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A DIFFUSIVE WEAK ALLEE EFFECT MODEL WITH U-SHAPED EMIGRATION AND MATRIX HOSTILITY

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Abstract. We study positive solutions to steady state reaction diffusion equations of the form:

$$\begin{aligned} -\Delta u &= \lambda f(u);\ \Omega \\ \alpha(u)\frac{\partial u}{\partial \eta} + \gamma\sqrt{\lambda}[1-\alpha(u)]u &= 0;\ \partial\Omega \end{aligned}$$

where u is the population density, $f(u) = \frac{1}{a}u(u+a)(1-u)$ represents a weak Allee effect type growth of the population with $a \in (0,1)$, $\alpha(u)$ is the probability of the population staying in the habitat Ω when it reaches the boundary, and positive parameters λ and γ represent the domain scaling and effective exterior matrix hostility, respectively. In particular, we analyze the case when $\alpha(s) = \frac{1}{[1+(A-s)^2+\epsilon]}$ for all $s \in [0,1]$, where $A \in (0,1)$ and $\epsilon \geq 0$. In this case $1-\alpha(s)$ represents a U-shaped relationship between density and emigration. Existence, nonexistence, and multiplicity results for this model are established via the method of sub-super solutions.

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1. **Introduction.** In this paper, we study the structure of the steady states of a reaction diffusion model governed by a weak Allee effect type growth reaction and a U-shaped relationship between density and emigration, also known as U-shaped density dependent emigration (UDDE). Namely, we consider:

$$\begin{cases} -\Delta u = \lambda f(u);\ \Omega \\ \alpha(u)\frac{\partial u}{\partial \eta} + \gamma\sqrt{\lambda}[1-\alpha(u)]u = 0;\ \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded region in $\mathbb{R}^N$; $N > 1$ with smooth boundary $\partial\Omega$, u is the population density normalized such that the carrying capacity is one, $\frac{\partial u}{\partial \eta}$ is the outward normal derivative of u on $\partial\Omega$, and $f(u) = \frac{1}{a}u(u+a)(1-u)$ represents a weak Allee effect type growth of the population with $a \in (0,1)$ being a parameter measuring the strength of the weak Allee effect (in the sense that per-capita growth rate is increasing for $u \in [0, \frac{1-a}{2})$). See [3] for a detailed derivation of the time-dependent model corresponding to (1), namely,

$$\begin{cases} u_t = \frac{1}{\lambda}\Delta u + f(u);\ t > 0, x \in \Omega \\ u(0,x) = u_0(x);\ x \in \Omega \\ \alpha(u)\frac{\partial u}{\partial \eta} + \gamma\sqrt{\lambda}[1-\alpha(u)]u = 0;\ t > 0, x \in \partial\Omega, \end{cases} \tag{2}$$

where $\lambda > 0$ is a domain scaling parameter, $\gamma > 0$ is a measure of the effective matrix hostility, and $\alpha(u)$ is the probability of the population staying in the habitat Ω when it reaches the boundary (see [4] for a detailed explanation of effective matrix hostility).

In particular, we study the case when $\alpha(s) = \frac{1}{[1+(A-s)^2+\epsilon]}$ for all $s \in [0,1]$, where $A \in (0,1)$ and $\epsilon > 0$. We note that A denotes the location of the maximum of $\alpha(s)$ with $\alpha(A) = \frac{1}{1+\epsilon}$. Thus, when $\epsilon \approx 0$ the probability of organisms remaining in the patch upon reaching the boundary is approximately 100%. Then (1) reduces to:

$$\begin{cases} -\Delta u = \lambda f(u);\ \Omega \\ \frac{\partial u}{\partial \eta} + \gamma\sqrt{\lambda}[(A-u)^2+\epsilon]u = 0;\ \partial\Omega, \end{cases} \tag{3}$$

where the relationship between emigration rate and population density, given by $1-\alpha(s) = \frac{(A-s)^2+\epsilon}{1+[(A-s)^2+\epsilon]}$ for all $s \in [0,1]$, is U-shaped (see Figure 1). Even though a recent literature review ([10]) found that U-shaped density dependent emigration was ecologically important, little is known about the population dynamical effects of such an emigration relationship. Notwithstanding, the authors have studied the effects of UDDE on populations governed by a logistic growth in [7] and [5].

