On the Images of Generalized Polynomials Evaluated on Matrices over an Algebraically Closed Skew Field

Aria Beaupré * Emily Hoopes-Boyd† Grace O'Brien ‡

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Abstract

Let K be a Makar-Limanov algebraically closed skew field. In the first part of this paper, we prove that the image of a generalized multilinear polynomial, with coefficients in K, evaluated over $M_m(K)$, is $M_m(K)$. In the second part, we show that any matrix in $M_m(K)$ may be written as the sum of three or fewer elements from the image of a generalized polynomial, with coefficients in K, evaluated over $M_m(K)$.

Key words and phrases. images of generalized polynomials, matrix algebra, algebraically closed skew field

1 Introduction

The celebrated L'vov-Kaplansky conjecture states that the set of values of a multilinear polynomial evaluated on a matrix algebra over a field is a vector space. So far, this statement has been proven only for the case of 2×2 matrices [3]. Recently, this research area has been active, with many partial results published. We refer the reader to the survey paper, [6], for an overview. Primarily, research has been focused on matrices over fields. However, some researchers have shown interest in the images of generalized polynomials on certain algebras [1, 2]. Throughout this paper, let K be a Makar-Limanov algebraically closed skew field [4], and $M_m(K)$ be the ring of $m \times m$ matrices over K. Let $X = \{x_1, x_2, \dots\}$ be an infinite set of noncommuting indeterminates. Let $K\{X\}$ be the free K-algebra in the indeterminates of X. Throughout the paper, we

*Department of Mathematics, Harvey Mudd College, Claremont, CA 91711, U.S.A.
E-mail address: abeaupre@g.hmc.edu

†Department of Mathematics, Kent State University, Kent, OH 44242, U.S.A.
E-mail address: ehoopes1@kent.edu

‡Department of Mathematics, The University of Michigan, Ann Arbor, MI 48109, U.S.A.
E-mail address: graceob@umich.edu

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will refer to the elements of $K\{X\}$ as generalized polynomials. We will study the images of these generalized polynomials evaluated over $M_m(K)$.

In [5], Rowen writes a generalized polynomial as

$$F(X_1, \ldots, X_n) = \sum r_{i1} X_{i_1} r_{i2} \ldots r_{it} X_{i_t} r_{i,t+1},$$

where $r_{i1}, \ldots, r_{i,t+1} \in K$ and $X_{i_1}, \ldots, X_{i_t} \in \{X_1, \ldots, X_n\}$. For our purposes, we wish to be more explicit concerning the indices of summation. So, we adjust Rowen's notation, and write a non-constant generalized polynomial as

$$F(X_1, \ldots, X_n) = b + \sum_{k=1}^{p} b_{k1} X_{k_1} b_{k2} \ldots b_{k,q_k} X_{k_{(q_k)}} b_{k,q_k+1},$$

where $b \in K$, $b_{k1}, \ldots, b_{k,q_k+1} \in K \setminus \{0\}$, and $X_{k_1}, \ldots, X_{k_{(q_k)}} \in \{X_1, \ldots, X_n\}$. Note, q_k denotes the degree of the k-th monomial of $F(X_1, \ldots, X_n)$, and q_k is fixed for each $1 \leq k \leq p$. Also, we define $q_k \geq 1$ for all $1 \leq k \leq p$. Thus, p determines the number of non-constant monomials of $F(X_1, \ldots, X_n)$.

A generalized polynomial, $F(X_1, \ldots, X_n)$, is called *multilinear* provided that each term of the generalized polynomial is of exactly order one for every X_i.

Considering polynomials over K, the Makar-Limanov construction provides an interesting property [4, Lemma 2]:

Remark 1.1. *For a non-constant generalized polynomial, $F(X_1, \ldots, X_n)$, with coefficients in K, the image of $F(X_1, \ldots, X_n)$ in K, $F(K)$, is K.*

Proof. Let $F(X_1, \ldots, X_n)$ be a generalized polynomial with coefficients in K. Since $F(X_1, \ldots, X_n)$ is non-constant, by Lemma 2 from [4], there exist $\overline{X}_1, \ldots, \overline{X}_n$ such that $F(\overline{X}_1, \ldots, \overline{X}_n) \neq 0$. Then there exists a variable X_i such that $F(\overline{X}_1, \ldots, \overline{X}_{i-1}, X_i, \overline{X}_{i+1}, \ldots, \overline{X}_n)$ is a non-constant generalized polynomial in one variable.

