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HADAMARD–BABICH ANSATZ FOR POINT-SOURCE ELASTIC WAVE EQUATIONS IN VARIABLE MEDIA AT HIGH FREQUENCIES*

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Abstract. Starting from Hadamard's method, we develop Babich's ansatz for the frequency-domain point-source elastic wave equations in an inhomogeneous medium in the high-frequency regime. First, we develop a novel asymptotic series, dubbed Hadamard's ansatz, to form the fundamental solution of the Cauchy problem for the time-domain point-source elastic wave equations in the region close to the source. Using the properties of generalized functions, we derive governing equations for the unknown asymptotics of the ansatz including the travel time functions and dyadic coefficients. In order to derive the initial data of the unknowns at the point source, we further propose a condition for matching Hadamard's ansatz with the homogeneous-medium fundamental solution at the point source. To treat singularity of dyadic coefficients at the source, we then introduce smoother dyadic coefficients. Directly taking the Fourier transform of Hadamard's ansatz in time, we obtain a new ansatz, dubbed Hadamard–Babich ansatz, for the frequency-domain point-source elastic wave equations. To verify the feasibility of the new ansatz, we truncate the ansatz to keep only the first two terms, and we further develop partial-differential-equation–based Eulerian approaches to compute the resulting asymptotic solutions. Numerical examples demonstrate the accuracy of our method.

Key words. eikonal equations, Hadamard–Babich ansatz, elastic wave equations, Eulerian geometrical optics

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1. Introduction.

Green's tensors for elastic wave equations are fundamental in many applications. For a generic inhomogeneous medium, an analytic form of the Green's tensor is in general not available. Direct methods such as finite-difference or finite-element methods are usually employed to numerically compute such tensors. However, at high frequencies, the direct methods suffer from the so-called pollution errors [4], so that it is very costly to directly resolve these highly oscillatory waves. Therefore, we seek alternative methods, such as asymptotic methods or methods of geometrical optics (GO), to carry out scale separation so that we can solve (1.1) at large frequencies. In this article, we propose a novel Hadamard–Babich GO ansatz consisting of an infinite series of dyadics and generalized functions for solving point-source elastic wave equations in an inhomogeneous medium at high frequencies.

We consider the following frequency-domain point-source (FDPS) elastic wave equations in $\mathbb{R}^3$:

$$\rho\omega^2\boldsymbol{G}+(\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{G})+\mu\nabla^2\boldsymbol{G}+\nabla\lambda(\nabla\cdot\boldsymbol{G})+\nabla\mu\times(\nabla\times\boldsymbol{G}))+2(\nabla\mu\cdot\nabla)\boldsymbol{G} = -\boldsymbol{I}\delta(\boldsymbol{r}-\boldsymbol{r}_0), \tag{1.1}$$

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where $\boldsymbol{G} = \boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0, \omega)$ is the so-called Green's tensor at the source $\boldsymbol{r}_0$, $\boldsymbol{I}$ is the 3×3 identity dyad, ω is the frequency, $\rho(\boldsymbol{r})$ is the mass density, $\lambda(\boldsymbol{r})$ and $\mu(\boldsymbol{r})$ are so-called Lamé's stiffness parameters, and $\boldsymbol{r} = (x_1, x_2, x_3)^T$. To avoid cluttered notations, we will suppress ω in the notation of $\boldsymbol{G}$. In the following derivations, we assume that the density and Lamé's parameters are smooth and are constants at infinity, which allow us to impose the Sommerfeld radiation condition at infinity so as to compute outgoing waves only.

An intuitive approach is to use the following Wentzel–Kramers–Brillouin GO ansatz

$$\boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0) = \sum_{l=0}^{\infty} \frac{\bar{\boldsymbol{A}}^l}{\omega^l}(\boldsymbol{r}; \boldsymbol{r}_0) e^{i\omega\tau(\boldsymbol{r}; \boldsymbol{r}_0)}, \tag{1.2}$$

where the unknowns $\bar{\boldsymbol{A}}^l$ and τ are independent of ω. Karal and Keller [13] derived the governing equations for $\bar{\boldsymbol{A}}^l$ and τ without specifying how to initialize these quantities in computation. However, since the overall isotropic elastic waves consist of superpositions of two waves, the compressional (P) wave and the shear (S) wave, the common GO wisdom of keeping only the leading-order term does not suffice to capture correct singularities of the Green's tensor at the source which in turn will affect the overall accuracy of the asymptotic solution, as will be seen in our numerical examples. Our recent work [22] for Maxwell's equations confirmed such defects for inhomogeneous media as well.

Moreover, such drawbacks cannot be easily resolved by using two or more terms in (1.2), since a critical challenge is how to initialize $\bar{\boldsymbol{A}}^l$ at the source $\mathbf{r}_0$. To resolve these issues for solving the scalar point-source Helmholtz equation, Babich in [2] proposed an asymptotic series based on Hankel functions, dubbed Babich's ansatz, to expand the highly oscillatory wavefield. This new ansatz yields a uniform asymptotic solution as $\omega \to \infty$ in the region of space containing the point source but no other caustics. It is worth mentioning that his method of finding such an ansatz is closely bound up with Hadamard's method of forming the fundamental solution of the Cauchy problem for the time-domain point-source acoustic wave equation; details were given in [10] and then were outlined by Courant and Hilbert [5]. While Babich's ansatz allows us to systematically initialize scalar amplitude coefficients for scalar Helmholtz equations, in [25, 16] we have developed systematic computational approaches to implement this ansatz for solving the point-source Helmholtz equation. Nevertheless, as stated by Babich himself on page 105 in [6], his ansatz cannot be trivially extended to the FDPS elastic wave equations (1.1).

To address these initialization issues for the FDPS Maxwell's equations, in [15] we proposed a novel ansatz, dubbed the Babich-like ansatz, based on the spherical Hankel functions, and we have demonstrated that our Babich-like ansatz gives a uniform asymptotic expansion of the underlying solution in the region of space containing a point source but no other caustics. Later on, by using Hadamard's method, we further showed in [17] that our Babich-like ansatz in fact is Babich's ansatz for the FDPS Maxwell's equations, and we have dubbed it the Hadamard–Babich ansatz for the FDPS Maxwell's equations. We remark in passing that, knowing about our work in [15], Babich [3] tried to develop his own asymptotic ansatz for the FDPS Maxwell's equations, but his construction of second- and higher-order approximations faces significant difficulties: his second- and higher-order amplitude coefficients have singularities at the source point, and he further admitted that he could not overcome these difficulties. With the above as the backdrop, we are motivated to develop the Hadamard–Babich ansatz for the FDPS elastic wave equations (1.1).

Consequently, we start by developing a new ansatz based on generalized functions, dubbed Hadamard's ansatz, for solving the time-domain point-source elastic wave equations using Hadamard's method in a region close to the source but no other caustics. Since the singularity induced by the point source is absorbed into the basis functions, this new ansatz enables us to develop systematic approaches to initialize the GO ingredients easily so that source singularities of Green's functions can be captured faithfully.

Based on the properties of generalized functions, we derive eikonal equations for both S and P waves and recursive systems of advection equations for dyadic coefficients. Next, by comparing Hadamard's ansatz with the homogeneous-medium fundamental solution, we propose a matching condition at the source which in turn gives the initial data for the unknown asymptotic coefficients. By taking the Fourier transform of Hadamard's ansatz in time, we immediately obtain the ansatz in the frequency domain, dubbed the Hadamard–Babich ansatz for the FDPS elastic wave equations (1.1). To verify the feasibility of the newly obtained ansatz, we truncate the ansatz to keep only the first two terms so that we are able to compute the resulting asymptotic solutions.

The rest of the article is organized as follows. In section 2, we develop Hadamard's method for elastic wave equations. In this long section, we first derive the fundamental solution for elastic wave equations in a homogeneous medium, and this fundamental solution motivates us to propose a novel generalized-function–based Hadamard's ansatz for solving elastic wave equations. This new ansatz leads us to solve governing equations for asymptotic ingredients of S and P waves. We further propose matching conditions to derive initial conditions so as to initialize those asymptotic ingredients. Section 3 gives the Hadamard–Babich ansatz for the frequency-domain elastic wave equations. Section 4 discusses numerical implementations. Section 5 gives some numerical examples to validate our methodology. Section 6 concludes the paper with some remarks. In Appendix A, we give derivations of some needed ingredients for S and P waves. In Appendix B, we develop Hadamard's method for anisotropic elastic wave equations, and we further show that Hadamard's method for isotropic elastic wave equations developed in the context is a special case of the anisotropic one.

2. Hadamard's method. To apply Hadamard's method, we consider the following Cauchy problem in time of the point-source elastic wave equations:

$$\begin{aligned}(2.1)\qquad &\rho\ddot{\boldsymbol{G}} - (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{G}) - \mu\nabla^2\boldsymbol{G} - \nabla\lambda(\nabla\cdot\boldsymbol{G}) \\ &\qquad -\nabla\mu\times(\nabla\times\boldsymbol{G})) - 2(\nabla\mu\cdot\nabla)\boldsymbol{G} = \boldsymbol{I}\delta(\boldsymbol{r}-\boldsymbol{r}_0)\delta(t),\\ (2.2)\qquad &\boldsymbol{G}(\boldsymbol{r},t;\boldsymbol{r}_0)|_{t<0} = 0,\end{aligned}$$

and we may write it in the form

$$\begin{aligned}(2.3)\qquad &\rho\ddot{G}_{ij} - (\lambda+\mu)G_{kj,ki} - \mu G_{ij,kk} - \lambda_{,i}G_{kj,k} - \mu_{,k}G_{ij,k} - \mu_{,k}G_{kj,i} = \delta_{ij}\delta(\boldsymbol{r}-\boldsymbol{r}_0)\delta(t),\\ (2.4)\qquad &G_{ij} = 0, \text{ for } t<0.\end{aligned}$$

Here, the subscript and comma notation are employed, and $\boldsymbol{r} = (x_1, x_2, x_3)^T$. Specifically, G_{ij} denotes the ijth entry of $\boldsymbol{G}$, $G_{ij,k}$ denotes the x_k-derivative of G_{ij}, so $G_{ij,kl}$ denotes $\partial^2_{x_k x_l}G_{ij}$ for $1 \le i,j,k,l \le 3$. Furthermore, the Einstein summation convention is assumed so that $G_{ij,kk} = \Delta G_{ij}$, and δ_{ij} is the Kronecker delta.

We seek the solution to (2.3) and (2.4) in terms of the generalized functions $f_+^{(k)}$ defined to be

$$f_+^{(k)}(x) = \frac{x_+^k}{k!} \text{ with } x_+ = \begin{cases} x & \text{for } x \geq 0, \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

for $k > -1$; for other values of k, $f_+^{(k)}$is defined by analytic continuation. Also, since

$$f_+^{(k)\prime} = f_+^{(k-1)}, \tag{2.6}$$

$f_+^{(k)}$ can be defined for negative integer values of k by successive differentiation in the sense of distribution. Since $f_+^{(0)}(x) = H(x)$, the Heaviside unit function, $f_+^{(-k-1)}(x) = \delta^{(k)}(x)$, the kth derivative of the δ-function for $k = 0, 1, 2, \ldots$. For further discussion of $f_+^{(k)}$ and related functions, see Chapter I, sections 3.4 and 3.5 of [9].

Using Duhamel's principle (pp. 202 and 729–730 in [5]), we may rewrite (2.3) and (2.4) for $t > 0$ as

$$\rho \ddot{G}_{ij} - (\lambda + \mu) G_{kj,ki} - \mu G_{ij,kk} - \lambda_{,i} G_{kj,k} - \mu_{,k} G_{ij,k} - \mu_{,k} G_{kj,i} = 0 \tag{2.7}$$

with initial conditions

$$G_{ij}(\boldsymbol{r}, 0; \boldsymbol{r}_0) = 0, \tag{2.8}$$

$$\dot{G}_{ij}(\boldsymbol{r}, 0; \boldsymbol{r}_0) = \frac{1}{\rho_0} \delta_{ij} \delta(\boldsymbol{r} - \boldsymbol{r}_0), \tag{2.9}$$

where ρ_0 is the density at the source point $\boldsymbol{r}_0$.

2.1. The case of a homogeneous medium. In the case when all elastic parameters are constants, we may write the isotropic elastic wave equations in a homogeneous medium in the form

$$\rho \ddot{G}_{ij} - (\lambda + \mu) G_{kj,ki} - \mu G_{ij,kk} = 0 \tag{2.10}$$

with initial conditions

$$G_{ij}(\boldsymbol{r}, 0; \boldsymbol{r}_0) = 0, \tag{2.11}$$

$$\dot{G}_{ij}(\boldsymbol{r}, 0; \boldsymbol{r}_0) = \frac{1}{\rho} \delta_{ij} \delta(\boldsymbol{r} - \boldsymbol{r}_0). \tag{2.12}$$

Following a suggestion of Friedlander [8] as we have done in [17], we will seek a solution to (2.10)–(2.12) in the form

$$G_{ij} = \varphi_{,ij} + \delta_{ij} \psi_{,kk} - \psi_{,ij} = \delta_{ij} \psi_{,kk} + (\varphi_{,ij} - \psi_{,ij}), \tag{2.13}$$

where φ and ψ are to be determined. On substituting this into (2.10), we obtain

$$\partial_i \partial_j [\rho \ddot{\varphi} - (\lambda + 2\mu) \Delta \varphi] + [\delta_{ij} \Delta - \partial_i \partial_j][\rho \ddot{\psi} - \mu \Delta \psi] = 0. \tag{2.14}$$

Let us now decompose the initial condition (2.12) in a similar way using the fact

$$\delta(\boldsymbol{r} - \boldsymbol{r}_0) = \Delta \left(\frac{-1}{4\pi r} \right), \tag{2.15}$$

where $r = |\boldsymbol{r} - \boldsymbol{r}_0|$. Thus we may write

$$\delta_{ij}\delta(\boldsymbol{r} - \boldsymbol{r}_0) = \partial_i\partial_j\left(\frac{-1}{4\pi r}\right) + [\delta_{ij}\Delta - \partial_i\partial_j]\left(\frac{-1}{4\pi r}\right). \tag{2.16}$$

Consequently, we see that G_{ij} given by (2.13) will satisfy (2.10)–(2.12) if

$$\rho\ddot{\varphi} - (\lambda + 2\mu)\Delta\varphi = 0, \quad t > 0, \tag{2.17}$$

$$\varphi = 0, \ \dot{\varphi} = \frac{-1}{4\rho\pi r}, \quad t = 0, \tag{2.18}$$

and

$$\rho\ddot{\psi} - \mu\Delta\psi = 0, \quad t > 0, \tag{2.19}$$

$$\psi = 0, \ \dot{\psi} = \frac{-1}{4\rho\pi r}, \quad t = 0. \tag{2.20}$$

We easily verify that (2.17)–(2.20) are satisfied by

$$\varphi = \frac{R(t - \gamma^P r)}{4\pi\rho r} - \frac{t}{4\pi\rho r} = \begin{cases} \frac{-\gamma^P}{4\pi\rho}, & \gamma^P r \le t, \\ \frac{-t}{4\pi\rho r}, & \gamma^P r > t, \end{cases} \tag{2.21}$$

$$\psi = \frac{R(t - \gamma^S r)}{4\pi\rho r} - \frac{t}{4\pi\rho r} = \begin{cases} \frac{-\gamma^S}{4\pi\rho}, & \gamma^S r \le t, \\ \frac{-t}{4\pi\rho r}, & \gamma^S r > t, \end{cases} \tag{2.22}$$

where

$$R(\xi) = \begin{cases} \xi, & \xi > 0, \\ 0, & \xi \le 0, \end{cases} \tag{2.23}$$

and

$$\gamma^S = \sqrt{\frac{\rho}{\mu}} \quad \text{and} \quad \gamma^P = \sqrt{\frac{\rho}{\lambda + 2\mu}}. \tag{2.24}$$

Carrying out the differentiations indicated in (2.13), we finally arrive at

$$\begin{aligned} G_{ij} =& \frac{{\gamma^S}^2(\delta_{ij} - t_i t_j)}{4\pi\rho r}\delta(t - \gamma^S r) + \frac{(\delta_{ij} - 3t_i t_j)}{4\pi\rho}\left[\frac{\gamma^S}{r^2}H(t - \gamma^S r) + \frac{1}{r^3}R(t - \gamma^S r)\right] \\ &+ \frac{{\gamma^P}^2 t_i t_j}{4\pi\rho r}\delta(t - \gamma^P r) - \frac{(\delta_{ij} - 3t_i t_j)}{4\pi\rho}\left[\frac{\gamma^P}{r^2}H(t - \gamma^P r) + \frac{1}{r^3}R(t - \gamma^P r)\right], \end{aligned} \tag{2.25}$$

where $r = |\boldsymbol{r} - \boldsymbol{r}_0|$ and $\boldsymbol{t} = (\boldsymbol{r} - \boldsymbol{r}_0)/r$.

Now we need to figure out the relation between G_{ij} and the generalized functions $f_+^{(k)}$. It turns out that the second-order time derivative of G_{ij} has a simple relation with $f_+^{(-3)}$ and $f_+^{(-2)}$. In fact, adopting similar techniques as used in proving Proposition 2.1 in [17], we can obtain

$$\ddot{G}_{ij} = \ddot{G}_{ij}^S + \ddot{G}_{ij}^P, \tag{2.26}$$

where

$$\ddot{G}_{ij}^S = \frac{{\gamma^S}^3}{\rho\pi}\left[2{\gamma^S}^2 r^2(\delta_{ij} - t_i t_j)f_+^{(-3)}(t^2 - {\gamma^S}^2 r^2) - 2\delta_{ij}f_+^{(-2)}(t^2 - {\gamma^S}^2 r^2)\right] \tag{2.27}$$

and

$$\ddot{G}_{ij}^P = \frac{{\gamma^P}^3}{\rho\pi}\left[2{\gamma^P}^2 r^2 t_i t_j f_+^{(-3)}(t^2 - {\gamma^P}^2 r^2) - \delta_{ij} f_+^{(-2)}(t^2 - {\gamma^P}^2 r^2)\right]. \tag{2.28}$$

Here the superscripts S and P refer to the S and P waves, respectively, and γ^S and γ^P are the S and P wave slowness, respectively, as will become apparent later; the vector $\boldsymbol{t} = (t_1, t_2, t_3)^T$ is the unit tangent vector to the S or P ray as the case may be, and the S and P rays coincide in this special case of a uniform medium.

To be convenient, we rewrite the above two functions in the following concise forms:

$$\ddot{\boldsymbol{G}}_{hom}^S = \frac{{\gamma^S}^3}{\rho\pi}\left[2{\gamma^S}^2 r^2 (\boldsymbol{I} - \boldsymbol{t}\boldsymbol{t}^T) f_+^{(-3)}(t^2 - {\gamma^S}^2 r^2) - 2\boldsymbol{I} f_+^{(-2)}(t^2 - {\gamma^S}^2 r^2)\right],$$

where $\boldsymbol{t}$ is the unit tangent vector to the S ray, and

$$\ddot{\boldsymbol{G}}_{hom}^P = \frac{{\gamma^P}^3}{\rho\pi}\left[2{\gamma^P}^2 r^2 \boldsymbol{t}\boldsymbol{t}^T f_+^{(-3)}(t^2 - {\gamma^P}^2 r^2) - \boldsymbol{I} f_+^{(-2)}(t^2 - {\gamma^P}^2 r^2)\right],$$

where $\boldsymbol{t}$ is the unit tangent vector to the P ray.

We remark in passing that in the above derivation, we have constructed functions φ and ψ to represent the P-wave and the S-wave related information, respectively, in the spatial-temporal domain, so that we are computing outgoing waves at infinity. In the time domain, since γ_S and γ_P are positive, (2.25) indicates that, as $t > 0$ increases, the singularities of G move along either the outgoing cone $\{r = \gamma_S^{-1} t\}$ or along the outgoing cone $\{r = \gamma_P^{-1} t\}$ so that the waves are indeed going outward. Taking the Fourier transform in time of the Green's function G_{ij} in (2.25), we can check that the usual Sommerfeld radiation condition for elastic waves holds so that outgoing waves are guaranteed at infinity [1].

2.2. The case of an inhomogeneous medium. When the elastic coefficients vary with $\boldsymbol{r}$, the formulas (2.27) and (2.28) for the constant case motivate us to seek the solution of (2.1) in the following form of asymptotic series

$$\ddot{\boldsymbol{G}}(\boldsymbol{r}, t; \boldsymbol{r}_0) = \sum_{l=0}^{\infty} \boldsymbol{A}^l(\boldsymbol{r}; \boldsymbol{r}_0) f_+^{(-3+l)}(t^2 - \mathcal{T}(\boldsymbol{r}; \boldsymbol{r}_0)), \tag{2.29}$$

or in subscript notation

$$\ddot{G}_{ij} = \sum_{l=0}^{\infty} A_{ij}^l f_+^{(-3+l)}(t^2 - \mathcal{T}), \tag{2.30}$$

where $\mathcal{T}(\boldsymbol{r}; \boldsymbol{r}_0) = \tau^2(\boldsymbol{r}; \boldsymbol{r}_0)$, and $\tau(\boldsymbol{r}; \boldsymbol{r}_0)$ is the travel time from the source $\boldsymbol{r}_0$ to $\boldsymbol{r}$. We shall define $\boldsymbol{A}^l(\boldsymbol{r}; \boldsymbol{r}_0) \equiv 0$ for $l < 0$.

Here, since the medium properties do not depend upon time, $\boldsymbol{A}^l(\boldsymbol{r}; \boldsymbol{r}_0)$ are assumed to be time independent. This assumption can be justified from two perspectives. On one hand, we could prove this by arguing that one would get the same solution delayed if the source were delayed and so the wave operator commutes with time shifts, where it is clear that ∂_t can be passed through the wave operator when the medium properties do not depend upon time. On the other hand, since $\ddot{\boldsymbol{G}}$ in a homogeneous medium can be expressed as the sum of terms which separates time and spatial variables, we believe that $\ddot{\boldsymbol{G}}$ in an inhomogeneous medium in a neighborhood of the source point should inherit such a property so that we seek unknowns $\boldsymbol{A}^l$ depending upon spatial variables only.

To derive the governing equations for $\boldsymbol{A}^l$ and $\mathcal{T}$, we need to use two simple properties of $f_+^{(k)}(x)$,

$$f_+^{(k)\prime}(x) = f_+^{(k-1)}(x), \tag{2.31}$$

$$x f_+^{(k)}(x) = (k+1) f_+^{(k+1)}(x). \tag{2.32}$$

Taking double time derivatives of (2.7) yields

$$\rho \partial_t^4 G_{ij} = (\lambda+\mu)\ddot{G}_{kj,ki} + \mu \ddot{G}_{ij,kk} + \lambda_{,i}\ddot{G}_{kj,k} + \mu_{,k}\ddot{G}_{ij,k} + \mu_{,k}\ddot{G}_{kj,i}. \tag{2.33}$$

To write (2.33) in terms of the generalized functions $f_+^{(k)}$, we compute the following double time derivative of $\ddot{\boldsymbol{G}}$,

$$\begin{aligned} \partial_t^4 G_{ij} &= \sum_{l=0}^{\infty} A_{ij}^l \partial_t^2 f_+^{(-3+l)}(t^2-\mathcal{T}) \\ &= \sum_{l=0}^{\infty} 4t^2 A_{ij}^l f_+^{(-5+l)}(t^2-\mathcal{T}) + 2A_{ij}^l f_+^{(-4+l)}(t^2-\mathcal{T}) \\ &= \sum_{l=0}^{\infty} 4\mathcal{T} A_{ij}^l f_+^{(-5+l)}(t^2-\mathcal{T}) + 2(2l-7)A_{ij}^l f_+^{(-4+l)}(t^2-\mathcal{T}) \\ &= \sum_{l=0}^{\infty} f_+^{(-5+l)}(t^2-\mathcal{T})[4\mathcal{T}A_{ij}^l + 2(2l-9)A_{ij}^{l-1}], \end{aligned} \tag{2.34}$$

and the following spatial derivatives of $\ddot{\boldsymbol{G}}$,

$$\begin{aligned} \ddot{G}_{ij,k} &= \sum_{l=0}^{\infty} -\mathcal{T}_{,k} A_{ij}^l f_+^{(-4+l)}(t^2-\mathcal{T}) + A_{ij,k}^l f_+^{(-3+l)}(t^2-\mathcal{T}) \\ &= \sum_{l=0}^{\infty} f_+^{(-5+l)}(t^2-\mathcal{T})[-\mathcal{T}_{,k}A_{ij}^{l-1} + A_{ij,k}^{l-2}], \end{aligned} \tag{2.35}$$

$$\ddot{G}_{kj,i} = \sum_{l=0}^{\infty} f_+^{(-5+l)}(t^2-\mathcal{T})[-\mathcal{T}_{,i}A_{kj}^{l-1} + A_{kj,i}^{l-2}], \tag{2.36}$$

$$\ddot{G}_{kj,k} = \sum_{l=0}^{\infty} f_+^{(-5+l)}(t^2-\mathcal{T})[-\mathcal{T}_{,k}A_{kj}^{l-1} + A_{kj,k}^{l-2}], \tag{2.37}$$

(2.38)
$$\begin{aligned} \ddot{G}_{kj,ki} &= \sum_{l=0}^{\infty} \mathcal{T}_{,k}\mathcal{T}_{,i} A_{kj}^l f_+^{(-5+l)}(t^2-\mathcal{T}) - \mathcal{T}_{,k} A_{kj,i}^l f_+^{(-4+l)}(t^2-\mathcal{T}) \\ &\quad - \mathcal{T}_{,i} A_{kj,k}^l f_+^{(-4+l)}(t^2-\mathcal{T}) - \mathcal{T}_{,ki} A_{kj}^l f_+^{(-4+l)}(t^2-\mathcal{T}) + A_{kj,ki}^l f_+^{(-3+l)}(t^2-\mathcal{T}) \\ &= \sum_{l=0}^{\infty} f_+^{(-5+l)}(t^2-\mathcal{T})[\mathcal{T}_{,k}\mathcal{T}_{,i}A_{kj}^l - \mathcal{T}_{,k}A_{kj,i}^{l-1} - \mathcal{T}_{,i}A_{kj,k}^{l-1} - \mathcal{T}_{,ki}A_{kj}^{l-1} + A_{kj,ki}^{l-2}], \end{aligned}$$

(2.39)
$$\ddot{G}_{ij,kk} = \sum_{l=0}^{\infty} f_+^{(-5+l)}(t^2 - \mathfrak{T})[\mathfrak{T}_{,k}\mathfrak{T}_{,k}A_{ij}^l - \mathfrak{T}_{,k}A_{ij,k}^{l-1} - \mathfrak{T}_{,k}A_{ij,k}^{l-1} - \mathfrak{T}_{,kk}A_{ij}^{l-1} + A_{ij,kk}^{l-2}].$$

Inserting the above derivatives into (2.33) and according to the meaning of an asymptotic series being zero, we vanish the coefficients of $f_+^{(-5+l)}$ successively to obtain

$$\begin{aligned} 0 = {} & 4\rho\mathfrak{T}A_{ij}^l - (\lambda+\mu)\mathfrak{T}_{,k}\mathfrak{T}_{,i}A_{kj}^l - \mu\mathfrak{T}_{,k}\mathfrak{T}_{,k}A_{ij}^l \\ & + 2\rho(2l-9)A_{ij}^{l-1} + (\lambda+\mu)\mathfrak{T}_{,i}A_{kj,k}^{l-1} + (\lambda+\mu)\mathfrak{T}_{,k}A_{kj,i}^{l-1} + (\lambda+\mu)\mathfrak{T}_{,ki}A_{kj}^{l-1} \\ & + 2\mu\mathfrak{T}_{,k}A_{ij,k}^{l-1} + \mu\mathfrak{T}_{,kk}A_{ij}^{l-1} + \lambda_{,i}\mathfrak{T}_{,k}A_{kj}^{l-1} + \mu_{,k}\mathfrak{T}_{,k}A_{ij}^{l-1} + \mu_{,k}\mathfrak{T}_{,i}A_{kj}^{l-1} \\ & - (\lambda+\mu)A_{kj,ki}^{l-2} - \mu A_{ij,kk}^{l-2} - \lambda_{,i}A_{kj,k}^{l-2} - \mu_{,k}A_{ij,k}^{l-2} - \mu_{,k}A_{kj,i}^{l-2}. \end{aligned} \tag{2.40}$$

Setting $l = 0$ in (2.40) and remembering that A_{ij}^l is zero for $l < 0$, we obtain

$$4\rho\mathfrak{T}A_{ij}^0 - (\lambda+\mu)\mathfrak{T}_{,k}\mathfrak{T}_{,i}A_{kj}^0 - \mu\mathfrak{T}_{,k}\mathfrak{T}_{,k}A_{ij}^0 = 0 \tag{2.41}$$

or, in vector notation,

$$(4\rho\mathfrak{T} - \mu|\nabla\mathfrak{T}|^2)\boldsymbol{A}_j^0 = (\lambda+\mu)\nabla\mathfrak{T}(\nabla\mathfrak{T}\cdot\boldsymbol{A}_j^0). \tag{2.42}$$

Here $\boldsymbol{A}_j^0$ denotes the jth column of the coefficient matrix $\boldsymbol{A}^0$, which means $\boldsymbol{A}^0 = (\boldsymbol{A}_1^0, \boldsymbol{A}_2^0, \boldsymbol{A}_3^0)$.