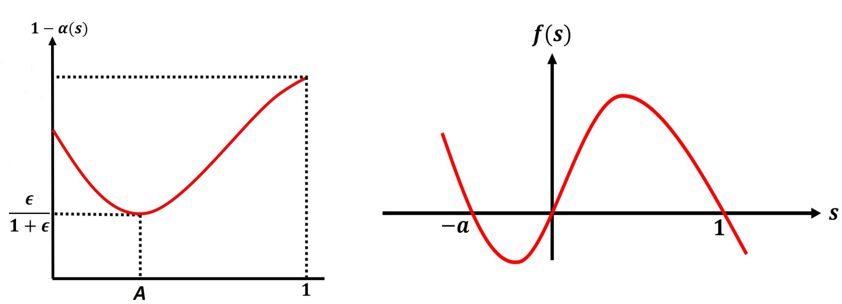

FIGURE 1. Illustration of $1-\alpha(s)$ and $f(s)$.

In order to state our main results, we first consider the following eigenvalue problem. For fixed $\gamma > 0$ and $D > 0$, let $E_1(\gamma, D)$ be the principal eigenvalue of:

$$\begin{cases} -\Delta v = Ev; \ \Omega \\ \frac{\partial v}{\partial \eta} + \gamma\sqrt{E}Dv = 0; \ \partial\Omega. \end{cases} \tag{4}$$

The existence of $E_1(\gamma, D)$ is established in [8]. Further, results in [8] imply that $E_1(\gamma, D) < \lambda_1$, $E_1(\gamma, D)$ is increasing in γ and D, and $\lim_{\gamma\to\infty} E_1(\gamma, D) = \lim_{D\to\infty} E_1(\gamma, D) = \lambda_1$, where λ_1 is the principal eigenvalue of $-\Delta$ with Dirichlet boundary conditions. Let $\overline{E}_1 = E_1(\gamma, A^2 + \epsilon)$. Motivated by the stability results in [13], we first prove the following theorem which connects $\overline{E}_1$ to the stability of the trivial solution $(u(x) \equiv 0)$ of (3).

Theorem 1.1. *The trivial solution of* (3) *is asymptotically stable if $\lambda < \overline{E}_1$, and it is unstable if $\lambda > \overline{E}_1$.*

Our next focus is to determine whether the solution structure of (3) has *Property $\mathcal{A}$*, defined as:
Property $\mathcal{A}$. *There exists $\overline{\lambda}(A, \gamma, \epsilon) < \overline{E}_1$ such that* (3) *has at least one positive solution u_λ for $\lambda \geq \overline{\lambda}$ such that $\|u_\lambda\|_\infty \longrightarrow 1$ as $\lambda \longrightarrow \infty$, has at least two positive solutions for $\lambda \in [\overline{\lambda}, \overline{E}_1)$, and no positive solutions for $\lambda \approx 0$ (see Figure 2).*

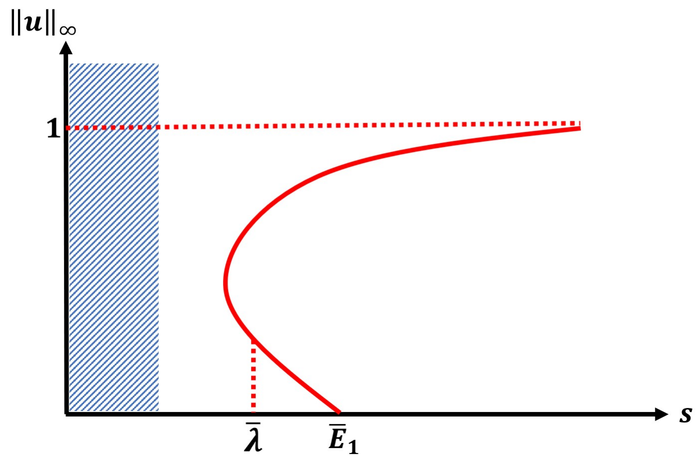

FIGURE 2. Bifurcation diagram for the solution set of (3).

A patch-level Allee effect is predicted by a reaction diffusion model when a version of bi-stable population dynamics occurs such that the trivial steady state and a positive steady state are both stable. In this case, there will exist a threshold for which the initial population density must exceed in order for the model to predict persistence (see [3], [10], [7], and [2]). Lemma 2.3 in [3] allows for determination of the existence of a patch-level Allee effect based solely upon the existence of a positive solution of (3) for $\lambda < \overline{E}_1$. Clearly, when *Property $\mathcal{A}$* is satisfied the model (2) will exhibit a patch-level Allee effect for $\lambda \in [\overline{\lambda}, \overline{E}_1)$. We first prove that (3) has *Property $\mathcal{A}$*, and thus a patch-level Allee effect, in the following theorem:

Theorem 1.2. *Let $A \in (0, 1)$, $\epsilon > 0$, and $\gamma > 0$. Then the solution set of* (3) *has Property $\mathcal{A}$.*

Since our growth rate $f(u)$ is taken to be of a weak Allee effect form, it is not surprising to find model predictions of an Allee effect at the patch-level. However, we are particularly interested in confirming model predictions of bi-stability scenarios other than a patch-level Allee effect in the case of UDDE as seen when $N = 1$ in [3]. To that end, we establish a multiplicity result for a range of λ to the right of $\overline{E}_1$ that guarantees a model prediction of bi-stability other than a patch-level Allee effect.