Let a be any element of K. Then, $F(\overline{X}_1, \ldots, \overline{X}_{i-1}, X_i, \overline{X}_{i+1}, \ldots, \overline{X}_n) - a$ is also a non-constant generalized polynomial with coefficients in K. Since K is algebraically closed, there exists a solution $\overline{X}_i$ to this polynomial. This holds for all $a \in K$, so the image of any generalized polynomial, with coefficients in K, is K. $\square$

Remark 1.1 implies that if $F(X_1, \ldots, X_n)$ is a generalized polynomial with coefficients in K and is non-constant evaluated over $M_m(K)$, then $F(X_1, \ldots, X_n)$ is non-constant evaluated over the scalar matrices of $M_m(K)$.

This paper contains two main results. First, Theorem 2.2 shows that, for a non-constant generalized multilinear polynomial, $F(X_1, \ldots, X_n)$, with coefficients in K, $F(M_m(K)) = M_m(K)$. Second, Theorem 3.7 states that if $F(X_1, \ldots, X_n)$ is a non-constant generalized polynomial with coefficients in K, then every $D \in M_m(K)$ is the sum of three or fewer elements of $F(M_m(K))$.

2 Generalized Multilinear Polynomials over K

In this section, we will show that for any non-constant generalized multilinear polynomial, $F(X_1, \ldots, X_n)$, with coefficients in K, $F(M_m(K)) = M_m(K)$. We will first prove this result for generalized multilinear polynomials in one variable. Note that if $F(X)$ is a generalized multilinear polynomial in one variable, it will have the form $F(X) = \sum_{k=1}^{p} a_k X b_k$, where $a_k, b_k \in K$, and $p \in \mathbb{N}$.

Lemma 2.1. *Let $F(X)$ be a non-constant generalized multilinear polynomial with coefficients in K. Then, for any matrix $D \in M_m(K)$, there exists a matrix $\overline{X} \in M_m(K)$ such that $F(\overline{X}) = D$.*

Proof. Let $F(X) = \sum_{k=1}^{p} a_k X b_k$ be a non-constant generalized multilinear polynomial, and let $D \in M_m(K)$. We will show that there exists an $\overline{X} \in M_m(K)$ such that $F(\overline{X}) = D$. Note that

$$\begin{aligned}\sum_{k=1}^{p} a_k X b_k &= \sum_{k=1}^{p} a_k \begin{bmatrix} x_{11} & \ldots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{m1} & \ldots & x_{mm} \end{bmatrix} b_k \\ &= \begin{bmatrix} \sum_{k=1}^{p} a_k x_{11} b_k & \ldots & \sum_{k=1}^{p} a_k x_{1m} b_k \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{p} a_k x_{m1} b_k & \ldots & \sum_{k=1}^{p} a_k x_{mm} b_k \end{bmatrix}.\end{aligned}$$

We will find a solution for the system of equations given by

$$d_{ij} = \sum_{k=1}^{p} a_k x_{ij} b_k,$$

for all $1 \leq i, j \leq m$. By Remark 1.1, we know that each equation has a solution, $\overline{x}_{ij} \in K$. As each variable, x_{ij}, only appears in one equation, we can construct $\overline{X}$ to have ij-th entry $\overline{x}_{ij}$. Thus, for any $D \in M_m(K)$, we have found a matrix $\overline{X} \in M_m(K)$ such that $F(\overline{X}) = D$. □

Now we will consider a non-constant generalized multilinear polynomial in n variables, $F(X_1, \ldots, X_n)$.

Theorem 2.2. *Let $F(X_1, \ldots, X_n)$ be a non-constant generalized multilinear polynomial with coefficients in K. Then, for any matrix $D \in M_m(K)$, there exist matrices $\overline{X}_1, \ldots, \overline{X}_n \in M_m(K)$ such that $F(\overline{X}_1, \ldots, \overline{X}_n) = D$.*

Proof. Let $F(X_1, \ldots, X_n)$ be a non-constant generalized multilinear polynomial, and let $D \in M_m(K)$. By Remark 1.1, we know that for any $d \in K$, there exist $\overline{x}_1, \ldots, \overline{x}_n \in K$ such that $F(\overline{x}_1 I, \ldots, \overline{x}_n I) = dI$. Set $\overline{X}_j = \overline{x}_j I$ for $1 \leq j < n$. Therefore, $F(\overline{X}_1, \ldots, \overline{X}_{n-1}, X_n)$ is a non-constant generalized multilinear polynomial in one variable evaluated over $M_m(K)$. So, by Lemma 2.1, there exists $\overline{X}_n \in M_m(K)$ such that $F(\overline{X}_1, \ldots, \overline{X}_{n-1}, \overline{X}_n) = D$.

