Journal of Combinatorial Theory, Series B 151 (2021) 148–222

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Packing and covering immersions in 4-edge-connected graphs

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ARTICLE INFO

Article history:
Received 6 May 2015
Available online xxxx

Keywords:
Graph immersions
Erdős-Pósa property

ABSTRACT

A graph G contains another graph H as an immersion if H can be obtained from a subgraph of G by splitting off edges and removing isolated vertices. In this paper, we prove an edge-variant of the Erdős-Pósa property with respect to the immersion containment in 4-edge-connected graphs. More precisely, we prove that for every graph H, there exists a function f such that for every 4-edge-connected graph G, either G contains k pairwise edge-disjoint subgraphs each containing H as an immersion, or there exists a set of at most $f(k)$ edges of G intersecting all such subgraphs. This theorem is best possible in the sense that the 4-edge-connectivity cannot be replaced by the 3-edge-connectivity.

1. Introduction

In this paper *graphs* are finite and are permitted to have loops and parallel edges. Many questions in graph theory or combinatorial optimization can be formulated as follows. Given a set of graphs $\mathcal{F}$ and a graph G, what is the maximum number of disjoint

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[2] Partially supported by NSF under Grant No. DMS-1664593, DMS-1929851 and DMS-1954054.

https://doi.org/10.1016/j.jctb.2021.06.005

subgraphs of G where each isomorphic to a member of $\mathcal{F}$ or what is the minimum number of vertices that are required to meet all such subgraphs? We call the former problem the *packing problem* and the maximum number the *packing number*, and we call the latter problem the *covering problem* and the minimum the *covering number*. For example, if $\mathcal{F}$ consists of the graph K_2, then the packing number is the maximum size of a matching and the covering number is the minimum size of a vertex cover; if $\mathcal{F}$ is the set of cycles, the covering number is the minimum size of a feedback vertex set.

In view of combinatorial optimization, the packing problem and the covering problem can be formulated as integer programming problems. And the covering problem is the dual of the packing problem. Furthermore, it is easy to see that the packing number is at most the covering number. On the other hand, it is natural to ask when the covering number can be bounded by a function of the packing number from above. In other words, we hope that the optimal solutions of the packing problem and covering problem are bounded by functions of each other.

Formally, a set of graphs $\mathcal{F}$ has the *Erdős-Pósa property* if for every integer k, there exists a number $f(k)$ such that for every graph G, either G contains k disjoint subgraphs each isomorphic to a member of $\mathcal{F}$, or there exists $Z \subseteq V(G)$ with $|Z| \leq f(k)$ such that $G - Z$ does not contain a subgraph isomorphic to a member of $\mathcal{F}$. A classical result of Erdős and Pósa [6] states that the set of cycles has the Erdős-Pósa property. Hence the packing number and the covering number for the set of cycles are tied together.

This theorem was later generalized by Robertson and Seymour in terms of graph minors. A graph is a *minor* of another graph if the former can be obtained from a subgraph of the latter by contracting edges. For every graph H, define $\mathcal{M}(H)$ to be the set of graphs containing H as a minor. Robertson and Seymour [16] proved that $\mathcal{M}(H)$ has the Erdős-Pósa property if and only if H is planar. So the aforementioned result of Erdős and Pósa follows from the case that H is the one-vertex graph with one loop.

A variant of the minor containment is the topological minor containment. We say that a graph is a *topological minor* of another graph if the former can be obtained from a subgraph of the latter by repeatedly contracting edges incident with at least one vertex of degree two. Minor containment and topological minor containment are closely related. For example, they are equivalent for characterizing planar graphs.

However, minors and topological minors behave much differently with respect to the Erdős-Pósa property. For every graph H, define $\mathcal{TM}(H)$ to be the set of graphs containing H as a topological minor. Unlike graph minors, the Erdős-Pósa property for $\mathcal{TM}(H)$ is not equivalent with the planarity of H. The author, Postle and Wollan [12] provided a characterization of graphs H in which $\mathcal{TM}(H)$ has the Erdős-Pósa property and proved that it is NP-hard to decide whether $\mathcal{TM}(H)$ has the Erdős-Pósa property for the input graph H.

The topological minor relation can be equivalently defined as follows. A graph H with no isolated vertices is a topological minor of another graph G if there exist an injection π_V from $V(H)$ to $V(G)$ and a function π_E that maps the edges e, say with ends u, v, of H to paths in G from $\pi_V(u)$ to $\pi_V(v)$ (if e is a loop with the end v, then $\pi_E(e)$ is a cycle in G containing $\pi_V(v)$) such that $\pi_E(e_1)$ and $\pi_E(e_2)$ are internally disjoint for distinct edges e_1, e_2 of H. Note that we consider a loop as a cycle as well.

We say that a graph H (allowing isolated vertices) is an *immersion* of a graph G if the mentioned internally disjoint property is replaced by the edge-disjoint property. Formally, an H-*immersion* in G is a pair of functions $\Pi = (\pi_V, \pi_E)$ such that the following hold.

- π_V is an injection from $V(H)$ to $V(G)$.
- π_E maps $E(H)$ to the set of subgraphs of G such that for every edge e of H, if e has distinct ends x, y, then $\pi_E(e)$ is a path with ends $\pi_V(x)$ and $\pi_V(y)$, and if e is the loop with end v, then $\pi_E(e)$ is a cycle containing $\pi_V(v)$.
- If e_1, e_2 are distinct edges of H, then $\pi_E(e_1)$ and $\pi_E(e_2)$ are edge-disjoint.

We denote the subgraph $\bigcup_{e\in E(H)} \pi_E(e) \cup \bigcup_{v\in V(H)} \pi_V(v)$ of G by $\Pi(H)$. We say that two H-immersions $\Pi = (\pi_V, \pi_E)$ and $\Pi' = (\pi'_V, \pi'_E)$ are *edge-disjoint* if $\bigcup_{W\in\pi_E(E(H))} E(W)$ is disjoint from $\bigcup_{W\in\pi'_E(E(H))} E(W)$. (In this paper, for any function f and any subset X of its domain, we define $f(X)$ to be the set $\{f(x) : x \in X\}$.) Equivalently, Π and Π' are edge-disjoint if and only if $\Pi(H)$ and $\Pi'(H)$ are edge-disjoint subgraphs of G.

As immersions consist of edge-disjoint paths, it is reasonable to ask for packing edge-disjoint copies of immersions instead of disjoint copies. Furthermore, one vertex can meet more than one edge-disjoint immersion, so it is more natural to cover these edge-disjoint subgraphs by edges instead of by vertices. This motivates an edge-variant of the Erdős-Pósa property.

We say that a set $\mathcal{F}$ of graphs has the *edge-variant of the Erdős-Pósa property* if for every integer k, there exists an integer $f(k)$ such that for every graph G, either G contains k edge-disjoint subgraphs each isomorphic to a member of $\mathcal{F}$, or there exists $Z \subseteq E(G)$ with $|Z| \leq f(k)$ such that $G - Z$ has no subgraph isomorphic to a member of $\mathcal{F}$. Raymond, Sau and Thilikos [14] proved that $\mathcal{M}(\theta_r)$ has the edge-variant of the Erdős-Pósa property, where θ_r is the loopless graph on two vertices with r edges.

For every graph H, define $\mathcal{I}(H)$ to be the set of graphs containing H as an immersion. $\mathcal{I}(H)$ does not have the edge-variant of the Erdős-Pósa property for every graph H. The necessary conditions for graphs H for which $\mathcal{TM}(H)$ has the Erdős-Pósa property mentioned in [12] are necessary for graphs H for which $\mathcal{I}(H)$ has the edge-variant of the Erdős-Pósa property. On the other hand, even though a family of graphs does not have the edge-variant of the Erdős-Pósa property, this family possibly has this property if we restrict the host graphs to be members of a smaller class of graphs. For example, the set of odd cycles does not have the edge-variant of the Erdős-Pósa property, but Kawarabayashi and Kobayashi [9] proved that it has the edge-variant of the Erdős-Pósa property in 4-edge-connected graphs. We address the same direction in this paper and prove that for every graph H, $\mathcal{I}(H)$ has the edge-variant of the Erdős-Pósa property in 4-edge-connected graphs. In fact, we prove the following theorem that is slightly stronger than the previous statement.

Theorem 1.1. *For every graph H, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G whose every component is 4-edge-connected and for every positive integer k, either G contains k edge-disjoint subgraphs each containing H as an immersion, or there exists $Z \subseteq E(G)$ with $|Z| \leq f(k)$ such that $G - Z$ has no H-immersion.*

Recall that the necessary conditions for which $\mathcal{TM}(H)$ has the Erdős-Pósa property mentioned in [12] provide necessary conditions for which $\mathcal{I}(H)$ has the edge-variant of the Erdős-Pósa property. The constructions in [12] show that the 4-edge-connectivity cannot be replaced by the 3-edge-connectivity. In addition, Kakimura and Kawarabayashi [8] proved that Theorem 1.1 is true if G is 4-edge-connected and H is a complete graph; they also provided some example showing that 3-edge-connectivity is not enough. On the other hand, Giannopoulou, Kwon, Raymond and Thilikos [7] proved that the requirement of being 4-edge-connected can be dropped if H is a loopless connected planar subcubic graph.

We remark that statements analogous to Theorem 1.1 with respect to minors and topological minors are not true. In other words, there does not exist a constant c such that for every graph H, $\mathcal{M}(H)$ and $\mathcal{TM}(H)$ have the Erdős-Pósa property even if the host graphs are c-connected. Let H be a graph such that $\mathcal{M}(H)$ (or $\mathcal{TM}(H)$, respectively) does not have the Erdős-Pósa property. So there exists a positive integer k such that for every positive integer N, there exists a graph G_N that does not contain k disjoint subgraphs where each of them is a member of $\mathcal{M}(H)$ (or $\mathcal{TM}(H)$, respectively) such that there exists no $Z \subseteq V(G)$ with $|Z| \leq N$ hitting all such subgraphs. For positive integers N, c, let $G_{c,N}$ be the graph obtained from G_N by adding c new vertices and adding edges from these c vertices to all vertices in G_N. Then for any positive integers c, N, the graph $G_{c,N}$ is c-connected but does not contain $k + c$ disjoint subgraphs where each of them is a member of $\mathcal{M}(H)$ (or $\mathcal{TM}(H)$, respectively), and there exists no hitting set in $G_{c,N}$ with size at most N. So there is no absolute constant c that would ensure that $\mathcal{M}(H)$ and $\mathcal{TM}(H)$ have the Erdős-Pósa property in c-connected graphs.

In fact, we prove a stronger version of Theorem 1.1 (see Theorem 7.5). Theorem 7.5 states that the 4-edge-connectivity can be replaced by the condition of having no edge-cut of order exactly three. (In fact, Theorem 7.5 is even slightly stronger than this.) Note that every Eulerian graph has no edge-cut of order three, so Theorem 7.5 implies that the edge-variant of the Erdős-Pósa property holds if the host graphs are Eulerian graphs. Recall that Kakimura and Kawarabayashi [8] proved that Theorem 1.1 is true if G is 4-edge-connected and H is a complete graph; we remark that what they actually proved is stronger: the 4-edge-connectivity of G can be replaced by the condition that no "minimal edge-cut" has size three. Our Theorem 7.5 also implies the stronger setting in [8]. See the remark after the proof of Theorem 7.5 for the details.

We also consider the following version of the half-integral packing problem in this paper. For every loopless graph H, an *H-half-integral immersion* in G is a pair of functions (π_V, π_E) such that the following hold.

- π_V is an injection from $V(H)$ to $V(G)$.
- π_E maps every edge e with ends u, v of G to a path in G from $\pi_V(u)$ to $\pi_V(v)$.
- For every edge e of G, there exist at most two edges e_1, e_2 of H such that $e \in \pi_E(e_1)$ and $e \in \pi_E(e_2)$.

The following theorem shows that the 4-edge-connectivity can be dropped if we consider the following version of half-integral packing of half-integral immersions.

Theorem 1.2. *For every loopless graph H, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G and for every positive integer k, either*

1. *G contains k H-half-integral immersions $(\pi_V^{(1)}, \pi_E^{(1)}), ..., (\pi_V^{(k)}, \pi_E^{(k)})$ such that for every edge e of G, there exist at most two pairs (i, e') with $1 \leq i \leq k$ and $e' \in E(H)$ such that $e \in \pi_E^{(i)}(e')$, or*
2. *there exists $Z \subseteq E(G)$ with $|Z| \leq f(k)$ such that $G - Z$ has no H-half-integral immersion.*

We remark that a result about half-integral packing of topological minors was proved by the author [11], but the notion of the half-integral packing in [11] is different from the one in above theorem.

More recent developments about the Erdős-Pósa property can be found in a survey of Raymond and Thilikos [15].

1.1. Overview of this paper

Now we roughly sketch the proof of Theorem 1.1 and describe the organization of this paper. More detailed sketches of proofs will be included in later sections when we are ready to prove them. We need following ingredients for the proof of Theorem 1.1.

1.1.1. Ingredient 1: edge-tangles

Tangle is one important notion in Robertson and Seymour's Graph Minors series. It defines a "consistent orientation" for each separation of small order in a graph and has been proven to be useful in dealing with problems related to the Erdős-Pósa property. If one can delete a small number of vertices from a graph such that the packing number of every component of the remaining graph is smaller than packing number of the original graph, then one can obtain a hitting set of small size by an easy inductive argument. So we can assume that there must exist a component whose packing number is not smaller than the packing number of the original graph. Note that such a component is unique as we cannot have two components whose packing number are not smaller than the packing number of the original graph. Hence, as long as we delete a small number of vertices, we know which component of the graph obtained by vertex-deletion is the "most important". Given a separation of a graph, by simply seeing which side has this "important" component, we obtain an orientation for separations to define a tangle. As we address the edge-variant of the Erdős-Pósa property in this paper, we develop a similar machinery called "edge-tangles" which gives an orientation for each edge-cut of small order in Section 2. This section is a preparation of many results of this paper and includes formal definitions of tangles and edge-tangles.

The notion of edge-tangles is natural but its explicit form seems unnoticed by the community until the first version of this paper was submitted for publication. We remark that Diestel and Oum [4,5] extended the ideas of "consistent orientations" for separations of graphs to a much more general setting for "abstract separation systems" to prove a general strong duality theorem. After a version of this paper was submitted for publication, it was pointed out by Reinhard Diestel (via private communication with the author) that the concept of edge-tangles defined in this paper coincides with a special case of their abstract separation systems when graphs are loopless. In particular, applying their duality theorem for abstract separation systems in [4] to edge-tangles, they [5, Section 5.2] noted that edge-tangles are dual to low "carving width" as defined by Seymour and Thomas [21]. The existence of such a duality theorem for edge-tangles was independently asked by Robin Thomas via private communication with the author around 2013 when the author introduced the notion of edge-tangles to develop a structure theorem for excluding immersions. In addition, Diestel, Hundertmark and Lemanczyk [3] applied their general work to edge-tangles to derive the classical Gomory-Hu tree theorem. Besides those developments for abstract separation systems, in this paper we consider different aspects for edge-tangles, mainly on developing structure theorems with respect to immersions and edge-tangles and its application to Erdős-Pósa type problems. We omit the details and formal definitions of the terms mentioned in this paragraph, as this paper does not rely on them.

In Section 2, we develop basic theory related to edge-tangles. In particular, in Sections 2.1 and 2.2, we show some basic properties for edge-tangles and build a relationship between edge-tangles in a graph G and tangles in the line graph of G. Those results will be used in later sections of this paper.

It is known that if a graph G contains a graph H as a minor, then tangles in H "induce" tangles in G; and every tangle of large order has a "subtangle" induced by a large grid minor. We show analogous results in Section 2.3: if a graph G contains a graph H as an immersion, then edge-tangles in H "induce" edge-tangles in G (Lemma 2.10); every edge-tangle of large order has a "sub-edge-tangle" determined by a large degree vertex or induced by a large wall immersion (Lemma 2.16).

We prove other lemmas about edge-tangles in Section 2.4. Those lemmas will be used in later sections.

1.1.2. Ingredient 2: a structure theorem for excluding immersions

The next step to prove Theorem 1.1 is to study the structure of minimum counterexamples to Theorem 1.1. If G is a graph that does not contain k edge-disjoint H-immersions, then G does not contain an H'-immersion for some larger graph H'.

So we shall prove a structure theorem for excluding a fixed graph as an immersion. Such a structure theorem was developed by the author in an unpublished manuscript in 2013. In this paper we include a proof of part of this theorem which is sufficient for proving Theorem 1.1. More specifically, we will prove in this paper a structure theorem (Theorem 4.6) for graphs that forbids a fixed graph as an immersion with respect to an edge-tangle that "grasps" a large set of pairwise edge-disjoint but pairwise intersecting subgraphs (called "thorns"). The formal definition of thorns is included in Section 4.

Roughly speaking, Theorem 4.6 states that if an H-immersion free graph G has an edge-tangle of large order grasping a large thorns, then we can sweep all except few vertices into the non-important side of edge-cuts in this edge-tangle so that the "resulting graph" is "simpler" than H in terms of the supply of large degree vertices. To prove Theorem 4.6, we first prove strengthenings of some Menger-type results of Robertson and Seymour [19] and Marx and Wollan [13] in Section 3, and then we complete the proof of this structure theorem in Section 4 by using results proved in Sections 2 and 3.

We remark that a structure theorem about excluding a fixed graph as an immersion was proved by Wollan [22]. But it seems to us that Wollan's structure theorem is not sufficiently informative to be applied in this paper. In addition, the structure theorem proved in this paper is a "local version" and is a critical component of the proof of a "global version" of an excluding immersion structure theorem proved in a later paper of the author [10]. The global version has other applications, see [10] for more details.

1.1.3. Ingredient 3: 4-edge-connectivity and thorns

Then, in Section 6 we will show that the 4-edge-connectivity of "sufficiently large" graphs can ensure that every edge-tangle "grasps" such a large thorns and hence the aforementioned structure theorem can be applied to minimum counterexamples to Theorem 1.1. This is achieved by Theorem 6.4. Recall that Lemma 2.16 shows that every edge-tangle of large order has a sub-edge-tangle determined by a large degree vertex or induced by a large wall immersion. Both a large degree vertex and a large grid immersion define a large thorns. So the remaining key step in proving Theorem 6.4 is to show that in any 4-edge-connected graph, one can obtain a large grid immersion from a large wall immersion.

1.1.4. Ingredient 4: preserving edge-connectivity

Finally, we prove Theorems 1.1 and 1.2 in Section 7.