Recalling that λ_1 is the principal eigenvalue of $-\Delta$ with Dirichlet boundary conditions, we establish:

Theorem 1.3. *Let $\tilde{\lambda} > \lambda_1$. Then there exists a $\gamma^*(\tilde{\lambda})$ and for $\gamma > \gamma^*$ there exists an $\epsilon^*(\tilde{\lambda}, \gamma) > 0$ such that (3) has at least three positive solutions for $\lambda \in [\overline{E}_1, \tilde{\lambda}]$ when $\epsilon < \epsilon^*$ (see Figure 3).*

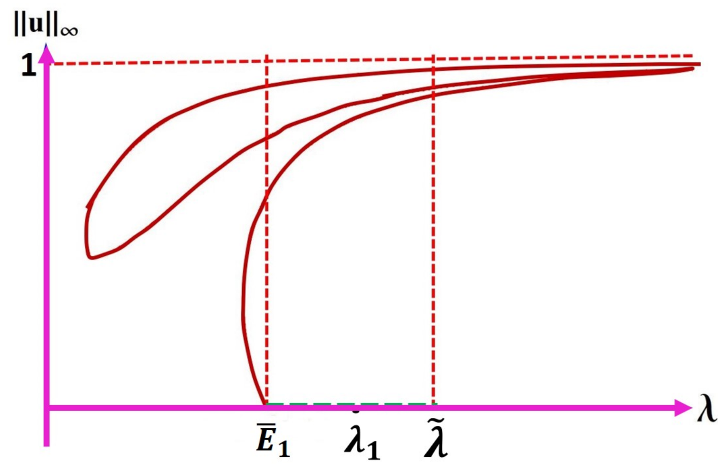

Figure 3. Bifurcation diagram for the solution set of (3) for $\gamma \gg 1$ and $\epsilon \approx 0$.

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Remark 1.1.

(1) The hypotheses of Theorem 1.3 give sufficient conditions on model prediction of a non-Allee effect type bi-stability for any $A \in (0, 1)$ (recall that A is the density for which the maximum $\alpha(s)$ is achieved) and satisfied when γ is sufficiently large, i.e. the matrix is sufficiently hostile, and $\epsilon \approx 0$, i.e. the probability of remaining in the patch upon reaching the boundary is approximately 100% when the population density is equal to A.
(2) See [3] for a more detailed study of (3) in the $N = 1$ case.

In Section 2, we introduce a sub-supersolution theorem and a three solution theorem that will be used to prove our existence and multiplicity results. In Section 3, we present the proofs of Theorems 1.1 - 1.3.

2. Preliminaries.

In this section, we first define a subsolution and a supersolution of (3). Next we state a sub-supersolution theorem and a three solution theorem that are used to prove existence and multiplicity results for positive solutions.

A function $\psi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is called a subsolution of (3) if ψ satisfies

$$\begin{gathered} -\Delta\psi \leq \lambda f(\psi);\ \Omega \\ \frac{\partial\psi}{\partial\eta} + \gamma\sqrt{\lambda}[(A-\psi)^2+\epsilon]\psi \leq 0;\ \partial\Omega. \end{gathered}$$

A function $Z \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is called a supersolution of (3) if Z satisfies

$$\begin{gathered} -\Delta Z \geq \lambda f(Z);\ \Omega \\ \frac{\partial Z}{\partial\eta} + \gamma\sqrt{\lambda}[(A-Z)^2+\epsilon]Z \geq 0;\ \partial\Omega. \end{gathered}$$

A strict subsolution of (3) is a subsolution which is not a solution. A strict supersolution of (3) is a supersolution which is not a solution. Then the following results hold (see [1], [11], and [15]).

Lemma 2.1. *Let ψ and Z be a subsolution and a supersolution of (3) respectively such that $\psi \leq Z$. Then (3) has a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $u \in [\psi, z]$.*

Lemma 2.2. *Let ψ_1 and Z_2 be a subsolution and a supersolution of (3) respectively such that $\psi_1 \leq Z_2$. Also, let ψ_2 and Z_1 be a strict subsolution and a strict supersolution of (3) respectively such that $\psi_2,\ Z_1 \in [\psi_1, Z_2]$ and $\psi_2 \nleq Z_1$. Then (3) has at least three solutions u_1, u_2 and u_3 where $u_i \in [\psi_i, Z_i]$; $i = 1, 2$ and $u_3 \in [\psi_1, Z_2] \backslash ([\psi_1, Z_1] \cup [\psi_2, Z_2])$.*

3. **Proofs of Theorems 1.1 - 1.3.** We first recall the following results from [7] and [9].

