CALDERON-ZYGMUND TYPE ESTIMATES FOR NONLOCAL PDE WITH HÖLDER CONTINUOUS KERNEL

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Abstract. We study interior L^p-regularity theory, also known as Calderon-Zygmund theory, of the equation

$$\langle \mathcal{L}^s u, \varphi \rangle := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\, dx\, dy = \langle f, \varphi \rangle, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n).$$

We prove that for $s \in (0,1)$, $t \in [s,2s]$, $p \in [2,\infty)$, K an elliptic, symmetric, and $K(\cdot,y)$ is *uniformly Hölder continuous*, the solution u belongs to $H^{2s-t,p}_{loc}(\Omega)$ as long as $2s-t<1$ and $f \in \left(H^{t,p'}_{00}(\Omega)\right)^*$.

The increase in differentiability and integrability is independent of the Hölder coefficient of K. For example, in the event that $f \in L^p_{loc}$, we can deduce that the solution $u \in H^{2s-\delta,p}_{loc}$ for any $\delta \in (0,s]$ as long as $2s-\delta<1$. This regularity result is different from its classical analogue for divergence-form equations $\operatorname{div}(\bar{K}\nabla u) = f$ where a C^γ-Hölder continuous coefficient $\bar{K}$ only allows solutions in $H^{1+\gamma}$. In fact, the regularity estimates we prove are another manifestation of the differential stability effects of nonlocal equations of the above that are observed by many authors – only that in our case we do not get a "small" differentiability improvement, but all the way up to $\min\{2s-t,1\}$.

The proof argues by comparison with the (much simpler) equation

$$\langle L^{s,t}_{diag} u, \varphi \rangle := \int_{\mathbb{R}^n} K(z,z)(-\Delta)^{\frac{t}{2}} u(z)\, (-\Delta)^{\frac{2s-t}{2}} \varphi(z)\, dz = \langle g, \varphi \rangle, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n).$$

and showing that as long as K is Hölder continuous and $s,t,2s-t \in (0,1)$ then the "commutator" $\mathcal{L}^s u - L^{s,t}_{diag} u$ behaves like a lower order operator.

Contents

1. Introduction and statement of main results

In this article, we develop the Calderon-Zygmund theory for a popular nonlocal equation

$$\mathcal{L}^s_\Omega u = f, \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, $s \in (0,1)$, and the operator $\mathcal{L}^s_\Omega$ is formally given by

$$\mathcal{L}^s_\Omega u(x) := P.V. \int_\Omega 2K(x,y) \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy.$$

The "coefficient of $\mathcal{L}^s_\Omega$" is $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and it is assumed to be measurable, symmetric, and bounded. Moreover we assume K to be bounded from below on the diagonal by a positive number, $\inf_x K(x,x) > 0$, which corresponds to ellipticity.

In the event that $K = 1$ and $\Omega = \mathbb{R}^n$, the operator $\mathcal{L}^s_\Omega$ corresponds to the well-known fractional Laplacian operator $(-\Delta)^s$.

The main objective of this paper is to address the question of regularity of such a solution u relative to the data f.

Before we state our main theorem, Theorem 1.2, we need some definitions. We say that K satisfies a uniform Hölder continuity assumption if there exists $\alpha \in (0,1)$, $\Lambda > 0$ such that

$$\sup_{z \in \mathbb{R}^n} |K(z,y) - K(z,x)| \leq \Lambda\, |x-y|^\alpha, \quad \text{for } x,y \in \mathbb{R}^n. \tag{1.2}$$

For given positive numbers λ, Λ and $\alpha \in (0,1)$, define the class of elliptic coefficients

$$\mathcal{K}(\alpha,\lambda,\Lambda) = \left\{ K : K(x,y) = K(y,x), \inf_{x \in \mathbb{R}^n} K(x,x) > \lambda, \|K\|_{L^\infty} < \frac{1}{\lambda} \text{ and satisfies (1.2)} \right\}.$$

We also need to introduce relevant differential operators as well as function spaces. Let $\mathcal{F}$ denote the Fourier transform. For $s > 0$ the fractional Laplacian $(-\Delta)^{\frac{s}{2}}$ is defined as the operator that for f in the Schwartz class acts as multiplier with symbol $c|\xi|^s$

$$\mathcal{F}((-\Delta)^{\frac{s}{2}} f)(\xi) = c\, |\xi|^s \mathcal{F} f(\xi). \tag{1.3}$$

The Riesz potential $I^s = (-\Delta)^{-\frac{s}{2}}$ is the inverse of the fractional Laplacian, i.e. the multiplier operator with symbol $(c|\xi|^s)^{-1}$,

$$\mathcal{F}(I^s f)(\xi) := \frac{1}{c}\, |\xi|^{-s} \mathcal{F} f(\xi). \tag{1.4}$$

This operator makes sense (for f a function in the Schwartz class) if $0 \le s < n$, because $|\xi|^{-s}$ is then locally integrable. In the definitions the constant c depends on n and s and plays no deeper role in the theory that we consider.

Next we will introduce two types of fractional Sobolev spaces that we need to state the main result: Bessel potential spaces $H^{s,p}$ and Besov spaces $W^{s,p}$. For $1 < p < \infty$, the Bessel potential spaces $H^{s,p}(\mathbb{R}^n)$ are defined as follows: $f \in H^{s,p}(\mathbb{R}^n)$ if $f \in L^p(\mathbb{R}^n)$ and $(-\Delta)^{\frac{s}{2}} f \in L^p(\mathbb{R}^n)$. The associated norm is

$$\|f\|_{H^{s,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{R}^n)}.$$

The Besov spaces $W^{s,p}(\Omega)$, for $s \in (0,1)$, are induced by the semi-norm (called Sobolev-Slobodeckij or Gagliardo norm)

$$[f]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}},$$

and $\|\cdot\|_{W^{s,p}(\Omega)} = \|\cdot\|_{L^p(\Omega)} + [\cdot]_{W^{s,p}(\Omega)}$ serves as a norm. For $p = 2$, $W^{s,2}(\mathbb{R}^n) = H^{s,2}(\mathbb{R}^n)$, for $p < 2$ we have $W^{s,p}(\mathbb{R}^n) \subsetneq H^{s,p}(\mathbb{R}^n)$ and for $p > 2$ we have $H^{s,p}(\mathbb{R}^n) \subsetneq W^{s,p}(\mathbb{R}^n)$. These spaces are particular examples of the more general Triebel-Lizorkin spaces and $F^s_{pp}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$ and $F^s_{p,2}(\mathbb{R}^n) = H^{s,p}(\mathbb{R}^n)$, see [33].

For $u \in W^{s,2}(\Omega)$, we define the map $\mathcal{L}^s_\Omega$ by

$$\langle \mathcal{L}^s_\Omega u, \varphi \rangle := \int_\Omega \int_\Omega K(x,y) \frac{(u(x) - u(y))\,(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, dx \, dy. \tag{1.5}$$

for any $\varphi \in W^{s,2}(\Omega)$. It is not difficult to show that if $K \in L^\infty(\Omega \times \Omega)$, then for any $u \in W^{s,2}(\Omega)$, $\mathcal{L}^s_\Omega u \in (W^{s,2}(\Omega))^*$.

We now define precisely what we mean by a solution to our equation of interest, (1.1).

Definition 1.1. Let $s \in (0,1)$ and $t \in [s, 2s)$. Suppose that $f_1, f_2 \in L^2(\mathbb{R}^n)$. We say $u \in W^{s,2}(\Omega)$ is a distributional solution of

$$\mathcal{L}^s_\Omega u = (-\Delta)^{\frac{2s-t}{2}} f_1 + f_2 \quad \text{in } \Omega_1 \tag{1.6}$$

for some $\Omega_1 \subseteq \Omega$ if for any $\varphi \in C^\infty_c(\Omega_1)$, it holds that

$$\langle \mathcal{L}^s_\Omega u, \varphi \rangle = \int_{\mathbb{R}^n} f_1 (-\Delta)^{\frac{2s-t}{2}} \varphi \, dx + \int_{\mathbb{R}^n} f_2 \varphi \, dx.$$

If Ω is bounded or $\Omega = \mathbb{R}^n$, the notion of solution introduced in Definition 1.1 coincides with the classical notion of weak solution. Moreover, for $\Omega = \mathbb{R}^n$ and for any bounded open subset Ω_1, given $f_1, f_2 \in L^2(\mathbb{R}^n)$, a solution to (1.6) exists with additional assumption on u. For example, a minimizer of the energy

$$\mathcal{E}(u) := \frac{1}{2} \langle \mathcal{L}^s_{\mathbb{R}^n} u, u \rangle - \int_{\mathbb{R}^n} f_1 \, (-\Delta)^{\frac{2s-t}{2}} u dx - \int_{\mathbb{R}^n} f_2 \, u dx$$

over $\{u \in H^{s,2}(\mathbb{R}^n) : u = 0 \quad \text{on } \mathbb{R}^n \setminus \Omega_1\}$ exists and is a solution to (1.6) in the sense of Definition 1.1.

We also notice that (1.6) is often thought as the nonlocal (fractional) analogue of the weak formulation of the elliptic differential equation

$$\operatorname{div}(A(\cdot)\nabla u) = \operatorname{div} h + g. \tag{1.7}$$

The question of regularity of weak solutions u to (1.7) in relation to the regularity of data (the coefficient A, the right-hand sides h and g) is decades old. One line of regularity theory is the Calderon-Zygmund regularity theory where higher integrability of the gradient ∇u of the solution u is sought in relation to higher integrability of h and g. The now well-known $W^{1,p}$-theory proves that for a possibly rough coefficient $A(x)$ but with small mean oscillation, for any $1 < p < \infty$, if $h \in L^p_{loc}$ and g is, say, smooth, then $\nabla u \in L^p_{loc}(\mathbb{R}^n)$ [24]. Another line of regularity focuses on the differentiability of ∇u and this is intimately related to the smoothness of the coefficient $A(x)$ in (1.7). In fact, the $W^{2,p}$-theory states that if A is Lipschitz continuous, and $g \in L^p_{loc}(\mathbb{R}^n)$, say h is smooth, then the weak solution u of (1.7) is twice differentiable and $D^2u \in L^p_{loc}$, [20, Theorem 9.11][1]

The main objective of this paper is to prove regularity results of the above type for distributional solutions u of nonlocal equations such as (1.6). Although the conditions we put are different, the spirit of the results is similar in the sense that we are looking for higher differentiability in the fractional Sobolev scale and higher integrability of the solution u as a function of data f_1 and f_2 in (1.6). The following theorem states the main result of the paper.

Theorem 1.2. *Let $s \in (0,1)$ and $s \leq t < \min\{2s, 1\}$. If for $2 \leq q < \infty$, $f_1, f_2 \in L^q(\Omega) \cap L^2(\mathbb{R}^n)$, and $u \in W^{s,2}(\Omega)$ is a distributional solution of*

$$\langle \mathcal{L}^s_\Omega u, \varphi \rangle = \int_{\mathbb{R}^n} f_1 (-\Delta)^{\frac{2s-t}{2}} \varphi \, dx + \int_{\mathbb{R}^n} f_2 \varphi \, dx \quad \forall \varphi \in C^\infty_c(\Omega_1),$$

for some $\Omega_1 \subseteq \Omega$ in the sense of Definition 1.1 with $\mathcal{L}^s_\Omega$ corresponding to $K \in \mathcal{K}(\alpha, \lambda, \Lambda)$ for some given $\alpha \in (0,1)$ and $\lambda, \Lambda > 0$, then for any $W^{s,2}$-extension $\tilde{u}$ of u to $\mathbb{R}^n$ we have $(-\Delta)^{\frac{t}{2}} \tilde{u} \in L^q_{loc}(\Omega_1)$ and for any $\Omega' \subset\subset \Omega_1$ we have

$$\|(-\Delta)^{\frac{t}{2}} \tilde{u}\|_{L^q(\Omega')} \leq C \left(\|u\|_{W^{s,2}(\Omega)} + \sum_{i=1}^{2} \|f_i\|_{L^q(\Omega_1)} + \|f_i\|_{L^2(\mathbb{R}^n)} \right).$$

The constant C depends only on s, t, q, α, λ, Λ, Ω, and Ω'.

We used the notation $A \subset\subset B$ when A and B are open and the closure of A is a compact subset of B. In this work we focus on the case $q \geq 2$, $t \geq s$. This corresponds to the natural setting of variational solutions (which by construction already belong to $W^{s,2}$). We believe

[1]Observe that the statement of [20, Theorem 9.11] is in non-divergence form with bounded order coefficients. To transform a divergence form equation to an nondivergence form equation with bounded coefficients, the original coefficients should be Lipschitz.

it requires only minor *conceptional* changes to treat *very weak solutions* which a priori lie in $W^{\tilde{s},\tilde{q}}$ for suitable $\tilde{s} < s$ or $\tilde{q} < 2$ – but adapting the already technical argument to very weak solutions would even further blur the conceptional elegance and simplicity of our approach. We will, therefore, postpone that to a future work.

Let us highlight some corollaries of Theorem 1.2 that might appear in applications. For the proofs we refer to Section 7.

Corollary 1.3. *Let $s \in (0,1)$ and $s \leq t < \min\{1, 2s\})$. If for $q \geq 2$, $f \in L^q(\Omega)$, $u \in W^{s,2}(\Omega)$ is a distributional solution of*

$$\langle \mathcal{L}^s_\Omega u, \varphi \rangle = \int_\Omega f\, \varphi\, dx, \quad \forall \varphi \in C_c^\infty(\Omega),$$

with $\mathcal{L}^s_\Omega$ corresponding to $K \in \mathcal{K}(\alpha, \lambda, \Lambda)$ for some given $\alpha \in (0,1)$ and $\lambda, \Lambda > 0$. Then for any $W^{s,2}$-extension $\tilde{u}$ of u to $\mathbb{R}^n$, $(-\Delta)^{\frac{t}{2}} \tilde{u} \in L^q_{loc}(\Omega)$, and for any $\Omega' \subset\subset \Omega$ we have

$$\|(-\Delta)^{\frac{t}{2}} \tilde{u}\|_{L^q(\Omega')} \leq C \left(\|f\|_{L^q(\Omega)} + \|\tilde{u}\|_{W^{s,2}(\mathbb{R}^n)} \right).$$

In particular, if $\gamma := t - \frac{n}{q} > 0$ then $u \in C^\gamma_{loc}(\Omega)$.

Corollary 1.4. *Let $s \in (0,1)$ and $s \leq t < \min\{1, 2s\}$. For any open set $\Omega \subset \mathbb{R}^n$, $2 \leq q < \infty$ the following holds.*

If $f \in (H^{2s-t,q'}(\Omega))^*$ and $u \in W^{s,2}(\Omega)$ is a distributional solution of

$$\langle \mathcal{L}^s_\Omega u, \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in C_c^\infty(\Omega)$$

in the sense of Definition 1.1 with $\mathcal{L}^s_\Omega$ corresponding to $K \in \mathcal{K}(\alpha, \lambda, \Lambda)$ for some given $\alpha \in (0,1)$ and $\lambda, \Lambda > 0$. Then for any $W^{s,2}$-extension $\tilde{u}$ of u to $\mathbb{R}^n$ we have $(-\Delta)^{\frac{t}{2}} \tilde{u} \in L^q_{loc}(\Omega)$ and for any $\Omega' \subset\subset \Omega$ we have

$$\|(-\Delta)^{\frac{t}{2}} \tilde{u}\|_{L^q(\Omega')} \leq C \left(\|u\|_{W^{s,2}(\Omega)} + \|f\|_{(H^{2s-t,q'}(\Omega))^*} \right)$$

The constant C depends only s, t, q, α, λ, Λ, Ω, and Ω'.

We can also change the metric in Corollary 1.4 via a diffeomorphism (a setup suggested by M. Fall in [16])

Corollary 1.5. *Let $s \in (0,1)$ and $p \geq 2$. Let $\Omega, \Omega_2 \subset\subset \mathbb{R}^n$ be two open sets and $\Phi : \Omega \to \Omega_2$ a $C^{1,\alpha}$-diffeomorphism for some $\alpha > 0$ with strictly positive Jacobian $\det(D\Phi) > 0$. Assume that $f \in (H^{2s-t,q'}(\Omega))^*$ and $u \in W^{s,2}(\Omega)$ is a distributional solution of*

$$\int_\Omega \int_\Omega \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|\Phi(x) - \Phi(y)|^{n+2s}} = \langle f, \varphi \rangle \quad \forall \varphi \in C_c^\infty(\Omega),$$

Then the conclusions of Corollary 1.4 still hold true.

Corollary 1.6. *Let $s \in (0,1)$ and $p \geq 2$. Let $\Omega \subset\subset \mathbb{R}^n$ be a smoothly bounded set, and let $\Omega_1 \subset\subset \Omega$ be open. Assume that $u \in W^{s,2}(\Omega)$ satisfies*

$$\langle \mathcal{L}^s_\Omega u, \varphi \rangle = \int_\Omega \int_\Omega \frac{(f(x) - f(y))\,(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}}\, dx\, dy \tag{1.8}$$

for any $\varphi \in C_c^\infty(\Omega_1)$, where $\mathcal{L}^s_\Omega$ corresponds to $K \in \mathcal{K}(\alpha, \lambda, \Lambda)$ for some given $\alpha \in (0,1)$ and $\lambda, \Lambda > 0$. Then if for $s < t_0 < \min\{2s, 1\}$, $f \in W^{t_0,p}(\Omega)$ then for any $s \leq t < t_0$, $u \in W^{t,p}_{loc}(\Omega)$, and for any $\Omega_1 \subset \Omega$ we have the estimate

$$[u]_{W^{t,p}(\Omega_1)} \leq C\, \left([f]_{W^{t_0,p}(\Omega)} + [u]_{W^{s,2}(\Omega)}\right) + \|u\|_{L^2(\Omega)}.$$

We observe that Corollary 1.3 is to some extent an analogue of the local $W^{2,p}$-theory for divergence-form equations such as (1.7). However, there is one major difference: while the higher fractional differentiability of solutions for local equations of the form (1.7) is *closely related* to the smoothness of the coefficient A, for nonlocal equations of the form (1.6) it is only *loosely related* to the smoothness of the coefficient K.

Namely, for local equations, if $\operatorname{div}(A\nabla u) \in L^p$ and $A \in C^\alpha$, then $u \in W^{s,q}_{loc}$ for any $s < 1 + \alpha$ and $2 - s - \frac{n}{p} > -\frac{n}{q}$. That is, the increase in differentiability of the solution depends on the relative smoothness of the coefficient, the α-Hölder continuity of A. This is, however, not so much an effect that highlights the differences of "nonlocal vs local", but it is rather a structural feature of the nonlocal equation as in Definition 1.1. More precisely, denote by $\nabla^\alpha := \nabla I^{1-\alpha}$ the Riesz- fractional gradient and by $\operatorname{div}_\alpha := \operatorname{div} I^{1-\alpha}$ the Riesz-fractional divergence, [42, 37, 43, 40]. Consider the nonlocal equation $\operatorname{div}_\alpha(A\nabla^\alpha u) = f \in L^p$. The improvement in differentiability of a solution u heavily depends on the differentiability of A – indeed there is a one-to-one correspondence relationship between solutions for equations of type $\operatorname{div}(A\nabla u)$ and $\operatorname{div}_\alpha(A\nabla_\alpha u)$, cf. [41]. Compare also the discussion after Theorem 1.7.

In the case of solutions to the nonlocal equation as in Definition 1.1, the increase on differentiability of u is independent of the measure of Hölder continuity of the coefficient K. In other words, as long as K is Hölder continuous of any order $\alpha \in (0,1)$, the solution can be proved to be differentiable up to the order of $\min\{1, 2s\}$.

This presents one of the distinctions of our work from that of the regularity result obtained in [10] (which considers L^2-regularity). In [10], the almost optimal regularity of solution to (1.1) corresponding to $f_1 = 0$, and $f_2 \in L^2_{loc}(\mathbb{R}^n)$ is obtained under the assumption that $K \in C^s(\mathbb{R}^n \times \mathbb{R}^n)$. Using this smoothness assumption on K, which allows the application of the "difference quotient" method of proving higher differentiability, in [10] the solution u is shown to belong to $H^{2s-\epsilon,2}_{loc}(\mathbb{R}^n)$ for any $\epsilon > 0$.

For right-hand sides in L^2 we get similar *differentiability* results to [10], but at most up to differential order 1. However, we merely assume K to be C^α-Hölder continuous for some $\alpha > 0$ possibly much smaller than s, and K only needs to be be positive on the diagonal. An example for a kernel that belongs to $\mathcal{K}(\alpha, \lambda, \Lambda)$ but does not fit the framework given in [10] is $K(x,y) = \frac{2\lambda+|x|^\alpha+|y|^\alpha}{\lambda+|x|^\alpha+|y|^\alpha} + 10^6(\sin x + \sin y)\frac{|x-y|^\alpha}{(1+|x-y|^\alpha)}$. Observe that for small $\lambda > 0$, K could be negative off the diagonal $\{x = y\}$.

Optimal local elliptic regularity theory for weak solutions to the Dirichlet problem associated with the fractional Laplacian is also investigated in [3] by extending the nonlocal equation to be posed in $\mathbb{R}^n$ via a careful cutoff analysis and using optimal regularity estimates for nonlocal equations posed in the whole space. Similar results are also obtained in [23, 22] by methods from pseudodifferential theory for equations that involve fractional Laplacian or its pseudodifferential generalizations which corresponds to K that is translation invariant and C^∞.

Let us also mention the recent work [30], where nonlocal equations of the type (1.6) are studied for translation invariant coefficients, $K(x,y) = K(x-y)$. In this work, without imposing any smoothness assumption on $K(x-y)$, and using a real-analytic perturbation argument pioneered in [8] and expanded in [7] to obtain $W^{1,p}$-estimates, it was shown that if $f_1 \in L^p_{loc}(\mathbb{R}^n)$, and $f_2 \in L^{\frac{pn}{n+sp}}_{loc}(\mathbb{R}^n)$, then any weak solution u to (1.6) is in $H^{s,p}_{loc}(\mathbb{R}^n)$. This result in [30] concerns only the higher integrability of $(-\Delta)^{\frac{s}{2}}u$, whereas, in comparison, our work presents results on both higher differentiability and higher integrability of $(-\Delta)^{\frac{s}{2}}u$ for solutions of nonlocal equations corresponding to coefficients that are not necessarily translation invariant. Cf. also [31].

We should also mention that for "strong solutions" of nonlocal equations of the type $(\mathcal{L}^s_{\mathbb{R}^n} + \gamma\mathcal{I})u = f$ corresponding to translation invariant coefficients, $K(x,y) = K(x-y)$, and $\gamma > 0$ the optimal regularity theory of $f \in L^p(\mathbb{R}^n) \implies u \in H^{2s,p}(\mathbb{R}^n)$ is obtained in [15]. Similar to the previous paper discussed, the result in [15] requires no smoothness assumption on $K(x-y)$ and relies on a priori mean-oscillation estimates and maximal function theorem.

Other types of improved regularity results have also been observed for weak solution of nonlocal equations of type (1.6) with coefficients $K(x,y)$ that are just measurable, elliptic and bounded from above. What is called a self-improvement property of such solutions, which was first obtained in [25] via a generalized Gehring lemma, states that for $f_1 \in H^{s+\epsilon}$ and $f_2 \in L^{\frac{2n}{n+2s}}_{loc}$, a weak solution $u \in H^{s,2}(\mathbb{R}^n)$ is in fact in $W^{s+\delta,2+\delta}_{loc}(\mathbb{R}^n)$. While the improvement in integrability of the solutions is expected, the incremental improvement in differentiability without requiring any smoothness assumption on the coefficient K is unique to nonlocal equations of this type. Intuitively, one can see why such improvement can be possible. In fact, that for any $s_1, s_2 \in (0,1)$ with $s_1 + s_2 = 2s$ we have

$$\langle \mathcal{L}^s_{\mathbb{R}^n} u, \varphi\rangle \leq \|K\|_{L^\infty} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+s_1p}}\,dx\,dy\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\varphi(x)-\varphi(y)|^{p'}}{|x-y|^{n+s_2p'}}\,dx\,dy\right)^{\frac{1}{p'}}$$

That is, there is a possibility that one can distribute derivatives freely on test functions or the solution. This is clearly false for the local case unless $s_1 = s_2 = 1$.

$$\int_{\mathbb{R}^n} A(x)\partial_\alpha u\, \partial_\alpha \varphi \not\lesssim \|A\|_{L^\infty}\, \|u\|_{\dot{H}^{s_1,p}}\, \|\varphi\|_{\dot{H}^{s_2,p'}}.$$

The self-improving property of nonlocal equations have also been demonstrated via other approaches: via functional analytic approach in [1] and via comparison and commutator estimates in [38]. This kind of δ-differential flexibility of nonlocal equations has also been observed and crucially used in the regularity theory of geometric equations [36, 4]. For non-translation invariant kernels $K(x,y)$, under a different Hölder continuity assumption, Fall proved in [16] Schauder estimates. There he also observed that the gain Hölder regularity below the differential order 1 is independent of the Hölder regularity of the kernel and starts to depend on the Hölder continuity of the kernel for differential orders above 1. We will treat the question of differentiability above 1 in the Sobolev-space context in a future work [17].

