

Riemann-Hilbert Approach to the Elastodynamic Equation. Half plane

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Abstract

We show, how the Riemann-Hilbert approach to the elastodynamic equations, which have been suggested in our preceding papers, works in the half-plane case. We pay a special attention to the emergence of the Rayleigh waves within the scheme.

1 Introduction

This paper is a complement to our previous work [11] where, following the general ideas of Fokas' method [4] -[8], we started to develop the Riemann-Hilbert scheme for solving the elastodynamic equations in the wedge-type domains. In [11], we show that the problem can be reduced to the solution of a certain matrix, 2×2 Riemann-Hilbert problem with a shift posed on a torus. A detail analysis of this problem is our ultimate goal which we hope to be able to present in our further publications. The aim of this paper is much more modest. We want to show how the basic ingredients of the elasticity theory, such as the Rayleigh waves, are produced in the framework of Fokas' method. To this end we shall consider the simplest case, the problem in the half-plane. Of course, the problem can be solved via the standard separation of variables. However, its analysis in the framework of the Riemann-Hilbert method shows many of the features which are also present in the more interesting and important case of the quarter plane.

The quarter plane case has already been outlined in [11]. In the next section we shall remind the principal ingredients of the approach that has been developed there.

2 Lax Pair for the elastodynamic equation

The elastodynamic equation in an isotropic medium defined by the Lamé parameters λ, μ , density ρ and frequency ω can be written as the following system of two scalar equations:

$$u_{xx} + \frac{h^2}{l^2}u_{zz} + \frac{l^2 - h^2}{l^2}w_{xz} + h^2u = 0, \quad (2.1)$$

$$w_{zz} + \frac{h^2}{l^2}w_{xx} + \frac{l^2 - h^2}{l^2}u_{xz} + h^2w = 0, \quad (2.2)$$

where $h^2 = \frac{\rho\omega^2}{\lambda+2\mu}$, $l^2 = \frac{\rho\omega^2}{\mu}$. Note that

$$l > h \quad (2.3)$$

The problem is two - dimensional in xz plane, and u and w are the x and z components of displacement, respectively. For the half plane problem ($z \geq 0$) on the surfaces $z = 0$ the stress free boundary conditions are :

$$T_{xz} = \mu(u_z + w_x) = -T_{xz}^{(0)}, \quad T_{zz} = \lambda u_x + (\lambda + 2\mu)w_z = -T_{zz}^{(0)}, \quad z = 0, \quad (2.4)$$

where $T_{zz}^{(0)}$ and $T_{xz}^{(0)}$ denote the given stresses which could be interpreted for example as the stresses of the incident Rayleigh wave. The solution should also satisfy Sommerfeld's radiation conditions [17] which we shall specify latter on (see equation (2.23) below). In fact, to make the problem well-posed one also has to add the surface wave radiation conditions. However, we will postpone doing this until Section 4. In that section we will show how one arrives at these condition, together with the Rayleigh surface waves, just following the logic of the method we are presenting in this paper. Indeed, as we have already indicated in the introduction, the appearance of the Rayleigh waves within Fokas's method is one of the principal methodological goals of this paper.

In [11], following the methodology of [6], we showed that equations (2.1), (2.2) are the compatibility conditions of the following two *Lax pairs*, written for the auxiliary scalar functions, $\phi \equiv \phi(z, x; k)$ and $\psi \equiv \psi(z, x; k)$ (see [11] and [10] for details),

$$\phi_z - ik\phi = \frac{1}{h^2}(\sqrt{k^2 - h^2} - k)\tau_1 - \frac{1}{h^2}(\tau_{1x} + i\tau_{1z}), \quad (2.5)$$

$$\phi_x + \sqrt{k^2 - h^2}\phi = \frac{i}{h^2}(k - \sqrt{k^2 - h^2})\tau_1 - \frac{i}{h^2}(\tau_{1x} + i\tau_{1z}),$$

and

$$\psi_z - ik\phi = \frac{1}{l^2}(\sqrt{k^2 - l^2} - k)\tau_2 - \frac{1}{l^2}(\tau_{2x} + i\tau_{2z}), \quad (2.6)$$

$$\psi_x + \sqrt{k^2 - l^2}\phi = \frac{i}{l^2}(k - \sqrt{k^2 - l^2})\tau_2 - \frac{i}{l^2}(\tau_{2x} + i\tau_{2z}),$$

where $\tau_1(z, x)$ and $\tau_2(z, x)$ are the Lamé potentials given by the equations

$$\tau_1 = \frac{1}{2}(u_x + w_z), \quad \tau_2 = \frac{1}{2}(w_x - u_z), \quad (2.7)$$

and ϕ and ψ satisfy the following asymptotic conditions ,

$$\phi, \psi = O\left(\frac{1}{k}\right), \quad k \rightarrow \infty^+ \quad (2.8)$$

$$\phi, \psi = O(1), \quad k \rightarrow \infty^-.$$

The last condition in conjunction with systems (2.5), (2.6) yields in fact the more specific asymptotic representation of the solutions ϕ and ψ as $k \rightarrow \infty^-$. Indeed we have (cf. [10]), that

$$\phi = -\frac{2i}{h^2}\tau_1 + O\left(\frac{1}{k}\right), \quad \psi = -\frac{2i}{l^2}\tau_2 + O\left(\frac{1}{k}\right) \quad k \rightarrow \infty^-. \quad (2.9)$$

In these formulae, $k \rightarrow \infty^\pm$ means that $k \rightarrow \infty$ and $\sqrt{k^2 - h^2}, \sqrt{k^2 - l^2} \rightarrow \pm k + ..$

Introducing the new spectral parameter ζ as follows,

$$k = \frac{h}{2} \left(\zeta + \frac{1}{\zeta} \right), \quad \sqrt{k^2 - h^2} = \frac{h}{2} \left(\zeta - \frac{1}{\zeta} \right), \quad (2.10)$$

so that,

$$\zeta \rightarrow \infty \text{ as } k \rightarrow \infty^+ \text{ and } \zeta \rightarrow 0 \text{ as } k \rightarrow \infty^-,$$

one can rewrite the first Lax pair (2.5) as

$$\phi_z - \frac{ih}{2} \left(\zeta + \frac{1}{\zeta} \right) \phi = Q_1, \quad (2.11)$$

$$\phi_x + \frac{h}{2} \left(\zeta - \frac{1}{\zeta} \right) \phi = \tilde{Q}_1, \quad (2.12)$$

where Q_1, \tilde{Q}_1 are the right-hand side parts of (2.5). In terms of ζ they are :

$$Q_1 = -\frac{\tau_1}{\zeta h} - \frac{1}{h^2}(\tau_{1x} + i\tau_{1z}); \quad \tilde{Q}_1 = \frac{i\tau_1}{\zeta h} - \frac{i}{h^2}(\tau_{1x} + i\tau_{1z}). \quad (2.13)$$

The normalization conditions (2.8) and (2.9) in terms of the new variable ζ read,

$$\phi = O\left(\frac{1}{\zeta}\right), \quad \zeta \rightarrow \infty, \quad (2.14)$$

$$\phi = -\frac{2i}{h^2}\tau_1 + O(\zeta) \quad \zeta \rightarrow 0. \quad (2.15)$$