□

Thus, the L'vov-Kaplansky conjecture holds if we replace the infinite field with a Makar-Limanov algebraically closed skew field.

3 Generalized Polynomials over K

In general, the images of (not necessarily multilinear) generalized polynomials evaluated over matrices are more difficult to describe than the multilinear case. We will prove that $F(M_m(K)) \neq M_m(K)$ for some non-constant generalized polynomial, $F(X_1, \ldots, X_n)$, with coefficients in K, but $\operatorname{span} F(M_m(K)) = M_m(K)$ for all such polynomials. The following lemmas will be useful in proving these results.

Lemma 3.1. *Let $F(X_1, \ldots, X_n)$ be a non-constant generalized polynomial with coefficients in K. Then, there exists an $X_i \in \{X_1, \ldots, X_n\}$ such that for some $\overline{x}_1, \ldots, \overline{x}_{i-1}, \overline{x}_{i+1}, \ldots, \overline{x}_n \in K$, $F(\overline{x}_1 I, \ldots, \overline{x}_{i-1} I, X_i, \overline{x}_{i+1} I, \ldots, \overline{x}_n I)$ is a non-constant generalized polynomial in one variable with coefficients in K, evaluated over $M_m(K)$.*

Proof. Let $F(X_1, \ldots, X_n)$ be a non-constant generalized polynomial with coefficients in K evaluated over $M_m(K)$. By Remark 1.1, $F(X_1, \ldots, X_n)$ is non-constant as a function on K^n. Hence, there exist X_i and $\overline{x}_1, \ldots, \overline{x}_{i-1}, \overline{x}_{i+1}, \ldots, \overline{x}_n \in K$ such that $F(\overline{x}_1 I, \ldots, \overline{x}_{i-1} I, X_i, \overline{x}_{i+1} I, \ldots, \overline{x}_n I)$ is a non-constant generalized polynomial in one variable with coefficients in K, evaluated over $M_m(K)$. □

Hence, for ease of matrix computation, many of our proofs will reduce a given polynomial in n variables to a non-constant generalized polynomial in one variable. These polynomials will be of the form:

$$G(X) = a + \sum_{k=1}^{p} a_{k1} X a_{k2} \ldots a_{k,z_k} X a_{k,z_k+1},$$

where $a \in K$, $a_{k1}, \ldots, a_{k,z_k+1} \in K \setminus \{0\}$, and $z_k, p \geq 1$.

In the next lemma, we learn that, for any system of nonzero generalized polynomials in n variables over K, there exists a solution in K^n that ensures each polynomial is nonzero.

Lemma 3.2. *For $h \geq 1$, let $\{H_1(x_1, \ldots, x_n), \ldots, H_h(x_1, \ldots, x_n)\}$ be a set of h nonzero generalized polynomials with coefficients in K. Note, $H_i(x_1, \ldots, x_n)$ may be trivially dependent on any x_j, for $1 \leq i \leq h$ and $1 \leq j \leq n$. Then, there exist $\overline{x}_1, \ldots, \overline{x}_n \in K$ such that $H_i(\overline{x}_1, \ldots, \overline{x}_n) \neq 0$ for all $1 \leq i \leq h$.*

Proof. Given that all $H_i(x_1, \ldots, x_n)$ are nonzero,

$$H(x_1, \ldots, x_n) := H_1(x_1, \ldots, x_n) \cdots H_h(x_1, \ldots, x_n)$$

is a nonzero polynomial. Therefore, by Remark 1.1, there exist $\overline{x}_1, \ldots, \overline{x}_n \in K$ such that $H(\overline{x}_1, \ldots, \overline{x}_n) \neq 0$. Thus, it must be the case that $H_i(\overline{x}_1, \ldots, \overline{x}_n) \neq 0$ for all $1 \leq i \leq h$. □

We will now turn our attention to the image of any non-constant generalized polynomial, $F(X_1, \ldots, X_n)$. First, we will show that all diagonal matrices are in $F(M_m(K))$.