Lemma 3.1. *Let σ_1 be the principal eigenvalue of the linearized equation associated with (3), namely*

$$\begin{cases} -\Delta\phi - \lambda f_u(u)\phi = \sigma\phi;\ \Omega \\ \frac{\partial\phi}{\partial\eta} + \gamma\sqrt{\lambda}[g_u(u)u + g(u)]\phi = \sigma\phi;\ \partial\Omega, \end{cases} \tag{5}$$

where u is any solution of (3) and $g(u) = (A-u)^2 + \epsilon$. Then the followings hold.
a) If $\sigma_1 > 0$, then u is stable. Furthermore, if u is isolated then it is asymptotically stable.
b) If $\sigma_1 < 0$, then u is unstable.

Lemma 3.2. *Let u be a solution of (3) and σ_1^* be the principal eigenvalue of the following boundary value problem*

$$\begin{cases} -\Delta\phi - \lambda f_u(u)\phi = \sigma\phi;\ \Omega \\ \frac{\partial\phi}{\partial\eta} + \gamma\sqrt{\lambda}[g_u(u)u + g(u)]\phi = 0;\ \partial\Omega. \end{cases} \tag{6}$$

Then, $sign(\sigma_1^*) = sign(\sigma_1)$ for $\sigma_1^*, \sigma_1 \neq 0$.

Now, we present a proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.2, it suffices to study the relationship between σ_1^* and λ in order to prove Theorem 1.1. Let $\lambda < \overline{E}_1$. Note that, for $\lambda < \overline{E}_1$, the trivial solution is isolated since $(\lambda, 0)$ is not a bifurcation point. By Lemma 3.2, we see that the trivial solution is asymptotically stable if the principal eigenvalue σ_1^* of (6) with $u \equiv 0$ is positive.

Let $\kappa = \gamma\sqrt{\lambda}g(0)$ and $B_1(\kappa)$ be the principal eigenvalue of $-\Delta\phi = B\phi; \Omega$, $\frac{\partial\phi}{\partial\eta} + \kappa\phi = 0; \partial\Omega$. Since $B_1(\kappa)$ is a strictly increasing concave function of κ (see [12]) and $\frac{\kappa^2}{\gamma^2 g(0)^2}$ is a strictly increasing convex function of κ which passes through the origin, they intersect at two points, namely at $(0, 0)$, and say at $(\kappa^*, B_1(\kappa^*))$ for $\kappa^* > 0$

(see Figure 4). We can easily see that $B_1(\kappa^*) = \overline{E}_1$ and $\kappa^* = \gamma\sqrt{\overline{E}_1}g(0)$. Further, $\lambda + \sigma_1^* = B_1(\gamma\sqrt{\lambda}g(0))$, where σ_1^* is the principal eigenvalue of (6). Thus, if $\lambda < \overline{E}_1$ then $\gamma\sqrt{\lambda}g(0) < \kappa^*$ and $B_1(\gamma\sqrt{\lambda}g(0)) > \lambda$, implying $\sigma_1^* > 0$. By Lemma 3.2 the trivial solution is asymptotically stable if $\lambda < \overline{E}_1$.

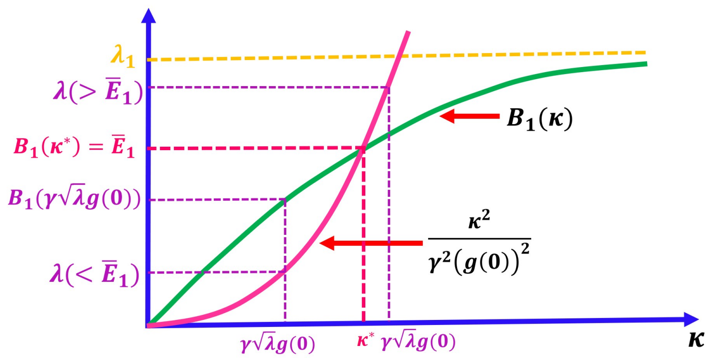

FIGURE 4. Graphs of κ vs $B_1(\kappa)$ and $\frac{\kappa^2}{\gamma^2(g(0))^2}$.