The new spectral parameter for the second Lax pair we shall denote $\tilde{\zeta}$. The variable $\tilde{\zeta}$ is defined by the relations,

$$k = \frac{l}{2} \left(\tilde{\zeta} + \frac{1}{\tilde{\zeta}} \right), \quad \sqrt{k^2 - l^2} = \frac{l}{2} \left(\tilde{\zeta} - \frac{1}{\tilde{\zeta}} \right), \quad (2.16)$$

so that,

$$\tilde{\zeta} \rightarrow \infty \text{ as } k \rightarrow \infty^+ \text{ and } \tilde{\zeta} \rightarrow 0 \text{ as } k \rightarrow \infty^-.$$

The second Lax pair reads as follows

$$\psi_z - \frac{il}{2} \left(\tilde{\zeta} + \frac{1}{\tilde{\zeta}} \right) \psi = Q_2, \quad (2.17)$$

$$\psi_x + \frac{l}{2} \left(\tilde{\zeta} - \frac{1}{\tilde{\zeta}} \right) \psi = \tilde{Q}_2, \quad (2.18)$$

where

$$Q_2 = -\frac{\tau_2}{\tilde{\zeta}l} - \frac{1}{l^2}(\tau_{2x} + i\tau_{2z}); \quad \tilde{Q}_2 = \frac{i\tau_2}{\tilde{\zeta}l} - \frac{i}{l^2}(\tau_{2x} + i\tau_{2z}), \quad (2.19)$$

and it is supplemented by the normalization conditions

$$\phi = O\left(\frac{1}{\tilde{\zeta}}\right), \quad \tilde{\zeta} \rightarrow \infty, \quad (2.20)$$

$$\phi = -\frac{2i}{l^2}\tau_2 + O(\tilde{\zeta}) \quad \tilde{\zeta} \rightarrow 0. \quad (2.21)$$

The potentials τ_1 and τ_2 can be taken as the basic objects instead of the original displacements u and w . Indeed, as it follows from (2.1) and (2.2), the functions u and w can be reconstructed via τ_1 and τ_2 with the help of the following equations

$$u = -\frac{2}{h^2}\tau_{1x} + \frac{2}{l^2}\tau_{2z}, \quad \text{and} \quad w = -\frac{2}{h^2}\tau_{1z} - \frac{2}{l^2}\tau_{2x}, \quad (2.22)$$

respectively. Also, in terms of potentials τ_1 and τ_2 , Sommerfeld's radiation conditions can be written as

$$\lim_{R \rightarrow \infty} R \left(\frac{\partial \tau_1}{\partial R} - ih\tau_1 \right) = 0, \quad \lim_{R \rightarrow \infty} R \left(\frac{\partial \tau_2}{\partial R} - il\tau_2 \right) = 0, \quad R = \sqrt{x^2 + z^2}. \quad (2.23)$$

The potentials $\tau_{1,2}$ satisfy the Helmholtz equations,

$$\tau_{1xx} + \tau_{1zz} + h^2\tau_1 = 0, \quad (2.24)$$

$$\tau_{2xx} + \tau_{2zz} + l^2\tau_2 = 0. \quad (2.25)$$

This fact follows again from the basic elastodynamic system (2.1) - (2.2). It is important to notice that the reverse statement is also true. That is, if the displacements u and w are determined by (2.22), then equations (2.24) and (2.25) for τ_1 and τ_2 imply the elastodynamic equations (2.1) and (2.2) for u and w . Moreover, the inverse formulae expressing τ_1 and τ_2 in terms of u and w are given by (2.7).

The linear systems (2.11)-(2.12) and (2.17)-(2.18) can be thought of as the Lax pairs for the equations (2.24) and (2.25), respectively. This Lax pair representation of the Helmholtz equation has already been known and used for the analysis of the boundary value problem for the Helmholtz equation in [7] and [10]. The very important novelty of

the situation we are dealing with in this paper is that the boundary conditions, which relations (2.4) impose on the functions τ_1 and τ_2 are completely different from the ones which appear in the pure Helmholtz problem. The most distinct feature of these conditions is that they mix the two Lax pairs together, and this in turn complicates dramatically the analysis of the global relation (the main ingredient of Fokas' method [8]) in the case of the quarter space. In the half space, however, the solution of the global relation can be obtained in the closed form and by simple algebraic means.

3 Half space problem

The considerations of the previous section were general. We now apply the Lax pair representation of the elastodynamic equation to the half plane problem. We will basically repeat the constructions of the Section 3 of [11].

3.1 Integration of the Lax Pairs. The integral representation for the potential functions.

Rewriting (2.11, 2.12) as

$$e^{\frac{ih}{2}(\zeta+\frac{1}{\zeta})z-\frac{h}{2}(\zeta-\frac{1}{\zeta})x}(\phi e^{-\frac{ih}{2}(\zeta+\frac{1}{\zeta})z+\frac{h}{2}(\zeta-\frac{1}{\zeta})x})_z = Q_1, \quad (3.26)$$

$$e^{\frac{ih}{2}(\zeta+\frac{1}{\zeta})z-\frac{h}{2}(\zeta-\frac{1}{\zeta})x}(\phi e^{-\frac{ih}{2}(\zeta+\frac{1}{\zeta})z+\frac{h}{2}(\zeta-\frac{1}{\zeta})x})_x = \tilde{Q}_1, \quad (3.27)$$

and then integrating, yields the following general formula for the solution of (2.11, 2.12):

$$\phi(\zeta, x, z) = e^{\frac{ih}{2}(\zeta+\frac{1}{\zeta})z-\frac{h}{2}(\zeta-\frac{1}{\zeta})x} \int_{(x^*, z^*)}^{(x, z)} e^{-\frac{ih}{2}(\zeta+\frac{1}{\zeta})z'+\frac{h}{2}(\zeta-\frac{1}{\zeta})x'} [Q_1 dz' + \tilde{Q}_1 dx']. \quad (3.28)$$

It is worth noticing, that the path independence of the line integral in the right hand side is equivalent to the elastodynamic equation.

