta(\{J(i)_m\}) \subset n_1\Delta(\{J(i)_m\}) + n_2\Delta(\{J(i)_m\}) = \Delta(\{J(i)_{mn_1}\}) + \Delta(\{J(i)_{mn_2}\})\\
&\subset \Delta(n_1,n_2) = \Delta(\{J(i)_{mn_1}J(i)_{mn_2}\}) \subset \Delta(\{J(i)_{m(n_1+n_2)}\}) = (n_1+n_2)\Delta(\{J(i)_m\})
\end{aligned}
$$

so that $\Delta(n_1, n_2) = (n_1 + n_2)\Delta(\{J(i)_m\})$.

Let $\Phi = (1, 1, \varphi)$ with $\varphi$ sufficiently large. Then

$$
\Delta_\Phi(n_1,n_2) = [(n_1+n_2)\Delta(\{J(i)_m\})] \cap H^-_{(n_1+n_2)\varphi} = (n_1+n_2)[\Delta(\{J(i)_m\}) \cap H^-_\varphi]. \tag{63}
$$

By (55),

$$
Q_i(n_1,n_2) = \delta[\mathrm{Vol}(\tilde{\Delta}_\Phi(n_1,n_2)) - \mathrm{Vol}(\Delta_\Phi(n_1,n_2))]
$$

and by (59),

$$
\mathrm{Vol}(\tilde{\Delta}_\Phi(n_1,n_2)) = (n_1+n_2)^d \mathrm{Vol}(\Delta(R_i) \cap H^-_\varphi).
$$

By (63),

$$
\mathrm{Vol}(\Delta_\Phi(n_1,n_2)) = (n_1+n_2)^d \mathrm{Vol}(\Delta(\{J(i)_m\}) \cap H^-_\varphi),
$$

so that

$$
Q_i(n_1,n_2) = c_i(n_1+n_2)^d
$$

where

$$
c_i = \delta[\mathrm{Vol}(\Delta(R_i) \cap H^-_\varphi) - \mathrm{Vol}(\Delta(\{J(i)_m\}) \cap H^-_\varphi)].
$$

Thus letting $c = \sum_{i=1}^s c_i$, we have that $Q(n_1, n_2) = c(n_1 + n_2)^d$.

Now $P(am_1, bm_2) = Q(m_1, m_2)$ so

$$
P(n_1,n_2) = c(\frac{n_1}{a} + \frac{n_2}{b})^d,
$$

and substituting the values $(n_1, n_2) = (1, 1)$, $(n_1, n_2) = (1, 0)$ and $(n_1, n_2) = (0, 1)$ we get that $\mathcal{I}(1)$ and $\mathcal{I}(2)$ satisfy Minkowski's equality, establishing the theorem if $R[\mathcal{I}(1)]$ and $R[\mathcal{I}(2)]$ are integrally closed.

Now suppose that $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are arbitrary $m_R$-filtrations satisfying (62). Define $m_R$-filtrations $\mathcal{J}(1)$ and $\mathcal{J}(2)$ by setting $\mathcal{J}(1) = \{J(1)_n\}$ and $\mathcal{J}(2) = \{J(2)_n\}$ where

$$
\sum_{n\ge 0} J(1)_n t^n = \overline{\sum_{n\ge 0} I(1)_n t^n} \text{ and } \sum_{n\ge 0} J(2)_n t^n = \overline{\sum_{n\ge 0} I(2)_n t^n}.
$$

Now

$$
\overline{\sum I(1)_{an} t^{an}} = \sum J(1)_{an} t^{an} \text{ and } \overline{\sum I(2)_{bn} t^{an}} = \sum J(2)_{bn} t^{an}
$$

so

$$
\sum_{n\ge 0} J(1)_{an} t^n = \overline{\sum_{n\ge 0} I(1)_{an} t^n} = \overline{\sum_{n\ge 0} I(1)_{bn} t^n} = \sum_{n\ge 0} J(2)_{bn} t^n.
$$

By the first part of the proof, $\mathcal{J}(1)$ and $\mathcal{J}(2)$ satisfy Minkowski's equality, and there is an expression

$$
\lim_{m\to\infty} \frac{\ell_R(R/J(1)_{mn_1}J(2)_{mn_2})}{m^d} = c(\frac{n_1}{a} + \frac{n_2}{b})^d.
$$

By Proposition 5.11,

$$
P(n_1,n_2) = \lim_{m\to\infty} \frac{\ell_R(R/J(1)_{mn_1}J(2)_{mn_2})}{m^d} = c(\frac{n_1}{a} + \frac{n_2}{b})^d,
$$

and substituting the values $(n_1, n_2) = (1, 1)$, $(n_1, n_2) = (1, 0)$ and $(n_1, n_2) = (0, 1)$ we get that $\mathcal{I}(1)$ and $\mathcal{I}(2)$ satisfy Minkowski's equality. ☐

## 9. Equality in the Minkowski inequality for mixed multiplicities

Let $R$ be a $d$-dimensional analytically unramified local domain and $\mathcal{I}(1)$, $\mathcal{I}(2)$ be $m_R$-filtrations.

The polynomial $f(n_1, n_2)$ of (57) has an expansion

$$f(n_1,n_2)=\sum_{d_1+d_2=d}\frac{1}{d_1!d_2!}e_R(\mathcal{I}(1)^{[d_1]},\mathcal{I}(2)^{[d_2]})n_1^{d_1}n_2^{d_2}$$

where $e_R(\mathcal{I}(1)^{[d_1]},\mathcal{I}(2)^{[d_2]})\in\mathbb{R}$ are the mixed multiplicities of $\mathcal{I}(1)$ and $\mathcal{I}(2)$ by (2) and (3). Set $e_i=e_R(\mathcal{I}(1)^{[d-i]},\mathcal{I}(2)^{[i]})$ for $0\le i\le d$. Then

$$f(n_1,n_2)=\sum_{i=0}^{d}\frac{1}{(d-i)!i!}e_in_1^{d-i}n_2^{i}.$$

We have by (5) that

$$e_0=e_R(\mathcal{I}(1))\text{ and }e_d=e_R(\mathcal{I}(2)).$$

By Formulas 3) and 1) of Theorem 1.3,

$$e_i^d\le e_0^{d-i}e_d^i\text{ for }0\le i\le d \tag{64}$$

and

$$e_i^2\le e_{i-1}e_{i+1} \tag{65}$$

for $1\le i\le d-1$. We expand

$$e_R(\mathcal{I}(1)\mathcal{I}(2))=d!f(1,1)=\sum_{i=0}^{d}\binom{d}{i}e_i\le\sum_{i=0}^{d}\binom{d}{i}e_0^{\frac{d-i}{d}}e_d^{\frac{i}{d}}=(e_0^{\frac{1}{d}}+e_d^{\frac{1}{d}})^d$$

obtaining the Minkowski inequality (7). Observe that

(66)

Equality holds in the Minkowski inequality if and only if equality holds in (64) for all $i$.

In this case,

$$f(n_1,n_2)=\sum_{i=0}^{d}\frac{1}{(d-i)!i!}e_0^{\frac{d-i}{d}}e_d^{\frac{i}{d}}n_1^{d-i}n_2^{i}=\frac{1}{d!}(\sqrt[d]{e_0}n_1+\sqrt[d]{e_d}n_2)^d. \tag{67}$$

We now show that when all $e_j$ are positive, the Minkowski equality holds between $\mathcal{I}(1)$ and $\mathcal{I}(2)$ if and only if equality holds in (65). We use an argument from [36]. Applying (65), we have

$$\left(\frac{e_i}{e_{i-1}}\right)^{d-i}\cdots\left(\frac{e_1}{e_0}\right)^{d-i}\le\left(\frac{e_d}{e_{d-1}}\right)^{i}\cdots\left(\frac{e_{i+1}}{e_i}\right)^{i} \tag{68}$$

where there are $i(d-i)$ terms on each side. We have equality in (68) for $1\le i\le d-1$ if and only if equality holds in (65) for all $i$. Now the LHS of (68) is $\frac{e_i^{d-i}}{e_0^{d-i}}$ and the RHS is $\frac{e_d^i}{e_i^i}$ so

(69) Equality holds in (65) for all $i$ if and only if equality holds in (64) for all $i$.

## 10. An analysis of the Minkowski equality

In this section, suppose that $R$ is a $d$-dimensional excellent analytically irreducible local domain and $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are $m_R$-filtrations. We further assume that equality holds in Minkowski's inequality (7) for $\mathcal{I}(1)$ and $\mathcal{I}(2)$. We also make the additional assumptions that $e_0 = e_R(\mathcal{I}(1)) > 0$ and $e_d = e_R(\mathcal{I}(2)) > 0$. We use the notation of Sections 9, 6 and 8.

Set $\alpha_1 = \sqrt[d]{e_0}$ and $\alpha_2 = \sqrt[d]{e_d}$. Choose $\varphi$ so that (51) is satisfied for these values of $\alpha_1$ and $\alpha_2$. Set $\gamma = \varphi^d \mathrm{Vol}(\Delta(R) \cap H_1^-)$. We then define the function $h(n_1, n_2) = \mathrm{Vol}(\Delta_\Phi(n_1, n_2))$. We have that for all $n_1, n_2 \in \mathbb{N}$,

$$h(n_1,n_2) = \mathrm{Vol}(\Delta_\Phi(n_1,n_2)) = (\gamma - \frac{1}{\delta d!})(\alpha_1 n_1 + \alpha_2 n_2)^d \tag{70}$$

where $\delta$ is the constant of (56), by (60) and (67).