Recall that if one can find an edge-cut of small order such that the subgraph induced on each side has smaller packing number for H-immersions, then one can apply induction on the packing number to obtain a small hitting set of each of these two subgraphs, and one can obtain a hitting set of the whole graph by further collecting the edges between the two sides of the edge-cut. Hence in the minimum counterexample to Theorem 1.1, no edge-cut of small order has this property. In particular, at most one side can contain an H-immersion. Furthermore, when H is connected, at least one side must contain an H-immersion, for otherwise the edges between the two sides is a small hitting set.

Hence we obtain an orientation of each edge-cut of small order by indicating the side having an H-immersion is important, so we obtain an edge-tangle. Our structure theorem (Theorem 4.6) ensures that for any H-immersion, one can find an edge-cut of small order such that the non-important side contains part of this H-immersion. By repeatedly applying machinery developed in Section 3, we can repeatedly enlarge the portion of this H-immersion contained in the non-important side until the entire H-immersion is contained in the non-important side, which is a contradiction by the definition of the edge-tangle. This is the purpose of Section 5 and is achieved by Lemma 5.6.

In fact, careful readers might notice that there are issues with the above strategy connected to induction on the packing number and defining an edge-tangle. One concern is that we require H to be connected. This concern can be solved relatively easily by induction on the number of components of H, and Lemma 5.6 actually already takes care of it. The other concern is more substantial: we tried to apply induction hypothesis to the subgraphs induced by each side of an edge-cut. Note that such a subgraph is not necessarily 4-edge-connected, so we cannot apply induction to it. The key strategy is to apply induction to the graph obtained from contracting one side of the edge-cut. Such a graph preserves the 4-edge-connectivity, but the packing number is not necessarily smaller than the original graph even though the other side of the edge-cut contains an H-immersion. To solve this issue, we actually prove a stronger version of Theorem 1.1 by allowing some vertices having labels, where the label of each vertex can be roughly considered the number of H-immersions that this vertex represents. More details are included in Section 7.

Section 7 formally proves this stronger setting of Theorem 1.1 and solves the aforementioned concern about preserving edge-connectivity. Theorem 7.5 is the strongest version in this paper and it implies Theorem 1.1. We remark that we do not require the edge-connectivity in Section 5 and Lemma 5.6, as they only require the edge-tangles to grasp a thorns. Moreover, Theorem 1.2 is a simple corollary of this stronger version of Theorem 1.1, and its proof is included in Section 7.

1.2. Notations

We define some notations to conclude this section. Given a subset X of the vertex-set $V(G)$ of a graph G, the subgraph of G induced by X is denoted by $G[X]$, and the set of vertices that are not in X but adjacent to some vertices in X is denoted by $N_G(X)$. When $X = \{v\}$, we write $N_G(\{v\})$ as $N_G(v)$ for simplicity. We define $N_G[X] = N_G(X) \cup X$ and $N_G[v] = N_G(v) \cup \{v\}$. A graph is *simple* if it does not contain parallel edges and loops. The *line graph* of a graph G, denoted by $L(G)$, is the simple graph with $V(L(G)) = E(G)$, and every pair of vertices $x, y \in V(L(G))$ are adjacent in $L(G)$ if and only if x, y are two edges having a common end in G. For every $v \in V(G)$, define $\mathrm{cl}(v)$ to be the clique in $L(G)$ consisting of the edges of G incident with v. The *degree* of a vertex v in a graph G, denoted by $\deg_G(v)$, is the number of edges of G incident with v, where each loop is counted twice. A vertex of G is an *isolated vertex* in G if it is not incident with any edge. Note that a vertex is non-isolated even if all edges incident with it are loops. If G is a graph and $Y \subseteq V(G)$, then $G - Y$ is the graph $G[V(G) - Y]$; if $Y \subseteq E(G)$, then $G - Y$ is the graph with $V(G - Y) = V(G)$ and $E(G - Y) = E(G) - Y$. For a positive integer k, a graph G is *k-edge-connected* if G contains at least two vertices and $G - F$ is connected for every $F \subseteq E(G)$ with $|F| < k$. For every positive integer n, we denote the set $\{1, 2, ..., n\}$ by $[n]$ for short.

2. Tangles and edge-tangles

2.1. Edge-cuts and separations of line graphs

A *separation* of a graph G is an ordered pair (A, B) of edge-disjoint subgraphs of G with $A \cup B = G$, and the *order* of (A, B) is $|V(A) \cap V(B)|$.

A separation (A, B) of G is *normalized* if every vertex $v \in V(A) \cap V(B)$ is adjacent to a vertex of $A - V(B)$ and adjacent to a vertex in $B - V(A)$. The *normalization* of a separation (A, B) of a graph G is the separation (A^*, B^*) of G defined as follows.

- Let S_1 be the set of all non-isolated vertices v in G contained in $V(A) \cap V(B)$ with $N_A(v) \subseteq V(B)$. Let A' be the graph $A - S_1$ and let B' be the subgraph of G such that (A', B') is a separation of G with $V(A') \cap V(B') = V(A) \cap V(B) - S_1$. In other words, (A', B') is obtained from (A, B) by removing all non-isolated vertices $v \in V(A) \cap V(B)$ of G with $N_A(v) \subseteq V(B)$ from A and putting all edges of G incident with v into B. Note that $V(B') = V(B)$.
- Let S_2 be the set of all edges in B' whose every end is in $V(A') \cap V(B')$. Let $A'' = A' \cup S_2$ and $B'' = B' - S_2$. Note that (A'', B'') is a separation of G, and every loop of G incident with some vertex in $V(A'') \cap V(B'')$ belongs to A''.
- Let S_3 be the set of all isolated vertices of B''. Note that S_3 consists of the isolated vertices of G contained in $V(B'')$ and some vertices contained in $V(A'') \cap V(B'')$ that are not adjacent to any vertex in $V(B'') - V(A'')$ by the definition of A''. Define A^* to be the graph obtained from A'' by adding $S_3 - V(A'')$, and define $B^* = B'' - S_3$. That is, we remove all isolated vertices of B'' from B'' and put them into A''.

The following lemma shows some basic properties of the normalization of a separation and will be used in the rest of the section.

Lemma 2.1. *Let G be a graph and (A, B) a separation of G. If (A^*, B^*) is the normalization of (A, B), then the following hold.*

1. *(A^*, B^*) is normalized.*
2. *The order of (A^*, B^*) is at most the order of (A, B).*
3. *If $e \in E(B) - E(B^*)$, then every end of e belongs to $V(A) \cap V(B)$.*

4. *If $e \in E(B^*) - E(B)$, then e is incident with some vertex v in $V(A) \cap V(B)$ with $N_A(v) \subseteq V(B)$.*

Proof. Let G be a graph, (A, B) a separation of G, and (A^*, B^*) the normalization of (A, B). Let S_1, S_2, S_3 be the sets and let $(A', B'), (A'', B'')$ be the separations mentioned in the definition of (A^*, B^*), respectively. It is clear that $V(A^*) \cap V(B^*) \subseteq V(A'') \cap V(B'') = V(A') \cap V(B') = V(A) \cap V(B) - S_1$, and $V(A) - V(B) \subseteq V(A^*) - V(B^*)$.

We first prove Statement 1. Let $v \in V(A^*) \cap V(B^*)$. So v is a non-isolated vertex of G and $v \in V(A) \cap V(B) - S_1$. Hence $N_A(v) \not\subseteq V(B)$. That is, v is adjacent to some vertex in $V(A) - V(B) \subseteq V(A^*) - V(B^*)$. Since $v \in V(B^*)$, v is not an isolated vertex of B''. So v is adjacent in B'' to some vertex $u \in V(B'')$. Note that u is not an isolated vertex of B''. Hence $u \in V(B'') - S_3 = V(B^*)$. Since $v \in V(A^*) \cap V(B^*) \subseteq V(A') \cap V(B')$, $u \notin V(A') \cap V(B')$, for otherwise every edge incident with both u, v belongs to S_2 and u is not adjacent to v in B''. Since $V(A'') \cap V(B'') = V(A') \cap V(B')$, $u \in V(B'') - V(A'')$. Since $u \notin S_3$, $u \in V(B^*) - V(A^*)$. So v is adjacent to a vertex in $V(B^*) - V(A^*)$. This shows that (A^*, B^*) is normalized.

Statement 2 immediately follows from the fact that $V(A^*) \cap V(B^*) \subseteq V(A) \cap V(B) - S_1$.

Now we prove Statement 3. Let $e \in E(B) - E(B^*)$. Since $e \in E(B)$, $e \notin E(A)$. So $e \notin E(A')$ and hence $e \in E(B')$. Since every vertex in S_3 is an isolated vertex in B'' and $e \notin E(B^*)$, $e \notin E(B'')$. So $e \in E(B') - E(B'') \subseteq S_2$. Hence every end of e belongs to $V(A') \cap V(B') \subseteq V(A) \cap V(B)$. This shows Statement 3.

Finally, we prove Statement 4. Let $e \in E(B^*) - E(B)$. Since every vertex in S_3 is an isolated vertex of B'', $e \in E(B^*) = E(B'') \subseteq E(B')$. So $e \notin E(A')$. Since $e \notin E(B)$, $e \in E(A)$. Hence e is incident with some vertex w in S_1. But every vertex in S_1 satisfies that $w \in V(A) \cap V(B)$ and $N_A(w) \subseteq V(B)$. This completes the proof. □

An *edge-cut* of a graph G is an ordered partition $[A, B]$ of $V(G)$, where some of A and B is allowed to be empty. The *order* of an edge-cut $[A, B]$, denoted by $|[A, B]|$, is the number of edges with one end in A and one end in B. For an edge e of G, we write $e \in [A, B]$ if e has one end in A and one end in B.

The *partner* of a normalized separation (A, B) of the line graph $L(G)$ of G is the edge-cut $[A', B']$ of G satisfying that A' is the union of the set of isolated vertices of G and the set $\{v \in V(G) : \mathrm{cl}(v) \subseteq V(A)\}$, and $B' = \{v \in V(G) : \mathrm{cl}(v) \subseteq V(B), \mathrm{cl}(v) \neq \emptyset\}$.

Lemma 2.2. *Let G be a graph, and let (A, B) be a separation of $L(G)$. If (A, B) is normalized, then the partner $[A', B']$ of (A, B) is a well-defined edge-cut of G, and the order of (A, B) equals the order of $[A', B']$.*

Proof. We first show that the partner $[A', B']$ of (A, B) is a well-defined edge-cut of G. That is, $A' \cup B' = V(G)$ and $A' \cap B' = \emptyset$. Let $v \in V(G)$. If v is an isolated vertex of G, then $v \in A'$; otherwise, $\mathrm{cl}(v)$ is a non-empty clique, so $\mathrm{cl}(v) \subseteq V(A)$ or $\mathrm{cl}(v) \subseteq V(B)$, and hence $v \in A' \cup B'$. So $A' \cup B' = V(G)$. Suppose to the contrary that $v \in A' \cap B'$. Then $\text{cl}(v) \subseteq V(A) \cap V(B)$ and $\text{cl}(v) \neq \emptyset$. So there exist $e_0 \in \text{cl}(v)$ and a set $X \subseteq V(G) - \{v\}$ with $|X| \leq 1$ such that $N_{L(G)}(e_0) \subseteq \text{cl}(v) \cup \bigcup_{u \in X} \text{cl}(u)$. But since (A, B) is normalized and $e_0 \in \text{cl}(v) \subseteq V(A) \cap V(B)$, e_0 is adjacent in $L(G)$ to a vertex in $A - V(B)$ and a vertex in $B - V(A)$. Since $\text{cl}(v) \subseteq V(A) \cap V(B)$, $\bigcup_{u \in X} \text{cl}(u)$ intersects both $V(A) - V(B)$ and $V(B) - V(A)$. But it is impossible since $|X| \leq 1$, a contradiction. This shows that $[A', B']$ is an edge-cut of G.

Now we show that the order of (A, B) equals the order of $[A', B']$. Let $e \in [A', B']$ with ends u, v, where $u \in A'$ and $v \in B'$. So $\text{cl}(u) \subseteq V(A)$ and $\text{cl}(v) \subseteq V(B)$. Hence, $e \in \text{cl}(u) \cap \text{cl}(v) \subseteq V(A) \cap V(B)$. This implies that the order of (A, B) is at least the order of $[A', B']$.

On the other hand, let $e \in V(A) \cap V(B)$. Since (A, B) is normalized, e is adjacent to a vertex e_A of $L(G)$ in $V(A) - V(B)$ and a vertex e_B of $L(G)$ in $V(B) - V(A)$. So e and e_A have a common end x in G, and e and e_B have a common end y of G. Since $e_A \notin V(B)$, $\text{cl}(x) \subseteq V(A)$ and hence $x \in A'$. Similarly, $\text{cl}(y) \subseteq V(B)$ and $y \in B'$. So $x \neq y$ and they are the ends of e. This proves that $e \in [A', B']$ and the order of (A, B) is at most the order of $[A', B']$. □

2.2. Basic properties of edge-tangles

Let θ be an integer. A *tangle* $\mathcal{T}$ in a graph G of *order* θ is a set of separations of G of order less than θ such that

(T1) for every separation (A, B) of G of order less than θ, either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$;
(T2) if $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$, then $A_1 \cup A_2 \cup A_3 \neq G$;
(T3) if $(A, B) \in \mathcal{T}$, then $V(A) \neq V(G)$.

The notion of tangles was first defined by Robertson and Seymour in [17]. We call (T1), (T2) and (T3) the *first*, *second* and *third tangle axioms*, respectively. Note that (T2) implies that if $(A, B) \in \mathcal{T}$, then $(B, A) \notin \mathcal{T}$.

An *edge-tangle* $\mathcal{E}$ in a graph G of *order* θ is a set of edge-cuts of G of order less than θ such that the following hold.

(E1) For every edge-cut $[A, B]$ of G of order less than θ, either $[A, B] \in \mathcal{E}$ or $[B, A] \in \mathcal{E}$;
(E2) If $[A_1, B_1], [A_2, B_2], [A_3, B_3] \in \mathcal{E}$, then $B_1 \cap B_2 \cap B_3 \neq \emptyset$.
(E3) If $[A, B] \in \mathcal{E}$, then G has at least θ edges incident with vertices in B.

We call (E1), (E2) and (E3) the *first*, *second* and *third edge-tangle axioms*, respectively. Note that if an edge-tangle $\mathcal{E}$ of order $\theta \geq 1$ in G exists, then $[\emptyset, V(G)] \in \mathcal{E}$ by (E1) and (E2), so $|E(G)| \geq \theta$ by (E3). Furthermore, for every $[A, B] \in \mathcal{E}$, there exists an edge of G whose every end is in B by (E3).

The following lemma is simple but useful. It shows that the "orientation" of the edge-cuts given by an edge-tangle is "consistent".

Lemma 2.3. *Let θ be a positive integer. Let G be a graph and $\mathcal{E}$ an edge-tangle of order θ in G. If $[A, B], [C, D] \in \mathcal{E}$, then the following hold.*

1. *If the order of $[A \cup C, B \cap D]$ is less than θ, then $[A \cup C, B \cap D] \in \mathcal{E}$.*
2. *If $A' \subseteq A$ and $[A', V(G)-A']$ is an edge-cut of G of order less than θ, then $[A', V(G)-A'] \in \mathcal{E}$.*

Proof. We first prove Statement 1. Assume that $[A \cup C, B \cap D]$ has order less than θ. By (E1), either $[A \cup C, B \cap D] \in \mathcal{E}$ or $[B \cap D, A \cup C] \in \mathcal{E}$. Since $[A, B], [C, D] \in \mathcal{E}$ and $B \cap D \cap (A \cup C) = \emptyset$, $[B \cap D, A \cup C] \notin \mathcal{E}$ by (E2). So $[A \cup C, B \cap D] \in \mathcal{E}$. This shows Statement 1.

Now we prove Statement 2. Let $A' \subseteq A$, and assume that $[A', V(G) - A']$ is an edge-cut of G of order less than θ. By (E1), either $[A', V(G) - A'] \in \mathcal{E}$ or $[V(G) - A', A'] \in \mathcal{E}$. Since $[A, B] \in \mathcal{E}$ and $B \cap A' = \emptyset$, $[V(G) - A', A'] \notin \mathcal{E}$ by (E2). So $[A', V(G) - A'] \in \mathcal{E}$. This shows Statement 2. □

The following lemma shows that the vertex-set of any component with at most one edge can be moved to either side of an edge-cut without flipping the "orientation" given by an edge-tangle.

Lemma 2.4. *Let θ be an integer with $\theta \geq 2$ and let G be a graph. Then the following hold.*

1. *Let S be the vertex-set of a component of G with at most one edge. If $[A, B] \in \mathcal{E}$, then $[A \cup S, B - S] \in \mathcal{E}$.*
2. *Let D be the union of the vertex-sets of the components of G with at most one edge. If $[A, B] \in \mathcal{E}$, then $[A \cup D, B - D] \in \mathcal{E}$.*

Proof. We first prove Statement 1. Suppose to the contrary that $[A \cup S, B - S] \notin \mathcal{E}$. Note that every edge with one end in $A \cup S$ and one end in $B - S$ is an edge with one end in A and one end in B. Hence the order of $[A \cup S, B - S]$ is at most the order of $[A, B]$. By (E1), $[B - S, A \cup S] \in \mathcal{E}$. The edge-cut $[S, V(G) - S]$ has order 0, so either $[S, V(G) - S] \in \mathcal{E}$ or $[V(G) - S, S] \in \mathcal{E}$ by (E1). Since $\theta \geq 2$ and there exists at most one edge incident with S, (E3) implies that $[V(G) - S, S] \notin \mathcal{E}$. So $[S, V(G) - S] \in \mathcal{E}$. Hence $[A, B], [B-S, A \cup S], [S, V(G)-S]$ are edge-cuts in $\mathcal{E}$ such that $B \cap (A \cup S) \cap V(G) - S = \emptyset$, contradicting (E2). This proves Statement 1.

Now we prove Statement 2. Let $S_1, S_2, ..., S_k$ be the subsets of $V(G)$ such that each S_i is the vertex-set of some component of G with at most one edge. So $D = \bigcup_{i=1}^k S_i$. For every $j \in [k]$, let $D_j = \bigcup_{i=1}^j S_i$. We shall prove that $[A \cup D_j, B - D_j] \in \mathcal{E}$ for $j \in [k]$ by induction on j. The case $j = 1$ immediately follows from Statement 1 of this lemma. So we may assume that $j \geq 2$ and $[A \cup D_{j-1}, B - D_{j-1}] \in \mathcal{E}$. Applying Statement 1 of this lemma by taking $[A, B] = [A \cup D_{j-1}, B - D_{j-1}]$ and $S = S_j$, we know that $[A \cup D_j, B - D_j] = [(A \cup D_{j-1}) \cup S_j, (B - D_{j-1}) - S_j] \in \mathcal{E}$. This shows that $[A \cup D_j, B - D_j] \in \mathcal{E}$ for every $j \in [k]$. Hence $[A \cup D, B - D] = [A \cup D_k, B - D_k] \in \mathcal{E}$. This proves the lemma. □

The next step is to build a relationship between edge-tangles in G and tangles in its line graph $L(G)$.