Choosing the contours of integration as shown in Figure 1, one obtains two distinct solutions:

$$\phi_1(\zeta, x, z) = \int_{-\infty}^x e^{\frac{h}{2}(\zeta-\frac{1}{\zeta})(x'-x)} \tilde{Q}_1(\zeta, x', z) dx' \quad (3.29)$$

$$\phi_2(\zeta, x, z) = \int_{\infty}^x e^{\frac{h}{2}(\zeta-\frac{1}{\zeta})(x'-x)} \tilde{Q}_1(\zeta, x', z) dx'. \quad (3.30)$$

The functions ϕ_1, ϕ_2 are analytic in the regions of the complex ζ plane which are shown in Figure 2. The difference,

$$\phi_1 - \phi_2,$$

is the solution of the homogeneous version of system (2.11, 2.12). Therefore,

$$\phi_1 - \phi_2 = e^{ih(\zeta+\frac{1}{\zeta})z/2-h(\zeta-\frac{1}{\zeta})x/2} \rho_{12}, \quad (3.31)$$

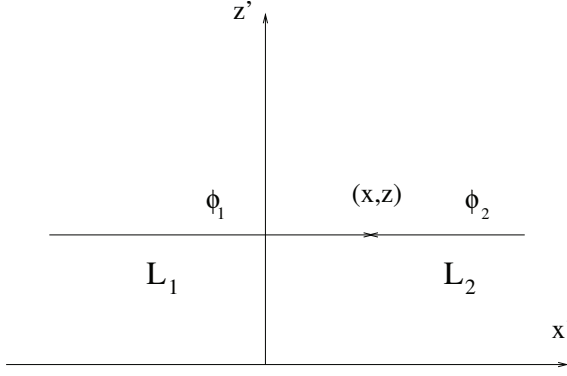


Figure 1: Contours L_i and the solutions $\phi_i(x, z)$, $i = 1, 2$ of the first scalar Lax pair

where the jump function $\rho_{12}(\zeta)$, $j, k = 1, 2$ does not depend on x and z and, as a function of ζ , is well defined on the boundaries of the regions in Figure 2, that is on the oriented contour K also depicted in Figure 2.

A key point now is to look at relation (3.31) as at the *Riemann-Hilbert problem* of finding the piecewise analytic function $\phi(\zeta)$ whose boundary values on the contour K , i.e. $\phi_+ = \phi_1$, $\phi_- = \phi_2$, satisfy the jump relation (3.31). Solving this Riemann-Hilbert problem we obtain the following integral representation for the piece-wise analytic function ϕ_ζ)

$$\phi(\zeta) = \frac{1}{2\pi i} \int_K \frac{e^{\frac{ih}{2}(s+1/s)z - \frac{h}{2}(s-1/s)x}}{s - \zeta} \rho_{12}(s) ds. \quad (3.32)$$

Taking into account (2.15), we derive from (3.32) the integral representation for τ_1

$$\tau_1 \equiv \frac{1}{2}(u_x + w_z) = \frac{h^2}{4\pi} \int_K \frac{e^{\frac{ih}{2}(\zeta+1/\zeta)z - \frac{h}{2}(\zeta-1/\zeta)x}}{\zeta} \rho_{12}(\zeta) d\zeta. \quad (3.33)$$

Similar representation we obtain for the potential τ_2 using the second Lax pair

$$\tau_2 \equiv \frac{1}{2}(w_x - u_z) = \frac{l^2}{4\pi} \int_{\tilde{K}} \frac{e^{\frac{il}{2}(\tilde{\zeta}+1/\tilde{\zeta})z - \frac{l}{2}(\tilde{\zeta}-1/\tilde{\zeta})x}}{\tilde{\zeta}} \tilde{\rho}_{12}(\tilde{\zeta}) d\tilde{\zeta}. \quad (3.34)$$

To complete the solution of the half space problem, we need to express the jump function $\rho_{12}(\zeta)$ and the similar function, $\tilde{\rho}_{12}(\tilde{\zeta})$ (coming from the second Lax pair), in terms of the given boundary data, i.e. in terms of the stresses $T_{xz}^{(0)}$ and $T_{zz}^{(0)}$. To this end, we notice that equation (3.31) holds for all x and z and that $\rho_{12}(\zeta)$ does not depend on x and z ; therefore, using this equation for $x = 0$ and $z = 0$ and remembering the definitions (3.29), (3.30) of the solutions $\phi_{1,2}$, one obtains the following formula for the jump function ρ_{12} :

$$\rho_{12}(\zeta) = \int_{-\infty}^{\infty} e^{\frac{h}{2}(\zeta - \frac{1}{\zeta})x'} \tilde{Q}_1(\zeta, x', 0) dx'. \quad (3.35)$$

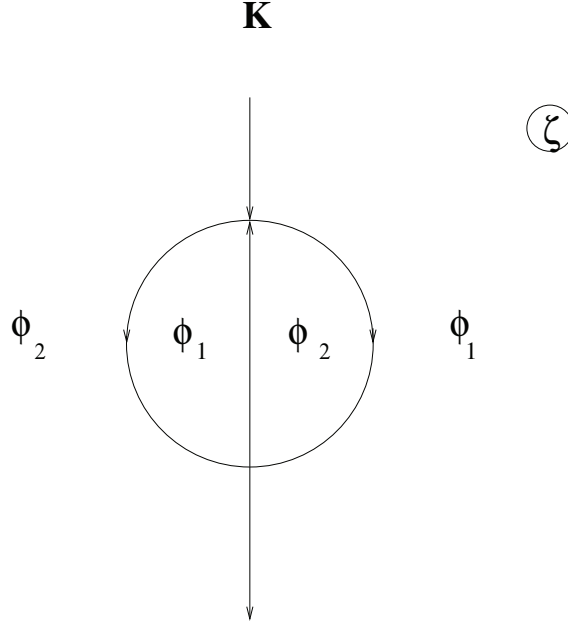


Figure 2: Regions of analyticity of functions ϕ_i , $i = 1, 3$ of the first scalar Lax pair

The integrand, $\tilde{Q}_1(\zeta, 0, x')$, involves the boundary values of the potential function τ_1 and its derivatives. However, not all of them can be determined by the boundary relations (2.4). In order to determine the remaining data, we have to appeal to the central ingredient of Fokas' method, i.e. to derive the relevant *global relation* for the jump function $\rho_{12}(\zeta)$.

Formula (3.35) can be rewritten in the form of the line integral of the conservative vector field,

$$\rho_{12}(\zeta) = \int_{-\infty < x' < \infty, z'=0} e^{-\frac{ih}{2}(\zeta + \frac{1}{\zeta})z' + \frac{h}{2}(\zeta - \frac{1}{\zeta})x'} [Q_1 dz' + \tilde{Q}_1 dx'].$$

Assuming that either $\zeta = -it, t > 1$ or $\zeta = it, t < 1$, the contour can be closed in the upper plane $z' \geq 0$. Therefore, $\rho_{12}(\zeta)$ is zero on these parts of the complex axis ζ ,

$$\rho_{12}(\zeta) = 0, \quad \zeta = -it, t > 1, \quad \text{and} \quad \zeta = it, 0 < t < 1, \quad (3.36)$$

which constitutes the global relation for our problem. Furthermore, the circular part of the contour has to be analyzed taking into account the radiation condition. Applying the stationary phase estimate as $R \rightarrow \infty$ ($x = R \cos \theta$, $z = R \sin \theta$, $0 \leq \theta \leq \pi$) to τ_1 (3.33) yields two stationary phase points

$$\zeta_1 = \sin \theta - i \cos \theta, \quad \zeta_2 = -(\sin \theta - i \cos \theta). \quad (3.37)$$

They belong respectively to C_r and C_l parts of K (see Fig3). These points provide the following asymptotic estimates

$$I_{C_r} \sim \frac{1}{2\pi i} \frac{\rho(\zeta_1)}{\zeta_1} e^{iRh} e^{i\theta} \sqrt{\frac{2}{Rh}} e^{-i\pi/4} \sqrt{\pi}, \quad (3.38)$$