Recall the polynomial $g(n_1, n_2)$ of (58). We have that $\mathrm{Vol}(\Delta_\Phi(1,0)) = g(1,0) = h(1,0) > 0$ and $\mathrm{Vol}(\Delta_\Phi(0,1)) = g(0,1) = h(0,1) > 0$ and $g(n_1,n_2) \le h(n_1,n_2)$ for all $n_1, n_2 \in \mathbb{N}$ by (61). Since $g$ and $h$ are homogeneous polynomials of the same degree, we have that $g(a_1,a_2) \le h(a_1,a_2)$ for all $a_1, a_2 \in \mathbb{Q}_{\ge 0}$. Thus, by continuity of polynomials,

$$g(a_1,a_2) \le h(a_1,a_2) \text{ for all } a_1, a_2 \in \mathbb{R}_{\ge 0}. \tag{71}$$

For $0 < t < 1$, we have that

$$\begin{aligned} h(1-t,t)^{\frac{1}{d}} &= (1-t)h(1,0)^{\frac{1}{d}} + th(0,1)^{\frac{1}{d}} = (1-t)g(1,0)^{\frac{1}{d}} + tg(0,1)^{\frac{1}{d}} \\ &\le g(1-t,t)^{\frac{1}{d}} \le h(1-t,t)^{\frac{1}{d}}. \end{aligned}$$

by (70), Theorem 4.3 and (71). Thus $g(1-t,t)^{\frac{1}{d}} = (1-t)g(1,0)^{\frac{1}{d}} + tg(0,1)^{\frac{1}{d}}$ for $0 < t < 1$ and so $\Delta_\Phi(1,0)$ and $\Delta_\Phi(0,1)$ are homothetic by Theorem 4.3.

We have $\mathrm{Vol}(\Delta_\Phi(0,1)) = \frac{e_d}{e_0}\mathrm{Vol}(\Delta(1,0))$ by (70). Let $T : \mathbb{R}^d \to \mathbb{R}^d$, given by

$$T(\vec{x}) = c\vec{x} + \vec{\gamma}$$

be the homothety such that $T(\Delta_\Phi(1,0)) = \Delta_\Phi(0,1)$. We have that

$$\frac{e_d}{e_0}\mathrm{Vol}(\Delta_\Phi(1,0) = \mathrm{Vol}(\Delta_\Phi(0,1)) = \mathrm{Vol}(T(\Delta_\Phi(1,0)) = c^d \mathrm{Vol}(\Delta_\Phi(1,0)$$

so

$$c = \sqrt{\frac{e_d}{e_0}}.$$

By (53), (52) and (29), we have that

$$\Delta_\Phi(1,0) \cap H_\psi = \begin{cases} \emptyset & \text{for } \psi > \sqrt[d]{e_0}\varphi \\ \Delta(R) \cap H_{\sqrt[d]{e_0}\varphi} & \text{for } \psi = \sqrt[d]{e_0}\varphi \end{cases} \tag{72}$$

and

$$\Delta_\Phi(0,1) \cap H_\psi = \begin{cases} \emptyset & \text{for } \psi > \sqrt[d]{e_d}\varphi \\ \Delta(R) \cap H_{\sqrt[d]{e_d}\varphi} & \text{for } \psi = \sqrt[d]{e_d}\varphi. \end{cases} \tag{73}$$

Writing $\vec{\gamma} = (\gamma_1, \ldots, \gamma_d)$, we have $T(H_{\sqrt[d]{e_0}\varphi}) = H_{\sqrt[d]{e_d}\varphi + (\gamma_1 + \cdots + \gamma_d)}$. Comparing equations (72) and (73), we see that $\gamma_1 + \cdots + \gamma_d = 0$.

Now $\Delta(R) \cap H_\psi = \psi(\Delta(R) \cap H_1)$ for all $\psi \in \mathbb{R}_{>0}$ by (32). Thus we may factor the homeomorphism $T : \Delta(R) \cap H_{\sqrt[d]{e_0}\varphi} \to \Delta(R) \cap H_{\sqrt[d]{e_d}\varphi}$ by homeomorphisms

$$\Delta(R) \cap H_{\sqrt[d]{e_0}\varphi} \xrightarrow{c} \Delta(R) \cap H_{\sqrt[d]{e_d}\varphi} \xrightarrow{+\vec{\gamma}} \Delta(R) \cap H_{\sqrt[d]{e_d}\varphi}.$$

But $\Delta(R)\cap H_{\sqrt[d]{e_d}\varphi}$ is a nonempty compact set, so the second map cannot be well defined unless $\vec{\gamma}=0$.

In summary, we have established the following theorem.

**Theorem 10.1.** *Suppose that $R$ is a $d$-dimensional analytically irreducible excellent local ring and $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are $m_R$-filtrations which have positive multiplicity $e_0=e_R(\mathcal{I}(1))>0$ and $e_d=e_R(\mathcal{I}(2))>0$ and that Minkowski's equality*

$$e_R(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}}=e_R(\mathcal{I}(1))^{\frac{1}{d}}+e_R(\mathcal{I}(2)^{\frac{1}{d}}$$

*holds. Let notation be as in sections 6, 8 and 9. Then*

$$\sqrt[d]{e_d}\Delta_\Phi(1,0)=\sqrt[d]{e_0}\Delta_\Phi(0,1)$$

*where $\Phi=(\sqrt[d]{e_0},\sqrt[d]{e_d},\varphi)$ in (53) and $\varphi$ is sufficiently large.*

We also obtain a partial converse to Theorem 10.1.

**Theorem 10.2.** *Suppose that $R$ is a $d$-dimensional analytically irreducible excellent local ring and $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are $m_R$-filtrations which have positive multiplicity $e_0=e_R(\mathcal{I}(1))>0$ and $e_d=e_R(\mathcal{I}(2))>0$. Suppose that $\sqrt[d]{e_d}\Delta_\Phi(1,0)=\sqrt[d]{e_0}\Delta_\Phi(0,1)$ for $\Phi=(\sqrt[d]{e_0},\sqrt[d]{e_d},\varphi)$ in (53) with $\varphi$ sufficiently large and that for the functions of (58) and (70) $g(n_1,n_2)=h(n_1,n_2)$ for all $n_1,n_2\in\mathbb{Z}^2$. Then Minkowski's equality*

$$e_R(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}}=e_R(\mathcal{I}(1))^{\frac{1}{d}}+e_R(\mathcal{I}(2))^{\frac{1}{d}}$$

*holds between $\mathcal{I}(1)$ and $\mathcal{I}(2)$.*

*Proof.* The convex bodies $\Delta_\Phi(1,0)$ and $\Delta_\Phi(0,1)$ are homothetic, so by Theorem 4.3,

$$g(1-t,t)^{\frac{1}{d}}=(1-t)g(1,0)^{\frac{1}{d}}+tg(0,1)^{\frac{1}{d}}$$

for $0\le t\le 1$. Taking $t=\frac{1}{2}$ and since $g$ is a homogeneous polynomial of degree $d$, we obtain that $g(1,1)^{\frac{1}{d}}=g(1,0)^{\frac{1}{d}}+g(0,1)^{\frac{1}{d}}$. Thus $h(1,1)^{\frac{1}{d}}=h(1,0)^{\frac{1}{d}}+h(0,1)^{\frac{1}{d}}$. By equations (59) and (60),

$$\begin{aligned} f(n_1,n_2) &= \delta[\mathrm{Vol}(\tilde{\Delta}_\Phi(n_1,n_2))-\mathrm{Vol}(\Delta_\Phi(n_1,n_2))]\\ &= \delta\varphi^d\mathrm{Vol}(\Delta(R)\cap H_1^-)(\sqrt[d]{e_0}n_1+\sqrt[d]{e_d}n_2)^d-\delta h(n_1,n_2).\end{aligned}$$

Set $\xi=d!\delta\varphi^d\mathrm{Vol}(\Delta(R)\cap H_1^-)$. We have that

$$e_R(\mathcal{I}(1)\mathcal{I}(2))=d!f(1,1)=\xi(\sqrt[d]{e_0}+\sqrt[d]{e_d})^d-d!\delta h(1,1),$$

$$e_0=e_R(\mathcal{I}(1))=d!f(1,0)=\xi e_0-d!\delta h(1,0),$$

$$e_d=e_R(\mathcal{I}(2))=d!f(0,1)=\xi e_d-d!\delta h(0,1).$$

Let $\chi=\frac{\xi-1}{d!\delta}$, so that $h(1,0)=\chi e_0$ and $h(0,1)=\chi e_d$. We have

$$h(1,1)=(h(1,0)^{\frac{1}{d}}+h(0,1)^{\frac{1}{d}})^d=\chi(\sqrt[d]{e_0}+\sqrt[d]{e_d})^d$$

and

$$e_R(\mathcal{I}(1)\mathcal{I}(2))=(\xi-d!\delta\chi)(\sqrt[d]{e_0}+\sqrt[d]{e_d})^d.$$

Now $\xi-d!\delta\chi=1$ so the Minkowski equality

$$e_R(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}}=e_R(\mathcal{I}(1))^{\frac{1}{d}}+e_R(\mathcal{I}(2))^{\frac{1}{d}}$$

holds. □

**Theorem 10.3.** *Suppose that $R$ is a $d$-dimensional analytically irreducible excellent local ring and $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are $m_R$-filtrations such that $e_R(\mathcal{I}(1))$ and $e_R(\mathcal{I}(2))$ are both non zero and equality holds in the Minkowski inequality (7) for $\mathcal{I}(1)$ and $\mathcal{I}(2)$. Then for all $m_R$-valuations $\mu$ of $R$, we have that*

$$e(\mathcal{I}(2))^{\frac{1}{d}}\gamma_\mu(\mathcal{I}(1)) = e(\mathcal{I}(1))^{\frac{1}{d}}\gamma_\mu(\mathcal{I}(2)).$$