Next, let $\lambda > \overline{E}_1$. By Lemma 3.2, the trivial solution is unstable if the principal eigenvalue σ_1^* of (6) is negative. But when $\lambda > \overline{E}_1$, $\gamma\sqrt{\lambda}g(0) > \kappa^*$ and $B_1(\gamma\sqrt{\lambda}g(0)) < \lambda$ implying $\sigma_1^* < 0$ (see Figure 4). Hence, the proof is complete. □

Proof of Theorem 1.2. First let $0 < \lambda < \overline{E}_1$. Consider the eigenvalue problem (see [8]):

$$\begin{cases} -\Delta\theta - \lambda\theta = \sigma\theta; \ \Omega \\ \frac{\partial\theta}{\partial\eta} + \gamma\sqrt{\lambda}K\theta = 0; \ \partial\Omega, \end{cases} \tag{7}$$

where $K > 0$ is a constant. Let σ_λ be the principal eigenvalue and θ_λ be the normalized eigenfunction such that $\theta_\lambda > 0$; $\overline{\Omega}$. Choose $K = A^2 + \epsilon$ and $\delta_\lambda = \frac{2\sigma_\lambda}{\lambda A^* \min_{\overline{\Omega}} \theta_\lambda}$, where $A^* > 0$ is such that $f''(s) > A^*$ for $s \approx 0$. Note that $\delta_\lambda > 0$ (since $\sigma_\lambda > 0$) for $\lambda < \overline{E}_1$ and $\delta_\lambda \longrightarrow 0$ (since $\sigma_\lambda \longrightarrow 0$ and $\min_{\overline{\Omega}} \theta_\lambda \not\to 0$) as $\lambda \longrightarrow \overline{E}_1$. Now, define $\psi := \delta_\lambda\theta_\lambda$. Clearly $\|\psi\|_\infty \longrightarrow 0$ when $\lambda \longrightarrow \overline{E}_1$. Further, by Taylor's Theorem, in Ω, we obtain (for some $\zeta \in [0, \psi]$):

$$-\Delta\psi - \lambda f(\psi) = (\sigma_\lambda + \lambda)\psi - \lambda\Big[\psi + \frac{f''(\zeta)}{2}\psi^2\Big] < \psi\Big[\sigma_\lambda - \frac{\lambda A^*}{2}\delta_\lambda \min_{\overline{\Omega}} \theta_\lambda\Big] = 0$$

for $\lambda < \overline{E}_1$ and $\lambda \approx \overline{E}_1$. Also, on $\partial\Omega$, we obtain (assuming $\lambda \approx \overline{E}_1$ so that $\|\psi\|_\infty < 2A$):

$$\frac{\partial\psi}{\partial\eta} + \gamma\sqrt{\lambda}[(A - \psi)^2 + \epsilon]\psi < \frac{\partial\psi}{\partial\eta} + \gamma\sqrt{\lambda}[A^2 + \epsilon]\psi = 0.$$

Thus, ψ is a strict subsolution of (3) for $\lambda < \overline{E}_1$ and $\lambda \approx \overline{E}_1$. It is easy to verify that $Z \equiv 1$ is a supersolution for any λ and hence by Lemma 2.1 there exists $\overline{\lambda} = \overline{\lambda}(A, \gamma, \epsilon) < \overline{E}_1$ such that (3) has a positive solution for $\lambda \in [\overline{\lambda}, \overline{E}_1)$.

Next, let $\lambda \geq \overline{E}_1$. Consider the eigenvalue problem (see [8]):

$$\begin{cases} -\Delta\phi - \lambda\phi = \mu\phi; \ \Omega \\ \frac{\partial\phi}{\partial\eta} + \gamma\sqrt{\lambda}(A^2 + \epsilon)\phi = \mu\phi; \ \partial\Omega. \end{cases} \tag{8}$$

Denote μ_λ as the principal eigenvalue and ϕ_λ as the corresponding normalized eigenfunction such that $\phi_\lambda > 0;\ \overline{\Omega}$. Then $\mu_\lambda \leq 0$ for $\lambda \geq \overline{E}_1$. Let $\tilde{\psi} := \beta\phi_\lambda$ for $\beta \in (0,1)$. Recall that $f(0) = 0$, $f'(0) = 1$ and $f''(0) > 0$. Hence, for $\beta \approx 0$, in Ω, we have:

$$-\Delta\tilde{\psi} - \lambda f(\tilde{\psi}) = (\lambda + \mu_\lambda)\tilde{\psi} - \lambda f(\tilde{\psi}) \leq 0$$

since $H(s) := (\lambda + \mu_\lambda)s - \lambda f(s)$ satisfies $H(0) = 0$, $H'(0) = \mu_\lambda \leq 0$ and $H''(0) = -\lambda f''(0) < 0$. Also, on $\partial\Omega$, assuming $\beta \approx 0$ so that $\|\tilde{\psi}\|_\infty < 2A$, we have:

$$\frac{\partial\tilde{\psi}}{\partial\eta} + \gamma\sqrt{\lambda}[(A - \tilde{\psi})^2 + \epsilon]\tilde{\psi} \leq \frac{\partial\tilde{\psi}}{\partial\eta} + \gamma\sqrt{\lambda}[A^2 + \epsilon]\tilde{\psi} = \mu_\lambda\tilde{\psi} \leq 0.$$