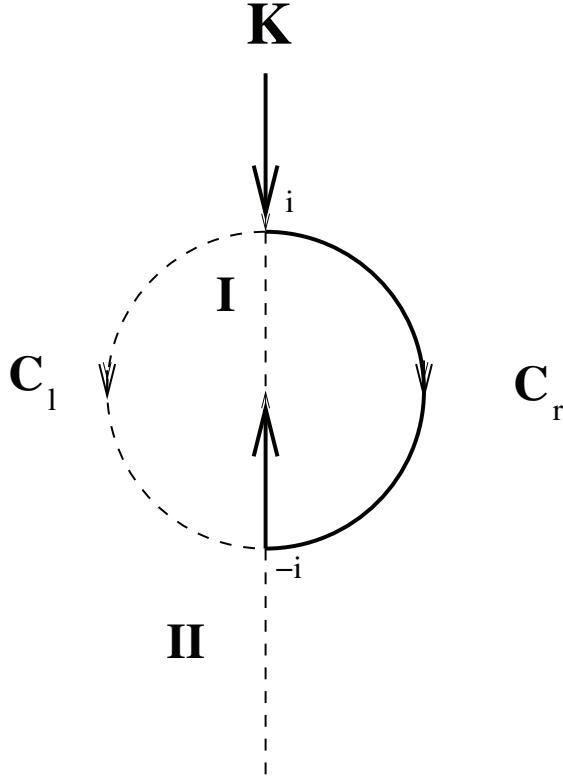


Figure 3: Contour of integration K of the first scalar Lax pair; sections of the contour where the jump function is zero are given by the dashed line.

$$I_{C_l} \sim \frac{1}{2\pi i} \frac{\rho(\zeta_2)}{\zeta_2} e^{-iRh} e^{i\theta} \sqrt{\frac{2}{Rh}} e^{-i\pi/4} \sqrt{\pi}. \quad (3.39)$$

The second asymptotic solution (3.39) does not satisfy the radiation condition (2.23); therefore, in addition to the global relation (3.36), we have that

$$\rho_{12}(\zeta) = 0, \quad \zeta \in C_l. \quad (3.40)$$

Taking into account (3.36) and (3.40), one finally obtains that the jump functions should be defined on the “non zero” parts of the contour K which are indicated in Figure 3 by solid lines.

3.2 Analysis of the global relation

In this section we use the global relation (3.36) and the radiation condition (3.40) to determine the jump function $\rho_{12}(\zeta)$ in terms of the known functions $T_{xz}^{(0)}$, $T_{zz}^{(0)}$.

Let us rewrite (3.35) changing x' to x and substituting \tilde{Q}_1 from (2.13):

$$\rho_{12}(\zeta) = \int_{-\infty}^{\infty} e^{\frac{h}{2}(\zeta - \frac{1}{\zeta})x} \left[\left(\frac{i}{h\zeta} \tau_1(x, 0) - \frac{i}{h^2} (\tau_{1x}(x, 0) + i\tau_{1z}(x, 0)) \right) \right] dx. \quad (3.41)$$

After integration of τ_{1x} by parts one obtains

$$\rho_{12}(\zeta) = \int_{-\infty}^{\infty} e^{\frac{h}{2}(\zeta - \frac{1}{\zeta})x} \left[\frac{i}{2h} \left(\zeta + \frac{1}{\zeta} \right) \tau_1(x, 0) + \frac{1}{h^2} \tau_{1z}(x, 0) \right] dx. \quad (3.42)$$

Then using conditions (2.4) at $z = 0$, equations (2.1), (2.2) and again integrating by parts, one finally arrives at the formula,

$$\rho_{12}(\zeta) = -b(\zeta)\Phi_1(\zeta) - d(\zeta)\Phi_2(\zeta) + F_1(\zeta), \quad (3.43)$$

where F_1 is defined by the given boundary data,

$$F_1 = -\frac{i}{4h(\lambda + 2\mu)} \left(\zeta + \frac{1}{\zeta} \right) \int_{-\infty}^{\infty} e^{\frac{h}{2}(\zeta - \frac{1}{\zeta})x} T_{zz}^{(0)}(x, 0) dx \\ - \frac{1}{2l^2\mu} \int_{-\infty}^{\infty} e^{\frac{h}{2}(\zeta - \frac{1}{\zeta})x} T_{xx}^{(0)}(x, 0) dx, \quad (3.44)$$

and Φ_1, Φ_2 are the following integrals of the unknown u and w :

$$\Phi_1(\zeta) = \int_{-\infty}^{\infty} e^{\frac{h}{2}(\zeta - \frac{1}{\zeta})x} u(x, 0) dx, \quad \Phi_2(\zeta) = \int_{-\infty}^{\infty} e^{\frac{h}{2}(\zeta - \frac{1}{\zeta})x} w(x, 0) dx. \quad (3.45)$$

The coefficient functions, $b(\zeta)$ and $d(\zeta)$, are given by the formulas:

$$b(\zeta) = \frac{ih^2}{4l^2} \left(\zeta^2 - \frac{1}{\zeta^2} \right), \quad (3.46)$$

$$d(\zeta) = \frac{l^2 - h^2}{2l^2} + \frac{h^2}{4l^2} \left(\zeta^2 + \frac{1}{\zeta^2} \right). \quad (3.47)$$

In terms of these functions the global relation on the parts I and II of the imaginary ζ - axis reads

$$b(\zeta)\Phi_1(\zeta) + d(\zeta)\Phi_2(\zeta) = F_1(\zeta) \quad (3.48)$$

Changing ζ to $-\frac{1}{\zeta}$ and using symmetries yields

$$-b(\zeta)\Phi_1(\zeta) + d(\zeta)\Phi_2(\zeta) = F_1\left(-\frac{1}{\zeta}\right) \quad (3.49)$$

on the parts of the imaginary axis which are included into non zero ρ sections of K . Hence the boundary conditions applied to the first Lax pair produces one equation to relate the two unknown functions, i.e. Φ_1 and Φ_2 , on these parts of the contour K .

Equation (3.48) also holds on C_l where $\rho_{12} = 0$. Changing ζ to $-\frac{1}{\zeta}$ and using symmetries yields

$$-b(\zeta)\Phi_1(\zeta) + d(\zeta)\Phi_2(\zeta) = F_1\left(-\frac{1}{\zeta}\right), \quad (3.50)$$

and hence we obtain an equation (actually the same as (3.49)) relating the two unknown functions on the arc C_r as well.

