ITERATED DIFFERENTIAL POLYNOMIAL RINGS OVER LOCALLY NILPOTENT RINGS

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Abstract. We study iterated differential polynomial rings over a locally nilpotent ring and show that a large class of such rings are Behrens radical. This extends results of Chebotar and Chen, Hagan, and Wang.

1. Introduction

Let R be a ring. An additive map $d : R \to R$ that satisfies Leibniz's rule is called a *derivation* of R. For a derivation d, the *differential polynomial ring* $R[X; d]$ is given by all polynomials of form $a_nX^n + \cdots + a_1X + a_0$ with $n \geq 0$ and $a_0, \dots, a_n \in R$. Multiplication is given by $Xa = aX + d(a)$ for all $a \in R$ and extending via associativity and linearity.

Recall that a ring is called *Brown-McCoy radical* if it cannot be mapped onto a simple ring with identity. Similarly, a ring is called *Behrens radical* if it cannot be mapped onto a ring with a non-zero idempotent.

In 1972, Krempa [8] showed that the Köthe conjecture is equivalent to the statement that every polynomial ring over a nil ring is Jacobson radical. The problem remains open, but this equivalent formulation motivated the investigation of parallel questions for more general radical classes. For example, Puczylowski and Smoktunowicz [10] proved in 1998 that a polynomial ring over a nil ring is Brown-McCoy radical. This result was strengthened in 2001 by Beidar, Fong, and Puczylowski [1], who proved that a polynomial ring over a nil ring is Behrens radical. The corresponding questions for multivariate polynomial rings were open until recently. Then, in 2018, Chebotar, Ke, Lee, and Puczylowski [4] employed techniques from convex geometry to prove that a multivariate polynomial ring over a nil ring is Brown-McCoy radical. It is still unknown whether such a ring need be Behrens radical.

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After restricting the class of base rings from nil rings to locally nilpotent rings, one can formulate analogous questions for differential polynomial rings. At a 2011 conference in Coimbra, Portugal, Shestakov asked whether a differential polynomial ring over a locally nilpotent ring is necessarily Jacobson radical. This can in some sense be viewed as the analog of the Köthe conjecture for differential polynomial rings. Curiously, this statement turned out to be false; a 2014 result of Smoktunowicz and Ziembowski [12] yields a constructive counterexample. Nonetheless, pursuing a similar line of investigation as in the non-differential case, Greenfeld, Smoktunowicz, and Ziembowski [7] asked whether a differential polynomial ring over a locally nilpotent must be Behrens radical. This question was promptly resolved in the affirmative by Chebotar [3] in 2018.

Extending the results of Chebotar in two different directions, Chen, Hagan, and Wang [5] proved the following two theorems:

Theorem 1. [5, Theorem 1] *Let $d_1, \ldots, d_p$ be derivations of a locally nilpotent ring R. Let $X_1, \ldots, X_p$ be commuting variables. Then the differential polynomial ring $R[X_1, \ldots, X_p; d_1, \ldots, d_p]$ is Behrens radical.*

Theorem 2. [5, Theorem 2] *Let δ be a derivation of a locally nilpotent ring R and let d be a derivation of $R[X; \delta]$ such that:*
(i) $d(R) \subseteq R$,
(ii) $d|_R$ is locally nilpotent, and
(iii) $d^n(aX) - Xd^n(a) \in R$ for all $a \in R$ and positive integers n.
Then $R[X; \delta][Y; d]$ is Behrens radical.

We remark that the proof of the latter theorem relies heavily on the assumption that $d|_R$ is locally nilpotent.

We wish to expand upon this line of investigation. First we establish a definition.

Definition 3. Let R be a ring. For all $1 \leq i \leq n$, suppose that d_i is a derivation of $R[X_1; d_1] \ldots [X_{i-1}; d_{i-1}]$. We denote $R[X_1; d_1] \ldots [X_n; d_n]$ as $R[\bar{X}_n, \bar{d}_n]$. We call such a ring an *iterated differential polynomial ring* over R.

If R is a ring without identity, let R^* denote the ring given by adjoining an identity element to R.

