ON MAPS PRESERVING SQUARE ROOTS OF IDEMPOTENT AND RANK-ONE NILPOTENT MATRICES

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Abstract. We characterize bijective linear maps on $M_n(\mathbb{C})$ that preserve the square roots of an idempotent matrix (of any rank). Every such map can be presented as a direct sum of a map preserving involutions and a map preserving square-zero matrices. Next, we consider bijective linear maps that preserve the square roots of a rank-one nilpotent matrix. These maps do not have standard forms when compared to similar linear preserver problems.

Introduction

Linear preserver problems (LPPs) concern the characterization of linear maps acting on matrix spaces that leave invariant certain functions, subsets, relations, etc. Let $M_n(\mathbb{C})$ denote the full matrix algebra of $n \times n$ complex matrices. Given a relation $\sim$ on $M_n(\mathbb{C})$, one may study linear maps $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfying $f(A) \sim f(B)$ whenever $A \sim B$. Given any matrix product $\star$ on $M_n(\mathbb{C})$, one can define a relation via zero $\star$-products; that is, declare $A \sim B$ if $A \star B = 0$. Then linear maps preserving the relation $\sim$ amounts to characterizing linear maps $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that $f(A) \star f(B) = 0$ whenever $A \star B = 0$. In the cases of the usual product $A \star B = AB$, the Lie product $[A, B] = AB - BA$, and the Jordan product $A \circ B = AB + BA$, bijective linear maps preserving the zero product are well-understood; see [15], [14], and [6], respectively (also note that a map that preserves the zero Lie product equivalently preserves commuting pairs of matrices). Motivated by similar questions in the setting of ring theory, the authors in [5] took a much more general approach concerning additive maps preserving products (that is, $f(x)f(y) = f(u)f(v)$ whenever $xy = uv$, where x, y, u, and v are elements of a ring satisfying some technical but reasonable conditions). Many of the above results conclude that such maps must be homomorphisms or antihomomorphisms, up to multiplication by a scalar. In other words, by only preserving a multiplicative property on certain pairs of elements, it turns out that all multiplicative structure must be preserved. See [12] for a more detailed discussion of LPPs and their solutions.

Recently, several authors have considered preserving relations induced by products equal to *fixed* nonzero matrices. We say that a linear map $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *preserves $\star$-products equal to C* if $f(A) \star f(B) = C$ whenever $A \star B = C$. One technique is to show that if f preserves $\star$-products equal to C, then f also preserves the zero $\star$-products. We collect some results concerning the three matrix products mentioned above.

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Suppose $f(A)f(B) = C$ whenever $AB = C$. Under the additional hypothesis of bijectivity, the case $C = e_{ij}$ was investigated in the two papers [1] and [2]. In both cases, f must be a scalar times a homomorphism (note that antihomomorphisms are not permissible). Likewise, if C is diagonalizable, a recent result appearing in [4] characterized all such maps and found that if C is singular, then f must be a scalar times a homomorphism and if C is invertible, then f may take a slightly different form. For example, if $AB = I_n$, the $n \times n$ identity matrix, then $BA = I_n$. This suggests that the transpose map $X \mapsto X^T$ (an antihomomorphism) preserves products equal to the identity. The case for arbitrary C is unknown; in particular, for nilpotents of high rank.

Suppose $[f(A), f(B)] = C$ whenever $[A, B] = C$. With the added hypotheses of bijectivity, the case $C = e_{12}$ was addressed in [7] while the case $C = e_{11} - e_{22}$ was addressed by the second author in [9]. Maps preserving the solutions to $AB - BA = e_{12}$ do not preserve commutativity while maps preserving the solutions to $AB - BA = e_{11} - e_{22}$ do preserve commutativity. The description is currently unknown for arbitrary C. On the other hand, when considering the Jordan product $A \circ B = AB + BA$, a complete description of bijective linear maps such that $f(A) \circ f(B) = M$ whenever $A \circ B = K$ has been obtained by the authors in [3], where M and K are arbitrary (in other words, maps preserving equal Jordan products have been completely classified). The usual product and Lie product seem to be more complex.

In this paper, we wish to understand bijective linear maps $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that

$$f(A)^2 = B \quad \text{whenever} \quad A^2 = B, \tag{1}$$

where B is a fixed nonzero idempotent or a rank-one nilpotent matrix. The matrix A is called a square root of B and we say f preserves the square roots of B. The conditions on f remind us of the equally deep subset of LPPs concerning maps preserving the roots of a matrix polynomial; that is, given a complex polynomial $p(x)$, describe bijective linear maps $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfying $p(f(A)) = 0$ whenever $p(A) = 0$. When $p(x)$ has at least two distinct roots, a complete description of f was obtained by Howard [8]. If $p(x)$ has only repeated roots, the results may be deduced from [Lemma 2.5, [11]]. As a special case, maps preserving the square-zero matrices (zeros of $p(x) = x^2$) is handled in [13].

The results in this paper may be considered a generalization of the results in [1] and [2] since every map preserving products equal to e_{ij} also preserve the square roots of e_{ij}. However, the square roots of a nontrivial idempotent or a nilpotent matrix do not span $M_n(\mathbb{C})$, and so we only obtain a meaningful description of f on the proper subspace of $M_n(\mathbb{C})$ generated by the square roots. Hence in some sense the solutions obtained here are "nonstandard" compared to other LPPs.

