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THE CYCLIC GRAPH OF A Z-GROUP

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Abstract

For a group G, we define a graph $\Delta(G)$ by letting $G^\# = G \setminus \{1\}$ be the set of vertices and by drawing an edge between distinct elements $x, y \in G^\#$ if and only if the subgroup $\langle x, y\rangle$ is cyclic. Recall that a Z-group is a group where every Sylow subgroup is cyclic. In this short note, we investigate $\Delta(G)$ for a Z-group G.

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1. Introduction

The groups under consideration in this note are finite. Let G be a group and define a graph $\Delta(G)$ associated with G as follows. Take $G^\# = G \setminus \{1\}$ as the vertex set. Then draw an edge between distinct vertices $x, y \in G^\#$ if and only if the subgroup $\langle x, y\rangle$ is cyclic. We shall refer to $\Delta(G)$ as the *cyclic graph* of G, although we note that the graph $\Delta(G)$ has also been called the *deleted enhanced power graph*. See, for example, [2]. The *enhanced power graph* includes the identity element as a vertex and so the enhanced power graph of a group is always connected. A brief investigation of this graph was undertaken in [1].

The cyclic graph of a group G was investigated in [4, 5]. In those papers, classification results were obtained under the assumption that the connected components of $\Delta(G)$ were complete graphs. In our previous paper [3], we studied the cyclic graph of a direct product.

Next, we mention another graph that can be attached to a group. Let G be a nonabelian group. The *commuting graph* of G, denoted by $\Gamma(G)$, is the graph whose vertices are the noncentral elements of G and whose edges connect distinct vertices x and y if and only if $xy = yx$. The commuting graph of a finite solvable group with trivial centre was classified in [6].

Recall that a group is called a *Z-group* if every Sylow subgroup is cyclic. Observe that a Frobenius complement of odd order is a Z-group and so is any group of square-free order. Our focus in this short note is the graph $\Delta(G)$ for a Z-group G. We have been able to characterise the disconnectedness of $\Delta(G)$.

THEOREM 1.1. *Let G be a Z-group. Then $\Delta(G)$ is disconnected if and only if G is a Frobenius group.*

If the graph $\Delta(G)$ is connected for a Z-group G, then a diameter bound follows.

THEOREM 1.2. *If G is a Z-group and $\Delta(G)$ is connected, then* $\mathrm{diam}(\Delta(G)) \leq 4$.

The next result describes a relationship between the graph $\Delta(G)$ and the subgroup $\mathbf{Z}(G)$ for a Z-group G.

THEOREM 1.3. *If G is a Z-group, then* $\mathrm{diam}(\Delta(G)) \leq 2$ *if and only if* $\mathbf{Z}(G) \neq \{1\}$.

Following [2], a vertex z in $\Delta(G)$ is called a *dominating vertex* if z is adjacent to every vertex in $\Delta(G) \setminus \{z\}$. The terms *complete vertex*, *cone vertex* and *universal vertex* have also been used as synonyms for a dominating vertex. If the graph $\Delta(G)$ has a dominating vertex, we shall say that $\Delta(G)$ is *dominatable*. In the proof of the previous theorem, we end up establishing the existence of a dominating vertex. We point out a necessary and sufficient condition for a dominating vertex in $\Delta(G)$ to exist, which answers a request in [2] for a characterisation of a group with a dominatable cyclic graph.

THEOREM 1.4. *Let G be a group, $g \in G$ and $\pi = \pi(o(g))$. Write $g = \prod_{p\in\pi} g_p$, where each g_p is a p-element for $p \in \pi$ and $g_p g_q = g_q g_p$ for all $p, q \in \pi$. Then g is a dominating vertex for $\Delta(G)$ if and only if, for each $p \in \pi$, a Sylow p-subgroup P of G is cyclic or generalised quaternion and $\langle g_p \rangle \leq P \cap \mathbf{Z}(G)$.*

As a corollary, we offer a generalisation of Theorem 3.2 in [2].

COROLLARY 1.5. *For a nilpotent group G, the graph $\Delta(G)$ is dominatable if and only if G has a cyclic or generalised quaternion Sylow subgroup.*

Let G be a Z-group and let $x, y \in G^{\#}$ be distinct. If x is adjacent to y in $\Delta(G)$, then $xy = yx$. In fact, the converse is true too. So, in particular, if $\mathbf{Z}(G) = \{1\}$, then $\Gamma(G)$ and $\Delta(G)$ are the same graph. In light of the previous results, we obtain the following corollary concerning the commuting graph of a Z-group with trivial centre.

COROLLARY 1.6. *If G is a Z-group with $\mathbf{Z}(G) = \{1\}$ and G is not a Frobenius group, then $\Gamma(G)$, the commuting graph of G, is connected with diameter* 3 *or* 4.