In this paper, we will prove the following result:

Theorem 4. *Suppose $R[\bar{X}_n, \bar{d}_n]$ is an iterated differential polynomial ring over a locally nilpotent ring R. Suppose that for all i each d_i can be extended to a derivation on $R^*[X_1; d_1] \ldots [X_{i-1}, d_{i-1}]$ such that d_i restricts to a derivation on R and further $d_i(X_j) \in R^*$ for all $0 < j < i$. Then $R[\bar{X}_n, \bar{d}_n]$ is Behrens radical.*

We remark that one may view Theorem 4 as a unification of the results in Theorems 1 and 2. If $n = 1$, we recover Chebotar's original theorem [3, Theorem 1]. If n is arbitrary and the derivations d_i are taken to be trivial off of R, we recover Theorem 1. If we set $n = 2$, we retrieve a strengthened version of Theorem 2; namely, hypothesis (ii) has been removed and hypothesis (iii) has been weakened. In particular, the key ingredients used in the proof of [5, Theorem 2] are shown to be unnecessary.

The results of this paper notwithstanding, there arises naturally the following question:

Question 5. Let R be a locally nilpotent ring and $R[\bar{X}_n, \bar{d}_n]$ an iterated differential polynomial ring. Is $R[\bar{X}_n, \bar{d}_n]$ Behrens radical?

2. Results

We first set notation. For elements a and b of a ring R, we define $[a, b]_0 = a$, $[a, b]_1 = [a, b] = ab - ba$, and $[a, b]_k = [[a, b]_{k-1}, b]$ for $k > 1$. Given elements $b_1, \ldots, b_p \in R$ and non-negative integers $k_1, \ldots, k_p$, we denote by $[a, \bar{b}]_{k_1,\ldots,k_p}$ the expression $[\ldots[a, b_1]_{k_1}, \ldots, b_p]_{k_p}$ and denote by $\bar{b}^{k_1,\ldots,k_p}$ the expression $b_1^{k_1} \ldots b_p^{k_p}$.

Additionally, suppose that $c_{i'_1,\ldots,i'_r} \in R$ for $0 \leq i'_q \leq i_q$ where $1 \leq q \leq r$ and the i_q are non-negative integers. Then, we write

$$\sum_{i'_1,\ldots,i'_r=0}^{i_1,\ldots,i_r} c_{i'_1,\ldots,i'_r} := \sum_{i'_1=0}^{i_1} \cdots \sum_{i'_r=0}^{i_r} c_{i'_1,\ldots,i'_r}.$$

Alternatively, if $i_1 = \cdots = i_r = s$, then we write

$$\sum_{i'_1,\ldots,i'_r}^{s} c_{i'_1,\ldots,i'_r} := \sum_{i'_1,\ldots,i'_r=0}^{s,\ldots,s} c_{i'_1,\ldots,i'_r} = \sum_{i'_1=0}^{s} \cdots \sum_{i'_r=0}^{s} c_{i'_1,\ldots,i'_r}.$$

We will now establish some preliminary lemmata. Our first lemma is an easy consequence of the Leibniz rule:

Lemma 6. *Let a, b, c be elements of a ring R. For any non-negative integer k, we have*

$$[ab, c]_k = \sum_{i=0}^{k} D_i [a, c]_i [b, c]_{k-i}$$

for some $D_i \in \mathbb{Z}$. □

Other useful results include the following:

Lemma 7. *For elements a and b in a ring R and non-negative integers r and s, we have*

$$[a^r, b]_s = \sum_{w_1,\dots,w_r=0}^{s} E_{w_1,\dots,w_r}[a,b]_{w_1}\dots[a,b]_{w_r}$$

for some $E_{w_1,\dots,w_r} \in \mathbb{Z}$.

Proof. The cases $r = 0$ and $r = 1$ are trivial. The first nontrivial case is Lemma 6. We induct on r. By applying Lemma 6, we can see that

$$[a^{r+1}, b]_s = \sum_{i=0}^{s} D_i[a^r, b]_i[a,b]_{s-i}.$$

for some $D_i \in \mathbb{Z}$. Now we may apply the inductive hypothesis:

$$\sum_{i=0}^{s} D_i[a^r, b]_i[a,b]_{s-i} = \sum_{i=0}^{s}\sum_{w_1,\dots,w_r=0}^{i} D_iE_{w_1,\dots,w_r}[a,b]_{w_1}\dots[a,b]_{w_r}[a,b]_{s-i}$$
$$= \sum_{w_1,\dots,w_{r+1}=0}^{s} E_{w_1,\dots,w_{r+1}}[a,b]_{w_1}\dots[a,b]_{w_{r+1}}.$$