Theorem 3.3. *Let $D_m(K) \subseteq M_m(K)$ be the set of diagonal matrices, and let $F(X_1, \ldots, X_n)$ be a non-constant generalized polynomial with coefficients in K. Then, $D_m(K) \subseteq F(M_m(K))$.*

Proof. By Lemma 3.1, we know there exists a set, $\overline{x}_1, \ldots, \overline{x}_{i-1}, \overline{x}_{i+1}, \ldots, \overline{x}_n \in K$, such that $G(X_i) = F(\overline{x}_1 I, \ldots, \overline{x}_{i-1} I, X_i, \overline{x}_{i+1} I, \ldots, \overline{x}_n I)$ is a non-constant generalized polynomial with coefficients in K. Thus, if $D_m(K) \subseteq G(M_m(K))$, then $D_m(K) \subseteq F(M_m(K))$. Set $X_i = X$. So,

$$G(X) = a + \sum_{k=1}^{p} a_{k1} X a_{k2} \ldots a_{k,z_k} X a_{k,z_k+1},$$

where $a \in K$, $a_{k1}, \ldots, a_{k,z_k+1} \in K \setminus \{0\}$, and $z_k, p \geq 1$.

Let $G(X) = B$ for some $X \in D_m(K)$. Then, $B \in D_m(K)$, where

$$b_{ii} = a + \sum_{k=1}^{p} a_{k1} x_{ii} a_{k2} \ldots a_{k,z_k} x_{ii} a_{k,z_k+1}$$

for $1 \leq i \leq m$. As each b_{ii} is the output of a nonzero polynomial, by Remark 1.1, we know, for any $d_{ii} \in K$, there exists a $\overline{x}_{ii} \in K$ such that $b_{ii} = d_{ii}$. Thus, for any $D \in D_m(K)$, there exists an $\overline{X} \in D_m(K)$ such that $F(\overline{X}) = D$. □

Next, we will further describe the image of any non-constant generalized polynomial, $F(X_1, \ldots, X_n)$, with coefficients in K, by showing the diagonal matrices are not the only upper triangular matrices contained in $F(M_m(K))$. First, we will build some notation. Let $T \in M_m(K)$ be upper triangular. For nonzero $c_i \in K$, rename

$$c_1 T c_2 \ldots c_z T c_{z+1} = W^{(z)}(T),$$

and let $w_{ij}^{(z)}$ be the ij-th entry of $W^{(z)}(T)$. Recall that the product of upper triangular matrices is itself upper triangular, so for all $i > j$, $w_{ij}^{(z)} = 0$. Therefore, we will focus on the entries where $i \leq j$.

Through matrix multiplication, we see that $w_{ij}^{(z)}$ is a polynomial in the set of variables, $V_{ij} := \{t_{\mu\gamma} : i \leq \mu, \gamma \leq j\}$.

Notice that we can rewrite a non-constant generalized polynomial, $G(T)$, with coefficients in K, as

$$G(T) = a + \sum_{k=1}^{p} a_{k1} T a_{k2} \ldots a_{k,z_k} T a_{k,z_k+1} = a + \sum_{k=1}^{p} W^{(z_k,k)}(T).$$

As seen in the duple of the superscript, $W^{(z_k,k)}$ is now indexed by k to denote the specific set of nonzero constants $\{a_{k1}, \ldots, a_{k,z_k+1}\}$ in the term $W^{(z_k,k)} = a_{k1} T a_{k2} \ldots a_{k,z_k} T a_{k,z_k+1}$.

If $B = G(T)$, we have $b_{ij} = a + \sum_{k=1}^{p} w_{ij}^{(z_k,k)}$. Let u_{ij} be the sum of the terms of $a + \sum_{k=1}^{p} w_{ij}^{(z_k,k)}$ that are non-trivially dependent on t_{ij}. Recall, that $z_k \geq 1$. Given the properties of matrix multiplication, for $i < j$, each u_{ij} is of the form:

$$u_{ij} = \sum_{k=1}^{p} \sum_{s=1}^{z_k} a_{k1} t_{ii} a_{k2} \dots a_{k,s-1} t_{ii} a_{k,s} t_{ij} a_{k,s+1} t_{jj} a_{k,s+2} \dots a_{k,z_k} t_{jj} a_{k,z_k+1}.$$

Thus, we can denote u_{ij} as a polynomial of $\{t_{ii}, t_{ij}, t_{jj}\}$, $u_{ij}(t_{ii}, t_{ij}, t_{jj})$.