Given an edge-tangle $\mathcal{E}$ of order θ in G, the *conjugate* $\overline{\mathcal{E}}$ of $\mathcal{E}$ is the set of separations of $L(G)$ of order less than $\lceil \theta/3 \rceil$ such that $(A, B) \in \overline{\mathcal{E}}$ if and only if the partner of the normalization of (A, B) is in $\mathcal{E}$.

One reason for considering separations of $L(G)$ of order less than $\lceil \theta/3 \rceil$ only instead of considering separations of $L(G)$ of order less than θ is due to a technicality in the proof of the following lemma which shows a relationship between $\mathcal{E}$ and $\overline{\mathcal{E}}$.

Lemma 2.5. *Let θ be an integer with $\theta \geq 2$ and G a graph. If $\mathcal{E}$ is an edge-tangle of order $3\theta - 2$ of G, then $\overline{\mathcal{E}}$ is a tangle of order θ in $L(G)$.*

Proof. Observe that every member of $\overline{\mathcal{E}}$ has order less than $\lceil \frac{3\theta-2}{3} \rceil = \theta$. We shall prove that $\overline{\mathcal{E}}$ satisfies tangle axioms (T1), (T2) and (T3). Note that $|E(G)| \geq 3\theta - 2$ since G has an edge-tangle of order $3\theta - 2$.

Claim 1: $\overline{\mathcal{E}}$ *satisfies (T1).*

Proof of Claim 1: Let (A, B) be a separation of $L(G)$ of order less than θ. Let (A_1, B_1) and (B_2, A_2) be the normalizations of (A, B) and (B, A), respectively. And let $[A_1', B_1']$ and $[B_2', A_2']$ be the partners of (A_1, B_1) and (B_2, A_2), respectively. If any of $[A_1', B_1']$ and $[B_2', A_2']$ is in $\mathcal{E}$, then (A, B) or (B, A) is in $\overline{\mathcal{E}}$, and we are done. So we may assume that none of $[A_1', B_1']$ and $[B_2', A_2']$ is in $\mathcal{E}$. By Lemmas 2.1 and 2.2, the order of $[A_1', B_1']$ and $[B_2', A_2']$ are less than θ, so $[B_1', A_1']$ and $[A_2', B_2']$ are in $\mathcal{E}$ by (E1). Let D be the union of the vertex-sets of the components of G with at most one edge. Let $[B_1'', A_1''] = [B_1' \cup D, A_1' - D]$ and let $[A_2'', B_2''] = [A_2' \cup D, B_2' - D]$. By Statement 2 of Lemma 2.4, $[B_1'', A_1'']$ and $[A_2'', B_2'']$ are in $\mathcal{E}$.

Let $v \in A_1'' \cap B_2''$. So $v \notin D$ and hence v does not belong to any component of G with at most one edge. In particular, v is not an isolated vertex in G and $\mathrm{cl}(v) \neq \emptyset$. Note that $\mathrm{cl}(v)$ is not a set consisting of one isolated vertex in $L(G)$, for otherwise the vertex in $\mathrm{cl}(v)$ is the only edge of some component of G and $v \in D$. Since $v \in A_1'' \subseteq A_1'$, $\mathrm{cl}(v) \subseteq V(A_1)$ as v is not an isolated vertex in G. Similarly, $\mathrm{cl}(v) \subseteq V(B_2)$. So $\mathrm{cl}(v) \subseteq V(A_1) \cap V(B_2)$.

Suppose that $\mathrm{cl}(v) \not\subseteq V(A)$. Then some vertex in $\mathrm{cl}(v)$ is in $V(B) - V(A)$ and is not an isolated vertex of B, so this vertex is in $V(B_1) - V(A_1)$. Hence $\mathrm{cl}(v) - V(A_1) \neq \emptyset$, a contradiction.

So $\mathrm{cl}(v) \subseteq V(A)$. Similarly, $\mathrm{cl}(v) \subseteq V(B)$, for otherwise $\mathrm{cl}(v) - V(B_2) \neq \emptyset$. Hence $\mathrm{cl}(v) \subseteq V(A) \cap V(B)$.

This shows that $\bigcup_{u\in A_1''\cap B_2''} \mathrm{cl}(u) \subseteq V(A)\cap V(B)$. Therefore, the number of edges of G incident with some vertex in $A_1''\cap B_2''$ is at most $|\bigcup_{u\in A_1''\cap B_2''} \mathrm{cl}(u)| \leq |V(A)\cap V(B)| < \theta$. In particular, the number of edges with one end in $A_1''\cap B_2''$ and one end in $B_1''\cup A_2''$ is less than θ. Hence, (E1) implies that either $[A_1''\cap B_2'', B_1''\cup A_2''] \in \mathcal{E}$ or $[B_1''\cup A_2'', A_1''\cap B_2''] \in \mathcal{E}$. But (E3) excludes the latter case, so $[A_1''\cap B_2'', B_1''\cup A_2''] \in \mathcal{E}$. However, $[B_1'', A_1'']$, $[A_2'', B_2'']$ and $[A_1''\cap B_2'', B_1''\cup A_2'']$ belong to $\mathcal{E}$, but $A_1''\cap B_2''\cap (B_1''\cup A_2'') = \emptyset$, contradicting (E2). This proves that $\overline{\mathcal{E}}$ satisfies (T1). □

Next, we show that $\overline{\mathcal{E}}$ satisfies (T3). Suppose that $(A,B)\in\overline{\mathcal{E}}$ with $V(A) = V(L(G))$. So the partner $[A', B']$ of the normalization of (A,B) is in $\mathcal{E}$. Note that $\bigcup_{v\in B'} \mathrm{cl}(v) \subseteq V(B) = V(A)\cap V(B)$, so the number of edges incident with vertices in B' is at most $|V(A)\cap V(B)| < \theta \leq 3\theta - 2$, contradicting (E3). Consequently, $\overline{\mathcal{E}}$ satisfies (T3).

Now we suppose that $\overline{\mathcal{E}}$ does not satisfy (T2). So there exist separations (A_1, B_1), (A_2, B_2), (A_3, B_3) in $\overline{\mathcal{E}}$ such that $A_1\cup A_2\cup A_3 = L(G)$. For each $i\in[3]$, let (A_i^*, B_i^*) be the normalization of (A_i, B_i), and let $[A_i', B_i']$ be the partner of (A_i^*, B_i^*). By the definition of $\overline{\mathcal{E}}$, $[A_i', B_i']\in\mathcal{E}$ for $i\in[3]$. The number of edges of G incident with vertices in $\bigcap_{i=1}^3 B_i'$ is at most $|\bigcup_{v\in\bigcap_{i=1}^3 B_i'} \mathrm{cl}(v)| \leq |\bigcap_{i=1}^3 V(B_i^*)| \leq |\bigcap_{i=1}^3 V(B_i)|$. However, $\bigcap_{i=1}^3 V(B_i) \subseteq \bigcup_{i=1}^3 V(A_i\cap B_i)$, as $A_1\cup A_2\cup A_3 = L(G)$. So the number of edges of G incident with vertices in $\bigcap_{i=1}^3 B_i'$ is at most $|\bigcup_{i=1}^3 V(A_i\cap B_i)| \leq 3(\theta-1)$. In addition, $|[A_1'\cup A_2', B_1'\cap B_2']| \leq \sum_{i=1}^2 |[A_i', B_i']| \leq 2(\theta-1) < 3\theta-2$, so $[A_1'\cup A_2', B_1'\cap B_2']\in\mathcal{E}$ by Lemma 2.3. Similarly, $|[A_1'\cup A_2'\cup A_3', B_1'\cap B_2'\cap B_3']| \leq \sum_{i=1}^3 |[A_i', B_i']| \leq 3(\theta-1) < 3\theta-2$, so $[A_1'\cup A_2'\cup A_3', B_1'\cap B_2'\cap B_3']\in\mathcal{E}$ by Lemma 2.3. Hence by (E3), the number of edges of G incident with vertices in $\bigcap_{i=1}^3 B_i'$ is at least $3\theta-2$, a contradiction. This proves that $\overline{\mathcal{E}}$ satisfies (T3). Consequently, $\overline{\mathcal{E}}$ is a tangle of order θ in $L(G)$. □

Let G be a graph and $\mathcal{E}$ a collection of edge-cuts of G of order less than a positive number θ, and let $X\subseteq E(G)$. Define $\mathcal{E}-X$ to be the set of edge-cuts of $G-X$ of order less than $\theta-|X|$ such that $[A,B]\in\mathcal{E}-X$ if and only if $[A,B]\in\mathcal{E}$.

Lemma 2.6. *Let G be a graph and θ be a positive integer. If $\mathcal{E}$ is an edge-tangle in G of order θ and X is a subset of $E(G)$ with $|X|<\theta$, then $\mathcal{E}-X$ is an edge-tangle in $G-X$ of order $\theta-|X|$.*

Proof. If $[A,B]$ is an edge-cut of order less than $\theta-|X|$ in $G-X$, then $[A,B]$ is an edge-cut in G of order less than θ. So for every edge-cut $[A,B]$ of $G-X$ of order less than $\theta-|X|$, since $\mathcal{E}$ is an edge-tangle in G of order θ, either $[A,B]\in\mathcal{E}$ or $[B,A]\in\mathcal{E}$, and hence either $[A,B]\in\mathcal{E}-X$ or $[B,A]\in\mathcal{E}-X$ by the definition of $\mathcal{E}-X$. This shows that $\mathcal{E}-X$ satisfies (E1).

Since $\mathcal{E}$ satisfies (E2), $B_1\cap B_2\cap B_3\neq\emptyset$, for any edge-cuts $[A_1,B_1],[A_2,B_2],[A_3,B_3]\in\mathcal{E}$. So for any members $[A_1,B_1],[A_2,B_2],[A_3,B_3]$ of $\mathcal{E}-X$, we have $[A_1,B_1],[A_2,B_2]$, $[A_3.B_3]\in\mathcal{E}$ by the definition of $\mathcal{E}-X$, so $B_1\cap B_2\cap B_3\neq\emptyset$. Hence $\mathcal{E}-X$ satisfies (E2).

If $[A, B] \in \mathcal{E} - X$, then $[A, B] \in \mathcal{E}$, so G contains at least θ edges incident with vertices in B. Hence, $G - X$ contains at least $\theta - |X|$ edges incident with vertices in B. So $\mathcal{E} - X$ satisfies (E3). This proves that $\mathcal{E} - X$ is an edge-tangle of order $\theta - |X|$. $\square$

Let $\mathcal{E}$ be an edge-tangle in a graph G. We say that a subset Y of $E(G)$ is *free* with respect to $\mathcal{E}$ if there exist no $Z \subseteq Y$ and $[A, B] \in \mathcal{E} - Z$ of order less than $|Y - Z|$ such that every edge in $Y - Z$ has every end in A.

Lemma 2.7. *Let G be a graph and $\mathcal{E}$ an edge-tangle in G of order $\theta \geq 1$. Let Z be a subset of $E(G)$ with $|Z| < \theta$ and let X be a subset of $E(G) - Z$ such that X is free with respect to $\mathcal{E} - Z$. If $|X| \leq \theta - |Z|$, then for every $X' \subseteq X$ and $Z' \subseteq Z$, X' is free with respect to $\mathcal{E} - Z'$.*

Proof. Suppose to the contrary that X' is not free with respect to $\mathcal{E} - Z'$. Then there exist $W' \subseteq X'$ and $[A, B] \in (\mathcal{E} - Z') - W' = \mathcal{E} - (Z' \cup W')$ of order less than $|X' - W'|$ such that every edge in $X' - W'$ has every end in A. Let $W = W' \cup (X - X')$. So $X' - W' = X - W$. Since $Z' \subseteq Z$ and $W' \subseteq X'$, $[A, B]$ is an edge-cut of $G - Z$ of order less than $|X' - W'| + |W'| \leq |X'| \leq |X| \leq \theta - |Z|$. Since $\mathcal{E} - Z$ has order $\theta - |Z|$ by Lemma 2.6, by (E1), either $[A, B] \in \mathcal{E} - Z$ or $[B, A] \in \mathcal{E} - Z$.

Since $W \subseteq X$ and X is free with respect to $\mathcal{E} - Z$, either $[A, B] \notin (\mathcal{E} - Z) - W$ or the order of $[A, B]$ in $G - (Z \cup W)$ is at least $|X - W| = |X' - W'|$. Since $Z \cup W \supseteq Z' \cup W'$, the order of $[A, B]$ in $G - (Z \cup W)$ is at most the order of $[A, B]$ in $G - (Z' \cup W')$, which is less than $|X' - W'|$. So $[A, B] \notin (\mathcal{E} - Z) - W$. Hence $[A, B] \notin \mathcal{E} - Z$ by the definition of $(\mathcal{E} - Z) - W$. So $[B, A] \in \mathcal{E} - Z$. By the definition of $\mathcal{E} - Z$, $[B, A] \in \mathcal{E}$. Since $[A, B] \in \mathcal{E} - (Z' \cup W')$, the order of $[B, A]$ in $G - (Z' \cup W')$ is less than $\theta - |Z' \cup W'|$. Since $[B, A] \in \mathcal{E}$, $[B, A] \in \mathcal{E} - (Z' \cup W')$ by the definition of $\mathcal{E} - (Z' \cup W')$. So $[A, B], [B, A] \in \mathcal{E} - (Z' \cup W')$, contradicting (E2). $\square$

Let $\mathcal{T}$ be a tangle in a graph G. We say that a subset X of $V(G)$ is *free* with respect to $\mathcal{T}$ if there does not exist $(A, B) \in \mathcal{T}$ of order less than $|X|$ such that $X \subseteq V(A)$.

Note that for every graph G, $V(L(G)) = E(G)$. So for every subset of $E(G)$, we can consider whether it is free with respect to an edge-tangle $\mathcal{E}$ in G and whether it is free with respect to the conjugate $\overline{\mathcal{E}}$ of $\mathcal{E}$.

Lemma 2.8. *Let $\mathcal{E}$ be an edge-tangle in a graph G, and let $\overline{\mathcal{E}}$ be the conjugate of $\mathcal{E}$. Let X, Z be disjoint subsets of $E(G)$. If X is free with respect to $\mathcal{E} - Z$, then X is free with respect to $\overline{\mathcal{E}} - Z$.*

Proof. Suppose that X is not free with respect to $\overline{\mathcal{E}} - Z$. Then there exists a separation $(A, B) \in \overline{\mathcal{E}} - Z$ of $L(G) - Z$ of order less than $|X|$ such that $X \subseteq V(A)$. We may assume that the order of (A, B) is as small as possible, and subject to that, $V(B)$ is inclusion-wise minimal. So every vertex in $V(A) \cap V(B) - X$ is adjacent to a vertex in $V(A) - V(B)$ and adjacent to a vertex in $V(B) - V(A)$; every vertex in $V(A) \cap V(B) \cap X$

is adjacent to a vertex in $V(B) - V(A)$. Furthermore, B has no isolated vertices by the minimality of (A, B). Let (A', B') be a separation of $L(G)$ with $V(A') = V(A) \cup Z$ and $V(B') = V(B) \cup Z$. Since $(A, B) \in \overline{\mathcal{E}} - Z$, $(A', B') \in \overline{\mathcal{E}}$. Let (A^*, B^*) be the normalization of (A', B'). So $V(B^*) - Z = V(B') - Z$, $V(A^*) \cap V(B^*) - (X \cup Z) = V(A') \cap V(B') - (X \cup Z)$ and $|V(A^*) \cap V(B^*) - (X \cup Z)| = |V(A') \cap V(B')| - |V(A') \cap V(B') \cap (X \cup Z)|$.

Suppose that $(A^*, B^*) \notin \overline{\mathcal{E}}$. Then $(B^*, A^*) \in \overline{\mathcal{E}}$ by (T1). By (T1) and (T3), $(G[V(A') \cap V(B')], G - E(G[V(A') \cap V(B')])) \in \overline{\mathcal{E}}$. Since $V(B') - Z = V(B^*) - Z$, we know $(A', B'), (B^*, A^*), (G[V(A') \cap V(B')], G - E(G[V(A') \cap V(B')]))$ are members of $\overline{\mathcal{E}}$ such that $A' \cup B^* \cup G[V(A') \cap V(B')] = L(G)$, contradicting (T2). So $(A^*, B^*) \in \overline{\mathcal{E}}$.

Let $[C, D]$ be the partner of (A^*, B^*). So $[C, D] \in \mathcal{E}$. Let $W = (V(A') \cap V(B')) \cap (X \cup Z)$. Note that W is a subset of $V(L(G)) = E(G)$. Every edge e in $X - W$ has every end in C since it is a vertex in $V(A') - V(B')$. And the order of $[C, D]$ in $G - W$ equals $|V(A^*) \cap V(B^*) - (X \cup Z)| = |V(A') \cap V(B')| - |W| < |X| - |W - Z| \leq |X - (W \cap X)|$.

Let θ be the order of $\mathcal{E}$. So the order of $\overline{\mathcal{E}}$ is at most $\lceil \theta/3 \rceil$. Since $(A', B') \in \overline{\mathcal{E}}$, $|V(A') \cap V(B')| < \theta/3$. So the order of $[C, D]$ in $G - W$ is at most $|V(A') \cap V(B')| - |W| < \theta/3 - |W| < \theta - |W|$. Hence $[C, D] \in \mathcal{E} - W$.

Therefore, $[C, D] \in (\mathcal{E} - Z) - (W \cap X)$ is an edge-cut of $(G - Z) - (W \cap X)$ of order less than $|X - (X \cap W)|$ such that every edge in $X - (X \cap W)$ has every end in C. So X is not free with respect to $\mathcal{E} - Z$, a contradiction. □

The converse of Lemma 2.8 is also true when $Z = \emptyset$, subject to a requirement on the size of the set, as shown in the following lemma.

Lemma 2.9. *Let $\mathcal{E}$ be an edge-tangle in a graph G of order at least two, and let $\overline{\mathcal{E}}$ be the conjugate of $\mathcal{E}$. Let X be a subset of $E(G)$. Denote the order of $\overline{\mathcal{E}}$ by θ. If X is free with respect to $\overline{\mathcal{E}}$ and $|X| \leq \theta$, then X is free with respect to $\mathcal{E}$.*

Proof. Suppose to the contrary that X is not free with respect to $\mathcal{E}$. So there exist $W \subseteq X$ and $[A, B] \in \mathcal{E} - W$ of order less than $|X - W|$ such that every edge in $X - W$ has every end in A. We further assume that the order of $[A, B]$ is as small as possible, and subject to that, $|A|$ is as large as possible.