for some D_i, $E_{w_1,\dots,w_r}$, $E_{w_1,\dots,w_{r+1}} \in \mathbb{Z}$. □

Lemma 8. *For elements $a_1,\dots,a_n$ and b in a ring R and non-negative integers $i_1,\dots,i_n$ and s, we have*

$$[\bar{a}^{i_1,\dots,i_n}, b]_s = \sum_{w_1^{(1)},\dots,w_{i_1}^{(1)}=0}^{s} \cdots \sum_{w_1^{(n)},\dots,w_{i_n}^{(n)}=0}^{s} E^{(1)}_{w_1^{(1)},\dots,w_{i_1}^{(1)}}\cdots E^{(n)}_{w_1^{(n)},\dots,w_{i_n}^{(n)}}$$
$$[a_1,b]_{w_1^{(1)}}\dots[a_1,b]_{w_{i_1}^{(1)}}\dots[a_n,b]_{w_1^{(n)}}\dots[a_n,b]_{w_{i_n}^{(n)}}$$

for $E^{(j)}_{w_1^{(j)},\dots w_{i_j}^{(j)}} \in \mathbb{Z}$ for $1 \le j \le n$.

Proof. Induct on n. The base step is Lemma 7. Applying Lemma 6, observe that

$$[\bar{a}^{i_1,\dots,i_{n+1}}, b]_s = \sum_{j=0}^{s} D_j[\bar{a}^{i_1,\dots,i_n}, b]_j[a_{n+1}^{i_{n+1}}, b]_{s-j}.$$

Then, applying the inductive hypothesis to $[\bar{a}^{i_1,\dots,i_n}, b]_j$ and the basis step to $[a_{n+1}^{i_{n+1}}, b]_{s-j}$, we are done. □

We will also take advantage of [5, Lemma 4 and Lemma 5]. We recite these here for completeness.

Lemma 9. [5, Lemma 4] *Let $e, x_1, \ldots, x_p$ be elements of a ring R and $n_1, \ldots, n_p$ be non-negative integers. Then*

$$e\bar{x}^{n_1,\ldots,n_p} = \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} \binom{n_1}{i_1} \cdots \binom{n_p}{i_p} \bar{x}^{i_1,\ldots,i_p}[e,\bar{x}]_{n_1-i_1,\ldots,n_p-i_p}.$$

□

Lemma 10. [5, Lemma 5] *Let $e, x_1, \ldots, x_p$ be elements of a ring R with $e^2 = e$. Then for any non-negative integers $k_1, \ldots, k_p$, we have*

$$[e,\bar{x}]_{k_1,\ldots,k_p} = \sum_{i_1=0}^{k_1} \cdots \sum_{i_p=0}^{k_p} r_{i_1,\ldots,i_p} e[e,\bar{x}]_{i_1,\ldots,i_p}$$

for some $r_{i_1,\ldots,i_p} \in R$. □

An easy application of Lemma 9 and Lemma 10 yields the following fact:

Lemma 11. *Suppose $e, x_1, \ldots, x_n$ are elements of a ring. Then $e\bar{x}^{i_1,\ldots,i_n}$ can be written as a sum of terms each ending in $e[e,\bar{x}]_{k_1,\ldots,k_n}$ where $0 \le k_j \le i_j$ and $1 \le j \le n$.* □

Finally, the following two lemmata are the technical heart of the proof of Theorem 4.

Let V be a K-vector space. Then, denote by $\mathrm{End}_K(V)$ the K-algebra of all linear transformation of V.

Lemma 12. *Let N be a subalgebra of $End_K(V)$. Let $a, x_1, \ldots, x_n \in End_K(V)$. Let $i_1, \ldots, i_n, k$ be non-negative integers. First, define the following sets:*

(i) Let A be the set of all $[a, x_j]_i$ for $1 \le j \le n$ and $0 \le i \le k$.