*Proof.* Starting with $\nu = (\mu, \omega)$ in the construction of Section 6, construct $\Delta_\Phi(n_1, n_2)$ as in Section 8, so that the conclusions of Theorem 10.1 hold. Let $\pi : \mathbb{R}^d \to \mathbb{R}$ be the projection onto the first factor. By definition of $\gamma_\mu(\mathcal{I}(1))$, $\gamma_\mu(\mathcal{I}(1))$ is in the compact set $\pi(\Delta_\Phi(1,0))$, $\pi^{-1}(\gamma_\mu(\mathcal{I}(1)) \cap \Delta_\Phi(1,0) \neq \emptyset$ and $\pi^{-1}(a) \cap \Delta_\Phi(1,0) = \emptyset$ if $a < \gamma_\mu(\mathcal{I}(1))$. In the same way, we have that $\pi^{-1}(\gamma_\mu(\mathcal{I}(2)) \cap \Delta_\Phi(0,1) \neq \emptyset$ and $\pi^{-1}(a) \cap \Delta_\Phi(0,1) = \emptyset$ if $a < \gamma_\mu(\mathcal{I}(2))$.

Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be the homothety $T(\vec{x}) = c\vec{x}$ where $c = \frac{\sqrt[d]{e_d}}{\sqrt[d]{e_0}}$ which takes $\Delta_\Phi(1,0)$ to $\Delta_\Phi(0,1)$. Now since $T$ multiplies the first coefficient of an element of $\Delta_\Phi(1,0)$ by $c$, and the smallest first coefficient of an element of $\Delta_\Phi(1,0)$ is $\gamma_\mu(\mathcal{I}(1)$, the smallest first coefficient of an element of $\Delta_\Phi(0,1)$ is $\gamma_\mu(\mathcal{I}(2)) = c\gamma_\mu(\mathcal{I}(1))$. □

Let us verify that these equalities do in fact hold in the classical case of $m_R$-primary ideals $I(1)$ and $I(2)$ satisfying the Minkowski equality. In this case, we have the (Noetherian) $m_R$-filtrations $\mathcal{I}(1) = \{I(1)^i\}$ and $\mathcal{I}(2) = \{I(2)^i\}$. Since the Minkowski equality holds, we have that there exists $m, n \in \mathbb{Z}_+$ such that $\overline{I(1)^m} = \overline{I(2)^n}$ where $\overline{I(1)^m}$ and $\overline{I(2)^n}$ are the respective integral closures of ideals by the Teissier, Rees and Sharp, Katz Theorem [39], [34], [21] recalled in Subsection 1.2. Now

$$e(\overline{I(1)^m}) = m^d e(I(1)) = m^d e(\mathcal{I}(1)),$$

$$e(\overline{I(2)^n}) = n^d e(I(2)) = n^d e(\mathcal{I}(2))$$

and

$$m\gamma_\mu(\mathcal{I}(1)) = m\mu(I(1)) = \mu(\overline{I(1)^m}),$$

$$n\gamma_\mu(\mathcal{I}(2)) = n\mu(I(2)) = \mu(\overline{I(2)^n}),$$

giving the desired formula

$$e(\mathcal{I}(2))^{\frac{1}{d}}\gamma_\mu(\mathcal{I}(1)) = e(\mathcal{I}(1))^{\frac{1}{d}}\gamma_\mu(\mathcal{I}(2)).$$

## 11. Equality of mixed multiplicities on normal excellent local rings

**Theorem 11.1.** *Let $R$ be a $d$-dimensional normal excellent local domain and let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be real divisorial $m_R$-filtrations. Let $X \to Spec(R)$ and $D_1 = \sum_{i=1}^r a_iE_i$, $D_2 = \sum_{i=1}^r b_iE_i$ be a representation of $D_1$ and $D_2$. Suppose that Minkowski's equality holds for $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$. Then there exists an effective real Weil divisor $\sum_{i=1}^r \gamma_iE_i$ such that*

$$\gamma_{E_j}(D_i) = \gamma_j e(\mathcal{I}(D_i))^{\frac{1}{d}}$$

*for all $j$ and $i$ and*

$$I(mD_i) = \Gamma(X, \mathcal{O}_X(-\lceil \sum_{j=1}^r m\gamma_j e(\mathcal{I}(D_i))^{\frac{1}{d}} E_j \rceil)$$

*for $i = 1$ and $2$ and all $m \in \mathbb{N}$.*

*Proof.* We have that both $e_R(\mathcal{I}(D_1))$ and $e_R(\mathcal{I}(D_2))$ are positive by Proposition 5.4. We have that $E_1, E_2, \ldots, E_r$ are the irreducible exceptional divisors of $\varphi : X \to \mathrm{Spec}(R)$. For $1 \le j \le r$, let

$$\gamma_j = \frac{\gamma_{E_j}(D_1)}{e(\mathcal{I}(D_1))^{\frac{1}{d}}}.$$

By Theorem 10.3, taking $\mu = \mu_{E_j}$,

$$\gamma_j = \frac{\gamma_{E_j}(D_2)}{e(\mathcal{I}(D_2))^{\frac{1}{d}}}$$

for $1 \le j \le r$. Now for $i = 1, 2$ and $m \in \mathbb{N}$, we have by Lemma 5.2 that

$$I(mD_i) = \Gamma(X, \mathcal{O}_X(-\lceil \sum_{j=1}^{r} m\gamma_{E_j}(D_i)E_j \rceil) = \Gamma(X, \mathcal{O}_X(-\lceil \sum_{j=1}^{r} me(\mathcal{I}(D_i))^{\frac{1}{d}}\gamma_j E_j \rceil).$$

□

**Corollary 11.2.** *Let $R$ be a normal $d$-dimensional excellent local domain and let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be real divisorial $m_R$-filtrations. Thus $e_i = e_R(\mathcal{I}(D_1)^{[d-i]}, \mathcal{I}(D_2)^{[i]}) > 0$ for $0 \le i \le d$ by Proposition 5.4. Let $X \to \mathrm{Spec}(R)$ and $D_1 = \sum_{i=1}^r a_iE_i$, $D_2 = \sum_{i=1}^r b_iE_i$ be a representation of $D_1$ and $D_2$. Suppose that Minkowski's equality holds for $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ and that for some $i$,*

$$\frac{e_i}{e_{i-1}} = \frac{a}{b} \in \mathbb{Q}$$

*where $a, b \in \mathbb{Z}_{>0}$. Then*

$$I(maD_1) = I(mbD_2)$$

*for all $m \in \mathbb{N}$.*

*Proof.* With our assumption that the Minkowski inequality is an equality, we have from the observation before (67) that $e_j = e_0^{\frac{d-j}{d}} e_d^{\frac{j}{d}}$ for all $j$. Since $e_0 = e_R(\mathcal{I}(D_1)), e_d = e_R(\mathcal{I}(D_2)) > 0$, we have that $e_j^2 = e_{j-1}e_{j+1}$ for $1 \le j \le d-1$ by (69). Thus $\frac{e_j}{e_{j-1}} = \frac{e_{j+1}}{e_j}$ for $1 \le j \le d-1$, and so $\frac{e_j}{e_{j-1}} = \frac{a}{b}$ for $1 \le j \le d$ and so

$$\left(\frac{a}{b}\right)^d = \frac{e_d}{e_0}.$$

We have $e_d^{\frac{1}{d}}\gamma_{E_j}(D_1) = e_0^{\frac{1}{d}}\gamma_{E_j}(D_2)$ for all $j$ by Theorem 10.3. Now for all $j$,

$$\frac{a}{b}\frac{\gamma_{E_j}(D_1)}{\gamma_{E_j}(D_2)} = \frac{e_d^{\frac{1}{d}}\gamma_{E_j}(D_1)}{e_0^{\frac{1}{d}}\gamma_{E_j}(D_2)} = 1$$

so that $a\gamma_{E_j}(D_1) = b\gamma_{E_j}(D_2)$ for all $j$. The conclusions of the corollary now follow from Lemma 5.2. □

Suppose that $R$ is a local domain and $\mathcal{I} = \{I_m\}$ is a filtration of $m_R$-primary ideals. We define a function $w_{\mathcal{I}}$ on $R$ by

$$w_{\mathcal{I}}(f) = \max\{m \mid f \in I_m\} \tag{74}$$

for $f \in R$. We have that $w_{\mathcal{I}}(f)$ is either a natural number or $\infty$.

**Lemma 11.3.** *Suppose that $R$ is a normal excellent local domain and that $\mathcal{I} = \mathcal{I}(D)$ is a rational divisorial $m_R$-filtration. Suppose that $f \in m_R$ is nonzero. Then $w_{\mathcal{I}(D)}(f) < \infty$ and there exists $d \in \mathbb{Z}_{>0}$ such that $w_{\mathcal{I}(D)}(f^{nd}) = nw_{\mathcal{I}(D)}(f^d)$ for all $n \in \mathbb{Z}_{>0}$.*

*Proof.* By assumption, there exists a representation $X \to \mathrm{Spec}(R)$ and $D = \sum_{i=1}^r a_iE_i$ of $\mathcal{I}(D)$ where the $a_i$ are all nonnegative rational numbers and some $a_i > 0$. Let $b_i = \mu_{E_i}(f)$ for $1 \le i \le r$. Then $f \in I(mD)$ if and only if $\mu_{E_i}(f) = b_i \ge ma_i$ for $1 \le i \le r$. Since some $a_i > 0$ we have that $w_{\mathcal{I}(D)}(f) < \infty$.