Although the setup of the equation is different, Brasco–Lindgren [5, 6] have obtained a higher regularity results for solutions of the fractional p-Laplacian. They developed a discrete differentiation scheme that was successfully used to obtain a higher differentiation result which essentially says if the right hand side is differentiable then the solution will have improved differentiability as well. The equation they studied, the fractional p-Laplacian, amounts to having the kernel $K(x,y) = \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{(p-2)s}}$ in our setting, and the regularity of this kernel improves as the solution improves in regularity, which is a situation quite different from ours.

Finally, we comment on our strategy of proving Theorem 1.2. Our argument relies on comparing the leading order operator in (1.1), $\mathcal{L}^s_{\mathbb{R}^n}$ with that of the simpler operator $L^{s,t}_{diag}$ defined as

$$\langle L^{s,t}_{diag}u,\varphi\rangle := \int_{\mathbb{R}^n} K(z,z)(-\Delta)^{\frac{t}{2}}u(z)\,(-\Delta)^{\frac{2s-t}{2}}\varphi(z)\,dz, \tag{1.9}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $s \le t < 1$. To facilitate comparison of the operators, let us define the difference function

$$D_{s,t}(u,\varphi) := \langle \mathcal{L}^s_{\mathbb{R}^n}u,\varphi\rangle - \Gamma\langle L^{s,t}_{diag}u,\varphi\rangle.$$

Here Γ is the constant (depending on s, t, and n) such that $D_{s,t}(u,\varphi) \equiv 0$ for all u and φ admissible whenever the coefficient K is a constant map. In this sense, $D_{s,t}(u,\varphi)$ can be seen as a commutator $\int [T,K]u\,\varphi$ which is the main intuition in what follows. Indeed we obtain in Theorem 3.1 a quantitative estimate for $D_{s,t}$ that shows that in the case of Hölder continuous K, the commutator is *of lower order*. Intuitively, the operator $D_{s,t}(u,\varphi)$ gives us the mechanism to 'transfer derivatives' to K which along the way reduces the number of derivatives on u and φ. The commutator estimate we state in Theorem 3.1 is similar in spirit to the Coifman-Rochberg-Weiss commutator $[T,K](f)$ where T is a Calderon-Zygmund operator. If K is Hölder continuous of order γ, then $[T,K](f)$ can be estimated by a Riesz potential $I^\gamma f$ of f (i.e. a fractional antiderivative) – this is exactly what we obtain for our commutator $D_{s,t}$ in Theorem 3.1. While such a quantitative estimate is almost obvious for the Coifman-Rochberg-Weiss commutator it is already involved for our situation. Observe, however that a consequence of the famous work [9] Coifman, Rochberg, Weiss is that the operator $f \mapsto [T,K](f)$ is a compact operator for K in VMO, [45]. This suggests that with some work there could be a version of our theorem for K in VMO (in a suitable sense yet to be defined).

Let us remark that after the completion of this work, Simon Nowak [32] obtained some higher differentiability and integrability under merely VMO-assumptions on the kernel K – for $t \in [s, t_0)$ where t_0 depends on s and p, and in general is strictly smaller that $\min\{2s, 1\}$.

Once we identify $D_{s,t}(u, \varphi)$ as a lower-order operator, we can essentially read the regularity theory for the operator in Theorem 1.2 from the regularity theory of equations of the type

$$\langle L^{s,t}_{diag} u, \varphi \rangle = \int g\varphi \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n), \tag{1.10}$$

which is relatively easy to handle. Notice that (1.10) is a distributional formulation of the elliptic equation $(-\Delta)^{\frac{2s-t}{2}}(K(z,z)(-\Delta)^{\frac{t}{2}} u) = g$. Thus, formally, (1.10) is equivalent to

$$(-\Delta)^{\frac{t}{2}} u(x) = \frac{1}{K(x,x)} I^{2s-t} g(x),$$

and thus one expects the estimate

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^n)} \leq \frac{1}{\inf_x K(x,x)} \|I^{2s-t} g\|_{L^p(\mathbb{R}^n)}.$$

In particular, if $g \in L^q(\mathbb{R}^n)$ for some $q \in (1, \infty)$ with $p := \frac{nq}{n-(2s-t)q} \in (1, \infty)$, then by Sobolev embedding $I^{2s-t} g \in L^p(\mathbb{R}^n)$; that is, if u solves (1.10) and $g \in L^q(\mathbb{R}^n)$, then $u \in H^{t,p}_{loc}(\mathbb{R}^n)$ which is the optimal regularity result we expect.

The precise argument is based on a duality argument and a bit tedious, but in the end we obtain the following result in Section 4.

Theorem 1.7. *Let $s \in (0,1)$ and $t \in (0, 2s)$. Assume that for some $q \in (1, \infty)$, $(-\Delta)^{\frac{t}{2}} u \in L^q(\mathbb{R}^n)$ is a distributional solution to*

$$\int_{\mathbb{R}^n} \bar{K}(z)(-\Delta)^{\frac{t}{2}} u \, (-\Delta)^{\frac{2s-t}{2}} \varphi = \int_{\mathbb{R}^n} f_1 \, (-\Delta)^{\frac{2s-t}{2}} \varphi + \int_{\mathbb{R}^n} f_2 \, \varphi \quad \forall \varphi \in C_c^\infty(\Omega).$$

Here $\bar{K} : \mathbb{R}^n \to \mathbb{R}$ is a positive, measurable, and bounded from above and below, i.e.

$$\Lambda^{-1} \leq \bar{K}(z) \leq \Lambda \quad a.e. \ x \in \mathbb{R}^n.$$

Then for any $\Omega' \subset\subset \Omega \subset\subset \mathbb{R}^n$, $p \in (1, \infty)$, if $f_1, f_2 \in L^q(\mathbb{R}^n) \cap L^p(\Omega)$, then $(-\Delta)^{\frac{t}{2}} u \in L^p(\Omega')$ with

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\Omega')} \lesssim \|f_1\|_{L^p(\Omega)} + \|f_2\|_{L^p(\Omega)} + \|f_1\|_{L^q(\mathbb{R}^n)} + \|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\mathbb{R}^n)}.$$

Let us remark that Theorem 1.7 holds with minor modifications for $s > 1$, for simplicity we restrict it to the realm we are working in. It might seem surprising at first that in Theorem 1.7 there is no assumption on the kernel being continuous or belonging to VMO

– and still we are able to obtain L^p-estimates for any $p > 1$ if the right-hand side of the equation is good enough. For classical divergence form equations,

$$\mathrm{div}(\bar{K}\nabla u) = f \tag{1.11}$$

if $\bar{K}$ is only bounded measurable, the best one can hope for is an $W^{1,2+\varepsilon}$-type estimate (if f is nice enough) – this is known as a Meyers-type estimate, [28, 29]. To emphasize the role the type of equation we are studying plays on the regularity result, the reason that we get a (seemingly) better result in Theorem 1.7 is not because of the fractional order, but rather of the fact that ∇ and div are non-elliptic operators, while $(-\Delta)^{\frac{2s-t}{2}}$ is invertible.

An argument such as the one described before Theorem 1.7 does not work for solutions to (1.11), because we cannot invert the div-operator (and indeed for merely bounded measurable kernels only Meyers' $2+\varepsilon$-estimate remains true). So in Theorem 1.7 we make crucial use of the fact that the equation involved is structurally substantially different from (and for our purposes: simpler than) (1.11) – even if $s = 1$.

As we discussed earlier, a more proper 'nonlocal analogue' of the equation (1.11) (in the sense that it has generally comparable regularity properties as (1.11)) is

$$\mathrm{div}_{2s-t}[\bar{K}\nabla^t u] = f \tag{1.12}$$

where we recall that ∇^t denotes the Riesz-fractional gradient ∇I^{1-t}, and $\mathrm{div}_{2s-t} = \mathrm{div}\, I^{1+2s-t}$. Indeed, if $\bar{K}$ is merely bounded, measurable then for solutions to (1.12) only Meyers-type estimates are known, [2, Section 9]; and one needs $\bar{K}$ in VMO to conclude L^p-estimates, [40]. See also [42, 43].

Let us remark on previous arguments that inspired this work: for regularity theory via an harmonic analysis approach in the local case with an elliptic matrix $A_{\alpha,\beta}$ instead of the scalar A see [24]. This was applied to nonlocal equations different from (1.1) in [40]. Commutator operator similar to $D_{s,t}$ have also been proved to be very useful in the harmonic analysis of harmonic-type maps between manifolds [36] and nonlocal equations arising in topological calculus of variations, [4].

The remainder of this work is as follows: in Section 3 we prove the commutator estimate for $D_{s,t}$. This essentially reduces the desired Calderon-Zygmund theory to that of the theory of a weighted fractional Laplacian which we treat in Section 4 where the proof of Theorem 1.7. Since we only obtain local estimates, we will repeatedly employ cutoff arguments that are obtained in Section 5. In Section 6, the proof of the main result Theorem 1.2 is presented. And finally, the corollaries of Theorem 1.2 are proved Section 7.

2. Preliminaries and notation

Some notation and convention we will use throughout the paper. Domains of integrals are always open sets. We use the symbol $\subset\subset$ to say compactly contained, e.g. $\Omega_1 \subset\subset \Omega_2$ if $\overline{\Omega_1}$ is compact and $\overline{\Omega_1} \subset \Omega_2$.

Constants change from line to line, and generally depend on the dimension. We will make frequent use of $\lesssim$, $\gtrsim$ and $\approx$, which denotes inequalities with multiplicative constants (depending on non-essential data). For example we say $A \lesssim B$ if for some constant $C > 0$ we have $A \leq CB$.

We work with fractional Laplacians, Sobolev spaces, and related operators. Below we introduce the notation but refer the interested reader to surveys, e.g. [14, 19], or monographs [34]. We will use many techniques from harmonic analysis, such as Sobolev inequalities, embeddings etc. – these are all well-known in the abstract framework of Triebel-Lizorkin or Besov-space theory – see e.g. in [21]. Generally we like to refer to [33] for the identification of Triebel-Lizorkin and Besov-spaces with the "usual" function spaces. While we try to make as little as possible use of such abstract arguments sometimes they are unavoidable.

For $s \in (0,2)$ the fractional Laplacian $(-\Delta)^{\frac{s}{2}}$, defined in (1.3) via Fourier transform, has a useful integral representation. Namely, for a function f in the Schwartz class

$$(-\Delta)^{\frac{s}{2}} f(x) = c \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+s}}\, dy,$$

where the integral is defined in the *principal value* sense, although we do not explicitly state it. For the Riesz potential defined in (1.4), for $s \in (0,n)$, we have the representation

$$(-\Delta)^{-\frac{s}{2}} f(x) \equiv I^s f(x) = c \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}}\, dy$$

for a function f in the Schwartz class. The constants c are different in each definition, they only play an analytic role when considering stability $s \to 1^{\pm}$, $s \to 2^-$ or $s \to 0^+$. Below we will choose it to be $c = 2$.

For functions f and g in the Schwartz class, the L^2-inner product of $(-\Delta)^{\frac{s}{2}} f(x)$ and $g(x)$ can be represented as, for $s \in (0,2)$,

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} f(x)\, g(x) dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(y) - f(x))(g(y) - g(x))}{|x-y|^{n+s}}\, dx\, dy \tag{2.1}$$

see e.g. [35, Proposition 2.36.] or [14]. We also use the Leibniz's rule for fractional Laplacian frequently: for $u, v \in W^{s,2}(\mathbb{R}^n)$, one can easily show using the integral formulation of the fractional Laplacian that

$$(-\Delta)^{\frac{s}{2}}(uv) = u(-\Delta)^{\frac{s}{2}} v + v(-\Delta)^{\frac{s}{2}} u - \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+s}} dy \tag{2.2}$$

for almost all $x \in \mathbb{R}^n$. See e.g. [3, Proposition 1.5], [11, Appendix A], [12]. Fractional Laplacians and gradients are related via Riesz transforms and Riesz potentials. The Riesz transform, $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n) := \nabla I^1$, has the Fourier symbol $ci\frac{\xi}{|\xi|}$, and a potential representation

$$\mathcal{R} f(x) = \int_{\mathbb{R}^n} \frac{\frac{x-y}{|x-y|}}{|x-y|^n}\, f(y)\, dy.$$

Riesz transforms are most prominent examples of Calderon-Zygmund operators and are L^p-bounded. That is, for $1 < p < \infty$, there exists a constant $C = C(n,p) > 0$ such that

$$\|\mathcal{R} f\|_{L^p} \leq C \|f\|_{L^p}, \quad \text{for all } f \in L^p.$$

The now classical L^p-regularity theory for linear second-order PDEs is called Calderon-Zygmund theory because it (secretly or explicitly) relies on estimates of Calderon-Zygmund-operators (in most of the cases: the Riesz transforms).

We will frequently use Sobolev inequalities for Riesz potential.

Proposition 2.1 (Sobolev inequalities). *Suppose that $s \in (0,n)$ and $p \in (1,\infty)$. Then,*

(a) if $sp < n$, then there exists a constant $C = C(s,p,n) > 0$ such that

$$\|I^s g\|_{L^{\frac{np}{n-sp}}(\mathbb{R}^n)} \leq C \|g\|_{L^p(\mathbb{R}^n)} \quad \text{for any } g \in L^p(\mathbb{R}^n). \tag{2.3}$$

In addition, if $\Omega \subset \mathbb{R}^n$ is bounded, then corresponding to any $q \in [1, \frac{np}{n-sp}]$, there is a constant $C = C(s,p,n,\Omega) > 0$ such that

$$\|I^s g\|_{L^q(\Omega)} \leq C \|g\|_{L^p(\mathbb{R}^n)} \quad \text{for any } g \in L^p(\mathbb{R}^n). \tag{2.4}$$

(b) If $sp \geq n$ and $\Omega \subset \mathbb{R}^n$ is bounded domain, then for any $q \in [1,\infty)$, and $r \in [1, \frac{n}{s})$, there exists a constant $C = C(s,p,n,\Omega) > 0$ such that

$$\|I^s g\|_{L^q(\Omega)} \leq C \left(\|g\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^r(\mathbb{R}^n)} \right). \tag{2.5}$$

Proof. The proof of (2.3) can be found in [44]. (2.4) follows easily from (2.3). As for (2.5), observe that for any $q \in (1,\infty)$ there exists some $\theta \in (r, \frac{n}{s})$ such that $\frac{\theta n}{n - s\theta} > q$. Observe that $\theta < p$ so that we have the interpolation inequality

$$\|g\|_{L^\theta(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^r(\mathbb{R}^n)}$$

By (2.4) we have

$$\|I^s g\|_{L^q(\Omega)} \lesssim \|g\|_{L^\theta(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^r(\mathbb{R}^n)}.$$

□

We also need the following characterization of the dual space of the function spaces $H^{s,p}(\mathbb{R}^n)$ and $\dot{H}^{s,p}(\mathbb{R}^n)$. The homogeneous space $\dot{H}^{s,p}(\mathbb{R}^n)$ is the set of tempered distributions u such that $(-\Delta)^{s/2} u \in L^p(\mathbb{R}^n)$, with the semi-norm $\|(-\Delta)^{s/2} u\|_{L^p}$.

By definition, $T \in (H^{s,p}(\mathbb{R}^n))^*$, the dual space of $H^{s,p}(\mathbb{R}^n)$, if T is linear on $\varphi \in C_c^\infty(\mathbb{R}^n)$ and

$$|T[\varphi]| \leq \Lambda \left(\|\varphi\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^{\frac{s}{2}} \varphi\|_{L^p(\mathbb{R}^n)} \right) \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n).$$

The operator norm of T, $\|T\|$, is defined to be the infimum of all such Λ. Similarly, $T \in \left(\dot{H}^{s,p}(\mathbb{R}^n)\right)^*$, if T is linear on $\varphi \in C_c^\infty(\mathbb{R}^n)$ and

$$|T[\varphi]| \leq \Lambda \|(-\Delta)^{\frac{s}{2}} \varphi\|_{L^p(\mathbb{R}^n)} \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n).$$

Proposition 2.2. *(Dual Spaces)*

(1) *If* $T \in (H^{s,p}(\mathbb{R}^n))^*$, *then there exists* $g_1, g_2 \in L^{p'}(\mathbb{R}^n)$,

$$\|g_1\|_{L^{p'}(\mathbb{R}^n)} + \|g_2\|_{L^{p'}(\mathbb{R}^n)} \approx \|T\|$$

such that

$$T[\varphi] = \int g_1 (-\Delta)^{\frac{s}{2}} \varphi \, dx + \int g_2 \varphi \, dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n).$$

(2) *If* $T \in \left(\dot{H}^{s,p}(\mathbb{R}^n)\right)^*$, *then there exists* $g \in L^{p'}(\mathbb{R}^n)$,

$$\|g\|_{L^{p'}(\mathbb{R}^n)} \approx \|T\|$$

such that

$$T[\varphi] = \int g (-\Delta)^{\frac{s}{2}} \varphi \, dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n).$$

Proof. Let $T \in (H^{s,p}(\mathbb{R}^n))^*$. Denoting $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$, using the equivalence of the norms (cf. [44, Chapter V§3])

$$\|f\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{R}^n)} \quad \text{and} \quad \|\mathcal{F}^{-1}(\langle \xi \rangle^s \mathcal{F}(f))\|_{L^p(\mathbb{R}^n)}$$

we have

$$|T(\varphi)| \leq \|T\| \|\mathcal{F}^{-1}(\langle \xi \rangle^s \mathcal{F}(\varphi))\|_{L^p}, \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

We then introduce the linear function $\tilde{T} : L^p(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$\tilde{T}(v) = T(\mathcal{F}^{-1}(\langle \xi \rangle^{-s} \mathcal{F}(v))).$$

Then from the estimate for T, we have that $|\tilde{T}(v)| \leq \|T\| \|v\|_{L^p}$ for all $v \in L^p(\mathbb{R}^n)$. By the characterization of the dual of L^p spaces we have $u_0 \in L^{p'}(\mathbb{R}^n)$ such that

$$\tilde{T}(v) = \int_{\mathbb{R}^n} u_0(x) v(x) dx, \quad \text{for all } v \in L^p(\mathbb{R}^n).$$

Define now $g = \mathcal{F}^{-1}(\langle \xi \rangle^s \mathcal{F}(u_0))$. Then $g \in H^{-s,p'}(\mathbb{R}^n)$ and for any $\varphi \in \mathcal{S}$, the Schwartz space, we have by applying Plancherel's theorem repeatedly that

$$\begin{aligned} \langle g, \varphi \rangle = \langle \mathcal{F}^{-1}(\langle \xi \rangle^s \mathcal{F}(u_0)), \mathcal{F}^1(\mathcal{F}\varphi) \rangle &= \langle \langle \xi \rangle^s \mathcal{F}(u_0), \mathcal{F}\varphi \rangle \\ &= \langle \mathcal{F}(u_0), \langle \xi \rangle^s (\mathcal{F}\varphi) \rangle \\ &= \langle u_0, \mathcal{F}^{-1}(\langle \xi \rangle^s (\mathcal{F}\varphi)) \rangle = \tilde{T}(\mathcal{F}^{-1}(\langle \xi \rangle^s (\mathcal{F}\varphi))) = T(\varphi) \end{aligned}$$

We next characterize g further. Using [44, Lemma 2 of Chapter 5, §3], that describes the relationship between Riesz and Bessel potentials, there exists a pair of finite measures ν_s and λ_s so that

$$g = \mathcal{F}^{-1}(\langle \xi \rangle^s \mathcal{F}(u_0)) = \nu_s * u_0 + \mathcal{F}^{-1}(|\xi|^s \mathcal{F}(\lambda_s * u_0))$$

Define $g_1 = \nu_s * u_0$ and $g_2 = \lambda_s * u_0$. Then both g_1 and g_2 are in $L^{p'}(\mathbb{R}^n)$. Moreover, by applying Plancherel's theorem again

$$T(\varphi) = \langle g, \varphi \rangle = \langle g_1, \varphi \rangle + \langle \mathcal{F}^{-1}(|\xi|^s \mathcal{F}(g_2)), \varphi \rangle = \langle g_1, \varphi \rangle + \langle g_2, \mathcal{F}^{-1}(|\xi|^s \mathcal{F}(\varphi)) \rangle$$

as desired.

As for the second part, observe that since $\dot{H}^{s,q}(\mathbb{R}^n) \approx F^s_{q,2}(\mathbb{R}^n)$ we have that $(\dot{H}^{s,q}(\mathbb{R}^n))^* \approx F^{-s}_{q',2}(\mathbb{R}^n)$ ([18, Remark 5.14]). Since I^s is an isomorphism from $F^{-s}_{q',2}(\mathbb{R}^n)$ to $F^0_{q',2} \approx L^{q'}$, see [33, §2.6, Proposition 2, p. 95], we find that for any $(\dot{H}^{s,q}(\mathbb{R}^n))^*$ there must be $g \in L^{q'}(\mathbb{R}^n)$ with $\|g\|_{L^{q'}(\mathbb{R}^n)} \approx \|T\|_{(\dot{H}^{s,q}(\mathbb{R}^n))^*}$ such that $(-\Delta)^{\frac{s}{2}} g[\varphi] = T[\varphi]$, that is $T[\varphi] = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} g\, \varphi$. □

Let us also mention two technical results that we will employ frequently. They fall under the notion of "cutoff argument", and the techniques are mainly based on estimating nonlocal quantities for functions with disjoint support.

Lemma 2.3. *Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \mathbb{R}^n$, and $u, v \in H^{s,2}(\mathbb{R}^n)$ with $u \equiv v$ in Ω_2, $s \in [0,1)$.*

Then for any $p \in (1,\infty)$ we have

$$\|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\Omega_1)} \lesssim \|(-\Delta)^{\frac{s}{2}} v\|_{L^p(\Omega_2)} + \|u\|_{L^p(\mathbb{R}^n)} + \|v\|_{L^p(\mathbb{R}^n)}.$$

Proof. Let $\eta \in C_c^\infty(\Omega_2)$ with $\eta \equiv 1$ in a neighborhood of Ω_1.

We have

$$u = \eta v + (1-\eta) u.$$

Then

$$\chi_{\Omega_1} (-\Delta)^{\frac{s}{2}} u = \chi_{\Omega_1} (-\Delta)^{\frac{s}{2}} (\eta v) + \chi_{\Omega_1} \left((-\Delta)^{\frac{s}{2}} (1-\eta) u \right),$$

and by the usual disjoint support argument

$$\left| \chi_{\Omega_1} \left((-\Delta)^{\frac{s}{2}} (1-\eta) u \right) \right| (x) \lesssim \int_{|x-y| \gtrsim 1} |x-y|^{-n-s} |u(y)|\, dy$$

By Young's inequality for convolutions we conclude

$$\|\chi_{\Omega_1} \left((-\Delta)^{\frac{s}{2}} (1-\eta) u \right) \|_{L^\infty} \lesssim \|u\|_{L^p(\mathbb{R}^n)}.$$

And thus in particular,

$$\|\chi_{\Omega_1} \left((-\Delta)^{\frac{s}{2}} (1-\eta) u \right) \|_{L^p} \lesssim \|u\|_{L^p(\mathbb{R}^n)}.$$

Now we use commutator notation $[T, m](f) = T(mf) - mTf$,

$$(-\Delta)^{\frac{s}{2}} (\eta v) = \eta (-\Delta)^{\frac{s}{2}} v + [(-\Delta)^{\frac{s}{2}}, \eta](v).$$

Since $s \in (0,1)$ we can use the Coifman–McIntosh–Meyer commutator estimate, e.g. in the formulation in [26, Theorem 6.1.] or the Leibniz rule, [26, Theorem 7.1.], and conclude that

$$\|[(-\Delta)^{\frac{s}{2}}, \eta](v)\|_{L^p(\Omega_1)} \lesssim \|\eta\|_{\mathrm{Lip}}\, \|v\|_{L^p(\mathbb{R}^n)}.$$

This concludes the proof. □

Proposition 2.4. *Suppose that* $\eta_1, \eta_2 \in C_c^\infty(\mathbb{R}^n)$, *and* $\eta_2 \equiv 1$ *in the neighborhood of the support of* η_1. *Suppose that* $p \in (1,\infty)$, $\tau \in (0,2)$ *and*

$$r > \frac{np}{n+\tau p} > 1 \text{ if } \tau \leq 1 \quad \text{and } r > \frac{np}{n+p} > 1 \text{ if } \tau \geq 1$$

Then we have the following estimates which holds for any $\psi \in C_c^\infty(\mathbb{R}^n)$ *with* $\operatorname{supp}\psi \subset \{x : \eta_1(x) = 1\}$.