(ii) Suppose that we can write any $[x_1, x_j]_{w_1^{(1)}} \ldots [x_1, x_j]_{w_{i_1}^{(1)}} \ldots [x_n, x_j]_{w_{i_n}^{(n)}}$ in the form $\sum_{i'_1,\ldots,i'_n=0}^{i_1,\ldots,i_n} \bar{x}^{i'_1,\ldots,i'_n} b_{i'_1,\ldots,i'_n}$ for any $1 \le j \le n$ and for some $b_{i'_1,\ldots,i'_n} \in End_K(V)$. Let B be the set of the $b_{i'_1,\ldots,i'_n}$ that arise in this way for $0 \le w_s^{(t)} \le k$ for all s, t.

(iii) Let C be the set of all elements of form $\beta\alpha$ where $\alpha \in A$ and $\beta \in B$.

Suppose that $C \subseteq N$. Then $[\bar{x}^{i_1,\ldots,i_n} a, x_j]_k$ can be written in the form

$$\sum_{i'_1,\ldots,i'_n=0}^{i_1,\ldots,i_n} \bar{x}^{i'_1,\ldots,i'_n} c_{i'_1,\ldots,i'_n}$$

for some $c_{i'_1,\ldots,i'_n} \in N$ *for all* $1 \le j \le n$.

Proof. By applying Lemma 6 and 8, we obtain

$$\begin{aligned}
[\bar{x}^{i_1,\ldots,i_n} a, x_j]_k &= \sum_{i=0}^{k} D_i [\bar{x}^{i_1,\ldots,i_n}, x_j]_i [a, x_j]_{k-i} \\
&= \sum_{i=0}^{k} \sum_{w_1^{(1)},\ldots,w_{i_1}^{(1)}=0}^{i} \cdots \sum_{w_1^{(n)},\ldots,w_{i_n}^{(n)}=0}^{i} D_i E^{(1)}_{w_1^{(1)},\ldots,w_{i_1}^{(1)}} \ldots E^{(n)}_{w_1^{(n)},\ldots,w_{i_n}^{(n)}} \\
&\quad [x_1, x_j]_{w_1^{(1)}} \ldots [x_1, x_j]_{w_{i_1}^{(1)}} \ldots [x_n, x_j]_{w_{i_n}^{(n)}} [a, x_j]_{k-i}.
\end{aligned}$$

for some $D_i, E^{(1)}_{w_1^{(1)},\ldots,w_{i_1}^{(1)}} \ldots E^{(n)}_{w_1^{(n)},\ldots,w_{i_n}^{(n)}} \in \mathbb{Z}$.

A single term of this sum is of form

$$\begin{aligned}
&D_i E^{(1)}_{w_1^{(1)},\ldots,w_{i_1}^{(1)}} \ldots E^{(n)}_{w_1^{(n)},\ldots,w_{i_n}^{(n)}} [x_1, x_j]_{w_1^{(1)}} \ldots [x_1, x_j]_{w_{i_1}^{(1)}} \ldots [x_n, x_j]_{w_{i_n}^{(n)}} [a, x_j]_{k-i} \\
&= \sum_{i'_1,\ldots,i'_n=0}^{i_1,\ldots,i_n} D_i E^{(1)}_{w_1^{(1)},\ldots,w_{i_1}^{(1)}} \ldots E^{(n)}_{w_1^{(n)},\ldots,w_{i_n}^{(n)}} \bar{x}^{i'_1,\ldots,i'_n} b_{i'_1,\ldots,i'_n} [a, x_j]_{k-i} \\
&= \sum_{i'_1,\ldots,i'_n=0}^{i_1,\ldots,i_n} D_i E^{(1)}_{w_1^{(1)},\ldots,w_{i_1}^{(1)}} \ldots E^{(n)}_{w_1^{(n)},\ldots,w_{i_n}^{(n)}} \bar{x}^{i'_1,\ldots,i'_n} c_{i'_1,\ldots,i'_n}.
\end{aligned}$$

for some $b_{i'_1,\ldots,i'_n} \in B$ and some $c_{i'_1,\ldots,i'_n} \in C \subseteq N$. Since for any $c \in N$, we have that $zc \in N$ for all $z \in \mathbb{Z}$, this concludes. □

Lemma 13. *Let* N *be a locally nilpotent subalgebra of* $End_K(V)$. *Let* $a_{i_1,\ldots,i_n}, x_1, \ldots, x_n \in End_K(V)$. *Suppose* $e = \sum_{i_1,\ldots,i_n=0}^{m_1,\ldots,m_n} \bar{x}^{i_1,\ldots,i_n} a_{i_1,\ldots,i_n}$ *is an idempotent. Define the following sets:*