In Section 1, we characterize the square roots of an idempotent matrix $e_{11} + e_{22} + \cdots + e_{tt}$. The subspace of $M_n(\mathbb{C})$ generated by all such square roots can be written as a direct sum of two square subspaces of $M_n(\mathbb{C})$, and consequently the maps that preserve the square roots can be written as a direct sum of two maps, each acting on a direct summand, that preserve different subsets of matrices. This is a rather surprising conclusion when compared to other LPPs. For the precise statement, see Theorem 1.5.

In Section 2, we characterize the square roots of e_{12}, a rank-one nilpotent matrix. The space generated by all square roots of e_{12} can be written as a direct sum of a square subspace and a nonsquare subspace of $M_n(\mathbb{C})$. Thus maps that preserve the square roots of e_{12} may take nonstandard forms, and we give some examples. Under certain technical conditions, at least a partial description of f can be obtained via Šemrl's square-zero matrix preserver result [13]. See Theorem 2.6 for the precise statement.

1. Maps preserving the square roots of an idempotent matrix

Let $E_t = e_{11} + \cdots + e_{tt}$, a rank-$t$ idempotent matrix. We wish to study bijective linear maps $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ that preserve the square roots of E_t. The square roots of E_t are described in Lemma 1.1, the proof of which is straightforward.

Lemma 1.1. *Given an $A \in M_n(\mathbb{C})$, we have that $A^2 = E_t$ if and only if*

$$A = \begin{pmatrix} C & 0 \\ 0 & S \end{pmatrix},$$

for some complex $t \times t$ matrix C and a complex $(n-t) \times (n-t)$ matrix S such that $C^2 = E_t$ and $S^2 = 0$.

Consequently the subspace generated by the square roots of E_t can be written as a direct sum of two matrix spaces, as follows.

Lemma 1.2. *If $\mathcal{A} = \langle A \in M_n(\mathbb{C}) : A^2 = E_t \rangle$, then*

$$\mathcal{A} = \begin{pmatrix} M_t(\mathbb{C}) & 0 \\ 0 & sl_{n-t} \end{pmatrix}.$$

Proof. Note that $M_t(\mathbb{C})$ is generated by $t \times t$ involutions and sl_{n-t} is generated by $(n-t)\times(n-t)$ square-zero matrices. Since the direct sum of an arbitrary $t \times t$ involution with an arbitrary $(n-t)\times(n-t)$ square-zero matrix is a square root of E_t, the result follows from Lemma 1.1. □

Let

$$\mathcal{U}_t = \begin{pmatrix} M_t(\mathbb{C}) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{L}_{n-t} = \begin{pmatrix} 0 & 0 \\ 0 & M_{n-t}(\mathbb{C}) \end{pmatrix}$$

denote the upper and lower block subspaces of $M_n(\mathbb{C})$. Let $sl(\mathcal{L}_{n-t})$ denote the trace-zero matrices in $\mathcal{L}_{n-t}$. Lemma 1.2 asserts that $\mathcal{A} = \mathcal{U}_t \oplus sl(\mathcal{L}_{n-t})$.

If $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a bijective linear map such that $f(A)^2 = E_t$ whenever $A^2 = E_t$, then $f(\mathcal{A}) = \mathcal{A}$ and $\mathcal{U}_t \oplus sl(\mathcal{L}_{n-t}) = f(\mathcal{U}_t) \oplus f(sl(\mathcal{L}_{n-t}))$. This is an unconventional situation for a linear preserver problem, in the sense that the span of the preserved subset is not all of $M_n(\mathbb{C})$. Thus, it only makes sense to describe f on its restriction to $\mathcal{A}$ (i.e., we cannot determine how f behaves outside of $\mathcal{A}$).

Lemma 1.3. *Let $f : \mathcal{A} \to \mathcal{A}$ be a bijective linear map such that $f(A)^2 = E_t$ whenever $A^2 = E_t$. Then $f(\mathcal{U}_t) \circ f(sl(\mathcal{L}_{n-t})) = 0$ and for all $T \in \mathcal{A}$ with $T^2 = 0$, we have $f(T)^2 = 0$.*

Proof. Given a $C \in \mathcal{U}_t$ with $C^2 = E_t$ and $S \in \mathcal{L}_{n-t}$ with $S^2 = 0$, clearly $(C + xS)^2 = E_t$ for all $x \in \mathbb{C}$ and

$$f(C + xS)^2 = (f(C) + xf(S))^2 = f(C)^2 + xf(C) \circ f(S) + x^2 f(S)^2 = E_t.$$

Cancelling $f(C)^2 = E_t$, the system of equations reduces to

$$xf(C) \circ f(S) + x^2 f(S)^2 = 0. \tag{2}$$

Subtracting the two equations obtained by taking $x = 1$ and $x = -1$, it follows that $f(C) \circ f(S) = 0$. Using the fact that the set of such C span $\mathcal{U}_t$ and the set of such S span $sl(\mathcal{L}_{n-t})$, as well as the fact that $f(C) \circ f(S)$ is bilinear in C and S, we conclude that $f(\mathcal{U}_t) \circ f(sl(\mathcal{L}_{n-t})) = 0$.