(a) *There exists a constant* $C > 0$ *such that*

$$\|(1-\eta_2)(-\Delta)^{\frac{\tau}{2}}\left((1-\eta_1)I^\tau\psi\right)\|_{L^{r'}(\mathbb{R}^n)} \leq C\|\psi\|_{L^{p'}(\mathbb{R}^n)}. \tag{2.6}$$

(b) *For any bounded set* $\Sigma \subset\subset \mathbb{R}^n$, *there exists a constant* $C = C(\Sigma)$ *such that*

$$\|(-\Delta)^{\frac{\tau}{2}}\left((1-\eta_1)I^\tau\psi\right)\|_{L^{r'}(\Sigma)} \leq C\|\psi\|_{L^{p'}(\mathbb{R}^n)}. \tag{2.7}$$

In either case the constant C *may depend on* r, τ, p, n, *and on* η_1, η_2, Σ, *but not on* ψ.

Proof. We prove part (b) first. Fix a large ball $B \subset\subset \mathbb{R}^n$ that compactly contains Σ. Since $(-\Delta)^{\frac{\tau}{2}} I^\tau\psi = \psi$, it follows from Leibniz's rule for fractional Laplacian, (2.2), that for $x \in \mathbb{R}^n$,

$$\begin{aligned}&(-\Delta)^{\frac{\tau}{2}}\left((1-\eta_1)I^\tau\psi\right)(x)\\ =&[(-\Delta)^{\frac{\tau}{2}}(1-\eta_1)]\, I^\tau\psi(x) + \underbrace{(1-\eta_1)\,\psi(x)}_{=0} + c\int_{\mathbb{R}^n} \frac{(\eta_1(x)-\eta_1(y))(I^\tau\psi(x)-I^\tau\psi(y))}{|x-y|^{n+\tau}}\,dy\end{aligned}$$

since the support of $1-\eta_1$ and ψ do not intersect. The right-hand side can be rewritten as

$$\begin{aligned}&(-\Delta)^{\frac{\tau}{2}}\left((1-\eta_1)I^\tau\psi\right)(x)\\ =&-[(-\Delta)^{\frac{\tau}{2}}\eta_1]I^\tau\psi(x) + c\int_B \frac{(\eta_1(x)-\eta_1(y))(I^\tau\psi(x)-I^\tau\psi(y))}{|x-y|^{n+\tau}}\,dy\\ &+ c\int_{\mathbb{R}^n\setminus B} \frac{(\eta_1(x)-\eta_1(y))(I^\tau\psi(x)-I^\tau\psi(y))}{|x-y|^{n+\tau}}\,dy.\end{aligned}$$

We will estimate each term in the right-hand side. We begin with the first one. To that end, since $\eta_1 \in C_c^\infty(\mathbb{R}^n)$, $(-\Delta)^{\frac{\tau}{2}}\eta_1 \in L^\infty(\mathbb{R}^n)$. Moreover, in view of Proposition 2.1, for any $1 < r' < \frac{np'}{n-\tau p'}$

$$\|I^\tau\psi\|_{L^{r'}(\Sigma)} \lesssim \|\psi\|_{L^{p'}(\mathbb{R}^n)}. \tag{2.8}$$

Notice also that since we assumed $\frac{np}{n+\tau p} > 1$, we have $n > \tau p'$. Thus we have

$$\|(-\Delta)^{\frac{\tau}{2}}\eta_1\, I^\tau\psi\|_{L^{r'}(\Sigma)} \lesssim \|\psi\|_{L^{p'}(\mathbb{R}^n)}.$$

For the second term, we use, see e.g. [39, Proposition 6.6.], that for any $\alpha < 1$,

$$|u(x)-u(y)| \lesssim |x-y|^\alpha\left(\mathcal{M}(-\Delta)^{\frac{\alpha}{2}}u(x) + \mathcal{M}(-\Delta)^{\frac{\alpha}{2}}u(y)\right),$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function. Then, applying this inequality for $u = I^\tau\psi$, by the Lipschitz continuity of η_1, for any $x \in \Sigma$ (observe that also B is bounded) for any $\alpha \in (0, \min\{\tau, 1\})$ with $1 + \alpha - \tau > 0$, we get

$$\begin{aligned}
&\int_B \frac{\big(\eta_1(x) - \eta_1(y)\big)\big(I^\tau\psi(x) - I^\tau\psi(y)\big)}{|x-y|^{n+\tau}}\, dy \\
&\quad\lesssim C \int_B |x-y|^{1+\alpha-\tau-n} \, \big(\mathcal{M}I^{\tau-\alpha}\psi(x) + \mathcal{M}I^{\tau-\alpha}\psi(y)\big)\, dy \\
&\quad\lesssim C\, \left(\mathcal{M}I^{\tau-\alpha}\psi(x) + I^{1+\alpha-\tau}\left(\mathcal{M}I^{\tau-\alpha}\psi\right)(x)\right),
\end{aligned}$$

where C depends only on $\|\eta_1\|_{\mathrm{Lip}}, \alpha, 2s, t, \mathrm{diam}\,(B)$, and $\mathrm{diam}\,(\Sigma)$. If $\tau \leq 1$, $1 + \alpha - \tau > 0$ is equivalent to $\alpha > 0$. In that case, whenever $1 < r' < \frac{np'}{n-\tau p'}$ we can choose α above so that $r' < \frac{np'}{n-(\tau-\alpha)p'}$. If $\tau \geq 1$, we need to choose $\alpha > \tau - 1 > 0$, so whenever $1 < r' < \frac{np'}{n-p'}$ we can find an α satisfying this condition so that $1 < r' < \frac{np'}{n-(\tau-\alpha)p'}$. Now to estimate the $L^{r'}$ norm of the map $x \mapsto \int_B \frac{\big(\eta_1(x)-\eta_1(y)\big)(I^\tau\psi(x)-I^\tau\psi(y))}{|x-y|^{n+\tau}}\, dy$ we estimate the norms of $\mathcal{M}I^{\tau-\alpha}\psi(x)$ and $I^{1+\alpha-\tau}\left(\mathcal{M}I^{\tau-\alpha}\psi\right)(x)$ separately. To that end, using maximal function theorem first and then Proposition 2.1 we have

$$\|\mathcal{M}I^{\tau-\alpha}\psi\|_{L^{r'}(\Sigma)} \lesssim \|\mathcal{M}I^{\tau-\alpha}\psi\|_{L^{\frac{np'}{n-(\tau-\alpha)p'}}(\Sigma)} \lesssim \|I^{\tau-\alpha}\psi\|_{L^{\frac{np'}{n-(\tau-\alpha)p'}}(\mathbb{R}^n)} \lesssim \|\psi\|_{L^{p'}(\mathbb{R}^n)}.$$

Also, using Proposition 2.1 first and then maximal function we have

$$\|I^{1+\alpha-\tau}\left(\mathcal{M}I^{\tau-\alpha}\psi\right)\|_{L^{r'}(\Sigma)} \lesssim \|\mathcal{M}I^{\tau-\alpha}\psi\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|I^{\tau-\alpha}\psi\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|\psi\|_{L^{p'_*}(\mathbb{R}^n)}$$

where $p'_* = \frac{np'}{n+(\tau-\alpha)p')} \leq p'$. Finally notice that since ψ is compactly supported, we have $\|\psi\|_{L^{p'_*}(\mathbb{R}^n)} \lesssim \|\psi\|_{L^{p'}(\mathbb{R}^n)}$. In summary, we have shown

$$\left\| x \mapsto \int_B \frac{\big(\eta_1(x) - \eta_1(y)\big)\big(I^\tau\psi(x) - I^\tau\psi(y)\big)}{|x-y|^{n+\tau}}\, dy \right\|_{L^{r'}(\Sigma)} \lesssim \|\psi\|_{L^{p'}(\mathbb{R}^n)}.$$

It remains to estimate for $x \in \Sigma$,

$$\begin{aligned}
\left|\int_{\mathbb{R}^n\setminus B} \frac{\big(\eta_1(x) - \eta_1(y)\big)\big(I^\tau\psi(x) - I^\tau\psi(y)\big)}{|x-y|^{n+\tau}}\, dy\right| &\lesssim \int_{\mathbb{R}^n\setminus B} \frac{I^\tau|\psi|(x) + |I^\tau|\psi|(y)}{1+|y|^{n+\tau}}\, dy \\
&\lesssim I^\tau|\psi|(x) + \|I^\tau|\psi|\|_{L^\infty(\mathbb{R}^n\setminus B)}.
\end{aligned}$$

The first term we have already estimated, (2.8). For the second term, observe that by the integral representation of I^τ and the support of ψ, we have

$$\|I^\tau|\psi|\|_{L^\infty(\mathbb{R}^n\setminus B)} \lesssim \|\psi\|_{L^1(\mathbb{R}^n)} \lesssim \|\psi\|_{L^{p'}(\mathbb{R}^n)}.$$

Thus, for any $1 < r' < \frac{np'}{n-\tau p'}$,

$$\left\| x \mapsto \int_{\mathbb{R}^n \setminus B} \frac{\big(\eta_1(x) - \eta_1(y)\big)(I^\tau \psi(x) - I^\tau \psi(y))}{|x-y|^{n+\tau}}\, dy \right\|_{L^{r'}(\mathbb{R}^n)} \lesssim \|\psi\|_{L^{p'}(\mathbb{R}^n)}.$$

This concludes the proof of part (b).

Next we prove part (a). As before, we split as

$$\begin{aligned}&(-\Delta)^{\frac{\tau}{2}} \left((1-\eta_1) I^\tau \psi\right)\\ =& \left((-\Delta)^{\frac{\tau}{2}} (1-\eta_1)\right) I^\tau \psi + \underbrace{(1-\eta_1)\,\psi}_{=0} + c \int_{\mathbb{R}^n} \frac{(\eta_1(\cdot) - \eta_1(y))(I^\tau\psi(\cdot) - I^\tau \psi(y))}{|\cdot - y|^{n+\tau}}\, dy\end{aligned}$$

Observe that by the disjoint support of ψ and $1-\eta_2$,

$$\|(1-\eta_2) I^\tau \psi\|_{L^\infty} \lesssim \|\psi\|_{L^1(\mathbb{R}^n)} \lesssim \|\psi\|_{L^{p'}(\mathbb{R}^n)}.$$

Moreover, $(-\Delta)^{\frac{\tau}{2}}(1-\eta_1) = (-\Delta)^{\frac{\tau}{2}} \eta_1 \in L^1 \cap L^\infty(\mathbb{R}^n)$ since $\eta_1 \in C_c^\infty(\mathbb{R}^n)$. Consequently, for any $r' \in [1,\infty]$,

$$\|(1-\eta_2) \left((-\Delta)^{\frac{\tau}{2}} (1-\eta_1)\right) I^\tau \psi\|_{L^{r'}(\mathbb{R}^n)} \lesssim \|\psi\|_{L^{p'}(\mathbb{R}^n)}.$$

On the other hand, for $x \in \operatorname{supp}(1-\eta_2)$ and $y \in \operatorname{supp}(\eta_1)$ we have $\eta_1(x) = 0$ and $|x-y| \gtrsim 1 + |x|$. Thus,

$$\begin{aligned}&\left|(1-\eta_2(x)) \int_{\mathbb{R}^n} \frac{\big(\eta_1(x) - \eta_1(y)\big)(I^\tau \psi(x) - I^\tau \psi(y))}{|x-y|^{n+\tau}}\, dy\right|\\ \lesssim& \frac{1}{1+|x|^{n+\tau}} \,|1-\eta_2(x)|\, \left(|I^\tau \psi(x)| + \int_\Omega |I^\tau \psi(y)| dy\right)\\ \lesssim& \frac{1}{1+|x|^{n+\tau}} \,|1-\eta_2(x)|\, \|\psi\|_{L^1(\mathbb{R}^n)}\\ \lesssim& \frac{1}{1+|x|^{n+\tau}} \,\|\psi\|_{L^{p'}(\mathbb{R}^n)}.\end{aligned}$$

The right-hand side is now integrable for any $r' \geq 1$, and (2.6) is established.

□

3. A commutator estimate

As we described in the introduction, the crucial idea of this work is to compare the two differential operators: $\mathcal{L}^s_{\mathbb{R}^n}$ defined by, for $s \in (0,1)$,

$$\langle \mathcal{L}^s_{\mathbb{R}^n} u, \varphi \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) \frac{(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\, dx\, dy \tag{3.1}$$

and $L^{s,t}_{diag}$ defined by, for $t \in [s, 2s)$, to the operator

$$\langle L^{s,t}_{diag} u, \varphi\rangle = \int_{\mathbb{R}^n} K(z,z)(-\Delta)^{\frac{t}{2}} u(z)\,(-\Delta)^{\frac{2s-t}{2}}\varphi(z)\,dz, \tag{3.2}$$

for u and φ in appropriate spaces. Observe that if K is constant, then the two operators are the same up to a multiplicative constant. Indeed, for $t \in [s, 2s)$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy = C\int_{\mathbb{R}^n}(-\Delta)^{\frac{2s}{2}} u\,\varphi dx = C\int_{\mathbb{R}^n}(-\Delta)^{\frac{t}{2}} u(-\Delta)^{\frac{2s-t}{2}}\varphi dx.$$

The first equality follows from the characterization of the fractional Laplacian, (2.1) and Fubini's theorem. The second "integration by parts" equality follows from the Fourier transform characterization of fractional Laplacian, (1.3) and Plancherel's theorem.

In this section, we prove a fundamental estimate for $D_{s,t}(u,\varphi)$, introduced as,

$$D_{s,t}(u,\varphi) = \langle L^{s,t}_{diag} u, \varphi\rangle - \Gamma\langle \mathcal{L}^s_{\mathbb{R}^n} u, \varphi\rangle \tag{3.3}$$

that establishes the difference $\mathcal{L}^s_{\mathbb{R}^n} - \Gamma L^{s,t}_{diag} u$ is a lower order differential operator when K is bounded and uniformly Hölder continuous. In (3.3), Γ is the universal constant that ensures that $D_{s,t}(u,\varphi) = 0$ whenever K is a constant kernel,

This allows us to obtain estimates for the operator in (3.1) from estimates for the operator (3.2), for which corresponding estimates are relatively easy to obtain as we will see in Section 4. The main theorem of this section is the following.

Theorem 3.1. *Let $s \in (0,1)$, $t \in (0,1)$ such that $2s - t \in (0,1)$. Suppose also that $\alpha \in (0,1)$ and $\Lambda > 0$ are given. Then, there exist constants $\sigma_0 \in (0,\alpha]$ and $\Gamma = \Gamma(n,s,t) \in \mathbb{R}$ such that the following holds. Let $K = K(x,y) \in \mathcal{C}(\alpha,\Lambda)$, where*

$$\mathcal{C}(\alpha,\Lambda) = \{K : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R} : |K(x,y)| \le \Lambda,\ \text{and (1.2) is satisfied}\}$$

Then for all $\sigma \in (0,\sigma_0)$ and all $\varepsilon \in (0,\frac{\sigma}{4})$ there exists a constant $C = C(\Lambda,\sigma,\varepsilon)$ such that,

$$|D_{s,t}(u,\varphi)| \le C\int_{\mathbb{R}^n} I^{\sigma-\varepsilon}|(-\Delta)^{\frac{t-\varepsilon}{2}} u|(x)\,|(-\Delta)^{\frac{2s-t}{2}}\varphi|(x)\,dx$$

and

$$|D_{s,t}(u,\varphi)| \le C\int_{\mathbb{R}^n} I^{\sigma-\varepsilon}|(-\Delta)^{\frac{t}{2}} u|(x)\,|(-\Delta)^{\frac{2s-t-\varepsilon}{2}}\varphi|(x)\,dx$$

for all $u \in H^{t,p}(\mathbb{R}^n)$ and any $\varphi \in C^\infty_c(\mathbb{R}^n)$. The constant $\sigma_0 \in (0,\alpha]$ depends on s and t in the following way: for any $\theta > 0$, if

$$s \in (\theta, 1-\theta),\ t \in (\theta, 1-\theta),\ 2s-t \in (\theta, 1-\theta)$$

then σ_0 can be chosen dependent only on θ and α (but not further depending on s and t).

Observe that $\mathcal{K}(\alpha,\lambda,\Lambda) \subset \mathcal{C}(\alpha, \max\{\Lambda, \frac{1}{\lambda}\})$, so Theorem 3.1 is applicable in our situation.

3.1. **Some preliminary estimates.** In this subsection we present some preliminary estimates that will be used in the proof of Theorem 3.1.

First we observe that the exponent of the Hölder continuity of K can be chosen to be very small, namely

Lemma 3.2. *Let $0 < \alpha < \beta$ and $\Lambda > 0$ then there exists $\Lambda' > 0$ such that whenever $K \in \mathcal{C}(\beta, \Lambda)$ then $K \in \mathcal{C}(\alpha, \Lambda')$*

This is an easy exercise which we leave to the reader.

Secondly we recall a quite useful application of the fundamental theorem of calculus.

Lemma 3.3. *For any $r \in \mathbb{R}$, there exists a constant $C = C(r)$ such that the following holds. Let $a, b \in \mathbb{R}^n \backslash \{0\}$ with $|a - b| \lesssim \min\{|a|, |b|\}$. Then for any $\sigma \in [0, 1]$ we have*

$$||a|^r - |b|^r| \leq C\,|a-b|^\sigma\, \min\left\{|a|^{r-\sigma}, |b|^{r-\sigma}\right\}.$$

Proof. We may assume that $r \neq 0$ otherwise the inequality it trivial.

If $|a - b| \lesssim \min\{|a|, |b|\}$, then $|a| \approx |b|$ (with a uniform constant), that is

$$\min\left\{|a|^{r-\sigma}, |b|^{r-\sigma}\right\} \approx |a|^{r-\sigma}.$$

Also for any $\sigma \in [0, 1]$ we have

$$|a - b| \lesssim |a - b|^\sigma |a|^{1-\sigma}.$$

Using the above inequality, to complete the proof it suffices to show that

$$||a|^r - |b|^r| \lesssim |a-b|\,|b|^{r-1}.$$

To that end, dividing by $|b|^r$, the above is equivalent to showing

$$\left|\left|\frac{a}{|b|}\right|^r - \left|\frac{b}{|b|}\right|^r\right| \lesssim \left|\frac{a}{|b|} - \frac{b}{|b|}\right|$$

Observe that since $|a| \approx |b|$, there are uniform constants $0 < R_1 < 1 < R_2 < \infty$ such that both $\frac{a}{|b|}$ and $\frac{b}{|b|}$ are in $A := B_{R_2}(0)\backslash B_{R_1}(0)$. So, the problem is now reduced to showing

$$||u|^r - |v|^r| \leq C\ |u - v| \quad \forall u, v \in A.$$

Since A is an annulus, for any $u, v \in A$ there exists a curve $\gamma \subset A$ with $\gamma(0) = u$, $\gamma(1) = v$, $|\gamma'| \approx |u - v|$ – with constants depending only on r_1 and r_2 (and thus uniform). Set

$$\eta(t) := |\gamma(t)|^\alpha.$$

Then, the fundamental theorem of calculus implies

$$||u|^\alpha - |v|^\alpha| \leq \sup_{t\in[0,1]} |\eta'(t)| \lesssim |\gamma(t)|^{\alpha-1}|\gamma'(t)| \lesssim |u - v|.$$

□

The following Lemma was essentially proven in [36, Proposition 6.3.].

Lemma 3.4. *Let $m \in (0,n)$, $\alpha \in (0,1)$, and $\lambda, \Lambda > 0$ are given. Then for any β such that*

$$\alpha < \beta < \min\{m+\alpha, 1\}$$

and any $K \in \mathcal{C}(\alpha, \Lambda)$ we have

$$|K(x,y) - K(z,z)| \, \left||x-z|^{m-n} - |y-z|^{m-n}\right| \lesssim |x-y|^\beta \left(|x-z|^{m+\alpha-\beta-n} + |y-z|^{m+\alpha-\beta-n}\right).$$

Proof. We first observe that we can estimate the difference $|K(x,y) - K(z,z)|$ in three different ways

$$|K(x,y) - K(z,z)| \lesssim \begin{cases} |x-z|^\alpha + |y-z|^\alpha \\ |x-y|^\alpha + |x-z|^\alpha \\ |x-y|^\alpha + |y-z|^\alpha. \end{cases} \tag{3.4}$$

The first one can be obtained by adding and subtracting $K(x,z)$:

$$|K(x,y) - K(z,z)| \leq |K(x,y) - K(x,z)| + |K(x,z) - K(z,z)| \lesssim |y-z|^\alpha + |x-z|^\alpha.$$

The second and third forms are obtained in similar ways as

$$\begin{aligned} |K(x,y) - K(z,z)| \leq& |K(x,y) - K(x,x)| + |K(x,x) - K(x,z)| + |K(x,z) - K(z,z)| \\ \lesssim& |x-y|^\alpha + 2|x-z|^\alpha, \end{aligned}$$

and

$$\begin{aligned} |K(x,y) - K(z,z)| \leq& |K(x,y) - K(y,y)| + |K(y,y) - K(y,z)| + |K(y,z) - K(z,z)| \\ \lesssim& |x-y|^\alpha + 2|y-z|^\alpha. \end{aligned}$$

The entire expression $|K(x,y) - K(z,z)| \, ||x-z|^{m-n} - |y-z|^{m-n}|$ can now be estimated by considering these three cases. To that end, first, if $|x-y| < \frac{1}{2}|x-z|$ or $|x-y| < \frac{1}{2}|y-z|$ then

$$|x-z| \approx |y-z|,$$

and thus by the mean value theorem, Lemma 3.3,

$$\left||x-z|^{m-n} - |y-z|^{m-n}\right| \lesssim |x-y| \, |x-z|^{m-1-n}.$$

So we take the first option in the estimate for K (3.4) and have under our assumptions on x, y, z (since $\beta \leq 1$

$$|K(x,y) - K(z,z)| \, \left||x-z|^{m-n} - |y-z|^{m-n}\right| \lesssim |x-y||x-z|^{m+\alpha-1-n} \lesssim |x-y|^\beta |x-z|^{m+\alpha-\beta-n}$$

Second, if $|x-y| \geq \frac{1}{2}|x-z|$ and $|x-y| \geq \frac{1}{2}|y-z|$ and $|x-z| < |y-z|$, we have

$$\left||x-z|^{m-n} - |y-z|^{m-n}\right| \lesssim |x-z|^{m-n}.$$

In this case we choose the second estimate for the estimate of K (3.4) and obtain (since $\beta \in (\alpha, m+\alpha)$,

$$\begin{aligned} |K(x,y) - K(z,z)| \, \left||x-z|^{m-n} - |y-z|^{m-n}\right| \lesssim& |x-y|^\alpha \, |x-z|^{m-n} + |x-z|^{\alpha+m-n} \\ \lesssim& |x-y|^\beta \, |x-z|^{\alpha+m-\beta-n}. \end{aligned}$$

Finally, if $|x-y| \geq \frac{1}{2}|x-z|$ and $|x-y| \geq \frac{1}{2}|y-z|$ but $|x-z| \geq |y-z|$, we have by a symmetric argument

$$|K(x,y)-K(z,z)| \left||x-z|^{m-n}-|y-z|^{m-n}\right| \lesssim |x-y|^{\beta}\, |y-z|^{\alpha+m-\beta-n}.$$

□

Lemma 3.5. *Let $\lambda, \Lambda > 0$ be given. Suppose also that $s,t \in (0,1)$ with $2s-t \in (0,1)$ in the following form: assume that for some $\theta \in (0,1)$,*

$$s \in (\theta, 1-\theta),\ t \in (\theta, 1-\theta),\ 2s-t \in (\theta, 1-\theta). \tag{3.5}$$

Then there exists $\alpha_0 = \alpha_0(\theta)$ such that for any $\alpha \in (0,\alpha_0)$, $\varepsilon \in (0,\frac{\alpha}{3})$, and $K \in \mathcal{C}(\alpha,\Lambda)$ the following holds. For $i,j = 1,2$ set

$$M^{\varepsilon}_{i,j}(z_1,z_2) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(x,y)-K(z_j,z_j)|\, |\kappa^{\varepsilon}_i(x,y,z_1,z_2)|\, dx\, dy.$$

where

$$\kappa^{\epsilon}_1(x,y,z_1,z_2) := \frac{\left(|x-z_1|^{t-\varepsilon-n}-|y-z_1|^{t-\varepsilon-n}\right)\,\left(|x-z_2|^{2s-t-n}-|y-z_2|^{2s-t-n}\right)}{|x-y|^{n+2s}},$$

$$\kappa^{\varepsilon}_2(x,y,z_1,z_2) := \frac{\left(|x-z_1|^{t-n}-|y-z_1|^{t-n}\right)\,\left(|x-z_2|^{2s-t-\varepsilon-n}-|y-z_2|^{2s-t-\varepsilon-n}\right)}{|x-y|^{n+2s}}$$

Then for any $f,g \in C^{\infty}_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(z_1)\, g(z_2) M^{\epsilon}_{i,j}(z_1,z_2) dz_1 dz_2 \leq C(\Lambda,\theta)\int_{\mathbb{R}^n} I^{\alpha-\varepsilon}|f|(x)\, |g|(x)\, dx, \quad i,j=1,2.$$

Proof. We prove the lemma by taking

$$\alpha_0 := \frac{1}{10}\min\{\theta, 1-\theta\}. \tag{3.6}$$

To that end, assume that $\alpha < \alpha_0$, $\varepsilon < \frac{\alpha}{3}$ from now on. We will only consider the case of M^{ε}_{12}; the estimate of the other M^{ε}_{ij} is analogous. To simplify notation we write $\kappa^{\epsilon} := \kappa^{\epsilon}_1$ and $M^{\varepsilon} := M^{\varepsilon}_{12}$.