Claim 1: *The following hold.*

- *Every non-isolated vertex of G is incident with an edge of $G - W$ whose every end is contained in A or an edge of $G - W$ whose every end is contained in B.*
- *A contains the vertex-set of every component of G with at most one edge.*

Proof of Claim 1: The first statement of this claim immediately follows from the minimality of the order of $[A, B]$. Furthermore, the minimality of $|[A, B]|$, the maximality of A, and Statement 2 of Lemma 2.4 imply the second statement of this claim. □

Define B' to be the induced subgraph of $L(G-W)$ such that $V(B')$ is the set of edges of $G-W$ incident with vertices of B. Define (A', B') to be a separation of $L(G-W)$ such that $V(A') \cap V(B')$ is the set of edges of $G-W$ with one end in A and one end in B. Note that the order of (A', B') is at most the order of $[A, B]$ in $G-W$. So $|V(A') \cap V(B')| < |X-W|$.

Let (A^*, B^*) be a separation of $L(G)$ with $V(A^*) = V(A') \cup W$ and $V(B^*) = V(B') \cup W$. So $X \subseteq V(A^*)$ and $|V(A^*) \cap V(B^*)| < |X| \leq \theta$. By (T1), $(A^*, B^*) \in \overline{\mathcal{E}}$ or $(B^*, A^*) \in \overline{\mathcal{E}}$.

Let (A'', B'') be the normalization of (A^*, B^*). Let $[C, D]$ be the partner of (A'', B'') in G.

Claim 2: $[C, D] = [A, B]$.

Proof of Claim 2: Suppose that there exists a vertex $v \in C - A$. Since A contains all isolated vertices in G by Claim 1, v is not an isolated vertex in G. Since $v \in C$, $\mathrm{cl}(v) \subseteq V(A'')$. Since $v \notin A$, $v \in B$. By Claim 1, v is incident with an edge e of $G-W$ whose every end is contained in B. So $e \in V(B') - V(A') = V(B^*) - V(A^*)$. Since $v \in B$, e is not the only edge of some component of G by Claim 1. So e is not an isolated vertex in B^*. Hence $e \in V(B'') - V(A'')$. But $e \in \mathrm{cl}(v) - V(A'')$, a contradiction. This shows $C \subseteq A$.

Suppose that there exists a vertex $u \in A - C$. Since C contains all isolated vertices of G, u is not an isolated vertex of G. Since $u \in A$, u is incident with an edge f of $G-W$ whose every end in A by Claim 1. Hence $f \in V(A') - V(B') = V(A^*) - V(B^*)$. So $f \in V(A'') - V(B'')$ and hence $\mathrm{cl}(u) \subseteq V(A'')$. This implies that $u \in C$, a contradiction.

Therefore, $A = C$. Since $\{A, B\}$ and $\{C, D\}$ are partitions of $V(G)$, $[A, B] = [C, D]$. $\square$

By Claim 2, since $[C, D] = [A, B] \in \mathcal{E}$, $(A^*, B^*) \in \overline{\mathcal{E}}$. But $X \subseteq V(A^*)$ and $|V(A^*) \cap V(B^*)| < |X|$, so X is not free with respect to $\overline{\mathcal{E}}$, a contradiction. This proves that X is free with respect to $\mathcal{E}$. $\square$

2.3. Immersions and edge-tangles

The following lemma provides a way to obtain an edge-tangle from an immersion.

Lemma 2.10. *Let H be a graph and $\mathcal{E}'$ an edge-tangle of order θ in H. Let G be a graph that contains an H-immersion (π_V, π_E). If $\mathcal{E}$ is the set of all edge-cuts $[A, B]$ of G of order less than θ such that there exists $[A', B'] \in \mathcal{E}'$ with $\pi_V(A') = A \cap \pi_V(V(H))$, then $\mathcal{E}$ is an edge-tangle of order θ in G.*

Proof. We shall show that $\mathcal{E}$ satisfies the edge-tangle axioms (E1), (E2) and (E3). Note that for every edge-cut $[A, B]$ of G, $A \cap \pi_V(V(H))$ and $B \cap \pi_V(V(H))$ are two disjoint subsets of $\pi_V(V(H))$ such that their union is $\pi_V(V(H))$, so there exists an edge-cut $[A', B']$ of H such that $\pi_V(A') = A \cap \pi_V(V(H))$ and $\pi_V(B') = B \cap \pi_V(V(H))$.

We first prove that $\mathcal{E}$ satisfies (E1). Let $[A, B]$ be an edge-cut of G of order less than θ. Let $[A', B']$ an edge-cut of H such that $\pi_V(A') = A \cap \pi_V(V(H))$ and $\pi_V(B') = B \cap \pi_V(V(H))$. Since (π_V, π_E) is an H-immersion, there are at least $|[A', B']|$ edge-disjoint paths in G from $A \cap \pi_V(V(H))$ to $B \cap \pi_V(V(H))$. So $|[A', B']| \leq |[A, B]| < \theta$. Hence, one of $[A', B']$ and $[B', A']$ is in $\mathcal{E}'$, and hence one of $[A, B]$ and $[B, A]$ is in $\mathcal{E}$. So $\mathcal{E}$ satisfies (E1).

Now we prove that $\mathcal{E}$ satisfies (E2). For each $i \in [3]$, let $[A_i, B_i] \in \mathcal{E}$ be an edge-cut of G. By the definition of $\mathcal{E}$, for each $i \in [3]$, there exists $[A'_i, B'_i] \in \mathcal{E}'$ such that $\pi_V(A'_i) = A_i \cap \pi_V(V(H))$ and hence $\pi_V(B'_i) = B_i \cap \pi_V(V(H))$. Since $\mathcal{E}'$ is an edge-tangle in H, $B'_1 \cap B'_2 \cap B'_3$ contains a vertex v of H. So $\pi_V(v) \in B_1 \cap B_2 \cap B_3$. This proves that $\mathcal{E}$ satisfies (E2).

Finally, we prove that $\mathcal{E}$ satisfies (E3). Let $[A, B] \in \mathcal{E}$. By the definition of $\mathcal{E}$, there exists $[A', B'] \in \mathcal{E}'$ such that $\pi_V(A') = A \cap \pi_V(V(H))$ and hence $\pi_V(B') = B \cap \pi_V(V(H))$. Since $\mathcal{E}'$ satisfies (E3), H contains at least θ edges incident with vertices in B'. So there are at least θ edge-disjoint subgraphs of G each containing a vertex in $\pi_V(B') \subseteq B$ and containing an edge of G. Therefore, G contains at least θ edges incident with vertices in B. Consequently, $\mathcal{E}$ is an edge-tangle in G. $\square$

We call the edge-tangle $\mathcal{E}$ defined in Lemma 2.10 the *edge-tangle induced by the H-immersion (π_V, π_E) and the edge-tangle $\mathcal{E}'$ in H*.

The $m \times n$ *wall* is the simple graph with vertex-set $\{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ and edge-set $\{(i, j)(i + 1, j) : 1 \leq i \leq n - 1, 1 \leq j \leq m\} \cup \{(2a - 1, 2b - 1)(2a - 1, 2b) : 1 \leq a \leq \lceil n/2 \rceil, 1 \leq b \leq \lfloor m/2 \rfloor\} \cup \{(2a, 2b)(2a, 2b + 1) : 1 \leq a \leq \lfloor n/2 \rfloor, 1 \leq b \leq \lfloor (m - 1)/2 \rfloor\}$. The *$i$-th row* of the $m \times n$ wall is the subgraph induced by $\{(x, i) : 1 \leq x \leq n\}$. The *$k$-th column* of the $m \times n$ wall is the subgraph induced by $\{(x, y) : 2k - 1 \leq x \leq \min\{2k, n\}, 1 \leq y \leq m\}$. Hence, the $m \times n$ wall contains m rows and $\lceil n/2 \rceil$ columns.

It was proved by Robertson, Seymour and Thomas [20] (see Theorem 2.11 below) that every tangle is "equivalent" to a subdivision of a wall. The next objective of this section is to prove an analogous result (Lemma 2.16) about edge-tangles and immersions, which will be used in Section 6.

For a graph H, an H-immersion (π_v, π_e) is an *H-subdivision* if

- for every pair of distinct edges e_1, e_2 of H, $V(\pi_E(e_1) \cap \pi_E(e_2)) \subseteq \pi_V(S)$, where S is the set of the common ends of e_1, e_2, and
- for every $e \in E(H)$, $\pi_V(V(H)) \cap V(\pi_E(e)) = \pi_V(S)$, where S is the set of ends of e.

We say that a *tangle $\mathcal{T}$ is induced by a $r \times r$ wall-subdivision* (π_V, π_E) if for every $(A, B) \in \mathcal{T}$, there exists a row of the wall such that $E(B)$ intersects $\pi_E(e)$ for every edge e of this row.

One corollary of the following restatement of [20, (2.3)] is that every graph with a tangle of large order contains a subdivision of a large wall.

Theorem 2.11. *[20, (2.3)] Let $\theta \geq 2$, and let $\mathcal{T}$ be a tangle in G of order at least $20^{\theta^4(2\theta-1)}$. If $\mathcal{T}' \subseteq \mathcal{T}$ is a tangle of order θ, then $\mathcal{T}'$ is induced by a $\theta \times \theta$ wall-subdivision.*

Lemma 2.12. *Let r and θ be positive integers with $\theta \leq r$. Let G be the $r \times 2r$ wall. If $[A, B]$ is an edge-cut of order less than θ of G, then*

1. *exactly one of A and B contains all vertices of a column of G,*
2. *exactly one of A and B contains all vertices of a row of G,*
3. *A contains all vertices of a column if and only if A contains all vertices of a row,*
4. *exactly one of A and B intersects vertices in at least θ columns of G,*
5. *exactly one of A and B intersects vertices in at least θ rows of G,*
6. *A intersects vertices in at least θ columns of G if and only if A contains all vertices of a column of G, and*
7. *A intersects vertices in at least θ rows of G if and only if A contains all vertices of a column of G.*

Proof. Note that G has r rows and r columns. Suppose that A contains all vertices of a column and B contains all vertices of another column. Then every row must contain an edge in $[A, B]$, so the order of $[A, B]$ is at least r, a contradiction. Suppose that none of A and B contains all vertices of a column. Then every column must contain an edge in $[A, B]$, so the order of $[A, B]$ is at least r, a contradiction. So exactly one of A and B contains all vertices of a column. Similarly, exactly one of A and B contains all vertices of a row. Furthermore, if A contains all vertices of a column, then B cannot contain all vertices of a row, so A contains all vertices of a row as well. Similarly, if B contains all vertices of a column, then B contains all vertices of a row. This shows Statements 1-3.

Since $r \geq \theta$, every column intersects vertices in at least θ rows. By Statement 1, at least one of A and B intersects vertices in at least θ rows. If one of A and B contains all vertices of a column and the other intersects vertices in at least θ rows, then there are at least θ rows containing both vertices in A and in B, so there are at least θ edges between A and B, a contradiction. So A contains all vertices of a column if and only if A intersects vertices in at least θ rows. This shows Statement 7. Then Statements 1 and 7 imply Statement 5.

Similarly, A contains all vertices of a row if and only if A intersects vertices in at least θ columns. So Statement 6 follows from Statement 3, and Statement 4 follows from Statements 5-7. □

Lemma 2.13. *Let r and θ be positive integers. Let G be the $r \times 2r$ wall. Let $\mathcal{E}$ be the set of all edge-cuts $[A, B]$ of order less than θ of G satisfying that B intersects vertices in at least θ columns of G. If $r \geq 2\theta$, then $\mathcal{E}$ is an edge-tangle of G of order θ.*

Proof. Let $[A,B]$ be an edge-cut of G of order less than θ. By Lemma 2.12, $[A,B]$ or $[B,A]$ is in $\mathcal{E}$. Hence, $\mathcal{E}$ satisfies (E1).

Let $[A_i,B_i] \in \mathcal{E}$ be edge-cuts of G for $i \in [3]$. For each $i \in [3]$, B_i intersects vertices in at least θ columns of G, so B_i contains all vertices of a column of G by Lemma 2.12. For each $i \in [3]$, let c_i be a column of G contained in B_i. Suppose that $B_1 \cap B_2 \cap B_3 = \emptyset$. If $c_1 = c_2$, then B_3 is disjoint from c_1, so A_3 contains c_1, a contradiction. So by symmetry, we may assume that c_1, c_2, c_3 are pairwise distinct. Since c_1 is contained in $A_2 \cup A_3$, we know that A_2 or A_3, say A_2, contains at least one half vertices of c_1, so A_2 intersects at least $r/2$ rows. But B_2 contains c_2, so there are at least $r/2$ edges with one end in A_2 and one end in B_2. Therefore, $[A_2,B_2]$ has order at least $r/2 \geq \theta$, a contradiction. Hence, $\mathcal{E}$ satisfies (E2).

For every $[A,B] \in \mathcal{E}$, B contains a column in G by Lemma 2.12, so there are at least $r \geq \theta$ edges in G incident with some vertices in B. This proves that $\mathcal{E}$ is an edge-tangle. $\square$

We call the edge-tangle $\mathcal{E}$ mentioned in Lemma 2.13 the *natural edge-tangle* in the $r \times 2r$ wall of order θ. Here is a short summary about natural edge-tangles in a wall.

Lemma 2.14. *Let r and θ be positive integers with $r \geq 2\theta$. Let W be the $r \times 2r$ wall. Let G be a graph and $\Pi = (\pi_V, \pi_E)$ a W-immersion in G. If $\mathcal{E}$ is the edge-tangle in G induced by Π and the natural edge-tangle of order θ in W, then for every edge-cut $[A,B]$ of G of order less than θ, the following are equivalent.*

1. $[A,B] \in \mathcal{E}$.
2. *B intersects the image of π_V of vertices in at least θ columns of W.*
3. *B contains the image of π_V of all vertices of some column of W.*
4. *B contains the image of π_V of all vertices of some row of W.*

Proof. Statements 1 and 2 are equivalent by the definition of $\mathcal{E}$. Statements 2-4 are equivalent by Lemma 2.12. $\square$

We need the following lemma to prove Lemma 2.16. It states that whenever each "vertex" of a large "grid" is labeled with a bounded number of labels in a way that every label is used by a bounded number of times, one can find a large "subgrid" such that the sets of labels used in this "subgrid" are pairwise disjoint.

Lemma 2.15. *For any nonnegative integers s,t,p,q, there exist integers $s^* = s^*(s,t,p,q)$, $t^* = t^*(s,t,p,q)$ such that the following holds. Let I, J be sets with $|I| = t^*$ and $|J| = s^*$. Let U be a set, and let f be a function that maps each pair $(i,j) \in I \times J$ to a subset of U of size at most p. If for every $u \in U$, $|\{(i,j) \in I \times J : u \in f((i,j))\}| \leq q$, then there exist $I' \subseteq I$ with $|I'| = t$ and $J' \subseteq J$ with $|J'| = s$ such that $f((x,y)) \cap f((x',y')) = \emptyset$ for distinct $(x,y),(x',y') \in I' \times J'$.*

Proof. We shall prove this lemma by induction on s. When $s = 0$, the lemma holds obviously. So we may assume that $s \geq 1$ and this lemma holds for every smaller s.

Let $s_1 = s^*(s-1, t, p, q)$ and $t_1 = t^*(s-1, t, p, q)$. Define $s^*(s, t, p, q) = s_1 + pqt_1 + 1$ and $t^*(s, t, p, q) = pqt_1$.

Without loss of generality, we may assume that $I = [t^*]$ and $J = [s^*]$. Note that for every $x \in [t^*]$, there are at most pq elements $x' \in [t^*]$ such that $f((x, 1)) \cap f((x', 1)) \neq \emptyset$. So there exists a subset I_1 of I with $|I_1| = \frac{|I|}{pq} = t_1$ such that $f((x, 1)) \cap f((x', 1)) = \emptyset$ for distinct $x, x' \in I_1$. Note that $|\bigcup_{x \in I_1} f((x, 1))| \leq pt_1$. So there exists a subset J_1 of $J - \{1\}$ with $|J_1| \geq |J| - 1 - pt_1 q = s_1$ such that $f((x, y)) \cap f((i, 1)) = \emptyset$ for every $(x, y) \in I_1 \times J_1$ and $i \in I_1$. By the induction hypothesis, there exist $I_2 \subseteq I_1$ with $|I_2| = t$ and $J_2 \subseteq J_1$ with $|J_2| = s - 1$ such that $f((x, y)) \cap f((x', y')) = \emptyset$ for distinct $(x, y), (x', y') \in I_2 \times J_2$. Define $I' = I_2$ and $J' = \{1\} \cup J_2$. Then $f((x, y)) \cap f((x', y')) = \emptyset$ for distinct $(x, y), (x', y') \in I' \times J'$. $\square$

Lemma 2.16. *For every positive integers θ and d with $\theta \geq 2$, there exists an integer $w = w(\theta, d)$ such that if $\mathcal{E}$ is an edge-tangle in a graph G of order at least w, then either there exists $v \in V(G)$ incident with at least d edges in G such that $v \in B$ for every $[A, B] \in \mathcal{E}_\theta$, or $\mathcal{E}_\theta$ is induced by a $2\theta \times 4\theta$ wall-immersion and the natural edge-tangle of order θ in the $2\theta \times 4\theta$ wall, where $\mathcal{E}_\theta$ is the edge-tangle in G of order θ such that $\mathcal{E}_\theta \subseteq \mathcal{E}$.*

Now we sketch the proof of Lemma 2.16. Since $\mathcal{E}$ is an edge-tangle in G of large order, $\overline{\mathcal{E}}$ is a tangle in $L(G)$ of large order, so it is induced by a very large wall subdivision Π in $L(G)$ by Theorem 2.11. If there exists a vertex v of G such that $\mathrm{cl}(v)$ contains many branch vertices of Π, then it is not hard to show that this vertex v satisfies the conclusion of Lemma 2.16. If there exists no such vertex v exists, then there exists a smaller (but still sufficiently large) wall subdivision Π^* such that every branch vertex of Π^* is a branch vertex of Π, and $\mathrm{cl}(u)$ contains at most one branch vertex of Π^* for every $u \in V(G)$. Such a wall subdivision Π^* in $L(G)$ defines a wall immersion Π' in G in an obvious way. Then one can show that the wall immersion Π' satisfies the conclusion of Lemma 2.16 by using the relationship between $\mathcal{E}$ and $\overline{\mathcal{E}}$.

Proof of Lemma 2.16. Let θ and d be positive integers with $\theta \geq 2$. Let $\theta' = s_{2.15}(2\theta, 4\theta, 4, (2\theta d)^2) + t_{2.15}(2\theta, 4\theta, 4, (2\theta d)^2) + 5\theta$, where $s_{2.15}$ and $t_{2.15}$ are the integers s^* and t^* mentioned in Lemma 2.15. Define $w = 20^{64\theta'^5}$.