Let

$$t = \min\left\{\frac{b_i}{a_i} \mid 1 \le i \le r \text{ and } a_i \ne 0\right\}.$$

Since all $b_i > 0$ and $D$ is a rational divisor, $t$ is a positive rational number, so we can write $t = \frac{c}{d}$ with $c, d \in \mathbb{Z}_{>0}$. Let $i_0$ be an index such that $t = \frac{b_{i_0}}{a_{i_0}}$. For all $n \in \mathbb{Z}_{>0}$ we have that $\mu_{E_i}(f^{nd}) = nd\mu_{E_i}(f) = ndb_i \ge nca_i$ for all $i$ and $\nu_{E_{i_0}}(f^{nd}) = ndb_{i_0} = nca_{i_0}$ so that $w_{\mathcal{I}(D)}(f^{dn}) = nc = nw_{\mathcal{I}(D)}(f^d)$. ☐

**Theorem 11.4.** *Suppose that $R$ is a $d$-dimensional normal excellent local ring. Let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be rational divisorial $m_R$-filtrations. Let $X \to \mathrm{Spec}(R)$, $D_1 = \sum_{i=1}^r a_iE_i$ and $D_2 = \sum_{i=1}^r b_iE_i$ be a representation of $D_1$ and $D_2$ (the $a_i$ and $b_i$ are nonnegative rational numbers). Suppose that the Minkowski equality holds between $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$. Then*

$$\xi = \frac{\sqrt[d]{e_R(\mathcal{I}(D_2))}}{\sqrt[d]{e_R(\mathcal{I}(D_1))}} \in \mathbb{Q}_{>0}.$$

*Writing*

$$\frac{\sqrt[d]{e_R(\mathcal{I}(D_2))}}{\sqrt[d]{e_R(\mathcal{I}(D_1))}} = \frac{a}{b}$$

*with $a, b \in \mathbb{Z}_{>0}$, we have that*

$$I(maD_1) = I(mbD_2)$$

*for all $m \in \mathbb{N}$.*

*Proof.* Minkowski's inequality holds by assumption, and $e_R(\mathcal{I}(D_i)) > 0$ for $i = 1, 2$ by Proposition 5.4. Thus by Corollary 11.1 we have that there exist $\overline{\gamma}_j \in \mathbb{R}_{>0}$ such that $\gamma_{E_j}(D_i) = e(\mathcal{I}(D_i))^{\frac{1}{d}}\overline{\gamma}_j$ for $i = 1, 2$ and $1 \le j \le r$. Let $\gamma_j = \gamma_{E_j}(D_1)$ for $1 \le j \le r$. Thus, with $\xi$ as defined in the statement of the theorem,

$$\gamma_{E_i}(D_2) = \xi\gamma_{E_i}(D_1) = \xi\gamma_i$$

for all $i$ and for all $m \in \mathbb{N}$,

$$I(mD_1) = \Gamma(X, \mathcal{O}_X(-\lceil\sum_{i=1}^r m\gamma_iE_i\rceil)$$

and

$$I(mD_2) = \Gamma(X, \mathcal{O}_X(-\lceil\sum_{i=1}^r m\xi_i\gamma_iE_i\rceil).$$

Let $g \in m_R$ be nonzero. By Lemma 11.3, there exists $\overline{d} \in \mathbb{Z}_+$ such that $w_{\mathcal{I}(D_i)}(g^{\overline{d}l}) = lw_{\mathcal{I}(D_i)}(g^{\overline{d}})$ for $i = 1, 2$ and all $l \in \mathbb{Z}_{>0}$. Let $f = g^{\overline{d}}$.

Let $m = w_{\mathcal{I}(D_2)}(f) > 0$ and $n = w_{\mathcal{I}(D_1)}(f) > 0$. Let $\delta \in \mathbb{R}_{>0}$. Now by Lemma 4.1, there exists $\alpha \in \mathbb{R}_{>0}$ such that $\alpha < \frac{\delta}{m}$ and there exists $\alpha' \in \mathbb{R}_{>0}$ such that $\alpha' < \frac{\delta}{m}$ and there exist positive integers $p_0, q_0, p_0', q_0'$ such that

$$\xi - \frac{\alpha}{q_0} < \frac{p_0}{q_0} \le \xi$$

and

$$\xi \le \frac{p_0'}{q_0'} < \xi + \frac{\alpha'}{q_0'}.$$

Let $p = p_0 m$, $q = q_0 m$. Then $\gamma_i p \le \xi \gamma_i q$ for all $i$ so that $I(qD_2) \subset I(pD_1)$. We have that $f^{q_0} \in I(qD_2) \subset I(pD_1)$ so that $w_{\mathcal{I}(D_1)}(f^{q_0}) \ge p$. Thus since $w_{\mathcal{I}(D_1)}(f^{q_0}) = q_0 w_{\mathcal{I}(D_1)}(f)$ (by our choice of $f$),

$$w_{\mathcal{I}(D_1)}(f) \ge \frac{p_0}{q_0} m = \frac{p_0}{q_0} w_{\mathcal{I}(D_2)}(f) > (\xi - \frac{\alpha}{q_0}) w_{\mathcal{I}(D_2)}(f). \tag{75}$$

Let $p' = p_0' n$, $q' = q_0' n$. Then $\gamma_i \xi q' \le \gamma_i p'$ for all $i$ so that $I(p'D_1) \subset I(q'D_2)$. We have that $f^{p_0'} \in I(p'D_1) \subset I(q'D_2)$. Thus since $w_{\mathcal{I}(D_2)}(f^{p_0'}) = p_0' w_{\mathcal{I}(D_2)}(f)$ (by our choice of $f$), we have that $w_{\mathcal{I}(D_2)}(f) \ge \frac{q_0' n}{p_0'} = \frac{q_0'}{p_0'} w_{\mathcal{I}(D_1)}(f)$. So

$$w_{\mathcal{I}(D_1)}(f) \le \frac{p_0'}{q_0'} w_{\mathcal{I}(D_2)}(f) < (\xi + \frac{\alpha'}{q_0'}) w_{\mathcal{I}(D_2)}(f). \tag{76}$$

Combining equations (75) and (76), we have that

$$\begin{array}{lll} w_{\mathcal{I}(D_1)}(f) & \le & (\xi + \frac{\alpha'}{q_0'}) w_{\mathcal{I}(D_2)}(f) \\ & = & (\xi - \frac{\alpha}{q_0}) w_{\mathcal{I}(D_2)}(f) + (\frac{\alpha}{q_0} + \frac{\alpha'}{q_0'}) w_{\mathcal{I}(D_2)}(f) \\ & \le & w_{\mathcal{I}(D_1)}(f) + (\frac{\alpha}{q_0} + \frac{\alpha'}{q_0'}) w_{\mathcal{I}(D_2)}(f) \\ & < & w_{\mathcal{I}(D_1)}(f) + 2\delta. \end{array}$$

All these inequalities approach equalities when the limit is taken as $\delta \mapsto 0$. Thus $w_{\mathcal{I}(D_1)}(f) = \xi w_{\mathcal{I}(D_2)}(f)$, and so

$$\xi = \frac{w_{\mathcal{I}(D_1)}(f)}{w_{\mathcal{I}(D_2)}(f)} \in \mathbb{Q}_{>0}.$$

Now we prove the last statement of the theorem. By Theorem 10.3, we have that

$$\gamma_{E_i}(D_1) = \frac{e_R(\mathcal{I}(D_1))^{\frac{1}{d}}}{e_R(\mathcal{I}(D_2))^{\frac{1}{d}}} \gamma_{E_i}(D_2)$$

for $1 \le i \le r$. Substituting into $I(maD_1) = \Gamma(X, \mathcal{O}_X(-\lceil \sum_{i=1}^r ma\gamma_{E_i}(D_1)E_i \rceil)$, we obtain that

$$I(maD_1) = \Gamma(X, \mathcal{O}_X(-\lceil \sum_{i=1}^r mb\gamma_{E_i}(D_2)E_i \rceil) = I(mbD_2)$$

for all $m \in \mathbb{N}$. ☐

**Theorem 11.5.** *Suppose that $R$ is a $d$-dimensional normal excellent local ring. Let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be real divisorial $m_R$-filtrations. Let $X \to Spec(R)$, $D_1 = \sum_{i=1}^r a_i E_i$ and $D_2 = \sum_{i=1}^r b_i E_i$ be a representation of $D_1$ and $D_2$. Then the following are equivalent*

1) *The Minkowski equality holds for $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$*

2)

$$\frac{\gamma_{E_i}(D_2)}{\gamma_{E_i}(D_1)} = \frac{e_R(\mathcal{I}(D_2))^{\frac{1}{d}}}{e_R(\mathcal{I}(D_1))^{\frac{1}{d}}}$$

*for all $i$.*

3) *For all $i$ and $j$ we have that*

$$\frac{\gamma_{E_i}(D_2)}{\gamma_{E_i}(D_1)} = \frac{\gamma_{E_j}(D_2)}{\gamma_{E_j}(D_1)}. \tag{77}$$

*Proof.* We have that both $e_R(\mathcal{I}(D_1))$ and $e_R(\mathcal{I}(D_2))$ are positive by Proposition 5.4.