We begin writing $M^{\epsilon}(z_1,z_2)$ as

$$\begin{aligned} M^{\epsilon}(z_1,z_2) &\leq \sum_{i,j=1}^{3} \iint_{\mathcal{O}_i\cap\mathcal{P}_j} |K(x,y)-K(z_2,z_2)|\, |\kappa^{\epsilon}(x,y,z_1,z_2)|\, dx\, dy \\ &=: \sum_{i,j=1}^{3} J^{(i,j)}_{\epsilon}(z_1,z_2), \end{aligned}$$

where the regions of integration are given by

$$\begin{aligned} \mathcal{O}_1 &= \{(x,y) : |x-y| \lesssim \min\{|x-z_1|, |y-z_1|\}\} \\ \mathcal{O}_2 &= \{(x,y) : |x-z_1| \lesssim \min\{|y-z_1|, |x-y|\}\} \\ \mathcal{O}_3 &= \{(x,y) : |y-z_1| \lesssim \min\{|x-z_1|, |x-y|\}\} \end{aligned}$$

and

$$\mathcal{P}_1 = \{(x,y) : |x-y| \lesssim \min\{|x-z_2|, |y-z_2|\}\}$$
$$\mathcal{P}_2 = \{(x,y) : |x-z_2| \lesssim \min\{|y-z_2|, |x-y|\}\}$$
$$\mathcal{P}_3 = \{(x,y) : |y-z_2| \lesssim \min\{|x-z_2|, |x-y|\}\}$$

Then we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(z_1)\, g(z_2) M^{\epsilon}(z_1,z_2) dz_1 dz_2 = \sum_{i,j} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(z_1)\, g(z_2) J^{i,j}_{\epsilon}(z_1,z_2) dz_1 dz_2.$$

We will estimate the integral that involves each of these terms.

Estimating terms involving $J_\epsilon^{(1,1)}$, $J_\epsilon^{(1,2)}$, $J_\epsilon^{(1,3)}$ and $J_\epsilon^{(2,1)}$:

We begin by noting that for $(x,y) \in \mathcal{O}_1$, from Lemma 3.3 by taking $r = t-\epsilon-n$ for ϵ small, for any $0 \le \sigma \le 1$ and any (x,y)

$$\left||x-z_1|^{t-\varepsilon-n} - |y-z_1|^{t-\varepsilon-n}\right| \lesssim |x-y|^{\sigma} \left(|x-z_1|^{t-\varepsilon-\sigma-n} + |y-z_1|^{t-\varepsilon-\sigma-n}\right).$$

Moreover, from Lemma 3.4 by taking $m = 2s-t$, $\alpha < \alpha_0$, for any $\beta < 2s-t+\alpha < 1-\frac{9\theta}{10} < 1$ and (x,y)

$$\begin{aligned}|K(x,y) - K(z_2,z_2)| & \left(|x-z_2|^{2s-t-n} - |y-z_2|^{2s-t-n}\right)\\ &\lesssim |x-y|^{\beta} \left(|x-z_2|^{2s-t+\alpha-\beta-n} + |y-z_2|^{2s-t+\alpha-\beta-n}\right).\end{aligned}$$

Combining the above two inequalities we obtain that for $0 < \epsilon < \frac{\alpha}{3}$, for any $\beta < 2s-t+\alpha$, any $\sigma \in [0,1]$ and any $(x,y) \in \mathcal{O}_1$

$$\begin{aligned}&|K(x,y) - K(z_2,z_2)|\, |\kappa_\epsilon(x,y,z_1,z_2)|\\ &\quad\lesssim |x-y|^{-2s-n+\beta+\sigma}(|x-z_1|^{t-\epsilon-\sigma-n} + |y-z_1|^{t-\epsilon-\sigma-n})\\ &\qquad\times (|x-z_2|^{2s-t+\alpha-\beta-n} + |y-z_2|^{2s-t+\alpha-\beta-n})\end{aligned} \tag{3.7}$$

Now for $(x,y) \in \mathcal{O}_1 \cap \mathcal{P}_1$, (3.7) reduces to

$$|K(x,y) - K(z_2,z_2)|\, |\kappa(x,y,z_1,z_2)| \lesssim |x-y|^{\beta+\sigma-2s-n}\, |x-z_2|^{2s-t+\alpha-\beta-n}\, |x-z_1|^{t-\varepsilon-\sigma-n}$$

after noting that in this case $|x-z_1| \approx |y-z_1|$ and $|x-z_2| \approx |y-z_2|$. In view of (3.5) and (3.6), we can choose β slightly smaller than $2s-t+\alpha$ and σ slightly smaller than $t-\varepsilon$ and still ensure $\beta+\sigma > 2s+\alpha-\varepsilon > 2\theta > 0$. For each x,

$$\int_{\{|x-y| \lesssim \min\{|x-z_1|, |x-z_2|\}\}} |x-y|^{\beta+\sigma-2s-n}\, dy \lesssim |x-z_1|^{\sigma-t+\frac{\alpha}{2}}\, |x-z_2|^{\beta-2s+t-\frac{\alpha}{2}}$$

and therefore,

$$\iint_{\mathcal{O}_1\cap\mathcal{P}_1} |K(x,y) - K(z_2,z_2)|\, |\kappa(x,y,z_1,z_2)| dx\, dy \lesssim \int_{\mathbb{R}^n} |x-z_2|^{\frac{\alpha}{2}-n}\, |x-z_1|^{\frac{\alpha}{2}-\varepsilon-n}\, dx.$$

From this we conclude that

$$\iint_{\mathbb{R}^n} J_\varepsilon^{(1,1)}(z_1,z_2)\, f(z_1)\, g(z_2) dz_1 dz_2 \lesssim \int_{\mathbb{R}^n} I^{\frac{\alpha}{2}-\varepsilon}|f|(x)\, I^{\frac{\alpha}{2}}|g|(x)\, dx$$
$$= \int_{\mathbb{R}^n} I^{\alpha-\varepsilon}|f|(x)\, |g|(x)\, dx,$$

where the last "integration by parts"-equality follows by an application of Plancherel's theorem.

For $(x,y) \in \mathcal{O}_1 \cap \mathcal{P}_2$, (3.7) reduces to

$$|K(x,y)-K(z_2,z_2)|\,|\kappa(x,y,z_1,z_2)| \lesssim |x-y|^{\beta+\sigma-2s-n}\,|x-z_2|^{2s-t+\alpha-\beta-n}\,|x-z_1|^{t-\varepsilon-\sigma-n}$$

for our choice of $\beta < 1$, $\sigma \in (0, t-\varepsilon)$. In view of (3.5) and (3.6), in fact we choose $\beta := 2s - t + \alpha/2 > \theta > \alpha$ to get the estimate that

$$|K(x,y)-K(z_2,z_2)|\,|\kappa(x,y,z_1,z_2)| \lesssim |x-y|^{\sigma-t+\alpha/2-n}\,|x-z_2|^{\frac{\alpha}{2}-n}\,|x-z_1|^{t-\varepsilon-\sigma-n}.$$

If σ is close enough to $t-\varepsilon$ and since $\varepsilon < \alpha/2$, we can integrate

$$\int_{|x-y|\lesssim|x-z_1|} |x-y|^{\sigma-t+\alpha/2-n}\,|x-z_2|^{\frac{\alpha}{2}-n}\,|x-z_1|^{t-\varepsilon-\sigma-n} dy \lesssim |x-z_2|^{\frac{\alpha}{2}-n}\,|x-z_1|^{\alpha/2-\varepsilon-n}$$

Arguing in the previous case, we obtain

$$\iint_{\mathbb{R}^n} J_\varepsilon^{(1,2)}(z_1,z_2)\, f(z_1)\, g(z_2) dz_1 dz_2 \lesssim \int_{\mathbb{R}^n} I^{\frac{\alpha}{2}-\varepsilon}|f|(x)\, I^{\frac{\alpha}{2}}|g|(x)\, dx$$
$$= \int_{\mathbb{R}^n} I^{\alpha-\varepsilon}|f|(x)\, |g|(x)\, dx,$$

For $(x,y) \in \mathcal{O}_1 \cap \mathcal{P}_3$ or $(x,y) \in \mathcal{O}_2 \cap \mathcal{P}_1$ and (3.7) reduces to

$$|K(x,y)-K(z_2,z_2)|\,|\kappa(x,y,z_1,z_2)| \lesssim |x-y|^{\beta+\sigma-2s-n}\,|y-z_2|^{2s-t+\alpha-\beta-n}\,|y-z_1|^{t-\varepsilon-\sigma-n}$$

As before we choose $\beta := 2s - t + \alpha/2$ (which is greater than α) to obtain that when $(x,y) \in \mathcal{O}_1 \cap \mathcal{P}_3$

$$\int_{|x-y|\lesssim|y-z_1|} |x-y|^{\sigma-t+\alpha/2-n}\,|y-z_2|^{\frac{\alpha}{2}-n}\,|y-z_1|^{t-\varepsilon-\sigma-n} dx \lesssim |y-z_2|^{\frac{\alpha}{2}-n}\,|y-z_1|^{\alpha/2-\varepsilon-n},$$

from which, we have

$$\iint_{\mathbb{R}^{2n}} J_\epsilon^{(1,3)}(z_1,z_2) f(z_1)\, g(z_2) dz_1 dz_2 \lesssim \int_{\mathbb{R}^d} I^{\frac{\alpha}{2}-\epsilon}(|f|)(y) I^{\frac{\alpha}{2}}(|g|)(y) dy = \int_{\mathbb{R}^n} I^{\alpha-\epsilon}|f|(y)|g|(y) dy.$$

For $(x,y) \in \mathcal{O}_2 \cap \mathcal{P}_1$, we have

$$\int_{|x-y|\lesssim|y-z_2|} |x-y|^{\sigma-t+\alpha/2-n}\,|y-z_2|^{\frac{\alpha}{2}-n}\,|y-z_1|^{t-\varepsilon-\sigma-n} dx \lesssim |y-z_2|^{\sigma-t+\alpha-n}\,|y-z_1|^{t-\epsilon-\sigma-n}$$

From this it follows that

$$\iint_{\mathbb{R}^{2n}} J_\epsilon^{(2,1)}(z_1,z_2) f(z_1) g(z_2) dz_1 dz_2 \lesssim \int_{\mathbb{R}^n} I^{t-\varepsilon-\sigma}|f|(y)\, I^{\sigma-t+\alpha}|g|(y)\, dy,$$

and integration by parts leads to the same estimate.

Estimating $J_\varepsilon^{(2,2)}$:

On the one hand, for $(x,y) \in \mathcal{O}_2 \cap \mathcal{P}_2$, we have (since $|x-z_2|^\alpha \lesssim |y-z_2|^\alpha$),

$$|K(x,y) - K(z_2,z_2)| \lesssim |x-z_2|^\alpha + |y-z_2|^\alpha \lesssim |y-z_2|^\alpha.$$

On the other hand, $|y-z_1|^{-1} \lesssim |x-z_1|^{-1}$ and $|y-z_2|^{-1} \lesssim |x-z_2|^{-1}$, and thus

$$\kappa^\varepsilon(x,y,z_1,z_2) \lesssim \frac{|x-z_1|^{t-\varepsilon-n}\,|x-z_2|^{2s-t-n}}{|x-y|^{n+2s}}.$$

This leads to

$$\begin{aligned}|K(x,y) &- K(z_2,z_2)|\,|\kappa_\varepsilon(x,y,z_1,z_2)| \\ &\lesssim |x-z_1|^{t-\varepsilon-n}\,|x-z_2|^{2s-t-n}\,|y-z_2|^\alpha\,|x-y|^{-2s-n} \\ &\lesssim |x-z_1|^{t-\varepsilon-n}\,|x-z_2|^{2s-t-n}\,|x-y|^{\alpha-2s-n},\end{aligned}$$

where in the last step we used that $|x-y| \approx |y-z_2|$.

In view of (3.5) and (3.6), $\alpha - 2s < \alpha - 2\theta < 0$, and we observe that

$$\begin{aligned}\int_{\{y:(x,y)\in\mathcal{O}_2\cap\mathcal{P}_2\}} |x-y|^{\alpha-2s-n}\,dy &\lesssim \int_{\{y:|x-y|\gtrsim\max\{|x-z_1|,|x-z_2|\}\}} |x-y|^{\alpha-2s-n}\,dy \\ &\lesssim \min\left\{|x-z_1|^{\alpha-2s},\,|x-z_2|^{\alpha-2s}\right\} \\ &\lesssim |x-z_1|^{\frac{\alpha}{2}-t}\,|x-z_2|^{\frac{\alpha}{2}+t-2s}.\end{aligned}$$

As a consequence for each x

$$\int_{\{y:(x,y)\in\mathcal{O}_2\cap\mathcal{P}_2\}} |K(x,y)-K(z_2,z_2)|\,|\kappa_\varepsilon(x,y,z_1,z_2)|dy \lesssim |x-z_1|^{\frac{\alpha}{2}-\varepsilon-n}\,|x-z_2|^{\frac{\alpha}{2}-n}$$

That is, in this particular case

$$\begin{aligned}\iint_{\mathbb{R}^{2n}} J_\varepsilon^{(2,2)}(z_1,z_2) f(z_1) g(z_2) dz_1 dz_2 &\lesssim \int_{\mathbb{R}^n} I^{\frac{\alpha}{2}-\varepsilon}|f|(x)\, I^{\frac{\alpha}{2}-\varepsilon}|g|(x)\,dx \\ &= \int_{\mathbb{R}^n} I^{\alpha-\varepsilon}|f|(x)\,|g|(x)\,dx.\end{aligned}$$

Estimating $J_\varepsilon^{(2,3)}$:

Since by (3.5) and (3.6) $\frac{\alpha}{3} < \theta - \frac{1}{10}\theta < t - \varepsilon$, $\frac{\alpha}{3} < \theta < 2s - t$ and $\varepsilon < \frac{\alpha}{3}$, we have for any $(x,y) \in \mathcal{O}_2 \cap \mathcal{P}_3$ that

$$\begin{aligned}|K(x,y) - K(z_2,z_2)|\,|\kappa_\varepsilon(x,y,z_1,z_2)| &\lesssim |x-z_1|^{t-\varepsilon-n}\,|y-z_2|^{2s-t-n}\,|x-z_2|^\alpha\,|x-y|^{-2s-n} \\ &\lesssim |x-z_1|^{\frac{\alpha}{3}-n}\,|y-z_2|^{\frac{\alpha}{3}-n}\,|x-y|^{2s-t-\frac{\alpha}{3}+t-\varepsilon-\frac{\alpha}{3}+\alpha-2s-n} \\ &\approx |x-z_1|^{\frac{\alpha}{3}-n}\,|y-z_2|^{\frac{\alpha}{3}-n}\,|x-y|^{\frac{\alpha}{3}-\varepsilon-n}\end{aligned}$$

Thus in this case, we have that

$$\iint_{\mathbb{R}^{2n}} J_\varepsilon^{(2,3)}(z_1,z_2)f(z_1)g(z_2)dz_1dz_2 \lesssim \int_{\mathbb{R}^n} I^{\frac{\alpha}{3}}|f|(x) I^{\frac{\alpha}{3}-\varepsilon}\left(I^{\frac{\alpha}{3}}|g|\right)(x) = \int_{\mathbb{R}^n} I^{\alpha-\varepsilon}|f|(x)\,|g|(x)\,dx,$$

where we use the semigroup property of the Riesz potential.

Estimating $J_\varepsilon^{(3,1)}$:

Here we get for any $\beta < 1$, $\beta < 2s - t + \alpha$ (in view of (3.5) and (3.6) $\alpha < \theta < 1 - (2s - t)$), for any $(x, y) \in \mathcal{O}_3 \cap \mathcal{P}_1$

$$\begin{aligned}
|K(x,y) - K(z_2,z_2)|\,|\kappa_\varepsilon(x,y,z_1,z_2)| \lesssim & |y-z_1|^{t-\varepsilon-n}\,|x-z_2|^{\alpha+2s-t-\beta-n}|x-y|^{\beta-2s-n} \\
\lesssim & |y-z_1|^{\frac{\alpha}{3}-n}\,|x-z_2|^{\alpha+2s-t-\beta-n}|x-y|^{\beta+t-\varepsilon-\frac{\alpha}{3}-2s-n}
\end{aligned}$$

Taking $\beta := 2s - t + \frac{2\alpha}{3}$ the above inequality simplifies to

$$|K(x,y) - K(z_2,z_2)|\,|\kappa_\varepsilon(x,y,z_1,z_2)| \lesssim |y-z_1|^{\frac{\alpha}{3}-n}\,|x-z_2|^{\frac{\alpha}{3}-n}|x-y|^{\frac{\alpha}{3}-\varepsilon-n}.$$

Since $\varepsilon < \frac{\alpha}{3}$, integrating we find that

$$\iint_{\mathbb{R}^{2n}} J_\varepsilon^{(3,1)}(z_1,z_2)f(z_1)g(z_2)dz_1dz_2 \lesssim \int_{\mathbb{R}^n} I^{\frac{\alpha}{3}-\varepsilon}I^{\frac{\alpha}{3}}|f|(x)I^{\frac{\alpha}{3}}|g|(x) = \int_{\mathbb{R}^n} I^{\alpha-\varepsilon}|f|(x)\,|g|(x)\,dx.$$

Estimating $J_\varepsilon^{(3,2)}$:

By (3.5) and (3.6), $\frac{\alpha}{3} < \theta - \frac{1}{10}\theta < t - \varepsilon$, and $\frac{\alpha}{3} < \theta < 2s - t$. Thus, for $(x, y) \in \mathcal{O}_3 \cap \mathcal{P}_2$

$$\begin{aligned}
|K(x,y) - K(z_2,z_2)|\,|\kappa_\varepsilon(x,y,z_1,z_2)| \lesssim & |y-z_1|^{t-\varepsilon-n}\,|x-z_2|^{2s-t-n}|y-z_2|^{\alpha}|x-y|^{-2s-n} \\
\approx & |y-z_1|^{t-\varepsilon-n}\,|x-z_2|^{2s-t-n}|x-y|^{\alpha-2s-n} \\
\lesssim & |y-z_1|^{\frac{\alpha}{3}-n}\,|x-z_2|^{\frac{\alpha}{3}-n}|x-y|^{\alpha-2s-n+t-\varepsilon-\frac{2\alpha}{3}+2s-t} \\
= & |y-z_1|^{\frac{\alpha}{3}-n}\,|x-z_2|^{\frac{\alpha}{3}-n}|x-y|^{\frac{\alpha}{3}-\varepsilon-n}.
\end{aligned}$$

As before, we can now estimate as

$$\int_{\mathbb{R}^{2n}} J_\epsilon^{(3,2)}(z_1,z_2)f(z_1)g(z_2)dz_1dz_2 \lesssim \int_{\mathbb{R}^n} I^{\frac{\alpha}{3}-\varepsilon}I^{\frac{\alpha}{3}}|f|(x)I^{\frac{\alpha}{3}}|g|(x) = \int_{\mathbb{R}^n} I^{\alpha-\varepsilon}|f|(x)\,|g|(x)\,dx.$$

Finally we estimate $J_\varepsilon^{(3,3)}$:

For $(x, y) \in \mathcal{O}_3 \cap \mathcal{P}_3$, we have that

$$\begin{aligned}
|K(x,y) - K(z_2,z_2)|\,|\kappa(x,y,z_1,z_2)| \lesssim & |y-z_1|^{t-\varepsilon-n}\,|y-z_2|^{2s-t-n}|x-z_2|^{\alpha}|x-y|^{-2s-n} \\
\approx & |y-z_1|^{t-\varepsilon-n}\,|y-z_2|^{2s-t-n}\,|x-y|^{\alpha-2s-n}
\end{aligned}$$

Observe that from (3.5) and (3.6), we have $\alpha < 2\theta < 2s$, $\frac{\alpha}{2} < \theta < t$ and $\frac{\alpha}{2} < \theta < 2s - t$. Moreover, for any y

$$\begin{aligned}\int_{\{x:(x,y)\in\mathcal{O}_3\cap\mathcal{P}_3\}} |x-y|^{\alpha-2s-n}dx \lesssim& \int_{\{x:|x-y|\gtrsim\max\{|y-z_1|,|y-z_2|\}\}} |x-y|^{\alpha-2s-n}dx\\ \lesssim& \min\left\{|y-z_1|^{\alpha-2s}, |y-z_2|^{\alpha-2s}\right\}\\ \le& |y-z_1|^{\frac{\alpha}{2}-t}|y-z_2|^{\frac{\alpha}{2}-2s+t}.\end{aligned}$$

Combining the previous two inequalities we have,

$$\int_{\{x:(x,y)\in\mathcal{O}_3\cap\mathcal{P}_3\}} |K(x,y)-K(z_2,z_2)|\,|\kappa_\varepsilon(x,y,z_1,z_2)|\,dx \lesssim |y-z_1|^{\frac{\alpha}{2}-\varepsilon-n}\,|y-z_2|^{\frac{\alpha}{2}-n}.$$

This implies in this case

$$\iint_{\mathbb{R}^{2n}} J_\varepsilon^{(33)}(z_1,z_2)f(z_1)g(z_2)dz_1dz_2 \lesssim \int_{\mathbb{R}^n} I^{\frac{\alpha}{2}-\varepsilon}|f|(y)\, I^{\frac{\alpha}{2}}|g|(y)\,dy = \int_{\mathbb{R}^n} I^{\alpha-\varepsilon}|f|(x)\,|g|(x)\,dx.$$

This completes the proof of Lemma 3.5. □

Lemma 3.6. *Set for $s \in (0,1)$ and $t \in (0,2s)$ with $2s-t \in (0,1)$,*

$$\kappa_0(x,y,z_1,z_2) := \frac{\left(|x-z_1|^{t-n}-|y-z_1|^{t-n}\right)\left(|x-z_2|^{2s-t-n}-|y-z_2|^{2s-t-n}\right)}{|x-y|^{n+2s}}$$

then there exists a constant $c = c(s,t)$ such that

$$\int_{\mathbb{R}^n} f(z)\,g(z)\,dz = c\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(z_1)g(z_2)\kappa_0(x,y,z_1,z_2)\,dz_1\,dz_2\,dx\,dy$$

holds for any $f \in L^p(\mathbb{R}^n)$, $p \in (1,\infty)$ and $g \in C^\infty(\mathbb{R}^n) \cap H^{1,p'}(\mathbb{R}^n)$.

Proof. Assume first that $f, g \in C_c^\infty(\mathbb{R}^n)$. Using the definitions of fractional Laplacian and Riesz potential via Fourier transform we have

$$\int_{\mathbb{R}^n} f(x)\,g(x)\,dx = c\int_{\mathbb{R}^n} ((-\Delta)^{\frac{2s}{2}} I^t f)(z)\,(I^{2s-t}g)(z)\,dz$$

In view of (2.1) we thus find

$$\int_{\mathbb{R}^n} f(x)\,g(x)\,dx = c\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left(I^tf(x)-I^tf(y)\right)\left(I^{2s-t}g(x)-I^{2s-t}g(y)\right)}{|x-y|^{n+2s}}\,dx\,dy,$$

which holds for any $f, g \in C_c^\infty(\mathbb{R}^n)$. By density, it also holds for $f \in L^p(\mathbb{R}^N)$ if e.g. $g \in H^{1,p'}(\mathbb{R}^n)$.

Recall that the Riesz potential is given by an explicit integral formula, and thus for almost every x and y in $\mathbb{R}^n$

$$I^tf(x) - I^tf(y) = C\int_{\mathbb{R}^n} \left(|z_1-x|^{t-n} - |z_1-y|^{t-n}\right)\,f(z_1)\,dz_1,$$

and

$$I^{2s-t}g(x) - I^{2s-t}g(y) = C\int_{\mathbb{R}^n} \left(|z_2-x|^{2s-t-n} - |z_2-y|^{2s-t-n}\right)\, g(z_2)\, dz_2$$

Again these formulas hold at first for $f, g \in C_c^\infty(\mathbb{R}^n)$ but by density they still hold for almost every x and y for our f and g. This proves the above formula. □

3.2. **Proof of the commutator estimate.** We are now ready to present the proof of the commutator estimate given in Theorem 3.1.