Denote the $2\theta' \times 2\theta'$ wall by W. Let G be a graph, and let $\mathcal{E}$ be an edge-tangle in G of order at least w. By Lemma 2.5, $\overline{\mathcal{E}}$ is a tangle of order at least $w/3 - 1 \geq 20^{(2\theta')^4(4\theta'-1)}$ in $L(G)$. For every integer t, let $\overline{\mathcal{E}}_t$ be the tangle in $L(G)$ of order t with $\overline{\mathcal{E}}_t \subseteq \overline{\mathcal{E}}$. By Theorem 2.11, $\overline{\mathcal{E}}_{2\theta'}$ is induced by a W-subdivision (π_V, π_E) in $L(G)$.

Claim 1: *If there exists $v \in V(G)$ such that $\mathrm{cl}(v)$ contains at least $(2\theta d)^2$ vertices in $\pi_V(V(W))$, then v is incident with at least d edges in G, and $v \in B$ for every $[A, B] \in \mathcal{E}_\theta$.*

Proof of Claim 1: Let v be a vertex of G such that $\mathrm{cl}(v)$ contains at least $(2\theta d)^2$ vertices in $\pi_V(V(W))$. So $|\mathrm{cl}(v)| \geq (2\theta d)^2$ and v is incident with at least $(2\theta d)^2 \geq d$ edges in G.

Suppose that there exists an edge-cut $[A, B] \in \mathcal{E}_\theta$ such that $v \in A$. We may assume that the order of $[A, B]$ is as small as possible, and subject to that, A is maximal. By Lemma 2.3 and the maximality of A, A contains all isolated vertices of G.

Since the order of $[A, B]$ is less than θ and $v \in A$ is incident with at least $(2\theta d)^2 \geq \theta$ edges in G, some edge of G incident with v has every end in A.

We define the following.

(i) Define B' to be a subgraph of $L(G)$ such that $V(B')$ consists of the edges of G incident with vertices in B.
(ii) Define (A', B') to be a separation of $L(G)$ such that $V(A') \cap V(B')$ consists of the edges of G with one end in A and one end in B.
(iii) Subject to (i) and (ii), $E(A')$ is maximal.

Since the order of $[A, B]$ is minimal and some edge of G incident with v has every end in A, (A', B') is normalized. Hence $[A, B]$ is the partner of (A', B'). Therefore, $(A', B') \in \overline{\mathcal{E}}$.

Since $|V(A') \cap V(B')| < \theta \leq 2\theta'$, $(A', B') \in \overline{\mathcal{E}}_{2\theta'}$. Since $\overline{\mathcal{E}}_{2\theta'}$ is induced by a W-subdivision (π_V, π_E) in $L(G)$, there exists a row r of W such that $E(B')$ intersects $\pi_E(e)$ for every edge e of the row r. Since $|V(A') \cap V(B')| < \theta$, there are at most θ vertices x of W in r such that $\pi_V(x) \in V(A') - V(B')$. That is, there are at least $2\theta' - \theta$ vertices x of W in r such that $\pi_V(x) \in V(B')$.

Suppose that there exists a row r' of W other than r such that there are at least 4θ vertices x of r' such that $\pi_V(x) \in V(A') - V(B')$. Then there exist 2θ columns $c_1, c_2, ..., c_{2\theta}$ of W such that $V(A') - V(B')$ intersects the image of π_V of some vertices of each of c_i. Since there are at least $2\theta' - \theta$ vertices x of W in r such that $\pi_V(x) \in V(B')$, there are at least θ columns c in $\{c_i : 1 \leq i \leq 2\theta\}$ such that both $V(A') - V(B')$ and $V(B')$ intersect the image of π_V of some vertices in c. Hence there are at least θ disjoint paths from $V(A')$ to $V(B')$, a contradiction.

Therefore, for every row of W, there are at least $2\theta' - 4\theta \geq 1$ vertices x of this row such that $\pi_V(x) \in V(B')$. In particular, $V(B')$ intersects the image of π_V of the vertices of each row. Since $|V(A') \cap V(B')| < \theta$, $V(A')$ intersects the image of π_V of vertices in at most $\theta - 1$ rows of W. Since $\theta' > \theta - 1$, $V(B')$ intersects the image of π_V of vertices in every column of W.

Since $v \in A$, $V(A') \supseteq \mathrm{cl}(v)$ contains at least $(2\theta d)^2$ vertices in $\pi_V(V(W))$. So $V(A')$ contains the image of π_V of some vertices in either at least $2\theta d$ rows of W or at least θd columns of W. Since $V(A')$ intersects the image of π_V of vertices in at most $\theta - 1$ rows of W, the former is impossible. So $V(A')$ contains the image of π_V of some vertices in at least θd columns of W. But $V(B')$ intersects the image of π_V of vertices in every column of W. So there exist $\theta d \geq \theta$ disjoint paths from $V(A')$ to $V(B')$, a contradiction. □

By Claim 1, we may assume that for every $v \in V(G)$, $\text{cl}(v)$ contains at most $(2\theta d)^2$ vertices in $\pi_V(V(W))$, for otherwise the lemma holds.

Denote the $2\theta \times 4\theta$ wall by W'.

Claim 2: *There exists a W'-subdivision $\Pi^* = (\pi_V^*, \pi_E^*)$ in $L(G)$ such that*

- $\pi_V^*(V(W')) \subseteq \pi_V(V(W))$ *and*
  - *if $x \in V(W')$ and $x' \in V(W)$ with $\pi_V^*(x) = \pi_V(x')$, then $x' = (i, j)$ for some i, j with $2\theta + 1 \leq i \leq 2\theta' - 2\theta$ and $\theta + 1 \leq j \leq \theta' - \theta$, and*
  - *if x, y are two vertices in the same row of W', then there exist two vertices x', y' in the same row of W such that $\pi_V^*(x) = \pi_V(x')$ and $\pi_V^*(y) = \pi_V(y')$,*
- $\bigcup_{e \in E(W')} \pi_E^*(e) \subseteq \bigcup_{e \in E(W)} \pi_E(e)$, *and*
- *for every $v \in V(G)$, $|\text{cl}(v) \cap \pi_V^*(V(W'))| \leq 1$.*

Proof of Claim 2: Let $I = \{i : \theta + 1 \leq i \leq \theta' - \theta\}$ and $J = \{j : \theta + 1 \leq j \leq \theta' - \theta\}$. For every $(i, j) \in I \times J$, define $f((i, j)) = \{v \in V(G) : \{\pi_V((2i - 1, j)), \pi_V((2i, j))\} \cap \text{cl}(v) \neq \emptyset\}$. Note that $|f((i, j))| \leq 4$ for each $(i, j) \in I \times J$. In addition, for each $v \in V(G)$, $|\{(i, j) \in I \times J : v \in f((i, j))\} \leq |\text{cl}(v) \cap \pi_V(V(W))| \leq (2\theta d)^2$. By Lemma 2.15, there exist $I' \subseteq I$ with $|I'| = 4\theta$ and $J' \subseteq J$ with $|J'| = 2\theta$ such that $f((i, j)) \cap f((i', j')) = \emptyset$ for distinct $(i, j), (i', j') \in I' \times J'$.

Denote the elements of I' by $x_1 < x_2 < ... < x_{4\theta}$ and denote the elements of J' by $y_1 < y_2 < ... < y_{2\theta}$. For each $i \in [2\theta]$ and $j \in [2\theta]$, define $\pi_V^*((2i-1, j)) = \pi_V((2x_{2i-1} - 1, y_j))$ and $\pi_V^*((2i, j)) = \pi_V((2x_{2i}, y_j))$. So $\pi_V^*(V(W')) \subseteq \pi_V(V(W))$, and for every $v \in V(G)$, $|\text{cl}(v) \cap \pi_V^*(V(W'))| \leq 1$. Furthermore, if x, y are two vertices in the same row of W', then there exist two vertices x', y' in the same row of W such that $\pi_V^*(x) = \pi_V(x')$ and $\pi_V^*(y) = \pi_V(y')$. Note $\theta + 1 \leq x_1 < x_{4\theta} \leq \theta' - \theta$ and $\theta + 1 \leq y_1 < y_{2\theta} \leq \theta' - \theta$, so if $x \in V(W')$ and $x' \in V(W)$ with $\pi_V^*(x) = \pi_V(x')$, then $x' = (i, j)$ for some i, j with $2\theta + 1 \leq i \leq 2\theta' - 2\theta$ and $\theta + 1 \leq j \leq \theta' - \theta$. It is obvious that one can define π_E^* such that (π_V^*, π_E^*) is a W'-subdivision in $L(G)$ such that $\bigcup_{e \in E(W')} \pi_E^*(e) \subseteq \bigcup_{e \in E(W)} \pi_E(e)$. □

Now we define a W'-immersion (π'_V, π'_E) in G.

- Define π'_V to be the function that maps each vertex of x of W' to a vertex v of G such that $\pi_V^*(x) \in \text{cl}(v)$ and $|N_{\Pi(W')}(\pi_V^*(x)) \cap \text{cl}(v)| = \max_{u \in V(G)} |N_{\Pi(W')}(\pi_V^*(x)) \cap \text{cl}(u)|$.
- Define π'_E to be the function that maps each edge e of W' to the path in G whose edge-set is the union of the set of internal vertices of $\pi_E^*(e)$ and the set $U_x \cup U_y$, where x, y are the ends of e and for each $u \in \{x, y\}$, the set U_u satisfies
  - if the vertex in $N_{\pi_E(e)}(u)$ is not in $\text{cl}(\pi'_V(u))$, then $U_u = \{\pi_V^*(u)\}$, and
  - if the vertex in $N_{\pi_E(e)}(u)$ is in $\text{cl}(\pi'_V(u))$, then $U_u = \emptyset$.

It is clear that (π'_V, π'_E) is a W'-immersion in G.

To prove this lemma, it is sufficient to show that $\mathcal{E}_\theta$ is induced by (π'_V, π'_E) and the natural edge-tangle of order θ in W'. Let $\mathcal{E}''$ be the edge-tangle induced by (π'_V, π'_E) and the natural edge-tangle of order θ in W'. Note that $\mathcal{E}''$ has order θ.

Suppose to the contrary that $\mathcal{E}_\theta \neq \mathcal{E}''$. So there exists $[A, B] \in \mathcal{E}_\theta - \mathcal{E}''$. We further assume that the order of $[A, B]$ is as small as possible, and subject to that, A is maximal.

Since $[\{v\}, V(G) - \{v\}] \in \mathcal{E}_\theta \cap \mathcal{E}''$ for every vertex v of G incident with less than θ edges of G, A contains all isolated vertices of G by Lemma 2.3 and the maximality of A. Similarly, by the minimality of the order of $[A, B]$ and Lemma 2.3, every vertex in A that is a non-isolated vertex of G is incident with an edge whose every end is in A, and every vertex in B that is a non-isolated vertex of G is incident with an edge whose every end is in B.

Define (A', B') to be the separation of $L(G)$ such that

(iv) $V(A') \cap V(B')$ consists of the vertices of $L(G)$ corresponding to the edges with one end in A and one end in B,

(v) $V(B')$ consists of the vertices of $L(G)$ corresponding to the edges of G incident with vertices in B, and

(vi) subject to (iv) and (v), $E(A')$ is maximal.

Since for every non-isolated vertex of G, it is not an isolated vertex in $G[A]$ or $G[B]$, so it has a neighbor in the same side of the edge-cut. Hence every vertex in $V(A') \cap V(B')$ is adjacent to a vertex in $V(A') - V(B')$ and a vertex in $V(B') - V(A')$. So (A', B') is normalized. Since A contains all isolated vertices of G, $[A, B]$ is the partner of (A', B'). Since $[A, B] \in \mathcal{E}_\theta$, $(A', B') \in \overline{\mathcal{E}}_\theta$. Since $\overline{\mathcal{E}}_{2\theta'}$ is induced by a W-subdivision (π_V, π_E) in $L(G)$, $E(B')$ intersects every path in the image of π_E of all edges of a row of W. Since the order of (A', B') is less than θ and W is a $2\theta' \times 2\theta'$ wall, $V(B') - V(A')$ contains at least $2\theta' - 2\theta$ vertices in the image of π_V of the vertices of a row of W. Furthermore, for each row of W, $V(A')$ contains at most 6θ vertices in the image of π_V of the vertices of this row, for otherwise there are at least θ disjoint paths from $V(A')$ to $V(B')$. Hence for every row of W, $V(B') - V(A')$ contains the image of π_V of some vertex of this row.

Since $[A, B] \notin \mathcal{E}''$, $[B, A] \in \mathcal{E}''$ by (E1). So A contains the image of π'_V of all vertices of a row of W' by Lemma 2.14. Recall that by the definition of π'_V, for every $v \in \pi'_V(V(W'))$, $\mathrm{cl}(v) \cap \pi^*_V(V(W')) \neq \emptyset$. And by Claim 2, for distinct $v_1, v_2 \in \pi'_V(V(W'))$, $\mathrm{cl}(v_1) \cap \mathrm{cl}(v_2) \cap \pi^*_V(V(W')) = \emptyset$. In addition, by Claim 2, if x, y are two vertices in the same row of W', then there exist two vertices x', y' in the same row of W such that $\pi^*_V(x) = \pi_V(x')$ and $\pi^*_V(y) = \pi_V(y')$. Hence, there exist a row r of W and a set S such that $S \subseteq V(A')$, and S consist of θ vertices in the image of π_V of vertices in the row r. By Claim 2, r is the j-th row of W, for some $\theta + 1 \leq j \leq \theta' - \theta$.

Let $S = \{v_1, v_2, ..., v_\theta\}$. For each $i \in [\theta]$, let x_i be the vertex of W such that $\pi_V(x_i) = v_i$. Since $\theta + 1 \leq j \leq \theta' - \theta$, there exist distinct rows $r_1, r_2, ..., r_\theta$ of W other than r such that for every permutation $\sigma : [\theta] \to [\theta]$, there exist disjoint paths $P_{\sigma,1}, P_{\sigma,2}, ..., P_{\sigma,\theta}$ in $L(G)$ such that for every $i \in [\theta]$, $P_{\sigma,i}$ is from v_i to $\pi_V(x'_{\sigma,i})$ for some vertex $x'_{\sigma,i}$ of

W in the row $r_{\sigma(i)}$ such that $P_{\sigma,i}$ is contained in the image of π_E of the column of W containing x_i. Since for each $i \in [\theta]$, $V(B') - V(A')$ contains the image of π_V of some vertex of each r_i, we know that for every permutation $\sigma : [\theta] \to [\theta]$ and for each $i \in [\theta]$, there exists a path $Q_{\sigma,i}$ contained in the image of π_E of r_i such that $Q_{\sigma,i}$ is from $\pi_V(x'_{\sigma,i})$ to a vertex in $V(B') - V(A')$. In addition, we say $Q_{\sigma,i}$ is *decreasing* if the index of the column containing $\pi_V(x'_{\sigma,i})$ is at least the index of the column containing the other end of $Q_{\sigma,i}$; otherwise, we say $Q_{\sigma,i}$ is *increasing*.

For each $i \in [\theta]$, assuming $\sigma^*(i')$ is defined for every $1 \leq i' \leq i-1$, define $\sigma^*(i)$ to be the element in $[\theta] - \{\sigma^*(i') : 1 \leq i' \leq i-1\}$ such that $Q_{\sigma^*,i}$ is decreasing if possible, and subject to this,

- if $Q_{\sigma^*,i}$ is decreasing, then $P_{\sigma^*,i}$ is as short as possible, and
- if $Q_{\sigma^*,i}$ is increasing, then $P_{\sigma^*,i}$ is as long as possible.

Then σ^* is a permutation on $[\theta]$. Since $P_{\sigma^*,1} \cup Q_{\sigma^*,1}, P_{\sigma^*,2} \cup Q_{\sigma^*,2}, ..., P_{\sigma^*,\theta} \cup Q_{\sigma^*,\theta}$ cannot be θ disjoint paths in $L(G)$ from $V(A')$ to $V(B') - V(A')$, there exist $1 \leq a < b \leq \theta$ such that $P_{\sigma^*,a} \cup Q_{\sigma^*,a}$ intersects $P_{\sigma^*,b} \cup Q_{\sigma^*,b}$.

Suppose that $Q_{\sigma^*,a}$ is decreasing. Since $P_{\sigma^*,a} \cup Q_{\sigma^*,a}$ intersects $P_{\sigma^*,b} \cup Q_{\sigma^*,b}$, $Q_{\sigma^*,b}$ is decreasing. But the choice of $P_{\sigma^*,a}$ implies that $P_{\sigma^*,a} \cup Q_{\sigma^*,a}$ and $P_{\sigma^*,b} \cup Q_{\sigma^*,b}$ are disjoint, a contradiction.

So $Q_{\sigma^*,a}$ is increasing. In particular, the ends of $Q_{\sigma^*,a}$ are not contained in the same column. If $Q_{\sigma^*,b}$ is increasing, then the choice of $P_{\sigma^*,a}$ implies that $P_{\sigma^*,a} \cup Q_{\sigma^*,a}$ and $P_{\sigma^*,b} \cup Q_{\sigma^*,b}$ are disjoint, a contradiction. So $Q_{\sigma^*,b}$ is decreasing. Since $P_{\sigma^*,a} \cup Q_{\sigma^*,a}$ intersects $P_{\sigma^*,b} \cup Q_{\sigma^*,b}$, the choice of $P_{\sigma^*,a}$ implies that the index of the column containing the end of $Q_{\sigma^*,b}$ other than $\pi_V(x'_{\sigma^*,b})$ is at most the index of the column containing $\pi_V(x'_{\sigma^*,a})$, so $Q_{\sigma^*,a}$ can be chosen to be decreasing, a contradiction. This proves the lemma. □

2.4. Other useful lemmas

The following two lemmas will be used in Section 5.

Lemma 2.17. *Let G be a graph and $\mathcal{E}$ an edge-tangle in G. Let p be a positive integer and let $[A_1, B_1], ..., [A_p, B_p] \in \mathcal{E}$. For each i with $1 \leq i \leq p$, let X_i be the set of edges of G between A_i and B_i. Assume that for every i with $1 \leq i \leq p$ and for every $v \in A_i$, there exists a path in $G[A_i]$ from v to an end of an edge in X_i. Assume the order of $\mathcal{E}$ is greater than $|\bigcup_{i=1}^p X_i|$. If $\bigcup_{i=1}^p X_i$ is free with respect to $\mathcal{E}$ and $X_i \cap X_j = \emptyset$ for every pair of distinct i, j, then $A_i \cap A_j = \emptyset$ for every pair of distinct i, j.*

Proof. There is nothing to prove if $p = 1$, so we may assume that $p \geq 2$. First, we suppose that there exists an edge $e \in X_1$ such that both ends of e are in A_2. Let $Z = (\bigcup_{i=1}^p X_i) - e$. Since the order of $\mathcal{E}$ is greater than $|\bigcup_{i=1}^p X_i|$, $[A_2, B_2] \in \mathcal{E} - Z$ has order zero in $G - Z$. But both ends of the unique member e of $\bigcup_{i=1}^{p} X_i - Z$ are in A_2. So $\bigcup_{i=1}^{p} X_i$ is not free with respect to $\mathcal{E}$, a contradiction. Hence, no edge in X_1 has both ends in A_2.