In the following lemma, we will show that each u_{ij} is a nonzero polynomial. This implies that $a + \sum_{k=1}^{p} w_{ij}^{(z_k,k)}$ is non-trivially a polynomial in t_{ij}.

Lemma 3.4. *Let $G(T)$ be a non-constant generalized polynomial evaluated over upper triangular $T \in M_m(K)$, with coefficients in K. For $i < j$, each u_{ij}, defined above, is a nonzero polynomial evaluated over K.*

Proof. First, we will partition the terms of $G(T)$ based on the degree of T. For all terms of $G(T)$, let $a_{k1} T a_{k2} \dots a_{k,z_k} T a_{k,z_k+1} \in \Omega_v^T$ provided that $v = z_k$.

Let T be an upper triangular matrix with the ij-th entry equal to t for $i \leq j$. Then,

$$\begin{aligned} u_{ij}(t,t,t) &= \sum_{k=1}^{p} \sum_{s=1}^{z_k} a_{k1} t a_{k2} \dots a_{k,s-1} t a_{k,s} t a_{k,s+1} t a_{k,s+2} \dots a_{k,z_k} t a_{k,z_k+1} \\ &= \sum_{k=1}^{p} z_k a_{k1} t a_{k2} \dots a_{k,z_k} t a_{k,z_k+1}. \end{aligned}$$

Let us partition the terms of $u_{ij}(t,t,t)$ based on degree of t. For all terms of $u_{ij}(t,t,t)$, let $z_k a_{k1} t a_{k2} \dots a_{k,z_k} t a_{k,z_k+1} \in \omega_v^T$ provided that $v = z_k$. For the sake of contradiction, assume $u_{ij}(t,t,t) = 0$ for some pair i, j with $1 \leq i < j \leq m$. Then $\sum_{y \in \omega_v^T} y = 0$ for all $v \in \mathbb{N}$. Note, for all $y \in \omega_v^T$,

$$y = z_k (a_{k1} t a_{k2} \dots a_{k,z_k} t a_{k,z_k+1}).$$

Let us consider any scalar matrix, aI, where $a \in K \setminus \{0\}$. Then, there exists an upper triangular matrix, A, such that its ij-th entry is equal to a for $i \leq j$. Thus, by above, $\omega_v^A = v \Omega_v^{aI}$ and $\sum_{y \in \omega_v^A} y = 0$. Therefore, as K is of characteristic zero [4], $\sum_{x \in \Omega_v^{aI}} x = 0$ for all $v \in \mathbb{N}$ and $a \in K$. So, $G(T)$ is a constant polynomial evaluated over scalar matrices, a contradiction by Remark 1.1.

Thus, u_{ij} is nonzero for $i < j$. □

Now, we are ready to show that the image of any non-constant generalized polynomial, with coefficients in K, contains some non-diagonal upper triangular matrices in $M_m(K)$.

Theorem 3.5. *Let $F(X_1, \dots, X_n)$ be a non-constant generalized polynomial with coefficients in K, and let $D \in M_m(K)$. Then, there exist upper triangular matrices $\overline{T}_1, \dots, \overline{T}_n \in M_m(K)$ such that $F(\overline{T}_1, \dots, \overline{T}_n) = B$, where $b_{ij} = d_{ij}$ for $i < j$.*

Proof. By Lemma 3.1, we know there exists a set, $\overline{x}_1, \dots, \overline{x}_{i-1}, \overline{x}_{i+1}, \dots, \overline{x}_n \in K$, such that $G(X_i) = F(\overline{x}_1 I, \dots, \overline{x}_{i-1} I, X_i, \overline{x}_{i+1} I, \dots, \overline{x}_n I)$ is a non-constant generalized polynomial with coefficients in K. Thus, if for any $D \in M_m(K)$, there exists an upper triangular $\overline{T} \in G(M_m(K))$ such that $d_{ij} = \bar{t}_{ij}$ for all $i < j$, then $\overline{T} \in F(M_m(K))$, as well.

So, for an upper triangular $T \in M_m(K)$, consider

$$G(T) = a + \sum_{k=1}^{p} a_{k1} T a_{k2} \dots a_{k,z_k} T a_{k,z_k+1} = a + \sum_{k=1}^{p} W^{(z_k,k)}(T).$$

If $G(T) = B$, then $b_{ij} = a + \sum_{k=1}^{p} w_{ij}^{(z_k,k)}$ is dependent on the set of variables $V_{ij} = \{t_{\mu\gamma} : i \leq \mu, \gamma \leq j\}$. Also, by Lemma 3.4, b_{ij} is non-trivially dependent on t_{ij}.