Given a square-zero matrix $T \in \mathcal{A}$, it may be written as a direct sum of square-zero matrices in $\mathcal{U}_t$ and $sl(\mathcal{L}_{n-t})$. In light of the observation $f(\mathcal{U}_t) \circ f(sl(\mathcal{L}_{n-t})) = 0$, equation (2) also implies that $f(S)^2 = 0$ whenever $S \in sl(\mathcal{L}_{n-t})$ is a square-zero matrix. To prove that f preserves square-zero matrices in $\mathcal{A}$ it suffices to show that $f(T)^2 = 0$ whenever $T \in \mathcal{U}_t$ is a square-zero matrix.

Working in $\mathcal{U}_t$, let $\hat{T}$ be the Jordan normal form of T, written as $T = P\hat{T}P^{-1}$ for some invertible $P \in \mathcal{U}_t$. It is easy to find a $t \times t$ involution, say $\hat{A}$, such that $\hat{A} \circ \hat{T} = 0$. Hence there is a matrix $A = P\hat{A}P^{-1}$ such that $(A + xT)^2 = E_t$ for all $x \in \mathbb{C}$. Using the same computations as above, we conclude that $f(A) \circ f(T) = 0$ and $f(T)^2 = 0$. □

We have seen that $\mathcal{U}_t \oplus sl(\mathcal{L}_{n-t}) = f(\mathcal{U}_t) \oplus f(sl(\mathcal{L}_{n-t}))$ as well $f(\mathcal{U}_t) \circ f(sl(\mathcal{L}_{n-t})) = 0$. It is not immediately obvious, however, if f preserves the direct summands. This turns out to be the case.

Lemma 1.4. *If $f : \mathcal{A} \to \mathcal{A}$ is a bijective linear map such that $f(A)^2 = E_t$ whenever $A^2 = E_t$, then $f(\mathcal{U}_t) = \mathcal{U}_t$ and $f(sl(\mathcal{L}_{n-t})) = sl(\mathcal{L}_{n-t})$.*

Proof. Write $E_t = f(U) + f(V)$, where $U \in \mathcal{U}_t$ and $V \in sl(\mathcal{L}_{n-t})$. Let $C_u, S_u \in \mathcal{U}_t$ and $C_l, S_l \in sl(\mathcal{L}_{n-t})$ be the matrices such that $f(U) = C_u + C_l$ and $f(V) = S_u + S_l$. So we have

$$(C_u + S_u) + (C_l + S_l) = E_t. \tag{3}$$

Clearly $C_l = -S_l$. By Lemma 1.3, $f(U) \circ f(V) = 0$ and so $C_u \circ S_u = C_l \circ S_l = 0$. Substituting in $C_l = -S_l$, we can conclude that $C_l^2 = S_l^2 = 0$. Squaring equation (3), we get

$$C_u^2 + S_u^2 = E_t. \tag{4}$$

Since $C_l = -S_l$, we can substitute $S_u = E_t - C_u$ into (4) to get

$$C_u^2 + (E_t - C_u)^2 = 2C_u^2 - 2C_u + E_t = E_t.$$

We have arrived at the conclusion that C_u is idempotent; i.e., $C_u^2 = C_u$.

Since C_t is trace-zero and $f(V)$, a linear combination of square-zero matrices by Lemma 1.3, is also trace-zero, we have

$$t = \mathrm{tr}(E_t) = \mathrm{tr}(f(U) + f(V)) = \mathrm{tr}(C_u + C_l) = \mathrm{tr}(C_u).$$

The trace of an idempotent matrix is its rank. Hence $\mathrm{rk}(C_u) = t$. Now, the only rank-t idempotent matrix in $\mathcal{U}_t$ is E_t. Consequently, $C_u = E_t$ and $S_u = 0$.

Hence $f(\mathcal{U}_t)$ contains a matrix whose $\mathcal{U}_t$-direct summand component is E_t. Since the Jordan product of $f(sl(\mathcal{L}_{n-t}))$ with E_t must be zero by Lemma 1.3, it follows that $f(sl(\mathcal{L}_{n-t})) = sl(\mathcal{L}_{n-t})$.

Now $f(\mathcal{U}_t) \circ f(sl(\mathcal{L}_{n-t})) = f(\mathcal{U}_t) \circ sl(\mathcal{L}_{n-t}) = 0$. The only matrix in $\mathcal{L}_{n-t}$ whose Jordan product with $sl(\mathcal{L}_{n-t})$ is identically zero is the $(n-t) \times (n-t)$ zero matrix. Hence $f(\mathcal{U}_t) = \mathcal{U}_t$. □

Theorem 1.5. *Let $n \geq 3$. If $f : \mathcal{A} \to \mathcal{A}$ is a bijective linear map such that $f(A)^2 = E_t$ whenever $A^2 = E_t$, with $1 \leq t \leq n-1$, then f is of the form $f = g \oplus h$, where $g : \mathcal{U}_t \to \mathcal{U}_t$ is a bijective linear map preserving involutions and $h : sl(\mathcal{L}_{n-t}) \to sl(\mathcal{L}_{n-t})$ is a bijective linear map preserving square-zero matrices.*

Proof. By Lemma 1.4, take g to be the restriction of f to $\mathcal{U}_t$ and h to be the restriction of f to $sl(\mathcal{L}_{n-t})$. The hypotheses on f ensure that any involution in $\mathcal{U}_t$ is mapped to an involution and that any square-zero matrix in $\mathcal{L}_{n-t}$ is mapped to a square-zero matrix. The description of g is due to Howard [8] and the description of h is due to Šemrl [13].