Similarly, for every pair of distinct i, j, no edge in X_i has both ends in A_j.

Now we suppose that there exist distinct i, j such that $A_i \cap A_j \neq \emptyset$. Let $v \in A_i \cap A_j$. Let P_i be a path in $G[A_i]$ from v to an end of an edge e' in X_i. Since X_i is disjoint from X_j and some end of e' is in B_j, e' has both ends in B_j. So P_i intersects X_j. But P_i is contained in $G[A_i]$, so every edge in $X_j \cap E(P_i)$ has both ends in A_i, a contradiction. This proves the lemma. □

Lemma 2.18. *Let ξ be a positive integer. Let G be a graph and $\mathcal{E}$ an edge-tangle in G of order at least $\xi + 2$. If Z is a subset of $E(G)$ with $|Z| \leq \xi$, then there exists a set W consisting of two edges of $G - Z$ with at least one common end such that W is free with respect to $\mathcal{E} - Z$.*

Proof. Suppose to the contrary that every set consisting of two edges of $G - Z$ with at least one common end is not free with respect to $\mathcal{E} - Z$. That is, for every edges $e_1, e_2 \in E(G) - Z$ sharing at least one common end, there exist $Y \subset \{e_1, e_2\}$ and $[A, B] \in \mathcal{E} - (Y \cup Z)$ of order at most $1 - |Y|$ such that every edge in $\{e_1, e_2\} - Y$ has every end in A. Let $G_1, ..., G_c$ be the components of $G - Z$.

Claim 1: *There exists a unique i with $1 \leq i \leq c$ such that $[V(G) - V(G_i), V(G_i)] \in \mathcal{E} - Z$.*

Proof of Claim 1: We first prove that there exists i with $1 \leq i \leq c$ such that $[V(G) - V(G_i), V(G_i)] \in \mathcal{E} - Z$. Suppose to the contrary that $[V(G) - V(G_i), V(G_i)] \notin \mathcal{E} - Z$ for every $i \in [c]$. Since $\mathcal{E} - Z$ has order at least two, by (E1), $[V(G_i), V(G) - V(G_i)] \in \mathcal{E} - Z$ for every $i \in [c]$. We prove that $[\bigcup_{j=1}^{k} V(G_j), V(G) - \bigcup_{j=1}^{k} V(G_j)] \in \mathcal{E} - Z$ for every $k \in [c]$ by induction on k. The case $k = 1$ is obviously true, so we may assume that $k \geq 2$ and $[\bigcup_{j=1}^{k-1} V(G_j), V(G) - \bigcup_{j=1}^{k-1} V(G_j)] \in \mathcal{E} - Z$. Then $[\bigcup_{j=1}^{k} V(G_j), V(G) - \bigcup_{j=1}^{k} V(G_j)] \in \mathcal{E} - Z$ by Lemma 2.3. Hence $[\bigcup_{j=1}^{k} V(G_j), V(G) - \bigcup_{j=1}^{k} V(G_j)] \in \mathcal{E} - Z$ for every $k \in [c]$. But when $k = c$, $[V(G), \emptyset] = [\bigcup_{j=1}^{c} V(G_j), V(G) - \bigcup_{j=1}^{c} V(G_j)] \in \mathcal{E} - Z$, contradicting (E3). This shows the existence of i.

Now we show the uniqueness of i. Suppose there exist distinct $a, b \in [c]$ such that $[V(G) - V(G_a), V(G_a)]$ and $[V(G) - V(G_b), V(G_b)]$ belong to $\mathcal{E} - Z$. But a, b are distinct, so $V(G_a) \cap V(G_b) = \emptyset$, contradicting (E2). This proves the claim. □

Without loss of generality, we may assume that $[V(G) - V(G_1), V(G_1)] \in \mathcal{E} - Z$. Define $\mathcal{E}'$ to be the set of edge-cuts of G_1 such that $[A, B] \in \mathcal{E}'$ if and only if $[A, B]$ has order less than two and $[A \cup \bigcup_{i=2}^{c} V(G_i), B] \in \mathcal{E} - Z$.

Suppose that $\mathcal{E}'$ is not an edge-tangle in G_1 of order two. It is easy to see that $\mathcal{E}'$ satisfies (E2) and (E3). So $\mathcal{E}'$ does not satisfy (E1). Hence there exists an edge-cut $[A, B]$ of G_1 such that $[B, A \cup \bigcup_{i=2}^{c} V(G_i)]$ and $[A, B \cup \bigcup_{i=2}^{c} V(G_i)]$ belong to $\mathcal{E} - Z$, but these two edge-cuts together with $[\bigcup_{i=2}^{c} V(G_i), V(G_1)]$ are three edge-cuts in $\mathcal{E} - Z$ such that $(A \cup \bigcup_{i=2}^{c} V(G_i)) \cap (B \cup \bigcup_{i=2}^{c} V(G_i)) \cap V(G_1) = \emptyset$, contradicting (E2).

Hence $\mathcal{E}'$ is an edge-tangle in G_1 of order two.

By considering the cut-edges of G_1, it is well-known that there exist a tree T and a partition $(X_t : t \in V(T))$ of $V(G_1)$ such that

- $G_1[X_t]$ either has only one vertex or is 2-edge-connected for every $t \in V(T)$,
- for every adjacent vertices t_1, t_2 of T, there exists uniquely one edge between X_{t_1} and X_{t_2}, and
- every edge of G_1 either has every end in X_t for some $t \in V(T)$, or has one end in X_{t_1} and one end in X_{t_2} for some adjacent vertices t_1, t_2 of T.

For each edge $e = t_1t_2$ of T, let T_{e,t_1} and T_{e,t_2} be the components of $T - e$ containing t_1 and t_2, respectively, and define $Y_{e,t_1} = \bigcup_{t \in V(T_{e,t_1})} X_t$ and $Y_{e,t_2} = \bigcup_{t \in V(T_{e,t_2})} X_t$. Since $\mathcal{E}'$ has order two, by (E1) and (E2), exactly one of $[Y_{e,t_1}, Y_{e,t_2}]$ and $[Y_{e,t_2}, Y_{e,t_1}] \in \mathcal{E} - Z$. If the former happens, then we orientate the edge e from t_1 to t_2; otherwise, we orientate the edge e from t_2 to t_1. So we obtain an orientation of $E(T)$ and hence T has a vertex t^* of out-degree zero.

Given two edges e, f of $G - Z$ with at least one common end, by the assumption, there exist $Y \subset \{e, f\}$ and $[A_{e,f}, B_{e,f}] \in \mathcal{E} - Z$ of order at most $1 - |Y|$ such that every end of the edges in $\{e, f\} - Y$ is in A. Let $[A'_{e,f}, B'_{e,f}]$ be the edge-cut $[A_{e,f} \cap V(G_1), B_{e,f} \cap V(G_1)]$ of G_1 of order at most $1 - |Y|$. If $[B'_{e,f}, A'_{e,f}] \in \mathcal{E}'$, then $[(B_{e,f} \cap V(G_1)) \cup \bigcup_{i=2}^{c} V(G_i), A_{e,f} \cap V(G_1)] \in \mathcal{E} - Z$ by the definition of $\mathcal{E}'$, but $[A_{e,f}, B_{e,f}]$ also belongs to $\mathcal{E} - Z$, contradicting (E2). So $[B'_{e,f}, A'_{e,f}] \notin \mathcal{E}'$. By (E1), $[A'_{e,f}, B'_{e,f}] \in \mathcal{E}'$. Since $[A'_{e,f}, B'_{e,f}]$ has order at most one, $B'_{e,f}$ contains X_{t^*}.

We first claim that X_{t^*} is a single vertex. Suppose X_{t^*} contains at least two vertices. Then $G_1[X_{t^*}]$ is 2-edge-connected. We choose e, f to be two edges of $G_1[X_{t^*}]$ sharing at least one common end. By the definition, one of e, f has every end in $A'_{e,f}$. But as proved in the previous paragraph, $B'_{e,f}$ contains X_{t^*} and hence contains the ends of e and f. So $A'_{e,f} \cap B'_{e,f} \neq \emptyset$, a contradiction.

Hence X_{t^*} contains exactly one vertex v. Since $\mathcal{E}'$ has order at least two, v is incident with at least two edges of G_1. Let e, f be two edges of G_1 incident with v. Since one of e, f has every end in $A'_{e,f}$, $v \in A'_{e,f}$, a contradiction. This proves the lemma. □

3. Spider theorems

The main result of this section is Lemma 3.3 which is an edge-version of a result (see Lemma 3.2 below) that is slightly stronger than a theorem implicitly proved by Robertson and Seymour [19] and explicitly proved by Marx and Wollan [13]. Lemma 3.3 enables us to show that given collections of "interesting sets" of edges, either we can extend a set free with respect to an edge-tangle by adding many sets from those given collections, or we can delete a bounded number of edges to make some collection of "interesting sets" containing no free sets. This lemma will be frequently used in this paper.

We need the following lemma, which is a slightly stronger form of [19, Theorem 7.2]. And it can be proved by simply modifying the proof in [19].

Lemma 3.1. *Let $h \geq 1$ and $w \geq 0$ be integers. Let $\mathcal{T}$ be a tangle in a graph G, and let $W \subseteq V(G)$ be free with respect to $\mathcal{T}$ with $|W| \leq w$. If $\mathcal{T}$ has order at least $(w+h)^{h+1}+h$, then there exists $W' \subseteq V(G)$ with $W \subseteq W'$ and $|W'| \leq (w+h)^{h+1}$ such that for every $(C, D) \in \mathcal{T}$ of order $|W| + h_C$ with $W \subseteq V(C)$, where h_C is an integer with $h_C < h$, there exists $(A^*, B^*) \in \mathcal{T}$ with $W' \subseteq V(A^* \cap B^*)$, $|V(A^* \cap B^*) - W'| \leq h_C$ and $C \subseteq A^*$.*

Proof. For every $(A, B) \in \mathcal{T}$ and every $v \in V(A) \cap V(B)$, the *$\mathcal{T}$-successor of (A, B) via v* is the separation (A', B') of G such that

(i) $v \notin V(B')$, $A \subseteq A'$ and $B' \subseteq B$,
(ii) subject to (i), the order of (A', B') is as small as possible, and
(iii) subject to (i) and (ii), B' is minimal.

Let A_0 be the graph such that $V(A_0) = W$ and $E(A_0) = \emptyset$. Let $\mathcal{T}_0 = \{(A_0, G)\}$, and for $i \geq 1$, let $\mathcal{T}_i$ be the set of all $\mathcal{T}$-successors (A', B') of members (A, B) of $\mathcal{T}_{i-1}$ via some vertex in $V(A) \cap V(B)$ with $|V(A') \cap V(B')| < |W| + h$. Let $W' = \bigcup_{0 \leq i \leq h} \bigcup_{(A,B) \in \mathcal{T}_i} (V(A) \cap V(B))$. It is proved in [19, Theorem 7.2] that $W \subseteq W'$, $|W'| \leq (w + h)^{h+1}$, and every member of $\mathcal{T}_i$ has order at least $|W| + i - 1$ for every $i \in [h + 1]$.

Let $(C, D) \in \mathcal{T}$ of order $|W| + h_C$ with $W \subseteq V(C)$ for some integer h_C with $h_C < h$. Note that $h_C \geq 0$ since W is free with respect to $\mathcal{T}$. Let $(C^*, D^*) \in \mathcal{T}$ be the separation of G such that

(iv) the order of (C^*, D^*) is at most $|W| + h_C$, $C \subseteq C^*$ and $D^* \subseteq D$,
(v) subject to (iv), the order of (C^*, D^*) is minimal, and
(vi) subject to (iv) and (v), C^* is maximal.

Let $(A, B) \in \mathcal{T}_i$ for some i with $0 \leq i \leq h$ such that $A \subseteq C^*$ and $D^* \subseteq B$. Note that such an (A, B) exists as (A_0, G) is a candidate. We assume that i is as large as possible.

Note that if $i \neq 0$, then (A, B) is a $\mathcal{T}$-successor of a member of $\mathcal{T}_{i-1}$, so either $(A, B) = (C^*, D^*)$ or the order of (A, B) is smaller than the order of (C^*, D^*), for otherwise (C^*, D^*) is a better candidate for being in $\mathcal{T}_i$ than (A, B) by (i)-(iii). Furthermore, if $(A, B) \neq (C^*, D^*)$, then $|W| + i - 1 \leq |V(A) \cap V(B)| < |V(C^*) \cap V(D^*)| \leq |W| + h_C$, so $i \leq h_C < h$.

Suppose that $V(A) \cap V(B) \not\subseteq V(C^*) \cap V(D^*)$. Let $v \in (V(A) \cap V(B)) - (V(C^*) \cap V(D^*))$. Let (A', B') be the $\mathcal{T}$-successor of (A, B) via v. Note that the order of (A', B') is at most the order of (C^*, D^*) as $A \subseteq C^*$ and $D^* \subseteq B$. Since $V(A) \cap V(B) \not\subseteq$

$V(C^*) \cap V(D^*)$, $(A, B) \neq (C^*, D^*)$, so $i \leq h-1$. By the maximality of i, either $A' \not\subseteq C^*$, or $D^* \not\subseteq B'$. By the maximality of C^*, the order of $(C^* \cup A', D^* \cap B')$ is greater than the order of (C^*, D^*). So the order of $(C^* \cap A', D^* \cup B')$ is smaller than the order of (A', B') by the submodularity. But $v \notin V(D^* \cup B')$ and $A \subseteq C^* \cap A'$ and $D^* \cup B' \subseteq B$, so (A', B') is not the $\mathcal{T}$-successor of (A, B) via v by (ii), a contradiction. Hence, $V(A) \cap V(B) \subseteq V(C^*) \cap V(D^*)$.

Let (A^*, B^*) be the separation of G such that $V(A^*) = V(C^*) \cup W'$ and $V(B^*) = V(D^*) \cup W'$, and subject to that, A^* is maximal. If $i = 0$, then the order of (A, B) is $|W|$; if $i \geq 1$, then the order of (A, B) is at least $|W| + i - 1$. So the order of (A, B) is at least $|W|$. Since $V(A) \cap V(B) \subseteq V(C^*) \cap V(D^*) \cap W'$, the order of (A^*, B^*) is at most $|V(C^*) \cap V(D^*)| - |V(A) \cap V(B)| + |W'| \leq (w+h)^{h+1} + h_C$. So $(A^*, B^*) \in \mathcal{T}$. In addition, $W' \subseteq V(A^*) \cap V(B^*)$ and $C \subseteq A^*$. And $|V(A^*) \cap V(B^*) - W'| \leq |V(C^*) \cap V(D^*)| - |V(A) \cap V(B)| \leq h_C$. □

For every tangle $\mathcal{T}$ of order θ in a graph G and every $Z \subseteq V(G)$ with $|Z| < \theta$, we define $\mathcal{T} - Z$ to be the set of separations (A, B) of $G - Z$ such that $(A', B') \in \mathcal{T}$ for some subgraphs A', B' of G with $V(A') = V(A) \cup Z$ and $V(B') = V(B) \cup Z$. Note that $\mathcal{T} - Z$ is a tangle in $G - Z$ of order $\theta - |Z|$ by [17, Theorem 6.2].

The following is a stronger form of [13, Theorem 3.3] and its proof uses ideas similar to that used in [13, Theorem 3.3].

Lemma 3.2. *Let G be a graph and $\mathcal{T}$ a tangle in G of order θ, and let c be a positive integer. For every $i \in [c]$, let d_i, k_i be positive integers, and let $\{X_{i,j} \subseteq V(G) : j \in J_i\}$ be a family of subsets of $V(G)$ indexed by a set J_i. Let d, k be integers such that $\theta \geq (kcd)^{d+1} + d$, $d_i \leq d$ and $k_i \leq k$ for $i \in [c]$. Let $J_i^* \subseteq J_i$ with $|J_i^*| \leq k_i$ for each $i \in [c]$, such that $\bigcup_{i=1}^{c} \bigcup_{j \in J_i^*} X_{i,j}$ is free with respect to $\mathcal{T}$ and $X_{i,j} \cap X_{i',j'} = \emptyset$ for distinct pairs $(i, j), (i', j')$ with $1 \leq i \leq i' \leq c$, $j \in J_i^*$ and $j' \in J_{i'}^*$. If $|X_{i,j}| \leq d_i$ for every $i \in [c]$ and $j \in J_i$, then either*

1. *there exist $J_1', J_2', ..., J_c'$ with $J_i^* \subseteq J_i' \subseteq J_i$ and $|J_i'| = k_i$ for each $i \in [c]$ such that $\bigcup_{i \in [c]} \bigcup_{j \in J_i'} X_{i,j}$ is free with respect to $\mathcal{T}$, and $X_{i,j} \cap X_{i',j'} = \emptyset$ for all distinct pairs $(i, j), (i', j')$ with $1 \leq i \leq i' \leq c$, $j \in J_i'$ and $j' \in J_{i'}'$, or*
2. *there exist $Z \subseteq V(G)$ with $|Z| \leq (kcd)^{d+1}$ and integer $i^* \in [c]$ with $|J_{i^*}^*| < k_{i^*}$ such that for every $j \in J_{i^*}$, either $X_{i^*,j} \cap Z \neq \emptyset$, or $X_{i^*,j}$ is not free with respect to $\mathcal{T} - Z$.*

Proof. For every $i \in [c]$, pick J_i' with $J_i^* \subseteq J_i' \subseteq J_i$ and $|J_i'| \leq k_i$ such that

(i) $\bigcup_{i \in [c]} \bigcup_{j \in J_i'} X_{i,j}$, denoted by W, is free with respect to $\mathcal{T}$,
(ii) $X_{i,j}$ and $X_{i',j'}$ are disjoint for all distinct pairs $(i, j), (i', j')$ with $1 \leq i \leq i' \leq c$, $j \in J_i'$ and $j' \in J_{i'}'$, and
(iii) subject to (i) and (ii), the sequence $(k_1 - |J_1'|, k_2 - |J_2'|, ..., k_c - |J_c'|)$, denoted by s, is lexicographically minimal.