Repeating computations for the second Lax pair on the $\tilde{\zeta}$ complex plane one obtains that the global relation has similar form as (3.48)

$$\delta(\tilde{\zeta})\tilde{\Phi}_1(\tilde{\zeta}) + \beta(\tilde{\zeta})\tilde{\Phi}_2(\tilde{\zeta}) = F_2(\tilde{\zeta}), \quad (3.51)$$

where

$$\delta(\tilde{\zeta}) = -\frac{1}{4}\left(\tilde{\zeta}^2 + \frac{1}{\tilde{\zeta}^2}\right), \quad (3.52)$$

$$\beta(\tilde{\zeta}) = \frac{i}{4}\left(\tilde{\zeta}^2 - \frac{1}{\tilde{\zeta}^2}\right), \quad (3.53)$$

$$\tilde{\Phi}_1(\tilde{\zeta}) = \int_{-\infty}^{\infty} e^{\frac{i}{2}(\tilde{\zeta} - \frac{1}{\tilde{\zeta}})x} u(x, 0) dx, \quad \tilde{\Phi}_2(\tilde{\zeta}) = \int_{-\infty}^{\infty} e^{\frac{i}{2}(\tilde{\zeta} - \frac{1}{\tilde{\zeta}})x} w(x, 0) dx, \quad (3.54)$$

$$\begin{aligned} F_2(\tilde{\zeta}) = & -\frac{1}{2\mu l^2} \int_{-\infty}^{\infty} e^{\frac{i}{2}(\tilde{\zeta} - \frac{1}{\tilde{\zeta}})x} T_{zzx}^{(0)}(x, 0) dx \\ & + \frac{i}{4l\mu} \left(\tilde{\zeta} + \frac{1}{\tilde{\zeta}}\right) \int_{-\infty}^{\infty} e^{\frac{i}{2}(\tilde{\zeta} - \frac{1}{\tilde{\zeta}})x} T_{xz}^{(0)}(x, 0) dx. \end{aligned} \quad (3.55)$$

Therefore, using the symmetries in the same way as for the first Lax pair, we can obtain another relation between the unknown functions on the non-zero parts of the contour \tilde{K} of $\tilde{\zeta}$ plane.

Summarizing our analysis of the global relation, we see that we have arrived at two algebraic linear equations - equations (3.49) and (3.51), for four unknown functions - the functions $\Phi_{1,2}(\zeta)$ and $\tilde{\Phi}_{1,2}(\tilde{\zeta})$. However, one can notice - see the definitions (3.45) and (3.54), that these four functions are actually depend only on two functional parameters - $u(x, 0)$ and $w(x, 0)$. This means, that just by counting the truly independent functions we have as many equations as we have unknowns. In order to make use of this observation, we are suggesting to transfer both Lax pairs onto the same complex plane.

3.3 Joint uniformization

Let us map the complex planes ζ and $\tilde{\zeta}$ to the complex plane ξ by the following formulae

$$\zeta = \frac{\xi}{a}, \quad l\left(\tilde{\zeta} - \frac{1}{\tilde{\zeta}}\right) = h\left(\zeta - \frac{1}{\zeta}\right), \quad (3.56)$$

where

$$a = \frac{l}{h} + \sqrt{\frac{l^2}{h^2} - 1}. \quad (3.57)$$

Note that $a > 1$. Transformation of the contour K from ζ plane to ξ is given in Figure 4.

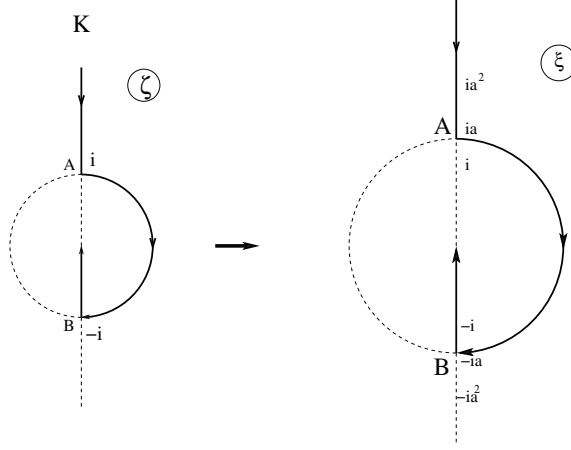


Figure 4: Transformation of contour K from ζ to ξ complex plane.

The explicit formula for the map $\tilde{\zeta}(\xi)$ is given by the equation,

$$\tilde{\zeta} = \frac{h}{2al} \left(\xi - \frac{a^2}{\xi} + \frac{1}{\xi} \sqrt{(\xi^2 + 1)(\xi^2 + a^4)} \right), \quad (3.58)$$

so that

$$\tilde{\zeta} + \frac{1}{\tilde{\zeta}} = \frac{h}{al\xi} \sqrt{(\xi^2 + 1)(\xi^2 + a^4)}.$$

Transformation of the contour \tilde{K} is presented in Figure 5.

The branch points of (3.58) $\xi = ia^2$ and $\xi = i$, presented in Figure 5 as A^t, A^b , have the same image $\tilde{\zeta} = \frac{h}{2al} \left(\xi - \frac{a^2}{\xi} \right) = i$. It is given as \tilde{A} point on the $\tilde{\zeta}$ plane. On the other hand, the intersection of the circle of radius a with the imaginary ξ axis (point A which is also marked as A_1 and A_2 to indicate left and right sides of the cut) has two different images \tilde{A}_1 and \tilde{A}_2 on $\tilde{\zeta}$ plane. At these points

$$\tilde{\zeta} = \frac{a^2 - 1 + 2ia}{a^2 + 1} \text{ and } \tilde{\zeta} = \frac{1 - a^2 + 2ia}{a^2 + 1},$$

and they belong to the unit circle: $|\tilde{\zeta}| = 1$. Symmetric low part of the contour has B notations.

Since,

$$\zeta - \frac{1}{\zeta} = \frac{1}{a} \left(\xi - \frac{a^2}{\xi} \right), \quad \tilde{\zeta} - \frac{1}{\tilde{\zeta}} = \frac{h}{al} \left(\xi - \frac{a^2}{\xi} \right), \quad (3.59)$$

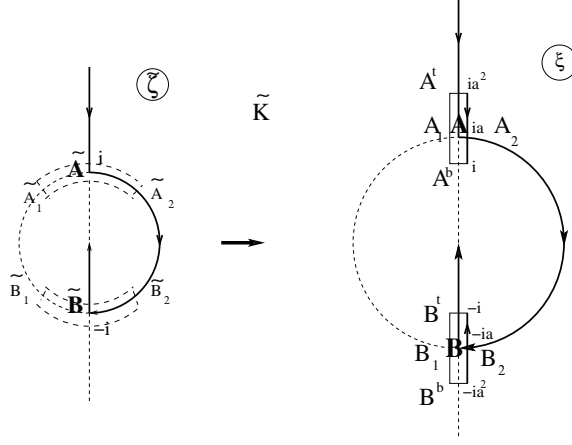


Figure 5: Transformation of contour \tilde{K} from ζ to ξ complex plane.

the both $\Phi_1(\zeta)$ and $\tilde{\Phi}_1(\tilde{\zeta})$ become $\Phi_1(\xi)$ while the both $\Phi_2(\zeta)$ and $\tilde{\Phi}_2(\tilde{\zeta})$ become $\Phi_2(\xi)$, where

$$\Phi_1(\xi) = \int_{-\infty}^{\infty} e^{\frac{h}{2a}(\xi - \frac{a^2}{\xi})x} u(0, x) dx, \quad \Phi_2(\xi) = \int_{-\infty}^{\infty} e^{\frac{h}{2a}(\xi - \frac{a^2}{\xi})x} w(0, x) dx. \quad (3.60)$$