Now, we will prove by construction that for any $D \in M_m(K)$, there exists an upper triangular matrix, $\overline{T}$, such that $G(\overline{T}) = B$, where $b_{ij} = d_{ij}$ for $i < j$. To accomplish this, we will index the diagonals of a matrix, $X \in M_m(K)$, such that $\sigma_c(X) := \{x_{rs} : s - r = c\}$ where $-(m-1) \leq c \leq m-1$. Therefore, $\sigma_0(X)$ is the main diagonal, $\sigma_1(X)$ is the superdiagonal, and so on.

We will use strong induction on c to show there exists $\overline{T}_c$ such that $G(\overline{T}_c) = B$, where B and D agree on every entry of the diagonals 1 through c for all $1 \leq c \leq m-1$.

Recall $u_{ij}(t_{ii}, t_{ij}, t_{jj})$ is the sum of the terms in b_{ij} that are dependent on t_{ij}. From Lemma 3.4, we know that $u_{ij}(t_{ii}, t_{ij}, t_{jj})$ is nonzero, so Lemma 3.2 ensures there is a set $Y_0 := \{\bar{t}_{hh} \in K : 1 \leq h \leq m\}$, such that $u_{ij}(\bar{t}_{ii}, t_{ij}, \bar{t}_{jj})$ is non-constant over t_{ij} for all $i < j$.

Consider our base case $c = 1$. We will construct a $\overline{T}_1$ such that $G(\overline{T}_1) = B$, where $b_{ij} = d_{ij}$ for $j = i+1$. First, set the hh-th entry of $\overline{T}_1$ equal to $\bar{t}_{hh} \in Y_0$ for all $1 \leq h \leq m$. We know for $j = i+1$,

$$b_{ij} = \sum_{k=1}^{p} \sum_{s=1}^{z_k} a_{k1} \bar{t}_{ii} a_{k2} \dots a_{k,s-1} \bar{t}_{ii} a_{k,s} t_{ij} a_{k,s+1} \bar{t}_{jj} a_{k,s+2} \dots a_{k,z_k} \bar{t}_{jj} a_{k,z_k+1}.$$

Thus, $b_{ij} = u_{ij}(\bar{t}_{ii}, t_{ij}, \bar{t}_{jj})$. Since $u_{ij}(\bar{t}_{ii}, t_{ij}, \bar{t}_{jj})$ is a non-constant polynomial in K, $u_{ij}(\bar{t}_{ii}, t_{ij}, \bar{t}_{jj}) = d_{ij}$ has a solution, by Remark 1.1. Denote this solution $\bar{t}_{ij}$, and set the ij-th entry of $\overline{T}_1$ to $\bar{t}_{ij}$. As $t_{ij} \notin V_{\mu\gamma}$ for any $t_{\mu\gamma} \in \sigma_1(T) \setminus \{t_{ij}\}$, we can fix $\bar{t}_{ij} \in \sigma_1(T)$ such that $G(\overline{T}_1) = B$, and $b_{ij} = d_{ij}$ for $b_{ij} \in \sigma_1(B)$. Thus, our base case holds.

Now, assume the induction hypothesis: for some $1 \leq q < m-1$, there exists $\overline{T}_q$ such that $G(\overline{T}_q) = B$, where $b_{ij} = d_{ij}$ for all $b_{ij} \in \sigma_c(B)$ such that

$1 \leq c \leq q < m-1$. Recall $b_{ij} \in \sigma_c(B)$ is dependent only on the variables $v \in V_{ij}$. Thus, there exists a set of fixed entries of $\overline{T}_q$, $Y_q = \{\bar{t}_{ij} : 0 \leq j-i \leq q\}$, such that $b_{ij} = d_{ij}$ for all b_{ij} where $1 \leq j-i \leq q$.

Consider the $q+1$ case. Let us construct $\overline{T}_{q+1}$. By the inductive hypothesis, for $0 \leq \gamma - \mu \leq q$, we can set the $\mu\gamma$-th element of $\overline{T}_{q+1}$ to $\bar{t}_{\mu\gamma} \in Y_q$. Thus, if $G(\overline{T}_{q+1}) = B$, then $b_{\mu\gamma} = d_{\mu\gamma}$ for $1 \leq \gamma - \mu \leq q$.