Note that such a set W exists since $\bigcup_{i=1}^{c} \bigcup_{j \in J_i^*} X_{i,j}$ is free and $X_{i,j} \cap X_{i',j'} = \emptyset$ for all distinct pairs $(i,j),(i',j')$ with $1 \leq i \leq i' \leq c'$, $j \in J_i^*$ and $j' \in J_{i'}^*$.

Assume that the first conclusion of this lemma does not hold. So s contains a non-zero entry. Let i^* be the smallest number i such that $k_i - |J_i'| > 0$. Note that $|W| \leq (kc-1)d$. Applying Lemma 3.1 by taking $(h,w) = (d,(kc-1)d)$, there exists $Z \subseteq V(G)$ with $W \subseteq Z$ and $|Z| \leq (kcd)^{d+1}$ such that for every $(C,D) \in \mathcal{T}$ of order $|W| + h_C$ with $W \subseteq V(C)$ for some $h_C < d$, there exists $(A',B') \in \mathcal{T}$ with $Z \subseteq V(A' \cap B')$, $|V(A' \cap B') - Z| \leq h_C$ and $C \subseteq A'$.

We shall prove that Z and i^* satisfy the second conclusion of this lemma. Assume that $j \in J_{i^*}$ such that $X_{i^*,j} \cap Z = \emptyset$. Since $W \subseteq Z$, $X_{i^*,j}$ is disjoint from W. By the maximality of W, $W \cup X_{i^*,j}$ is not free with respect to $\mathcal{T}$. So there exists a separation $(C,D) \in \mathcal{T}$ of order at most $|W| + |X_{i^*,j}| - 1$ with $W \cup X_{i^*,j} \subseteq V(C)$. By the choice of Z, there exists $(A',B') \in \mathcal{T}$ with $Z \subseteq V(A' \cap B')$, $|V(A' \cap B') - Z| \leq |X_{i^*,j}| - 1$ and $C \subseteq A'$. That is, $X_{i^*,j} \subseteq V(A') - Z$ and the order of $(A' - Z, B' - Z)$ is less than $|X_{i^*,j}|$. So $X_{i^*,j}$ is not free with respect to $\mathcal{T} - Z$. This proves the lemma. □

What we really need in this paper is a version of Lemma 3.2 with respect to edge-tangles.

Lemma 3.3. *Let G be a graph and $\mathcal{E}$ an edge-tangle in G of order θ, and let c be a positive integer. For every $i \in [c]$, let d_i, k_i be positive integers, and let $\{X_{i,j} \subseteq E(G) : j \in J_i\}$ be a family of subsets of $E(G)$ indexed by a set J_i. Let d,k be integers such that $\theta \geq 3(kcd)^{d+1} + 3d$, $d_i \leq d$ and $k_i \leq k$ for $i \in [c]$. Let $J_i^* \subseteq J_i$ with $|J_i^*| \leq k_i$ for each $i \in [c]$, such that $\bigcup_{i=1}^{c} \bigcup_{j \in J_i^*} X_{i,j}$ is free with respect to $\mathcal{E}$ and $X_{i,j} \cap X_{i',j'} = \emptyset$ for distinct pairs $(i,j),(i',j')$ with $1 \leq i \leq i' \leq c$, $j \in J_i^*$ and $j' \in J_{i'}^*$. If $|X_{i,j}| \leq d_i$ for every $i \in [c]$ and $j \in J_i$, then either*

1. *there exist $J_1', J_2', ..., J_c'$ with $J_i^* \subseteq J_i' \subseteq J_i$ and $|J_i'| = k_i$ for each $i \in [c]$ such that $\bigcup_{i=1}^{c} \bigcup_{j \in J_i'} X_{i,j}$ is free with respect to $\mathcal{E}$, and $X_{i,j}$ and $X_{i',j'}$ are disjoint for all distinct pairs $(i,j),(i',j')$ with $1 \leq i \leq i' \leq c$, $j \in J_i'$ and $j' \in J_{i'}'$, or*
2. *there exist $Z \subseteq E(G)$ with $|Z| \leq (kcd)^{d+1}$ and integer $i^* \in [c]$ with $|J_{i^*}^*| < k_{i^*}$ such that for every $j \in J_{i^*}$, either $X_{i^*,j} \cap Z \neq \emptyset$, or $X_{i^*,j}$ is not free with respect to $\mathcal{E} - Z$.*

Proof. Since $\mathcal{E}$ is an edge-tangle of order θ in G, $\overline{\mathcal{E}}$ is a tangle of order at least $\lfloor \theta/3 \rfloor \geq (kcd)^{d+1} + d$ in $L(G)$ by Lemma 2.5. Note that for every $i \in [c]$ and $j \in J_i$, $X_{i,j}$ is a subset of $E(G)$ so it is a subset of $V(L(G))$. Since $\bigcup_{i=1}^{c} \bigcup_{j \in J_i^*} X_{i,j}$ is free with respect to $\mathcal{E}$, $\bigcup_{i=1}^{c} \bigcup_{j \in J_i^*} X_{i,j}$ is free with respect to $\overline{\mathcal{E}}$ by Lemma 2.8. So by Lemma 3.2, either

(i) there exist $J_1', J_2', ..., J_c'$ with $J_i^* \subseteq J_i' \subseteq J_i$ and $|J_i'| = k_i$ for each $i \in [c]$ such that $\bigcup_{i=1}^{c} \bigcup_{j \in J_i'} X_{i,j}$ is free with respect to $\overline{\mathcal{E}}$, and $X_{i,j}$ and $X_{i',j'}$ are disjoint for all distinct pairs $(i,j),(i',j')$ with $1 \leq i \leq i' \leq c$, $j \in J_i'$ and $j' \in J_{i'}'$, or

(ii) there exist $Z \subseteq V(L(G)) = E(G)$ with $|Z| \leq (kcd)^{d+1}$ and integer $i^* \in [c]$ with $|J^*_{i^*}| < k_{i^*}$ such that for every $j \in J_{i^*}$, either $X_{i^*,j} \cap Z \neq \emptyset$, or $X_{i^*,j}$ is not free with respect to $\overline{\mathcal{E}} - Z$.

If (i) holds, then $\bigcup_{i=1}^{c} \bigcup_{j \in J'_i} X_{i,j}$ is a set of size at most $ckd \leq \lfloor \theta/3 \rfloor$ that is free with respect to $\overline{\mathcal{E}}$, so $\bigcup_{i=1}^{c} \bigcup_{j \in J'_i} X_{i,j}$ is free with respect to $\mathcal{E}$ by Lemma 2.9, and hence Statement 1 of this lemma holds.

So we may assume that Z and i^* mentioned in (ii) exist. We shall prove that Statement 2 of this lemma holds. Suppose to the contrary that there exists $j \in J_{i^*}$ such that $X_{i^*,j} \cap Z = \emptyset$ and $X_{i^*,j}$ is free with respect to $\mathcal{E} - Z$. So $X_{i^*,j} \cap Z = \emptyset$ and $X_{i^*,j}$ is free with respect to $\overline{\mathcal{E}} - Z$ by Lemma 2.8, contradicting (ii). This proves the lemma. □

4. Excluding immersions

Given a simple graph H, an *H-minor* of a graph G is a map α with domain $V(H)$ such that

- $\alpha(h)$ is a nonempty connected subgraph of G, for every $h \in V(H)$;
- if h_1 and h_2 are different vertices in H, then $\alpha(h_1)$ and $\alpha(h_2)$ are disjoint;
- if h_1h_2 is an edge in H, then there exists an edge of G with one end in $\alpha(h_1)$ and one end in $\alpha(h_2)$.

We say that G *contains an H-minor* if such a function α exists. And for every $h \in V(H)$, $\alpha(h)$ is called a *branch set* of α.

Given a simple graph H, an *H-thorns* of a graph G is a map α with domain $V(H)$ such that

- $\alpha(h)$ is a connected subgraph of G with at least one edge, for every $h \in V(H)$;
- if h_1 and h_2 are different vertices in H, then $\alpha(h_1)$ and $\alpha(h_2)$ are edge-disjoint;
- if h_1h_2 is an edge in H, then $V(\alpha(h_1)) \cap V(\alpha(h_2)) \neq \emptyset$;

We say that G *contains an H-thorns* if such a function α exists. And for every $h \in V(H)$, $\alpha(h)$ is called a *branch set* of α.

Note that if a graph contains a vertex v incident with d edges, then it contains a K_d-thorns whose branch sets are the edges incident with v. Another example of thorns is that every $r \times r$-grid contains a K_r-thorns by defining $\alpha(v_i)$ to be the union of the i-th row and the i-th column.

Lemma 4.1. *If H is a simple graph, then a graph G contains an H-thorns if and only if $L(G)$ contains an H-minor.*

Proof. Let α be an H-thorns in G. For every $h \in V(H)$, define $\beta(h)$ to be the subgraph of $L(G)$ induced by $E(\alpha(h))$. It is clear that β is an H-minor in $L(G)$.

Let β' be an H-minor in $L(G)$. For every $h \in V(H)$, define $\alpha'(h)$ to be the connected subgraph of G with $E(\alpha'(h)) = V(\beta'(h))$. Then it is obvious that α' is an H-thorns in G. $\square$

The following was proved by Robertson and Seymour [18].

Lemma 4.2 *([18, Theorem 5.4]). Let G be a graph, and let Z be a subset of $V(G)$ with $|Z| = \xi$. Let $k \geq \lfloor \frac{3}{2}\xi \rfloor$, and let α be a K_k-minor in G. If there is no separation (A, B) of G of order less than $|Z|$ such that $Z \subseteq V(A)$ and $A \cap \alpha(h) = \emptyset$ for some $h \in V(K_k)$, then for every partition $(Z_1, ..., Z_n)$ of Z into non-empty subsets, there are n connected subgraphs $T_1, ..., T_n$ of G, mutually disjoint and with $V(T_i) \cap Z = Z_i$ for $1 \leq i \leq n$.*

Now, we prove an edge-variant of Lemma 4.2.

Lemma 4.3. *Let G be a graph, and let X be a subset of $E(G)$ with $|X| = \xi$. Let $k \geq \lfloor \frac{3}{2}\xi \rfloor$, and let α be a K_k-thorns in G. If there exist no $Y \subseteq X$ and edge-cut $[A, B]$ of $G - Y$ of order less than $\xi - |Y|$ such that every edge in $X - Y$ is incident with some vertex in A and $A \cap V(\alpha(h)) = \emptyset$ for some $h \in V(K_k)$, then for every partition $(X_1, ..., X_n)$ of X into non-empty subsets, there are n connected subgraphs $T_1, ..., T_n$ of G, mutually edge-disjoint and with $E(T_i) \cap X = X_i$ for $1 \leq i \leq n$.*

Proof. Let β be the K_k-minor in $L(G)$ corresponding to α mentioned in Lemma 4.1.

Claim 1: *There does not exist a separation (A', B') of $L(G)$ of order less than ξ such that $X \subseteq V(A')$ and $A' \cap \beta(h) = \emptyset$ for some $h \in V(K_k)$.*

Proof of Claim 1: Suppose to the contrary that there exists a separation (A', B') of $L(G)$ of order less than ξ such that $X \subseteq V(A')$ and $A' \cap \beta(h) = \emptyset$ for some $h \in V(K_k)$. We may assume that the order of (A', B') is as small as possible. So every vertex in $V(A') \cap V(B') - X$ must have an neighbor in $V(A') - V(B')$ and a neighbor in $V(B') - V(A')$, and every vertex in $V(A') \cap V(B') \cap X$ has a neighbor in $V(B') - V(A')$. Define $B = \{v \in V(G) : \mathrm{cl}(v) \subseteq V(B')\}$ and $A = V(G) - B$. Then $[A, B]$ is an edge-cut of G. Let Y be the subset of X consisting of the edges in X with every end in B. Note that the order of $[A, B]$ equals $|V(A') \cap V(B')| - |\{v \in V(A') \cap V(B') : v$ has no neighbor in $V(A') - V(B')\}| = |V(A') \cap V(B')| - |Y| < \xi - |Y|$. Furthermore, every edge in $X - Y$ has an end in A. In addition, every vertex of $\beta(h)$ is in $V(B') - V(A')$, so every edge of $\alpha(h)$ has every end in B. That is, $V(\alpha(h)) \cap A = \emptyset$, a contradiction. $\square$

Let $(X_1, X_2, ..., X_n)$ be a partition of X into nonempty sets. By Lemma 4.2 and Claim 1, there exist mutually disjoint connected subgraphs $T'_1, ..., T'_n$ of $L(G)$ such that $V(T'_i) \cap X = X_i$ for every $1 \leq i \leq n$. For every $1 \leq i \leq n$, define T_i to be the connected subgraph of G with $E(T_i) = V(T'_i)$. Then $T_1, ..., T_n$ are mutually edge-disjoint and $E(T_i) \cap X = X_i$. $\square$

A tangle $\mathcal{T}$ in G *controls* an H-minor α if there do not exist $(A, B) \in \mathcal{T}$ of order less than $|V(H)|$ and $h \in V(H)$ such that $V(\alpha(h)) \subseteq V(A)$. An edge-tangle $\mathcal{E}$ in G *controls* an H-thorns α if $V(\alpha(h)) \cap B \neq \emptyset$ for every $h \in V(H)$ and every $[A, B] \in \mathcal{E}$ of order less than $|V(H)|$.

The *degree sequence* of a graph G is the non-increasing sequence of the degrees of the vertices of G.

Lemma 4.4. *Let G be a graph and H be a graph on h vertices with degree sequence $(d_1, d_2, ..., d_h)$. Let $d = d_1$ and $t \geq 3hd$. Let $V(H) = \{u_1, u_2, ..., u_h\}$, where $\deg_H(u_i) = d_i$ for every $i \in [h]$. Let $\mathcal{E}$ be an edge-tangle of order at least $2hd$ in G that controls a K_t-thorns. Let ℓ be the number of loops of H. Assume that there exist pairwise disjoint subsets $X_0, X_1, X_2, ..., X_h$ of $E(G)$ such that $\bigcup_{i=0}^{h} X_i$ is free with respect to $\mathcal{E}$, X_0 can be partitioned into ℓ 2-element subsets $S_1, S_2, ..., S_\ell$ where the two edges in each S_j share at least one common end s_j, and for each $i \in [h]$, X_i consists of d_i edges incident with a common vertex v_i. If $v_1, v_2, ..., v_h$ are distinct and there exists a partition of $\{S_1, S_2, ..., S_\ell\}$ into sets $D_1, D_2, ..., D_h$ such that for every $i \in [h]$, $|D_i|$ equals the number of loops incident with u_i and $v_i \notin \{s_j : S_j \in D_i\}$, then G has an H-immersion (π_V, π_E) with $\pi_V(V(H)) = \{v_1, v_2, ..., v_h\}$.*

Proof. Let α be a K_t-thorns in G controlled by $\mathcal{E}$, and let $X = \bigcup_{i=0}^{h} X_i$. Note that $|X| \leq 2\ell + hd \leq 2hd$.

Claim 1: *For every positive integer r and every partition $(Z_1, Z_2, ..., Z_r)$ of X into non-empty subsets, there exist pairwise edge-disjoint connected subgraphs $T_1, T_2, ..., T_r$ of G such that $E(T_i) \cap X = Z_i$ for every $1 \leq i \leq r$.*

Proof of Claim 1: Suppose that there exist $Y \subseteq X$ and an edge-cut $[A, B]$ of $G - Y$ of order less than $|X - Y|$ such that every edge in $X - Y$ is incident with some vertex in A and $A \cap V(\alpha(u)) = \emptyset$ for some $u \in V(K_t)$. We assume that Y is maximal, so every edge in $X - Y$ has every end in A. Since X is free with respect to $\mathcal{E}$, $[A, B] \notin \mathcal{E} - Y$. But the order of $[A, B]$ in $G - Y$ is less than $|X - Y| \leq 2hd - |Y|$. So $[B, A] \in \mathcal{E} - Y$ by (E1). Hence $[B, A] \in \mathcal{E}$ is an edge-cut of G of order less than $2hd \leq t$. However, $\mathcal{E}$ controls α, so $A \cap V(\alpha(u)) \neq \emptyset$, a contradiction. Therefore, this claim follows from Lemma 4.3. $\square$

Let $E(H) = \{e_1, e_2, ..., e_{|E(H)|}\}$, where e_j is not a loop for every $j \in [|E(H)| - \ell]$. For every $i \in [h]$, let Y_i be a subset of X_i such that $|Y_i|$ equals the number of non-loops incident with u_i. For every $i \in [h]$, define a bijection f_i from Y_i to the set of non-loop edges of H incident with u_i, and define an onto function f'_i from $X_i - Y_i$ to the set of loops of H incident with u_i such that the preimage of every loop incident with u_i has size two. Define f_E to be a bijection from the set of loops of H to $\{S_1, S_2, ..., S_\ell\}$ such that if e is a loop of H incident with u_i for some $i \in [h]$, then $f_E(e) \in D_i$.

For each $i \in [|E(H)| - \ell]$, define Z_i to be the subset of X consisting of the two edges in $\bigcup_{j=1}^{h} Y_j$ mapped to e_i by $f_1, f_2, ..., f_h$. For each loop e of H, let $\{Z_{e,1}, Z_{e,2}\}$ be a partition of the union of $f_E(e)$ and the preimage of e by $f'_1, f'_2, ..., f'_h$ into two sets of size two such that $|Z_{e,1} \cap f_E(e)| = |Z_{e,2} \cap f_E(e)| = 1$. So $\{Z_1, Z_2, ..., Z_{|E(H)|-\ell}, Z_{e,1}, Z_{e,2} : e$ is a loop of $H\}$ is a partition of X into non-empty sets.

By Claim 1, there exist pairwise edge-disjoint connected subgraphs $T_1, T_2, ..., T_{|E(H)|-\ell}, T_{e,1}, T_{e,2}$ of G (for every loop e of H) such that $E(T_i) \cap X = Z_i$ for every $1 \le i \le |E(H)| - \ell$, and $E(T_{e,j}) \cap X = Z_{e,j}$ for every loop e of H and $j \in [2]$. Note that for every $i \in [|E(H)| - \ell]$, there exists a path in T_i connecting v_j, v_k, where j, k are the indices such that e_i belongs to the image of f_j and f_k. For each loop e of H, there exists a cycle contained in $T_{e,1} \cup T_{e,2}$ containing v_i, where i is the index such that e is incident with u_i, since $v_i \notin \{s_j : S_j \in D_i\}$.