If $t \geq 3$, g takes the form $X \mapsto \pm PXP^{-1}$ or $X \mapsto \pm PX^TP^{-1}$, where $P \in \mathcal{U}_t$ is invertible. If $t \leq n-2$, h takes the form $X \mapsto cQXQ^{-1}$ or $X \mapsto cQX^TQ^{-1}$, where $c \in \mathbb{C}$ is nonzero and $Q \in \mathcal{L}_{n-t}$ is invertible. □

Remark 1.6. In general, g and h may be of opposite types; one could be a homomorphism and the other an antihomomorphism, so in general f need not be a homomorphism nor an antihomomorphism on $\mathcal{A}$.

There are some extremal cases of interest. If $t = 1$, then $g(e_{11}) = \pm e_{11}$. If $t = 2$, then g does not have a standard description (for example, the map $X \mapsto PXP^{-1} - \mathrm{tr}(X)I_2$ preserves involutions but is not of a standard form). If $t = n-1$, then $sl(\mathcal{L}_1) = 0$, and so h is simply the zero map.

Remark 1.7. If $n = 2$, then $\mathcal{A} = \langle e_{11} \rangle$. Hence any linear map $f : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ such that $f(e_{11}) = \pm e_{11}$ preserves the square roots of e_{11}.

Remark 1.8. One can use the theorem to obtain a description of maps preserving products of rank-t idempotents which differ in the image and preimage. Let $P, Q \in M_n(\mathbb{C})$ be rank-t idempotent matrices. Since P and Q are similar to E_t, we can write $P = UE_tU^{-1}$ and $Q = VE_tV^{-1}$ for some $U, V \in M_n(\mathbb{C})$, invertible. Thus the subspace generated by square roots of P and Q are conjugates of $\mathcal{A}$, denoted $\mathcal{A}_P$ and $\mathcal{A}_Q$, respectively. Let $\hat{f} : \mathcal{A}_Q \to \mathcal{A}_P$ be a bijective linear map satisfying $\hat{f}(A)^2 = P$ whenever $A^2 = Q$. Notice that

$$X \mapsto U^{-1}\hat{f}(VXV^{-1})U$$

is a bijective linear map from $\mathcal{A}$ to $\mathcal{A}$ preserving square roots of E_t. By deferring to Theorem 1.5, the description of $\hat{f}$ may be obtained. The description of $\hat{f}$ can then be extended to all of $M_n(\mathbb{C})$.

2. Maps preserving the square roots of a rank-one nilpotent matrix

We now wish to consider bijective linear maps $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ with $f(A)^2 = e_{12}$, whenever $A^2 = e_{12}$. First, we identify the square roots of e_{12}, which exist as long as $n \geq 3$. Since any square root of e_{12} must commute with e_{12}, it is easy to verify the following lemma.

Lemma 2.1. *Let $n \geq 3$ and $A \in M_n(\mathbb{C})$. If $A^2 = e_{12}$, then*

$$A = \begin{pmatrix} 0 & a & \mathbf{u}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{v} & S \end{pmatrix},$$

where $a \in \mathbb{C}$*,* $S \in M_{n-2}(\mathbb{C})$ *is a square-zero matrix,* $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{n-2}$ *satisfy* $\mathbf{v} \in \ker S$*,* $\mathbf{u} \in \ker S^T$*, and* $\mathbf{u}^T \mathbf{v} = 1$*.*

Can the S in Lemma 2.1 be arbitary? Consider the following equivalent matrix-theoretical question: given any square-zero matrix, S, does there exist $\mathbf{u} \in \ker S^T$ and $\mathbf{v} \in \ker S$ such that $\mathbf{u}^T \mathbf{v} = 1$? This turns out to be false when S has maximum rank and n is even, but true otherwise.

Lemma 2.2. *If* $S \in M_t(\mathbb{C})$ *is similar to* $\operatorname{diag}(J_2(0), J_2(0), \ldots, J_2(0))$*, and* $\mathbf{u}, \mathbf{v} \in \mathbb{C}^t$ *satisfy* $S\mathbf{v} = S^T \mathbf{u} = 0$*, then* $\mathbf{u}^T \mathbf{v} = 0$*. Otherwise, if* $\operatorname{rk}(S) < \frac{t}{2}$*, there exist* $\mathbf{u}, \mathbf{v} \in \mathbb{C}^t$ *such that* $S\mathbf{v} = S^T \mathbf{u} = 0$ *and* $\mathbf{u}^T \mathbf{v} = 1$*.*

Proof. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_t$ denote the collection of standard basis vectors of $\mathbb{C}^t$ and let $J_k(\lambda)$ denote the $k \times k$ Jordan block corresponding to the eigenvalue $\lambda \in \mathbb{C}$.