Proof of Theorem 3.1. Assume first that $u, \varphi \in C_c^\infty(\mathbb{R}^n)$. Fix, $s \in (0,1)$, and $t \in (0,1)$ such that $0 < 2s - t < 1$. Using the inverse relationship between the fractional Laplacian and the Riesz potential, for every $x \in \mathbb{R}^n$, we have that

$$u(x) = C\int_{\mathbb{R}^n} |x-z_1|^{t-n}\, (-\Delta)^{\frac{t}{2}} u(z_1)\, dz_1,$$

and

$$\varphi(x) = C\int_{\mathbb{R}^n} |x-z_2|^{2s-t-n}\, (-\Delta)^{\frac{2s-t}{2}} \varphi(z_2)\, dz_2.$$

Plugging in these equations in $\langle \mathcal{L}^s_{\mathbb{R}^n} u, \varphi\rangle$ and interchanging the integrals we obtain that

$$\begin{aligned}\langle \mathcal{L}^s_{\mathbb{R}^n} u, \varphi\rangle =& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\, dx\, dy\\ =&C^2 \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\,(-\Delta)^{\frac{t}{2}}u(z_1)\,(-\Delta)^{\frac{2s-t}{2}}\varphi(z_2)\kappa_0(x,y,z_1,z_2)\, dz_1 dz_2 dx\, dy\end{aligned}$$

where

$$\kappa_0(x,y,z_1,z_2) := \frac{\left(|x-z_1|^{t-n} - |y-z_1|^{t-n}\right)\,\left(|x-z_2|^{2s-t-n} - |y-z_2|^{2s-t-n}\right)}{|x-y|^{n+2s}}$$

is as defined in Lemma 3.6. Notice that the constant C^2 depends only on s, t, and n. Let us remark that this is related to the Calderon-Zygmund operator treated recently in [46]. Since $u, \varphi \in C_c^\infty(\mathbb{R}^n)$, $f(z_1) := K(z_1, z_1)(-\Delta)^{\frac{t}{2}}u(z_1)$ and $g(z_2) := (-\Delta)^{\frac{2s-t}{2}}\varphi(z_2)$ belong to $L^p(\mathbb{R}^n)$ for any $p \in [1, \infty]$, moreover g belongs to $H^{1,p}(\mathbb{R}^n)$ for any $p \in (1, \infty)$. Consequently, by Lemma 3.6,

$$\begin{aligned}\langle L^{s,t}_{diag} u, \varphi\rangle =& \int_{\mathbb{R}^n} K(z,z)\,(-\Delta)^{\frac{t}{2}}u(z)\,(-\Delta)^{\frac{2s-t}{2}}\varphi(z) dz\\ =& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(z_1,z_1)\,(-\Delta)^{\frac{t}{2}}u(z_1)\,(-\Delta)^{\frac{2s-t}{2}}\varphi(z_2)\kappa_0(x,y,z_1,z_2)\, dz_1 dz_2 dx\, dy.\end{aligned}$$

Thus for the choice of the constant $\Gamma = C^2$, we have

$$\begin{aligned}D_{s,t}(u,\varphi) =& \langle \mathcal{L}^s_{\mathbb{R}^n} u, \varphi\rangle - \Gamma \langle L^{s,t}_{diag} u, \varphi\rangle\\ =& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(x,y,z_1,z_2) dz_1 dz_2\, dx\, dy\end{aligned} \tag{3.8}$$

where

$$\Phi(x,y,z_1,z_2) := (K(x,y)-K(z_1,z_1))\ (-\Delta)^{\frac{t}{2}}u(z_1)\,(-\Delta)^{\frac{2s-t}{2}}\varphi(z_2)\kappa_0(x,y,z_1,z_2).$$

By the definition of the Riesz potential I^σ and the fact that $I^\sigma = \left((-\Delta)^{\frac{\sigma}{2}}\right)^{-1}$ for any $\sigma \in (0,n)$, we have for any $x,y\in\mathbb{R}^n$ and any $\varepsilon < 2s-t$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} (-\Delta)^{\frac{2s-t}{2}}\varphi(z_2)\left(|x-z_2|^{2s-t-n}-|y-z_2|^{2s-t-n}\right)\,dz_2\\
=&c_1\left(I^{2s-t}(-\Delta)^{\frac{2s-t}{2}}\varphi(x)-I^{2s-t}(-\Delta)^{\frac{2s-t}{2}}\varphi(y)\right)\\
=&c_1\left(\varphi(x)-\varphi(y)\right)\\
=&c_1\left(I^{2s-t-\varepsilon}(-\Delta)^{\frac{2s-t-\varepsilon}{2}}\varphi(x)-I^{2s-t-\varepsilon}(-\Delta)^{\frac{2s-t}{2}}\varphi(y)\right)\\
=&c_2\int_{\mathbb{R}^n} (-\Delta)^{\frac{2s-t-\varepsilon}{2}}\varphi(z_2)\left(|x-z_2|^{2s-t+\varepsilon-n}-|y-z_2|^{2s-t+\varepsilon-n}\right)\,dz_2
\end{aligned}$$

where c_2 will depend on ϵ. By Fubini's theorem we can thus rewrite the representation (3.8) for $D_{s,t}(u,\varphi)$ into

$$D_{s,t}(u,\varphi)=c\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Phi_\varepsilon(x,y,z_1,z_2)\,dx\,dydz_1dz_2,$$

where $\Phi_\varepsilon(x,y,z_1,z_2)=(K(x,y)-K(z_1,z_1))\ (-\Delta)^{\frac{t}{2}}u(z_1)\,(-\Delta)^{\frac{2s-t-\varepsilon}{2}}\varphi(z_2)\kappa_\epsilon(x,y,z_1,z_2)$ and

$$\kappa_\varepsilon(x,y,z_1,z_2) := \frac{\left(|x-z_1|^{t-n}-|y-z_1|^{t-n}\right)\,\left(|x-z_2|^{2s-t-\varepsilon-n}-|y-z_2|^{2s-t-\varepsilon-n}\right)}{|x-y|^{n+2s}}.$$

We can now estimate the latter to obtain that

$$|D_{s,t}(u,\varphi)| \precsim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |(-\Delta)^{\frac{t}{2}}u(z_1)|\,|(-\Delta)^{\frac{2s-t-\varepsilon}{2}}\varphi(z_2)|\,M^\varepsilon(z_1,z_2)dz_1dz_2.$$

where

$$M^\varepsilon(z_1,z_2)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)-K(z_1,z_1)|\,|\kappa_\varepsilon(x,y,z_1,z_2)|\,dx\,dy.$$

Now in view of Lemma 3.5 (when this M^ε correspond to $M^\varepsilon_{1,2}$ of the lemma) we have for small enough α, $\varepsilon < \alpha/3$, and $K\in\mathcal{C}(\alpha,\Lambda)$

$$|D_{s,t}(u,\varphi)| \precsim \int_{\mathbb{R}^n} I^{\alpha-\varepsilon}|(-\Delta)^{\frac{t}{2}}u|(x)\,|(-\Delta)^{\frac{2s-t-\varepsilon}{2}}\varphi|(x)\,dx.$$

The other estimate follows the same way by reversing the role of u and φ from the beginning and we conclude under the assumption that $u\in C_c^\infty(\mathbb{R}^n)$.

In the case that $u\in H^{t,p}(\mathbb{R}^n)$, but still $\varphi\in C_c^\infty(\mathbb{R}^n)$, take let $u_k\in C_c^\infty(\mathbb{R}^n)$

$$\|u_k-u\|_{H^{t,p}(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0.$$

Observe that since $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $2s-1<t$ we have

$$\lim_{k\to\infty} \langle \mathcal{L}^s_{\mathbb{R}^n} u_k, \varphi\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{(u_k(x)-u_k(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy = \langle \mathcal{L}^s_{\mathbb{R}^n} u, \varphi\rangle.$$

Similarly,

$$\lim_{k\to\infty} \langle L^{s,t}_{diag} u_k, \varphi\rangle = \int_{\mathbb{R}^n} K(z,z)\,(-\Delta)^{\frac{t}{2}} u_k(z)\,(-\Delta)^{\frac{2s-t}{2}}\varphi(z)dz$$
$$= \int_{\mathbb{R}^n} K(z,z)\,(-\Delta)^{\frac{t}{2}} u(z)\,(-\Delta)^{\frac{2s-t}{2}}\varphi(z)dz = \langle L^{s,t}_{diag} u, \varphi\rangle.$$

Combining the above we see that

$$\lim_{k\to\infty} D_{s,t}(u_k,\varphi) = D_{s,t}(u,\varphi).$$

Moreover, we have already shown that

$$|D_{s,t}(u_k,\varphi)| \lesssim \int_{\mathbb{R}^n} I^{\sigma-\varepsilon}|(-\Delta)^{\frac{t}{2}} u_k|(x)\,|(-\Delta)^{\frac{2s-t-\varepsilon}{2}}\varphi|(x)\,dx$$
$$\approx \int_{\mathbb{R}^n} |(-\Delta)^{\frac{t}{2}} u_k|(x)\, I^{\sigma-\varepsilon}|(-\Delta)^{\frac{2s-t-\varepsilon}{2}}\varphi|(x)\,dx$$

Again, from the $H^{t,p}$-convergence of u_k (and using once again that $\varphi \in C_c^\infty(\mathbb{R}^n)$ is fixed so that

$$\|I^{\sigma-\varepsilon}|(-\Delta)^{\frac{2s-t-\varepsilon}{2}}\varphi|\|_{L^{p'}(\mathbb{R}^n)} < \infty,$$

we find

$$\limsup_{k\to\infty} |D_{s,t}(u_k,\varphi)| \lesssim \int_{\mathbb{R}^n} |(-\Delta)^{\frac{t}{2}} u|(x)\, I^{\sigma-\varepsilon}|(-\Delta)^{\frac{2s-t-\varepsilon}{2}}\varphi|(x)\,dx$$

This concludes the proof of Theorem 3.1. □

4. Calderon-Zygmund theory for weighted fractional Laplace: Proof of Theorem 1.7

First, we prove the following intermediate result. Let us stress that the results in this section can be extended to $s \geq 1$ with only minor modifications, but since this is not a focus of this work we do not pursue this direction here.

Proposition 4.1. *Let $s\in(0,1)$ and $t\in(0,2s)$. Assume that for some $q\in(1,\infty)$, $(-\Delta)^{\frac{t}{2}}u \in L^q(\mathbb{R}^n)$ is a distributional solution to*

$$\int_{\mathbb{R}^n} \bar{K}(z)(-\Delta)^{\frac{t}{2}}u(z)\,(-\Delta)^{\frac{2s-t}{2}}\varphi(z)\,dz = \int_{\mathbb{R}^n} f_1(z)\,(-\Delta)^{\frac{2s-t}{2}}\varphi(z)\,dz + \int_{\mathbb{R}^n} f_2(z)\,\varphi(z)\,dz \quad \forall \varphi\in C_c^\infty(\Omega).$$

Here $\bar{K}:\mathbb{R}^n\to\mathbb{R}$ is a positive, measurable, and bounded from above and below, i.e.

$$\Lambda^{-1} \leq \bar{K}(z) \leq \Lambda \quad a.e.\ x\in\mathbb{R}^n.$$

Then for any $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega \subset \mathbb{R}^n$, $p > q$, *and* $r \in (1,p)$ *such that*

$$r > \frac{np}{n+(2s-t)p} \quad \text{if } 2s-t \leq 1, \text{ and } r > \frac{np}{n+p} \text{ if } 2s-t \geq 1$$

if $f_1, f_2 \in L^q(\mathbb{R}^n) \cap L^p(\Omega_2)$ *then* $(-\Delta)^{\frac{t}{2}} u \in L^p(\Omega_1)$ *with the estimate*

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\Omega_1)} \precsim \sum_{j=1}^{2} \left(\|f_j\|_{L^p(\Omega_2)} + \|f_j\|_{L^q(\mathbb{R}^n)}\right) + \|(-\Delta)^{\frac{t}{2}} u\|_{L^r(\Omega_2)} + \|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\mathbb{R}^n)}. \tag{4.1}$$

We delay the proof of the proposition. First we use the proposition to prove the optimal regularity of solutions to the weighted fractional equation.

Proof of Theorem 1.7. If $p \leq q$, there is nothing to prove. So, we assume $p > q$. We will use Proposition 4.1 to iterate the estimate on successive subdomains. Assume first that $2s - t < 1$. Let $\Omega_1 = \Omega'$, and $p_1 = p$. We introduce successive subdomains

$$\Omega' = \Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \Omega_L \subset\subset \Omega$$

and successive positive numbers

$$p_1 = p,\, p_{i+1} \in [q, p_i) \text{ with } p_{i+1} > \frac{np_i}{n+(2s-t)p_i}$$

in such a way that for some L, $p_L = q$. It is not difficult to see that such a finite L exists depending on p, q, n, s and t. By Proposition 4.1, in each step we have

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^{p_i}(\Omega_i)} \precsim \sum_{j=1}^{2} \left(\|f_j\|_{L^p(\Omega)} + \|f_j\|_{L^q(\mathbb{R}^n)}\right) + \|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\mathbb{R}^n)} + \|(-\Delta)^{\frac{t}{2}} u\|_{L^{p_{i+1}}(\Omega_{i+1})}.$$

Iterating the above inequality L number of times we get that

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\Omega')} \precsim \sum_{j=1}^{2} \left(\|f_j\|_{L^p(\Omega)} + \|f_j\|_{L^q(\mathbb{R}^n)}\right) + \|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\mathbb{R}^n)} + \|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\Omega)},$$

from which the desired inequality follows. If $2s - t \geq 1$, then an obvious modification of the above iteration lead to the inequality. □

We can now prove Proposition 4.1.

Proof of Proposition 4.1. To prove (4.1) we use a duality argument and show that

$$\sup_{\substack{\psi \in C_c^\infty(\Omega_1) \\ \|\psi\|_{L^{p'}} \leq 1}} \int_{\mathbb{R}^n} (-\Delta)^{\frac{t}{2}} u \psi \, dx \precsim \sum_{j=1}^{2} \left(\|f_j\|_{L^p(\Omega)} + \|f_j\|_{L^q(\mathbb{R}^n)}\right) + \|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\mathbb{R}^n)} + \|(-\Delta)^{\frac{t}{2}} u\|_{L^r(\Omega_2)}.$$

Using the ellipticity of $\bar{K}$, it suffices to show that for any $\psi \in C_c^\infty(\Omega_1)$,

$$\frac{1}{\|\psi\|_{L^{p'}(\mathbb{R}^n)}} \int_{\mathbb{R}^n} \bar{K}(z)\,(-\Delta)^{\frac{t}{2}} u\,\psi\,dz \lesssim \sum_{j=1}^{2} \left(\|f_j\|_{L^p(\Omega)} + \|f_j\|_{L^q(\mathbb{R}^n)}\right) + \|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\mathbb{R}^n)} + \|(-\Delta)^{\frac{t}{2}} u\|_{L^r(\Omega_2)}\,. \tag{4.2}$$

To simplify notation, we will write $\Omega_2 = \Omega$.

Let $\eta_1, \eta_2 \in C_c^\infty(\Omega)$, $\eta_1 \equiv 1$ in a neighborhood of Ω_1 and $\eta_2 \equiv 1$ in a neighborhood of $\operatorname{supp}\eta_1$. Set

$$\varphi := \eta_1 \left(I^{2s-t}\psi\right),$$

which is now an admissible test function for the equation. Then using the inverse relationship between $(-\Delta)^{\frac{2s-t}{2}}$ and I^{2s-t}, we have the identity

$$\psi = (-\Delta)^{\frac{2s-t}{2}}\varphi + \eta_2(-\Delta)^{\frac{2s-t}{2}}(1-\eta_1)I^{2s-t}\psi + (1-\eta_2)(-\Delta)^{\frac{2s-t}{2}}(1-\eta_1)I^{2s-t}\psi,$$

from which it follows that

$$\int_{\mathbb{R}^n} \bar{K}(z)\,(-\Delta)^{\frac{t}{2}} u\,\psi\,dz = I + II + III$$

where

$$I := \int_{\mathbb{R}^n} \bar{K}(z)\,(-\Delta)^{\frac{t}{2}} u\,(-\Delta)^{\frac{2s-t}{2}}\varphi\,dz,$$

$$II := \int_{\mathbb{R}^n} \bar{K}(z)\,\eta_2(-\Delta)^{\frac{t}{2}} u\,(-\Delta)^{\frac{2s-t}{2}}\left((1-\eta_1)I^{2s-t}\psi\right)\,dz, \text{ and}$$

$$III := \int_{\mathbb{R}^n} \bar{K}(z)\,(1-\eta_2)(-\Delta)^{\frac{t}{2}} u\,(-\Delta)^{\frac{2s-t}{2}}\left((1-\eta_1)I^{2s-t}\psi\right)\,dz.$$

Now using the equation, since φ is a valid test function, we have that

$$I = \int_{\mathbb{R}^n} \bar{K}(z)\,(-\Delta)^{\frac{t}{2}} u\,(-\Delta)^{\frac{2s-t}{2}}\varphi\,dz = \int_{\mathbb{R}^n} f_1(-\Delta)^{\frac{2s-t}{2}}\varphi\,dz + \int_{\mathbb{R}^n} f_2\varphi\,dz.$$

The right-hand side can now be rewritten using the identity between φ and ψ as

$$I = I_1 - I_2 - I_3,$$

where

$$I_1 := \int_{\mathbb{R}^n} f_1\,\psi + f_2\,\varphi\,dz$$

$$I_2 := \int_{\mathbb{R}^n} \eta_2\,f_1\,(-\Delta)^{\frac{2s-t}{2}}\left((1-\eta_1)\,I^{2s-t}\psi\right)\,dz$$

$$I_3 := \int_{\mathbb{R}^n} (1-\eta_2)\,f_1\,(-\Delta)^{\frac{2s-t}{2}}\left((1-\eta_1)\,I^{2s-t}\psi\right)\,dz.$$

Clearly,

$$\int_{\mathbb{R}^n} f_1\psi\,dz \lesssim \|f_1\|_{L^p(\Omega)}\,\|\psi\|_{L^{p'}(\Omega)}.$$

Sobolev embedding, Proposition 2.1, together with the fact that ψ is compactly supported implies

$$\int_{\mathbb{R}^n} f_2\varphi\, dz \lesssim \|f_2\|_{L^p(\Omega)}\, \|I^{2s-t}\psi\|_{L^{p'}(\Omega)} \lesssim \|f_2\|_{L^p(\Omega)}\, \|\psi\|_{L^{p'}(\Omega)}.$$

That is,

$$|I_1| \lesssim \left(\|f_1\|_{L^p(\Omega)} + \|f_2\|_{L^p(\Omega)}\right)\, \|\psi\|_{L^{p'}(\Omega)}.$$

Notice that by our choice of r,

$$r > \frac{np}{n+(2s-t)p} \Leftrightarrow r' < \frac{np'}{n-(2s-t)p'},$$

and therefore, Proposition 2.4 is applicable.

To estimate I_2, we apply Proposition 2.4 part (b)with $(\tau, r, p)_{P.2.4} = (2s-t, p, p)$, and obtain that

$$|I_2| \lesssim \|f\|_{L^p(\Omega)}\, \|\psi\|_{L^{p'}(\Omega)}.$$

Moreover, again apply Proposition 2.4 part (b), and estimate $|II|$ as

$$|II| \lesssim \|(-\Delta)^{\frac{t}{2}}u\|_{L^r(\Omega)}\|\psi\|_{L^{p'}(\mathbb{R}^n)}.$$

For the remaining cases III and I_3, we apply again Proposition 2.4 part (a) to estimate as

$$|I_3| \lesssim \|f\|_{L^q(\mathbb{R}^n)}\|\psi\|_{L^{p'}(\mathbb{R}^n)},$$

and

$$|III| \lesssim \|(-\Delta)^{\frac{t}{2}}u\|_{L^q(\mathbb{R}^n)}\, \|\psi\|_{L^{p'}(\mathbb{R}^n)}.$$

This was the last estimate needed for (4.2), and we can conclude the proof. □

We finish the section by proving regularity result for weighted fractional elliptic equation when the coefficient $\bar{K}$ is Hölder continuous. In this case, we can "differentiate the equation", which leads to estimates of the following form.

Proposition 4.2. *Let $s\in (0,1)$ and $t \in [s, 2s)$. Assume that for some $q \in (1,\infty)$ $(-\Delta)^{\frac{t}{2}}u \in L^q(\mathbb{R}^n)$ is a distributional solution to*

$$\int_{\mathbb{R}^n} \bar{K}(z)(-\Delta)^{\frac{t}{2}}u\, (-\Delta)^{\frac{2s-t}{2}}\varphi\, dz = \int_{\mathbb{R}^n} f_1\, (-\Delta)^{\frac{2s-t}{2}}\varphi\, dz + \int f_2\, \varphi\, dz \quad \forall \varphi \in C_c^\infty(\Omega).$$

Assume that K is positive, measurable, and bounded from above and below, i.e.

$$\Lambda^{-1} \leq \bar{K}(x) \leq \Lambda \quad a.e.\ x \in \mathbb{R}^n.$$

and $\bar{K}$ is moreover uniformly Hölder continuous, i.e. for some $\gamma \in (0,1]$,

$$\sup_{x,y,\mathbb{R}^n} \frac{|\bar{K}(x)-\bar{K}(y)|}{|x-y|^\gamma} \leq \Lambda.$$

Then for any $\beta < \min\{\gamma, 2s-t\}$, and any $\Omega' \subset\subset \Omega \subset\subset \mathbb{R}^n$

$$\|(-\Delta)^{\frac{t+\beta}{2}}u\|_{L^q(\Omega')} \leq C(\Omega, \Lambda, s, t, p, q)\, \left(\|(-\Delta)^{\frac{t}{2}}u\|_{L^q(\mathbb{R}^n)} + \|f_2\|_{L^q(\Omega)} + \|(-\Delta)^{\frac{\beta}{2}}f_1\|_{L^q(\mathbb{R}^n)}\right).$$

Proof. Let $\Omega_2 \subset \mathbb{R}^n$ be open such that $\Omega' \subset\subset \Omega_2 \subset\subset \Omega$. To prove the proposition, we will show that for any $\psi \in C_c^\infty(\Omega_2)$,

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{t}{2}} u (-\Delta)^{\frac{\beta}{2}} \psi \, dz \lesssim \left(\|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\mathbb{R}^n)} + \|f_2\|_{L^q(\Omega)} + \|(-\Delta)^{\frac{\beta}{2}} f_1\|_{L^q(\mathbb{R}^n)} \right) \|\psi\|_{L^{q'}(\mathbb{R}^n)}. \tag{4.3}$$

which by duality implies that $(-\Delta)^{\frac{t+\beta}{2}} u \in L^q(\Omega')$, with

$$\|(-\Delta)^{\frac{t+\beta}{2}} u\|_{L^q(\Omega_2)} \lesssim \|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\mathbb{R}^n)} + \|f_2\|_{L^q(\Omega)} + \|(-\Delta)^{\frac{\beta}{2}} f_1\|_{L^q(\mathbb{R}^n)}.$$

To establish (4.3) observe

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{t}{2}} u (-\Delta)^{\frac{\beta}{2}} \psi \, dz = \int_{\mathbb{R}^n} (-\Delta)^{\frac{t}{2}} u \, (-\Delta)^{\frac{\beta}{2}} \left(\bar{K} \left(\frac{1}{\bar{K}} \psi \right) \right) dz = I + II$$

where

$$I := \int_{\mathbb{R}^n} \bar{K} \, (-\Delta)^{\frac{t}{2}} u \, (-\Delta)^{\frac{\beta}{2}} \left(\frac{1}{\bar{K}} \psi \right) dz \text{ and } II := \int_{\mathbb{R}^n} (-\Delta)^{\frac{t}{2}} u \, [(-\Delta)^{\frac{\beta}{2}}, \bar{K}] \left(\frac{1}{\bar{K}} \psi \right) dz$$

where we used commutator notation

$$[(-\Delta)^{\frac{\beta}{2}}, f](g) = (-\Delta)^{\frac{\beta}{2}} (fg) - f \, (-\Delta)^{\frac{\beta}{2}} g.$$

Now since $\bar{K}$ is γ-Hölder continuous we can apply Coifman-McIntosh-Meyer estimate, e.g. as in [26, Theorem 6.1.], combined with Sobolev inequality to obtain

$$II \lesssim \|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\mathbb{R}^n)} \, [\bar{K}]_{C^\gamma} \, \|\frac{1}{\bar{K}} \psi\|_{L^{q'}(\mathbb{R}^n)} \lesssim \|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\mathbb{R}^n)} \, \|\psi\|_{L^{q'}(\mathbb{R}^n)}.$$