By our choice of $\bar{t}_{ii}$ and $\bar{t}_{jj}$, b_{ij} is non-trivially dependent on t_{ij}. Therefore, for the fixed set $\{\bar{t}_{\mu\gamma} : t_{\mu\gamma} \in V_{ij} \setminus \{t_{ij}\}\} \subseteq Y_q$, there exists a $\bar{t}_{ij} \in K$ such that $b_{ij} = d_{ij}$ for $j - i = q+1$. Recall, if $t_{ij}, t_{\mu\gamma} \in \sigma_{q+1}(T)$ and $t_{\mu\gamma} \in V_{ij}$, then $t_{\mu\gamma} = t_{ij}$. So, we can find a $\overline{T}_{q+1}$ such that $G(\overline{T}_{q+1}) = B$, and $b_{ij} = d_{ij}$ for $1 \leq j-i \leq q+1$. Thus, for any $D \in M_m(K)$, there exists a $B \in G(M_m(K))$ such that B is upper triangular, and $d_{ij} = b_{ij}$ for all $i < j$.

Therefore, for any $D \in M_m(K)$, there exists a $B \in F(M_m(K))$ such that B is upper triangular, and $d_{ij} = b_{ij}$ for all $i < j$. $\square$

The above argument also holds if we consider $\overline{T}_1, \ldots, \overline{T}_n \in M_m(K)$ to be lower triangular matrices.

Corollary 3.6. *Let $F(X_1, \ldots, X_n)$ be a non-constant generalized polynomial with coefficients in K, and let $D \in M_m(K)$. Then, there exist lower triangular matrices $\overline{T}_1, \ldots, \overline{T}_n \in M_m(K)$, such that $F(\overline{T}_1, \ldots, \overline{T}_n) = B$ where $b_{ij} = d_{ij}$ for $i > j$.*

Theorem 3.3, Theorem 3.5, and Corollary 3.6 all discuss types of matrices that are in the image of a non-constant generalized polynomial, $F(X_1, \ldots, X_n)$, with coefficients in K. One might wonder if $F(M_m(K)) = M_m(K)$. The following example shows that this is not the case. Consider the polynomial $F(X) = X^2$. Suppose there exists $\overline{X} \in M_2(K)$ such that

$$(\overline{X})^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{which implies } (\overline{X})^4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $\overline{X}$ is nilpotent with index greater than 2, a contradiction since the nilpotency index of an $m \times m$ matrix is at most m. Thus, we know $F(M_m(K)) \neq M_m(K)$ for some non-constant generalized polynomial, $F(X_1, \ldots, X_n)$.

With this in mind, Theorem 3.7 will prove that

$$\operatorname{span} F(M_m(K)) = M_m(K)$$

for any non-constant generalized polynomial, $F(X_1, \ldots, X_n)$, with coefficients in K. More specifically, we will show that any $D \in M_m(K)$ can be written as the sum of three or fewer elements of $F(M_m(K))$.

Theorem 3.7. *Let $F(X_1, \ldots, X_n)$ be a non-constant generalized polynomial with coefficients in K. Then, every $D \in M_m(K)$ is the sum of three or fewer elements in $F(M_m(K))$.*

Proof. Let us consider an arbitrary $D \in M_m(K)$. By Theorem 3.5 and Corollary 3.6, there exist $B, C \in F(M_m(K))$ such that B is an upper triangular matrix where $b_{ij} = d_{ij}$ for $1 \leq i < j \leq m$, and C is a lower triangular matrix where $c_{ij} = d_{ij}$ for $1 \leq j < i \leq m$. By Theorem 3.3, there exists a diagonal matrix, $A \in F(M_m(K))$, such that $a_{ii} = d_{ii} - (b_{ii} + c_{ii})$ for all $1 \leq i \leq m$. Therefore, by construction, $D = A + B + C$. So, every $D \in M_m(K)$ is the sum of three or fewer elements in $F(M_m(K))$. $\square$

It may be possible to reduce the number of elements in $F(M_m(K))$ needed to sum to any $D \in M_m(K)$ from three to two. We conclude this paper with the following question:

Question. *Let $F(X_1, \ldots, X_n)$ be a non-constant generalized polynomial with coefficients in K. Is it possible to write any $D \in M_m(K)$ as a sum of two or fewer elements from $F(M_m(K))$?*

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