There is an invertible $Q \in M_t(\mathbb{C})$ such that

$$S = Q \operatorname{diag}(J_2(0), J_2(0), \ldots, J_2(0)) Q^{-1}.$$

Let $\hat{S} = \operatorname{diag}(J_2(0), J_2(0), \ldots, J_2(0))$. If $\mathbf{v} \in \ker S$, then it may be written $\mathbf{v} = Q\hat{\mathbf{v}}$, where $\hat{\mathbf{v}} \in \langle \mathbf{e}_1, \mathbf{e}_3, \ldots, \mathbf{e}_{t-1} \rangle = \ker \hat{S}$. Similarly, $\mathbf{u} \in \ker S^T$ implies $\mathbf{u} = (Q^{-1})^T \hat{\mathbf{u}}$, where $\hat{\mathbf{u}} \in \langle \mathbf{e}_2, \mathbf{e}_4, \ldots, \mathbf{e}_t \rangle = \ker \hat{S}^T$. Then

$$\mathbf{u}^T \mathbf{v} = \hat{\mathbf{u}}^T Q^{-1} Q \hat{\mathbf{v}} = \hat{\mathbf{u}}^T \hat{\mathbf{v}} = 0.$$

If $\operatorname{rk}(S) < \frac{t}{2}$, then we may write

$$S = Q \operatorname{diag}(J_2(0), J_2(0), \ldots, J_2(0), J_1(0), \ldots, J_1(0)) Q^{-1},$$

with at least one copy of $J_1(0)$. Then the kernels of the Jordan form of S and its transpose have at least one vector in common, in particular, $\mathbf{e}_t$. Taking $\hat{\mathbf{v}} = \hat{\mathbf{u}} = \mathbf{e}_t$ and defining $\mathbf{v} = Q\hat{\mathbf{v}}$ and $\mathbf{u} = (Q^{-1})^T \hat{\mathbf{u}}$, we get

$$\mathbf{u}^T \mathbf{v} = \hat{\mathbf{u}}^T Q^{-1} Q \hat{\mathbf{v}} = (\mathbf{e}_t)^T \mathbf{e}_t = 1,$$

as claimed. □

As before, the square roots of e_{12} span a proper subspace of $M_n(\mathbb{C})$. Let $\mathcal{A}$ denote this subspace and let us restrict attention to bijective linear maps acting on $\mathcal{A}$.

Lemma 2.3. *Let* $n \geq 3$ *and* $\mathcal{A} = \langle A \in M_n(\mathbb{C}) : A^2 = e_{12} \rangle$*. If* $n = 4$*,* $\mathcal{A}$ *has the form*

$$\mathcal{A} = \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix}.$$

Otherwise, $\mathcal{A}$ has the form

$$\mathcal{A} = \begin{pmatrix} 0 & * & * & \dots & * \\ 0 & 0 & 0 & \dots & 0 \\ 0 & * & & & \\ 0 & * & & & \\ \vdots & \vdots & & sl_{n-2} & \\ 0 & * & & & \end{pmatrix}.$$

Proof. The fact that $e_{12} \in \mathcal{A}$ is clear from the Lemma 2.1. Observe

$$A = \begin{pmatrix} 0 & 0 & \mathbf{u}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{v} & 0 \end{pmatrix},$$

is a square root of e_{12} as long as $\mathbf{u}^T\mathbf{v} = 1$, where $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{n-2}$. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n-2}$ denote the collection of standard basis vectors of $\mathbb{C}^{n-2}$. Taking $\mathbf{u} = \mathbf{e}_j$ and $\mathbf{v} = \mathbf{e}_j$ for $1 \leq j \leq n-2$ yields $\mathbf{u}^T\mathbf{v} = 1$, which corresponds to the matrix $A = e_{1j} + e_{j2}$. Letting $\mathbf{i}$ denote the imaginary unit, the pair $\mathbf{u} = \mathbf{i}\,\mathbf{e}_j$ and $\mathbf{v} = -\mathbf{i}\,\mathbf{e}_j$ also has $\mathbf{u}^T\mathbf{v} = 1$ and corresponds to the matrix $A = \mathbf{i}e_{1j} - \mathbf{i}e_{j2}$. Taking linear combinations of the matrices $e_{1j} + e_{j2}$ and $\mathbf{i}e_{1j} - \mathbf{i}e_{j2}$ shows that $e_{1j}, e_{j2} \in \mathcal{A}$ for $j \geq 3$. The case when $n = 3$ is clear.

Let $n = 4$. Suppose

$$A = \begin{pmatrix} 0 & 0 & \mathbf{u}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{v} & S \end{pmatrix}$$

is a 4×4 square root of e_{12} with $\mathbf{u}, \mathbf{v} \in \mathbb{C}^2$ and $S^2 = 0$. By Lemma 2.2, we must have $S = 0$. Hence $\mathcal{A} = \langle e_{12}, e_{13}, e_{14}, e_{32}, e_{42} \rangle$.