For I, we argue similar to the proof of Proposition 4.1. To that end, let $\eta \in C_c^\infty(\Omega)$, $\eta \equiv 1$ in a neighborhood of Ω_2. Then, splitting I using η we get that,

$$\begin{aligned} I = & \int_{\mathbb{R}^n} \bar{K}(z) \, (-\Delta)^{\frac{t}{2}} u \, (-\Delta)^{\frac{2s-t}{2}} \left(\eta I^{2s-t-\beta} \left(\frac{1}{\bar{K}} \psi \right) \right) dz \\ & + \int_{\mathbb{R}^n} \bar{K}(z) \, (-\Delta)^{\frac{t}{2}} u \, (-\Delta)^{\frac{2s-t}{2}} \left((1-\eta) I^{2s-t-\beta} \left(\frac{1}{\bar{K}} \psi \right) \right) dz \end{aligned}$$

We now use the equation and $\eta I^{2s-t-\beta} \left(\frac{1}{\bar{K}} \psi \right)$ as a valid test function to conclude that

$$\begin{aligned} I = & \int_{\mathbb{R}^n} f_1 \, (-\Delta)^{\frac{2s-t}{2}} \left(\eta I^{2s-t-\beta} \left(\frac{1}{\bar{K}} \psi \right) \right) dz + \int_{\mathbb{R}^n} f_2 \left(\eta I^{2s-t-\beta} \left(\frac{1}{\bar{K}} \psi \right) \right) dz \\ & + \int_{\mathbb{R}^n} \bar{K}(z) \, (-\Delta)^{\frac{t}{2}} u \, (-\Delta)^{\frac{2s-t}{2}} \left((1-\eta) I^{2s-t-\beta} \left(\frac{1}{\bar{K}} \psi \right) \right) dz \\ = & I_1 + I_2 + I_3 \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \int_{\mathbb{R}^n} (-\Delta)^{\frac{\beta}{2}} f_1 \, (-\Delta)^{\frac{2s-t-\beta}{2}} \left(\eta I^{2s-t-\beta} \left(\frac{1}{\bar{K}} \psi \right) \right) dz \\ I_2 &:= \int_{\mathbb{R}^n} f_2 \left(\eta I^{2s-t-\beta} \left(\frac{1}{\bar{K}} \psi \right) \right) dz \\ I_3 &:= \int_{\mathbb{R}^n} \bar{K}(z) \, (-\Delta)^{\frac{t}{2}} u \, (-\Delta)^{\frac{2s-t}{2}} \left((1-\eta) I^{2s-t-\beta} \left(\frac{1}{\bar{K}} \psi \right) \right) dz \end{aligned}$$

The term I_1 can be estimates using we can estimate with the help of (2.7) and (2.6), in the same way we estimated I of the proof of Proposition 4.1, which imply

$$|I_1| \lesssim \|(-\Delta)^{\frac{\beta}{2}} f_1\|_{L^q(\mathbb{R}^n)} \, \|\psi\|_{L^{q'}}.$$

By Sobolev inequality, Proposition 2.1 part (b),

$$|I_2| \lesssim \|f_2\|_{L^q(\Omega)} \, \|\psi\|_{L^{q'}}.$$

Similarly by Proposition 2.1 part (a), we can estimate I_3 as

$$|I_3| \lesssim \|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\mathbb{R}^n)} \, \left\|(-\Delta)^{\frac{2s-t}{2}} \left((1-\eta) I^{2s-t-\beta} \left(\frac{1}{K}\psi\right)\right)\right\|_{L^{q'}(\mathbb{R}^n)}$$

Now observe that $1-\eta$ and ψ have disjoint support, so that we can argue similarly to (2.6) to obtain

$$\left\|(-\Delta)^{\frac{2s-t}{2}} \left((1-\eta) I^{2s-t-\beta} \left(\frac{1}{K}\psi\right)\right)\right\|_{L^{q'}(\mathbb{R}^n)} \lesssim \|\psi\|_{L^{q'}(\mathbb{R}^n)} + \left\|\left((1-\eta)(-\Delta)^{\frac{\beta}{2}} \left(\frac{1}{K}\psi\right)\right)\right\|_{L^{q'}(\mathbb{R}^n)}.$$

Observe that $\psi \in C_c^\infty(\Omega_2)$ and $1-\eta \equiv 0$ in a neighborhood of Ω_2. If $\beta = 0$ this implies

$$(1-\eta)(x)(-\Delta)^{\frac{\beta}{2}} \left(\frac{1}{K}\psi\right)(x) \equiv 0.$$

If $\beta > 0$ we use that for $y \in \Omega_2$ and $x \in \operatorname{supp}(1-\eta)$ we have $|y-x| \approx 1+|x|$, and estimate

$$\left|\left((1-\eta)(-\Delta)^{\frac{\beta}{2}} \left(\frac{1}{K}\psi\right)\right)(x)\right| \lesssim \int_{\mathbb{R}^n} (1+|x|)^{-n-\beta} \frac{1}{|K(y)|} |\psi(y)| \, dy,$$

and thus

$$\left\|\left((1-\eta)(-\Delta)^{\frac{\beta}{2}} \left(\frac{1}{K}\psi\right)\right)\right\|_{L^{q'}(\mathbb{R}^n)} \lesssim \Lambda \, \|\psi\|_{L^1(\mathbb{R}^n)} \lesssim \|\psi\|_{L^q(\mathbb{R}^n)}$$

We conclude that

$$\left\|(-\Delta)^{\frac{2s-t}{2}} \left((1-\eta) I^{2s-t-\beta} \left(\frac{1}{K}\psi\right)\right)\right\|_{L^{q'}(\mathbb{R}^n)} \lesssim \|\psi\|_{L^{q'}(\mathbb{R}^n)}.$$

This establishes (4.3) and that concludes the proof of the proposition. □

5. Local to global equation

The main idea for the proof of Theorem 1.2 is to use Theorem 3.1 to compare the equation of Theorem 1.2 with an easier equation to which we can apply Theorem 1.7 and Proposition 4.2. This works well on a local scale and the improvement of differentiability and integrability is each time incremental. So we apply this strategy repeatedly, which means that we repeatedly need to use cutoff arguments to restrict our equation to the set where we already have shown some improvement for differentiability and integrability. We describe this cutoff argument in this section. The next theorem states that if for a given $\Omega_1 \subset\subset \Omega$, u solves the equation

$$\mathcal{L}^s_\Omega u = F, \quad \text{in } \Omega_1,$$

then u can be extended in $\mathbb{R}^n$ in a controlled way. Namely, the extension v solves an equation of the form

$$\mathcal{L}^s_{\mathbb{R}^n} v = G, \quad \text{in } \mathbb{R}^n$$

and the norm of v is controlled by u, and the norm of data G is controlled by the norms u and F. To be precise, we have the following.

Theorem 5.1. *Let $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega \subseteq \mathbb{R}^n$ be open sets. Take $s \in (0,1)$, $t \in [s,1)$ and $p,q \in [2,\infty)$, $\tau \in (0,1)$ (if $n=1$ additionally, $\tau \leq s$) satisfying the following conditions:*

$$\frac{1}{q} \geq \frac{1}{p} - \frac{\tau}{n}, \quad \text{and} \quad \frac{1}{q} \geq \frac{1}{p} - \frac{t}{n} \quad \text{and} \quad \frac{1}{q} \geq \frac{1}{p} - \frac{1-2s+t}{n}, \tag{5.1}$$

$$\frac{1}{q} > \frac{1}{p} - \frac{\tau+1-2s+t}{n} \tag{5.2}$$

and

$$2s-1 < \tau. \tag{5.3}$$

Suppose that $K \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. For any $u \in H^{s,2}(\mathbb{R}^n)$ such that $(-\Delta)^{\frac{t}{2}} u \in L^p(\Omega_1)$ satisfies for some $f_1, f_2 \in L^q(\mathbb{R}^n)$ the equation

$$\langle \mathcal{L}^s_\Omega u, \varphi \rangle = \int_{\mathbb{R}^n} f_1 \, (-\Delta)^{\frac{\tau}{2}} \varphi \, dz + \int_{\mathbb{R}^n} f_2 \, \varphi, \, dz \quad \forall \varphi \in C^\infty_c(\Omega_1).$$

Then there exist $v \in H^{s,2} \cap H^{t,p}(\mathbb{R}^n)$, $\operatorname{supp} v \subset \Omega_1$, such that $u \equiv v$ in Ω_2 and $g_1, g_2 \in L^q(\mathbb{R}^n)$ such that

$$\langle \mathcal{L}^s_{\mathbb{R}^n} v, \varphi \rangle = \int_{\mathbb{R}^n} g_1 (-\Delta)^{\frac{\tau}{2}} \varphi \, dz + \int_{\mathbb{R}^n} g_2 \varphi \, dz \quad \forall \varphi \in C^\infty_c(\mathbb{R}^n). \tag{5.4}$$

Moreover,

$$\|v\|_{H^{t,2}(\mathbb{R}^n)} + \|(-\Delta)^{\frac{t}{2}} v\|_{L^p(\mathbb{R}^n)} \lesssim \|u\|_{H^{s,2}(\mathbb{R}^n)} + \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\Omega_1)}, \tag{5.5}$$

and

$$\|g_1\|_{L^q(\mathbb{R}^n)} + \|g_2\|_{L^q(\mathbb{R}^n)} \lesssim \|f_1\|_{L^q(\mathbb{R}^n)} + \|f_2\|_{L^q(\mathbb{R}^n)} + \|u\|_{H^{s,2}(\mathbb{R}^n)} + \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\Omega_1)}, \tag{5.6}$$

Additionally, for any $\beta \in (0,1)$ such that

$$2s-1+\beta < \tau \quad \text{and} \quad t + \frac{n}{q} > \frac{n}{p} - \tau - 1 + 2s + \beta, \tag{5.7}$$

we have, whenever the right-hand side is finite,

$$\|(-\Delta)^{\frac{\beta}{2}} g_1\|_{L^q(\mathbb{R}^n)} \lesssim \|f_1\|_{H^{\beta,p}(\mathbb{R}^n)} + \|u\|_{H^{s,2}(\mathbb{R}^n)} + \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\Omega_1)}. \tag{5.8}$$

Above, g_1 and g_2 and v are independent of q and β, in the sense that if we apply the statement above to $f_1, f_2 \in L^{q_1} \cap L^{q_2}$ then there is one set of functions g_1, g_2, v satisfying the equations and the estimates in L^{q_1} and L^{q_2}.

We split the proof of Theorem 5.1 into several steps.

The first step is a cutoff argument, essentially replacing u with ηu for a suitable cutoff function η.

Lemma 5.2. *Under the assumptions of Theorem 5.1, let $\Omega_2 \subset\subset \tilde{\Omega} \subset\subset \Omega_1$. Then there exist $w \in H^{s,2}(\mathbb{R}^n) \cap H^{t,p}(\mathbb{R}^n)$ with $\operatorname{supp} w \subset \Omega_1$, $w \equiv u$ in a neighborhood of Ω_2, and $g_1, g_2 \in L^q(\mathbb{R}^n)$ such that*

$$\langle \mathcal{L}^s_\Omega w, \varphi\rangle = \int_{\mathbb{R}^n} g_1\,(-\Delta)^{\frac{\tau}{2}}\varphi\,dz + \int_{\mathbb{R}^n} g_2\,\varphi\,dz \quad \forall \varphi \quad \forall \varphi \in C_c^\infty(\tilde{\Omega}). \tag{5.9}$$

such that (5.5) *(with v replaced by w),* (5.6)*, and* (5.8) *hold.*

Proof. Let $\tilde{\Omega} \subset\subset \tilde{\Omega}_{1,1} \subset\subset \Omega_1$ with $\Omega_2 \subset\subset \tilde{\Omega}$, and let $\eta \in C_c^\infty(\Omega_1)$, $\eta \equiv 1$ in $\tilde{\Omega}_{1,1}$.

Set $w := \eta u$, From Poincaré inequality and Sobolev embedding, we find that (5.5) holds. Moreover, for any $\varphi \in C_c^\infty(\tilde{\Omega})$, we have that

$$\begin{aligned}
\langle \mathcal{L}^s_\Omega w, \varphi\rangle &= \int_\Omega\int_\Omega K(x,y)\frac{\big(w(x)-w(y)\big)\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&= \int_\Omega\int_\Omega K(x,y)\frac{(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad + \int_{\Omega\setminus\tilde{\Omega}_{1,1}}\int_{\tilde{\Omega}} K(x,y)\frac{(1-\eta(y))u(y)\,\varphi(x)}{|x-y|^{n+2s}}\,dx\,dy.
\end{aligned}$$

Now, to show (5.9) holds, we set $g_1 := f_1$ and $g_2 := f_2 + \tilde{g}_2$ where

$$\tilde{g}_2(x) := \chi_{\tilde{\Omega}}(x)\int_{\Omega\setminus\tilde{\Omega}_{1,1}} K(x,y)\frac{(1-\eta(y))u(y)\,\varphi(x)}{|x-y|^{n+2s}}\,dy.$$

To obtain the estimate (5.6) we only need to estimate $\tilde{g}_2$. Observe that for any $y \in \operatorname{supp}(1-\eta)$ and $x \in \tilde{\Omega}$ we have $|x-y| \gtrsim c + |y|$. Consequently,

$$\|\tilde{g}_2\|_{L^\infty(\mathbb{R}^n)} \lesssim \sup_{x\in\mathbb{R}^n}\int_\Omega |u(y)|\,(c+|x-y|)^{-n-2s}\,dy \lesssim \|u\|_{L^2(\mathbb{R}^n)}.$$

Since $\tilde{\Omega}$ is bounded and $\operatorname{supp} g_2 \subset \tilde{\Omega}$ we find $g_2 \in L^1 \cap L^\infty(\mathbb{R}^n)$, in particular

$$\|\tilde{g}_2\|_{L^q(\mathbb{R}^n)} \lesssim \|u\|_{L^2(\mathbb{R}^n)}.$$

This concludes the proof of Lemma 5.2 □

In the second step we increase the domain of integration of (5.9) from Ω to $\mathbb{R}^n$.

Lemma 5.3. *Under the assumption of Theorem 5.1, let* $\Omega_2 \subset\subset \tilde{\Omega} \subset\subset \Omega_1$. *Then there exist* $w \in H^{s,2}(\mathbb{R}^n) \cap H^{t,p}(\mathbb{R}^n)$ *with* $\operatorname{supp} w \subset \Omega_1$, $w \equiv u$ *in a neighborhood of* Ω_2, *and* $h_1, h_2 \in L^q(\mathbb{R}^n)$ *such that*

$$\langle \mathcal{L}^s_{\mathbb{R}^n} w, \varphi \rangle = \int_{\mathbb{R}^n} h_1 \, (-\Delta)^{\frac{\tau}{2}} \varphi \, dz + \int_{\mathbb{R}^n} h_2 \, \varphi \, dz \quad \forall \varphi \quad \forall \varphi \in C_c^\infty(\tilde{\Omega}). \tag{5.10}$$

such that (5.5) *holds with* v *replaced by* w. *Moreover,* (5.6) *and* (5.8) *with* h_1, h_2 *instead of* g_1, g_2, *respectively.*

Proof. Take w, g_1, g_2 from Lemma 5.2 and let $\varphi \in C_c^\infty(\tilde{\Omega})$.

$$\begin{aligned} \langle \mathcal{L}^s_{\mathbb{R}^n} w, \varphi \rangle &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) \frac{\big(w(x) - w(y)\big)\,(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, dy \, dx \\ &= \int_{\Omega} \int_{\Omega} K(x,y) \frac{\big(w(x) - w(y)\big)\,(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dy \, dx \\ &\quad + 2 \int_{\mathbb{R}^n \setminus \Omega} \int_{\Omega} K(x,y) \frac{\big(w(x) - w(y)\big)\,(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dy \, dx \\ &\quad + \int_{\mathbb{R}^n \setminus \Omega} \int_{\mathbb{R}^n \setminus \Omega} K(x,y) \frac{\big(w(x) - w(y)\big)\,(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dy \, dx. \end{aligned} \tag{5.11}$$

The third term in right-hand side of (5.11) vanishes because of $\operatorname{supp} w \subset \Omega_1 \subset\subset \Omega$. Moreover, since $\operatorname{supp} w \subset \Omega_1 \subset\subset \Omega$ and $\operatorname{supp} \varphi \subset \tilde{\Omega} \subset\subset \Omega$, the second term in (5.11) becomes

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus \Omega} \int_{\Omega} K(x,y) \frac{\big(w(x) - w(y)\big)\,(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dy \, dx \\ &= \int_{\mathbb{R}^n \setminus \Omega} \int_{\tilde{\Omega}} K(x,y) \frac{w(y)\,\varphi(y)}{|x-y|^{n+2s}} \, dy \, dx \;= \int_{\mathbb{R}^n} \varphi(y) w(y) \, \chi_{\tilde{\Omega}}(y) \int_{\mathbb{R}^n \setminus \Omega} \frac{K(x,y)}{|x-y|^{n+2s}} \, dx \, dy. \end{aligned}$$

Now the conclusion of the lemma is satisfied if we set $h_1 := g_1$ and $h_2 := g_2 + \tilde{h}_2$, where

$$\tilde{h}_2(y) := w(y) \, \chi_{\tilde{\Omega}}(y) \int_{\mathbb{R}^n \setminus \Omega} \frac{K(x,y)}{|x-y|^{n+2s}} \, dx.$$

To see this, first, we obtain (5.10) from (5.9), (5.11) and the above observations. In addition, estimates (5.5) and (5.8) hold from Lemma 5.3 since w did not change and $h_1 = g_1$. In order to prove (5.6) for g_1, g_2 replaced by h_1, h_2 we only need an estimate for $\tilde{h}_2$, which we obtain by arguing in similar fashion as in the proof of Lemma 5.2. Since $\operatorname{dist}(\tilde{\Omega}, \mathbb{R}^n \setminus \Omega) > 0$, for all points $x \in \mathbb{R}^n \setminus \Omega$ and $y \in \tilde{\Omega}$ we have $|x - y| \gtrsim 1 + |x|$, and thus

$$|\tilde{h}_2(y)| \lesssim |w(y)| \, \chi_{\tilde{\Omega}}(y) \int_{\mathbb{R}^n \setminus \Omega} \frac{\Lambda}{1 + |x|^{n+2s}} \, dx \approx |w(y)| \, \chi_{\tilde{\Omega}}(y).$$

Thus,

$$\|\tilde{h}_2\|_{L^q(\mathbb{R}^n)} \lesssim \|w\|_{L^q(\tilde{\Omega}}. \tag{5.12}$$

Finally, since $w \in H^{t,p}(\mathbb{R}^n)$ with compact support, in view of (5.1) and Sobolev inequality, Proposition 2.1, we have

$$\|w\|_{L^q(\tilde{\Omega})} \precsim \|(-\Delta)^{\frac{t}{2}} w\|_{L^p(\mathbb{R}^n)}. \tag{5.13}$$

We conclude that $\tilde{h}_2$ satisfies the estimates (5.6) with g_2 replaced by $\tilde{h}_2$ in view of (5.12), (5.13) and (5.5). This concludes the proof of Lemma 5.3. □

In the last step of the proof, we increase the domain of the test functions in (5.10) from $\tilde{\Omega}$ to $\mathbb{R}^n$. This is where the the main influence of the conditions on p, q, τ etc. come into play.

Proof of Theorem 5.1. Take w, h_1, h_2 from Lemma 5.3, so that (5.10) holds.

Let $\eta \in C_c^\infty(\tilde{\Omega})$, $\eta \equiv 1$ in Ω_2 and set $v := \eta w$. Since we know from from Lemma 5.3 that w satisfies the estimates (5.5) (with v replaced by w), consequently in view of Poincaré and Sobolev inequality, so does v.

Fix any $\psi \in C_c^\infty(\mathbb{R}^n)$. Observe that

$$\begin{aligned}
\big(v(x) - v(y)\big)\,\big(\psi(x) - \psi(y)\big) =&\big(\eta(x)w(x) - \eta(y)w(y)\big)\,\big(\psi(x) - \psi(y)\big)\\
=&\big(w(x) - w(y)\big)\,\big(\eta(x)\psi(x) - \eta(y)\psi(y)\big)\\
&+ \big(w(x) - w(y)\big)\,\big(\eta(x) - \eta(y)\big)\,\psi(y)\big)\\
&+ \big(\eta(x) - \eta(y)\big)\,w(y)\,\big(\psi(x) - \psi(y)\big).
\end{aligned}$$

We can now use the map $\eta\psi \in C_c^\infty(\tilde{\Omega})$ as a test function for (5.10), and obtain

$$\langle \mathcal{L}^s_{\mathbb{R}^n} v, \varphi\rangle = I + II + III \tag{5.14}$$

where

$$\begin{aligned}
I :=& \int_{\mathbb{R}^n} h_1\,(-\Delta)^{\frac{\tau}{2}}(\eta\psi)\,dx + \int_{\mathbb{R}^n} h_2\,\eta\psi\,dx\\
II :=& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{\big(w(x) - w(y)\big)\,\big(\eta(x) - \eta(y)\big)\psi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy\\
III :=& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{\big(\eta(x) - \eta(y)\big)w(y)\,\big(\psi(x) - \psi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy.
\end{aligned}$$

Using the commutator notation $[T, m](g) = T(mg) - mTg$, we can rewrite the first term of I as

$$\begin{aligned}
\int_{\mathbb{R}^n} h_1\,(-\Delta)^{\frac{\tau}{2}}(\eta\psi)\,dx =& \int_{\mathbb{R}^n} h_1\,\eta(-\Delta)^{\frac{\tau}{2}}\psi\,dx + \int_{\mathbb{R}^n} h_1\,[(-\Delta)^{\frac{\tau}{2}}, \eta](\psi)\,dx\\
=& \int_{\mathbb{R}^n} \eta h_1\,(-\Delta)^{\frac{\tau}{2}}\psi\,dx - \int_{\mathbb{R}^n} [(-\Delta)^{\frac{\tau}{2}}, \eta](h_1)\,\psi\,dx.
\end{aligned}$$

In the last step we used an integration by parts, we can justify by approximation as follows: since $\tau \in (0,1)$ we can use the Coifman–McIntosh–Meyer commutator estimate, e.g. in the formulation in [26, Theorem 6.1.], and have

$$\|[(-\Delta)^{\frac{\tau}{2}}, \eta](h_1)\|_{L^q(\mathbb{R}^n)} \lesssim \|\eta\|_{\mathrm{Lip}}\, \|h_1\|_{L^q(\mathbb{R}^n)}.$$

Also, by Leibniz formula (2.2) (or Sobolev embedding) for any $\beta > 0$,

$$\|(-\Delta)^{\frac{\beta}{2}}(\eta h_1)\|_{L^q(\mathbb{R}^n)} \lesssim \|h_1\|_{H^{\beta,q}(\mathbb{R}^n)},$$

whenever the right-hand side is finite. So if we set

$$g_1^1 := \eta h_1 \quad \text{and} \quad g_2^1 := -[(-\Delta)^{\frac{\tau}{2}}, \eta](h_1) \quad \text{and} \quad g_2^2 := \eta h_2$$

we have shown that

$$I = \int_{\mathbb{R}^n} g_1^1 (-\Delta)^{\frac{\tau}{2}} \psi \, dz + \int_{\mathbb{R}^n} (g_2^1 + g_2^2)\psi \, dz,$$

and g_1^1, g_2^1, g_2^2 satisfy (5.8), (5.6) because h_1, h_2 satisfies those equations.