First suppose that Minkowski's equality holds between $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$. Then by Theorem 10.3,

$$\frac{\gamma_{E_i}(D_2)}{\gamma_{E_i}(D_1)} = \frac{e_R(\mathcal{I}(D_2))^{\frac{1}{d}}}{e_R(\mathcal{I}(D_1))^{\frac{1}{d}}}$$

for all $i$. Thus 2) holds. If 2) holds then 3) certainly holds.

Now suppose that 3) holds for all $i,j$. Let $\gamma_i = \gamma_{E_i}(D_1)$ and let $\xi \in \mathbb{R}_{>0}$ be such that

$$\xi = \frac{\gamma_{E_i}(D_2)}{\gamma_{E_i}(D_1)}$$

for all $i$. Then $\gamma_{E_i}(D_2) = \xi\gamma_{E_i}(D_i)$ for all $i$. For $\lambda \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$, define

$$K(\lambda)_n = \Gamma(X, \mathcal{O}_X(-\lceil n\lambda\gamma_1 E_1 + \cdots + n\lambda\gamma_r E_r\rceil)),$$

and a filtration of $m_R$-primary ideals $\mathcal{K}(\lambda) = \{K(\lambda)_n\}$. Observe that $\mathcal{K}(\lambda) = \mathcal{I}(\sum_{i=1}^r \lambda\gamma_i E_i)$.

For $n_1, n_2 \in \mathbb{N}$ define

$$J(n_1,n_2)_m = I(mn_1D_1)I(mn_2D_2)$$

and a filtration of $m_R$-primary ideals $\mathcal{J}(n_1,n_2) = \{J(n_1,n_2)_m\}$. We have that for all $n_1, n_2$, $J(n_1,n_2)_m \subset K(n_1+n_2\xi)_m$ for all $m$ so that

$$\Delta(\mathcal{J}(n_1,n_2)) \subset \Delta(\mathcal{K}(n_1+n_2\xi))$$

for all $n_1,n_2$. We further have that $n_1\Delta(\mathcal{I}(D_1)) + n_2\Delta(\mathcal{I}(D_2)) \subset \Delta(\mathcal{J}(n_1,n_2))$ for all $n_1,n_2$. We have that $\Delta(\mathcal{I}(D_1)) = \Delta(\mathcal{K}(1))$. Now by Proposition 7.7, we have that $\Delta(\mathcal{I}(D_2)) = \xi\Delta(\mathcal{K}(1))$ and $\Delta(\mathcal{K}(n_1+n_2\xi)) = (n_1+n_2\xi)\Delta(\mathcal{K}(1))$. So $n_1\Delta(\mathcal{I}(D_1)) + n_2\Delta(\mathcal{I}(D_2)) = (n_1+n_2\xi)\Delta(\mathcal{K}(1))$ and thus $\Delta(\mathcal{J}(n_1,n_2)) = n_1\Delta(\mathcal{I}(D_1)) + n_2\Delta(\mathcal{I}(D_2))$ for all $n_1,n_2 \in \mathbb{N}$. Thus the conditions of Theorem 10.2 are satisfied, and so Minkowski's equality holds between $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$. ☐

**Theorem 11.6.** *Suppose that $R$ is a normal excellent local ring. Let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be rational divisorial $m_R$-filtrations. Then $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ satisfy the Minkowski equality if and only if there exist $a,b \in \mathbb{Z}_{>0}$ such that $I(amD_1) = I(bmD_2)$ for all $m \in \mathbb{N}$.*

*Proof.* Let $X \to \mathrm{Spec}(R)$, $D_1 = \sum_{i=1}^r a_iE_i$ and $D_2 = \sum_{i=1}^r b_iE_i$ be a representation of $D_1$ and $D_2$.

If $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ satisfy the Minkowski equality then there exist $a,b \in \mathbb{Z}_{>0}$ such that $I(amD_1) = I(bmD_2)$ for all $m \in \mathbb{N}$ by Theorem 11.4.

Suppose that there exist $a,b \in \mathbb{Z}_{>0}$ such that $I(amD_1) = I(bmD_2)$ for all $m \in \mathbb{N}$. With this assumption, $\gamma_{E_i}(aD_1) = \gamma_{E_i}(bD_2)$ for $1 \le i \le r$. Now $\gamma_{E_i}(aD_1) = a\gamma_{E_i}(D_1)$ and $\gamma_{E_i}(bD_2) = b\gamma_{E_i}(D_2)$, so

$$\frac{\gamma_{E_i}(D_2)}{\gamma_{E_i}(D_1)} = \frac{a}{b}$$

for $1\le i\le r$. Thus the Minkowski equality holds for $D_1$ and $D_2$ by Theorem 11.5. $\square$

## 12. Excellent local domains and the Minkowski equality

**Theorem 12.1.** *Suppose that $R$ is a $d$-dimensional excellent local domain. Let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be integral divisorial $m_R$-filtrations. Then the Minkowski equality holds between $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ if and only if there exist $a,b\in\mathbb{Z}_{>0}$ such that $I(amD_1)=I(bmD_2)$ for all $m\in\mathbb{N}$.*

*Proof.* We use the notation of Subsection 5.3. Let $S$ be the normalization of $R$ with maximal ideals $m_i$. we have that $D_1=\sum_{i=1}^t D_1(i)$, $D_2=\sum_{i=1}^t D_2(i)$. Write

$$P(n_1,n_2)=\lim_{m\to\infty}\frac{\ell_R(R/I(mn_1D_1)I(mn_2D_2))}{m^d}=\sum_{j=0}^d\frac{1}{(d-j)!j!}e_jn_1^{d-j}n_2^j$$

and

$$P_i(n_1,n_2)=\lim_{m\to\infty}\frac{\ell_{S_{m_i}}(S_{m_i}/J(mn_1D_1(i))J(mn_2D_2(i)))}{m^d}=\sum_{j=0}^d\frac{1}{(d-j)!j!}e(i)_jn_1^{d-j}n_2^j.$$

We have that

$$P(n_1,n_2)=\sum_{i=1}^t a_iP_i(n_1,n_2) \tag{78}$$

with $a_i=[S/m_i:R/m_R]$ for $1\le i\le t$ by Lemma 5.3 and (18). Let $\mathcal{J}(D_k(i))$ be the filtration $\{J(mD_k(i))\}$ for $k=1,2$ and all $i$.

Since $D_1,D_2\ne 0$ we have that some $D_1(i)\ne 0$ and some $D_2(j)\ne 0$. Thus $e(i)_0>0$ and $e(j)_d>0$ by Proposition 5.4 and so $e_0>0$ and $e_d>0$ by (32). Since the Minkowski equality holds between $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ we have by (66) that equality holds in (65) for the $e_i$, so (67) holds which implies all $e_i>0$. Thus there exists $\xi\in\mathbb{R}_{>0}$ such that

$$\xi=\frac{e_1}{e_0}=\cdots=\frac{e_d}{e_{d-1}}. \tag{79}$$

By (78) we have that $e_j=\sum_{i=1}^t a_ie(i)_j$ for all $j$. By the inequality (65) and (79) we have that

$$\begin{array}{lll}0&\le&\sum_{i=1}^t a_i(e(i)_{j+1}^{\frac12}-\xi e(i)_{j-1}^{\frac12})^2=\sum_{i=1}^t a_i(e(i)_{j+1}-2\xi e(i)_{j+1}^{\frac12}e(i)_{j-1}^{\frac12}+\xi^2e(i)_{j-1})\\ &\le&\sum_{i=1}^t a_i(e(i)_{j+1}-2\xi e(i)_j+\xi^2e(i)_{j-1})\\ &=&e_{j+1}-2\xi e_j+\xi^2e_{j-1}\\ &=&\xi^2e_{j-1}-2\xi^2e_{j-1}+\xi^2e_{j-1}=0.\end{array}$$

Thus

$$e(i)_{j+1}^{\frac12}=\xi e(i)_{j-1}^{\frac12}\text{ and }e(i)_j^2=e(i)_{j-1}e(i)_{j+1}$$

for all $i$. Since this holds for all $j$, we have that equality holds in (65) for all $i$ and $j$. Further, we have that for a particular $i$, either

$$e(i)_j=0\text{ for all }j \tag{80}$$

or

$$e(i)_j>0\text{ for all }j. \tag{81}$$

If (80) holds for a particular $i$, then $e(i)_0 = e(i)_d = 0$ so we have the degenerate case $D(i)_1 = D(i)_2 = 0$ by Proposition 5.4, so that

$$J(mD_1(i)) = J(nD_2(i)) = S_{m_i} \text{ for all } m, n \in \mathbb{N}. \tag{82}$$

Suppose that (81) holds for a particular $i$. Then by (69), the Minkowski equality holds between $\mathcal{I}(D(i)_1)$ and $\mathcal{I}(D(i)_2)$ for this $i$. Thus there exists $\lambda_i \in \mathbb{R}_{>0}$ such that

$$\frac{e(i)_{j+1}}{e(i)_j} = \lambda_i$$

for all $j$. Thus

$$\xi^2 = \frac{e(i)_{j+1}}{e(i)_{j-1}} = \frac{e(i)_{j+1}}{e(i)_j}\frac{e(i)_j}{e(i)_{j-1}} = \lambda_i^2$$

so that $\lambda_i = \xi$ and so

$$\frac{e(i)_d^{\frac{1}{d}}}{e(i)_0^{\frac{1}{d}}} = \xi.$$

Since $D_1, D_2 \neq 0$, (81) holds for some $i$, so that $\xi \in \mathbb{Q}_{>0}$ by Theorem 11.4. Write $\xi = \frac{a}{b}$ with $a, b \in \mathbb{Z}_{>0}$. We have that $J(maD_1(i)) = J(mbD_2(i))$ for all $i$ such that (81) holds and $m \in \mathbb{N}$ by Theorem 11.4. Thus $J(maD_1) = J(mbD_2)$ for all $m \in \mathbb{N}$ by formula (17) and thus $I(maD_1) = I(mbD_2)$ for all $m \in \mathbb{N}$ since $I(maD_1) = J(maD_1) \cap R$ and $I(mbD_2) = J(mbD_2) \cap R$ for all $m$.