Hence, for $\beta \approx 0$, $\tilde{\psi}$ is a subsolution for $\lambda \geq \overline{E}_1$. Again using the supersolution $Z \equiv 1$ and Lemma 2.1, there exists a positive solution for (3) when $\lambda \geq \overline{E}_1$. Combining the above two cases, we conclude that (3) has a positive solution for all $\lambda \geq \overline{\lambda}$.

Now, we will show that there exists a positive solution u_λ of (3) for $\lambda \gg 1$ such that $\|u_\lambda\|_\infty \longrightarrow 1$ as $\lambda \longrightarrow \infty$. Consider the boundary value problem:

$$\begin{cases} -\Delta w = \lambda f(w);\ \Omega \\ \quad w = 0;\ \partial\Omega. \end{cases} \tag{9}$$

In [14], it was established that there exists $\lambda^* \in (0, \lambda_1)$ such that (9) has a positive solution $w_\lambda \in [0,1]$ for $\lambda \geq \lambda^*$, and $\|w_\lambda\|_\infty \longrightarrow 1$ as $\lambda \longrightarrow \infty$. Now by the Hopf's maximum principle $\frac{\partial w_\lambda}{\partial\eta} < 0$ on $\partial\Omega$, and hence w_λ is a strict subsolution for (3). Again using the supersolution $Z \equiv 1$ and Lemma 2.1, (3) has a positive solution $u_\lambda \in [w_\lambda, 1]$ for $\lambda \geq \lambda^*$, and since $\|w_\lambda\|_\infty \longrightarrow 1$ as $\lambda \longrightarrow \infty$, we obtain $\|u_\lambda\|_\infty \longrightarrow 1$ as $\lambda \longrightarrow \infty$.

Next, we will show that there exist at least two positive solutions for $\lambda \in [\overline{\lambda}, \overline{E}_1)$. Consider the eigenvalue problem (8) with μ_λ and $\phi_\lambda > 0;\ \overline{\Omega}$ as before. Then $\mu_\lambda > 0$ for $\lambda < \overline{E}_1$ (see [8]). Let $Z_1 := \beta_1\phi_\lambda$ with $\beta_1 > 0$. For $\beta_1 \approx 0$, in Ω, we have

$$-\Delta Z_1 - \lambda f(Z_1) = (\lambda + \mu_\lambda)Z_1 - \lambda f(Z_1) > 0$$

since $H_1(s) := (\lambda + \mu_\lambda)s - \lambda f(s)$ satisfies $H_1(0) = 0$ and $H_1'(0) = \mu_\lambda > 0$. Also on $\partial\Omega$, choosing $\beta_1 \approx 0$ so that $|\gamma\sqrt{\lambda}[(A - Z_1)^2 - A^2]| < \mu_\lambda$, we have:

$$\begin{aligned} \frac{\partial Z_1}{\partial\eta} + \gamma\sqrt{\lambda}[(A - Z_1)^2 + \epsilon]Z_1 &= \frac{\partial Z_1}{\partial\eta} + \gamma\sqrt{\lambda}[A^2 + \epsilon]Z_1 + \gamma\sqrt{\lambda}[(A - Z_1)^2 - A^2]Z_1 \\ &= \Big\{\mu_\lambda + \gamma\sqrt{\lambda}\big[(A - Z_1)^2 - A^2\big]\Big\}Z_1 > 0. \end{aligned}$$

Hence, for $\beta_1 \approx 0$, Z_1 is a strict supersolution for $\lambda < \overline{E}_1$. Now for $\lambda \in [\overline{\lambda}, \overline{E}_1)$ we have the solution $\psi_0 \equiv 0$ (hence a subsolution), strict subsolution $\psi = \delta_\lambda\theta_\lambda\ (\leq 1)$, strict supersolution $Z_1 = \beta_1\phi_\lambda$ (with $\beta_1 \approx 0$ so that $\psi \not\lesssim Z_1$ and $Z_1 \leq 1$), and the supersolution $Z \equiv 1$. By Lemma 2.2, for $\lambda \in [\overline{\lambda}, \overline{E}_1)$ there exist at least two positive solutions u_1, u_2 with $u_1 \in [\psi, Z]$ and $u_2 \in [0, Z]\backslash\{[0, Z_1] \cup [\psi, Z]\}$.