Similar to the argument in (5.11), by the support of w and η, we have for the remaining terms of (5.14)

$$\begin{aligned}
II + III =& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y) \frac{\big(w(x)-w(y)\big)\,\big(\eta(x)-\eta(y)\big)\psi(y)}{|x-y|^{n+2s}}\, dx\, dy \\
&+ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y) \frac{\big(\eta(x)-\eta(y)\big) w(y)\,\big(\psi(x)-\psi(y)\big)}{|x-y|^{n+2s}}\, dx\, dy \\
=& \int_{\Omega_1}\int_{\Omega_1} K(x,y) \frac{\big(w(x)-w(y)\big)\,\big(\eta(x)-\eta(y)\big)\psi(y))}{|x-y|^{n+2s}}\, dx\, dy \\
&+ \int_{\Omega_1}\int_{\Omega_1} K(x,y) \frac{\big(\eta(x)-\eta(y)\big) w(y)\,\big(\psi(x)-\psi(y)\big)}{|x-y|^{n+2s}}\, dx\, dy \\
&+ \int_{\tilde\Omega}\int_{\mathbb{R}^n\setminus\Omega_1} (K(x,y)+K(y,x)) \frac{w(y)\,\eta(y)\psi(y)}{|x-y|^{n+2s}}\, dx\, dy \\
&+ \int_{\tilde\Omega}\int_{\mathbb{R}^n\setminus\Omega_1} (K(y,x)-K(x,y)) \frac{\eta(y)\, w(y)\,\big(\psi(x)-\psi(y)\big)}{|x-y|^{n+2s}}\, dx\, dy.
\end{aligned}$$

We set

$$\begin{aligned}
g_2^3(y) :=& \chi_{\tilde\Omega} w(y)\,\eta(y) \int_{\mathbb{R}^n\setminus\Omega_1} \frac{K(x,y)+K(y,x)}{|x-y|^{n+2s}}\, dx \\
g_2^4(x) :=& -\chi_{\mathbb{R}^n\setminus\Omega_1}(x) \int_{\tilde\Omega} \frac{K(x,y)\eta(y)\,w(y)}{|x-y|^{n+2s}}\, dy + \chi_{\mathbb{R}^n\setminus\Omega_1}(x) \int_{\tilde\Omega} \frac{K(y,x)\eta(y)\,w(y)}{|x-y|^{n+2s}}\, dy \\
g_2^5(y) :=& -2\chi_{\tilde\Omega}(y)\eta(y)\,w(y) \int_{\mathbb{R}^n\setminus\Omega_1} \frac{K(x,y)}{|x-y|^{n+2s}}\, dx + 2\chi_{\tilde\Omega}(y)\eta(y)\,w(y) \int_{\mathbb{R}^n\setminus\Omega_1} \frac{K(y,x)}{|x-y|^{n+2s}}\, dx
\end{aligned}$$

Then

$$II + III = II_1 + III_1 + \int_{\mathbb{R}^n} \psi(y) g_2^3(y)\, dy + \int_{\mathbb{R}^n} \psi(x)\, g_2^4(x) dx + \int_{\mathbb{R}^n} \psi(y)\, g_2^5(y) dy$$

where

$$II_1 := \int_{\Omega_1} \int_{\Omega_1} K(x,y) \frac{\big(w(x)-w(y)\big)\, \big(\eta(x)-\eta(y)\big)\psi(y)\big)}{|x-y|^{n+2s}}\, dx\, dy, \text{ and}$$

$$III_1 := \int_{\Omega_1} \int_{\Omega_1} K(x,y) \frac{\big(\eta(x)-\eta(y)\big) w(y)\, \big(\psi(x)-\psi(y)\big)}{|x-y|^{n+2s}}\, dx\, dy.$$

As in the steps before,

$$\|g_2^3\|_{L^q(\mathbb{R}^n)} + \|g_2^5\|_{L^q(\mathbb{R}^n)} \precsim \|(-\Delta)^{\frac{s}{2}} w\|_{L^p(\mathbb{R}^n)},$$

As for g_2^4, by the distance of $x \in \mathbb{R}^n \backslash \Omega_1$ and $y \in \tilde{\Omega}$ we have $|x-y| \succsim c + |x|$, and thus

$$|g_2^4(x)| \precsim \frac{1}{1+|x|^{n+2s}}\, \|w\|_{L^1(\tilde{\Omega})} \precsim \frac{1}{1+|x|^{n+2s}}\, \|w\|_{L^p(\mathbb{R}^n)}.$$

Since $\frac{1}{1+|x|^{n+2s}}$ is integrable to any power, we find that

$$\|g_2^4\|_{L^q(\mathbb{R}^n)} \precsim \|w\|_{L^p(\mathbb{R}^n)} \precsim \|(-\Delta)^{\frac{s}{2}} w\|_{L^p(\mathbb{R}^n)} + \|w\|_{L^2(\mathbb{R}^n)}.$$

That is g_2^3, g_2^4, g_2^5 satisfy (5.6) because w satisfies (5.5).

Next we estimate II_1.

$$\int_{\Omega_1} \int_{\Omega_1} K(x,y) \frac{\big(w(x)-w(y)\big)\, \big(\eta(x)-\eta(y)\big)\psi(y)}{|x-y|^{n+2s}}\, dx\, dy = \int_{\mathbb{R}^n} \psi(y)\, g_2^6(y)\, dy$$

for

$$g_2^6(y) := \chi_{\Omega_1}(y) \int_{\Omega_1} K(x,y) \frac{\big(w(x)-w(y)\big)\, \big(\eta(x)-\eta(y)\big)}{|x-y|^{n+2s}}\, dx.$$

Now we have, see e.g. [39, Proposition 6.6.], for any $\alpha < 1$,

$$|w(x)-w(y)| \precsim |x-y|^\alpha\, \left(\mathcal{M}(-\Delta)^{\frac{\alpha}{2}} w(x) + \mathcal{M}(-\Delta)^{\frac{\alpha}{2}} w(y)\right),$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function. Using this, the Lipschitz continuity of η and the definition of the Riesz potential $I^{\alpha+1-2s}$, we find for any $\alpha \in (2s-1,1)$

$$|g_2^6| \precsim \chi_{\Omega_1} \left(\mathcal{M}(-\Delta)^{\frac{\alpha}{2}} w + \chi_{\Omega_1} I^{\alpha+1-2s} \left(\chi_{\Omega_1} \mathcal{M}(-\Delta)^{\frac{\alpha}{2}} w\right)\right)$$

Observe that $t \geq s > 2s-1$. In particular in view of (5.1) we can choose $\alpha \leq t$ such that

$$t - \frac{n}{p} \geq \alpha - \frac{n}{q},$$

and from Sobolev embedding (observe that Ω_1 is bounded) we obtain

$$\|g_2^6\|_{L^q(\mathbb{R}^n)} \precsim \|(-\Delta)^{\frac{t}{2}} w\|_{L^p(\mathbb{R}^n)}.$$

That is, we have shown that

$$II_1 = \int_{\mathbb{R}^n} g_2^6 \psi dx,$$

and g_2^6 satisfies (5.6) because w satisfies (5.5).

The last term it remains to estimate is III_1. Set

$$T[\psi] := \int_{\Omega_1} \int_{\Omega_1} \frac{K(x,y)\big(\eta(x)-\eta(y)\big)w(y)\,\big(\psi(x)-\psi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy$$

Clearly T is a linear operator acting on $\psi \in C_c^\infty(\mathbb{R}^n)$. Moreover, as above, for any $\alpha \in (2s-1,1)$,

$$|T[\psi]| \lesssim \int_{\Omega_1} |w|\,\left(\mathcal{M}(-\Delta)^{\frac{\alpha}{2}}\psi + I^{\alpha+1-2s}\left(\chi_{\Omega_1}(-\Delta)^{\frac{\alpha}{2}}\psi\right)\right)\,dx.$$

Under the assumption (5.3) we can take $\alpha < \tau$, and have

$$\|(-\Delta)^{\frac{\alpha}{2}}\psi\|_{L^{\frac{nq'}{n-(\tau-\alpha)q'}}(\mathbb{R}^n)} \lesssim \|(-\Delta)^{\frac{\tau}{2}}\psi\|_{L^{q'}(\mathbb{R}^n)}.$$

We repeat this argument for $T[(-\Delta)^{\frac{\beta}{2}}\psi]$. If $2s-1+\beta<\tau$, we can choose $\alpha \in (2s-1,1)$, $\alpha>0$, such that $\alpha+\beta<\tau$, (observe that since $q'\leq 2$, $\tau-\max\{2s-1,0\}-\beta<\frac{n}{q'}$ is certainly satisfied if $n\geq 2$, $\tau\in(0,1)$. If $n=1$, the condition $\tau\leq s$ implies $\tau-\max\{2s-1,0\}\leq \frac{1}{2}$ as well),

$$|T[(-\Delta)^{\frac{\beta}{2}}\psi]| \lesssim \|w\|_{L^{\frac{nq}{n+(\tau-\alpha-\beta)q}}(\Omega)}\,\|(-\Delta)^{\frac{\alpha+\beta}{2}}\psi\|_{L^{\frac{nq'}{n-(\tau-\alpha-\beta)q'}}(\mathbb{R}^n)}$$

If for $\beta \geq 0$ (5.7) is satisfied, then

$$\|w\|_{L^{\frac{nq}{n+(\tau-\alpha-\beta)q}}(\Omega)} \lesssim \|w\|_{H^{t,p}(\mathbb{R}^n)}$$

In particular for $\beta=0$, in view of (5.2),

$$|T[\psi]| \lesssim \|w\|_{H^{t,p}(\mathbb{R}^n)}\,\|(-\Delta)^{\frac{\tau}{2}}\psi\|_{L^{q'}(\mathbb{R}^n)}. \tag{5.15}$$

and if (5.7) is satisfied we also have

$$|T[(-\Delta)^{\frac{\beta}{2}}\psi]| \lesssim \|w\|_{H^{t,p}(\mathbb{R}^n)}\,\|(-\Delta)^{\frac{\tau}{2}}\psi\|_{L^{q'}(\mathbb{R}^n)}. \tag{5.16}$$

(5.15) implies that T is a linear bounded operator on $\dot{H}^{\tau,q'}(\mathbb{R}^n)$. By the characterization of dual spaces, Proposition 2.2 we find $g_1^7 \in L^q(\mathbb{R}^n)$ such that

$$III_1 = T[\psi] = \int_{\mathbb{R}^n} g_1^7 (-\Delta)^{\frac{\tau}{2}}\psi\,dx$$

and

$$\|g_1^7\|_{L^q(\mathbb{R}^n)} \lesssim \|w\|_{H^{t,p}(\mathbb{R}^n)}.$$

If (5.7) is satisfied, (5.16) implies that $\psi \mapsto T[(-\Delta)^{\frac{\beta}{2}}\psi]$ is a still linear bounded operator on $\dot{H}^{\tau,q'}(\mathbb{R}^n)$. From the characterization of dual spaces, Proposition 2.2 we thus find $g_{7,\beta} \in L^q(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} g_1^7 (-\Delta)^{\frac{\tau+\beta}{2}} \psi\, T[(-\Delta)^{\frac{\beta}{2}}\psi]\, dz = \int_{\mathbb{R}^n} g_{7,\beta} (-\Delta)^{\frac{\tau}{2}} \psi\, dz.$$

and

$$\|g_{7,\beta}\|_{L^q(\mathbb{R}^n)} \lesssim \|w\|_{H^{t,p}(\mathbb{R}^n)}.$$

This implies that $(-\Delta)^{\frac{\beta}{2}} g_1^7 = g_{7,\beta}$, and we have consequently the estimate needed for (5.8)

$$\|(-\Delta)^{\frac{\beta}{2}} g_1^7\|_{L^q(\mathbb{R}^n)} \lesssim \|w\|_{H^{t,p}(\mathbb{R}^n)}.$$

That is, g_1^7 satisfies (5.6) and (5.8).

In view (5.14) for

$$g_1 := g_1^1 + g_1^7$$

and

$$g_2 := g_2^1 + g_2^2 + g_2^3 + g_2^4 + g_2^5 + g_2^6$$

we have shown (5.4) holds, and g_1, g_2 satisfy the estimate (5.6) and (5.8). We have already observed that w and v satisfy the estimate (5.5), so the proof of Theorem 5.1 is completed. □

6. The Regularity theory: Proof of Theorem 1.2

In this section we prove the main result of the paper, Theorem 1.2. The argument of the proof is based on iterating the following incremental higher integrability result for a priori known smooth enough solution.

Theorem 6.1. *Fix $s \in (0,1)$, $t \in [s, 2s)$, $t < 1$. For given $\alpha \in (0,1)$, $\lambda, \Lambda > 0$, let $K \in \mathcal{K}(\alpha, \lambda, \Lambda)$. Suppose also that for any $2 \le p < \infty$, $u \in H^{s,2}(\mathbb{R}^n) \cap H^{t,p}(\mathbb{R}^n) \cap H^{t,2}(\mathbb{R}^n)$ with $\operatorname{supp} u \subset \Omega \subset\subset \mathbb{R}^n$ is a solution to*

$$\langle \mathcal{L}^s_{\mathbb{R}^n} u, \varphi\rangle = \int_{\mathbb{R}^n} f_1\, (-\Delta)^{\frac{2s-t}{2}} \varphi\, dz + \int_{\mathbb{R}^n} f_2\, \varphi\, dz \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n). \tag{6.1}$$

Then there exists $\bar{\varepsilon} > 0$ such that if $r \in [p, p+\bar{\varepsilon})$ and $f_1, f_2 \in L^r(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, then

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^r(\Omega)} \lesssim \sum_{i=1}^{2} \|f_i\|_{L^r(\mathbb{R}^n)} + \|f_i\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^n)}.$$

In addition, if $\beta \in [0, \bar{\varepsilon}]$, $(-\Delta)^{\frac{\beta}{2}} f_1 \in L^p(\mathbb{R}^n)$, and $f_1, f_2 \in L^p(\mathbb{R}^n)$, then $(-\Delta)^{\frac{t+\beta}{2}} u \in L^p_{loc}(\mathbb{R}^n)$ and for any $\Omega \subset\subset \mathbb{R}^n$ we have the estimate

$$\|(-\Delta)^{\frac{t+\beta}{2}} u\|_{L^p(\Omega)} \lesssim \|(-\Delta)^{\frac{\beta}{2}} f_1\|_{L^p(\mathbb{R}^n)} + \|f_1\|_{L^p(\mathbb{R}^n)} + \|f_2\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^n)}.$$

Here, $\bar{\varepsilon} > 0$ is uniform in the following sense: $\bar{\varepsilon}$ depends only on α and the number $\theta \in (0,1)$ which is such that

$$\theta < s, t, 2s - t < 1 - \theta, \text{ and } 2 \leq p < \frac{1}{\theta}.$$

Proof. First we observe that in view of Theorem 3.1 and (6.1) we have for any $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\begin{aligned}\int_{\mathbb{R}^n} K(z,z)(-\Delta)^{\frac{t}{2}} u\, (-\Delta)^{\frac{2s-t}{2}} \varphi\, dz &= \int_{\mathbb{R}^n} f_1\, (-\Delta)^{\frac{2s-t}{2}} \varphi\, dx + \int_{\mathbb{R}^n} f_2 \varphi\, dx - D_{s,t}(u,\varphi)\\ &= \int_{\mathbb{R}^n} (-\Delta)^{\frac{\beta}{2}} f_1\, (-\Delta)^{\frac{2s-t-\beta}{2}} \varphi\, dx + \int_{\mathbb{R}^n} f_2 \varphi\, dx - D_{s,t}(u,\varphi),\end{aligned} \tag{6.2}$$

where $D_{s,t}(u,\varphi)$ is as defined in (3.3) and where we have taken without loss of generality that the constant $\Gamma = 1$ in Theorem 3.1. Now we observe that the map T defined as

$$T[\varphi] := D_{s,t}(u,\varphi)$$

is linear in $\varphi \in C_c^\infty(\mathbb{R}^n)$. Choose $\sigma = 8\varepsilon$ from Theorem 3.1, for ε small enough so that $\frac{np'}{n+\sigma p'} \in (1,\infty)$ for all $p \in [2, \frac{1}{\theta}]$.

From Theorem 3.1 and Sobolev embedding, Proposition 2.1, we have the estimate for any $\beta \in [0,\varepsilon]$

$$\begin{aligned}T[\varphi] \lesssim& \int_{\mathbb{R}^n} |(-\Delta)^{\frac{t}{2}} u|(x)\, I^{\sigma-\varepsilon} |(-\Delta)^{\frac{2s-t-\varepsilon}{2}} \varphi|(x)\, dx\\ \lesssim& \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^n)}\, \|I^{\sigma-\varepsilon} |(-\Delta)^{\frac{2s-t-\varepsilon}{2}} \varphi|\|_{L^{p'}(\mathbb{R}^n)}\\ \lesssim& \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^n)}\, \|(-\Delta)^{\frac{2s-t-\varepsilon}{2}} \varphi\|_{L^{\frac{np'}{n+(\sigma-\varepsilon)p'}}(\mathbb{R}^n)}\\ =& \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^n)}\, \|I^{\varepsilon-\beta} (-\Delta)^{\frac{2s-t-\beta}{2}} \varphi\|_{L^{\frac{np'}{n+(\sigma-\varepsilon)p'}}(\mathbb{R}^n)}\\ \lesssim& \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^n)}\, \|(-\Delta)^{\frac{2s-t-\beta}{2}} \varphi\|_{L^{\frac{np'}{n+(\sigma-\beta)p'}}(\mathbb{R}^n)}.\end{aligned}$$

Here σ and ε can be chosen to depend only on θ, and since $p < \frac{1}{\theta}$ we can make that choice so that $\frac{np'}{n+(\sigma-2\varepsilon)p'} > 1$ and Sobolev embedding is applicable with a uniform constant.

That is, T belongs to $\left(\dot{H}^{2s-t-\beta, \frac{np'}{n+(\sigma-\beta)p'}}(\mathbb{R}^n)\right)^*$ for any $\beta \in [0,\varepsilon]$. By classification of the dual spaces, Proposition 2.2, and since $\left(\frac{np'}{n+(\sigma-\beta)p'}\right)' = \frac{np}{n-(\sigma-\beta)p}$ we find $g_\beta \in L^{\frac{np}{n-(\sigma-\beta)p}}(\mathbb{R}^n)$

$$\|g_\beta\|_{L^{\frac{np}{n-(\sigma-\beta)p}}(\mathbb{R}^n)} \lesssim \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^n)}, \tag{6.3}$$

and

$$T[\varphi] = \int_{\mathbb{R}^n} g_\beta\, (-\Delta)^{\frac{2s-t-\beta}{2}} \varphi dx.$$

That is, (6.2) becomes for any $\beta \in [0,\varepsilon]$

$$\int_{\mathbb{R}^n} K(z,z)(-\Delta)^{\frac{t}{2}} u\, (-\Delta)^{\frac{2s-t}{2}} \varphi dz = \int_{\mathbb{R}^n} \left((-\Delta)^{\frac{\beta}{2}} f_1 + g_\beta\right)\, (-\Delta)^{\frac{2s-t-\beta}{2}} \varphi\, dx + \int_{\mathbb{R}^n} f_2 \varphi\, dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$.

For $\beta = 0$ we obtain from (6.3) and Theorem 1.7 that for any $\Omega \subset\subset \mathbb{R}^n$, $r \in \left[p, \frac{np}{n-\sigma p}\right]$, we have

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^r(\Omega)} \lesssim \sum_{i=1}^2 \left(\|f_i\|_{L^r(\mathbb{R}^n)} + \|f_i\|_{L^p(\mathbb{R}^n)}\right) + \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^n)}.$$

Observe that we can find $\bar{\varepsilon}$ such that $\frac{np}{n-\sigma p} \geq p + \bar{\varepsilon}$ for all $p \in [2, \frac{1}{\theta}]$.

For $\beta \in [0,\varepsilon]$ from (6.3) and Proposition 4.2 for any

$$\|(-\Delta)^{\frac{t+\beta}{2}} u\|_{L^p(\Omega)} \lesssim \|(-\Delta)^{\frac{\beta}{2}} f_1\|_{L^p(\mathbb{R}^n)} + \sum_{i=1}^2 \|f_i\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\mathbb{R}^n)}.$$

This concludes the proof of Theorem 6.1. □

Iterating Theorem 6.1 and Theorem 5.1 leads to the proof of Theorem 1.2, namely

Theorem 6.2. *Fix $s \in (0,1)$, $t \in [s, 2s)$, $t < 1$. For given $\alpha \in (0,1)$, $\lambda, \Lambda > 0$, let $K \in \mathcal{K}(\alpha,\lambda,\Lambda)$. Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega \subseteq \mathbb{R}^n$ be two open sets. Assume that $u \in W^{s,2}(\Omega)$ satisfies the equation*

$$\langle \mathcal{L}^s_\Omega u, \varphi\rangle = \int_{\mathbb{R}^n} f_1 (-\Delta)^{\frac{2s-t}{2}} \varphi dx + \int f_2 \varphi dx \quad \forall \varphi \in C_c^\infty(\Omega''). \tag{6.4}$$

If $f_1, f_2 \in L^q(\Omega) \cap L^2(\mathbb{R}^n)$, $q \in [2,\infty)$, then $(-\Delta)^{\frac{t}{2}} u \in L^q(\Omega')$ and we have

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^q(\Omega')} \leq C(\Omega,\Omega',\Omega'',s,t,p,q) \left(\|u\|_{W^{s,2}(\Omega)} + \sum_{i=1}^2 \|f_i\|_{L^q(\Omega)} + \|f_i\|_{L^2(\mathbb{R}^n)}\right).$$

Proof. Fix $\theta \in (0,1)$ such that

$$t < 1 - 10\theta,\ 10\theta < s < 1 - 10\theta,\ 10\theta < 2s - t < 1 - 10\theta, 2 \leq q < \frac{1}{10\theta}. \tag{6.5}$$

We also fix $\varepsilon = \varepsilon(\theta,\gamma)$ from Theorem 6.1, and w.l.o.g. $\varepsilon < \frac{1}{10}\frac{\theta}{n}$.

Step 0: Rewriting the equation Take some cutoff function $\eta \in C_c^\infty(\Omega)$ with $\eta \equiv 1$ in a neighborhood of Ω''

$$\begin{aligned}\int_{\mathbb{R}^n} f_1 (-\Delta)^{\frac{2s-t}{2}} \varphi dx &= \int_{\mathbb{R}^n} \eta f_1 (-\Delta)^{\frac{2s-t}{2}} \varphi\, dx + \int_{\mathbb{R}^n} (1-\eta) f_1 (-\Delta)^{\frac{2s-t}{2}} \varphi\, dx\\ &= \int_{\mathbb{R}^n} \eta f_1 (-\Delta)^{\frac{2s-t}{2}} \varphi\, dx + \int_{\mathbb{R}^n} \chi_{\Omega''} (-\Delta)^{\frac{2s-t}{2}} \left((1-\eta) f_1\right) \varphi\, dx.\end{aligned}$$

Now observe that by the disjoint support of $\chi_{\Omega''}$ and $1-\eta$ we have

$$\|\chi_{\Omega''}(-\Delta)^{\frac{2s-t}{2}}\left((1-\eta)f_1\right)\|_{L^\infty} \precsim \|f_1\|_{L^2(\mathbb{R}^n)}$$

For $\sigma \in [s,t]$ we set

$$\tilde{f}_{1,\sigma} := I^{t-\sigma}(\eta f_1)$$

and

$$\tilde{f}_2 := \chi_{\Omega''} f_2 + \chi_{\Omega''}(-\Delta)^{\frac{2s-t}{2}}\left((1-\eta)f_1\right)$$

then we have for all $\varphi \in C_c^\infty(\Omega'')$,

$$\int_\Omega\int_\Omega K(x,y)\frac{(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_{\mathbb{R}^n}\tilde{f}_{1,\sigma}(-\Delta)^{\frac{2s-\sigma}{2}}\varphi\,dx + \int \tilde{f}_2\varphi\,dx.$$

Moreover $\tilde{f}_2 \in L^q(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and since $t-s \leq 1-s < 1-\theta$ we have by Sobolev embedding, Proposition 2.1,

$$\|I^{t-s}(\eta f_1)\|_{L^q(\mathbb{R}^n)} \leq C(\theta)\|\eta f_1\|_{L^{\frac{nq}{n+(t-s)q}}} \precsim \|\eta f_1\|_{L^q(\mathbb{R}^n)}$$

If $n \geq 2$ we also have

$$\|I^{t-s}(\eta f_1)\|_{L^2(\mathbb{R}^n)} \leq C(\theta)\|\eta f_1\|_{L^{\frac{2n}{n+(t-s)2}}} \precsim \|\eta f_1\|_{L^2(\mathbb{R}^n)} \tag{6.6}$$

so that for $n\geq 2$ we have found $\tilde{f}_{1,\sigma}, \tilde{f}_2 \in L^q\cap L^2(\mathbb{R}^n)$ such that (6.4) holds for t replaced with σ and f_1, f_2 replaced with $\tilde{f}_{1,\sigma}, \tilde{f}_2$.

If $n = 1$ we need a slight adaptation to have (6.6) (if t is close to one and s is close to zero): Let $\eta_2 \in C_c^\infty(\mathbb{R}^n)$ with $\eta_2 \equiv 1$ in a neighborhood of Ω''. Then we set

$$\tilde{\tilde{f}}_{1,\sigma} := \eta_2 I^{t-\sigma}(\eta f_1)$$

and

$$\tilde{\tilde{f}}_{2,\sigma} := \tilde{f}_2 + \chi_{\Omega''}\left((-\Delta)^{\frac{2s-t}{2}}\left((1-\eta_2)I^{t-\sigma}(\eta f_1)\right)\right).$$

By the disjoint support we then get the same estimates as before.

In conclusion, for any $\sigma \in [s,t]$ we have $f_{1,\sigma}, f_{2,\sigma} \in L^2\cap L^q(\mathbb{R}^n)$ and

$$\int_\Omega\int_\Omega K(x,y)\frac{(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_{\mathbb{R}^n} f_{1,\sigma}(-\Delta)^{\frac{2s-\sigma}{2}}\varphi\,dx + \int f_{2,\sigma}\varphi\,dx, \tag{6.7}$$

for any $\varphi \in C_c^\infty(\Omega'')$.