The converse follows from Theorem 8.4, since $\overline{R[\mathcal{I}(D_j)]} = R[\mathcal{I}(D_j)]$ for $j = 1, 2$.

□

Theorem 12.1 is proven in dimension $d = 2$ in [12, Theorem 5.9] using the theory of relative Zariski decomposition, which requires dimension two. This theory is also used to prove the fact that the mixed multiplicities $e_i$ of integral divisorial filtrations are rational numbers in dimension two. This fact is used in the proof of [12, Theorem 5.9]. The mixed multiplicities of integral divisorial filtrations can be irrational numbers in dimension $\geq 3$, as is shown in the example of Section 15.

The following corollary is proven in the case that $d = 2$ in [12, Corollary 5.10].

**Corollary 12.2.** *Suppose that $R$ is an excellent local domain and $\mu_1$ and $\mu_2$ are $m_R$-valuations such that Minkowski's equality holds between the $m_R$-filtrations $\mathcal{I}(\mu_1) = \{I(\mu_1)_m\}$ and $\mathcal{I}(\mu_2) = \{I(\mu_2)_m\}$. Then $\mu_1 = \mu_2$.*

*Proof.* We have by Theorem 12.1 that $I(\mu_1)_{an} = I(\mu_2)_{bn}$ for all $n$ and some positive integers $a$ and $b$ which we can take to be relatively prime.

Suppose that $0 \neq f \in I(\mu_1)_n$. Then $f^a \in I(\mu_1)_{an} = I(\mu_2)_{bn}$ so that $a\mu_2(f) \geq bn$. If $f^a \in I(\mu_2)_{bn+1}$ then $f^{ab} \in I(\mu_2)_{b(bn+1)} = I(\mu_1)_{a(bn+1)}$ so that $\mu_1(f) > n$. Thus

$$\mu_1(f) = n \text{ if and only if } \mu_2(f) = \frac{b}{a}n. \tag{83}$$

Further, (83) holds for every nonzero $f \in \mathrm{QF}(R)$ since $f$ is a quotient of nonzero elements of $R$.

Now the maps $\mu_1 : \mathrm{QF}(R) \setminus \{0\} \to \mathbb{Z}$ and $\mu_2 : \mathrm{QF}(R) \setminus \{0\} \to \mathbb{Z}$ are surjective, so there exists $0 \neq f \in \mathrm{QF}(R)$ such that $\mu_1(f) = 1$ and there exists $0 \neq g \in \mathrm{QF}(R)$ such that $\mu_2(g) = 1$ which implies that $a = b = 1$ since $a, b$ are relatively prime. Thus $\mu_1 = \mu_2$.

□

**Remark 12.3.** *With the assumptions of the above corollary and further assuming that $R$ is normal, the functions $w_{\mathcal{I}(\mu_i)}$ of (74) are $w_{\mathcal{I}(\mu_i)} = \mu_i$. Thus*

$$w_{\mathcal{I}(\mu_i)}(f^n) = \mu_i(f^n) = n\mu_i(f) = nw_{\mathcal{I}(\mu_i)}(f)$$

*for all nonzero $f \in R$ and $i = 1, 2$. Thus the proof of Theorem 11.4 shows that*

$$\xi = \frac{w_{\mathcal{I}(\mu_1)}(f)}{w_{\mathcal{I}(\mu_2)}(f)} = \frac{\mu_1(f)}{\mu_2(f)}$$

*for all nonzero $f \in m_R$.*

## 13. Bounded $m_R$-Filtrations

Bounded $m_R$-filtrations are defined in Subsection 5.6.

**Theorem 13.1.** *Suppose that $R$ is an excellent local domain, $\mathcal{I}(1)$ is a real bounded $m_R$-filtration and $\mathcal{I}(2)$ is an arbitrary $m_R$-filtration such that $\mathcal{I}(1) \subset \mathcal{I}(2)$. Then the following are equivalent*

1) $e(\mathcal{I}(1)) = e(\mathcal{I}(2))$.
2) *There is equality of integral closures*

$$\overline{\sum_{m\ge 0} I(1)_m t^m} = \overline{\sum_{m\ge 0} I(2)_m t^m}$$

*in $R[t]$.*

*Proof.* 2) implies 1) follows from [14, Theorem 6.9] or [12, Appendix] as summarized in Subsection 1.1.

We now prove 1) implies 2). Let $\mathcal{I}(D_1)$ be the real divisorial $m_R$-filtrations such that $\overline{R(\mathcal{I}(1))} = R(\mathcal{I}(D_1))$. Thus $R(\mathcal{I}(D_1)) \subset \overline{R(\mathcal{I}(2))} = R[\overline{\mathcal{I}(2)}]$ so that $\mathcal{I}(D_1) \subset \overline{\mathcal{I}(2)}$. We have that $e(\mathcal{I}(1)) = e(\mathcal{I}(D_1))$ and $e(\mathcal{I}(2)) = e(\overline{\mathcal{I}(2)})$ by [14, Theorem 6.9] or [12, Appendix]. Thus $e(\mathcal{I}(D_1)) = e(\overline{\mathcal{I}(2)})$ and so $R(\mathcal{I}(D_1)) = R(\overline{\mathcal{I}(2)})$ by Theorem 7.5. Thus 2) holds for $\mathcal{I}(1)$ and $\mathcal{I}(2)$.

∎

**Theorem 13.2.** *Suppose that $R$ is a $d$-dimensional excellent local domain and $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are bounded $m_R$-filtrations. Then the following are equivalent*

1) *The Minkowski inequality*

$$e(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}} = e(\mathcal{I}(1))^{\frac{1}{d}} + e(\mathcal{I}(2))^{\frac{1}{d}}$$

*holds.*

2) *There exist positive integers $a, b$ such that there is equality of integral closures*

$$\overline{\sum_{n\ge 0} I(1)_{an} t^n} = \overline{\sum_{n\ge 0} I(2)_{bn} t^n}$$

*in $R[t]$.*

*Proof.* Let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be integral divisorial $m_R$-filtrations such that $\overline{R(\mathcal{I}(1))} = R(\mathcal{I}(D_1))$ and $\overline{R(\mathcal{I}(2))} = R(\mathcal{I}(D_2))$. By Proposition 5.11, we have equality of functions

$$\lim_{m\to\infty} \frac{\ell(R/I(i)_{mn_1} I(i)_{mn_2})}{m^d} = \lim_{m\to\infty} \frac{\ell(R/I(D_i)_{mn_1} I(D_i)_{mn_2})}{m^d}$$

for $i = 1, 2$ and all $n_1, n_2 \in \mathbb{N}$. Since 1) and 2) are equivalent for the integral divisorial filtrations $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ by Theorem 12.1, they are also equivalent for the bounded $m_R$-filtrations $\mathcal{I}(1)$ and $\mathcal{I}(2)$. $\square$

## 14. Analytically Irreducible local Rings

Let $R$ be an analytically irreducible local domain. A local ring $R$ is analytically irreducible if the $m_R$-adic completion $\hat{R}$ is a domain. The complete local ring $\hat{R}$ is then an excellent local domain.

**Lemma 14.1.** *(*[36, Proposition 9.3.5]*) Let $R$ be an analytically irreducible local domain. Then there is a 1-1 correspondence between $m_R$-valuations of $R$ and $m_{\hat{R}}$-valuations of $\hat{R}$.*

**Lemma 14.2.** *Let $R$ be a analytically irreducible local domain. Let $\mu_1, \ldots, \mu_s$ be $m_R$-valuations and $n_1, \ldots, n_s \in \mathbb{Z}_s$. Let $\hat{\mu}_i$ be the unique extension of $\mu_i$ to a $m_{\hat{R}}$-valuation for $1 \le i \le s$. Then*

$$I(\mu_1)_{n_1} \cap \cdots \cap I(\mu_s)_{n_s}\hat{R} = I(\hat{\mu}_1)_{n_1} \cap \cdots \cap I(\hat{\mu}_s)_{n_s}$$

*and*

$$(I(\hat{\mu}_1)_{n_1} \cap \cdots \cap I(\hat{\mu}_s)_{n_s}) \cap R = I(\mu_1)_{n_1} \cap \cdots \cap I(\mu_s)_{n_s}.$$

*Proof.* We certainly have that $I(\mu_1)_{n_1} \cap \cdots \cap I(\mu_s)_{n_s}\hat{R} \subset I(\hat{\mu}_1)_{n_1} \cap \cdots \cap I(\hat{\mu}_s)_{n_s}$. Suppose that $f \in I(\hat{\mu}_1)_{n_1} \cap \cdots \cap I(\hat{\mu}_s)_{n_s}$. There exists $a > 0$ such that $m_{\hat{R}}^a \subset I(\hat{\mu}_1)_{n_1} \cap \cdots \cap I(\hat{\mu}_s)_{n_s}$ and $a\hat{\mu}_i(m_{\hat{R}}) > n_i$ for all $i$. Since $\hat{R}/m_{\hat{R}}^a \cong R/m_R^a$, there exists $g \in R$ and $h \in m_{\hat{R}}^a$ such that $f = g + h$. For all $i$, we have

$$\mu_i(g) = \hat{\mu}_i(f - h) \ge \min\{\hat{\mu}_i(f), \hat{\mu}_i(h)\} \ge n_i.$$

Thus $g \in I(\mu_1)_{n_1} \cap \cdots \cap I(\mu_s)_{n_s}$. Now $h = \sum a_j b_j$ with $a_j \in m_R^a$ and $b_j \in \hat{R}$. We have that

$$\mu_i(a_j) = \hat{\mu}_i(a_j) \ge a\hat{\mu}_i(m_{\hat{R}}) > n_i$$

for all $i$ and $j$ so that $a_j \in I(\mu_1)_{n_1} \cap \cdots \cap I(\mu_s)_{n_s}$ for all $j$. Thus $f \in (I(\mu_1)_{n_1} \cap \cdots \cap I(\mu_s)_{n_s})\hat{R}$.