Finally, we will show that for $\lambda \approx 0$, (3) has no positive solutions. Recall $\sigma_\lambda, \theta_\lambda$ in (7) with $K = \epsilon$. Suppose u is a positive solution of (3), by Green's second identity we obtain:

$$L = \int_\Omega [(\Delta u)\theta_\lambda - (\Delta\theta_\lambda)u]dx = \int_{\partial\Omega} -\gamma\sqrt{\lambda}(A - u)^2 u\theta_\lambda ds \leq 0.$$

However, $L = \int_\Omega [-\lambda f(u) + (\lambda + \sigma_\lambda)u]\theta_\lambda dx \geq \int_\Omega [\sigma_\lambda - (M-1)\lambda]u\theta_\lambda dx$ where $M > 0$ is such that $f(s) \leq Ms$ for $s \in [0, \infty)$. But $\frac{\sigma_\lambda}{\lambda} \longrightarrow \infty$ as $\lambda \longrightarrow 0$ (see [6]) and hence

$L > 0$ for $\lambda \approx 0$, which is a contradiction. Thus (3) has no positive solutions for $\lambda \approx 0$. Thus, Theorem 1.2 is proven. □

Proof of Theorem 1.3. Here, we will discuss the existence of three positive solutions for $\lambda \in [\overline{E}_1, \tilde{\lambda}]$.

Let $\Gamma \supset \Omega$, $\Gamma \approx \Omega$ be such that the boundary value problem (see [14])

$$-\Delta w = \tilde{\lambda} f(w);\ \Gamma$$
$$w = 0;\ \partial\Gamma,$$

has a positive solution $w_{\tilde{\lambda}} = Z_1$ (say) such that $Z_1 \in (0, \frac{A}{2});\ \partial\Omega$. This is possible since (9) has a positive solution for $\lambda \geq \lambda^* \in (0, \lambda_1)$. Also let $C = \min_{\partial\Omega} Z_1$ and choose $\gamma^*(\tilde{\lambda}) > 0$ such that for $\gamma > \gamma^*$

$$E_1(\gamma, A^2) > \tfrac{\lambda_1}{2} \tag{10}$$

($\Rightarrow E_1(\gamma, A^2 + \epsilon) > \frac{\lambda_1}{2}$) and

$$\tfrac{\partial Z_1}{\partial \eta} + \gamma\sqrt{\tfrac{\lambda_1}{2}}\tfrac{A^2}{4}C > 0;\ \partial\Omega \tag{11}$$

($\Rightarrow \frac{\partial Z_1}{\partial \eta} + \gamma\sqrt{\frac{\lambda_1}{2}}(\frac{A^2}{4} + \epsilon)C > 0;\ \partial\Omega$) hold. Now for $\lambda \in [\frac{\lambda_1}{2}, \tilde{\lambda}]$ we have:

$$-\Delta Z_1 = \tilde{\lambda} f(Z_1) \geq \lambda f(Z_1);\ \Omega$$

and

$$\frac{\partial Z_1}{\partial \eta} + \gamma\sqrt{\lambda}[(Z_1 - A)^2 + \epsilon]Z_1 \geq \frac{\partial Z_1}{\partial \eta} + \gamma\sqrt{\frac{\lambda_1}{2}}\Big(\frac{A^2}{4} + \epsilon\Big)C > 0;\ \partial\Omega.$$

Thus, Z_1 is a strict supersolution for (3) when $\lambda \in [\frac{\lambda_1}{2}, \tilde{\lambda}]$. Next, consider the boundary value problem:

$$\begin{cases} -\Delta v = \lambda v(1 - v);\ \Omega \\ \frac{\partial v}{\partial \eta} + 2\gamma\sqrt{\lambda}(A - v)^2 v = 0;\ \partial\Omega. \end{cases} \tag{12}$$

For each $\lambda > 0$, (12) has a unique solution $v_\lambda \in [A, 1];\ \overline{\Omega}$ (see [7]). Further, by Hopf's maximum principle $v_\lambda > A;\ \partial\Omega$. Let $c_\lambda = \min_{\partial\Omega} v_\lambda$ and $\epsilon^*(\tilde{\lambda}, \gamma) = \min_{\lambda \in [\overline{E}_1, \tilde{\lambda}]} (c_\lambda - A)^2$. Let $\psi_2 = v_\lambda$. Then for $\epsilon < \epsilon^*$, ψ_2 satisfies (for $\lambda \in [\overline{E}_1, \tilde{\lambda}]$):

$$-\Delta \psi_2 = \lambda \psi_2(1 - \psi_2) \leq \lambda f(\psi_2);\ \Omega$$

(since $\frac{a + \psi_2}{a} > 1$), and

$$\frac{\partial \psi_2}{\partial \eta} + \gamma\sqrt{\lambda}[(\psi_2 - A)^2 + \epsilon]\psi_2 = \gamma\sqrt{\lambda}[\epsilon - (\psi_2 - A)^2]\psi_2 < \gamma\sqrt{\lambda}[\epsilon - \epsilon^*]\psi_2 < 0;\ \partial\Omega$$

(since $\frac{\partial \psi_2}{\partial \eta} + 2\gamma\sqrt{\lambda}(\psi_2 - A)^2\psi_2 = 0;\ \partial\Omega$). Thus ψ_2 is a strict subsolution for $\lambda \in [\overline{E}_1, \tilde{\lambda}]$.