Let $L \in \mathbb{N}$ a number which we shall define later, and choose nested open sets

$$\Omega' := \Omega_{2L} \subset\subset \ldots \subset\subset \Omega_1 \subset\subset \Omega'' \subset\subset \Omega. \tag{6.8}$$

<u>Step 1: First improvement</u>

$$\frac{1}{q_1} := \max\left\{\frac{1}{2}-\frac{\theta}{n}, \theta\right\} \tag{6.9}$$

then (5.1) and (5.2) are satisfied in view of (6.5).

$$\beta_1 := \min\{\frac{1}{2}\varepsilon, s-t\} \tag{6.10}$$

$$p_1 := \min\{2+\frac{1}{2}\varepsilon, q_1\} \tag{6.11}$$

We claim that

$$\begin{aligned} &\|(-\Delta)^{\frac{s+\beta_1}{2}} u\|_{L^{p_1}(\Omega_2)} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^{p_1}(\Omega_2)} \\ &\lesssim \sum_{i=1}^{2} \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}. \end{aligned} \tag{6.12}$$

We apply Theorem 5.1 for $\tilde{s} = \tilde{t} = \tilde{r} = s$, $\tilde{p} = 2$ and the equation to (6.7) with $\sigma = s$. Then (5.3) is satisfied since $s < 1$. We also choose $\tilde{q} := q_1 \in (2, q]$ then (5.1) and (5.2) are satisfied in view of (6.5).

Observe that $f_{1,\sigma}, f_{2,\sigma} \in L^q \cap L^2(\mathbb{R}^n) \subset L^{q_1}(\mathbb{R}^n)$, so from Theorem 5.1 we obtain $v_1 \in H^{s,2}(\mathbb{R}^n)$, $\operatorname{supp} v_1 \subset\subset \Omega''$

$$v_1 \equiv u \quad \text{in a neighborhood of } \Omega_1$$

and for any $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y) \frac{(v_1(x)-v_1(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\, dx\, dy \\ &= \int_{\mathbb{R}^n} g_{1,s} (-\Delta)^{\frac{s}{2}} \varphi\, dx + \int_{\mathbb{R}^n} g_{2,s}\varphi\, dx, \end{aligned} \tag{6.13}$$

for some $g_1, g_2 \in L^{q_1}(\mathbb{R}^n)$ with the estimate

$$\|g_{1,s}\|_{L^{q_1}(\mathbb{R}^n)} + \|g_{2,s}\|_{L^{q_1}(\mathbb{R}^n)} \lesssim \sum_{i=1}^{2} \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}$$

and

$$\|(-\Delta)^{\frac{s}{2}} v_1\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|_{H^{s,2}(\mathbb{R}^n)}$$

and in view of (5.8), for any $0 \le \alpha \le \min\{\theta, t-s\}$ we have (for $\tilde{\beta} := \alpha$) that (5.7) is satisfied

$$\|(-\Delta)^{\frac{\alpha}{2}} g_{1,\sigma}\|_{L^2(\mathbb{R}^n)} \lesssim \|f_1\|_{L^2(\mathbb{R}^n)} + \|u\|_{H^{s,2}(\mathbb{R}^n)}.$$

In view of Theorem 6.1 (applied to $\tilde{t} := s$ and the equation (6.13)) we have the estimate

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}} v_1\|_{L^{p_1}(\Omega'')} &\lesssim \sum_{i=1}^{2} \left(\|g_i\|_{L^{p_1}(\mathbb{R}^n)} + \|g_i\|_{L^2(\mathbb{R}^n)}\right) + \|(-\Delta)^{\frac{s}{2}} v_1\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \sum_{i=1}^{2} \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}. \end{aligned}$$

Moreover, since we applied Theorem 6.1 to the equation (6.13), we have

$$\|(-\Delta)^{\frac{s+\beta_1}{2}} v_1\|_{L^2(\Omega'')} \precsim \sum_{i=1}^2 \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}.$$

Since $u \equiv v_1$ in a neighborhood of Ω_1, by Lemma 2.3, we find that this implies

$$\|(-\Delta)^{\frac{s}{2}} u\|_{L^{p_1}(\Omega_1)} \precsim \sum_{i=1}^2 \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}. \tag{6.14}$$

and

$$\|(-\Delta)^{\frac{s+\beta_1}{2}} u\|_{L^2(\Omega_1)} \precsim \sum_{i=1}^2 \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}. \tag{6.15}$$

In order to obtain (6.12) we need to have an L^{p_1}-estimate in (6.15). For this we repeat this argument for the equation (6.7) with $\sigma = s + \beta_1$ (this is only necessary if $t > s$, otherwise $\beta_1 = 0$ and we are done with (6.14)).

We apply Theorem 5.1 for $\tilde{s} = s$, $\tilde{t} = s + \beta_1$ and $\tilde{r} = s - \beta_1$, $\tilde{p} = 2$, $\tilde{q} := q_1$ to (6.7) with $\sigma = s + \beta_1$. Again (5.3) is satisfied, since $s + \beta_1 < s + t - s = t < 1$. (5.1), and (5.2) are satisfied in view of (6.5). Then Theorem 5.1 implies the existence of $v_2 \in H^{s+\beta_1,2}(\mathbb{R}^n)$, $\operatorname{supp} v_2 \subset\subset \Omega''$

$$v_2 \equiv u \quad \text{in a neighborhood of } \Omega_2$$

and for all $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) \frac{(v_2(x) - v_2(y))\,(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}}\, dx\, dy \\ &\qquad = \int_{\mathbb{R}^n} g_{1,s+\beta_1} (-\Delta)^{\frac{s+\beta_1}{2}} \varphi\, dx + \int_{\mathbb{R}^n} g_{2,s+\beta_1} \varphi\, dx, \end{aligned} \tag{6.16}$$

for some $g_{1,s+\beta_1}, g_{2,s+\beta_1} \in L^{q_1}(\mathbb{R}^n)$ with the estimate

$$\|g_{1,s+\beta_1}\|_{L^{q_1}(\mathbb{R}^n)} + \|g_{2,s+\beta_1}\|_{L^{q_1}(\mathbb{R}^n)} \precsim \sum_{i=1}^2 \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}$$

and (with the additional help of (6.15)),

$$\|(-\Delta)^{\frac{s+\beta}{2}} v_2\|_{L^2(\mathbb{R}^n)} \precsim \sum_{i=1}^2 \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}.$$

Applying Theorem 6.1 (for $\tilde{t} := s + \beta_1$ and the equation (6.16), observe that ε does not change) we have

$$\|(-\Delta)^{\frac{s+\beta}{2}} v_2\|_{L^{p_1}(\Omega_2)} \precsim \sum_{i=1}^2 \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}.$$

By Lemma 2.3, since $u \equiv v_2$ in a neighborhood of Ω_2 we find

$$\|(-\Delta)^{\frac{s+\beta_1}{2}} u\|_{L^{p_1}(\Omega_2)} \precsim \sum_{i=1}^{2} \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}. \tag{6.17}$$

Together, (6.17) and (6.14) imply (6.12).

<u>Step 2: Iteration</u> We define for $k \in \mathbb{N}$,

$$\frac{1}{q_{k+1}} := \max\left\{\frac{1}{p_k} - \frac{\theta}{n}, \frac{1}{q}\right\}, \tag{6.18}$$

$$p_{k+1} := \min\left\{p_k + \frac{1}{2}\varepsilon, q_{k+1}\right\}, \tag{6.19}$$

$$\beta_{k+1} := \beta_k + \min\left\{\frac{1}{2}\varepsilon, s - t - \beta_k\right\}. \tag{6.20}$$

starting from q_1, p_1, β_1 as in (6.9), (6.11), (6.10), respectively.

Our goal is to show that for any $k \in \mathbb{N}$,

$$\begin{aligned} &\|(-\Delta)^{\frac{s+\beta_k}{2}} u\|_{L^{p_k}(\Omega_{2k})} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^{p_k}(\Omega_{2k})} \\ &\qquad \precsim \sum_{i=1}^{2} \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}. \end{aligned} \tag{6.21}$$

We prove this by induction induction. We already have shown (6.21) to hold for $k = 1$, (6.12).

So assume as induction hypothesis that for some $k \in \mathbb{N}$ (6.21) holds. We need to show

$$\|(-\Delta)^{\frac{s}{2}} u\|_{L^{p_{k+1}}(\Omega_{2k+2})} \precsim \sum_{i=1}^{2} \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}. \tag{6.22}$$

and

$$\|(-\Delta)^{\frac{s+\beta_{k+1}}{2}} u\|_{L^{p_{k+1}}(\Omega_{2k+2})} \precsim \sum_{i=1}^{2} \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}. \tag{6.23}$$

<u>First we treat (6.22)</u>. If $q_k = q$ there is nothing to show, because then $p_k = p_{k+1} = q$. If not, we apply Theorem 5.1 for $\tilde{s} = \tilde{t} = \tilde{r} = s$, $\tilde{p} = p_k$, $\tilde{q} := q_k$ to (6.7) with $\sigma = s$. Again (5.3) is satisfied since $s < 1$. (5.1) and (5.2) are satisfied in view of (6.5) and the fact that since $\varepsilon < \frac{1}{10}\frac{\theta}{n}$ we have that $\left|\frac{1}{p_k} - \frac{1}{p_{k-1}}\right| \leq \frac{\theta}{n}$.

Then Theorem 5.1 implies the existence of $v_1 \in H^{s,p_k}(\mathbb{R}^n)$, $\operatorname{supp} v_1 \subset\subset \Omega''$

$$v_1 \equiv u \quad \text{in a neighborhood of } \Omega_{2k+1}$$

and for all $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{(v_1(x)-v_1(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy\\ = \int_{\mathbb{R}^n} g_{1,s}(-\Delta)^{\frac{s}{2}}\varphi\,dx + \int_{\mathbb{R}^n} g_{2,s}\varphi\,dx,\end{aligned} \tag{6.24}$$

for some $g_{1,s}, g_{2,s} \in L^{q_k}(\mathbb{R}^n)$ with the estimate

$$\|g_{1,s}\|_{L^{q_k}(\mathbb{R}^n)} + \|g_{2,s}\|_{L^{q_k}(\mathbb{R}^n)} \precsim \sum_{i=1}^2 \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}.$$

and additionally (using the induction hypothesis (6.21))

$$\|(-\Delta)^{\frac{s}{2}} v_1\|_{L^{p_k}(\mathbb{R}^n)} \precsim \sum_{i=1}^2 \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}.$$

Applying Theorem 6.1 for $\tilde{t} := s$ and the equation (6.24) we obtain

$$\begin{aligned}\|(-\Delta)^{\frac{s}{2}} v_1\|_{L^{p_{k+1}}(\Omega'')} &\precsim \sum_{i=1}^2 \left(\|g_i\|_{L^{p_{k+1}}(\mathbb{R}^n)} + \|g_i\|_{L^2(\mathbb{R}^n)}\right) + \|(-\Delta)^{\frac{s}{2}} v_1\|_{L^{p_k}(\mathbb{R}^n)}\\ &\precsim \sum_{i=1}^2 \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}.\end{aligned}$$

Since $u \equiv v_1$ in a neighborhood of Ω_{2k+1} this implies (6.22).

Now we treat (6.23). We apply Theorem 5.1 to $\tilde{s} = s$, $\tilde{t} = s+\beta_k$, $\tilde{p} = p_k$, $\tilde{q} = q_k$, $\tilde{r} = s-\beta_k$ and to the equation (6.7) with $\sigma = s + \beta_k$. (5.3) is satisfied since $s + \beta_k \leq t < 1$. As before, (5.1), (5.2) are satisfied in view of the choice of θ, q_k, p_k, β_{k+1}. Since we have by assumption (6.21), we find $v_2 \in H^{s,2} \cap H^{s+\beta_k}(\mathbb{R}^n)$, $v_2 \equiv u$ in a neighborhood of Ω_{2k+1}, $g_1, g_2 \in L^{q_k}(\mathbb{R}^n)$ such that for all $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{(v_2(x)-v_2(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_{\mathbb{R}^n} g_1(-\Delta)^{\frac{s-\beta_k}{2}}\varphi\,dx + \int_{\mathbb{R}^n} g_2\varphi\,dx.$$

We apply Theorem 6.1 for $\tilde{t} = s + \beta_k$ to this equation, and find that

$$\|(-\Delta)^{\frac{s+\beta_{k+1}}{2}} v_2\|_{L^{p_k}(\Omega_{2k})} \precsim \sum_{i=1}^2 \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}.$$

Since $v_2 \equiv u$ in a neighborhood of Ω_{2k+1} we conclude that

$$\|(-\Delta)^{\frac{s+\beta_{k+1}}{2}} u\|_{L^{p_k}(\Omega_{2k+1})} \precsim \sum_{i=1}^2 \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}. \tag{6.25}$$

If $p_{k+1} = p_k$ we have (6.23). Otherwise, we need to apply this chain of arguments one more time: This time, we apply Theorem 5.1 to $\tilde{s} = s$, $\tilde{t} = s + \beta_{k+1}$, $\tilde{p} = p_k$, $\tilde{q} = q_k$, $\tilde{r} = s - \beta_k$ and to the equation (6.7) with $\sigma = s+\beta_{k+1}$. Again, (5.3) is satisfied since $s+\beta_{k+1} \leq t < 1$, and (5.2), (5.1) are satisfied in view of the choice of θ, q_k, p_k, β_{k+1}. Since we have (6.25), we obtain from Theorem 5.1 $v_3 \in H^{s,2} \cap H^{s+\beta_{k+1}}(\mathbb{R}^n)$, $v_2 \equiv u$ in a neighborhood of Ω_{2k+2}, $g_1, g_2 \in L^{q_k}(\mathbb{R}^n)$ such that for all $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{(v_2(x)-v_2(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_{\mathbb{R}^n} g_1(-\Delta)^{\frac{s-\beta_{k+1}}{2}}\varphi\,dx + \int_{\mathbb{R}^n} g_2\varphi\,dx.$$

We apply Theorem 6.1 for $\tilde{t} = s+\beta_{k+1}$ to this equation, and find that

$$\|(-\Delta)^{\frac{s+\beta_{k+1}}{2}} v_3\|_{L^{p_{k+1}}(\Omega_{2k+1})} \lesssim \sum_{i=1}^2 \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}.$$

Since $v_3 \equiv u$ in a neighborhood of Ω_{2k+2}, we finally conclude (6.23).

Conclusion: From the definition of p_{k+1}, q_{k+1}, β_{k+1} as in (6.19), (6.18), (6.20) starting from p_1, q_1, β_1 as in (6.11), (6.9), (6.10) we see that there is a large number (depending on ε and θ, s, t, and q – all of which are fixed numbers in this proof) there is a finite number $L \in \mathbb{N}$ such that $p_L = q_L = q$, $\beta_L = t-s$. Thus, from we have (6.21) we obtain

$$\|(-\Delta)^{\frac{t}{2}} u\|_{L^p(\Omega_{2L})} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\Omega_{2L})} \lesssim \sum_{i=1}^2 \left(\|f_i\|_{L^2(\mathbb{R}^n)} + \|f_i\|_{L^q(\mathbb{R}^n)}\right) + \|u\|_{H^{s,2}(\mathbb{R}^n)}.$$

Since $\Omega' \subset \Omega_{2L}$ (see (6.8)), and taking into account the arguments from Step 0 of this proof, we conclude. □

7. Proof of the corollaries of Theorem 1.2

Corollary 1.3 is an immediate consequence of Theorem 1.2 and its $H^{t,q}_{loc}$-estimates.

Proof of Corollary 1.4. Let $\Omega'' \subset\subset \Omega$ with $\Omega' \subset\subset \Omega''$. Let $\eta \in C_c^\infty(\Omega)$ with $\eta \equiv 1$ in Ω''. Since $f \in \left(H^{2s-t,q'}(\Omega)\right)^*$ we have that $\tilde{f} = \eta f \in \left(H^{2s-t,q'}(\mathbb{R}^n)\right)^*$, since for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle \tilde{f}, \varphi\rangle := \langle f, \eta\varphi\rangle$$

Then u is a solution of

$$\langle \mathcal{L}^s_\Omega u, \varphi\rangle = \langle \tilde{f}, \varphi\rangle \quad \forall \varphi \in C_c^\infty(\Omega'').$$

Observe that

$$\langle \tilde{f}, \varphi\rangle \lesssim \|f\|_{\left(H^{2s-t,q'}(\Omega)\right)^*} \|\eta\varphi\|_{H^{2s-t,q'}(\mathbb{R}^n)}$$

By the fractional Leibniz rule, we also have

$$\|\eta\varphi\|_{H^{2s-t,q'}(\mathbb{R}^n)} \lesssim \|\varphi\|_{H^{2s-t,q'}(\mathbb{R}^n)}$$

Moreover since $q' \le 2$ and η has compact support,

$$\|\eta\varphi\|_{H^{2s-t,2}(\mathbb{R}^n)} \lesssim \|\varphi\|_{H^{2s-t,2}(\mathbb{R}^n)}.$$

In view of Proposition 2.2 we find $f_1, f_2 \in L^q \cap L^2(\mathbb{R}^n)$ such that

$$\langle \tilde{f}, \varphi \rangle = \int_{\mathbb{R}^n} f_1 (-\Delta)^{\frac{2s-t}{2}} \varphi \, dx + \int_{\mathbb{R}^n} f_2 \varphi \, dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n),$$

and

$$\|f_1\|_{L^q(\mathbb{R}^n)} + \|f_2\|_{L^q(\mathbb{R}^n)} + \|f_1\|_{L^2(\mathbb{R}^n)} + \|f_2\|_{L^2(\mathbb{R}^n)} \lesssim \|\tilde{f}\|_{\left(H^{2s-t,q'}(\mathbb{R}^n)\right)^*} \lesssim \|f\|_{\left(H^{2s-t,q'}(\Omega)\right)^*}.$$

Thus, u is a solution of

$$\langle \mathcal{L}^s_\Omega u, \varphi \rangle = \int_{\mathbb{R}^n} f_1 (-\Delta)^{\frac{2s-t}{2}} \varphi \, dx + \int_{\mathbb{R}^n} f_2 \varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega'').$$

Applying Theorem 1.2 to this equation in $\Omega' \subset\subset \Omega''$ we obtain the claim.

□

Lastly, we show the following corollary of Theorem 1.2 for equations of the type $\mathcal{L}^s_\Omega u = \operatorname{div}_{s,\Omega} F$, where div_s denotes a fractional divergence as treated e.g. in [13, 27]. Observe that Corollary 1.6 is a direct consequence of Corollary 7.1 if we set $F(x,y) := \frac{f(x)-f(y)}{|x-y|^s}$.

Corollary 7.1. *Let $s \in (0,1)$ and $p \geq 2$. Let $\Omega \subset\subset \mathbb{R}^n$ be a smoothly bounded set, and let $\Omega_1 \subset\subset \Omega$ be open. Assume that $u \in W^{s,2}(\Omega)$ satisfies*

$$\langle \mathcal{L}^s_\Omega u, \varphi \rangle = \int_\Omega \int_\Omega \frac{F(x,y)\,(\varphi(x) - \varphi(y))}{|x-y|^{n+s}} \, dx \, dy$$

for any $\varphi \in C_c^\infty(\Omega)$, where $\mathcal{L}^s_\Omega$ corresponds to $K \in \mathcal{K}(\alpha, \lambda, \Lambda)$ for some given $\alpha \in (0,1)$ and $\lambda, \Lambda > 0$. Then if for any $t > 0$ we have

$$\int_\Omega \int_\Omega \frac{|F(x,y)|^p}{|x-y|^{n+tp}} \, dx \, dy < \infty$$

then for any $r \in [s, s+t)$ we have $u \in W^{t,p}_{loc}(\Omega)$, and for any $\Omega_1 \subset \Omega$ we have the estimate

$$[u]_{W^{r,p}(\Omega_1)} \leq C \left(\left(\int_\Omega \int_\Omega \frac{|F(x,y)|^p}{|x-y|^{n+tp}} \, dx \, dy \right)^{\frac{1}{p}} + \|u\|_{L^2(\Omega)} + [u]_{W^{s,2}(\Omega)} \right).$$

Proof. Set

$$\Lambda := \left(\int_\Omega \int_\Omega \frac{|F(x,y)|^p}{|x-y|^{n+tp}} \, dx \, dy \right)^{\frac{1}{p}}$$

Observe that since Ω is bounded, we have for any $\tilde{t} \in [0,t]$,

$$\left(\int_\Omega \int_\Omega \frac{|F(x,y)|^p}{|x-y|^{n+\tilde{t}p}} \right)^{\frac{1}{p}} \lesssim \Lambda.$$

Let $\Omega_2 \subset\subset \Omega_3 \subset \mathbb{R}^n$ be an open set such that $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$. Take $\eta \in C_c^\infty(\Omega)$ such that $\eta \equiv 1$ in a neighborhood of Ω_3. Then for any $\varphi \in C_c^\infty(\Omega_3)$,

$$\begin{aligned}&\int_\Omega \int_\Omega \frac{F(x,y)\,(\varphi(x) - \varphi(y))}{|x-y|^{n+s}}\,dx\,dy\\ =&\int_\Omega \int_\Omega \frac{F(x,y)\,(\eta(x)\varphi(x) - \eta(y)\varphi(y))}{|x-y|^{n+s}}\,dx\,dy.\end{aligned}$$

Moreover we have for any $\varphi \in C_c^\infty(\mathbb{R}^n)$, and any $\tilde{t} \in [0,t]$,

$$\begin{aligned}T[\varphi] :=&\int_\Omega \int_\Omega \frac{F(x,y)\,(\eta(x)\varphi(x) - \eta(y)\varphi(y))}{|x-y|^{n+s}}\,dx\,dy\\ \lesssim&\Lambda [\varphi]_{W^{s-\tilde{t},p'}(\mathbb{R}^n)}.\end{aligned}$$

By Sobolev embedding, for any $r > s - \tilde{t}$,

$$[\varphi]_{W^{s-\tilde{t},p'}(\mathbb{R}^n)} \lesssim \|\varphi\|_{H^{r,p'}(\mathbb{R}^n)}.$$

That is, T is an element of $(H^{r,p'}(\mathbb{R}^n))^*$, and by Proposition 2.2 we find $f_1, f_2 \in L^p(\mathbb{R}^n)$ such that

$$T[\varphi] = \int_{\mathbb{R}^n} f_1 (-\Delta)^{\frac{s}{2}} \varphi\,dx + \int_{\mathbb{R}^n} f_2 \varphi\,dx,$$

with

$$\|f_1\|_{L^p(\mathbb{R}^n)} + \|f_2\|_{L^p(\mathbb{R}^n)} \lesssim \Lambda.$$

In particular we have for any $\varphi \in C_c^\infty(\Omega_2)$

$$\langle \mathcal{L}^s_\Omega u, \varphi \rangle = \int_{\mathbb{R}^n} f_1 (-\Delta)^{\frac{s}{2}} \varphi\,dx + \int_{\mathbb{R}^n} f_2 \varphi\,dx,$$

and from Theorem 1.2 we conclude that for any $W^{s,2}$-extension $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ of $u\Big|_\Omega$ we have

$$\|(-\Delta)^{\frac{r}{2}} \tilde{u}\|_{L^p(\Omega_2)} \lesssim \Lambda + [\tilde{u}]_{W^{s,2}(\mathbb{R}^n)} + \|\tilde{u}\|_{L^2(\mathbb{R}^n)}$$

Again from Sobolev embedding this implies for any $0 < \tilde{r} < r$

$$[u]_{W^{\tilde{r},p}(\Omega_1)} \lesssim \Lambda + [\tilde{u}]_{W^{s,2}(\mathbb{R}^n)} + \|\tilde{u}\|_{L^2(\mathbb{R}^n)},$$

Since Ω is an extension domain we can find an extension $\tilde{u}$ such that

$$[\tilde{u}]_{W^{s,2}(\mathbb{R}^n)} + \|\tilde{u}\|_{L^2(\mathbb{R}^n)} \lesssim [u]_{W^{s,2}(\Omega)} + \|u\|_{L^2(\Omega)},$$

and conclude the theorem. □

Proof of Corollary 1.5. Observe that for a $C^{1,\alpha}$-diffeomorphism Φ the maps u and $u \circ \Phi$ belong to the same Sobolev spaces $H^{s,p}$ and $W^{s,p}$ as long as $s \leq 1$.

By the transformation rule

$$\int_\Omega \int_\Omega \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|\Phi(x)-\Phi(y)|^{n+2s}}\,dx\,dy$$

$$=\int_{\Omega_2}\int_{\Omega_2} K(x,y)\frac{(u\circ\Phi(x)-u\circ\Phi(y))(\varphi\circ\Phi(x)-\varphi\circ\Phi(y))}{|x-y|^{n+2s}}dx\,dy$$

where $K(x,y)=\det(D\Phi(x))\det(D\Phi(y))$ is still Hölder continuous. Now we can apply Corollary 1.4 to this K. □

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