Since $A \to \hat{A}$ is faithfully flat, we have that

$$(I(\hat{\mu}_1)_{n_1} \cap \cdots \cap I(\hat{\mu}_s)_{n_s}) \cap R = (I(\mu_1)_{n_1} \cap \cdots \cap I(\mu_s)_{n_s}\hat{R}) \cap R = I(\mu_1)_{n_1} \cap \cdots \cap I(\mu_s)_{n_s}.$$

$\square$

By Lemma 5.8, if $D = a_1\mu_1 + \cdots + a_s\mu_s$ where $\mu_1, \ldots, \mu_s$ are $m_R$-valuations and $a_1, \ldots, a_s \in \mathbb{R}_{>0}$, then $R[\mathcal{I}(D)]$ is integrally closed in $R[t]$. Let $\hat{D} = a_1\hat{\mu}_1 + \cdots + a_s\hat{\mu}_s$ and $\mathcal{I}(\hat{D})$ be the induced $m_R$-filtration on $\hat{R}$.

**Lemma 14.3.** *Let $R$ be an analytically irreducible local domain and suppose that $\mathcal{I} = \{I_m\}$ is a (real) bounded $m_R$-filtration; that is, there exists a (real) divisorial $m_R$-filtration $\mathcal{I}(D)$ such that the integral closure $\overline{R[\mathcal{I}]}$ of $R[\mathcal{I}]$ in $R[t]$ is $R[\mathcal{I}(D)]$. Let $\hat{\mathcal{I}} = \{I_m\hat{R}\}$. Then $\hat{\mathcal{I}} = \mathcal{I}(\hat{D})$ is a (real) bounded $m_{\hat{R}}$-filtration and the integral closure $\overline{\hat{R}[\hat{\mathcal{I}}]}$ of $\hat{R}[\hat{\mathcal{I}}]$ in $\hat{R}[t]$ is $\hat{R}[\mathcal{I}(\hat{D})]$.*

*Proof.* $R[\mathcal{I}(D)] = \sum_{m \ge 0} I(mD)t^m$ is integral over $R[\mathcal{I}] = \sum_{n \ge 0} I_m t^m$ so the integrally closed ring $\hat{R}[\mathcal{I}(\hat{D})] = \sum_{m \ge 0} I(mD)\hat{R}t^m$ is integral over $\hat{R}[\hat{\mathcal{I}}] = \sum_{n \ge 0} I_m\hat{R}t^m$. $\square$

**Theorem 14.4.** *Suppose that $R$ is an analytically irreducible local ring, $\mathcal{I}(1)$ is real bounded $m_R$-filtration and $\mathcal{I}(2)$ is an arbitrary $m_R$-filtration such that $\mathcal{I}(1) \subset \mathcal{I}(2)$. Then the following are equivalent*

1) $e(\mathcal{I}(1)) = e(\mathcal{I}(2))$.
2) *There is equality of integral closures*

$$\overline{\sum_{m\ge 0} I(1)_m t^m} = \overline{\sum_{m\ge 0} I(2)_m t^m}$$

*in $R[t]$.*

*Proof.* We have that $\ell_{\hat R}(\hat R/I(j)_m\hat R) = \ell_R(R/I(j)_m)$ for $j = 1,2$ and all $m \in \mathbb{N}$. Thus $e_R(\mathcal{I}(j)) = e_{\hat R}(\hat{\mathcal{I}}(j))$ for $j = 1,2$.

Let $\mathcal{I}(D_1)$ be the real divisorial filtration on $R$ such that $\overline{R[\mathcal{I}(1)]} = R[\mathcal{I}(D_1)]$. By Lemma 14.3, $\mathcal{I}(D_1)\hat R = \mathcal{I}(\hat D_1)$ is a real bounded $m_R$-filtration and $\overline{\hat R[\mathcal{I}(1)\hat R]} = R[\mathcal{I}(\hat D_1)]$.

We have that $\hat R[\mathcal{I}(\hat D_1)] = \overline{\hat R[\mathcal{I}(2)\hat R]}$ if and only if $R[\mathcal{I}(D_1)] = \overline{R[\mathcal{I}(2)]}$ by Lemmas 14.2 and 14.3. Theorem 14.4 thus follows from Theorem 13.1, since $R \to \hat R$ is faithfully flat. □

**Theorem 14.5.** *Suppose that $R$ is a $d$-dimensional analytically irreducible local ring and $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are bounded $m_R$-filtrations. Then the following are equivalent*

1) *The Minkowski inequality*

$$e(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}} = e(\mathcal{I}(1))^{\frac{1}{d}} + e(\mathcal{I}(2))^{\frac{1}{d}}$$

*holds.*

2) *There exist positive integers $a, b$ such that there is equality of integral closures*

$$\overline{\sum_{n\ge 0} I(1)_{an} t^n} = \overline{\sum_{n\ge 0} I(2)_{bn} t^n}$$

*in $R[t]$.*

*Proof.* Since $\ell_{\hat R}(\hat R/I(1)_{mn_1}I(2)_{mn_2}\hat R) = \ell_R(R/I(1)_{mn_1}I(2)_{mn_2})$ for all $m, n_1, n_2 \in \mathbb{N}$, we have that

$$e_R(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}} = e_R(\mathcal{I}(1))^{\frac{1}{d}} + e_R(\mathcal{I}(2))^{\frac{1}{d}}$$

if and only if

$$e_{\hat R}(\hat{\mathcal{I}}(1)\hat{\mathcal{I}}(2))^{\frac{1}{d}} = e_{\hat R}(\hat{\mathcal{I}}(1))^{\frac{1}{d}} + e_{\hat R}(\hat{\mathcal{I}}(2))^{\frac{1}{d}}.$$

There exist integral divisorial $m_R$-filtrations $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ such that $\overline{R[\mathcal{I}(1)]} = R[\mathcal{I}(D_1)]$ and $\overline{R[\mathcal{I}(2)]} = R[\mathcal{I}(D_2)]$. By Lemma 14.2, we have that $\sum_{n\ge 0} I(D_1)_{an}\hat R t^n = \sum_{n\ge 0} I(D_2)_{bn}\hat R t^n$ if and only if $\sum_{n\ge 0} I(D_1)_{an} t^n = \sum_{n\ge 0} I(D_2)_{bn} t^n$. Since $\overline{\sum_{n\ge 0} I(j)_{cn} t^n} = \sum_{n\ge 0} I(D_j)_{cn} t^n$ and

$$\overline{\sum_{n\ge 0} I(j)_{cn}\hat R t^n} = \sum_{n\ge 0} I(D_j)_{cn}\hat R t^n$$

for all $c \in \mathbb{Z}_{>0}$ and $j = 1,2$, we have that $\overline{\sum_{n\ge 0} I(1)_{an} t^n} = \overline{\sum_{n\ge 0} I(2)_{bn} t^n}$ if and only if $\overline{\sum_{n\ge 0} I(1)_{an}\hat R t^n} = \overline{\sum_{n\ge 0} I(2)_{bn}\hat R t^n}$.

By Lemma 14.3 and Theorem 13.2, we have that the conclusions of Theorem 14.5 holds. □

## 15. An Example

In Theorem 1.4 [13], the following example is constructed. Let $k$ be an algebraically closed field. A 3-dimensional normal algebraic local ring $R$ over $k$ is constructed, and the blow up $\varphi : X \to \mathrm{Spec}(R)$ of an $m_R$-primary ideal such that $X$ is nonsingular with two irreducible exceptional divisors $E_1$ and $E_2$ is constructed.

The resolution of singularities of a three dimensional normal local ring which we construct is similar to the one constructed in [15, Example 6] which is used to give an example of a divisorial filtration with irrational multiplicity.

**Theorem 15.1.** *([13, Theorem 1.4]) Let $D = n_1E_1 + n_2E_2$ with $n_1, n_2 \in \mathbb{N}$. Then*

$$\lim_{m\to\infty} \frac{\ell_R(R/I(mD))}{m^3} = \begin{cases} 33n_1^3 & \text{if } n_2 < n_1 \\ 78n_1^3 - 81n_1^2n_2 + 27n_1n_2^2 + 9n_2^3 & \text{if } n_1 \le n_2 < n_1\left(3 - \frac{\sqrt{3}}{3}\right) \\ \left(\frac{2007}{169} - \frac{9\sqrt{3}}{338}\right)n_2^3 & \text{if } n_1\left(3 - \frac{\sqrt{3}}{3}\right) < n_2. \end{cases}$$

We compute the functions $\gamma_{E_1}$ and $\gamma_{E_2}$ in [13, Theorem 4.1].