Two cases arise, and we will call them Case I and Case II.

Case I, both sides of (2.42) are zero; since $\boldsymbol{A}_j^0$ for $1 \le j \le 3$ is assumed to be nonzero, we have

$$\nabla\mathfrak{T}\cdot\boldsymbol{A}_j^0 = 0, \text{ and } 4\rho\mathfrak{T} - \mu|\nabla\mathfrak{T}|^2 = 0; \tag{2.43}$$

Case II, both sides of (2.42) are nonzero; then we have

$$\boldsymbol{A}_j^0 \parallel \nabla\mathfrak{T}, \text{ and } 4\rho\mathfrak{T} - (\lambda+2\mu)|\nabla\mathfrak{T}|^2 = 0, \tag{2.44}$$

where the latter equation can be obtained by taking the scalar (or dot) product of (2.42) with $\nabla\mathfrak{T}$.

2.3. Case I, the S wave: $\boldsymbol{\nabla\mathfrak{T}\cdot A_j^0 = 0}$ and $\boldsymbol{4\rho\mathfrak{T} - \mu|\nabla\mathfrak{T}|^2 = 0}$.

Here

$$|\nabla\mathfrak{T}|^2 = \frac{4\rho\mathfrak{T}}{\mu}. \tag{2.45}$$

Let us set $\mathfrak{T} = \tau^2$. Then we see that

$$|\nabla\tau|^2 = \frac{\rho}{\mu}, \tag{2.46}$$

the usual eikonal equation for the S-wave travel time, and so $\mathfrak{T}$ is the square of the S-wave travel time τ. The rays are obtained in the process of solving (2.45) or (2.46) by the method of characteristics, which gives

$$\frac{dx_i}{p_i} = \frac{d\tau}{\frac{\rho}{\mu}} = \frac{dp_i}{\frac{1}{2}(\frac{\rho}{\mu})_{,i}}, \tag{2.47}$$

where $p_i = \tau_{,i}$ and from (2.46),

$$\frac{1}{2}\left(|p|^2 - \frac{\rho}{\mu}\right) = 0. \tag{2.48}$$

In our development whenever convenient we shall use $\mathcal{T}$ rather than the more usual τ. The advantage is that the argument $\Gamma = t^2 - \mathcal{T} = t^2 - \tau^2$ is linear in $\mathcal{T}$ but quadratic in τ so that in using $\mathcal{T}$, differentiation with respect to x_j does not generate factors of 2 and τ, leaving the resulting formula easier to read.

For reference here we repeat (2.40), omitting the first and third terms as they cancel by (2.45). Thus

$$\begin{aligned}0 =& -(\lambda+\mu)\mathcal{T}_{,k}\mathcal{T}_{,i}A^l_{kj}\\ &+ 2\rho(2l-9)A^{l-1}_{ij} + (\lambda+\mu)\mathcal{T}_{,i}A^{l-1}_{kj,k} + (\lambda+\mu)\mathcal{T}_{,k}A^{l-1}_{kj,i} + (\lambda+\mu)\mathcal{T}_{,ki}A^{l-1}_{kj}\\ &+ 2\mu\mathcal{T}_{,k}A^{l-1}_{ij,k} + \mu\mathcal{T}_{,kk}A^{l-1}_{ij} + \lambda_{,i}\mathcal{T}_{,k}A^{l-1}_{kj} + \mu_{,k}\mathcal{T}_{,k}A^{l-1}_{ij} + \mu_{,k}\mathcal{T}_{,i}A^{l-1}_{kj}\\ &- (\lambda+\mu)A^{l-2}_{kj,ki} - \mu A^{l-2}_{ij,kk} - \lambda_{,i}A^{l-2}_{kj,k} - \mu_{,k}A^{l-2}_{ij,k} - \mu_{,k}A^{l-2}_{kj,i}.\end{aligned} \tag{2.49}$$

2.3.1. S-wave transport equations for $\boldsymbol{A}^0$ and $\boldsymbol{A}^1$. First, we derive the governing equation for $\boldsymbol{A}^0$. Setting $l = 1$ in (2.49) and knowing that $A^{-1}_{ij} = 0$, we get

$$\begin{aligned}0 =& -(\lambda+\mu)\mathcal{T}_{,k}\mathcal{T}_{,i}A^1_{kj} - 14\rho A^0_{ij} + (\lambda+\mu)\mathcal{T}_{,i}A^0_{kj,k}\\ &+ 2\mu\mathcal{T}_{,k}A^0_{ij,k} + \mu\mathcal{T}_{,kk}A^0_{ij} + \mu_{,k}\mathcal{T}_{,k}A^0_{ij} + \mu_{,k}\mathcal{T}_{,i}A^0_{kj},\end{aligned} \tag{2.50}$$

where we have made use of $\mathcal{T}_{,k}A^0_{kj} = 0$ by $\nabla\mathcal{T}\cdot\boldsymbol{A}^0_j = 0$, and its derivative

$$(\mathcal{T}_{,k}A^0_{kj})_{,i} = \mathcal{T}_{,k}A^0_{kj,i} + \mathcal{T}_{,ki}A^0_{kj} = 0.$$

Form the scalar product of (2.50) with $\mathcal{T}_{,i}$ to get

$$0 = -(\lambda+\mu)\mathcal{T}_{,i}\mathcal{T}_{,i}\mathcal{T}_{,k}A^1_{kj} + (\lambda+\mu)\mathcal{T}_{,i}\mathcal{T}_{,i}A^0_{kj,k} + 2\mu\mathcal{T}_{,k}\mathcal{T}_{,i}A^0_{ij,k} + \mu_{,k}\mathcal{T}_{,i}\mathcal{T}_{,i}A^0_{kj}. \tag{2.51}$$

But

$$\begin{aligned}0 &= 2(\mathcal{T}_{,k}\mathcal{T}_{,i}A^0_{ij})_{,k} = 2(\mathcal{T}_{,kk}\mathcal{T}_{,i}A^0_{ij} + \mathcal{T}_{,k}\mathcal{T}_{,ik}A^0_{ij} + \mathcal{T}_{,k}\mathcal{T}_{,i}A^0_{ij,k})\\ &= (\mathcal{T}_{,k}\mathcal{T}_{,k})_{,i}A^0_{ij} + 2\mathcal{T}_{,k}\mathcal{T}_{,i}A^0_{ij,k}\\ &= \left(\frac{4\rho\mathcal{T}}{\mu}\right)_{,i}A^0_{ij} + 2\mathcal{T}_{,k}\mathcal{T}_{,i}A^0_{ij,k},\end{aligned} \tag{2.52}$$

so that

$$2\mu\mathcal{T}_{,i}\mathcal{T}_{,k}A^0_{ij,k} = -\mu\left(\frac{4\rho\mathcal{T}}{\mu}\right)_{,i}A^0_{ij}. \tag{2.53}$$

Thus (2.51) becomes

$$\begin{aligned}(\lambda+\mu)\frac{4\rho\mathcal{T}}{\mu}\mathcal{T}_{,k}A^1_{kj} &= (\lambda+\mu)\frac{4\rho\mathcal{T}}{\mu}A^0_{kj,k} - \mu\left(\frac{4\rho\mathcal{T}}{\mu}\right)_{,i}A^0_{ij} + \mu_{,k}\frac{4\rho\mathcal{T}}{\mu}A^0_{kj}\\ &= (\lambda+\mu)\frac{4\rho\mathcal{T}}{\mu}A^0_{kj,k} - \mu^2\left(\frac{4\rho\mathcal{T}}{\mu^2}\right)_{,k}A^0_{kj},\end{aligned} \tag{2.54}$$

so that we can obtain the following relationship between A^0_{kj} and $\mathcal{T}_{,k}A^1_{kj}$,

$$\mathcal{T}_{,k}A^1_{kj} = A^0_{kj,k} - \frac{\mu^3}{\rho(\lambda+\mu)}\left(\frac{\rho}{\mu^2}\right)_{,k} A^0_{kj}. \tag{2.55}$$

Inserting (2.55) into (2.50) results in the following transport equation for A^0_{ij},

$$\mathcal{T}_{,k}A^0_{ij,k} + \frac{1}{2\mu}((\mu\mathcal{T}_{,k})_{,k} - 14\rho)A^0_{ij} + \frac{\mu}{2\rho}\left(\frac{\rho}{\mu}\right)_{,k} A^0_{kj}\mathcal{T}_{,i} = 0. \tag{2.56}$$

Or, in vector notation,

$$(\nabla\mathcal{T}\cdot\nabla)\boldsymbol{A}^0 + \frac{1}{2\mu}\left(\nabla\cdot(\mu\nabla\mathcal{T}) - 14\rho\right)\boldsymbol{A}^0 + \frac{\mu}{2\rho}\nabla\mathcal{T}\left[\nabla\left(\frac{\rho}{\mu}\right)\cdot\boldsymbol{A}^0\right] = 0, \tag{2.57}$$

where $\nabla(\frac{\rho}{\mu})\cdot\boldsymbol{A}^0 = ((\frac{\rho}{\mu})_{,i}A^0_{i1}, (\frac{\rho}{\mu})_{,i}A^0_{i2}, (\frac{\rho}{\mu})_{,i}A^0_{i3})$ results in a row vector.

Next, we will derive the governing equation for the amplitude coefficient $\boldsymbol{A}^1$. Setting $l = 2$ in (2.49), all terms exist, so that we can get a relationship between $\nabla\mathcal{T}\cdot\boldsymbol{A}^2$, $\boldsymbol{A}^1$, and $\boldsymbol{A}^0$:

$$\begin{aligned}0 = {} & (\lambda+\mu)(-\mathcal{T}_{,k}A^2_{kj} + A^1_{kj,k})\mathcal{T}_{,i} - 10\rho A^1_{ij} + (\lambda+\mu)(\mathcal{T}_{,k}A^1_{kj})_{,i} \\ & + 2\mu\mathcal{T}_{,k}A^1_{ij,k} + \mu\mathcal{T}_{,kk}A^1_{ij} + \lambda_{,i}\mathcal{T}_{,k}A^1_{kj} + \mu_{,k}\mathcal{T}_{,k}A^1_{ij} + \mu_{,k}\mathcal{T}_{,i}A^1_{kj} \\ & - (\lambda+\mu)A^0_{kj,ki} - \mu A^0_{ij,kk} - \lambda_{,i}A^0_{kj,k} - \mu_{,k}A^0_{ij,k} - \mu_{,k}A^0_{kj,i}.\end{aligned} \tag{2.58}$$

To make our formulas easy to read, we introduce some notations. By the relation (2.55), we define

$$a^0_j \equiv \mathcal{T}_{,k}A^1_{kj} = A^0_{kj,k} - \frac{\mu^3}{\rho(\lambda+\mu)}\left(\frac{\rho}{\mu^2}\right)_{,k} A^0_{kj}, \tag{2.59}$$

so that $\boldsymbol{a}^0 = (a^0_1, a^0_2, a^0_3)$ is a row vector depending only on A^0_{kj}, and

$$B^0_{ij} \equiv (\lambda+\mu)A^0_{kj,ki} + \mu A^0_{ij,kk} + \lambda_{,i}A^0_{kj,k} + \mu_{,k}A^0_{ij,k} + \mu_{,k}A^0_{kj,i}. \tag{2.60}$$

Next we take the scalar product of (2.58) with $\mathcal{T}_{,i}$ to get

$$\begin{aligned}0 = {} & (\lambda+\mu)(-\mathcal{T}_{,k}A^2_{kj} + A^1_{kj,k})\mathcal{T}_{,i}\mathcal{T}_{,i} - 10\rho a^0_j + (\lambda+\mu)a^0_{j,i}\mathcal{T}_{,i} + 2\mu\mathcal{T}_{,k}A^1_{ij,k}\mathcal{T}_{,i} \\ & + \mu\mathcal{T}_{,kk}a^0_j + \lambda_{,i}a^0_j\mathcal{T}_{,i} + \mu_{,k}\mathcal{T}_{,k}a^0_j + \mu_{,k}\mathcal{T}_{,i}A^1_{kj}\mathcal{T}_{,i} - B^0_{ij}\mathcal{T}_{,i}.\end{aligned} \tag{2.61}$$

Introduce

$$b^0_j \equiv 10\rho a^0_j - (\lambda+\mu)\mathcal{T}_{,i}a^0_{j,i} - \mu\mathcal{T}_{,kk}a^0_j - \lambda_{,i}\mathcal{T}_{,i}a^0_j - \mu_{,i}\mathcal{T}_{,i}a^0_j + B^0_{ij}\mathcal{T}_{,i}, \tag{2.62}$$

where $\boldsymbol{b}^0 = (b^0_1, b^0_2, b^0_3)$ is also a row vector. Consequently, (2.61) becomes

$$\begin{aligned}0 = {} & (\lambda+\mu)(-\mathcal{T}_{,k}A^2_{kj})\mathcal{T}_{,i}\mathcal{T}_{,i} + (\lambda+\mu)A^1_{kj,k}\mathcal{T}_{,i}\mathcal{T}_{,i} \\ & + 2\mu\mathcal{T}_{,k}A^1_{ij,k}\mathcal{T}_{,i} + \mu_{,k}\mathcal{T}_{,i}A^1_{kj}\mathcal{T}_{,i} - b^0_j.\end{aligned} \tag{2.63}$$

Inserting (2.63) into (2.58) and employing the following relationship,

$$2\mu\mathcal{T}_{,k}A^1_{ij,k}\mathcal{T}_{,i} = 2\mu\mathcal{T}_{,k}(\mathcal{T}_{,i}A^1_{ij})_{,k} - 4\rho\mathcal{T}_{,i}A^1_{ij} - 4\mu\mathcal{T}\left(\frac{\rho}{\mu}\right)_{,i} A^1_{ij}, \tag{2.64}$$

we obtain the following transport equation for A^1_{ij},

$$\mathcal{T}_{,k}A^1_{ij,k} + \frac{1}{2\mu}((\mu\mathcal{T}_{,k})_{,k} - 10\rho)A^1_{ij} + \frac{\mu}{2\rho}A^1_{kj}\left(\frac{\rho}{\mu}\right)_{,k}\mathcal{T}_{,i} = R^0_{ij}, \tag{2.65}$$

where

$$R^0_{ij} \equiv -\frac{a^0_j\mathcal{T}_{,i}}{2\mathcal{T}} + \frac{\mu a^0_{j,k}\mathcal{T}_{,k}\mathcal{T}_{,i}}{4\rho\mathcal{T}} - \frac{b^0_j\mathcal{T}_{,i}}{8\rho\mathcal{T}} - \frac{(\lambda+\mu)a^0_{j,i}}{2\mu} - \frac{\lambda_{,i}a^0_j}{2\mu} + \frac{B^0_{ij}}{2\mu}. \tag{2.66}$$

Or, in vector notation,

$$(\nabla\mathcal{T}\cdot\nabla)\boldsymbol{A}^1 + \frac{1}{2\mu}[\nabla\cdot(\mu\nabla\mathcal{T}) - 10\rho]\boldsymbol{A}^1 + \frac{\mu}{2\rho}\nabla\mathcal{T}\left[\nabla\left(\frac{\rho}{\mu}\right)\cdot\boldsymbol{A}^1\right] = \boldsymbol{R}^0, \tag{2.67}$$

where

$$\boldsymbol{R}^0 \equiv -\frac{\nabla\mathcal{T}\boldsymbol{a}^0}{2\mathcal{T}} + \frac{\mu\nabla\mathcal{T}[(\nabla\mathcal{T}\cdot\nabla)\boldsymbol{a}^0]}{4\rho\mathcal{T}} - \frac{\nabla\mathcal{T}\boldsymbol{b}^0}{8\rho\mathcal{T}} - \frac{\nabla\lambda\boldsymbol{a}^0}{2\mu} - \frac{(\lambda+\mu)\nabla\boldsymbol{a}^0}{2\mu} + \frac{\boldsymbol{B}^0}{2\mu}. \tag{2.68}$$

Here $\nabla\mathcal{T}\boldsymbol{a}^0 = (\mathcal{T}_{,i}a^0_j)_{3\times3}$ and $\nabla\boldsymbol{a}^0 = (a^0_{j,i})_{3\times3}$ are 3 by 3 matrices.

2.3.2. S-wave transport equations for general $\boldsymbol{A^l}$. By proceeding in a similar manner to the above, we can derive the governing equations for other $\boldsymbol{A}^l$ when $l \geq 2$. Hence we may conclude that the general amplitude coefficients $\boldsymbol{A}^l$ satisfy

$$(\nabla\mathcal{T}\cdot\nabla)\boldsymbol{A}^l + \frac{1}{2\mu}[\nabla\cdot(\mu\nabla\mathcal{T}) + (4l-14)\rho]\boldsymbol{A}^l + \frac{\mu}{2\rho}\nabla\mathcal{T}\left[\nabla\left(\frac{\rho}{\mu}\right)\cdot\boldsymbol{A}^l\right] = \boldsymbol{R}^{l-1}, \tag{2.69}$$

where

$$\begin{aligned}\boldsymbol{R}^{l-1} \equiv & -\frac{\nabla\mathcal{T}\boldsymbol{a}^{l-1}}{2\mathcal{T}} + \frac{\mu\nabla\mathcal{T}[(\nabla\mathcal{T}\cdot\nabla)\boldsymbol{a}^{l-1}]}{4\rho\mathcal{T}} - \frac{\nabla\mathcal{T}\boldsymbol{b}^{l-1}}{8\rho\mathcal{T}} - \frac{\nabla\lambda\boldsymbol{a}^{l-1}}{2\mu} \\ & - \frac{(\lambda+\mu)\nabla\boldsymbol{a}^{l-1}}{2\mu} + \frac{\boldsymbol{B}^{l-1}}{2\mu}.\end{aligned} \tag{2.70}$$

Here $\boldsymbol{B}^{l-1}$, $\boldsymbol{a}^{l-1}$, and $\boldsymbol{b}^{l-1}$ are given by

$$\begin{aligned}\boldsymbol{B}^{l-1} \equiv & (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{A}^{l-1}) + \mu\nabla^2\boldsymbol{A}^{l-1} + \nabla\lambda(\nabla\cdot\boldsymbol{A}^{l-1}) \\ & + \nabla\mu\times(\nabla\times\boldsymbol{A}^{l-1}) + 2(\nabla\mu\cdot\nabla)\boldsymbol{A}^{l-1},\end{aligned} \tag{2.71}$$

$$\begin{aligned}\boldsymbol{a}^{l-1} & \equiv \nabla\mathcal{T}\cdot\boldsymbol{A}^l \\ & = \nabla\cdot\boldsymbol{A}^{l-1} + \frac{1}{\lambda+\mu}\nabla\mu\cdot\boldsymbol{A}^{l-1} + \frac{2\mu^2}{(\lambda+\mu)4\rho\mathcal{T}}\nabla\mathcal{T}\cdot[(\nabla\mathcal{T}\cdot\nabla)\boldsymbol{A}^{l-1}] - \frac{\mu\boldsymbol{b}^{l-2}}{(\lambda+\mu)4\rho\mathcal{T}},\end{aligned} \tag{2.72}$$

$$\begin{aligned}\boldsymbol{b}^{l-1} \equiv & 2\rho(-2l+7)\boldsymbol{a}^{l-1} - (\lambda+\mu)(\nabla\mathcal{T}\cdot\nabla)\boldsymbol{a}^{l-1} - \mu\nabla^2\mathcal{T}\boldsymbol{a}^{l-1} \\ & - (\nabla\mathcal{T}\cdot\nabla\lambda)\boldsymbol{a}^{l-1} - (\nabla\mathcal{T}\cdot\nabla\mu)\boldsymbol{a}^{l-1} + \nabla\mathcal{T}\cdot\boldsymbol{B}^{l-1},\end{aligned} \tag{2.73}$$

where $\boldsymbol{b}^l$, $\boldsymbol{a}^l$, $\boldsymbol{B}^l$, and $\boldsymbol{R}^l$ are zero for all $l < 0$.

2.3.3. S wave: Verification in a homogeneous medium. In a homogeneous medium, the parameters ρ, λ, and μ are constants so that we have the exact solution (2.27) for the S-wave, which we rewrite here as

$$\ddot{G}^S_{ij} = \frac{{\gamma^S}^3}{\rho\pi}[2{\gamma^S}^2 r^2(\delta_{ij} - t_i t_j) f_+^{(-3)}(t^2 - {\gamma^S}^2 r^2) - 2\delta_{ij} f_+^{(-2)}(t^2 - {\gamma^S}^2 r^2)]. \tag{2.74}$$

To simplify the derivation, we assume that $\mathbf{r}_0 = \mathbf{0}$ in this subsection. From the exact solution we know that

$$A^0_{ij} = 2\frac{{\gamma^S}^3}{\rho\pi}{\gamma^S}^2 r^2(\delta_{ij} - t_i t_j), \tag{2.75}$$

$$A^1_{ij} = -2\frac{{\gamma^S}^3}{\rho\pi}\delta_{ij}. \tag{2.76}$$

In this case, the last term of (2.57) disappears, and also, since in a homogeneous medium $\mathcal{T} = {\gamma^S}^2 r^2 = \frac{\rho}{\mu} r^2$, we see that $\nabla \cdot (\mu \nabla \mathcal{T}) = 6\rho$. Hence (2.56) reduces to

$$\mathcal{T}_{,k} A^0_{ij,k} - 4\frac{\rho}{\mu} A^0_{ij} = 0. \tag{2.77}$$

From the exact formula for A^0_{ij} in (2.75), we see that

$$\begin{aligned}
\mathcal{T}_{,k} A^0_{ij,k} &= 2\frac{\rho}{\mu} x_k \left(2\frac{{\gamma^S}^3}{\rho\pi}{\gamma^S}^2 r^2(\delta_{ij} - t_i t_j)\right)_{,k} \\
&= 2\frac{\rho}{\mu} x_k 2\frac{{\gamma^S}^3}{\rho\pi}{\gamma^S}^2(2x_k\delta_{ij} - x_j\delta_{ik} - x_i\delta_{jk}) \\
&= 2\frac{\rho}{\mu} 4\frac{{\gamma^S}^3}{\rho\pi}{\gamma^S}^2 r^2(\delta_{ij} - t_i t_j) \\
&= 4\frac{\rho}{\mu} A^0_{ij}.
\end{aligned} \tag{2.78}$$

This verifies (2.56).