Taking into account these transformations and changing $b(\zeta), d(\zeta), F_1(\zeta), \beta(\tilde{\zeta}), \delta(\tilde{\zeta}), F_2(\tilde{\zeta})$ to $b(\xi), d(\xi), F_1(\xi), \beta(\xi), \delta(\xi), F_2(\xi)$ yields the system of two algebraic equations for the two unknown functions $\Phi_1(\xi)$ and $\Phi_2(\xi)$ on all parts of the contour $K(\xi)$. Indeed, we have that

$$-b(\xi)\Phi_1(\xi) + d(\xi)\Phi_2(\xi) = F_1\left(-\frac{a^2}{\xi}\right), \quad (3.61)$$

$$\delta(\xi)\Phi_1(\xi) - \beta(\xi)\Phi_2(\xi) = F_2\left(-\frac{a^2}{\xi}\right),$$

if $\xi \in [ia^2, i\infty) \cup [ia, ia^2]_+ \cup [-ia, -i]_+ \cup [-i, i0) \cup C_r$, and

$$b(\xi)\Phi_1(\xi) + d(\xi)\Phi_2(\xi) = F_1(\xi), \quad (3.62)$$

$$\delta(\xi)\Phi_1(\xi) - \beta(\xi)\Phi_2(\xi) = F_2\left(-\frac{a^2}{\xi}\right),$$

if $\xi \in [i, ia]_+ \cup [-ia, -ia^2]_+$. Here $[\dots]_+$ means the right side of the cut $[\dots]$, and the functions $b(\xi), d(\xi), \beta(\xi), \delta(\xi)$ are given by the formulae.

$$b(\xi) = \frac{ih^2}{4l^2} \left(\frac{\xi^2}{a^2} - \frac{a^2}{\xi^2} \right), \quad d(\xi) = \frac{l^2 - h^2}{2l^2} + \frac{h^2}{4l^2} \left(\frac{\xi^2}{a^2} + \frac{a^2}{\xi^2} \right), \quad (3.63)$$

$$\delta(\xi) = -\frac{h^2}{4l^2} \left[\left(\frac{\xi^2}{a^2} + \frac{a^2}{\xi^2} \right) + \frac{1}{2} \left(a - \frac{1}{a} \right)^2 \right], \quad \beta(\xi) = \frac{ih^2}{4a^2l^2} \left(\frac{\xi}{a} - \frac{a}{\xi} \right) \Omega(\xi), \quad (3.64)$$

where

$$\Omega(\xi) = \frac{a}{\xi} \sqrt{(\xi^2 + 1)(\xi^2 + a^4)}. \quad (3.65)$$

It is worth noticing that

$$\Omega(-\xi) = -\Omega(\xi), \quad \Omega\left(\frac{a^2}{\xi}\right) = \Omega(\xi). \quad (3.66)$$

4 Analysis of the solution. Rayleigh waves

Summarizing our derivations, we see that on all parts of the ξ - image of the contour K , the functions $\Phi_1(\xi)$ and $\Phi_2(\xi)$ can be defined by solving a simple algebraic system. Changing variable ξ back to the variables ζ and $\tilde{\zeta}$, we obtain the jump functions $\rho_{12}(\zeta)$ and $\tilde{\rho}_{12}(\tilde{\zeta})$, respectively. This would complete the solution of the half space problem. Let us look at the solutions of the algebraic systems more carefully. Note that we need to know functions $\Phi_1(\xi)$ and $\Phi_2(\xi)$ on the image of the contours K and \tilde{K} on the ξ - plane only, i.e. for $\xi \in [ia^2, i\infty) \cup [ia, ia^2]_+ \cup [-ia, -i]_+ \cup [-i, i0) \cup C_r$. Hence, we need to consider the system (3.61) only. It follows then that,

$$\Phi_1(\xi) = \frac{\beta(\xi)F_1\left(-\frac{a^2}{\xi}\right) + d(\xi)F_2\left(-\frac{a^2}{\xi}\right)}{D(\xi)} \text{ and } \Phi_2(\xi) = \frac{\delta(\xi)F_1\left(-\frac{a^2}{\xi}\right) + b(\xi)F_2\left(-\frac{a^2}{\xi}\right)}{D(\xi)}, \quad (4.67)$$

where

$$D(\xi) = d(\xi)\delta(\xi) - \beta(\xi)b(\xi) \quad (4.68)$$

is the determinant of system (3.61). Our task now is to analyze its zeros.

By a straightforward calculation, we have that

$$D(\xi) = -\left(a + \frac{1}{a}\right)^{-4} D_0(\xi),$$

$$D_0(\xi) = \frac{1}{4} \left[\left(a - \frac{1}{a}\right)^2 + 2 \left(\frac{\xi^2}{a^2} + \frac{a^2}{\xi^2}\right) \right]^2 - \frac{1}{a^2} \left(\frac{\xi^2}{a^2} - \frac{a^2}{\xi^2}\right) \left(1 - \frac{a^2}{\xi^2}\right) \sqrt{(\xi^2 + 1)(\xi^2 + a^4)}. \quad (4.69)$$

Going back to the original spectral parameter k ,

$$k = \frac{h}{2} \left(\frac{\xi}{a} + \frac{a}{\xi} \right), \quad (4.70)$$

and recalling the definition of the parameter a , one can check that

$$\frac{h}{2} \left(\frac{\xi}{a} - \frac{a}{\xi} \right) = \sqrt{k^2 - h^2}, \quad \frac{1}{\xi} \sqrt{(\xi^2 + 1)(\xi^2 + a^4)} = \frac{2a}{h} \sqrt{k^2 + l^2 - h^2}.$$

From this, it is easy to see that

$$\frac{h^4}{16}D_0(\xi) = \left(k^2 - h^2 + \frac{l^2}{2}\right)^2 - k(k^2 - h^2)\sqrt{k^2 + l^2 - h^2}. \quad (4.71)$$

Introducing the physical quantities (see [3]),

$$c^2 = \frac{\omega^2}{h^2 - k^2}, \quad \alpha^2 = \frac{\omega^2}{h^2}, \quad \beta^2 = \frac{\omega^2}{l^2}, \quad (4.72)$$

we arrive at the final formula for the determinant D_0 ,

$$\frac{c^4 h^4}{4\omega^4}D_0(\xi) = \left(2 - \frac{c^2}{\beta^2}\right)^2 - 4\sqrt{1 - \frac{c^2}{\alpha^2}}\sqrt{1 - \frac{c^2}{\beta^2}}, \quad (4.73)$$

which means that

$$D(\xi) = 0 \iff \left(2 - \frac{c^2}{\beta^2}\right)^2 = 4\sqrt{1 - \frac{c^2}{\alpha^2}}\sqrt{1 - \frac{c^2}{\beta^2}}. \quad (4.74)$$

Equation in the right hand side of this equivalence relation is the classical equation for the velocity c of the Rayleigh wave - see e.g., [3].