Now, let $\psi_1 = \tilde{\psi}(= \beta\phi)$ where $\tilde{\psi}$ is as in the proof of Theorem 1.2. Note that when $\beta \approx 0$, ψ_1 is a subsolution for $\lambda \geq \overline{E}_1$. Finally, take $Z_2 \equiv 1$ which is a supersolution for $\lambda > 0$. Now choosing $\beta \approx 0$, we can make sure $Z_1, \psi_2 \in [\psi_1, Z_2]$. Further, note that $\psi_2 \geq A;\ \overline{\Omega}$ while $Z_1 < \frac{A}{2};\ \partial\Omega$. By Lemma 2.2, (3) has at least three positive solutions when $\lambda \in [\overline{E}_1, \tilde{\lambda}]$ and Theorem 1.3 is proven. □

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REFERENCES

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.*, **18** (1976), 620–709.

[2] R. S. Cantrell and C. Cosner, Density dependent behavior at habitat boundaries and the Allee effect, *Bull. Math. Biol.*, **69** (2007), 2339–2360.

[3] J. T. Cronin, N. Fonseka, J. Goddard II, J. Leonard and R. Shivaji, Modeling the effects of density dependent emigration, weak Allee effects, and matrix hostility on patch-level population persistence, *Math. Biosci. Eng.*, **17** (2020), 1718–1742.

[4] J. T. Cronin, J. Goddard II, and R. Shivaji, Effects of patch matrix-composition and individual movement response on population persistence at the patch-level, *Bull. Math. Biol.*, **81** (2019), 3933–3975.

[5] N. Fonseka, J. Goddard, Q. Morris, R. Shivaji and B. Son, On the effects of the exterior matrix hostility and a U-shaped density dependent dispersal on a diffusive logistic growth model, *Discrete Contin. Dyn. Syst. Ser. S*, **13** (2020), 3401–3415.

[6] N. Fonseka, A. Muthunayake, R. Shivaji and B. Son, Singular reaction diffusion equations where a parameter influences the reaction term and the boundary condition, *Topol. Methods Nonlinear Anal.*, Accepted.

[7] J. Goddard II, Q. Morris, C. Payne and R. Shivaji, A diffusive logistic equation with U-shaped density dependent dispersal on the boundary, *Topol. Methods Nonlinear Anal.*, **53** (2019), 335–349.

[8] J. Goddard II, Q. A. Morris, S. B. Robinson and R. Shivaji, An exact bifurcation diagram for a reaction diffusion equation arising in population dynamics, *Bound. Value Probl.*, **2018** (2018), Paper No. 170, 17 pp.

[9] J. Goddard II and R. Shivaji, Stability analysis for positive solutions for classes of semilinear elliptic boundary-value problems with nonlinear boundary conditions, *Proc. Roy. Soc. Edinburgh Sect. A*, **147** (2017), 1019–1040.

[10] R. R. Harman, J. Goddard II, R. Shivaji and J. T. Cronin, Frequency of occurrence and population-dynamic consequences of different forms of density-dependent emigration, *Am. Nat.*, **195** (2019), 851–867.

[11] F. Inkmann, Existence and multiplicity theorems for semilinear elliptic equations with nonlinear boundary conditions, *Indiana Univ. Math. J.*, **31** (1982), 213–221.

[12] M. A. Rivas and S. B. Robinson, Eigencurves for linear elliptic equations, *ESAIM Control Optim. Calc. Var.*, **25** (2019), Paper No. 45, 25 pp.

[13] D. H. Sattinger, Monotone methods in nonlinear elliptic and parabolic boundary value problems, *Indiana Univ. Math. J.*, **21** (1971/72), 979–1000.

[14] J. Shi and R. Shivaji, Persistence in reaction diffusion models with weak Allee effect, *J. Math. Biol.*, **52** (2006), 807–829.

[15] R. Shivaji, A remark on the existence of three solutions via sub-super solutions, *Nonlinear Analysis and Applications*, Lecture notes in pure and applied mathematics, **109** (1987), 561–566, Ed. V. Lakshmikantham.

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