Now we verify (2.55) in a homogeneous medium, i.e., $\mathcal{T}_{,i} A^1_{ij} = A^0_{ij,i}$. We have

$$\begin{aligned}
A^0_{ij,i} &= \left(2\frac{{\gamma^S}^5}{\rho\pi} r^2(\delta_{ij} - t_i\ t_j)\right)_{,i} \\
&= 2\frac{{\gamma^S}^5}{\rho\pi}(r^2\delta_{ij} - x_i x_j)_{,i} \\
&= 2\frac{{\gamma^S}^5}{\rho\pi}(2x_i\delta_{ij} - 3x_j - x_j) \\
&= -4\frac{{\gamma^S}^5}{\rho\pi} x_j.
\end{aligned} \tag{2.79}$$

Again using $\mathcal{T} = \frac{\rho}{\mu} r^2$ and $\mathcal{T}_{,i} = 2\frac{\rho}{\mu} x_i$, we obtain

$$\mathcal{T}_{,i} A^1_{ij} = -4\frac{{\gamma^S}^5}{\rho\pi} x_j, \tag{2.80}$$

which is the same as (2.79), thus verifying (2.55).

Finally we may check the governing equation (2.67) for $\boldsymbol{A}^1$ in a homogeneous medium. Using formulas (2.75) and (2.79), we obtain

$$a_j^0 = -4\frac{{\gamma^S}^5}{\rho\pi}x_j, \tag{2.81}$$

$$a_{j,k}^0 = -4\frac{{\gamma^S}^5}{\rho\pi}\delta_{jk}, \tag{2.82}$$

$$B_{ij}^0 = -4\frac{(\lambda+\mu){\gamma^S}^5}{\rho\pi}\delta_{ij} + \frac{8\mu{\gamma^S}^5}{\rho\pi}\delta_{ij}. \tag{2.83}$$

Also using the fact that $\mathcal{T} = \frac{\rho}{\mu}r^2$, $\mathcal{T}_{,i} = 2\frac{\rho}{\mu}x_i$, and $\mathcal{T}_{,kk} = 6\frac{\rho}{\mu}$, we can get

$$\begin{aligned} b_j^0 &= 10\rho a_j^0 - (\lambda+\mu)\mathcal{T}_{,i}a_{j,i}^0 - \mu\mathcal{T}_{,kk}a_j^0 + B_{ij}^0\mathcal{T}_{,i} \\ &= -\frac{40{\gamma^S}^5}{\pi}x_j + \frac{8(\lambda+\mu){\gamma^S}^5 x_j}{\mu\pi} + \frac{24{\gamma^S}^5 x_j}{\pi} - \frac{8(\lambda+\mu){\gamma^S}^5 x_j}{\mu\pi} + \frac{16{\gamma^S}^5 x_j}{\pi} \\ &= 0. \end{aligned} \tag{2.84}$$

Hence we can obtain R_{ij}^0 as the following,

$$\begin{aligned} R_{ij}^0 &= -\frac{a_j^0\mathcal{T}_{,i}}{2\mathcal{T}} + \frac{\mu}{4\rho\mathcal{T}}a_{j,k}^0\mathcal{T}_{,k}\mathcal{T}_{,i} - \frac{b_j^0\mathcal{T}_{,i}}{8\rho\mathcal{T}} - \frac{(\lambda+\mu)a_{j,i}^0}{2\mu} + \frac{B_{ij}^0}{2\mu} \\ &= 4\frac{{\gamma^S}^5 x_i x_j}{\rho\pi r^2} - 4\frac{{\gamma^S}^5 x_i x_j}{\rho\pi r^2} + 2\frac{(\lambda+\mu){\gamma^S}^5\delta_{ij}}{\mu\rho\pi} - 2\frac{(\lambda+\mu){\gamma^S}^5\delta_{ij}}{\mu\rho\pi} + 4\frac{{\gamma^S}^5\delta_{ij}}{\rho\pi} \\ &= 4\frac{{\gamma^S}^5\delta_{ij}}{\rho\pi}. \end{aligned} \tag{2.85}$$

For the left-hand side of (2.67), we use formula (2.76) to get

$$\mathcal{T}_{,k}A_{ij,k}^1 + \frac{1}{2\mu}((\mu\mathcal{T}_{,k})_{,k} - 10\rho)A_{ij}^1 = -2\frac{\rho}{\mu}A_{ij}^1 = 4\frac{{\gamma^S}^5}{\rho\pi}\delta_{ij}, \tag{2.86}$$

which is the same as R_{ij}^0 derived above. Similarly, we can also verify the amplitude coefficients $A_{ij}^l = 0$ for $l \geq 2$ in a homogeneous medium.

2.3.4. S wave: Initial conditions for $\boldsymbol{A}^l$. In this section, we shall initialize those amplitude coefficients $\boldsymbol{A}^l$ for the S wave at the source $\boldsymbol{r}_0$ in the three dimensional space. In a center-deleted neighborhood of $\boldsymbol{r}_0$, say $0 < |\boldsymbol{r} - \boldsymbol{r}_0| < x_0$, where x_0 is a positive constant, we propose the following matching condition

$$(\ddot{\boldsymbol{G}}^S(\boldsymbol{r};t) - \ddot{\boldsymbol{G}}_{hom}^S(\boldsymbol{r};t))t^n \in L^1(\{t;(0,T)\}) \tag{2.87}$$

over any finite time period $(0,T)$ for any nonnegative integer n, where

$$\ddot{\boldsymbol{G}}_{hom}^S = \frac{{\gamma_0^S}^3}{\rho_0\pi}[2{\gamma_0^S}^2 r^2(\boldsymbol{I} - \boldsymbol{t}\boldsymbol{t}^T)f_+^{(-3)}(t^2 - {\gamma_0^S}^2 r^2) - 2\boldsymbol{I}f_+^{(-2)}(t^2 - {\gamma_0^S}^2 r^2)] \tag{2.88}$$

with $\boldsymbol{t} = \frac{\boldsymbol{r}-\boldsymbol{r}_0}{|\boldsymbol{r}-\boldsymbol{r}_0|}$ being the unit tangent vector to the S ray. Here $\boldsymbol{t}\boldsymbol{t}^T = (t_i t_j)_{3\times 3}$ is a matrix. And $\gamma_0^S = \sqrt{\frac{\rho_0}{\mu_0}}$ is the slowness for the S wave with $\rho_0 = \rho(\boldsymbol{r}_0)$ and $\mu_0 = \mu(\boldsymbol{r}_0)$.

Since $f_+^{(-3+l)}(t^2-\mathcal{T}) \in L^1(\{t;(0,T)\})$ for $l \geq 3$, the above matching condition can be reduced to

$$
\begin{aligned}
\ddot{\boldsymbol{G}}^n_{\text{diff}} &= [\ddot{\boldsymbol{G}}^S(\boldsymbol{r};t) - \ddot{\boldsymbol{G}}^S_{hom}(\boldsymbol{r};t)]t^n \\
&= \boldsymbol{A}^0 f_+^{(-3)}(t^2-\mathcal{T}(\boldsymbol{r}))t^n + \boldsymbol{A}^1 f_+^{(-2)}(t^2-\mathcal{T}(\boldsymbol{r}))t^n \\
&\quad + \boldsymbol{A}^2 f_+^{(-1)}(t^2-\mathcal{T}(\boldsymbol{r}))t^n - \ddot{\boldsymbol{G}}^S_{hom}t^n \\
&= \boldsymbol{A}^0\delta^{(2)}(t^2-\mathcal{T}(\boldsymbol{r}))t^n + \boldsymbol{A}^1\delta^{(1)}(t^2-\mathcal{T}(\boldsymbol{r}))t^n + \boldsymbol{A}^2\delta(t^2-\mathcal{T}(\boldsymbol{r}))t^n \\
&\quad - \frac{{\gamma_0^S}^3}{\rho_0\pi}[2{\gamma_0^S}^2 r^2(\boldsymbol{I}-\boldsymbol{t}\boldsymbol{t}^T)\delta^{(2)}(t^2-{\gamma^S}^2 r^2) - 2\boldsymbol{I}\delta^{(1)}(t^2-{\gamma_0^S}^2 r^2)]t^n \\
&\in L^1(\{t;(0,T)\})
\end{aligned} \tag{2.89}
$$

for any $\boldsymbol{r}$ satisfying $0 < |\boldsymbol{r}-\boldsymbol{r}_0| < x_0$.

For $\boldsymbol{r}$ sufficiently close to $\boldsymbol{r}_0$ such that all δ-related functions do not vanish, one gets

$$
\begin{aligned}
\int_0^T \delta^{(k)}[t^2-\mathcal{T}(\boldsymbol{r})]t^n dt &= \int_{-\mathcal{T}(\boldsymbol{r})}^{T^2-\mathcal{T}(\boldsymbol{r})} \delta^{(k)}(\tilde{t})\frac{1}{2}(\tilde{t}+\mathcal{T}(\boldsymbol{r}))^{\frac{n}{2}-\frac{1}{2}}d\tilde{t} \\
&= \frac{(-1)^k}{2}[(\tilde{t}+\mathcal{T}(\boldsymbol{r}))^{\frac{n-1}{2}}]^{(k)}|_{\tilde{t}=0} \\
&= \frac{(-1)^k}{2}\prod_{j=0}^{k-1}\left(\frac{n}{2}-\frac{1}{2}-j\right)[\mathcal{T}(\boldsymbol{r})]^{\frac{n}{2}-\frac{1}{2}-k}.
\end{aligned} \tag{2.90}
$$

Since k is at most 2 in our situation, $\frac{n}{2}-\frac{1}{2}-k \geq 0$ for $n \geq 5$ so that we are concerned about the five cases $n = 0,1,2,3,4$ only. Therefore, we enforce

$$
\begin{aligned}
\int_0^T \ddot{\boldsymbol{G}}^0_{\text{diff}}dt &= \frac{3}{8}\left[\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{5}{2}}} - \frac{2{\gamma_0^S}^5 r^2}{\rho_0\pi(\gamma_0^S r)^5}(\boldsymbol{I}-\boldsymbol{t}\boldsymbol{t}^T)\right] \\
&\quad + \frac{1}{4}\left[\frac{\boldsymbol{A}^1}{\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}}} + \frac{2{\gamma_0^S}^3}{\rho_0\pi(\gamma_0^S r)^3}\boldsymbol{I}\right] + \frac{\boldsymbol{A}^2}{2\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} \\
&= \frac{3}{8}\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{5}{2}}} + \frac{1}{4}\frac{\boldsymbol{A}^1}{\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}}} + \frac{\boldsymbol{A}^2}{2\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} - \frac{1}{4\rho_0\pi r^3}(\boldsymbol{I}-3\boldsymbol{t}\boldsymbol{t}^T) = O(1),
\end{aligned} \tag{2.91}
$$

$$
\int_0^T \ddot{\boldsymbol{G}}^1_{\text{diff}}dt = \frac{\boldsymbol{A}^2}{2} = O(1), \tag{2.92}
$$

$$
\begin{aligned}
\int_0^T \ddot{\boldsymbol{G}}^2_{\text{diff}}dt &= -\frac{1}{8}\left[\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}}} - \frac{2{\gamma_0^S}^5 r^2}{\rho_0\pi(\gamma_0^S r)^3}(\boldsymbol{I}-\boldsymbol{t}\boldsymbol{t}^T)\right] \\
&\quad - \frac{1}{4}\left[\frac{\boldsymbol{A}^1}{\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} + \frac{2{\gamma_0^S}^3}{\rho_0\pi(\gamma_0^S r)}\boldsymbol{I}\right] + \frac{\boldsymbol{A}^2}{2}\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}} \\
&= -\frac{1}{8}\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}}} - \frac{1}{4}\frac{\boldsymbol{A}^1}{\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} + \frac{\boldsymbol{A}^2}{2}\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}} - \frac{{\gamma_0^S}^2}{4\rho_0\pi r}(\boldsymbol{I}+\boldsymbol{t}\boldsymbol{t}^T) = O(1),
\end{aligned} \tag{2.93}
$$

$$\int_0^T \ddot{\boldsymbol{G}}^3_{\text{diff}} dt = -\frac{1}{2}\left[\boldsymbol{A}^1 + \frac{2{\gamma_0^S}^3 \boldsymbol{I}}{\rho_0 \pi}\right] + \frac{1}{2}\boldsymbol{A}^2 \mathcal{T}(\boldsymbol{r}) = O(1), \tag{2.94}$$

$$\begin{aligned}\int_0^T \ddot{\boldsymbol{G}}^4_{\text{diff}} dt &= \frac{3}{8}\left[\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} - \frac{2{\gamma_0^S}^5 r^2}{\rho_0\pi(\gamma_0^S r)}(\boldsymbol{I} - \boldsymbol{t}\boldsymbol{t}^T)\right] \\ &\quad - \frac{3}{4}\left[\boldsymbol{A}^1 \mathcal{T}(\boldsymbol{r})^{\frac{1}{2}} + \frac{2{\gamma_0^S}^3}{\rho_0\pi}(\gamma_0^S r)\boldsymbol{I}\right] + \frac{\boldsymbol{A}^2}{2}\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}} \\ &= \frac{3}{8}\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} - \frac{3}{4}\boldsymbol{A}^1\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}} + \frac{\boldsymbol{A}^2}{2}\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}} - \frac{3{\gamma_0^S}^4 r}{4\rho_0\pi}(3\boldsymbol{I} - \boldsymbol{t}\boldsymbol{t}^T) = O(1)\end{aligned} \tag{2.95}$$

for $0 < |\boldsymbol{r} - \boldsymbol{r}_0| < x_0$.

Hiding $O(1)$ terms above, we get

$$\frac{3}{8}\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{5}{2}}} + \frac{1}{4}\frac{\boldsymbol{A}^1}{\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}}} + \frac{\boldsymbol{A}^2}{2\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} - \frac{1}{4\rho_0\pi r^3}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T) = O(1), \tag{2.96}$$

$$-\frac{1}{8}\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}}} - \frac{1}{4}\frac{\boldsymbol{A}^1}{\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} - \frac{{\gamma_0^S}^2}{4\rho_0\pi r}(\boldsymbol{I} + \boldsymbol{t}\boldsymbol{t}^T) = O(1), \tag{2.97}$$

$$\frac{3}{8}\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} = O(1). \tag{2.98}$$

Considering (2.96) $\times\, \mathcal{T}(\boldsymbol{r})$ + (2.97), we can obtain the initial condition for $\boldsymbol{A}^0$ as $\boldsymbol{r} \to \boldsymbol{r}_0$,

$$\boldsymbol{A}^0 = \frac{\mathcal{T}^{\frac{5}{2}}}{\rho_0\pi r^3}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T) + \frac{{\gamma_0^S}^2\mathcal{T}^{\frac{3}{2}}}{\rho_0\pi r}(\boldsymbol{I} + \boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}^{\frac{3}{2}}). \tag{2.99}$$

Moreover, since τ is the travel time from the source $\boldsymbol{r}_0$ to $\boldsymbol{r}$ which is proportional to $r = |\boldsymbol{r} - \boldsymbol{r}_0|$, $\mathcal{T} = \tau^2$ is proportional to r^2 as $\boldsymbol{r} \to \boldsymbol{r}_0$, leading to $\boldsymbol{A}^0 = O(1)$ as $\boldsymbol{r} \to \boldsymbol{r}_0$.

Next (2.96) $\times\, 4\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}}$ gives rise to the initial condition for $\boldsymbol{A}^1$ as $\boldsymbol{r} \to \boldsymbol{r}_0$,

$$\begin{aligned}\boldsymbol{A}^1 &= -\frac{3}{2}\frac{\boldsymbol{A}^0}{\mathcal{T}} + \frac{\mathcal{T}^{\frac{3}{2}}}{\rho_0\pi r^3}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T) + 2\boldsymbol{A}^2\mathcal{T}(\boldsymbol{x}) + O(\mathcal{T}^{\frac{3}{2}}) \\ &= -\frac{3}{2}\frac{\boldsymbol{A}^0}{\mathcal{T}} + \frac{\mathcal{T}^{\frac{3}{2}}}{\rho_0\pi r^3}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}).\end{aligned} \tag{2.100}$$

Substituting (2.99) into (2.100), we could obtain

$$\boldsymbol{A}^1 = -\frac{1}{2}\frac{\mathcal{T}^{\frac{3}{2}}}{\rho_0\pi r^3}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T) - \frac{3}{2}\frac{{\gamma_0^S}^2\mathcal{T}^{\frac{1}{2}}}{\rho_0\pi r}(\boldsymbol{I} + \boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}^{\frac{1}{2}}). \tag{2.101}$$

Similarly to the case for $\boldsymbol{A}^0$, we may conclude that $\boldsymbol{A}^1 = O(1)$ as $\boldsymbol{r} \to \boldsymbol{r}_0$.

Finally, (2.96) $\times\, 2\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}$ gives rise to the initial condition for $\boldsymbol{A}^2$ as $\boldsymbol{r} \to \boldsymbol{r}_0$,

$$\boldsymbol{A}^2 = -\frac{3}{4}\frac{\boldsymbol{A}^0}{\mathcal{T}^2} - \frac{1}{2}\frac{\boldsymbol{A}^1}{\mathcal{T}} + \frac{\mathcal{T}^{\frac{1}{2}}}{2\rho_0\pi r^3}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}^{\frac{1}{2}}). \tag{2.102}$$

When substituting (2.100) into (2.102), the first three terms on the right-hand side of (2.102) will be canceled. Hence we may conclude that $\boldsymbol{A}^2 = O(1)$ as $\boldsymbol{r} \to \boldsymbol{r}_0$ which is also consistent with (2.92).

2.3.5. S wave: Governing equations for desingularizing $\boldsymbol{A}^l$. We enforce the condition that $\boldsymbol{A}^l$ is bounded at the source for all $l \geq 0$, which may not be amenable to numerical computation. Therefore, we define new dyadic coefficients to be

$$\tilde{\boldsymbol{A}}^l = \boldsymbol{A}^l \tau^{2l} \tag{2.103}$$

so that

$$\boldsymbol{A}^l = O(1) \text{ as } \boldsymbol{r} \to \boldsymbol{r}_0,$$
$$\tilde{\boldsymbol{A}}^l = O(\tau^{2l}) \text{ as } \boldsymbol{r} \to \boldsymbol{r}_0.$$

We introduce $\tilde{\boldsymbol{A}}^l$ here since $\boldsymbol{A}^l$ may not be smooth at the source $\boldsymbol{r}_0$ for an inhomogeneous medium; this can be seen from the initial conditions (2.99)–(2.102). It is easy to derive governing equations for $\tilde{\boldsymbol{A}}^l$ from those for $\boldsymbol{A}^l$ for $l = 0, 1$ as the following,

$$(\nabla\mathcal{T} \cdot \nabla)\tilde{\boldsymbol{A}}^0 + \frac{1}{2\mu}[\nabla \cdot (\mu\nabla\mathcal{T}) - 14\rho]\tilde{\boldsymbol{A}}^0 + \frac{\mu}{2\rho}\nabla\mathcal{T}\left[\nabla\left(\frac{\rho}{\mu}\right) \cdot \tilde{\boldsymbol{A}}^0\right] = 0, \tag{2.104}$$

$$(\nabla\mathcal{T} \cdot \nabla)\tilde{\boldsymbol{A}}^1 + \frac{1}{2\mu}[\nabla \cdot (\mu\nabla\mathcal{T}) - 18\rho]\tilde{\boldsymbol{A}}^1 + \frac{\mu}{2\rho}\nabla\mathcal{T}\left[\nabla\left(\frac{\rho}{\mu}\right) \cdot \tilde{\boldsymbol{A}}^1\right] = \tilde{\boldsymbol{R}}^0, \tag{2.105}$$

where

$$\tilde{\boldsymbol{R}}^0 \equiv -\frac{\nabla\mathcal{T}\tilde{\boldsymbol{a}}^0}{2} + \frac{\mu\nabla\mathcal{T}[(\nabla\mathcal{T} \cdot \nabla)\tilde{\boldsymbol{a}}^0]}{4\rho} - \frac{\nabla\mathcal{T}\tilde{\boldsymbol{b}}^0}{8\rho} - \frac{\mathcal{T}\nabla\lambda\tilde{\boldsymbol{a}}^0}{2\mu} - \frac{(\lambda+\mu)\mathcal{T}\nabla\tilde{\boldsymbol{a}}^0}{2\mu} + \frac{\mathcal{T}\tilde{\boldsymbol{B}}^0}{2\mu}. \tag{2.106}$$

Here $\tilde{\boldsymbol{B}}^0$, $\tilde{\boldsymbol{a}}^0$, and $\tilde{\boldsymbol{b}}^0$ are given by

$$\begin{aligned}\tilde{\boldsymbol{B}}^0 \equiv\ & (\lambda+\mu)\nabla(\nabla \cdot \tilde{\boldsymbol{A}}^0) + \mu\nabla^2\tilde{\boldsymbol{A}}^0 + \nabla\lambda(\nabla \cdot \tilde{\boldsymbol{A}}^0) \\ & + \nabla\mu \times (\nabla \times \tilde{\boldsymbol{A}}^0) + 2(\nabla\mu \cdot \nabla)\tilde{\boldsymbol{A}}^0,\end{aligned} \tag{2.107}$$

$$\tilde{\boldsymbol{a}}^0 \equiv \nabla \cdot \tilde{\boldsymbol{A}}^0 + \frac{1}{\lambda+\mu}\nabla\mu \cdot \tilde{\boldsymbol{A}}^0 + \frac{2\mu}{(\lambda+\mu)|\nabla\mathcal{T}|^2}\nabla\mathcal{T}[(\nabla\mathcal{T} \cdot \nabla)\tilde{\boldsymbol{A}}^0], \tag{2.108}$$

$$\begin{aligned}\tilde{\boldsymbol{b}}^0 \equiv\ & 10\rho\tilde{\boldsymbol{a}}^0 - (\lambda+\mu)(\nabla\mathcal{T} \cdot \nabla)\tilde{\boldsymbol{a}}^0 - \mu\nabla^2\mathcal{T}\tilde{\boldsymbol{a}}^0 \\ & - (\nabla\mathcal{T} \cdot \nabla\lambda)\tilde{\boldsymbol{a}}^0 - (\nabla\mathcal{T} \cdot \nabla\mu)\tilde{\boldsymbol{a}}^0 + \nabla\mathcal{T} \cdot \tilde{\boldsymbol{B}}^0.\end{aligned} \tag{2.109}$$

The governing equations for general $\tilde{\boldsymbol{A}}^l$ for $l \geq 2$ are

$$(\nabla\mathcal{T} \cdot \nabla)\tilde{\boldsymbol{A}}^l + \frac{1}{2\mu}[\nabla \cdot (\mu\nabla\mathcal{T}) - (4l+14)\rho]\tilde{\boldsymbol{A}}^l + \frac{\mu}{2\rho}\nabla\mathcal{T}\left[\nabla\left(\frac{\rho}{\mu}\right) \cdot \tilde{\boldsymbol{A}}^l\right] = \tilde{\boldsymbol{R}}^{l-1}, \tag{2.110}$$

where

$$\begin{aligned}\tilde{\boldsymbol{R}}^{l-1} \equiv\ & -\frac{\nabla\mathcal{T}\tilde{\boldsymbol{a}}^{l-1}}{2}\mathcal{T}^{l-1} + \frac{\mu\nabla\mathcal{T}[(\nabla\mathcal{T} \cdot \nabla)\tilde{\boldsymbol{a}}^{l-1}]}{4\rho}\mathcal{T}^{l-1} - \frac{\nabla\mathcal{T}\tilde{\boldsymbol{b}}^{l-1}}{8\rho}\mathcal{T}^{l-1} - \frac{\nabla\lambda\tilde{\boldsymbol{a}}^{l-1}}{2\mu}\mathcal{T}^l \\ & - \frac{(\lambda+\mu)\nabla\tilde{\boldsymbol{a}}^{l-1}}{2\mu}\mathcal{T}^l + \frac{\tilde{\boldsymbol{B}}^{l-1}}{2\mu}\mathcal{T}^l.\end{aligned} \tag{2.111}$$

Here the explicit formulas for $\tilde{\boldsymbol{B}}^{l-1}$, $\tilde{\boldsymbol{a}}^{l-1}$, and $\tilde{\boldsymbol{b}}^{l-1}$ are given in Appendix A.

Now we consider the initialization for $\tilde{\boldsymbol{A}}^l$. First, we can obtain the following initial conditions for $\tilde{\boldsymbol{A}}^0$, $\tilde{\boldsymbol{A}}^1$, and $\tilde{\boldsymbol{A}}^2$ from (2.99) to (2.103),

$$\tilde{\boldsymbol{A}}^0 = \frac{\mathcal{T}^{\frac{5}{2}}}{\rho_0 \pi r^3}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T) + \frac{{\gamma_0^S}^2 \mathcal{T}^{\frac{3}{2}}}{\rho_0 \pi r}(\boldsymbol{I} + \boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}^{\frac{3}{2}}), \tag{2.112}$$

$$\tilde{\boldsymbol{A}}^1 = -\frac{3}{2}\tilde{\boldsymbol{A}}^0 + \frac{\mathcal{T}^{\frac{5}{2}}}{\rho_0 \pi r^3}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}^2), \tag{2.113}$$

$$\tilde{\boldsymbol{A}}^2 = -\frac{3}{4}\tilde{\boldsymbol{A}}^0 - \frac{1}{2}\tilde{\boldsymbol{A}}^1 + \frac{\mathcal{T}^{\frac{5}{2}}}{2\rho_0 \pi r^3}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}^{\frac{5}{2}}). \tag{2.114}$$

Next, we give the details to initialize the general $\tilde{\boldsymbol{A}}^l$ from the governing equations (2.110) and the initial conditions (2.112)–(2.114) for $\tilde{\boldsymbol{A}}^l$ with $0 \le l \le 2$.

In an isotropic medium, the travel time function $\tau(\cdot;\boldsymbol{r}_0)$ solving the eikonal equation $|\nabla\tau(\boldsymbol{r};\boldsymbol{r}_0)| = \gamma^S(\boldsymbol{r})$ is locally smooth near the source except at the source point $\boldsymbol{r}_0$ itself [19, 27].