(i) Consider the set of all $[a_{i_1,\ldots,i_n}, x_j]_i$ *for all* $1 \le j \le n$, $0 \le i \le \max_s\{m_s\}$, *and* $0 \le i_r \le m_r$. *Call this set* A_1. *For any* $a_{i_1,\ldots,i_n}$, *suppose that we may write* $[\bar{x}^{i_1,\ldots,i_n} a_{i_1,\ldots,i_n}, x_1]_{k_1}$ *as* $\sum_{i'_1,\ldots,i'_n=0}^{i_1,\ldots,i_n} \bar{x}^{i'_1,\ldots,i'_n} c_{i'_1,\ldots,i'_n}$ *for some* $c_{i'_1,\ldots,i'_n} \in N$ *for all* $0 \le k_1 \le m_1$. *Let the set of all* $[c_{i'_1,\ldots i'_n}, x_j]_i$ *for all* $1 \le j \le n$, $0 \le i \le \max_s\{m_s\}$, *and* $0 \le i'_r \le m_r$ *be called* A_2. *In this way, inductively define* $A_1, \ldots, A_n$. *Let* $A = \bigcup_{i=0}^{n} A_i$.

(ii) Suppose that any $[x_1, x_j]_{w_1^{(1)}} \cdots [x_1, x_j]_{w_{i_1}^{(n)}} \cdots [x_n, x_j]_{w_{i_n}^{(n)}}$ *can be written in the form* $\sum_{i'_1,\ldots,i'_n=0}^{i_1,\ldots,i_n} \bar{x}^{i'_1,\ldots,i'_n} b_{i'_1,\ldots,i'_n}$ *for* $1 \le j \le n$ *and for some* $b_{i'_1,\ldots,i'_n} \in End_K(V)$. *Let* B *be the set of the* $b_{i'_1,\ldots,i'_n}$ *that arise in this way for* $0 \le w_s^{(t)} \le \max_j\{m_j\}$ *for all* s, t.

(iii) Let C be the set of all elements of form $\beta\alpha$ where $\alpha \in A$ and $\beta \in B$.

Suppose $C \subseteq N$. Then $e = 0$.

Proof. First, we remark that by Lemma 12, our assumption (i) is a valid hypothesis. Let S be the subalgebra of N generated by C. Then S is nilpotent, so there exists subspaces $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_h = V$ such that $S(V_i) = V_{i-1}$. We claim that for any $0 \le l \le h$ we have $e[e, \bar{x}]_{k_1,\ldots,k_n}(V_l) = 0$ for all $0 \le k_j \le m_j$ and $1 \le j \le n$.

We induct on l. When $l = 0$, the statement is clear. Before proceeding with the induction, we make the following intermediary assertion:

Claim. The element $[e, \bar{x}]_{k_1,\ldots,k_n}$ can be written in the form

$$\sum_{i_1,\ldots,i_n=0}^{m_1,\ldots,m_n} \bar{x}^{i_1,\ldots,i_n} c_{i_1,\ldots,i_n}$$

for some $c_{i_1,\ldots,i_n} \in S$.

Proof. For this claim, we perform a nested induction on n. When $n = 1$, we have

$$[e, x_1]_{k_1} = \sum_{i_1,\ldots,i_n=0}^{m_1,\ldots,m_n} [\bar{x}^{i_1,\ldots,i_n} a_{i_1,\ldots,i_n}, x_1]_{k_1}.$$

By condition (i) and the fact that C generates S, this concludes the basis. For the inductive step, observe that

$$[e, \bar{x}]_{k_1,\ldots,k_n} = \sum_{i_1,\ldots,i_n=0}^{m_1,\ldots,m_n} [[\bar{x}^{i_1,\ldots,i_n} a_{i_1,\ldots,i_n}, \bar{x}]_{k_1,\ldots,k_{n-1}}, x_n]_{k_n}.$$