2. Notation and preliminaries

Let G be a group and let $x, y \in G$. We write $x \approx y$ to indicate that the subgroup $\langle x, y\rangle$ is cyclic. If n is a positive integer, then $\pi(n)$ denotes the set of prime divisors of n. For a group G, set $\pi(G) = \pi(|G|)$. Fix a set of prime numbers π. An element $x \in G$ is called a *π-element* if every prime divisor of $o(x)$ is a member of π. If every prime divisor of $o(x)$ lies outside of π, then x is called a *π'-element*. In the case where $\pi = \{p\}$, we use the terms p-element and p'-element. The set of prime numbers is denoted by $\mathbb{P}$.

Let G be a group. Notice that if $x, y \in G^{\#}$ are commuting elements with coprime orders, then $x \approx y$. This fact gives us a useful way to build paths in $\Delta(G)$. We also mention that conjugation preserves adjacency in $\Delta(G)$: specifically, if $x, y \in G^{\#}$ with $x \approx y$, then $x^g \approx y^g$ for each $g \in G$.

A graph related to the cyclic graph is the *commuting graph*, which is defined as follows. Let G be a nonabelian group. The commuting graph $\Gamma(G)$ is the graph whose vertices are the noncentral elements of G and whose edges connect distinct nonidentity elements x and y if and only if $xy = yx$. Taking the noncentral elements of G as the vertices for $\Gamma(G)$ is fairly standard, although variations on the vertex set do exist. If G is a Z-group with a trivial centre, then the vertex set of $\Gamma(G)$ is the same as the vertex set of $\Delta(G)$. In fact, the edge sets are the same too; the following lemma also appears as a part of Theorem 30 in [1].

LEMMA 2.1. *If G is a Z-group with $\mathbf{Z}(G) = \{1\}$, then $\Delta(G) = \Gamma(G)$.*

PROOF. If $x, y \in G$ with $x \approx y$, then clearly $xy = yx$. But notice that if $xy = yx$, then $\langle x, y\rangle$ is an abelian Z-group, which is therefore cyclic. Hence, $x \approx y$. □

Next, we make a few remarks about Z-groups. Many properties of Z-groups are known. For example, if G is a Z-group, then G is p-nilpotent for the smallest prime divisor p of $|G|$. We also know that Z-groups are solvable. The specific results that we need in this paper are encapsulated in the following theorem.

THEOREM 2.2 [7, Theorem 10.26]. *If G is a Z-group, then the derived subgroup G' is cyclic and the factor group G/G' is cyclic. Moreover, G' is a Hall subgroup of G.*

Finally, we need to make an observation about Frobenius groups. Recall that a group G is a *Frobenius group* if G has a nontrivial proper subgroup H such that $H \cap H^g = \{1\}$ for each $g \in G \setminus H$. The subgroup H is called a *Frobenius complement*. Now, let G be a Frobenius group with Frobenius complement H. Frobenius groups are centreless and so $\Gamma(G)$ and $\Delta(G)$ have the same vertex set. In particular, $\Delta(G)$ is a spanning subgraph of $\Gamma(G)$. Because $\mathbf{C}_G(h) \le H$ for each $h \in H^{\#}$, the graph $\Gamma(G)$ is disconnected. (This fact appears as Lemma 3.1 in [6].) Hence, $\Delta(G)$ is disconnected as well.

3. Main results

Our first theorem provides a necessary and sufficient condition for the cyclic graph of a Z-group G to be disconnected. Additionally, a diameter bound of $\Delta(G)$ is available under the assumption that $\Delta(G)$ is connected.

THEOREM 3.1. *Let G be a Z-group. Then $\Delta(G)$ is disconnected if and only if G is a Frobenius group. Moreover, if $\Delta(G)$ is connected, then* $\mathrm{diam}(\Delta(G)) \le 4$.

PROOF. Frobenius groups have disconnected cyclic graphs. To prove the converse, assume that G is not a Frobenius group. We shall establish the connectedness of $\Delta(G)$.

Abelian Z-groups are cyclic and so we may assume that G is nonabelian. Hence, $\{1\} < G' < G$. If $\mathbf{C}_G(g) \le G'$ for each $g \in (G')^\#$, then G is a Frobenius group with kernel G', contrary to our hypothesis. Hence, there exists some $g_0 \in (G')^\#$ with $\mathbf{C}_G(g_0) \not\le G'$. Let H be a complement for G' in G. Fix $x \in \mathbf{C}_G(g_0) \setminus G'$ and write $x = yh$ for $y \in G'$ and $h \in H$. Then $g_0^{yh} = g_0^x = g_0$ and so $g_0^{h^{-1}} = g_0^y = g_0$. It follows that $h \in \mathbf{C}_H(g_0)$.