Assuming that the refractive index ${\gamma^S}^2(\boldsymbol{r})$ is analytic, ${\gamma^S}^2(\boldsymbol{r})$ can be written as the following power series centered at the source point,

$${\gamma^S}^2(\boldsymbol{r}) = \sum_{k=0}^{\infty} \Phi_k(\boldsymbol{r};\boldsymbol{r}_0), \tag{2.115}$$

where Φ_k is the homogeneous polynomial of degree-k term in the Taylor expansion of ${\gamma^S}^2$ about the source $\boldsymbol{r}_0$ and $\Phi_0 = {\gamma_0^S}^2$ with $\gamma_0^S = \gamma^S(\boldsymbol{r}_0)$.

Assuming that $\mathcal{T} = \tau^2$ is analytic, we have in the source neighborhood [15]

$$\tau^2 - \left(\Phi_0 + \frac{1}{2}\Phi_1 + \frac{1}{3}\Phi_2 - \frac{1}{48\Phi_0}|\nabla\Phi_1|^2 r^2\right) r^2 = O(r^5). \tag{2.116}$$

This in fact gives rise to the following estimate: for any $k \in \boldsymbol{R}$, as $r \to 0$,

$$\left(\frac{\tau}{\gamma_0^S r}\right)^k = 1 + \frac{k}{2\Phi_0}\left(\frac{\Phi_1}{2} + \frac{\Phi_2}{3} - \frac{|\nabla\Phi_1|^2}{48\Phi_0}r^2\right) + \frac{k}{4}\left(\frac{k}{2} - 1\right)\frac{\Phi_1^2}{4\Phi_0^2} + O(r^3). \tag{2.117}$$

Then using formulas (2.116) and (2.117), we can rewrite (2.112)–(2.114) as follows.

(a) The sequence $\{\tilde{\boldsymbol{A}}_{0k}\}$ is initialized by

$$\tilde{\boldsymbol{A}}_{00} = 0,\ \tilde{\boldsymbol{A}}_{01} = 0,\ \tilde{\boldsymbol{A}}_{02} = \frac{2{\gamma_0^S}^5 r^2}{\rho_0 \pi}(\boldsymbol{I} - \boldsymbol{t}\boldsymbol{t}^T). \tag{2.118}$$

(b) The sequence $\{\tilde{\boldsymbol{A}}_{1k}\}$ is initialized by

$$\begin{aligned} &\tilde{\boldsymbol{A}}_{10} = 0,\ \tilde{\boldsymbol{A}}_{11} = 0,\ \tilde{\boldsymbol{A}}_{12} = -\frac{2{\gamma_0^S}^5 r^2}{\rho_0 \pi}\boldsymbol{I}, \\ &\tilde{\boldsymbol{A}}_{13} = -\frac{3}{2}\tilde{\boldsymbol{A}}_{03} + \frac{5{\gamma_0^S}^3 \Phi_1 r^2}{4\rho_0 \pi}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T). \end{aligned} \tag{2.119}$$

(c) The sequence $\{\tilde{\boldsymbol{A}}_{2k}\}$ is initialized by

(2.120)

$$\begin{aligned} &\tilde{\boldsymbol{A}}_{20} = 0,\ \tilde{\boldsymbol{A}}_{21} = 0, \tilde{\boldsymbol{A}}_{22} = 0,\ \tilde{\boldsymbol{A}}_{23} = 0, \\ &\tilde{\boldsymbol{A}}_{24} = -\frac{3}{4}\tilde{\boldsymbol{A}}_{04} - \frac{1}{2}\tilde{\boldsymbol{A}}_{14} + \frac{{\gamma_0^S}^3}{2\rho_0 \pi}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T)\left[\frac{5\Phi_2}{6}r^2 + \frac{15\Phi_1^2}{32\Phi_0}r^2 - \frac{5|\nabla\Phi_1|^2}{96\Phi_0}r^4\right]. \end{aligned}$$

If we further assume that $\log\mu(\boldsymbol{r})$, $\tilde{\boldsymbol{A}}^l$, and $\tilde{\boldsymbol{R}}^{l-1}$ for all $l \geq 0$ are analytic at the source $\boldsymbol{r}_0$, then those asymptotic behaviors (2.112)–(2.114) for $\tilde{\boldsymbol{A}}^0$, $\tilde{\boldsymbol{A}}^1$, and $\tilde{\boldsymbol{A}}^2$, respectively, can be further clarified, and the exact solutions of $\tilde{\boldsymbol{A}}^l$ for general l near the source can be constructed recursively.

We first expand these ingredients as Taylor series about the source as

$$\log\mu(\boldsymbol{r}) = \sum_{k=0}^{\infty} U_k(\boldsymbol{r}), \tag{2.121}$$

$$\tilde{\boldsymbol{A}}^l(\boldsymbol{r}) = \sum_{k=0}^{\infty} \tilde{\boldsymbol{A}}_{lk}(\boldsymbol{r}), \tag{2.122}$$

$$\tilde{\boldsymbol{R}}^{l-1}(\boldsymbol{r}) = \sum_{k=0}^{\infty} \tilde{\boldsymbol{R}}_{l-1,k}(\boldsymbol{r}), \tag{2.123}$$

where the term with subscript k denotes a homogeneous polynomial of degree k. As $\mathcal{T} = \tau^2$ is analytic near the source neighborhood, we denote its Taylor series as

$$\mathcal{T}(\boldsymbol{r}) = \tau^2(\boldsymbol{r}) = \sum_{k=0}^{\infty} T_k(\boldsymbol{r}). \tag{2.124}$$

We see from the eikonal equation that $T_0 = T_1 = 0$ and $T_2 = \Phi_0 r^2$. Thus the transport equation (2.110) for the S wave can be rewritten as

$$\begin{aligned}
&\sum_{k=0}^{\infty}\Phi_k(\boldsymbol{r})\left(\sum_{k=2}^{\infty}\nabla T_k(\boldsymbol{r})\cdot\nabla\right)\sum_{k=0}^{\infty}\tilde{\boldsymbol{A}}_{lk}(\boldsymbol{r}) + \frac{1}{2}\sum_{k=2}^{\infty}\nabla T_k(\boldsymbol{r})\left[\sum_{k=0}^{\infty}\nabla\Phi_k(\boldsymbol{r})\cdot\sum_{k=0}^{\infty}\tilde{\boldsymbol{A}}_{lk}(\boldsymbol{r})\right]\\
&+\sum_{k=0}^{\infty}\Phi_k(\boldsymbol{r})\left[\frac{1}{2}\sum_{k=2}^{\infty}\Delta T_k(\boldsymbol{r}) + \frac{1}{2}\sum_{k=2}^{\infty}\nabla T_k(\boldsymbol{r})\cdot\sum_{k=0}^{\infty}\nabla U_k(\boldsymbol{r})\right]\sum_{k=0}^{\infty}\tilde{\boldsymbol{A}}_{lk}(\boldsymbol{r})\\
&+\sum_{k=0}^{\infty}\Phi_k(\boldsymbol{r})\left[-(2l+7)\sum_{k=0}^{\infty}\Phi_k(\boldsymbol{r})\right]\sum_{k=0}^{\infty}\tilde{\boldsymbol{A}}_{lk}(\boldsymbol{r})\\
&=\sum_{k=0}^{\infty}\Phi_k(\boldsymbol{r})\sum_{k=0}^{\infty}\tilde{\boldsymbol{R}}_{l-1,k}(\boldsymbol{r}).
\end{aligned} \tag{2.125}$$

Comparing the qth degree polynomial of both sides, we have

$$\begin{aligned}
&0 = \Phi_0(\nabla T_2\cdot\nabla)\tilde{\boldsymbol{A}}_{lq} + \Phi_0(\frac{1}{2}\Delta T_2 - (2l+7)\Phi_0)\tilde{\boldsymbol{A}}_{lq}\\
&+\frac{1}{2}\sum_{s=0}^{q-1}\sum_{j+k=q-s,j,k\geq 0}\nabla T_{2+k}(\nabla\Phi_j\cdot\tilde{\boldsymbol{A}}_{ls}) + \sum_{s=0}^{q-1}\sum_{j+k=q-s,j,k\geq 0}\Phi_j(\nabla T_{2+k}\cdot\nabla)\tilde{\boldsymbol{A}}_{ls}\\
&+\sum_{s=0}^{q-1}\sum_{j+k=q-s,j,k\geq 0}\Phi_j\left(\frac{\Delta T_{2+k}}{2} + \frac{1}{2}\sum_{m+n=k,m,n\geq 0}\nabla T_{2+m}\cdot\nabla U_n - (2l+7)\Phi_k\right)\tilde{\boldsymbol{A}}_{ls}\\
&-\sum_{j+k=q,j,k\geq 0}\Phi_j\tilde{\boldsymbol{R}}_{l-1,k}.
\end{aligned} \tag{2.126}$$

Using the fact that $(\nabla T_2 \cdot \nabla)\tilde{\boldsymbol{A}}_{lq} = 2q\Phi_0\tilde{\boldsymbol{A}}_{lq}$ and $\Delta T_2 = 6\Phi_0$, we get

$$\begin{aligned}
0 &= (2q-(2l+4))\Phi_0^2\tilde{\boldsymbol{A}}_{lq} \\
&+ \frac{1}{2}\sum_{s=0}^{q-1}\sum_{j+k=q-s,j,k\geq 0}\nabla T_{2+k}(\nabla\Phi_j\cdot\tilde{\boldsymbol{A}}_{ls}) + \sum_{s=0}^{q-1}\sum_{j+k=q-s,j,k\geq 0}\Phi_j(\nabla T_{2+k}\cdot\nabla)\tilde{\boldsymbol{A}}_{ls} \\
&+ \sum_{s=0}^{q-1}\sum_{j+k=q-s,j,k\geq 0}\Phi_j\left(\frac{\Delta T_{2+k}}{2} + \frac{1}{2}\sum_{m+n=k,m,n\geq 0}\nabla T_{2+m}\cdot\nabla U_n - (2l+7)\Phi_k\right)\tilde{\boldsymbol{A}}_{ls} \\
&- \sum_{j+k=q,j,k\geq 0}\Phi_j\tilde{\boldsymbol{R}}_{l-1,k}.
\end{aligned} \tag{2.127}$$

Hence we obtain a recursive formula to compute the qth degree polynomial $\tilde{\boldsymbol{A}}_{lq}$ of the lth amplitude coefficient $\tilde{\boldsymbol{A}}^l$.

It is clear that (2.127) is not valid for computing $\tilde{\boldsymbol{A}}_{lq^*}$ when $q^* = l+2$, since the coefficient of $\tilde{\boldsymbol{A}}_{lq^*}$ becomes 0. However, in this case, we first notice that when $l \geq 3$, $\tilde{\boldsymbol{A}}^l$ has been assumed to be $O(\mathcal{T}^l) = O(r^{2l}) = o(r^{l+2})$ due to $2l \geq l+2$, so that $\tilde{\boldsymbol{A}}_{lq^*} = 0$ when $q^* = l+2$. In addition, we have three special cases: (1) $l = 0$ so that $q^* = 2$, but $\tilde{\boldsymbol{A}}_{02}$ has been initialized according to (2.118); (2) $l = 1$ so that $q^* = 3$, but $\tilde{\boldsymbol{A}}_{13}$ has been initialized according to (2.119); (3) $l = 2$ so that $q^* = 4$, but $\tilde{\boldsymbol{A}}_{24}$ has been initialized according to (2.120).

In the recursive formula (2.127), $\tilde{\boldsymbol{R}}_{l-1,k}$ depends on $\tilde{\boldsymbol{A}}^{l-1}$ but does not depend on $\tilde{\boldsymbol{A}}^l$; therefore, the power series of $\tilde{\boldsymbol{A}}^l$ is computable via (2.127) once $\tilde{\boldsymbol{A}}^{l-1}$ is available near the source.

2.4. Case II, P wave: $\boldsymbol{A_j^0} \parallel \boldsymbol{\nabla\mathcal{T}}$ and $\boldsymbol{4\rho\mathcal{T} - (\lambda+2\mu)|\nabla\mathcal{T}|^2 = 0}$. Let us repeat (2.40) here for reference,

$$\begin{aligned}
0 &= 4\rho\mathcal{T}A_{ij}^l - (\lambda+\mu)\mathcal{T}_{,k}\mathcal{T}_{,i}A_{kj}^l - \mu\mathcal{T}_{,k}\mathcal{T}_{,k}A_{ij}^l \\
&+ 2\rho(2l-9)A_{ij}^{l-1} + (\lambda+\mu)\mathcal{T}_{,i}A_{kj,k}^{l-1} + (\lambda+\mu)\mathcal{T}_{,k}A_{kj,i}^{l-1} + (\lambda+\mu)\mathcal{T}_{,ki}A_{kj}^{l-1} \\
&+ 2\mu\mathcal{T}_{,k}A_{ij,k}^{l-1} + \mu\mathcal{T}_{,kk}A_{ij}^{l-1} + \lambda_{,i}\mathcal{T}_{,k}A_{kj}^{l-1} + \mu_{,k}\mathcal{T}_{,k}A_{ij}^{l-1} + \mu_{,k}\mathcal{T}_{,i}A_{kj}^{l-1} \\
&- (\lambda+\mu)A_{kj,ki}^{l-2} - \mu A_{ij,kk}^{l-2} - \lambda_{,i}A_{kj,k}^{l-2} - \mu_{,k}A_{ij,k}^{l-2} - \mu_{,k}A_{kj,i}^{l-2}.
\end{aligned} \tag{2.128}$$

Rewriting the first line using the fact that in Case II,

$$-\mu\mathcal{T}_{,k}\mathcal{T}_{,k}A_{ij}^l = (\lambda+\mu)\mathcal{T}_{,k}\mathcal{T}_{,k}A_{ij}^l - 4\rho\mathcal{T}A_{ij}^l, \tag{2.129}$$

we have

$$\begin{aligned}
0 &= (\lambda+\mu)(\mathcal{T}_{,k}\mathcal{T}_{,k}A_{ij}^l - \mathcal{T}_{,k}\mathcal{T}_{,i}A_{kj}^l) \\
&+ 2\rho(2l-9)A_{ij}^{l-1} + (\lambda+\mu)\mathcal{T}_{,i}A_{kj,k}^{l-1} + (\lambda+\mu)\mathcal{T}_{,k}A_{kj,i}^{l-1} + (\lambda+\mu)\mathcal{T}_{,ki}A_{kj}^{l-1} \\
&+ 2\mu\mathcal{T}_{,k}A_{ij,k}^{l-1} + \mu\mathcal{T}_{,kk}A_{ij}^{l-1} + \lambda_{,i}\mathcal{T}_{,k}A_{kj}^{l-1} + \mu_{,k}\mathcal{T}_{,k}A_{ij}^{l-1} + \mu_{,k}\mathcal{T}_{,i}A_{kj}^{l-1} \\
&- (\lambda+\mu)A_{kj,ki}^{l-2} - \mu A_{ij,kk}^{l-2} - \lambda_{,i}A_{kj,k}^{l-2} - \mu_{,k}A_{ij,k}^{l-2} - \mu_{,k}A_{kj,i}^{l-2}.
\end{aligned} \tag{2.130}$$

2.4.1. P-wave transport equations for $\boldsymbol{A^0}$ and $\boldsymbol{A^1}$. First, we derive the governing equation for $\boldsymbol{A}^0$. We set $l = 1$ in (2.130) and then take the scalar product of (2.130) with $\mathcal{T}_{,i}$ to obtain

$$0 = -14\rho\mathcal{T}_{,i}A^0_{ij} + (\lambda+\mu)\mathcal{T}_{,i}\mathcal{T}_{,i}A^0_{kj,k} + (\lambda+\mu)\mathcal{T}_{,i}\mathcal{T}_{,k}A^0_{kj,i} + (\lambda+\mu)\mathcal{T}_{,i}\mathcal{T}_{,ki}A^0_{kj} + 2\mu\mathcal{T}_{,i}\mathcal{T}_{,k}A^0_{ij,k} + \mu\mathcal{T}_{,i}\mathcal{T}_{,kk}A^0_{ij} + \lambda_{,i}\mathcal{T}_{,i}\mathcal{T}_{,k}A^0_{kj} + \mu_{,k}\mathcal{T}_{,i}\mathcal{T}_{,k}A^0_{ij} + \mu_{,k}\mathcal{T}_{,i}\mathcal{T}_{,i}A^0_{kj}, \tag{2.131}$$

where we have made use of the eikonal equation, $4\rho\mathcal{T} = (\lambda+2\mu)\mathcal{T}_{,i}\mathcal{T}_{,i}$. Also the last line does not contribute since $A^{-1}_{ij} = 0$.

Since $\boldsymbol{A}^0_j$ is parallel to $\nabla\mathcal{T}$ for $1 \le j \le 3$, we may write it as

$$A^0_{ij} = \alpha^0_j\mathcal{T}_{,i}, \tag{2.132}$$

where $\boldsymbol{\alpha}^0 = (\alpha^0_1, \alpha^0_2, \alpha^0_3)$ is a row vector.

Inserting (2.132) into (2.131), we get

$$0 = -14\rho\mathcal{T}_{,i}\alpha^0_j\mathcal{T}_{,i} + 2(\lambda+2\mu)\mathcal{T}_{,i}\mathcal{T}_{,i}\alpha^0_{j,k}\mathcal{T}_{,k} + 2(\lambda+2\mu)\mathcal{T}_{,i}\mathcal{T}_{,k}\alpha^0_j\mathcal{T}_{,ik} + (\lambda+2\mu)\mathcal{T}_{,i}\mathcal{T}_{,i}\mathcal{T}_{,kk}\alpha^0_j + (\lambda_{,k}+2\mu_{,k})\mathcal{T}_{,k}\mathcal{T}_{,i}\mathcal{T}_{,i}\alpha^0_j. \tag{2.133}$$

Employing the following relations

$$[(\lambda+2\mu)\mathcal{T}_{,i}\mathcal{T}_{,i}]_{,k}\mathcal{T}_{,k} = (\lambda+2\mu)_{,k}\mathcal{T}_{,i}\mathcal{T}_{,i}\mathcal{T}_{k} + 2(\lambda+2\mu)\mathcal{T}_{,i}\mathcal{T}_{,ik}\mathcal{T}_{,k}, \tag{2.134}$$

$$(4\rho\mathcal{T})_{,k}\mathcal{T}_{,k} = 4\rho_{,k}\mathcal{T}_{,k}\mathcal{T} + 4\rho\mathcal{T}_{,k}\mathcal{T}_{,k}, \tag{2.135}$$

we can get

$$0 = -14\rho\alpha^0_j\mathcal{T}_{,i}\mathcal{T}_{,i} + 2(\lambda+2\mu)\mathcal{T}_{,i}\mathcal{T}_{,i}\mathcal{T}_{,k}\alpha^0_{j,k} + (\lambda+2\mu)\alpha^0_j\mathcal{T}_{,kk}\mathcal{T}_{,i}\mathcal{T}_{,i} + 4\rho_{,k}\mathcal{T}_{,k}\mathcal{T}\alpha^0_j + 4\rho\mathcal{T}_{,k}\mathcal{T}_{,k}\alpha^0_j. \tag{2.136}$$

Simplifying the result, we obtain

$$\mathcal{T}_{,i}\alpha^0_{j,i} + \left[\frac{1}{2\rho}(\rho\mathcal{T}_{,i})_{,i} - \frac{5\rho}{\lambda+2\mu}\right]\alpha^0_j = 0. \tag{2.137}$$

Next, we can derive the governing equation for $\boldsymbol{A}^1$. Here we use vector notation instead of subscript notation for simplicity. To determine $\boldsymbol{A}^1$, we set $l = 1$ in (2.130) and form the cross product of the resulting equation with $\nabla\mathcal{T}$:

$$0 = -(\lambda+\mu)|\nabla\mathcal{T}|^2\nabla\mathcal{T}\times\boldsymbol{A}^1 - (\lambda+\mu)\nabla\mathcal{T}\times\nabla(\nabla\mathcal{T}\cdot\boldsymbol{A}^0) - 2\mu\nabla\mathcal{T}\times[(\nabla\mathcal{T}\cdot\nabla)\boldsymbol{A}^0] - (\nabla\mathcal{T}\times\nabla\lambda)(\nabla\mathcal{T}\cdot\boldsymbol{A}^0). \tag{2.138}$$

To go further, we need to use the following relations which can be verified directly,

$$\nabla\mathcal{T}\times[(\nabla\mathcal{T}\cdot\nabla)\boldsymbol{A}^0] = 2\mathcal{T}\nabla\mathcal{T}\times\left(\nabla\left(\frac{\rho}{\lambda+2\mu}\right)\boldsymbol{\alpha}^0\right) \tag{2.139}$$

and

$$\nabla\mathcal{T}\times\nabla(\nabla\mathcal{T}\cdot\boldsymbol{A}^0) = \frac{4\rho\mathcal{T}}{\lambda+2\mu}\nabla\mathcal{T}\times\nabla\boldsymbol{\alpha}^0 + 4\mathcal{T}\nabla\mathcal{T}\times\left(\nabla\left(\frac{\rho}{\lambda+2\mu}\right)\boldsymbol{\alpha}^0\right). \tag{2.140}$$

Inserting the above two relations into (2.138), we can obtain

$$\nabla\mathcal{T}\times\left(\boldsymbol{A}^1 + \nabla\boldsymbol{\alpha}^0 + \frac{(\lambda+2\mu)^2}{\rho(\lambda+\mu)}\nabla\left(\frac{\rho}{\lambda+2\mu}\right)\boldsymbol{\alpha}^0 + \frac{\nabla\lambda\boldsymbol{\alpha}^0}{\lambda+\mu}\right) = 0. \tag{2.141}$$

From (2.141) we may conclude that

$$\boldsymbol{A}^1 = \boldsymbol{C}^0 + \nabla\mathfrak{T}\boldsymbol{\alpha}^1, \tag{2.142}$$

where $\boldsymbol{\alpha}^1 = (\alpha_1^1, \alpha_2^1, \alpha_3^1)$ is a row vector to be determined, and

$$\boldsymbol{C}^0 = -\nabla\boldsymbol{\alpha}^0 - \frac{(\lambda+2\mu)^2}{\rho(\lambda+\mu)}\nabla\left(\frac{\rho}{\lambda+2\mu}\right)\boldsymbol{\alpha}^0 - \frac{\nabla\lambda\boldsymbol{\alpha}^0}{\lambda+\mu}. \tag{2.143}$$

To find $\boldsymbol{\alpha}^1$, we set $l = 2$ in (2.130) and form the scalar product of the resulting equation with $\nabla\mathfrak{T}$, yielding

$$\begin{aligned} 0 = {} & 10\rho(\nabla\mathfrak{T}\cdot\boldsymbol{A}^1) - (\lambda+\mu)\nabla\mathfrak{T}\cdot\nabla(\nabla\mathfrak{T}\cdot\boldsymbol{A}^1) - (\lambda+\mu)|\nabla\mathfrak{T}|^2(\nabla\cdot\mathbf{A}^1) \\ & - 2\mu\nabla\mathfrak{T}\cdot[(\nabla\mathfrak{T}\cdot\nabla)\boldsymbol{A}^1] - \mu\nabla^2\mathfrak{T}(\nabla\mathfrak{T}\cdot\boldsymbol{A}^1) - (\nabla\lambda\cdot\nabla\mathfrak{T})(\nabla\mathfrak{T}\cdot\boldsymbol{A}^1) \\ & - (\nabla\mu\cdot\mathbf{A}^1)|\nabla\mathfrak{T}|^2 - (\nabla\mu\cdot\nabla\mathfrak{T})(\nabla\mathfrak{T}\cdot\boldsymbol{A}^1) \\ & + (\lambda+\mu)\nabla\mathfrak{T}\cdot\nabla(\nabla\cdot\boldsymbol{A}^0) + \mu\nabla\mathfrak{T}\cdot(\nabla^2\boldsymbol{A}^0) + (\nabla\lambda\cdot\nabla\mathfrak{T})(\nabla\cdot\boldsymbol{A}^0) \\ & + \nabla\mathfrak{T}\cdot[\nabla\mu\times(\nabla\times\boldsymbol{A}^0)] + 2\nabla\mathfrak{T}\cdot[(\nabla\mu\cdot\nabla)\boldsymbol{A}^0]. \end{aligned} \tag{2.144}$$

Inserting (2.142) into (2.144), we obtain

$$\begin{aligned} 0 = {} & [10\rho - \mu\nabla^2\mathfrak{T} - \nabla\mathfrak{T}\cdot\nabla(\lambda+\mu)]|\nabla\mathfrak{T}|^2\boldsymbol{\alpha}^1 - (\lambda+\mu)\nabla\mathfrak{T}\cdot\nabla(|\nabla\mathfrak{T}|^2\boldsymbol{\alpha}^1) \\ & - (\lambda+\mu)|\nabla\mathfrak{T}|^2\nabla\cdot(\nabla\mathfrak{T}\boldsymbol{\alpha}^1) - 2\mu\nabla\mathfrak{T}\cdot[(\nabla\mathfrak{T}\cdot\nabla)(\nabla\mathfrak{T}\boldsymbol{\alpha}^1)] - (\nabla\mu\cdot\nabla\mathfrak{T})|\nabla\mathfrak{T}|^2\boldsymbol{\alpha}^1 \\ & + [10\rho - \mu\nabla^2\mathfrak{T} - \nabla\mathfrak{T}\cdot\nabla(\lambda+\mu)](\nabla\mathfrak{T}\cdot\boldsymbol{C}^0) - (\lambda+\mu)\nabla\mathfrak{T}\cdot\nabla(\nabla\mathfrak{T}\cdot\boldsymbol{C}^0) \\ & - (\lambda+\mu)|\nabla\mathfrak{T}|^2(\nabla\cdot\boldsymbol{C}^0) - 2\mu\nabla\mathfrak{T}\cdot[(\nabla\mathfrak{T}\cdot\nabla)\boldsymbol{C}^0] - (\nabla\mu\cdot\boldsymbol{C}^0)|\nabla\mathfrak{T}|^2 + \boldsymbol{d}^0, \end{aligned} \tag{2.145}$$

where

$$\begin{aligned} \boldsymbol{d}^0 = {} & (\lambda+\mu)\nabla\mathfrak{T}\cdot\nabla(\nabla\cdot\boldsymbol{A}^0) + \mu\nabla\mathfrak{T}\cdot\nabla^2\boldsymbol{A}^0 + (\nabla\lambda\cdot\nabla\mathfrak{T})(\nabla\cdot\boldsymbol{A}^0) \\ & + \nabla\mathfrak{T}\cdot[\nabla\mu\times(\nabla\times\boldsymbol{A}^0)] + 2\nabla\mathfrak{T}\cdot[(\nabla\mu\cdot\nabla)\boldsymbol{A}^0]. \end{aligned} \tag{2.146}$$