Suppose $n \geq 5$. Also by Lemma 2.2, for every rank-one square-zero matrix $S \in sl_{n-2}$ there exist vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{n-2}$ such that

$$A = \begin{pmatrix} 0 & 0 & \mathbf{u}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{v} & S \end{pmatrix}$$

is a square root of e_{12}. Since $\mathcal{A}$ contains the matrix units e_{1j} and e_{j2} for $j \geq 3$, it follows that

$$\begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & S \end{pmatrix} \in \mathcal{A}.$$

The rank-one square-zero matrices in the lower $(n-2) \times (n-2)$ block span a subspace of $M_n(\mathbb{C})$ isomorphic to sl_{n-2}, so $\mathcal{A}$ has the form as claimed. □

Let $\mathcal{C} = \langle e_{12}, e_{13}, \dots, e_{1n}, e_{32}, e_{42}, \dots, e_{n2} \rangle$ and, borrowing notation from Section 1, let $\mathcal{L}_{n-2}$ denote the space of lower $(n-2) \times (n-2)$ block matrices in $M_n(\mathbb{C})$. Since the size of the lower block is fixed in this preserver problem, we drop the subscript and just refer to the lower block as $\mathcal{L}$. The previous lemma demonstrates that $\mathcal{A} = \mathcal{C}$ for $n = 3$ and 4, and $\mathcal{A} = \mathcal{C} \oplus sl(\mathcal{L})$ for $n \geq 5$.

Lemma 2.4. *Let $f : \mathcal{A} \to \mathcal{A}$ be a bijective linear map such that $f(A)^2 = e_{12}$, whenever $A^2 = e_{12}$. Then*

(1) $f(e_{12}) \in \langle e_{12} \rangle$,
(2) $f(C) \circ f(S) = 0$ whenever $(C + S)^2 = e_{12}$, with $C \in \mathcal{C}$ and $S \in sl(\mathcal{L})$, and
(3) $f(S)^2 = 0$ whenever $S \in sl(\mathcal{L})$ is a square-zero matrix, except possibly if $\mathrm{rk}(S) = \frac{n-2}{2}$.

Proof. If $A^2 = e_{12}$, then $A + xe_{12}$ is a square root of e_{12} for all $x \in \mathbb{C}$ and

$$e_{12} = f(A + xe_{12})^2 = f(A)^2 + xf(e_{12}) \circ f(A) + x^2 f(e_{12})^2.$$

Using $f(A)^2 = e_{12}$ and the standard trick of varying over values of x, conclude that $f(e_{12})^2 = 0$ with $f(e_{12}) \in \mathcal{A}$, as well as $f(e_{12}) \circ f(A) = 0$. By linearity, it follows that $f(e_{12}) \circ \mathcal{A} = 0$. Writing

$$f(e_{12}) = \begin{pmatrix} 0 & * & \mathbf{u}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{v} & S \end{pmatrix} \in \mathcal{A},$$

we get that if $n = 3, 4$, then $S = 0$, and if $n \geq 5$, then $S \circ sl_{n-2} = 0$ implies $S = 0$. Using $e_{1j}, e_{j2} \in \mathcal{A}$ for $j \geq 3$, we get that $f(e_{12}) \circ e_{1j} = f(e_{12}) \circ e_{j2} = 0$ implies $\mathbf{u} = 0$ and $\mathbf{v} = 0$. Thus $f(e_{12}) \in \langle e_{12} \rangle$.

By Lemma 2.1, every square root of e_{12} may be written as $C + xS$, where $x \in \mathbb{C}$, subject to the constraints $C^2 = e_{12}, S^2 = 0$, and $C \circ S = 0$. Then $f(C)^2 = e_{12}$ and $f(C + xS)^2 = e_{12}$ imply that

$$xf(C) \circ f(S) + x^2 f(S)^2 = 0.$$

By the usual argument, it follows that $f(C) \circ f(S) = 0$ and $f(S)^2 = 0$. This proves (2).

By Lemma 2.2, one can always find a $C \in \mathcal{C}$ such that $(C + xS)^2 = e_{12}$ for all $x \in \mathbb{C}$, provided that $\mathrm{rk}(S) < \frac{n-2}{2}$. Hence (3) is proved as well. □

Lemma 2.4 demonstrates that f possesses some multiplicative structure. However, we can find several maps that preserve the square roots of e_{12} that are not simply automorphisms or antiautomorphisms. The first map exploits the special relationship between e_{12} and $\mathcal{A}$.

Example 1. Suppose $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective linear map such that $f(A)^2 = e_{12}$ whenever $A^2 = e_{12}$. Since $e_{12}\mathcal{A} = \mathcal{A}e_{12} = 0$, it follows that $A \mapsto f(A) + z(A)e_{12}$ also preserves square roots of e_{12} whenever $z : M_n(\mathbb{C}) \to \mathbb{C}$ is a linear functional.

One can also ask for a description on the quotient space $\mathcal{A}/\langle e_{12} \rangle$. In fact, every bijective linear map that preserves square roots of e_{12} restricts to a bijection on $\mathcal{A}/\langle e_{12} \rangle$. This can be seen as follows. Given such a preserver f, define $\zeta : \mathcal{A} \to \mathcal{A}$ by $A \mapsto f(A) - g(A)e_{12}$, where $g : M_n(\mathbb{C}) \to \mathbb{C}$ is linear, such that the $(1, 2)$-entry of $\zeta(A)$, denoted $[\zeta(A)]_{1,2}$, is zero for all $A \in \mathcal{A}$. Define

$$\hat{\mathcal{A}} = \{A \in \mathcal{A} : [A]_{1,2} = 0\} \cong \mathcal{A}/\langle e_{12} \rangle.$$

Clearly, $\zeta(\hat{\mathcal{A}}) \subseteq \hat{\mathcal{A}}$ by the definition of ζ. If $A \in \hat{\mathcal{A}}$ is such that $\zeta(A) = f(A) - g(A)e_{12} = 0$, then $f(A) \in \langle e_{12} \rangle$. Since $f(e_{12}) \in \langle e_{12} \rangle$ by Lemma 2.4, it follows that $A \in \langle e_{12} \rangle \cap \hat{\mathcal{A}} = 0$. Hence $\ker \zeta|_{\hat{\mathcal{A}}} = 0$ and ζ acts bijectively on $\hat{\mathcal{A}}$.