Now, let $g \in G^\#$. If $\pi(o(g)) \cap \pi(G') \neq \emptyset$, then let $p \in \pi(o(g)) \cap \pi(G')$. For a suitable integer n, $o(g^n) = p$; hence, $g^n \in G'$ and $g \approx g^n \approx g_0$. Otherwise, $\pi(o(g)) \cap \pi(G') = \emptyset$ and $g \in H^a$ for some $a \in G'$. Note that $h^a \approx g_0^a = g_0$ as $h \approx g_0$ and conjugation preserves adjacency. Hence, $g \approx h^a \approx g_0$. The result follows. □

The group `SmallGroup(60,7)` furnishes an example of a Z-group with connected cyclic graph of diameter 4 and so the bound in the previous theorem is sharp. The cyclic graph for `SmallGroup(60,7)` is displayed in Figure 1. We mention a few more examples. The group `SmallGroup(210,2)` is a Z-group with connected cyclic graph of diameter 3. The cyclic graph for `SmallGroup(210,2)` is displayed in Figure 2. Finally, `SmallGroup(60,3)` provides an example of a Z-group with connected cyclic graph of diameter 2. The cyclic graph for `SmallGroup(60,3)` is displayed in Figure 3. These three graphs were computed using GAP [9] and displayed using Mathematica.

The next theorem highlights a connection between the subgroup $\mathbf{Z}(G)$ and the graph $\Delta(G)$ for a Z-group G.

THEOREM 3.2. *If G is a Z-group, then* $\operatorname{diam}(\Delta(G)) \le 2$ *if and only if* $\mathbf{Z}(G) \neq \{1\}$.

PROOF. Assume that $\mathbf{Z}(G) \neq \{1\}$. Fix $z \in \mathbf{Z}(G)^\#$ with $o(z) = p$, a prime. Since $\langle z\rangle$ is a normal p-subgroup of G and every Sylow p-subgroup of G is cyclic, $\langle z\rangle$ is the unique subgroup of G with order p. If $g \in G^\#$ and p divides $o(g)$, then $\langle z\rangle \le \langle g\rangle$. Hence, $g \approx z$.

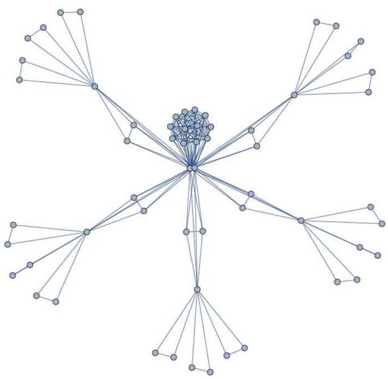

FIGURE 1. Cyclic graph of `SmallGroup(60,7)`.

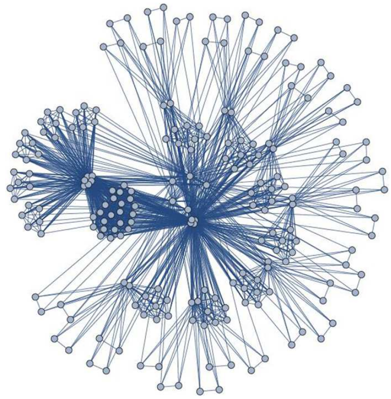

FIGURE 2. Cyclic graph of `SmallGroup(210,2)`.

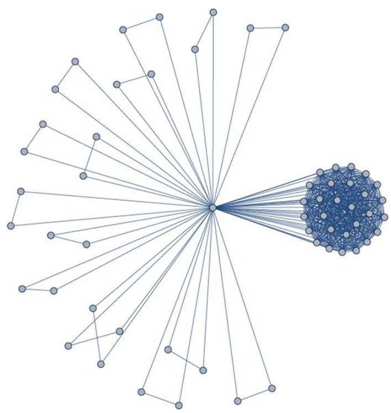

FIGURE 3. Cyclic graph of `SmallGroup(60,3)`.

Otherwise, $o(g)$ is a p'-number and so g and z are commuting elements with coprime orders. Again, $g \approx z$.

Assume that $\mathrm{diam}(\Delta(G)) \leq 2$. Let H be a complement of G'. Set $G' = \langle x \rangle$. As $G/G' \cong H$, the subgroup H is cyclic. Set $H = \langle h \rangle$. If $x \approx h$, then G is abelian and so $\mathbf{Z}(G) = G \neq \{1\}$. Otherwise, $x \approx z \approx h$ for some $z \in G^{\#}$. Now, $G' = \langle x \rangle \leq \mathbf{C}_G(z)$ and $H = \langle h \rangle \leq \mathbf{C}_G(z)$. It follows that $G = G'H \leq \mathbf{C}_G(z)$. Hence, $z \in \mathbf{Z}(G)^{\#}$. □

Let G be a group and recall that a vertex z in $\Delta(G)$ is called a *dominating vertex* if $z \approx g$ for each $g \in \Delta(G) \setminus \{z\}$. A dominating vertex appears in the previous proof and the following theorem highlights a necessary and sufficient condition for such a vertex to exist.