Using the following relations

$$\nabla\mathfrak{T}\cdot[(\nabla\mathfrak{T}\cdot\nabla)(\nabla\mathfrak{T}\boldsymbol{\alpha}^1)] = (\nabla\mathfrak{T}\cdot\nabla\boldsymbol{\alpha}^1)|\nabla\mathfrak{T}|^2 + \frac{1}{2}\nabla\mathfrak{T}\cdot\nabla(|\nabla\mathfrak{T}|^2)\boldsymbol{\alpha}^1 \tag{2.147}$$

and

$$\nabla\cdot(\nabla\mathfrak{T}\boldsymbol{\alpha}^1) = \nabla\mathfrak{T}\cdot\nabla\boldsymbol{\alpha}^1 + \nabla^2\mathfrak{T}\boldsymbol{\alpha}^1, \tag{2.148}$$

we may simplify (2.145) into

$$\nabla\mathfrak{T}\cdot\nabla\boldsymbol{\alpha}^1 + \left[\frac{1}{2\rho}\nabla\cdot(\rho\nabla\mathfrak{T}) - \frac{3\rho}{\lambda+2\mu}\right]\boldsymbol{\alpha}^1 = \boldsymbol{c}^0, \tag{2.149}$$

where $\boldsymbol{c}^0$ is defined to be

$$\begin{aligned} \boldsymbol{c}^0 \equiv {} & \frac{1}{8\rho\mathfrak{T}}\{[10\rho - \mu\nabla^2\mathfrak{T} - \nabla\mathfrak{T}\cdot\nabla(\lambda+\mu)](\nabla\mathfrak{T}\cdot\boldsymbol{C}^0) - (\lambda+\mu)\nabla\mathfrak{T}\cdot\nabla(\nabla\mathfrak{T}\cdot\boldsymbol{C}^0) \\ & - (\lambda+\mu)|\nabla\mathfrak{T}|^2(\nabla\cdot\boldsymbol{C}^0) - 2\mu\nabla\mathfrak{T}\cdot[(\nabla\mathfrak{T}\cdot\nabla)\boldsymbol{C}^0] - (\nabla\mu\cdot\boldsymbol{C}^0)|\nabla\mathfrak{T}|^2 + \boldsymbol{d}^0\}. \end{aligned} \tag{2.150}$$

2.4.2. P-wave transport equations for general $\boldsymbol{A}^l$. Similarly, we can derive governing equations for $\boldsymbol{A}^l$ when $l \geq 2$, which are stated as the following:

$$\boldsymbol{A}^l = \boldsymbol{C}^{l-1} + \nabla\mathcal{T}\boldsymbol{\alpha}^l, \tag{2.151}$$

where $\boldsymbol{\alpha}^l$ is to be determined, and $\boldsymbol{C}^{l-1}$ is available from already computed quantities by using the following formula,

$$\begin{aligned}\boldsymbol{C}^{l-1} \equiv \frac{1}{(\lambda+\mu)|\nabla\mathcal{T}|^2}\{&-\rho(4l-18)\boldsymbol{A}^{l-1}\\ &-(\lambda+\mu)\nabla(\nabla\mathcal{T}\cdot\boldsymbol{A}^{l-1}) - 2\mu(\nabla\mathcal{T}\cdot\nabla)\boldsymbol{A}^{l-1} - \mu\nabla^2\mathcal{T}\boldsymbol{A}^{l-1}\\ &-\nabla\lambda(\nabla\mathcal{T}\cdot\boldsymbol{A}^{l-1}) - (\nabla\mu\cdot\nabla\mathcal{T})\boldsymbol{A}^{l-1} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{A}^{l-2})\\ &+\mu\nabla^2\boldsymbol{A}^{l-2} + \nabla\lambda(\nabla\cdot\boldsymbol{A}^{l-2}) + \nabla\mu\times(\nabla\times\boldsymbol{A}^{l-2}) + 2(\nabla\mu\cdot\nabla)\boldsymbol{A}^{l-2}\}.\end{aligned} \tag{2.152}$$

Accordingly, we can derive the recursive equations for $\boldsymbol{\alpha}^l$,

$$\nabla\mathcal{T}\cdot\nabla\boldsymbol{\alpha}^l + \left[\frac{1}{2\rho}\nabla\cdot(\rho\nabla\mathcal{T}) + \frac{(2l-5)\rho}{\lambda+2\mu}\right]\boldsymbol{\alpha}^l = \boldsymbol{c}^{l-1}, \tag{2.153}$$

where $\boldsymbol{c}^{l-1}$ satisfies

$$\begin{aligned}\boldsymbol{c}^{l-1} \equiv \frac{1}{8\rho\mathcal{T}}\{&[(14-4l)\rho - \mu\nabla^2\mathcal{T} - \nabla\mathcal{T}\cdot\nabla(\lambda+\mu)](\nabla\mathcal{T}\cdot\boldsymbol{C}^{l-1})\\ &-(\lambda+\mu)\nabla\mathcal{T}\cdot\nabla(\nabla\mathcal{T}\cdot\boldsymbol{C}^{l-1}) - (\lambda+\mu)|\nabla\mathcal{T}|^2(\nabla\cdot\boldsymbol{C}^{l-1})\\ &-2\mu\nabla\mathcal{T}\cdot[(\nabla\mathcal{T}\cdot\nabla)\boldsymbol{C}^{l-1}] - (\nabla\mu\cdot\boldsymbol{C}^{l-1})|\nabla\mathcal{T}|^2\\ &+(\lambda+\mu)\nabla\mathcal{T}\cdot\nabla(\nabla\cdot\boldsymbol{A}^{l-1}) + \mu\nabla\mathcal{T}\cdot\nabla^2\boldsymbol{A}^{l-1} + (\nabla\lambda\cdot\nabla\mathcal{T})(\nabla\cdot\boldsymbol{A}^{l-1})\\ &+\nabla\mathcal{T}\cdot[\nabla\mu\times(\nabla\times\boldsymbol{A}^{l-1})] + 2\nabla\mathcal{T}\cdot[(\nabla\mu\cdot\nabla)\boldsymbol{A}^{l-1}]\}\end{aligned} \tag{2.154}$$

with $\boldsymbol{C}^l = 0$ and $\boldsymbol{c}^l = 0$ for $l < 0$.

2.4.3. P wave: Verification in a homogeneous medium. In a homogeneous medium, the parameters ρ, λ, and μ are constants. In this case, (2.137) reduces to

$$\mathcal{T}_{,i}\alpha^0_{j,i} - \frac{2\rho}{\lambda+2\mu}\alpha^0_j = 0. \tag{2.155}$$

Here we have $\mathcal{T} = {\gamma^P}^2 r^2 = \frac{\rho}{\lambda+2\mu}r^2$ in a homogeneous medium. We have the exact Green's function for the P wave, which we rewrite here as

$$\ddot{G}^P_{ij} = \frac{{\gamma^P}^3}{\rho\pi}[2{\gamma^P}^2 r^2 t_i t_j f_+^{(-3)}(t^2 - {\gamma^P}^2 r^2) - \delta_{ij} f_+^{(-2)}(t^2 - {\gamma^P}^2 r^2)]. \tag{2.156}$$

To simplify the derivation, we assume that the source is at the origin in this subsection. From this, we obtain $A^0_{ij} = 2\frac{{\gamma^P}^5}{\rho\pi}x_i x_j$ and $A^1_{ij} = -\frac{{\gamma^P}^3}{\rho\pi}\delta_{ij}$. So for any $1 \leq j \leq 3$, $A^0_{ij} = \frac{{\gamma^P}^3}{\rho\pi}\mathcal{T}_{,i}x_j$, and the corresponding $\alpha^0_j = \frac{{\gamma^P}^3}{\rho\pi}x_j$. Since $\alpha^0_{j,i} = \frac{{\gamma^P}^3}{\rho\pi}\delta_{ij}$, we have

$$\mathcal{T}_{,i}\alpha^0_{j,i} - \frac{2\rho}{\lambda+2\mu}\alpha^0_j = 2\frac{{\gamma^P}^5}{\rho\pi}x_i\delta_{ij} - 2\frac{{\gamma^P}^5}{\rho\pi}x_j = 0, \tag{2.157}$$

which verifies (2.155).

Now we verify (2.142) in a uniform medium, i.e., $A^1_{ij} = C^0_{ij} + \alpha^1_j \mathcal{T}_{,i}$. From (2.143), we have

$$C^0_{ij} = -\alpha^0_{j,i} = -\frac{{\gamma^P}^3}{\rho\pi}\delta_{ij}, \tag{2.158}$$

yielding

$$A^1_{ij} - C^0_{ij} = -\frac{{\gamma^P}^3}{\rho\pi}\delta_{ij} + \frac{{\gamma^P}^3}{\rho\pi}\delta_{ij} = 0, \tag{2.159}$$

which implies that $\alpha^1_j = 0$.

To see that $\alpha^1_j = 0$ satisfies (2.149) in a homogeneous medium, we just need to check that c^0_j defined in (2.150) vanishes,

$$c^0_j = \frac{1}{8\rho\mathcal{T}}\{(10\rho - \mu\mathcal{T}_{,kk})C^0_{ij}\mathcal{T}_{,i} - (\lambda+\mu)\mathcal{T}_{,i}(C^0_{kj}\mathcal{T}_{,k})_{,i} + d^0_j\}. \tag{2.160}$$

First we find that $\mathbf{d}^0$ in (2.146) is given by

$$\begin{aligned} d^0_j &= (\lambda+\mu)\mathcal{T}_{,i}A^0_{kj,ki} + \mu\mathcal{T}_{,i}A^0_{ij,kk} \\ &= 16(\lambda+\mu)\frac{{\gamma^P}^7}{\rho\pi}x_j + 8\mu\frac{{\gamma^P}^7}{\rho\pi}x_j. \end{aligned} \tag{2.161}$$

Inserting this into (2.160), we can obtain

$$\begin{aligned} c^0_j &= \frac{1}{8\rho\mathcal{T}}\left\{-\frac{20{\gamma^P}^5 x_j}{\pi} + \frac{12\mu{\gamma^P}^7 x_j}{\rho\pi} + \frac{4(\lambda+\mu){\gamma^P}^7 x_j}{\rho\pi} + \frac{16(\lambda+\mu){\gamma^P}^7 x_j}{\rho\pi} + \frac{8\mu{\gamma^P}^7 x_j}{\rho\pi}\right\} \\ &= 0. \end{aligned} \tag{2.162}$$

Similarly, we can verify that the coefficients A^l_{ij} are 0 for $l \geq 2$ in a homogeneous medium.

2.4.4. P wave: Initial conditions for $\boldsymbol{A}^l$. In this section, we shall initialize those amplitude coefficients $\boldsymbol{A}^l$ for the P wave at the source $\boldsymbol{r}_0$ in the three dimensional space. In a center-deleted neighborhood of $\boldsymbol{r}_0$, say $0 < |\boldsymbol{r} - \boldsymbol{r}_0| < x_0$, where x_0 is a positive constant, we propose the following matching condition,

$$(\ddot{\boldsymbol{G}}^P(\boldsymbol{r};t) - \ddot{\boldsymbol{G}}^P_{hom}(\boldsymbol{r};t))t^n \in L^1(\{t;(0,T)\}) \tag{2.163}$$

over any finite time period $(0,T)$ for any nonnegative integer n, where

$$\ddot{\boldsymbol{G}}^P_{hom} = \frac{{\gamma^P_0}^3}{\rho_0\pi}\left[2{\gamma^P_0}^2 r^2 \boldsymbol{t}\boldsymbol{t}^T f_+^{(-3)}(t^2 - {\gamma^P_0}^2 r^2) - \boldsymbol{I} f_+^{(-2)}(t^2 - {\gamma^P_0}^2 r^2)\right] \tag{2.164}$$

with $\boldsymbol{t} = \frac{\boldsymbol{r}-\boldsymbol{r}_0}{|\boldsymbol{r}-\boldsymbol{r}_0|}$ being the unit tangent vector to the P ray. Here $\boldsymbol{t}\boldsymbol{t}^T = (t_i t_j)_{3\times3}$ is a matrix, and $\gamma^P_0 = \sqrt{\frac{\rho_0}{\lambda_0+2\mu_0}}$ is the slowness for the P wave with $\rho_0 = \rho(\boldsymbol{r}_0)$, $\mu_0 = \mu(\boldsymbol{r}_0)$, and $\lambda_0 = \lambda(\boldsymbol{r}_0)$.

Since $f_+^{(-3+l)}(t^2-\mathfrak{T})\in L^1(\{t;(0,T)\})$ for $l\geq 3$, the above matching condition can be reduced to

$$\begin{aligned}(2.165)\qquad \ddot{\boldsymbol{G}}^n_{\rm diff}&=[\ddot{\boldsymbol{G}}^P(\boldsymbol{r};t)-\ddot{\boldsymbol{G}}^P_{hom}(\boldsymbol{r};t)]t^n\\&=\boldsymbol{A}^0f_+^{(-3)}(t^2-\mathfrak{T}(\boldsymbol{r}))t^n+\boldsymbol{A}^1f_+^{(-2)}(t^2-\mathfrak{T}(\boldsymbol{r}))t^n\\&\quad+\boldsymbol{A}^2f_+^{(-1)}(t^2-\mathfrak{T}(\boldsymbol{r}))t^n-\ddot{\boldsymbol{G}}^P_{hom}t^n\\&=\boldsymbol{A}^0\delta^{(2)}(t^2-\mathfrak{T}(\boldsymbol{r}))t^n+\boldsymbol{A}^1\delta^{(1)}(t^2-\mathfrak{T}(\boldsymbol{r}))t^n+\boldsymbol{A}^2\delta(t^2-\mathfrak{T}(\boldsymbol{r}))t^n\\&\quad-\frac{{\gamma_0^P}^3}{\rho_0\pi}[2{\gamma_0^P}^2r^2\boldsymbol{tt}^T\delta^{(2)}(t^2-{\gamma_0^P}^2r^2)-\boldsymbol{I}\delta^{(1)}(t^2-{\gamma_0^P}^2r^2)]t^n\\&\in L^1(\{t;(0,T)\})\end{aligned}$$

for any $\boldsymbol{r}$ satisfying $0<|\boldsymbol{r}-\boldsymbol{r}_0|<x_0$.

For $\boldsymbol{r}$ sufficiently close to $\boldsymbol{r}_0$ such that all δ-related functions do not vanish, one gets

$$\begin{aligned}(2.166)\qquad \int_0^T\delta^{(k)}[t^2-\mathfrak{T}(\boldsymbol{r})]t^ndt&=\int_{-\mathfrak{T}(\boldsymbol{r})}^{T^2-\mathfrak{T}(\boldsymbol{r})}\delta^{(k)}(\tilde{t})\frac{1}{2}(\tilde{t}+\mathfrak{T}(\boldsymbol{r}))^{\frac{n}{2}-\frac{1}{2}}d\tilde{t}\\&=\frac{(-1)^k}{2}\prod_{j=0}^{k-1}\left(\frac{n}{2}-\frac{1}{2}-j\right)[\mathfrak{T}(\boldsymbol{r})]^{\frac{n}{2}-\frac{1}{2}-k}.\end{aligned}$$

Since k is at most 2 in our situation, $\frac{n}{2}-\frac{1}{2}-k\geq 0$ for $n\geq 5$ so that we are concerned about the five cases $n=0,1,2,3,4$ only. Therefore, we enforce

$$\begin{aligned}(2.167)\qquad \int_0^T\ddot{\boldsymbol{G}}^0_{\rm diff}dt&=\frac{3}{8}\left[\frac{\boldsymbol{A}^0}{\mathfrak{T}(\boldsymbol{r})^{\frac{5}{2}}}-\frac{2{\gamma_0^P}^5r^2}{\rho_0\pi(\gamma_0^Pr)^5}\boldsymbol{tt}^T\right]\\&\quad+\frac{1}{4}\left[\frac{\boldsymbol{A}^1}{\mathfrak{T}(\boldsymbol{r})^{\frac{3}{2}}}+\frac{{\gamma_0^P}^3}{\rho_0\pi(\gamma_0^Pr)^3}\boldsymbol{I}\right]+\frac{\boldsymbol{A}^2}{2\mathfrak{T}(\boldsymbol{r})^{\frac{1}{2}}}\\&=\frac{3}{8}\frac{\boldsymbol{A}^0}{\mathfrak{T}(\boldsymbol{r})^{\frac{5}{2}}}+\frac{1}{4}\frac{\boldsymbol{A}^1}{\mathfrak{T}(\boldsymbol{r})^{\frac{3}{2}}}+\frac{\boldsymbol{A}^2}{2\mathfrak{T}(\boldsymbol{r})^{\frac{1}{2}}}+\frac{1}{4\rho_0\pi r^3}(\boldsymbol{I}-3\boldsymbol{tt}^T)=O(1),\end{aligned}$$

$$(2.168)\qquad \int_0^T\ddot{\boldsymbol{G}}^1_{\rm diff}dt=\frac{\boldsymbol{A}^2}{2}=O(1),$$

$$\begin{aligned}(2.169)\qquad \int_0^T\ddot{\boldsymbol{G}}^2_{\rm diff}dt&=-\frac{1}{8}\left[\frac{\boldsymbol{A}^0}{\mathfrak{T}(\boldsymbol{r})^{\frac{3}{2}}}-\frac{2{\gamma_0^P}^5r^2}{\rho_0\pi(\gamma_0^Pr)^3}\boldsymbol{tt}^T\right]\\&\quad-\frac{1}{4}\left[\frac{\boldsymbol{A}^1}{\mathfrak{T}(\boldsymbol{r})^{\frac{1}{2}}}+\frac{{\gamma_0^P}^3}{\rho_0\pi(\gamma_0^Pr)}\boldsymbol{I}\right]+\frac{\boldsymbol{A}^2}{2}\mathfrak{T}(\boldsymbol{r})^{\frac{1}{2}}\\&=-\frac{1}{8}\frac{\boldsymbol{A}^0}{\mathfrak{T}(\boldsymbol{r})^{\frac{3}{2}}}-\frac{1}{4}\frac{\boldsymbol{A}^1}{\mathfrak{T}(\boldsymbol{r})^{\frac{1}{2}}}+\frac{\boldsymbol{A}^2}{2}\mathfrak{T}(\boldsymbol{r})^{\frac{1}{2}}-\frac{{\gamma_0^P}^2}{4\rho_0\pi r}(\boldsymbol{I}-\boldsymbol{tt}^T)=O(1),\end{aligned}$$

$$(2.170)\qquad \int_0^T\ddot{\boldsymbol{G}}^3_{\rm diff}dt=-\frac{1}{2}\left[\boldsymbol{A}^1+\frac{{\gamma_0^P}^3}{\rho_0\pi}\right]+\frac{1}{2}\boldsymbol{A}^2\mathfrak{T}(\boldsymbol{r})=O(1),$$

$$
\begin{aligned}
(2.171)\quad \int_0^T \ddot{\boldsymbol{G}}^4_{\text{diff}} dt &= \frac{3}{8}\left[\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} - \frac{2{\gamma_0^P}^5 r^2}{\rho_0\pi(\gamma_0^P r)}\boldsymbol{t}\boldsymbol{t}^T\right] \\
&\quad - \frac{3}{4}\left[\boldsymbol{A}^1\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}} + \frac{{\gamma_0^P}^3}{\rho_0\pi}(\gamma_0^P r)\boldsymbol{I}\right] + \frac{\boldsymbol{A}^2}{2}\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}} \\
&= \frac{3}{8}\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} - \frac{3}{4}\boldsymbol{A}^1\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}} + \frac{\boldsymbol{A}^2}{2}\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}} - \frac{3{\gamma_0^P}^4 r}{4\rho_0\pi}(\boldsymbol{I}+\boldsymbol{t}\boldsymbol{t}^T) = O(1)
\end{aligned}
$$

for $0 < |\boldsymbol{r}-\boldsymbol{r}_0| < x_0$.

Hiding the $O(1)$ terms above, we get

$$
(2.172)\quad \frac{3}{8}\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{5}{2}}} + \frac{1}{4}\frac{\boldsymbol{A}^1}{\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}}} + \frac{\boldsymbol{A}^2}{2\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} + \frac{1}{4\rho_0\pi r^3}(\boldsymbol{I}-3\boldsymbol{t}\boldsymbol{t}^T) = O(1),
$$

$$
(2.173)\quad -\frac{1}{8}\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}}} - \frac{1}{4}\frac{\boldsymbol{A}^1}{\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} - \frac{{\gamma_0^P}^2}{4\rho_0\pi r}(\boldsymbol{I}-\boldsymbol{t}\boldsymbol{t}^T) = O(1),
$$

$$
(2.174)\quad \frac{3}{8}\frac{\boldsymbol{A}^0}{\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}} = O(1).
$$

Considering (2.172) $\times\, \mathcal{T}(\boldsymbol{r})$ + (2.173), we can obtain the initial condition for $\boldsymbol{A}^0$ as $\boldsymbol{r}\to\boldsymbol{r}_0$,

$$
(2.175)\quad \boldsymbol{A}^0 = -\frac{\mathcal{T}^{\frac{5}{2}}}{\rho_0\pi r^3}(\boldsymbol{I}-3\boldsymbol{t}\boldsymbol{t}^T) + \frac{{\gamma_0^P}^2\mathcal{T}^{\frac{3}{2}}}{\rho_0\pi r}(\boldsymbol{I}-\boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}^{\frac{3}{2}}).
$$

Moreover, since τ is the travel time from the source $\boldsymbol{r}_0$ to $\boldsymbol{r}$ which is proportional to $r = |\boldsymbol{r}-\boldsymbol{r}_0|$, $\mathcal{T}=\tau^2$ is proportional to r^2 as $\boldsymbol{r}\to\boldsymbol{r}_0$, leading to $\boldsymbol{A}^0 = O(1)$ as $\boldsymbol{r}\to\boldsymbol{r}_0$.

Next, (2.172) $\times\, 4\mathcal{T}(\boldsymbol{r})^{\frac{3}{2}}$ gives rise to the initial condition for $\boldsymbol{A}^1$ as $\boldsymbol{r}\to\boldsymbol{r}_0$,

$$
\begin{aligned}
\boldsymbol{A}^1 &= -\frac{3}{2}\frac{\boldsymbol{A}^0}{\mathcal{T}} - \frac{\mathcal{T}^{\frac{3}{2}}}{\rho_0\pi r^3}(\boldsymbol{I}-3\boldsymbol{t}\boldsymbol{t}^T) + 2\boldsymbol{A}^2\mathcal{T} + O(\mathcal{T}^{\frac{3}{2}}) \\
(2.176)\quad &= -\frac{3}{2}\frac{\boldsymbol{A}^0}{\mathcal{T}} - \frac{\mathcal{T}^{\frac{3}{2}}}{\rho_0\pi r^3}(\boldsymbol{I}-3\boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}).
\end{aligned}
$$

Substituting (2.175) into (2.176), we get

$$
(2.177)\quad \boldsymbol{A}^1 = \frac{1}{2}\frac{\mathcal{T}^{\frac{3}{2}}}{\rho_0\pi r^3}(\boldsymbol{I}-3\boldsymbol{t}\boldsymbol{t}^T) - \frac{3}{2}\frac{{\gamma_0^P}^2\mathcal{T}^{\frac{1}{2}}}{\rho_0\pi r}(\boldsymbol{I}-\boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}^{\frac{1}{2}}).
$$

Similar to the case for $\boldsymbol{A}^0$, we have $\boldsymbol{A}^1 = O(1)$ as $\boldsymbol{r}\to\boldsymbol{r}_0$.

Finally, (2.172) $\times\, 2\mathcal{T}(\boldsymbol{r})^{\frac{1}{2}}$ gives rise to the initial condition for $\boldsymbol{A}^2$ as $\boldsymbol{r}\to\boldsymbol{r}_0$,

$$
(2.178)\quad \boldsymbol{A}^2 = -\frac{3}{4}\frac{\boldsymbol{A}^0}{\mathcal{T}^2} - \frac{1}{2}\frac{\boldsymbol{A}^1}{\mathcal{T}} - \frac{\mathcal{T}^{\frac{1}{2}}}{2\rho_0\pi r^3}(\boldsymbol{I}-3\boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}^{\frac{1}{2}}).
$$

When substituting (2.176) into (2.178), the first three terms on the right-hand side of (2.178) will be canceled. So we have $\boldsymbol{A}^2 = O(1)$ as $\boldsymbol{r}\to\boldsymbol{r}_0$ which is also consistent with (2.168).