Hence our main conclusion: *The zeros of the determinant $D(\xi)$ of the linear system (3.61) representing the global relation of the half-plane problem coincide with the images ξ_c of the Rayleigh velocity c under the map chain $c \rightarrow k \rightarrow \xi$.*

Combining (4.70, 4.72) we obtain the value of ξ_c in terms of the Rayleigh wave velocity c as

$$\xi_c = ia \left(\frac{\alpha}{c} + \sqrt{\frac{\alpha^2}{c^2} - 1} \right), \quad (4.75)$$

or, taking into account the expression of the transformation parameter a , in terms of α and β as

$$\xi_c = i \left(\frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} - 1} \right) \left(\frac{\alpha}{c} + \sqrt{\frac{\alpha^2}{c^2} - 1} \right). \quad (4.76)$$

Due to symmetries (3.66), there are two zeros: ξ_c and $\frac{a^2}{\xi_c}$.

Since $0 < c < \beta < \alpha$, the roots lie on the intervals $(ia^2, i\infty)$ and $(-i, i0)$ of ξ plane. This means, that the densities $\rho_{12}(\zeta)$ and $\tilde{\rho}_{12}(\tilde{\zeta})$ do have poles on the contours K and \tilde{K} , respectively. This means we have to deform the contours K and \tilde{K} near the poles and go around them. *This is where the both - the Rayleigh surface waves and the surface wave radiation condition, will show up in our approach.* We are going now to explain this in details.

Let us consider, for example, the part, I_{ζ_c} , of the τ_1 - integral (3.33) that contains the top pole, $\zeta_c = \xi_c/a$, i.e.,

$$I_{\zeta_c} = \frac{h^2}{4\pi} \int_i^{i\infty} \frac{e^{\frac{ih}{2}(\zeta+1/\zeta)z - \frac{h}{2}(\zeta-1/\zeta)x}}{\zeta} \rho_{12}(\zeta) d\zeta.$$

As is written, this integral does not exist, of course. It needs to be regularized. To this end, let us denote ρ_c the residue of $\rho_{12}(\zeta)$ at the pole $\zeta = \zeta_c$ and rewrite I_{ζ_c} as

$$I_{\zeta_c} = I_{\zeta_c}^{(1)} + I_{\zeta_c}^{(2)}, \quad (4.77)$$

where

$$I_{\zeta_c}^{(1)} = \frac{h^2}{4\pi} \int_i^{i\infty} \frac{e^{\frac{ih}{2}(\zeta+1/\zeta)z - \frac{h}{2}(\zeta-1/\zeta)x}}{\zeta} \left(\rho_{12}(\zeta) - \frac{\rho_c}{\zeta - \zeta_c} \right) d\zeta, \quad (4.78)$$

and

$$I_{\zeta_c}^{(2)} = \frac{h^2}{4\pi} \int_i^{i\infty} \frac{e^{\frac{ih}{2}(\zeta+1/\zeta)z - \frac{h}{2}(\zeta-1/\zeta)x}}{\zeta} \frac{\rho_c}{\zeta - \zeta_c} d\zeta. \quad (4.79)$$

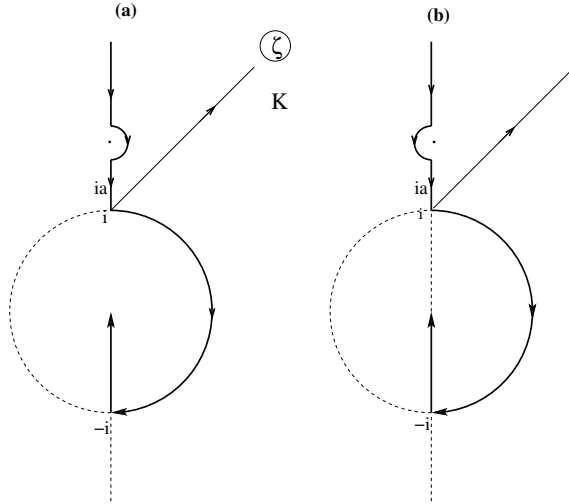


Figure 6: Top pole on the imaginary axes

We note that the integral $I_{\zeta_c}^{(1)}$ has no singularities. Moreover, it can be easily estimated as $R = \sqrt{x^2 + z^2} \rightarrow \infty$. Indeed, since the corresponding stationary point lies on the circle part of the contour K we immediately conclude that

$$I_{\zeta_c}^{(1)} = O\left(\frac{1}{R}\right), \quad R \rightarrow \infty.$$

Hence this part does not contribute either to radiation part or to the surface wave part of the potential $\tau_1(x, z)$. Let us then concentrate on the indeed singular integral $I_{\zeta_c}^{(2)}$.

A standard way to regularize the integral $I_{\zeta_c}^{(2)}$ would be to deform the contour around the pole. Let us assume that $x > 0$ and suppose that we go around the pole from the right (see Figure 6). Because of analyticity and exponential decay of the integrand in the first quadrant of the complex plane outside of the unit circle, this part K may be deformed to, for example, the ray $\arg \zeta = \pi/4$ and integration by part produces the $O\left(\frac{1}{R}\right)$ asymptotic behavior. Suppose now that we go around the pole from the left. We still have to close the contour in the first quadrant because of the analyticity and decay. This time, the leading term will be given by the residue of the integrand, i.e., we shall have,

$$I_{\zeta_c}^{(2)} = \frac{ih^2\rho_c}{2\xi_c} e^{\frac{ih}{2a}(\xi_c+a^2/\xi_c)z - \frac{h}{2a}(\xi_c-a^2/\xi_c)x} + O\left(\frac{1}{R}\right), \quad R \rightarrow \infty. \quad (4.80)$$

In the case $x < 0$, the integrand is exponentially decay in the second quadrant and hence the situation is reverse: if the contour goes around the pole from left we have the $O\left(\frac{1}{R}\right)$ asymptotic behavior of the integral $I_{\zeta_c}^{(2)}$ while going around the pole from the right would produce the residue term (4.80) with the opposite sign. In other words, for the part I_{ζ_c} of the τ_1 - integral (3.33) we have that

$$I_{\zeta_c} = \text{sign}(x) \frac{ih^2\rho_c}{2\xi_c} e^{\frac{ih}{2a}(\xi_c+a^2/\xi_c)z - \frac{h}{2a}(\xi_c-a^2/\xi_c)x} + O\left(\frac{1}{R}\right), \quad R \rightarrow \infty, \quad (4.81)$$

if $x > 0$ and the contour goes around the pole from the left or if $x < 0$ and the contour goes around the pole from the right. At the same time,

$$I_{\zeta_c} = O\left(\frac{1}{R}\right), \quad R \rightarrow \infty, \quad (4.82)$$

if $x > 0$ and the contour goes around the pole from the right or if $x < 0$ and the contour goes around the pole from the left.