THEOREM 3.3. *The cyclic graph of a group G has a dominating vertex if and only if G has a unique subgroup of order p for some prime p and this subgroup is central.*

Proof. Let c be a dominating vertex of $\Delta(G)$. For suitable integer t, $o(c^t) = p \in \mathbb{P}$. For each $g \in \Delta(G) \setminus \{c^t\}$,

$$\langle c^t, g\rangle \leq \langle c, g\rangle$$

and so c^t is a dominating vertex as well. Note that $\langle c^t\rangle$ is a central subgroup of prime order. Suppose that $\langle y\rangle$ has order p. The subgroup $\langle c^t, y\rangle$ is cyclic and therefore has a unique subgroup of order p. Hence, $\langle c^t\rangle = \langle y\rangle$.

Conversely, suppose that $\langle z\rangle$ is a central subgroup of order $p \in \mathbb{P}$ and, further, that $\langle z\rangle$ is the *unique* subgroup of order p. If $g \in G$ is a p'-element, then $z \approx g$ since z and g are commuting elements with coprime orders. If p divides $o(g)$, then $|\langle g^t\rangle| = p$ for a suitable integer t. Our uniqueness hypothesis forces $\langle z\rangle = \langle g^t\rangle$. Again, $z \approx g$. The element z is a dominating vertex. □

The relationship between the existence of a dominating vertex for the cyclic graph of a group and the Sylow subgroup structure of the group can be developed a bit further. Let G be a group, $g \in G$ and $\pi = \pi(o(g))$. Using Theorem 5.1.5 in [8], write $g = \prod_{p\in\pi} g_p$, where each g_p is a p-element for $p \in \pi$ and $g_p g_q = g_q g_p$ for all $p, q \in \pi$. Then g is a dominating vertex for $\Delta(G)$ if and only if, for each $p \in \pi$, a Sylow p-subgroup P of G is cyclic or generalised quaternion and $\langle g_p\rangle \leq P \cap \mathbf{Z}(G)$. We remark that this result strengthens Theorem 3.3 and has essentially the same proof.

Bera and Bhuniya [2] showed that if G is abelian, then $\Delta(G)$ is dominatable if and only if G has a cyclic Sylow subgroup. We generalise this result.

Corollary 3.4. *If G is a nilpotent group, then $\Delta(G)$ is dominatable if and only if G has a cyclic or generalised quaternion Sylow subgroup.*

Proof. If $\Delta(G)$ has a dominating vertex, then, by Theorem 3.3, G has a unique subgroup $\langle x\rangle$ of prime order, say p, that is contained in $\mathbf{Z}(G)$. It is easy to check that if P is the Sylow p-subgroup of G, then $\langle x\rangle$ is the unique subgroup of P of order p; hence, P is cyclic or generalised quaternion.

Conversely, suppose that G has a Sylow p-subgroup P that is cyclic or generalised quaternion. Let $\langle z\rangle$ be the unique subgroup of G of order p. Let $g \in G^\#$. If $o(g)$ is a p'-number, then $z \approx g$ as z and g are therefore commuting elements with coprime orders. If p divides $o(g)$, then $|\langle g^s\rangle| = p$ for a suitable integer s. Hence, $\langle z\rangle = \langle g^s\rangle$ and so z is a power of g. Again, $z \approx g$. We conclude that z is a dominating vertex. □

As mentioned previously, if G is a Z-group with $\mathbf{Z}(G) = \{1\}$, then $\Gamma(G) = \Delta(G)$. We now obtain information about the commuting graph $\Gamma(G)$ of a Z-group G with trivial centre.

Corollary 3.5. *Let G be a Z-group with $\mathbf{Z}(G) = \{1\}$. If G is not a Frobenius group, then $\Gamma(G)$ is connected with* $\mathrm{diam}(\Gamma(G)) \in \{3, 4\}$.

Proof. By Lemma 2.1, $\Gamma(G) = \Delta(G)$. Since G is not a Frobenius group, Theorem 3.1 yields that $\Gamma(G)$ is connected. Theorem 3.2 gives us that $\mathrm{diam}(\Gamma(G)) \geq 3$. Finally, an application of Theorem 3.1 implies that $\mathrm{diam}(\Gamma(G))$ is either 3 or 4. □

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