2.4.5. P wave: Governing equations for desingularizing $\boldsymbol{A}^l$. Although we enforce the condition that $\boldsymbol{A}^l$ are $O(1)$ near the source for all $l \geq 0$, they are not necessarily smooth at the source $\boldsymbol{r}_0$ in an inhomogeneous medium; this can be seen from the initial conditions (2.175)–(2.178). Therefore, we desingularize $\boldsymbol{A}^l$ by introducing a new set of dyadic coefficients,

$$\tilde{\boldsymbol{A}}^l = \boldsymbol{A}^l \tau^{2l}, \tag{2.179}$$

so that

$$\boldsymbol{A}^l = O(1) \text{ as } \boldsymbol{r} \to \boldsymbol{r}_0,$$
$$\tilde{\boldsymbol{A}}^l = O(\tau^{2l}) \text{ as } \boldsymbol{r} \to \boldsymbol{r}_0.$$

According to (2.151) and (2.179), we can set

$$\tilde{\boldsymbol{A}}^l = \tilde{\boldsymbol{C}}^{l-1} + \nabla \mathcal{T} \tilde{\boldsymbol{\alpha}}^l, \tag{2.180}$$

where $\tilde{\boldsymbol{C}}^{l-1} = \boldsymbol{C}^{l-1}\mathcal{T}^l$ and $\tilde{\boldsymbol{\alpha}}^l = \boldsymbol{\alpha}^l \mathcal{T}^l$. Hence, we can derive transport equations for $\tilde{\boldsymbol{\alpha}}^l$ from transport equations for $\boldsymbol{\alpha}^l$ for $l = 0, 1$ as the following,

$$\nabla \mathcal{T} \cdot \nabla \tilde{\boldsymbol{\alpha}}^0 + \left[\frac{1}{2\rho}\nabla \cdot (\rho \nabla \mathcal{T}) - \frac{5\rho}{\lambda + 2\mu}\right] \tilde{\boldsymbol{\alpha}}^0 = 0, \tag{2.181}$$

$$\nabla \mathcal{T} \cdot \nabla \tilde{\boldsymbol{\alpha}}^1 + \left[\frac{1}{2\rho}\nabla \cdot (\rho \nabla \mathcal{T}) - \frac{7\rho}{\lambda + 2\mu}\right] \tilde{\boldsymbol{\alpha}}^1 = \tilde{\boldsymbol{c}}^0, \tag{2.182}$$

where $\tilde{\boldsymbol{c}}^0$ can be computed by the following formula,

$$\begin{aligned} \tilde{\boldsymbol{c}}^0 \equiv & \frac{1}{8\rho}\{10\rho - \mu \nabla^2 \mathcal{T} - \nabla \mathcal{T} \cdot \nabla(\lambda + \mu)](\nabla \mathcal{T} \cdot \boldsymbol{C}^0) - (\lambda + \mu)\nabla \mathcal{T} \cdot \nabla(\nabla \mathcal{T} \cdot \boldsymbol{C}^0) \\ & - (\lambda + \mu)|\nabla \mathcal{T}|^2 (\nabla \cdot \boldsymbol{C}^0) - 2\mu \nabla \mathcal{T} \cdot [(\nabla \mathcal{T} \cdot \nabla)\boldsymbol{C}^0] - (\nabla \mu \cdot \boldsymbol{C}^0)|\nabla \mathcal{T}|^2 + \boldsymbol{d}^0\}, \end{aligned} \tag{2.183}$$

where $\boldsymbol{C}^0$ and $\boldsymbol{d}^0$ are defined in (2.143) and (2.146) since $\tilde{\boldsymbol{A}}^0 = \boldsymbol{A}^0$.

For general $l \geq 2$, we can similarly derive the following governing equations,

$$\nabla \mathcal{T} \cdot \nabla \tilde{\boldsymbol{\alpha}}^l + \left[\frac{1}{2\rho}\nabla \cdot (\rho \nabla \mathcal{T}) - \frac{(2l+5)\rho}{\lambda + 2\mu}\right] \tilde{\boldsymbol{\alpha}}^l = \tilde{\boldsymbol{c}}^{l-1}, \tag{2.184}$$

where $\tilde{\boldsymbol{c}}^{l-1} = \mathcal{T}^l \boldsymbol{c}^{l-1}$, and $\boldsymbol{c}^{l-1}$ is the same as (2.154). The detailed formula for $\tilde{\boldsymbol{c}}^{l-1}$ is given in Appendix A.

We are interested in the initialization for $\tilde{\boldsymbol{\alpha}}^l$, but we only have the relation (2.180) between $\tilde{\boldsymbol{\alpha}}^l$ and $\tilde{\boldsymbol{A}}^l$. It will become apparent later that we can initialize both $\tilde{\boldsymbol{\alpha}}^l$ and $\tilde{\boldsymbol{A}}^l$ based on the relation (2.180) and the governing equations (2.184) for $\tilde{\boldsymbol{\alpha}}^l$.

First, we consider the initialization for $\tilde{\boldsymbol{A}}^l$, $0 \leq l \leq 2$. We can get the following initial condition for $\tilde{\boldsymbol{A}}^0$, $\tilde{\boldsymbol{A}}^1$, and $\tilde{\boldsymbol{A}}^2$ from (2.175)–(2.179),

$$\tilde{\boldsymbol{A}}^0 = -\frac{\mathcal{T}^{\frac{5}{2}}}{\rho_0 \pi r^3}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T) + \frac{{\gamma^P}^2 \mathcal{T}^{\frac{3}{2}}}{\rho_0 \pi r}(\boldsymbol{I} - \boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}^{\frac{3}{2}}), \tag{2.185}$$

$$\tilde{\boldsymbol{A}}^1 = -\frac{3}{2}\tilde{\boldsymbol{A}}^0 - \frac{\mathcal{T}^{\frac{5}{2}}}{\rho_0 \pi r^3}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}^2), \tag{2.186}$$

$$\tilde{\boldsymbol{A}}^2 = -\frac{3}{4}\tilde{\boldsymbol{A}}^0 - \frac{1}{2}\tilde{\boldsymbol{A}}^1 - \frac{\mathcal{T}^{\frac{5}{2}}}{2\rho_0 \pi r^3}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T) + O(\mathcal{T}^{\frac{5}{2}}). \tag{2.187}$$

Assuming that the refractive index ${\gamma^P}^2(\boldsymbol{r})$ is analytic, ${\gamma^P}^2(\boldsymbol{r})$ can be written as the following power series

$$
{\gamma^P}^2(\boldsymbol{r}) = \sum_{k=0}^{\infty} \Psi_k(\boldsymbol{r};\boldsymbol{r}_0), \tag{2.188}
$$

where Ψ_k are homogeneous polynomials of degree k in the Taylor expansion of ${\gamma^P}^2$ about the source $\boldsymbol{r}_0$ and $\Psi_0 = {\gamma_0^P}^2$ with $\gamma_0^P = \gamma^P(\boldsymbol{r}_0)$.

In an isotropic medium, when γ^P is smooth, the travel time function $\tau(\cdot;\boldsymbol{r}_0)$ solving the eikonal equation $|\nabla\tau(\boldsymbol{r};\boldsymbol{r}_0)| = \gamma^P(\boldsymbol{r})$ is locally smooth in the source neighborhood except at the source itself [19, 27]. Therefore, we may assume that τ^2 is analytic, so that we have in the source neighborhood [15],

$$
\tau^2 - \left(\Psi_0 + \frac{1}{2}\Psi_1 + \frac{1}{3}\Psi_2 - \frac{1}{48\Psi_0}|\nabla\Psi_1|^2 r^2\right) r^2 = O(r^5). \tag{2.189}
$$

Similarly to the case for the S wave, we can derive the initial conditions for the first three dyadic coefficients.

(a) The sequence $\{\tilde{\boldsymbol{A}}_{0k}\}$ is initialized by

$$
\tilde{\boldsymbol{A}}_{00} = 0,\ \tilde{\boldsymbol{A}}_{01} = 0,\ \tilde{\boldsymbol{A}}_{02} = \frac{2{\gamma_0^P}^5 r^2}{\rho_0\pi}\boldsymbol{t}\boldsymbol{t}^T. \tag{2.190}
$$

(b) The sequence $\{\tilde{\boldsymbol{A}}_{1k}\}$ is initialized by

$$
\begin{aligned}
&\tilde{\boldsymbol{A}}_{10} = 0,\ \tilde{\boldsymbol{A}}_{11} = 0,\ \tilde{\boldsymbol{A}}_{12} = -\frac{{\gamma_0^P}^5 r^2}{\rho_0\pi}\boldsymbol{I},\\
&\tilde{\boldsymbol{A}}_{13} = -\frac{3}{2}\tilde{\boldsymbol{A}}_{03} - \frac{5{\gamma_0^P}^3\Psi_1 r^2}{4\rho_0\pi}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T).
\end{aligned} \tag{2.191}
$$

(c) The sequence $\{\tilde{\boldsymbol{A}}_{2k}\}$ is initialized by

(2.192)

$$
\begin{aligned}
&\tilde{\boldsymbol{A}}_{20} = 0,\ \tilde{\boldsymbol{A}}_{21} = 0, \tilde{\boldsymbol{A}}_{22} = 0,\ \tilde{\boldsymbol{A}}_{23} = 0,\\
&\tilde{\boldsymbol{A}}_{24} = -\frac{3}{4}\tilde{\boldsymbol{A}}_{04} - \frac{1}{2}\tilde{\boldsymbol{A}}_{14} - \frac{{\gamma_0^P}^3}{2\rho_0\pi}(\boldsymbol{I} - 3\boldsymbol{t}\boldsymbol{t}^T)\left[\frac{5\Psi_2}{6}r^2 + \frac{15\Psi_1^2}{32\Psi_0}r^2 - \frac{5|\nabla\Psi_1|^2}{96\Psi_0}r^4\right].
\end{aligned}
$$

If we further assume that $\log\rho(\boldsymbol{r})$, $\tilde{\boldsymbol{\alpha}}^l$, $\tilde{\boldsymbol{C}}^{l-1}$, and $\tilde{\boldsymbol{c}}^{l-1}$ for $l \geq 0$ are all analytic in a neighborhood of the source $\boldsymbol{r}_0$, we can expand these functions as Taylor series at the source,

$$
\log\rho(\boldsymbol{r}) = \sum_{k=0}^{\infty} P_k(\boldsymbol{r}), \tag{2.193}
$$

$$
\tilde{\boldsymbol{\alpha}}^l(\boldsymbol{r}) = \sum_{k=0}^{\infty} \tilde{\boldsymbol{\alpha}}_{lk}(\boldsymbol{r}), \tag{2.194}
$$

$$
\tilde{\boldsymbol{C}}^{l-1}(\boldsymbol{r}) = \sum_{k=0}^{\infty} \tilde{\boldsymbol{C}}_{l-1,k}(\boldsymbol{r}), \tag{2.195}
$$

$$\tilde{\boldsymbol{c}}^{l-1}(\boldsymbol{r}) = \sum_{k=0}^{\infty} \tilde{\boldsymbol{c}}_{l-1,k}(\boldsymbol{r}), \tag{2.196}$$

where the term with subscript k denotes a homogeneous polynomial of degree k.

As $\mathcal{T} = \tau^2$ is analytic near the source neighborhood, we denote its Taylor series as

$$\mathcal{T}(\boldsymbol{r}) = \tau^2(\boldsymbol{r}) = \sum_{k=0}^{\infty} T_k(\boldsymbol{r}). \tag{2.197}$$

Moreover, from the formula (2.180), $\tilde{\boldsymbol{\alpha}}^l$ can be expanded

$$\sum_{k=0}^{\infty} \tilde{\boldsymbol{A}}_{lk} = \sum_{k=0}^{\infty} \tilde{\boldsymbol{C}}_{l-1,k} + \sum_{k=0}^{\infty} \nabla \mathcal{T}_k \sum_{k=0}^{\infty} \tilde{\boldsymbol{\alpha}}_{lk}. \tag{2.198}$$

Equating the $(q+1)$th degree polynomials of both sides, we can get

$$\nabla \mathcal{T}_2 \tilde{\boldsymbol{\alpha}}_{lq} = \tilde{\boldsymbol{A}}_{l,q+1} - \tilde{\boldsymbol{C}}_{l-1,q+1} - \sum_{k=0}^{q-1} \nabla \mathcal{T}_{q+2-k} \tilde{\boldsymbol{\alpha}}_{lk}, \tag{2.199}$$

and forming the dot product of the above with $\nabla \mathcal{T}_2$ yields

$$\tilde{\boldsymbol{\alpha}}_{lq} = \frac{1}{4\Psi_0^2 r^2} \left(\nabla \mathcal{T}_2 \cdot \tilde{\boldsymbol{A}}_{l,q+1} - \nabla \mathcal{T}_2 \cdot \tilde{\boldsymbol{C}}_{l-1,q+1} - \sum_{k=0}^{q-1} (\nabla \mathcal{T}_2 \cdot \nabla \mathcal{T}_{q+2-k}) \tilde{\boldsymbol{\alpha}}_{lk} \right), \tag{2.200}$$

which is useful for initializing the computation of $\tilde{\boldsymbol{\alpha}}_l$.

On the other hand, the transport equations (2.184) for $\tilde{\boldsymbol{\alpha}}^l$ can be expanded,

$$\begin{aligned} &\sum_{k=2}^{\infty} \nabla T_k(\boldsymbol{r}) \cdot \sum_{k=0}^{\infty} \nabla \tilde{\boldsymbol{\alpha}}_{lk}(\boldsymbol{r}) \\ &+ \left[\frac{1}{2} \sum_{k=2}^{\infty} \Delta T_k(\boldsymbol{r}) + \frac{1}{2} \sum_{k=2}^{\infty} \nabla T_k(\boldsymbol{r}) \sum_{k=0}^{\infty} \nabla P_k(\boldsymbol{r}) - (2l+5) \sum_{k=0}^{\infty} \Psi_k(\boldsymbol{r}) \right] \sum_{k=0}^{\infty} \tilde{\boldsymbol{\alpha}}_{lk}(\boldsymbol{r}) \\ &= \sum_{k=0}^{\infty} \tilde{\boldsymbol{c}}_{l-1,k}(\boldsymbol{r}). \end{aligned} \tag{2.201}$$

Equating qth degree polynomials of both sides, we have

$$\begin{aligned} (2q-2l-2)\Psi_0 \tilde{\boldsymbol{\alpha}}_{lq} = \tilde{\boldsymbol{c}}_{l-1,q} &- \sum_{k=1}^{q-1} \nabla T_{2+q-k} \cdot \nabla \tilde{\boldsymbol{\alpha}}_{lk} - \frac{1}{2} \sum_{k=0}^{q-1} \Delta T_{2+q-k} \tilde{\boldsymbol{\alpha}}_{lk} \\ &+ (2l+5) \sum_{k=0}^{q-1} \Psi_{q-k} \tilde{\boldsymbol{\alpha}}_{lk} - \frac{1}{2} \sum_{k=0}^{q-1} \tilde{\boldsymbol{\alpha}}_{lk} \left(\sum_{j=1}^{q-k} \nabla T_{2+q-j-k} \cdot \nabla P_j \right). \end{aligned} \tag{2.202}$$

Hence we obtain a recursive formula to compute the qth degree polynomial $\tilde{\boldsymbol{\alpha}}_{lq}$. It is clear that (2.202) is not valid for computing $\tilde{\boldsymbol{\alpha}}_{lq^*}$ for $q^* = l+1$ since the coefficient of $\tilde{\boldsymbol{\alpha}}_{lq^*}$ vanishes. Two cases arise here: case (a) for $l \geq 3$ and case (b) for $0 \leq l \leq 2$.

Consider case (a) for $l \geq 3$ first; in this case, $\tilde{\boldsymbol{A}}^l$ have been assumed to be $O(\mathcal{T}^l) = O(r^{2l})$. From the formula (2.180), it implies that $\tilde{\boldsymbol{\alpha}}^l$ behaves at least $O(r^{2l-1}) = o(r^{l+1})$ due to $2l \geq l+2$, and we further have that $\tilde{\boldsymbol{\alpha}}_{lq^*} = 0$ for $q^* = l+1$.

Next, consider case (b) for $0 \leq l \leq 2$; in this case, $\tilde{\boldsymbol{\alpha}}_{01}$, $\tilde{\boldsymbol{\alpha}}_{12}$ and $\tilde{\boldsymbol{\alpha}}_{23}$ can be directly initialized from initial conditions (2.190)–(2.192) and the recursive formula (2.200), respectively.

In the recursive formula (2.202), $\tilde{\boldsymbol{c}}_{l-1,q}$ depends on $\tilde{\boldsymbol{\alpha}}^{l-1}$ but does not depend on $\tilde{\boldsymbol{\alpha}}^l$; therefore, the power series of $\tilde{\boldsymbol{\alpha}}^l$ is computable via (2.202) once $\tilde{\boldsymbol{\alpha}}^{l-1}$ is obtained near the source.

3. Hadamard–Babich ansatz in the frequency domain. The Fourier transform of (2.29) in time yields the FDPS elastic wave equations (1.1) with

$$\boldsymbol{G}(\boldsymbol{r};\boldsymbol{r}_0) = \int_0^\infty \boldsymbol{G}(\boldsymbol{r},t;\boldsymbol{r}_0)e^{i\omega t}dt, \tag{3.1}$$

so that by (2.29), (2.103), and (2.179), we get the following frequency-domain asymptotic ansatz,

$$\boldsymbol{G}(\boldsymbol{r};\boldsymbol{r}_0) = \sum_{l=0}^{\infty} \frac{\tilde{\boldsymbol{A}}^l(\boldsymbol{r};\boldsymbol{r}_0)}{-\omega^2\mathcal{T}^l} \int_0^\infty e^{i\omega t} f_+^{(-3+l)}(t^2 - \tau^2(\boldsymbol{r};\boldsymbol{r}_0))dt. \tag{3.2}$$

The integral in (3.2) has the following closed form [17],

$$\int_\tau^\infty e^{i\omega t} f_+^{(\nu-\frac{1}{2})}(t^2 - \tau^2(\boldsymbol{r};\boldsymbol{r}_0))dt = \frac{1}{2}i\sqrt{\pi}\left(\frac{2\tau}{\omega}\right)^\nu e^{i\omega\nu}H_\nu^{(1)}(\omega\tau) \equiv f_\nu(\omega,\tau), \tag{3.3}$$

where $f_\nu(\omega,\tau)$ is exactly the basis function used in Babich's ansatz [2].

By (3.2) and (3.3), we immediately find that the frequency-domain Hadamard's ansatz should be

$$\boldsymbol{G}(\boldsymbol{r};\boldsymbol{r}_0) = \sum_{l=0}^{\infty} \frac{\tilde{\boldsymbol{A}}^l(\boldsymbol{r};\boldsymbol{r}_0)}{-\omega^2\mathcal{T}^l} f_{-3+l+\frac{1}{2}}(\omega,\tau(\boldsymbol{r};\boldsymbol{r}_0)), \tag{3.4}$$

which we refer to as the Hadamard–Babich ansatz.

4. Numerical implementation.

4.1. Numerical computation for S waves. Numerically, it is impossible to construct $\{\tilde{\boldsymbol{A}}^l\}$ for all $l \geq 0$ so that we have to truncate the ansatz in our implementation. To make sure that this truncated ansatz is at least capable of reproducing the homogeneous fundamental solution (2.26), we truncate the formula to obtain the following ansatz,

$$\boldsymbol{G}^S(\boldsymbol{r};\boldsymbol{r}_0) = \frac{\tilde{\boldsymbol{A}}^0(\boldsymbol{r};\boldsymbol{r}_0)}{-\omega^2} f_{-3+\frac{1}{2}}(\omega,\tau(\boldsymbol{r};\boldsymbol{r}_0)) + \frac{\tilde{\boldsymbol{A}}^1(\boldsymbol{r};\boldsymbol{r}_0)}{-\omega^2\mathcal{T}(\boldsymbol{r};\boldsymbol{r}_0)} f_{-2+\frac{1}{2}}(\omega,\tau(\boldsymbol{r};\boldsymbol{r}_0)). \tag{4.1}$$

Therefore, we have to compute τ, $\tilde{\boldsymbol{A}}^0$, and $\tilde{\boldsymbol{A}}^1$ to approximate the Green's function $\boldsymbol{G}^S$.

For the sake of convenience, we restate the governing equations for τ_S, $\tilde{\boldsymbol{A}}^0$, and $\tilde{\boldsymbol{A}}^1$ along with their initial conditions. We have that τ_S satisfies

$$|\nabla\tau_S| = \gamma^S \tag{4.2}$$

with the initial condition $\tau_S(\boldsymbol{r}_0;\boldsymbol{r}_0) = 0$.

We have that $\tilde{\boldsymbol{A}}^0$ satisfies

$$(\nabla\mathcal{T}\cdot\nabla)\tilde{\boldsymbol{A}}^0+\frac{1}{2\mu}[\nabla\cdot(\mu\nabla\mathcal{T})-14\rho]\tilde{\boldsymbol{A}}^0+\frac{\mu}{2\rho}\nabla\mathcal{T}\left[\nabla\left(\frac{\rho}{\mu}\right)\cdot\tilde{\boldsymbol{A}}^0\right]=0 \tag{4.3}$$

with the initial condition

$$\tilde{\boldsymbol{A}}^0=\frac{\mathcal{T}^{\frac{5}{2}}}{\rho_0\pi r^3}(\boldsymbol{I}-3\boldsymbol{t}\boldsymbol{t}^T)+\frac{{\gamma_0^S}^2\mathcal{T}^{\frac{3}{2}}}{\rho_0\pi r}(\boldsymbol{I}+\boldsymbol{t}\boldsymbol{t}^T)+O(\mathcal{T}^{\frac{3}{2}}), \tag{4.4}$$

where $\boldsymbol{t}$ is the unit tangent vector to the S ray.

We also have that $\tilde{\boldsymbol{A}}^1$ satisfies

$$(\nabla\mathcal{T}\cdot\nabla)\tilde{\boldsymbol{A}}^1+\frac{1}{2\mu}[\nabla\cdot(\mu\nabla\mathcal{T})-18\rho]\tilde{\boldsymbol{A}}^1+\frac{\mu}{2\rho}\nabla\mathcal{T}\left[\nabla\left(\frac{\rho}{\mu}\right)\cdot\tilde{\boldsymbol{A}}^1\right]=\tilde{\boldsymbol{R}}^0 \tag{4.5}$$

with the initial condition

$$\tilde{\boldsymbol{A}}^1=-\frac{3}{2}\tilde{\boldsymbol{A}}^0+\frac{\mathcal{T}^{\frac{5}{2}}}{\rho_0\pi r^3}(\boldsymbol{I}-3\boldsymbol{t}\boldsymbol{t}^T)+O(\mathcal{T}^2), \tag{4.6}$$

where

$$\tilde{\boldsymbol{R}}^0=-\frac{\nabla\mathcal{T}\tilde{\boldsymbol{a}}^0}{2}+\frac{\mu\nabla\mathcal{T}[(\nabla\mathcal{T}\cdot\nabla)\tilde{\boldsymbol{a}}^0]}{4\rho}-\frac{\nabla\mathcal{T}\tilde{\boldsymbol{b}}^0}{8\rho}-\frac{\mathcal{T}\nabla\lambda\tilde{\boldsymbol{a}}^0}{2\mu}-\frac{(\lambda+\mu)\mathcal{T}\nabla\tilde{\boldsymbol{a}}^0}{2\mu}+\frac{\mathcal{T}\tilde{\boldsymbol{B}}^0}{2\mu}. \tag{4.7}$$

Here $\tilde{\boldsymbol{B}}^0$, $\tilde{\boldsymbol{a}}^0$, and $\tilde{\boldsymbol{b}}^0$ are given by

$$\begin{aligned}\tilde{\boldsymbol{B}}^0&=(\lambda+\mu)\nabla(\nabla\cdot\tilde{\boldsymbol{A}}^0)+\mu\nabla^2\tilde{\boldsymbol{A}}^0+\nabla\lambda(\nabla\cdot\tilde{\boldsymbol{A}}^0)\\&\quad+\nabla\mu\times(\nabla\times\tilde{\boldsymbol{A}}^0)+2(\nabla\mu\cdot\nabla)\tilde{\boldsymbol{A}}^0,\end{aligned} \tag{4.8}$$

$$\tilde{\boldsymbol{a}}^0=\nabla\cdot\tilde{\boldsymbol{A}}^0+\frac{1}{\lambda+\mu}\nabla\mu\cdot\tilde{\boldsymbol{A}}^0+\frac{2\mu}{(\lambda+\mu)|\nabla\mathcal{T}|^2}\nabla\mathcal{T}[(\nabla\mathcal{T}\cdot\nabla)\tilde{\boldsymbol{A}}^0], \tag{4.9}$$

$$\begin{aligned}\tilde{\boldsymbol{b}}^0&=10\rho\tilde{\boldsymbol{a}}^0-(\lambda+\mu)(\nabla\mathcal{T}\cdot\nabla)\tilde{\boldsymbol{a}}^0-\mu\nabla^2\mathcal{T}\tilde{\boldsymbol{a}}^0\\&\quad-(\nabla\mathcal{T}\cdot\nabla\lambda)\tilde{\boldsymbol{a}}^0-(\nabla\mathcal{T}\cdot\nabla\mu)\tilde{\boldsymbol{a}}^0+\nabla\mathcal{T}\cdot\tilde{\boldsymbol{B}}^0.\end{aligned} \tag{4.10}$$

To compute (4.1) numerically, we need to solve the eikonal equation to obtain τ_S, and the vectorial transport equations (4.3)–(4.5) to obtain $\tilde{\boldsymbol{A}}^0$ and $\tilde{\boldsymbol{A}}^1$ for S waves. To obtain a first-order accurate approximation of $\tilde{\boldsymbol{A}}^1$, we have to achieve a third-order accurate approximation of $\tilde{\boldsymbol{A}}^0$ due to the term $\Delta\tilde{\boldsymbol{A}}^0$ appearing in (4.5). However, to obtain a third-order accurate approximation of $\tilde{\boldsymbol{A}}^0$, we need a fifth-order accurate τ as $\Delta\tau^2$ appearing in (4.3) for $\tilde{\boldsymbol{A}}^0$. Therefore, we employ fifth-order weighted essentially nonoscillatory (WENO) [20, 14, 11] Lax–Friedrichs fast sweeping schemes developed in [12, 26, 29, 30, 25, 16] to solve the eikonal equation (4.2), where the upwind source singularity at the source point is treated by the factorization idea [21, 28, 7, 18, 25, 16]. As for $\tilde{\boldsymbol{A}}^0$ and $\tilde{\boldsymbol{A}}^1$, their components are coupled in the transport equations, and we cannot directly employ high-order Lax–Friedrichs schemes developed [25, 16]. However, since the transport equations for $\tilde{\boldsymbol{A}}^l$ are analogous to those of Maxwell's equations in [15], we will directly use the method developed in [15] to transform these strongly coupled systems into decoupled scalar equations and then use high-order Lax–Friedrichs WENO schemes to compute those solutions accordingly.