**Theorem 15.2.** *([13, Theorem 4.1]) Let $D = n_1E_1 + n_2E_2$ with $n_1, n_2 \in \mathbb{N}$, an effective exceptional divisor on $X$.*

1) *Suppose that $n_2 < n_1$. Then $\gamma_{E_1}(D) = n_1$ and $\gamma_{E_2}(D) = n_1$.*
2) *Suppose that $n_1 \le n_2 < n_1\left(3 - \frac{\sqrt{3}}{3}\right)$. Then $\gamma_{E_1}(D) = n_1$ and $\gamma_{E_2}(D) = n_2$.*
3) *Suppose that $n_1\left(3 - \frac{\sqrt{3}}{3}\right) < n_2$. Then $\gamma_{E_1}(D) = \frac{3}{9-\sqrt{3}}n_2$ and $\gamma_{E_2}(D) = n_2$.*

*In all three cases, $-\gamma_{E_1}(D)E_1 - \gamma_{E_2}(D)E_2$ is nef on $X$.*

**Corollary 15.3.** *Suppose that $D_1$ and $D_2$ are effective integral exceptional divisors on $X$. If $D_1$ and $D_2$ are in the first region of Theorem 15.1, then Minkowski's equality holds between them. If $D_1$ and $D_2$ are in the second region, then Minkowski's equality holds between them if and only if $D_2$ is a rational multiple of $D_1$. If $D_1$ and $D_2$ are in the third region, then Minkowski's equality holds between them. Minkowski's equality cannot hold between $D_1$ and $D_2$ in different regions.*

*Proof.* This follows from Theorems 11.5 and 15.2. □

The interpretation of mixed multiplicities as anti-positive intersection multiplicities is particularly useful in the calculation of examples. We quote some statements from [12] which, along with the calculations in Theorem 15.2 and the identities

$$(E_1^3) = 468, (E_1^2 \cdot E_2) = -162, (E_1 \cdot E_2^2) = 54, (E_2^3) = 54 \tag{84}$$

on page 15 of [13] allow us to compute the mixed multiplicities of any divisors $D_1 = a_1E + a_2E_2$ and $D_2 = b_1E_1 + b_2E_2$.

It is shown in [12, Theorem 8.3] that we have identities

$$e_R(\mathcal{I}(D_1)^{[d_1]}, \mathcal{I}(D_2)^{[d_2]}; R) = -\langle(-D_1)^{d_1} \cdot (-D_2)^{d_2}\rangle \tag{85}$$

where $\langle(-D_1)^{d_1} \cdot (-D_2)^{d_2}\rangle$ are the anti-positive intersection products defined in [12]. In particular, $e_R(\mathcal{I}(D); R) = -\langle(-D)^d\rangle$. Thus by (3), we have that [13, Formula (1.8)]

$$\begin{aligned} &\lim_{m\to\infty} \frac{\ell_R(R/I(mn_1D_1)I(mn_2D_2))}{m^d} \\ &= -\sum_{d_1+d_2=d} \frac{1}{d_1!d_2!}\langle(-D_1)^{d_1} \cdot (-D_2)^{d_2}\rangle n_1^{d_1}n_2^{d_2}. \end{aligned} \tag{86}$$

and [13, Formula (1.9)]

$$\lim_{m\to\infty}\frac{\ell_R(R/I(mD))}{m^d}=-\frac{\langle(-D)^d\rangle}{d!}. \tag{87}$$

**Proposition 15.4.** *(*[13, Proposition 2.4]*) Suppose that $D_1,\ldots,D_d$ are effective $\mathbb{Q}$-Cartier divisors with exceptional support such that the divisors $-\sum\gamma_{E_i}(D_j)E_i$ are nef for $1\le j\le d$. Then the positive intersection product $\langle -D_1\cdot,\ldots,\cdot-D_d\rangle$ is the ordinary intersection product $(-\sum\gamma_{E_i}(D_1)E_i\cdot\ldots\cdot-\sum\gamma_{E_i}(D_d)E_i)$.*

We now use this method to compute the mixed multiplicities of $\mathcal{I}(E_1)$ and $\mathcal{I}(E_2)$. By Theorem 15.2

$$\gamma_{E_1}(E_1)=1\ \gamma_{E_2}(E_1)=1,\ \gamma_{E_1}(E_2)=\frac{3}{9-\sqrt{3}},\ \gamma_{E_2}(E_2)=1.$$

By formulas (84) and (86) and Proposition 15.4,
(88)

$$\begin{aligned}
&\lim_{m\to\infty}\tfrac{\ell_R(R/I(mn_1E_1)I(mn_2E_2))}{m^3}\\
&=\textstyle\sum_{i_1+i_2=3}\frac{1}{i_1!i_2!}e_R(\mathcal{I}(E_1)^{[i_1]},\mathcal{I}(E_2)^{[i_2]})n_1^{i_1}n_2^{i_2}\\
&=\textstyle\sum_{i_1+i_2=3}-\frac{1}{i_1!i_2!}\left((-\gamma_{E_1}(E_1)E_1-\gamma_{E_2}(E_1)E_2)^{i_1}\cdot(-\gamma_{E_1}(E_2)E_1-\gamma_{E_2}(E_2)E_2)^{i_2})\right)n_1^{i_1}n_2^{i_2}\\
&=\textstyle\sum_{i_1+i_2=3}\frac{1}{i_1!i_2!}\left((E_1+E_2)^{i_1}\cdot(\frac{3}{9-\sqrt{3}}E_1+E_2)^{i_2}\right)n_1^{i_1}n_2^{i_2}\\
&=33n_1^3+(\tfrac{891}{26}+\tfrac{99}{26}\sqrt{3})n_1^2n_2+(\tfrac{12042}{338}-\tfrac{27}{338}\sqrt{3})n_1n_2^2+\left(\tfrac{2007}{169}-\tfrac{9\sqrt{3}}{338}\right)n_2^3,
\end{aligned}$$

in contrast to the function of Theorem 15.1.

We make a more detailed analysis in the third region.

**Example 15.5.** *Suppose that $D_1=a_1E_1+a_2E_2$ and $D_2=b_1E_1+b_2E_2$ are integral divisors in the third region of Theorem 15.1, $n_1(3-\frac{\sqrt{3}}{3})<n_2$. Then $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ satisfy equality in Minkowski's inequality. We have*

$$e_i=e_R(\mathcal{I}(D_1)^{[3-i]},\mathcal{I}(D_2)^{[i]})=a_2^{3-i}b_2^i\left(\frac{12042}{169}-\frac{27\sqrt{3}}{169}\right)$$

*for $0\le i\le 3$ and $\frac{e_i}{e_{i-1}}=\frac{b_2}{a_2}$ for $0\le i\le 3$. Thus*

$$I(mb_2D_1)=I(ma_2D_2)$$

*for all $m\in\mathbb{N}$.*

*Proof.* By Theorem 15.2

$$\gamma_{E_1}(D_1)=\frac{3}{9-\sqrt{3}}a_2,\ \gamma_{E_2}(D_1)=a_2,\ \gamma_{E_1}(D_2)=\frac{3}{9-\sqrt{3}}b_2,\ \gamma_{E_2}(D_2)=b_2.$$

By formula (86) and Proposition 15.4,

$$\begin{aligned}
&\lim_{m\to\infty}\tfrac{\ell_R(R/I(mD_1)I(mD_2))}{m^3}\\
&=\textstyle\sum_{i_1+i_2=3}\frac{1}{i_1!i_2!}e_R(\mathcal{I}(D_1)^{[i_1]},\mathcal{I}(D_2)^{[i_2]})n_1^{i_1}n_2^{i_2}\\
&=\textstyle\sum_{i_1+i_2=3}-\frac{1}{i_1!i_2!}\left((-\gamma_{E_1}(D_1)E_1-\gamma_{E_2}(D_1)E_2)^{i_1}\cdot(-\gamma_{E_1}(D_2)E_1-\gamma_{E_2}(D_2)E_2)^{i_2})\right)n_1^{i_1}n_2^{i_2}\\
&=\textstyle\sum_{i_1+i_2=3}-\frac{1}{i_1!i_2!}\left(\left(-\frac{3}{9-\sqrt{3}}a_2E_1-a_2E_2\right)^{i_1}\cdot\left(-\frac{3}{9-\sqrt{3}}b_2E_1-b_2E_2\right)^{i_2}\right)n_1^{i_1}n_2^{i_2}\\
&=-\left(-\tfrac{3}{9-\sqrt{3}}E_1-E_2\right)^3\left[\textstyle\sum_{i_1+i_2=3}\frac{1}{i_1!i_2!}a_2^{i_1}b_2^{i_2}n_1^{i_1}n_2^{i_2}\right]\\
&=\left(\tfrac{12042}{169}-\tfrac{27\sqrt{3}}{169}\right)\left[\textstyle\sum_{i_1+i_2=3}\frac{1}{i_1!i_2!}a_2^{i_1}b_2^{i_2}n_1^{i_1}n_2^{i_2}\right].
\end{aligned}$$

We obtain the formulas for the $e_i$ of the statement of the theorem from which we conclude that the Minkowski equality is satisfied. The identity $I(mb_2D_1) = I(ma_2D_2)$ now follows from Corollary 11.2 and Corollary 11.1. $\square$

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