In fact the subspace $sl(\mathcal{L})$ does not need to be mapped back into $sl(\mathcal{L})$. Hence maps preserving square roots of e_{12} do not necessarily preserve direct summands.

Example 2. Consider the linear map $f : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{pmatrix} 0 & a & \mathbf{u}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{v} & S \end{pmatrix} \mapsto \begin{pmatrix} 0 & a & \mathbf{u}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{v}+\mathbf{s}_1 & S \end{pmatrix},$$

where $\mathbf{s}_1$ is the first column of S. The map f is bijective. Suppose A is a square root of e_{12}. Then $\mathbf{u}^T\mathbf{v} = 1$, $S\mathbf{v} = S^T\mathbf{u} = \mathbf{0}$ and $S^2 = 0$. Then observe that

$$f(A)^2 = \begin{pmatrix} 0 & \mathbf{u}^T\mathbf{v} + \mathbf{u}^T\mathbf{s}_1 & (S^T\mathbf{u})^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & S\mathbf{v} + S\mathbf{s}_1 & S^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 + \mathbf{u}^T\mathbf{s}_1 & \mathbf{0}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & 0 \end{pmatrix}$$

and f preserves square roots of e_{12} because $\mathbf{u}^T\mathbf{s}_1$ is the first entry of the vector $S^T\mathbf{u} = \mathbf{0}$.

Analogously, adding any row of S to $\mathbf{u}^T$ yields a map that preserves the square roots of e_{12}. In fact, adding any fixed linear combination of the columns of S to $\mathbf{v}$ or the rows of S to $\mathbf{u}^T$ also yields such a map.

Note that $sl(\mathcal{L})$ is not mapped to a multiple of a conjugate of $sl(\mathcal{L})$. Indeed, suppose for all $X \in sl(\mathcal{L})$, $f(X) = cUXU^{-1}$ for some invertible $U \in M_n(\mathbb{C})$. Then $cUe_{ij}U^{-1} = e_{ij}$ for $i \geq 3, j \geq 3$, and $i \neq j$. This implies that for all such pairs (i,j), the off-diagonal entries of U in the ith column and jth row are zero. Hence U is a linear combination of the $n \times n$ identity matrix, e_{12}, and e_{21}. But then $cU\,sl(\mathcal{L})U^{-1} \subseteq sl(\mathcal{L})$ while $f(sl(\mathcal{L})) \not\subseteq sl(\mathcal{L})$, a contradiction. The proof for $X \mapsto cUX^TU^{-1}$ follows by similar reasoning. Hence f does not act as a standard map on $sl(\mathcal{L})$.

So maps preserving square roots of e_{12} can be very strange. Contrast this to the recent result of Catalano and Chang-Lee [2], who found standard solutions for maps preserving products equal to e_{12}.

Every square-zero matrix $S \in sl(\mathcal{L})$ satisfying $\mathrm{rk}(S) < \frac{n-2}{2}$ is mapped to a square-zero matrix. In principle, it may be that a square-zero matrix whose rank is exactly $\frac{n-2}{2}$ is not mapped to a square-zero matrix. However, we claim that if rank-one and rank-two square-zero matrices of are mapped to square-zero matrices, we can guarantee that all square-zero matrices are preserved. The following is a simplified argument of results appearing in [10] and is of independent interest.

Proposition 2.5. *Let $n \geq 4$. If $\phi : sl_n \to sl_n$ is a bijective linear map that sends rank-one and rank-two square-zero matrices to square-zero matrices, then ϕ takes the form*

(1) $\phi(X) = cUXU^{-1}$, *or*
(2) $\phi(X) = cUX^TU^{-1}$,

where $c \in \mathbb{C}$ and $U \in M_n(\mathbb{C})$ is invertible.

Proof. Given a square-zero matrix $S \in sl_n$, one can conclude using a Jordan normal form argument that there exist $k = \mathrm{rk}(S)$ orthogonal rank-one square-zero matrices $S_1, S_2, \ldots, S_k$ such that

$$S = S_1 + S_2 + \cdots + S_k.$$

By hypothesis, $\phi(S_i)^2 = 0$ and $\phi(S_i + S_j)^2 = \phi(S_i) \circ \phi(S_j) = 0$ for all $1 \leq i, j \leq k$. Hence

$$\phi(S)^2 = \phi(S_1 + S_2 + \cdots + S_k)^2 = 0,$$

and so ϕ preserves all square-zero matrices. By [13], ϕ takes a standard form. $\square$

Considering the examples above, there is little hope of obtaining a complete description of bijective linear maps preserving square roots of e_{12}. Despite this, observe that in each example, the space $\mathcal{C}$ was preserved and the image of $sl(\mathcal{L})$ completely contained $sl(\mathcal{L})$. In fact these two properties are equivalent, as the next theorem shows. Note that if $n = 3$ or $n = 4$, the theorem trivially holds since $\mathcal{A} = \mathcal{C}$ in both cases. We also discuss the case $n = 6$ in Remark 2.7.