4.2. Numerical computation for P waves. Similarly, to compute the Green's function for P waves, we can also truncate the formula to obtain the following ansatz

$$\boldsymbol{G}^P(\boldsymbol{r};\boldsymbol{r}_0) = \frac{\tilde{\boldsymbol{A}}^0(\boldsymbol{r};\boldsymbol{r}_0)}{-\omega^2} f_{-3+\frac{1}{2}}(\omega,\tau(\boldsymbol{r};\boldsymbol{r}_0)) + \frac{\tilde{\boldsymbol{A}}^1(\boldsymbol{r};\boldsymbol{r}_0)}{-\omega^2\mathfrak{T}(\boldsymbol{r};\boldsymbol{r}_0)} f_{-2+\frac{1}{2}}(\omega,\tau(\boldsymbol{r};\boldsymbol{r}_0)). \tag{4.11}$$

Therefore, we also have to compute τ, $\tilde{\boldsymbol{A}}^0$, and $\tilde{\boldsymbol{A}}^1$ to approximate the Green's function $\boldsymbol{G}^P$.

However, in the P-wave case, we can decouple related transport equations easily due to the property of P waves. Here we restate the governing equations for those related ingredients. We have

$$|\nabla\tau_P| = \gamma^P \tag{4.12}$$

with the initial condition $\tau_P(\boldsymbol{r}_0;\boldsymbol{r}_0) = 0$;

$$\nabla\mathfrak{T}\cdot\nabla\tilde{\boldsymbol{\alpha}}^0 + \left[\frac{1}{2\rho}\nabla\cdot(\rho\nabla\mathfrak{T}) - \frac{5\rho}{\lambda+2\mu}\right]\tilde{\boldsymbol{\alpha}}^0 = 0 \tag{4.13}$$

with the initial condition $\tilde{\boldsymbol{\alpha}}^0 = \frac{\gamma_0^{P\,3}}{\rho_0\pi}(\boldsymbol{r}-\boldsymbol{r}_0)$ as $\boldsymbol{r}\to\boldsymbol{r}_0$;

$$\nabla\mathfrak{T}\cdot\nabla\tilde{\boldsymbol{\alpha}}^1 + \left[\frac{1}{2\rho}\nabla\cdot(\rho\nabla\mathfrak{T}) - \frac{7\rho}{\lambda+2\mu}\right]\tilde{\boldsymbol{\alpha}}^1 = \tilde{\boldsymbol{c}}^0 \tag{4.14}$$

with the initial condition $\tilde{\boldsymbol{\alpha}}^1 = 0$ as $\boldsymbol{r}\to\boldsymbol{r}_0$, where $\tilde{\boldsymbol{c}}^0$ is

$$\begin{aligned}\tilde{\boldsymbol{c}}^0 =& \frac{1}{8\rho}\{10\rho - \mu\nabla^2\mathfrak{T} - \nabla\mathfrak{T}\cdot\nabla(\lambda+\mu)](\nabla\mathfrak{T}\cdot\boldsymbol{C}^0) - (\lambda+\mu)\nabla\mathfrak{T}\cdot\nabla(\nabla\mathfrak{T}\cdot\boldsymbol{C}^0)\\ &-(\lambda+\mu)|\nabla\mathfrak{T}|^2(\nabla\cdot\boldsymbol{C}^0) - 2\mu\nabla\mathfrak{T}\cdot[(\nabla\mathfrak{T}\cdot\nabla)\boldsymbol{C}^0] - (\nabla\mu\cdot\boldsymbol{C}^0)|\nabla\mathfrak{T}|^2 + \boldsymbol{d}^0\},\end{aligned} \tag{4.15}$$

where $\boldsymbol{C}^0$ and $\boldsymbol{d}^0$ are defined in (2.143) and (2.146).

Numerically, we apply similar considerations as those for computing S waves to compute the ingredients of P waves, and we omit the details here.

5. Numerical examples. In this section, we will study several numerical examples. To obtain a reference solution, if necessary, we apply the finite-difference time-domain (FDTD) method directly to the time-domain elastic wave equations to obtain a numerical solution in time, and then take the Fourier transform in time to obtain a numerical solution in the frequency domain.

In the following, $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ indicate the one-term and the two-term truncations of the Hadamard–Babich ansatz, respectively.

Example 1: Constant model. We take $\lambda = 1$, $\mu = 1$, and $\rho = 1$. The computational setup for our asymptotic method is the following:

- The computational domain is $[0, 0.5] \times [0, 0.5] \times [0, 0.5]$.
- The mesh size is $51 \times 51 \times 51$ with grid size $h = 0.01$.
- The angular frequency $\omega = 16\pi$ or $\omega = 32\pi$.
- The source point is $(0.25, 0.25, 0.25)^T$.

In this example, since all elastic parameters are constants, we use the exact solution to check the accuracy of our numerical solutions.

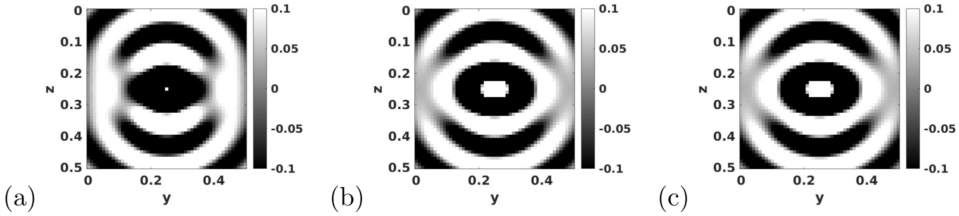

FIG. 5.1. *Example* 1 *with* $\boldsymbol{r}_0 = (0.25, 0.25, 0.25)^T$ *and* $\omega = 16\pi$. *The yy-component of* $\boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0)$ *at* $x = 0.25$ *computed via* (a) *one-term ansatz* $\boldsymbol{G}_1$; (b) *two-term ansatz* $\boldsymbol{G}_2$; (c) *exact solution* $\boldsymbol{G}_{hom}$.

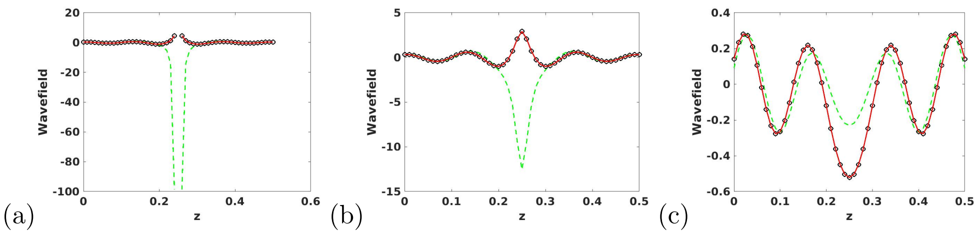

FIG. 5.2. *Example* 1 *with* $\boldsymbol{r}_0 = (0.25, 0.25, 0.25)^T$ *and* $\omega = 16\pi$. *The yy-component of* $\boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0)$ *at* $x = 0.25$ *and at* (a) $y = 0.25$; (b) $y = 0.23$; (c) $y = 0.15$. *Green dash line:* $\boldsymbol{G}_1$; *black circle line:* $\boldsymbol{G}_2$; *red solid line: exact solution* $\boldsymbol{G}_{hom}$.

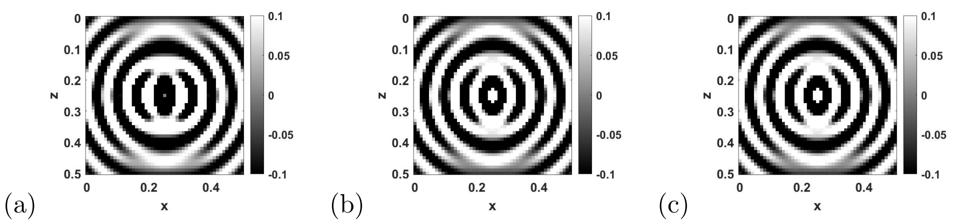

FIG. 5.3. *Example* 1 *with* $\boldsymbol{r}_0 = (0.25, 0.25, 0.25)^T$ *and* $\omega = 32\pi$. *The zz-component of* $\boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0)$ *at* $y = 0.25$ *computed via* (a) *one-term ansatz* $\boldsymbol{G}_1$; (b) *two-term ansatz* $\boldsymbol{G}_2$; (c) *exact solution* $\boldsymbol{G}_{hom}$.

We first compare the results at $\omega = 16\pi$. Figure 5.1 shows the contour plots of the yy-component of $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{G}_{hom}$ at $x = 0.25$. Figure 5.2 shows the detailed comparisons along three different lines. These results clearly show that the two-term ansatz can correctly capture singularities near the source.

Next, we consider frequency $\omega = 32\pi$ so that there are roughly eight waves propagating in the computational domain and about five to six points are used per wavelength. Figure 5.3 shows the zz-components of $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{G}_{hom}$ at $y = 0.25$. Figure 5.4 shows the detailed comparisons along three different lines. Therefore, in a homogeneous medium, the two-term approximation is able to faithfully reproduce source singularities of the exact solution.

In Table 5.1, we list numerical errors in the L^2-norm between our new ansatz-based solutions and exact solutions for the component G_{yy} in the computational domain excluding a neighborhood of the source at which the Green's functions $\boldsymbol{G}$ is singular. In terms of G_{yy}, although the one-term ansatz-based solution approximates the exact solution asymptotically in terms of $O(1/\omega)$, our two-term ansatz-based solution approximates the exact solution not only asymptotically in terms of $O(1/\omega)$ but also with very high accuracy.

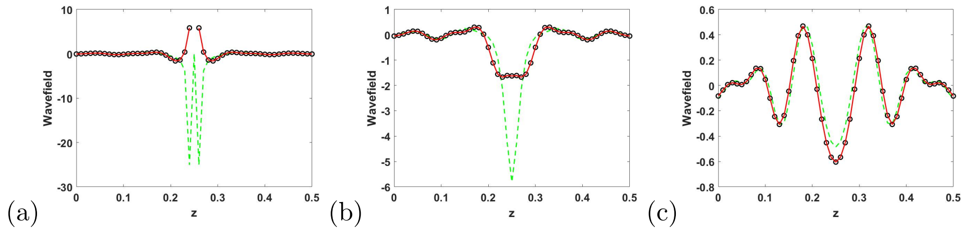

FIG. 5.4. *Example* 1 *with* $\boldsymbol{r}_0 = (0.25, 0.25, 0.25)^T$ *and* $\omega = 32\pi$*. The zz-component of* $\boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0)$ *at* $y = 0.25$ *and at* (a) $x = 0.25$*;* (b) $x = 0.23$*;* (c) $x = 0.15$*. Green dash line:* $\boldsymbol{G}_1$*; black circle line:* $\boldsymbol{G}_2$*; red solid line: exact solution* $\boldsymbol{G}_{hom}$*.*

TABLE 5.1

Example 1*:* L^2*-norm errors between the new ansatz-based solutions and the FDTD solution in a region excluding a neighborhood of the source.*

ω	8π	16π	32π	64π
$1/\omega$	4E-2	2E-2	1E-2	5E-3
one-term $L^2_{G_{yy}}$ error	3.4E-2	1.6E-2	8.9E-3	4.3E-3
two-term $L^2_{G_{yy}}$ error	3.2E-9	1.6E-9	8.3E-10	4.3E-10

Example 2: A variable density model. We take $\lambda = 1$ and $\mu = 1$ but variable ρ:

$$\rho = \frac{1}{(0.5 - (y - 0.25))^2}.$$

In this case, the exact solution is not available, so we compute the FDTD-based solution to check the accuracy of our method.

The computational setup for our asymptotic methods is the following:

- The computational domain is $[0, 0.5] \times [0, 0.5] \times [0, 0.5]$.
- The mesh size is $51 \times 51 \times 51$ with grid size $h = 0.01$.
- The angular frequency $\omega = 8\pi$ or $\omega = 16\pi$.
- The source point is $(0.25, 0.25, 0.25)^T$.

We first compare the results at frequency $\omega = 8\pi$. Figure 5.5 shows the contour plots of the yy-component of $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{G}_{\mathrm{FDTD}}$ at $x = 0.25$. Figure 5.6 shows the detailed comparisons along three different lines. We can see that $\boldsymbol{G}_2$ matches $\boldsymbol{G}_{\mathrm{FDTD}}$ much better than $\boldsymbol{G}_1$.

Next, we compare the results at frequency $\omega = 16\pi$. Figure 5.7 shows the contour plots of the yy-component of $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{G}_{\mathrm{FDTD}}$ at $x = 0.25$. Figure 5.8 shows the detailed comparisons along three different lines. At high frequencies, the discrepancy between $\boldsymbol{G}_1$ and $\boldsymbol{G}_{\mathrm{FDTD}}$ concentrates near the source, and such a discrepancy disappears between the two-term solution $\boldsymbol{G}_2$ and $\boldsymbol{G}_{\mathrm{FDTD}}$. Therefore, in an inhomogeneous medium the two-term approximation is able to faithfully reproduce source singularities of the exact solution and yields a uniform asymptotic expansion in the region of space containing the point source but no caustics.

Example 3: A Gaussian model. We take $\rho = 9$ but variable λ and μ. Here we set $\lambda = \mu$ and

$$\mu = \left(3.0 - 1.75 e^{-\frac{(x-0.25)^2 + (y-0.25)^2 + (z-0.125)^2}{0.125}}\right)^2.$$

In this example, we compute the FDTD-based solution to check the accuracy of our method.

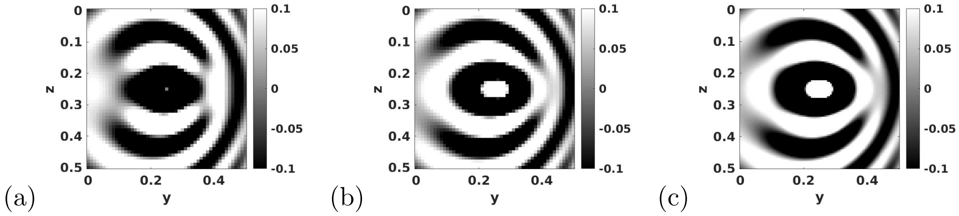

FIG. 5.5. *Example 2 with* $\boldsymbol{r}_0 = (0.25, 0.25, 0.25)^T$ *and* $\omega = 8\pi$. *The yy-component of* $\boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0)$ *at* $x = 0.25$ *computed via* (a) *one-term ansatz* $\boldsymbol{G}_1$*;* (b) *two-term ansatz* $\boldsymbol{G}_2$*;* (c) *FDTD-based solution.*

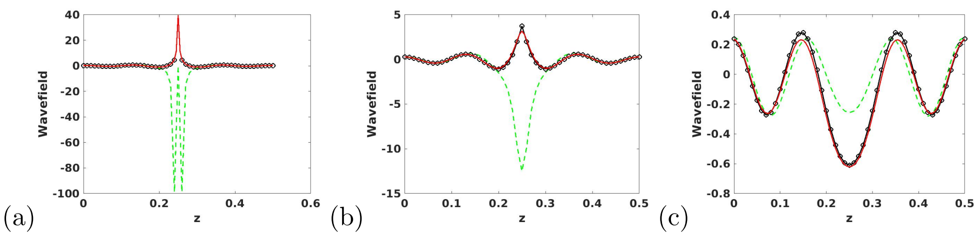

FIG. 5.6. *Example 2 with* $\boldsymbol{r}_0 = (0.25, 0.25, 0.25)^T$ *and* $\omega = 8\pi$. *The yy-component of* $\boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0)$ *at* $x = 0.25$ *and at* (a) $y = 0.25$*;* (b) $y = 0.23$*;* (c) $y = 0.15$. *Green dash line:* $\boldsymbol{G}_1$*; black circle line:* $\boldsymbol{G}_2$*; red solid line: FDTD-based solution.*

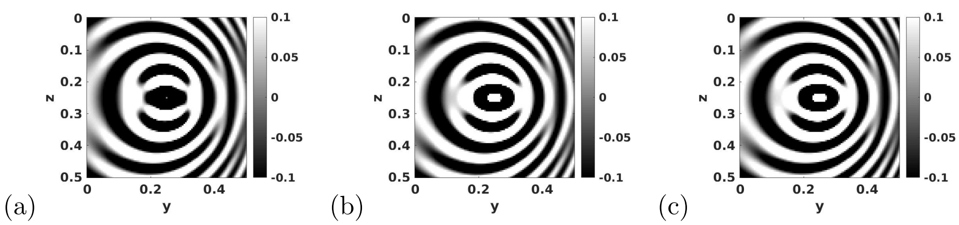

FIG. 5.7. *Example 2 with* $\boldsymbol{r}_0 = (0.25, 0.25, 0.25)^T$ *and* $\omega = 16\pi$. *The yy-component of* $\boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0)$ *at* $x = 0.25$ *computed via* (a) *one-term ansatz* $\boldsymbol{G}_1$*;* (b) *two-term ansatz* $\boldsymbol{G}_2$*;* (c) *FDTD-based solution.*

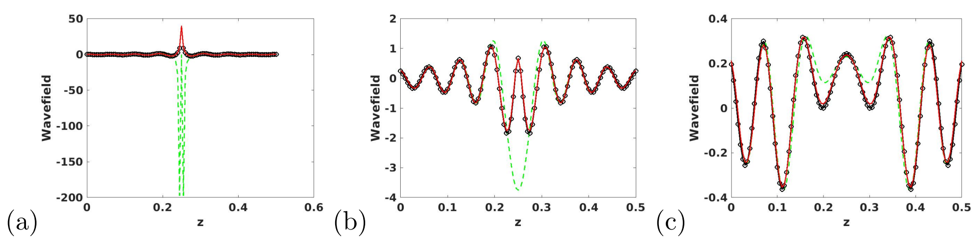

FIG. 5.8. *Example 2 with* $\boldsymbol{r}_0 = (0.25, 0.25, 0.25)^T$ *and* $\omega = 16\pi$. *The yy-component of* $\boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0)$ *at* $x = 0.25$ *and at* (a) $y = 0.25$*;* (b) $y = 0.23$*;* (c) $y = 0.15$. *Green dash line:* $\boldsymbol{G}_1$*; black circle line:* $\boldsymbol{G}_2$*; red solid line: FDTD-based solution.*

The computational setup for our asymptotic methods is the following:
- The computational domain is $[0, 0.5] \times [0, 0.5] \times [0, 0.5]$.
- The mesh size is $51 \times 51 \times 51$ with grid size $h = 0.01$.
- The angular frequency $\omega = 16\pi$ or $\omega = 32\pi$.
- The source point is $(0.25, 0.25, 0.25)^T$.

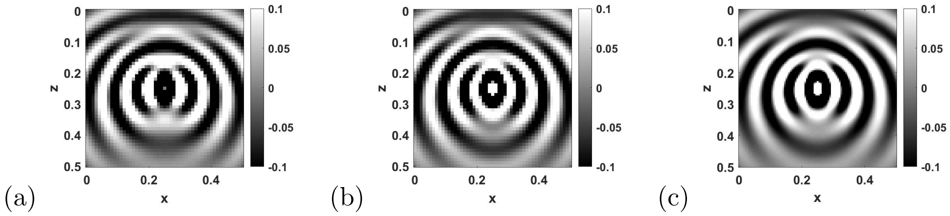

FIG. 5.9. *Example 3 with* $\boldsymbol{r}_0 = (0.25, 0.25, 0.25)^T$ *and* $\omega = 16\pi$. *The zz-component of* $\boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0)$ *at* $y = 0.25$ *computed via* (a) *one-term ansatz* $\boldsymbol{G}_1$*;* (b) *two-term ansatz* $\boldsymbol{G}_2$*;* (c) *FDTD method. Mesh in* (a) *and* (b): $51 \times 51 \times 51$*; mesh in* (c): $101 \times 101 \times 101$.

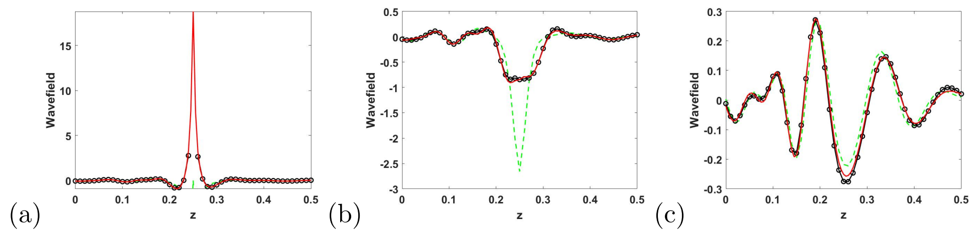

FIG. 5.10. *Example 3 with* $\boldsymbol{r}_0 = (0.25, 0.25, 0.25)^T$ *and* $\omega = 16\pi$. *The zz-component of* $\boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0)$ *at* $y = 0.25$ *and at* (a) $x = 0.25$*;* (b) $x = 0.23$*;* (c) $x = 0.15$. *Green dash line:* $\boldsymbol{G}_1$*; black circle line:* $\boldsymbol{G}_2$*; red solid line: FDTD-based solution.*

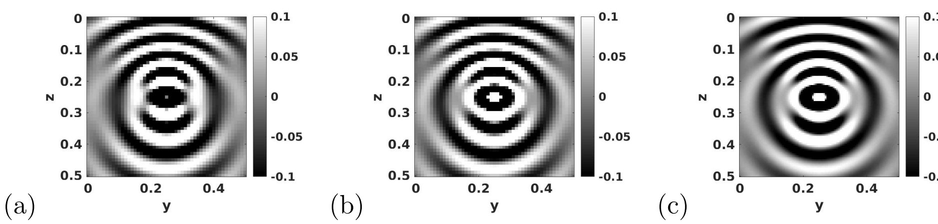

FIG. 5.11. *Example 3 with* $\boldsymbol{r}_0 = (0.25, 0.25, 0.25)^T$ *and* $\omega = 16\pi$. *The yy-component of* $\boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0)$ *at* $x = 0.25$ *computed via* (a) *one-term ansatz* $\boldsymbol{G}_1$*;* (b) *two-term ansatz* $\boldsymbol{G}_2$*;* (c) *FDTD method. Mesh in* (a) *and* (b): $51 \times 51 \times 51$*; mesh in* (c): $101 \times 101 \times 101$.

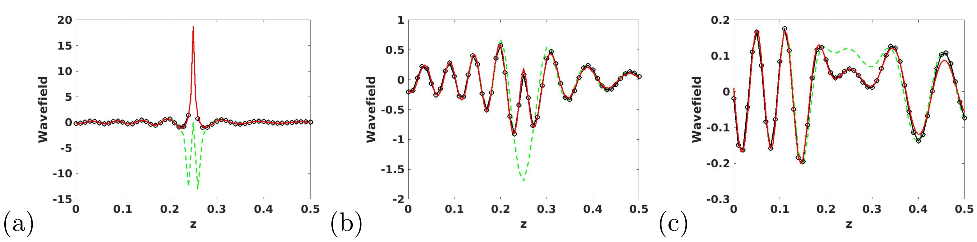

FIG. 5.12. *Example 3 with* $\boldsymbol{r}_0 = (0.25, 0.25, 0.25)^T$ *and* $\omega = 16\pi$. *The yy-component of* $\boldsymbol{G}(\boldsymbol{r}; \boldsymbol{r}_0)$ *at* $x = 0.25$ *and at* (a) $y = 0.25$*;* (b) $y = 0.23$*;* (c) $y = 0.15$. *Green dash line:* $\boldsymbol{G}_1$*; black circle line:* $\boldsymbol{G}_2$*; red solid line: FDTD-based solution.*

We compare the results at frequency $\omega = 16\pi$. Figures 5.9 and 5.10 show the solutions of the zz-component of $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{G}_{\text{FDTD}}$ at $y = 0.25$. Figures 5.11 and 5.12 show the solutions of the yy-component of $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{G}_{\text{FDTD}}$ at $x = 0.25$.

Table 5.2

Example 3: L^2-norm errors between the two-term ansatz solution and the FDTD solution in a region excluding a neighborhood of the source.

ω	8π	16π	32π
$1/\omega$	4E-2	2E-2	1E-2
$L^2_{G_{xy}}$	2.4E-3	2.4E-3	2.9E-3
$L^2_{G_{yy}}$	6.3E-3	3.6E-3	6.3E-3

Table 5.2 shows numerical differences between our two-term ansatz-based solution and the FDTD solution for the two particular components, $\boldsymbol{G}_{xy}$ and $\boldsymbol{G}_{yy}$. Since the computed FDTD solution is a numerical rather than "exact" solution which is not accurate enough, the numerical comparison did not indicate a clear asymptotic convergence pattern in terms of $O(1/\omega)$, but we can still conclude that our numerical solutions agree well with the FDTD solution with accuracy on the order of $O(1/\omega)$.

6. Conclusion. Based on Hadamard's method, we develop a novel ansatz for the vectorial three dimensional point-source elastic wave equation in inhomogeneous media at high frequencies. We develop a new Hadamard's ansatz to form the fundamental solution of the Cauchy problem for the time-domain point-source elastic wave equations (2.3) and (2.4) in the region close to the point source. We derive the governing equations for asymptotics involved in the ansatz, such as travel time and the dyadic coefficients. We further propose matching conditions to deduce initial conditions of amplitude coefficients at the source. Consequently, the Fourier transform of Hadamard's ansatz in time directly yields the Hadamard–Babich ansatz for the FDPS elastic wave equations (1.1). Finally, to validate our method, we truncate the ansatz after two terms and apply high-order Lax–Friedrichs WENO schemes to compute the involved asymptotic ingredients. Numerical results demonstrate that our new ansatz is capable of producing accurate asymptotic solutions in the region of space containing the point source but no other caustics. Incorporating this new ansatz into Huygens' principle to treat caustics consists of an ongoing project. Another ongoing project is to implement the Hadamard–Babich ansatz for anisotropic elastic wave equations.

Appendix A. Derivation of some ingredients for governing equations of S and P waves.