When we do the similar analysis (see Figure 7) with the part $I_{\zeta_c^{-1}}$ of the τ_1 - integral (3.33) containing the bottom pole ζ_c^{-1} , i.e. with the integral

$$I_{\zeta_c^{-1}} = \frac{h^2}{4\pi} \int_{-i}^{-i0} \frac{e^{\frac{ih}{2}(\zeta+1/\zeta)z - \frac{h}{2}(\zeta-1/\zeta)x}}{\zeta} \rho_{12}(\zeta) d\zeta,$$

we would arrive at the estimates,

$$I_{\zeta_c^{-1}} = \text{sign}(x) \frac{ih^2\rho_c}{2\xi_c} e^{\frac{ih}{2a}(\xi_c+a^2/\xi_c)z + \frac{h}{2a}(\xi_c-a^2/\xi_c)x} + O\left(\frac{1}{R}\right), \quad R \rightarrow \infty, \quad (4.83)$$

if $x < 0$ and the contour goes around the pole from the left or if $x > 0$ and the contour goes around the pole from the right, and

$$I_{\zeta_c^{-1}} = O\left(\frac{1}{R}\right), \quad R \rightarrow \infty, \quad (4.84)$$

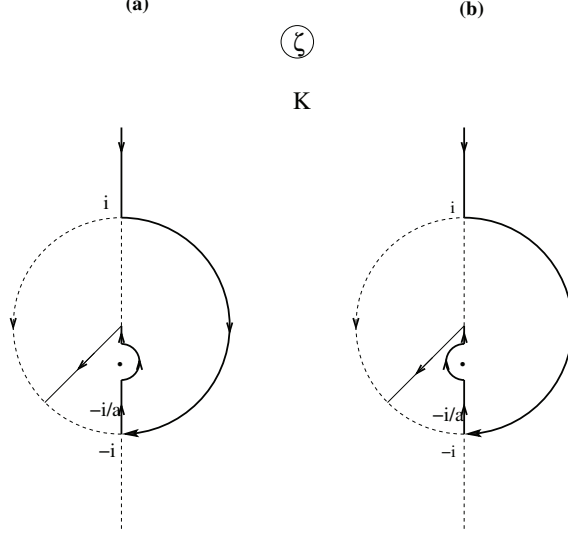


Figure 7: Bottom pole on the imaginary axes

if $x < 0$ and the contour goes around the pole from the right or if $x > 0$ and the contour goes around the pole from the left.

Observe, that

$$\frac{h}{2a} \left(\xi_c - \frac{a^2}{\xi_c} \right) = i \frac{\omega}{c} \equiv i k_c,$$

while

$$i \frac{h}{2a} \left(\xi_c - \frac{a^2}{\xi_c} \right) = -h \sqrt{\frac{\alpha^2}{c^2} - 1} = -\sqrt{\frac{\omega^2}{c^2} - h^2} \equiv -\sqrt{k_c^2 - h^2},$$

where k_c is the Rayleigh wave number (note that $k_c > h$). Hence, equations (4.81) and (4.83) can be written as equations,

$$I_{\zeta_c} = \text{sign}(x) \frac{i h^2 \rho_c}{2 \xi_c} e^{-\sqrt{k_c^2 - h^2} z - i \frac{\omega}{c} x} + O\left(\frac{1}{R}\right), \quad R \rightarrow \infty, \quad (4.85)$$

and

$$I_{\zeta_c^{-1}} = \text{sign}(x) \frac{i h^2 \rho_c}{2 \xi_c} e^{-\sqrt{k_c^2 - h^2} z + i \frac{\omega}{c} x} + O\left(\frac{1}{R}\right), \quad R \rightarrow \infty, \quad (4.86)$$

respectively, and written in this form they clearly represent the Rayleigh surface waves propagating along the surface $z = 0$. The direction of their propagation depends on the particular choice of the way we are going around the poles ζ_c and ζ_c^{-1} . Let us choose the contour of integration as it is indicated in Figure 8. From the above analysis it follows that the only bottom pole will contribute and the potential τ_1 will then exhibit the Rayleigh surface wave behavior described by the equation

$$\tau_1(x, z) = \text{sign}(x) \frac{i h^2 \rho_c}{2 \xi_c} e^{-\sqrt{k_c^2 - h^2} z + i \frac{\omega}{c} x} + O\left(\frac{1}{R}\right), \quad R \rightarrow \infty.$$

This formula ensures that the potential τ_1 satisfies the surface wave radiation condition

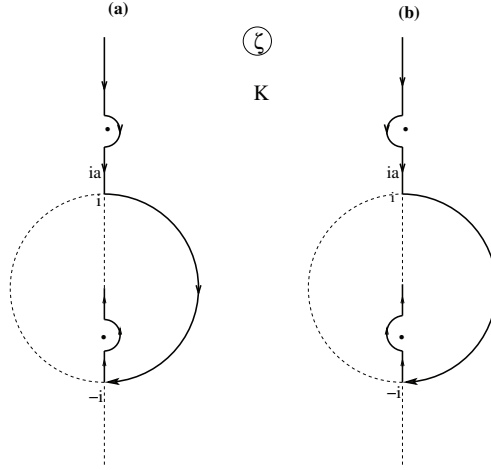


Figure 8: Final contour of integration: choice (a) corresponds $x > 0$, choice (b) corresponds $x < 0$

(see [18]) in a small parabolic sector near the surface depicted in Figure 9 :

$$\lim_{x \rightarrow \infty} x \left(\frac{\partial \tau_1}{\partial x} - ik_c \tau_1 \right) = 0. \quad (4.87)$$

Together with the radiation condition which, as we have already shown, the solution τ_1 also satisfies, (4.87) guarantees that the solution we just constructed is exactly the one whose existence and uniqueness are proven in [18].

The similar analysis of the potential $\tau_2(x, z)$ yields the presence of the Rayleigh surface waves described this time by the formula

$$\tau_2(x, z) = \text{sign}(x) \frac{ih^2 \rho_c}{2\xi_c} e^{\frac{ih}{a^2} \Omega(\xi_c) z + i \frac{\omega}{c} x} + O\left(\frac{1}{R}\right), \quad R \rightarrow \infty.$$

Note, that since $\xi_c \in (ia^2, i\infty)$,

$$\frac{ih}{a^2} \Omega(\xi_c) < 0.$$

Remark. We have explained the intrinsic reason of appearance of the Rayleigh waves within Fokas's scheme. The Sommerfeld radiation condition, which we imposed at the very beginning in the setting of the boundary value problem we are studying, can be also motivated entirely by the method's logic. Indeed, as we saw in Sections 3.1 and 3.2, the radiation condition allows us to set the algebraic equations for the unknown functions $\Phi_{1,2}$ in the circular part of the contour K which is not covered by the global relation.

In conclusion, we want to mention that in the quarter-space problem the oriented contour consists of the contour K appearing in this paper and a similar, but rotated by

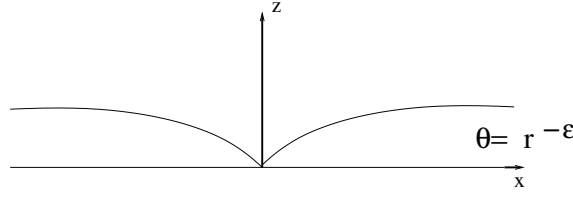


Figure 9: Narrow sector near the surface $z = 0$

ninety degree contour which corresponds to half space problem for $x \geq 0$. As a result , the no-jump section of the circular part of quarter-space contour becomes only a second-quadrant part C_2 instead of C_l (which corresponds to $C_2 + C_3$) in this paper. But a fourth-quadrant part C_4 in the quarter-space problem has a "double" jump because it is included in both $x \geq 0$ and $z \geq 0$ problems. Moreover, we expect Rayleigh wave contributions in the quarter-space problem as "pole/residues" contributions located on non-zero horizontal and vertical parts of the quarter-space problem contour.

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