Applying the inductive hypothesis, this is

$$\sum_{i_1,\ldots,i_n=0}^{m_1,\ldots,m_n} \sum_{i'_1,\ldots,i'_n=0}^{i_1,\ldots,i_n} [\bar{x}^{i'_1,\ldots,i'_n} c_{i'_1,\ldots,i'_n}, x_n]_{k_n}.$$

for some $c_{i'_1,\ldots,i'_n} \in S$. Applying condition (i) to $[\bar{x}^{i'_1,\ldots,i'_n} c_{i'_1,\ldots,i'_n}, x_n]_{k_n}$, this proves our intermediary claim. □

Now we proceed with the outer induction. Let $v \in V_l$. Then

$$\begin{aligned} e[e,\bar{x}]_{k_1,\dots,k_n}(v) &= \sum_{i_1,\dots,i_n=0}^{m_1,\dots,m_n} e\bar{x}^{i_1,\dots,i_n} c_{i_1,\dots,i_n}(v) \\ &= \sum_{i_1,\dots,i_n=0}^{m_1,\dots,m_n} e\bar{x}^{i_1,\dots,i_n}(u_{i_1,\dots i_n}) \end{aligned}$$

for $u_{i_1,\dots i_n} \in V_{l-1}$. By Lemma 11 and the inductive hypothesis, we are done. □

Proof of Theorem 4. We follow the approach of [3] and [5]. Suppose $R[\bar{X}_n, \bar{d}_n]$ as in the theorem is Behrens radical. Then there exists a surjective homomorphism φ from $R[\bar{X}_n, \bar{d}_n]$ onto a subdirectly irreducible ring A such that there is a nonzero idempotent in the heart of A. Note that A must be a prime ring whose extended centroid K is a field. Let Q be the Martindale right ring of quotients of A.

Let $x_i : A \to A$ be maps given by $x_i(\varphi(t)) := \varphi(X_i t)$ for all $t \in R[\bar{X}_n, \bar{d}_n]$ where $1 \le i \le n$. We claim that the x_i are well-defined. Suppose $\varphi(t) = 0$ and $\varphi(X_i t) \neq 0$. Since A is prime, there must be $t' \in R[\bar{X}_n, \bar{d}_n]$ such that $\varphi(t')\varphi(X_i t) \neq 0$. We also have

$$\begin{aligned} \varphi(t')\varphi(X_i t) &= \varphi(t' X_i t) \\ &= \varphi(t' X_i)\varphi(t) \\ &= 0, \end{aligned}$$

which is a contradiction. Note that the x_i are endomorphisms of right A-modules, so all x_i are in Q. Let the subring of Q generated by A and the x_i be denoted A'. Let R' be the subring of $R^*[\bar{X}_n, \bar{d}_n]$ generated by $R[\bar{X}_n, \bar{d}_n]$ and X_i^j for all $1 \le i \le n$ and all $0 \le j$. Let $\psi : R' \to A'$ be an additive map such that $\psi(X_i^j) = x_i^j$ and $\psi(t) = \varphi(t)$ for all $t \in R[\bar{X}_n, \bar{d}_n]$. Note that ψ is a homomorphism extending φ. We can write a nonzero idempotent $e \in A \subseteq A'$ as

$$\begin{aligned} e &= \varphi\left(\sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} X_1^{i_1} \dots X_n^{i_n} r_{i_1,\dots,i_n}\right) \\ &= \psi\left(\sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} X_1^{i_1} \dots X_n^{i_n} r_{i_1,\dots,i_n}\right) \\ &= \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \bar{x}^{i_1,\dots,i_n} a_{i_1,\dots,i_n} \end{aligned}$$

where the m_j are non-negative integers, $r_{i_1,\dots,i_n} \in R$, and $\psi(r_{i_1,\dots i_n}) = a_{i_1,\dots,i_n}$. Let D be the subring of A' generated by all x_i and all $a_{i_1,\dots,i_n}$. Let $B = D \cap \psi(R)$. Note that B and the subalgebra BK of Q are locally nilpotent. The subalgebra DK of $A'K$ is finitely generated, so it can embedded into $\mathrm{End}_K(V)$ for some K-vector space V. Then we can assume that $x_i \in \mathrm{End}_K(V)$. Finally, we have that $N = BK$ is locally nilpotent and $e = \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \bar{x}^{i_1,\dots,i_n} a_{i_1,\dots,i_n}$ is a nonzero idempotent. Applying Lemma 13, we have a contradiction. □

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