A.1. The formula of $\tilde{\boldsymbol{R}}^{l-1}$ for the S wave in (2.110). Let us rewrite the governing equation for $\tilde{\boldsymbol{A}}^l$ as the following:

$$(\nabla\mathcal{T}\cdot\nabla)\tilde{\boldsymbol{A}}^l + \frac{1}{2\mu}[\nabla\cdot(\mu\nabla\mathcal{T}) - (4l+14)\rho]\tilde{\boldsymbol{A}}^l + \frac{\mu}{2\rho}\nabla\mathcal{T}\left[\nabla\left(\frac{\rho}{\mu}\right)\cdot\tilde{\boldsymbol{A}}^l\right] = \tilde{\boldsymbol{R}}^{l-1}, \tag{A.1}$$

where

$$\begin{aligned}\tilde{\boldsymbol{R}}^{l-1} = &-\frac{\nabla\mathcal{T}\tilde{\boldsymbol{a}}^{l-1}}{2}\mathcal{T}^{l-1} + \frac{\mu\nabla\mathcal{T}[(\nabla\mathcal{T}\cdot\nabla)\tilde{\boldsymbol{a}}^{l-1}]}{4\rho}\mathcal{T}^{l-1} - \frac{\nabla\mathcal{T}\tilde{\boldsymbol{b}}^{l-1}}{8\rho}\mathcal{T}^{l-1} - \frac{\nabla\lambda\tilde{\boldsymbol{a}}^{l-1}}{2\mu}\mathcal{T}^l\\ &-\frac{(\lambda+\mu)\nabla\tilde{\boldsymbol{a}}^{l-1}}{2\mu}\mathcal{T}^l + \frac{\tilde{\boldsymbol{B}}^{l-1}}{2\mu}\mathcal{T}^l.\end{aligned} \tag{A.2}$$

Here $\tilde{\boldsymbol{B}}^{l-1}$, $\tilde{\boldsymbol{a}}^{l-1}$, and $\tilde{\boldsymbol{b}}^{l-1}$ are defined as

$$
\begin{aligned}
\tilde{\boldsymbol{B}}^{l-1} \equiv\ & (\lambda+\mu)\left[\frac{l(l-1)}{\mathcal{T}^{l+1}}\nabla\mathcal{T}(\nabla\mathcal{T}\cdot\tilde{\boldsymbol{A}}^{l-1}) - \frac{l-1}{\mathcal{T}^{l}}\nabla(\nabla\mathcal{T}\cdot\tilde{\boldsymbol{A}}^{l-1})\right. \\
& \left. - \frac{l-1}{\mathcal{T}^{l}}\nabla\mathcal{T}(\nabla\cdot\tilde{\boldsymbol{A}}^{l-1}) + \frac{1}{\mathcal{T}^{l-1}}\nabla(\nabla\cdot\tilde{\boldsymbol{A}}^{l-1})\right] \\
& + \mu\left[\frac{l(l-1)4\rho}{\mu\mathcal{T}^{l}}\tilde{\boldsymbol{A}}^{l-1} - \frac{(l-1)\nabla^2\mathcal{T}}{\mathcal{T}^{l}}\tilde{\boldsymbol{A}}^{l-1}\right. \\
& \left. - \frac{2(l-1)}{\mathcal{T}^{l}}(\nabla\mathcal{T}\cdot\nabla)\tilde{\boldsymbol{A}}^{l-1} + \frac{1}{\mathcal{T}^{l-1}}\nabla^2\tilde{\boldsymbol{A}}^{l-1}\right] \\
& + \nabla\lambda\left[\frac{-l+1}{\mathcal{T}^{l}}\nabla\mathcal{T}\cdot\tilde{\boldsymbol{A}}^{l-1} + \frac{1}{\mathcal{T}^{l-1}}\nabla\cdot\tilde{\boldsymbol{A}}^{l-1}\right] \\
& + \frac{1}{\mathcal{T}^{l-1}}\nabla\mu\times(\nabla\times\tilde{\boldsymbol{A}}^{l-1}) - \frac{l-1}{\mathcal{T}^{l}}\nabla\mu\times(\nabla\mathcal{T}\times\tilde{\boldsymbol{A}}^{l-1}) \\
& + 2\left[-\frac{l-1}{\mathcal{T}^{l}}(\nabla\mu\cdot\nabla\mathcal{T})\tilde{\boldsymbol{A}}^{l-1} + \frac{1}{\mathcal{T}^{l-1}}(\nabla\mu\cdot\nabla)\tilde{\boldsymbol{A}}^{l-1}\right],
\end{aligned} \tag{A.3}
$$

$$
\begin{aligned}
\tilde{\boldsymbol{a}}^{l-1} \equiv\ & \frac{-(l-1)}{\mathcal{T}^{l}}\nabla\mathcal{T}\cdot\tilde{\boldsymbol{A}}^{l-1} + \frac{1}{\mathcal{T}^{l-1}}\nabla\cdot\tilde{\boldsymbol{A}}^{l-1} + \frac{1}{(\lambda+\mu)\mathcal{T}^{l-1}}\nabla\mu\cdot\tilde{\boldsymbol{A}}^{l-1} \\
& - \frac{2\mu(l-1)}{(\lambda+\mu)\mathcal{T}^{l}}\nabla\mathcal{T}\cdot\tilde{\boldsymbol{A}}^{l-1} + \frac{\mu^2}{(\lambda+\mu)2\rho\mathcal{T}^{l}}\nabla\mathcal{T}\cdot[(\nabla\mathcal{T}\cdot\nabla)\tilde{\boldsymbol{A}}^{l-1}] - \frac{\mu\tilde{\boldsymbol{b}}^{l-2}}{(\lambda+\mu)4\rho\mathcal{T}},
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
\tilde{\boldsymbol{b}}^{l-1} \equiv\ & 2\rho(-2l+7)\tilde{\boldsymbol{a}}^{l-1} - (\lambda+\mu)(\nabla\mathcal{T}\cdot\nabla)\tilde{\boldsymbol{a}}^{l-1} - \mu\nabla^2\mathcal{T}\tilde{\boldsymbol{a}}^{l-1} \\
& - (\nabla\mathcal{T}\cdot\nabla\lambda)\tilde{\boldsymbol{a}}^{l-1} - (\nabla\mathcal{T}\cdot\nabla\mu)\tilde{\boldsymbol{a}}^{l-1} + \nabla\mathcal{T}\cdot\tilde{\boldsymbol{B}}^{l-1}.
\end{aligned} \tag{A.5}
$$

A.2. The formula of $\tilde{\boldsymbol{c}}^{l-1}$ for the P wave in (2.184). We rewrite the governing equation for $\tilde{\boldsymbol{\alpha}}^{l}$ as the following:

$$
\nabla\mathcal{T}\cdot\nabla\tilde{\boldsymbol{\alpha}}^{l} + \left[\frac{1}{2\rho}\nabla\cdot(\rho\nabla\mathcal{T}) - \frac{(2l+5)\rho}{\lambda+2\mu}\right]\tilde{\boldsymbol{\alpha}}^{l} = \tilde{\boldsymbol{c}}^{l-1}, \tag{A.6}
$$

and $\tilde{\boldsymbol{c}}^{l-1} = \mathcal{T}^{l}\boldsymbol{c}^{l-1}$, where $\boldsymbol{c}^{l-1}$ is the same as (2.154). The formula for $\tilde{\boldsymbol{c}}^{l-1}$ is

$$
\begin{aligned}
\tilde{\boldsymbol{c}}^{l-1} \equiv\ & \frac{1}{8\rho}\{[(14-4l)\rho - \mu\nabla^2\mathcal{T} - \nabla\mathcal{T}\cdot\nabla(\lambda+\mu)]\frac{1}{\mathcal{T}}(\nabla\mathcal{T}\cdot\tilde{\boldsymbol{C}}^{l-1}) \\
& + \frac{(\lambda+\mu)4\rho l}{(\lambda+2\mu)\mathcal{T}}(\nabla\mathcal{T}\cdot\tilde{\boldsymbol{C}}^{l-1}) - \frac{(\lambda+\mu)}{\mathcal{T}}\nabla\mathcal{T}\cdot\nabla(\nabla\mathcal{T}\cdot\tilde{\boldsymbol{C}}^{l-1}) \\
& + \frac{(\lambda+\mu)4\rho l}{(\lambda+2\mu)\mathcal{T}}(\nabla\mathcal{T}\cdot\tilde{\boldsymbol{C}}^{l-1}) - \frac{(\lambda+\mu)4\rho}{(\lambda+2\mu)}(\nabla\cdot\tilde{\boldsymbol{C}}^{l-1}) \\
& + \frac{8\mu\rho l}{(\lambda+2\mu)\mathcal{T}}(\nabla\mathcal{T}\cdot\tilde{\boldsymbol{C}}^{l-1}) - \frac{2\mu}{\mathcal{T}}\nabla\mathcal{T}\cdot[(\nabla\mathcal{T}\cdot\nabla)\tilde{\boldsymbol{C}}^{l-1}] \\
& - \frac{4\rho}{\lambda+2\mu}(\nabla\mu\cdot\tilde{\boldsymbol{C}}^{l-1}) + \mathcal{T}^{l-1}\nabla\mathcal{T}\cdot\tilde{\boldsymbol{D}}^{l-1}\},
\end{aligned} \tag{A.7}
$$

where $\tilde{\boldsymbol{C}}^{l-1} = \boldsymbol{C}^{l-1}\mathcal{T}^l$ satisfies

$$\begin{aligned}\tilde{\boldsymbol{C}}^{l-1} \equiv \frac{\lambda+2\mu}{(\lambda+\mu)4\rho}\{&-\rho(4l-18)\tilde{\boldsymbol{A}}^{l-1} + \frac{(l-1)(\lambda+\mu)}{\mathcal{T}}\nabla\mathcal{T}(\nabla\mathcal{T}\cdot\tilde{\boldsymbol{A}}^{l-1})\\ &-(\lambda+\mu)\nabla(\nabla\mathcal{T}\cdot\tilde{\boldsymbol{A}}^{l-1}) + \frac{8(l-1)\rho\mu}{\lambda+2\mu}\tilde{\boldsymbol{A}}^{l-1} - 2\mu(\nabla\mathcal{T}\cdot\nabla)\tilde{\boldsymbol{A}}^{l-1}\\ &-\mu\nabla^2\mathcal{T}\tilde{\boldsymbol{A}}^{l-1} - \nabla\lambda(\nabla\mathcal{T}\cdot\tilde{\boldsymbol{A}}^{l-1}) - (\nabla\mu\cdot\nabla\mathcal{T})\tilde{\boldsymbol{A}}^{l-1} + \mathcal{T}^{l-1}\tilde{\boldsymbol{D}}^{l-2}\},\end{aligned} \tag{A.8}$$

and $\tilde{\boldsymbol{D}}^{l-1}$ satisfies

$$\begin{aligned}\tilde{\boldsymbol{D}}^{l-1} \equiv (\lambda+\mu)&\left[\frac{l(l-1)}{\mathcal{T}^{l+1}}\nabla\mathcal{T}(\nabla\mathcal{T}\cdot\tilde{\boldsymbol{A}}^{l-1}) - \frac{l-1}{\mathcal{T}^l}\nabla(\nabla\mathcal{T}\cdot\tilde{\boldsymbol{A}}^{l-1})\right.\\ &\left.-\frac{l-1}{\mathcal{T}^l}\nabla\mathcal{T}(\nabla\cdot\tilde{\boldsymbol{A}}^{l-1}) + \frac{1}{\mathcal{T}^{l-1}}\nabla(\nabla\cdot\tilde{\boldsymbol{A}}^{l-1})\right]\\ &+\mu\left[\frac{l(l-1)4\rho}{\mu\mathcal{T}^l}\tilde{\boldsymbol{A}}^{l-1} - \frac{(l-1)\nabla^2\mathcal{T}}{\mathcal{T}^l}\tilde{\boldsymbol{A}}^{l-1}\right.\\ &\left.-\frac{2(l-1)}{\mathcal{T}^l}(\nabla\mathcal{T}\cdot\nabla)\tilde{\boldsymbol{A}}^{l-1} + \frac{1}{\mathcal{T}^{l-1}}\nabla^2\tilde{\boldsymbol{A}}^{l-1}\right]\\ &+\nabla\lambda\left[\frac{-l+1}{\mathcal{T}^l}\nabla\mathcal{T}\cdot\tilde{\boldsymbol{A}}^{l-1} + \frac{1}{\mathcal{T}^{l-1}}\nabla\cdot\tilde{\boldsymbol{A}}^{l-1}\right]\\ &+\frac{1}{\mathcal{T}^{l-1}}\nabla\mu\times(\nabla\times\tilde{\boldsymbol{A}}^{l-1}) - \frac{l-1}{\mathcal{T}^l}\nabla\mu\times(\nabla\mathcal{T}\times\tilde{\boldsymbol{A}}^{l-1})\\ &+2\left[-\frac{l-1}{\mathcal{T}^l}(\nabla\mu\cdot\nabla\mathcal{T})\tilde{\boldsymbol{A}}^{l-1} + \frac{1}{\mathcal{T}^{l-1}}(\nabla\mu\cdot\nabla)\tilde{\boldsymbol{A}}^{l-1}\right].\end{aligned} \tag{A.9}$$

Appendix B. Hadamard's ansatz for anisotropic wave equations.

B.1. Derivation of eikonal equations and transport equations. To develop Hadamard's ansatz for waves in anisotropic elastic solids, we start with the fundamental equations. Hooke's law states that the stress σ_{ij} is related to the strain e_{kl} through a stiffness tensor C_{ijkl} by the relation

$$\sigma_{ij} = C_{ijkl}e_{kl}. \tag{B.1}$$

Therefore, the motion equation without body force takes the form

$$\rho\frac{\partial^2\boldsymbol{U}}{\partial t^2} = \nabla\cdot\sigma, \tag{B.2}$$

where $\boldsymbol{U} = (U_i)$ is the displacement vector. By the relation between strain and displacement,

$$e_{kl} = \frac{1}{2}\left(\frac{\partial U_k}{\partial x_l} + \frac{\partial U_l}{\partial x_k}\right), \tag{B.3}$$

and the symmetry of the stiffness tensor, the motion equation leads to the wave equation

$$\rho\frac{\partial^2 U_j}{\partial t^2} = \frac{\partial}{\partial x_i}\left(C_{ijkl}\frac{\partial U_k}{\partial x_l}\right). \tag{B.4}$$

Inspired by our work for isotropic elastic waves, we shall seek an asymptotic solution of (B.4) in the following form:

$$\ddot{U}_i(\boldsymbol{x},t) = \sum_{l=0}^{\infty} U_i^l(\boldsymbol{x}) f_+^{(l-3)}(t^2 - \mathcal{T}(\boldsymbol{x})). \tag{B.5}$$

To derive the governing equations for the unknowns U_i^l and $\mathcal{T}$, we shall need $\partial_t^4 U_i$ and spatial derivatives of U_i. Since this is quite similar to our derivations for the isotropic elastic case, we can just use the formulas (2.34)–(2.39) directly. Inserting those formulas into the following equation

$$\rho \frac{\partial^4 U_j}{\partial t^4} = \frac{\partial}{\partial x_i}\left(C_{ijkl} \frac{\partial \ddot{U}_k}{\partial x_l} \right), \tag{B.6}$$

which is the second time derivative of (B.4), and equating individual coefficients of $f_+^{(l-5)}$ to zero successively, we can obtain

$$\begin{aligned} 0 &= 4\rho \mathcal{T} U_j^l - C_{ijkl}\mathcal{T}_{,l}\mathcal{T}_{,i}U_k^l + 2\rho(2l-9)U_j^{l-1} + C_{ijkl,i}\mathcal{T}_{,l}U_k^{l-1} \\ &\quad + C_{ijkl}\mathcal{T}_{,l}U_{k,i}^{l-1} + C_{ijkl}\mathcal{T}_{,i}U_{k,l}^{l-1} + C_{ijkl}\mathcal{T}_{,li}U_k^{l-1} \\ &\quad - C_{ijkl,i}U_{k,l}^{l-2} - C_{ijkl}U_{k,li}^{l-2}. \end{aligned} \tag{B.7}$$

Setting $l = 0$ and remembering that $\boldsymbol{U}^l = 0$ for $l < 0$, we obtain the Christoffel's equation

$$(C_{ijkl}\mathcal{T}_{,l}\mathcal{T}_{,i} - 4\rho\mathcal{T}\delta_{jk})U_k^0 = 0, \tag{B.8}$$

which leads to the anisotropic eikonal equation for $\mathcal{T}$,

$$\det(C_{ijkl}\mathcal{T}_{,l}\mathcal{T}_{,i} - 4\rho\mathcal{T}\delta_{jk}) = 0, \tag{B.9}$$

where $\mathcal{T}(\boldsymbol{x}) = \tau^2(\boldsymbol{x})$, and τ is the travel time. So we can obtain the eikonal equation for $\mathcal{T}$,

$$\det(C_{ijkl}\mathcal{T}_{,l}\mathcal{T}_{,i} - 4\rho\mathcal{T}\delta_{jk}) = \det(a_{ijkl}p_l p_i - \delta_{jk}) = 0, \tag{B.10}$$

which is the same as the eikonal equation in [23, 24]. Here $\boldsymbol{p} = (p_i) = \nabla\tau$, $a_{ijkl} = C_{ijkl}/\rho$ are the density-normalized elastic parameters.

Next setting $l = 1$ yields

$$\begin{aligned} 0 &= 4\rho\mathcal{T}U_j^1 - C_{ijkl}\mathcal{T}_{,l}\mathcal{T}_{,i}U_k^1 - 14\rho U_j^0 + C_{ijkl,i}\mathcal{T}_{,l}U_k^0 \\ &\quad + C_{ijkl}\mathcal{T}_{,l}U_{k,i}^0 + C_{ijkl}\mathcal{T}_{,i}U_{k,l}^0 + C_{ijkl}\mathcal{T}_{,li}U_k^0. \end{aligned} \tag{B.11}$$

By (B.8), $\boldsymbol{U}^0$ is a multiple of the normalized eigenvector $\boldsymbol{g}$ of the matrix $(a_{ijkl}p_l p_i)$, $\boldsymbol{U}^0 = U^0\boldsymbol{g}$. By the symmetry of the stiffness tensor C_{ijkl} and (B.8), we can obtain the equation for the amplitude U^0,

$$\begin{aligned} 0 &= -14\rho U^0 + C_{ijkl,i}\mathcal{T}_{,l}U^0 g_k g_j \\ &\quad + C_{ijkl}\mathcal{T}_{,l}(U^0 g_k)_{,i}g_j + C_{ijkl}\mathcal{T}_{,i}(U^0 g_k)_{,l}g_j + C_{ijkl}\mathcal{T}_{,li}U^0 g_k g_j. \end{aligned} \tag{B.12}$$

B.2. Verification of (B.9) in an isotropic medium. When the wave propagates in an isotropic medium, the stiffness tensor C_{ijkl} has the following form:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{B.13}$$

This involves only two independent parameters, λ and μ, known as Lamé's moduli. So (B.8) reduces to

$$((\lambda+\mu)\mathfrak{T}_{,j}\mathfrak{T}_{,k} + \mu\mathfrak{T}_{,l}\mathfrak{T}_{,l} - 4\rho\mathfrak{T}\delta_{jk})U_k^0 = 0 \tag{B.14}$$

or, in vector notation,

$$((\lambda+\mu)\nabla\mathfrak{T}\nabla\mathfrak{T}^T + (\mu\nabla\mathfrak{T}\cdot\nabla\mathfrak{T} - 4\rho\mathfrak{T})\boldsymbol{I})\boldsymbol{U}^0 = 0. \tag{B.15}$$

Since $\nabla\mathfrak{T}\nabla\mathfrak{T}^T$ is a rank-one matrix, the eigenvalues are $\nabla\mathfrak{T}\cdot\nabla\mathfrak{T}$, 0, and 0. Hence if $\det((\lambda+\mu)\nabla\mathfrak{T}\nabla\mathfrak{T}^T + (\mu\nabla\mathfrak{T}\cdot\nabla\mathfrak{T} - 4\rho\mathfrak{T})\boldsymbol{I}) = 0$, we can obtain two eikonal equations,

$$\mu|\nabla\mathfrak{T}|^2 = 4\rho\mathfrak{T} \tag{B.16}$$

and

$$(\lambda+2\mu)|\nabla\mathfrak{T}|^2 = 4\rho\mathfrak{T}. \tag{B.17}$$

The corresponding eigenvectors are $\boldsymbol{U}^0 \perp \nabla\mathfrak{T}$ for (B.16) and $\boldsymbol{U}^0 \parallel \nabla\mathfrak{T}$ for (B.17). We can see that these are actually the two cases that we find in section 2.2. Hence we have verified (B.9).

REFERENCES

[1] J. Achenbach, *Wave Propagation in Elastic Solids*, North-Holland, Amsterdam, 1993.
[2] V. M. Babich, *The short wave asymptotic form of the solution for the problem of a point source in an inhomogeneous medium*, USSR Comput. Math. Math. Phys., 5 (1965), pp. 247–251.
[3] V. M. Babich, *A point source of electromagnetic waves in an inhomogeneous medium: A high frequency ansatz and the dual nonstationary singular solution*, J. Math. Sci., 243 (2019), pp. 634–639.
[4] I. M. Babuška and S. A. Sauter, *Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers?*, SIAM J. Numer. Anal., 34 (1997), pp. 2392–2423.
[5] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. II, Wiley, New York, 1962.
[6] M. V. Fedoryuk, *Partial Differential Equations* V, Encyclopaedia Math. Sci. 34, Springer, Berlin, 1999.
[7] S. Fomel, S. Luo, and H. K. Zhao, *Fast sweeping method for the factored eikonal equation*, J. Comput. Phys., 228 (2009), pp. 6440–6455.
[8] F. G. Friedlander, *private communication*, 1966.
[9] I. Gelfand and G. Shilov, *Generalized Functions*, Vol. 1, Academic Press, New York, 1964.
[10] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Yale University Press, New Haven, CT, 1923.
[11] G.-S. Jiang and D. Peng, *Weighted ENO schemes for Hamilton–Jacobi equations*, SIAM J. Sci. Comput., 21 (2000), pp. 2126–2143.
[12] C. Y. Kao, S. J. Osher, and J. Qian, *Lax-Friedrichs sweeping schemes for static Hamilton-Jacobi equations*, J. Comput. Phys., 196 (2004), pp. 367–391.
[13] F. Karal Jr. and J. B. Keller, *Elastic wave propagation in homogeneous and inhomogeneous media*, J. Acoust. Soc. Amer., 31 (1959), pp. 694–705.
[14] X. D. Liu, S. J. Osher, and T. Chan, *Weighted essentially nonoscillatory schemes*, J. Comput. Phys., 115 (1994), pp. 200–212.
[15] W. Lu, J. Qian, and R. Burridge, *Babich-like ansatz for three-dimensional point-source Maxwell's equations in an inhomogeneous medium at high frequencies*, SIAM J. Multiscale Model. Simul., 14 (2016), pp. 1089–1122.

[16] W. Lu, J. Qian, and R. Burridge, *Babich's expansion and the fast Huygens sweeping method for the Helmholtz wave equation at high frequencies*, J. Comput. Phys., 313 (2016), pp. 478–510.

[17] W. Lu, J. Qian, and R. Burridge, *Extending Babich's ansatz for point-source Maxwell's equations using Hadamard's method*, SIAM J. Multiscale Model. Simul., 16 (2018), pp. 727–751.

[18] S. Luo, J. Qian, and R. Burridge, *High-order factorization based high-order hybrid fast sweeping methods for point-source eikonal equations*, SIAM J. Numer. Anal., 52 (2014), pp. 23–44.

[19] J. Milnor, *Morse Theory*, Ann. Math. 51, Princeton University Press, Princeton, NJ, 1973.

[20] S. J. Osher and C.-W. Shu, *High-order essentially nonoscillatory schemes for Hamilton–Jacobi equations*, SIAM J. Numer. Anal., 28 (1991), pp. 907–922.

[21] A. Pica, *Fast and accurate finite-difference solutions of the* 3*-D eikonal equation parameterized in celerity*, in Expanded Abstracts, Society of Exploration Geophysicists, Tulsa, OK, 1997, pp. 1774–1777.

[22] J. Qian, W. Lu, L. Yuan, S. Luo, and R. Burridge, *Eulerian geometrical optics and fast Huygens sweeping methods for three-dimensional time-harmonic high-frequency Maxwell's equations in inhomogeneous media*, SIAM J. Multiscale Model. Simul., 14 (2016), pp. 595–636.

[23] J. Qian and W. W. Symes, *Paraxial eikonal solvers for anisotropic quasi-P traveltimes*, J. Comput. Phys., 173 (2001), pp. 1–23.

[24] J. Qian and W. W. Symes, *Finite-difference quasi-P traveltimes for anisotropic media*, Geophysics, 67 (2002), pp. 147–155.

[25] J. Qian, L. Yuan, Y. Liu, S. Luo, and R. Burridge, *Babich's expansion and high-order Eulerian asymptotics for point-source Helmholtz equations*, J. Sci. Comput., 65 (2015), pp. 1–26.

[26] S. Serna and J. Qian, *A stopping criterion for higher-order sweeping schemes for static Hamilton-Jacobi equations*, J. Comput. Math., 28 (2010), pp. 552–568.

[27] W. W. Symes and J. Qian, *A slowness matching Eulerian method for multivalued solutions of eikonal equations*, J. Sci. Comput., 19 (2003), pp. 501–526.

[28] L. Zhang, J. W. Rector, and G. M. Hoversten, *Eikonal solver in the celerity domain*, Geophys. J. Internat., 162 (2005), pp. 1–8.

[29] Y. T. Zhang, H. K. Zhao, and J. Qian, *High order fast sweeping methods for static Hamilton-Jacobi equations*, J. Sci. Comput., 29 (2006), pp. 25–56.

[30] H. K. Zhao, *Fast sweeping method for eikonal equations*, Math. Comp., 74 (2005), pp. 603–627.