Theorem 2.6. *Let $n \geq 5$ and $n \neq 6$. If $f : \mathcal{A} \to \mathcal{A}$ is a bijective linear map such that $f(A)^2 = e_{12}$ whenever $A^2 = e_{12}$. Then $f(\mathcal{C}) = \mathcal{C}$ if and only if $f(sl(\mathcal{L}))|_{sl(\mathcal{L})} = sl(\mathcal{L})$. Moreover, if either situation holds, then there is a bijective linear map $\phi : sl(\mathcal{L}) \to sl(\mathcal{L})$ preserving square-zero matrices that extends to f.*

Proof. If $f(\mathcal{C}) = \mathcal{C}$, then f is bijective, and so $f(sl(\mathcal{L}))|_{sl(\mathcal{L})} = sl(\mathcal{L})$. Conversely, assume for a contradiction that $f(sl(\mathcal{L}))|_{sl(\mathcal{L})} = sl(\mathcal{L})$ but $f(\mathcal{C}) \neq \mathcal{C}$. Using this and the fact that $f(e_{12}) \in \langle e_{12} \rangle \subseteq \mathcal{C}$, there must be a $j \geq 3$ so that $A = e_{1j} + e_{j2}$ or $A = \mathbf{i}e_{1j} - \mathbf{i}e_{j2}$ satisfies $f(A)|_{sl(\mathcal{L})} = R \neq 0$. Since $f(A)^2 = e_{12}$, we get that R is a square-zero matrix. By hypothesis there is a bijective linear map $\phi : sl(\mathcal{L}) \to sl(\mathcal{L})$ that extends to f; that is, $\phi(X) = f(X)|_{sl(\mathcal{L})}$ for all $X \in sl(\mathcal{L})$. If $n = 5$, then ϕ sends rank-one square-zero matrices to square-zero matrices. Since every square-zero matrix in sl_3 is rank-one, it follows that ϕ preserves square-zero matrices. If $n \geq 7$, then rank-one and rank-two square-zero matrices in $sl(\mathcal{L})$ are mapped to square-zero matrices, and so by Proposition 2.5, ϕ preserves square-zero matrices as well. So we have

$$\phi(X) = cUXU^{-1} \quad \text{or} \quad \phi(X) = cUX^TU^{-1}$$

for all $X \in sl(\mathcal{L})$, where $U \in M_{n-2}(\mathbb{C})$ is invertible and $c \in \mathbb{C}\backslash\{0\}$.

By statement (2) of Lemma 2.4, we have that $f(A) \circ f(e_{kl}) = 0$ whenever $j \neq k, l$ and $k, l \geq 3$ are distinct. Hence $R \circ f(e_{kl})|_{sl(\mathcal{L})} = R \circ \phi(e_{kl}) = R \circ (cUe_{kl}U^{-1}) = 0$. Equivalently, $U^{-1}RU \circ e_{kl} = 0$, and so the off-diagonal entries of $U^{-1}RU$ in the kth column and lth row are zero. By varying k, l over all possible choices, we conclude that $U^{-1}RU$ is a diagonal square-zero matrix. The only diagonal square-zero matrix is the zero matrix, and so $R = 0$, a contradiction. Thus the assumption that $f(\mathcal{C}) \neq \mathcal{C}$ is false, proving the converse. $\square$

Remark 2.7. In the case $n = 3$ or $n = 4$, the equivalence is trivial since $\mathcal{A} = \mathcal{C}$ in both cases. In the case $n = 6$, the space $sl(\mathcal{L}_4)$ contains only rank-one and rank-two square-zero matrices; but the rank-two square-zero matrices have maximum rank in $sl(\mathcal{L}_4)$. By Lemma 2.2, there is no matrix $C \in \mathcal{C}$ such that $(C + S)^2 = e_{12}$ when $S \in sl(\mathcal{L}_4)$ and $\text{rk}(S) = 2$. Hence it may be that $f(S)^2 \neq 0$, which explains the above exclusion. Note that a description of bijective linear map sending rank-one square-zero matrices to square-zero matrices would be a significant generalization of Šemrl's result [13] (and by extension, Proposition 2.5) that would find application in many preserver problems. In particular, if such maps turn out to be standard, then there is no need to exclude $n = 6$ in Theorem 2.6 above.

One can use a similar method as in Remark 1.8 to extend to the arbitrary rank-one nilpotent cases. Let $N, M \in M_n(\mathbb{C})$ be rank-one nilpotents and $\hat{f} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective linear map such that $\hat{f}(A)^2 = N$, whenever $A^2 = M$. Then writing $N = Ue_{12}U^{-1}$ and $M = Ve_{12}V^{-1}$ for some invertible $U, V \in M_n(\mathbb{C})$, the map $X \mapsto U^{-1}\hat{f}(VXV^{-1})U$ is a bijective linear map preserving the square roots of e_{12}.

Based on the complicated structure of the square roots of e_{12} and the (nonstandard) maps that preserve them, we believe that the characterization of maps preserving square roots of arbitrary nilpotent matrices would also be quite challenging.

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