Define $\pi_V : V(H) \to V(G)$ such that $\pi_V(u_i) = v_i$ for every $i \in [h]$. Define π_E to be a function that maps each non-loop edge e_i of H (for some $i \in [|E(H)| - \ell]$)) to be a path in T_i from $\pi_V(u)$ to $\pi_V(u')$, where u, u' are the ends of e_i, and maps each loop e of H to a cycle in $T_{e,1} \cup T_{e,2}$ containing $\pi_V(u'')$, where u'' is the end of e. Then (π_V, π_E) is an H-immersion in G with $\pi_V(V(H)) = \{v_1, v_2, ..., v_h\}$. □

A family $\mathcal{D}$ of edge-cuts of a graph is *cross-free* if $A \cap C = \emptyset$ for every pair of distinct edge-cuts $[A, B], [C, D]$ in $\mathcal{D}$.

Lemma 4.5. *Let k, θ be integers. Let G be a graph and $\mathcal{E}$ an edge-tangle in G of order at least θ. If there exist $C \subseteq E(G)$ with $|C| \le \theta - k$ and a subset $\mathcal{D}$ of $\mathcal{E} - C$ such that every member of $\mathcal{D}$ is an edge-cut of $G - C$ of order less than k, then there exists a cross-free family $\mathcal{D}^* \subseteq \mathcal{E} - C$ such that every member of $\mathcal{D}^*$ is an edge-cut of $G - C$ of order less than k such that $\bigcup_{[A,B] \in \mathcal{D}} A = \bigcup_{[A,B] \in \mathcal{D}^*} A$.*

Proof. Define $\mathcal{D}^*$ to be a subset of $\mathcal{E} - C$ with $\bigcup_{[A,B] \in \mathcal{D}} A = \bigcup_{[A,B] \in \mathcal{D}^*} A$ such that every member of $\mathcal{D}^*$ is an edge-cut of $G - C$ of order less than k, and subject to that, $\sum_{[A,B] \in \mathcal{D}^*} |A|$ is as small as possible. Note that such a family $\mathcal{D}^*$ exists as $\mathcal{D}$ is a candidate. To prove this lemma, it suffices to show that $\mathcal{D}^*$ is cross-free.

Suppose that $\mathcal{D}^*$ is not cross-free. Then there exist $[A_1, B_1], [A_2, B_2] \in \mathcal{D}^*$ such that $A_1 \neq A_2$ and $A_1 \cap A_2 \neq \emptyset$. By the submodularity, $|[A_1 \cap B_2, B_1 \cup A_2]| + |[A_1 \cup B_2, B_1 \cap A_2]| \le |[A_1, B_1]| + |[B_2, A_2]| \le 2(k-1)$, so one of $[A_1 \cap B_2, B_1 \cup A_2]$ and $[B_1 \cap A_2, A_1 \cup B_2]$ has order at most $k-1$. By symmetry, we may assume that $[A_1 \cap B_2, B_1 \cup A_2]$ has order at most $k-1$. Note that the order of $\mathcal{E} - C$ is at least $\theta - |C| \ge k$. By Lemma 2.3, $[A_1 \cap B_2, B_1 \cup A_2] \in \mathcal{E} - C$, since $[A_1, B_1] \in \mathcal{E} - C$. Let $\mathcal{D}' = (\mathcal{D}^* - \{[A_1, B_1]\}) \cup \{[A_1 \cap B_2, A_2 \cup B_1]\}$. Since $A_1 \subseteq (A_1 \cap B_2) \cup A_2$, $\mathcal{D}'$ is contained in $\mathcal{E} - C$ and is a family of edge-cuts of $G - C$ of order at most $k - 1$ such that $\bigcup_{[A,B] \in \mathcal{D}'} A = \bigcup_{[A,B] \in \mathcal{D}^*} A = \bigcup_{[A,B] \in \mathcal{D}} A$. Hence, by the minimality of $\mathcal{D}^*$, $|A_1 \cap B_2| \geq |A_1|$. This implies that $A_1 = A_1 \cap B_2 \subseteq B_2$, so $A_1 \cap A_2 = \emptyset$, a contradiction. Therefore, $\mathcal{D}^*$ is cross-free. □

A graph is *exceptional* if it contains exactly one vertex of degree at least two, and this vertex is incident with a loop.

Theorem 4.6 is a structure theorem for excluding a fixed non-exceptional graph as an immersion in a graph with an edge-tangle controlling a big complete graph-thorns.

Theorem 4.6. *For any positive integers d, h, there exist positive integers $\theta = \theta(d, h)$ and $\xi = \xi(d, h)$ such that the following holds. If H is a non-exceptional graph with degree sequence $(d_1, d_2, ..., d_h)$, where $d_1 = d$, and G is a graph that does not contain an H-immersion, then for every edge-tangle $\mathcal{E}$ of order at least θ in G controlling a K_{3dh}-thorns, there exist $C \subseteq E(G)$ with $|C| \leq \xi$, $U \subseteq V(G)$ with $|U| \leq h - 1$ and a cross-free family $\mathcal{D} \subseteq \mathcal{E} - C$ such that for every vertex $v \in V(G) - U$, there exists $[A, B] \in \mathcal{D}$ of order at most $d_{|U|+1} - 1$ with $v \in A$.*

Now we sketch the proof of Theorem 4.6. We greedily pick a vertex v and a set X_v of sufficiently many edges incident with it such that v is not picked before, X_v is disjoint from all previously picked sets of edges, and the union of X_v and all previously picked sets is free with respect to $\mathcal{E}$, until we cannot find such a vertex or a such set of edges. We first assume that H has no loops. If we picked at least $|V(H)|$ vertices in the process, then we can construction an H-immersion by Lemma 4.4, a contradiction. So the set U of picked vertices has size at most $|V(H)| - 1$. If we can further repeatedly pick a vertex v and a set X_v of $d_{|U|+1}$ edges incident with v such that v is not picked before, X_v is disjoint from all previously picked sets of edges, and the union of X_v and all previously picked sets is free with respect to $\mathcal{E}$, until we get $|V(H)|$ vertices, then again we can construct an H-immersion, a contradiction. So Lemma 3.3 implies that one can delete a bounded number of edges such that each vertex in $V(G) - U$ is contained in the first entry of an edge-cut in the edge-tangle, and we are done. The case that H has loops is similar but takes extra work. Lemma 4.4 implies that we cannot further pick many disjoint set of two edges sharing a common end such that the union of those sets is free, so that Lemma 3.3 implies that for every vertex, there exists an edge-cut of order at most one such that this vertex belongs to the first entry of the edge-cut.

Proof of Theorem 4.6. For any positive integers d, h, define $\xi(d, h) = (h+1)((3hd^2)^{d+1} + dh)$ and $\theta(d, h) = 3(2hd^2)^{d+1} + 3dh + \xi$.

Let d, h be positive integers. Denote $\xi(d, h)$ and $\theta(d, h)$ by ξ and θ, respectively. Let H be a non-exceptional graph on h vertices with degree sequence $(d_1, d_2, ..., d_h)$ and $d_1 = d$. Since there exists no graph on one vertex with maximum degree one, this theorem holds if $d = h = 1$. Suppose that (d, h) is a pair of positive integers with $d + h$ minimum such that this theorem does not hold. That is, there exists a graph G that does not contain an H-immersion and there exists an edge-tangle $\mathcal{E}$ in G of order at least θ controlling a K_{3dh}-thorns such that there do not exist $C \subseteq E(G)$ with $|C| \leq \xi$, $U \subseteq V(G)$ with $|U| \leq h-1$ and a cross-free family $\mathcal{D} \subseteq \mathcal{E} - C$ such that for every vertex $v \in V(G) - U$, there exists $[A, B] \in \mathcal{D}$ of order at most $d_{|U|+1} - 1$ with $v \in A$.

Claim 1: *$|V(G)| \geq h$ and H does not contain an isolated vertex.*

Proof of Claim 1: It is clear that $|V(G)| \geq h$, for otherwise choosing $C = \emptyset$, $U = V(G)$ and $\mathcal{D} = \emptyset$ leads to a contradiction. Suppose that H contains an isolated vertex u. Let $H' = H - u$. Note that the degree sequence of H' is $(d_1, d_2, ..., d_{h-1})$, and H' is non-exceptional. Since $|V(G)| \geq h$ and G does not contain an H-immersion, G does not contain an H'-immersion. By the minimality of $d + h$, there exist $C \subseteq E(G)$ with $|C| \leq \xi(d, h-1) \leq \xi(d, h)$, $U \subseteq V(G)$ with $|U| \leq (h-1) - 1$ and a cross-free family $\mathcal{D} \subseteq \mathcal{E} - C$ such that for every vertex $v \in V(G) - U$, there exists $[A, B] \in \mathcal{D}$ of order at most $d_{|U|+1} - 1$ with $v \in A$, a contradiction. □

Claim 2: *There do not exist $C \subseteq E(G)$ with $|C| \leq \xi$ and $U \subseteq V(G)$ with $|U| \leq h-1$ such that for every $v \in V(G) - U$, there exists $[A_v, B_v] \in \mathcal{E} - C$ of order at most $d_{|U|+1} - 1$ such that $v \in A_v$.*

Proof of Claim 2: Suppose to the contrary that there exist $C \subseteq E(G)$ with $|C| \leq \xi$ and $U \subseteq V(G)$ with $|U| \leq h-1$ such that for every $v \in V(G) - U$, there exists $[A_v, B_v] \in \mathcal{E} - C$ of order at most $d_{|U|+1} - 1$ such that $v \in A_v$. That is, there exists a family $\mathcal{D}' \subseteq \mathcal{E} - C$ of edge-cuts of $G - C$ of order at most $d_{|U|+1} - 1$ such that for every $v \in V(G) - U$, there exists $[A, B] \in \mathcal{D}'$ such that $v \in A$. In particular, $V(G) - U \subseteq \bigcup_{[A,B] \in \mathcal{D}'} A$. By Lemma 4.5, there exists a cross-free family $\mathcal{D} \subseteq \mathcal{E} - C$ of edge-cuts of $G - C$ of order at most $d_{|U|+1} - 1$ such that $\bigcup_{[A,B] \in \mathcal{D}} A = \bigcup_{[A,B] \in \mathcal{D}'} A \supseteq V(G) - U$. Hence for every $v \in V(G) - U$, there exists $[A, B] \in \mathcal{D}$ with $v \in A$, a contradiction. □

For each $i \in [d]$, define U_i to be a subset of $V(G)$ and define $\mathcal{S}_i^*$ to be a collection of subsets of $E(G)$ such that U_i and $\mathcal{S}_i^*$ satisfy the following properties.

(i) For every $S \in \mathcal{S}_i^*$, S consists of $d-i+1$ edges of G with a common end $v_S \notin \bigcup_{j=1}^{i-1} U_j$, and S is disjoint from S' for every $S' \in \bigcup_{j=1}^{i-1} \mathcal{S}_j^*$.
(ii) For every pair of distinct sets $S, S' \in \mathcal{S}_i^*$, we have $S \cap S' = \emptyset$ and $v_S \neq v_{S'}$.
(iii) $\bigcup_{j=1}^{i} \bigcup_{S \in \mathcal{S}_j^*} S$ is free with respect to $\mathcal{E}$.
(iv) Subject to (i)-(iii), $\mathcal{S}_i^*$ is maximal.
(v) $U_i = \{v_S : S \in \mathcal{S}_i^*\}$.

If there exists $k \in [d]$ such that $|\bigcup_{i=1}^{k} U_i| < |\{u \in V(H) : \deg_H(u) \geq d-k+1\}|$, then define r to the minimum such k; if there does not exist such k, then $|\bigcup_{i=1}^{d} U_i| \geq |\{u \in V(H) : \deg_H(u) \geq 1\}| = h$ since H has no isolated vertex, and we define $r = d$.

If $|\bigcup_{i=1}^{r} U_i| \leq h$, then define $U^* = \bigcup_{i=1}^{r} U_i$; otherwise, $r = d$ and we define U^* to be a set with $\bigcup_{i=1}^{j-1} U_i \subseteq U^* \subseteq \bigcup_{i=1}^{j} U_i$ with $|U^*| = h$, where j is the minimum such that

$\sum_{i=1}^{j}|U_i| \geq h$. Define $\mathcal{S}^* = \{S \in \bigcup_{i=1}^{r} \mathcal{S}_i^* : v_S \in U^*\}$. Note that $|\mathcal{S}^*| = |U^*| \leq h$, and members of $\mathcal{S}^*$ are pairwise disjoint and $\bigcup_{S \in \mathcal{S}^*} S$ is free with respect to $\mathcal{E}$. For every $v \in V(G)$, define $\mathcal{S}_v$ to be the collection of the sets of $d_{|U^*|+1}$ edges of $G - \bigcup_{S \in \mathcal{S}^*} S$ incident with v, where we define $d_i = 0$ if $i > h$. Note that $\mathcal{S}_v = \emptyset$ if v is incident with less than $d_{|U^*|+1}$ edges.

Claim 3: *There exist distinct $v_1, v_2, ..., v_{h-|U^*|} \in V(G) - U^*$ and pairwise disjoint sets $X_1, X_2, ..., X_{h-|U^*|}$ such that $\bigcup_{S \in \mathcal{S}^*} S \cup \bigcup_{i=1}^{h-|U^*|} X_i$ is free with respect to $\mathcal{E}$, and for each $i \in [h - |U^*|]$, $X_i \in \mathcal{S}_{v_i}$.*

Proof of Claim 3: There is nothing to prove if $|U^*| \geq h$. So we may assume that $|U^*| < h$.

Let $\mathcal{S}_1 = \mathcal{S}^*$ and let $\mathcal{S}_2 = \bigcup_{v \in V(G)-U^*} \mathcal{S}_v$. For $i \in [2]$, let J_i be a set such that we can write $\mathcal{S}_i = \{X_{i,j} : j \in J_i\}$. Let $J_1^* = J_1$ and $J_2^* = \emptyset$. So $\bigcup_{i=1}^{2} \bigcup_{j \in J_i^*} X_{i,j} = \bigcup_{X \in \mathcal{S}^*} X$ is free with respect to $\mathcal{E}$, and $X_{i,j} \cap X_{i',j'} = \emptyset$ for every distinct pairs $(i,j), (i',j')$ with $1 \leq i \leq i' \leq 2$, $j \in J_i^*$ and $j' \in J_{i'}^*$. Let $k_1 = |\mathcal{S}_1|$ and $k_2 = (d-1)(h-1)+1$. Let $k = dh$. Note that $k_1 = |U^*| \leq h-1$. So $k \geq \max\{k_1, k_2\}$ and $\theta \geq 3(2kd)^{d+1} + 3d$. Since every member of $\mathcal{S}_1$ has size at most d and every member of $\mathcal{S}_2$ has size $d_{|U^*|+1} \leq d$, by Lemma 3.3, either

- (i') there exist J_1', J_2' with $J_i^* \subseteq J_i' \subseteq J_i$ and $|J_i'| = k_i$ for $i \in [2]$ such that $\bigcup_{i=1}^{2} \bigcup_{j \in J_i'} X_{i,j}$ is free with respect to $\mathcal{E}$, and $X_{i,j} \cap X_{i',j'} = \emptyset$ for every distinct pairs $(i,j), (i',j')$ with $1 \leq i \leq i' \leq 2$, $j \in J_i'$ and $j' \in J_{i'}'$, or
- (ii') there exist $Z \subseteq E(G)$ with $|Z| \leq (2dk)^{d+1}$ and integer $i^* \in [2]$ with $|J_{i^*}^*| < k_{i^*}$ such that for every $j \in J_{i^*}$, either $X_{i^*,j} \cap Z \neq \emptyset$, or $X_{i^*,j}$ is not free with respect to $\mathcal{E} - Z$.

Suppose that (ii') holds. Define $C = Z \cup \bigcup_{X \in \mathcal{S}^*} X$. Then $|C| \leq |Z| + d|\mathcal{S}^*| \leq (2dk)^{d+1} + dh \leq \xi$. Since $|J_1^*| = |J_1| = k_1$, $i^* = 2$. So every member X of $\mathcal{S}_1 \cup \mathcal{S}_2$ disjoint from C belongs to $\mathcal{S}_2 = \mathcal{S}_{i^*}$ and hence is not free with respect to $\mathcal{E} - Z$ and hence is not free with respect to $\mathcal{E} - C$ by Lemma 2.7. By Claim 2, there exists $v \in V(G) - U^*$ such that there does not exist $[A_v, B_v] \in \mathcal{E} - C$ of order at most $d_{|U^*|+1} - 1$ such that $v \in A_v$, for otherwise choosing $C = C$ and $U = U^*$ contradicts Claim 2. In particular, $v \in V(G) - U^*$ is incident with at least $d_{|U^*|+1}$ edges in $G - C$. Hence there exists $X \in \mathcal{S}_v \subseteq \mathcal{S}_2$ such that every edge in X is incident with v and $X \cap C = \emptyset$. So X is not free with respect to $\mathcal{E} - C$. Hence there exist $Y \subseteq X$ and an edge-cut $[A, B] \in \mathcal{E} - (C \cup Y)$ of $G - (C \cup Y)$ of order less than $|X - Y|$ such that every edge in $X - Y$ has every end in A. Since every edge in $X - Y$ is incident with v, we have that $v \in A$ and $[A, B] \in \mathcal{E} - C$ is an edge-cut of $G - C$ of order less than $|X| = d_{|U^*|+1}$, a contradiction.

So (i') holds. Note that $J_1 = J_1' = J_1^*$. If there exist $h - |U^*|$ distinct vertices $v_1, v_2, ..., v_{h-|U^*|} \in V(G) - U^*$ and $j_1, j_2, ..., j_{h-|U^*|} \in J_2'$ such that $X_{2,j_i} \in \mathcal{S}_{v_i}$ for each $i \in [h - |U^*|]$, then the claim holds.

So we may assume that there exist at most $h - |U^*| - 1 \leq h - 1$ vertices $v_1, v_2, ..., v_{h-|U^*|-1}$ in $V(G) - U^*$ such that $\{X_{2,j} : j \in J_2'\} \subseteq \bigcup_{i=1}^{h-|U^*|-1} \mathcal{S}_{v_i}$. Since $|J_2'| = (d-1)(h-1)+1$, there exists $j^* \in [h - |U^*| - 1]$ such that $\mathcal{S}_{v_{j^*}}$ contains at least d members of $\{X_{2,j} : j \in J_2'\}$. Let W be a subset of $\bigcup_{X \in \mathcal{S}_{v_{j^*}} \cap \{X_{2,j}:j \in J_2'\}} X$ of size d. Note that such a set W exists since members of $\{X_{2,j} : j \in J_2'\}$ are pairwise disjoint and non-empty. Since $\bigcup_{i=1}^{2} \bigcup_{j \in J_i'} X_{i,j}$ is free with respect to $\mathcal{E}$, $W \cup (\bigcup_{S \in \mathcal{S}_1^*} S)$ is free with respect to $\mathcal{E}$. But W is disjoint from $\bigcup_{S \in \mathcal{S}_1^*} S$ and consists of d edges incident with $v_{j^*} \notin U_1$, contradicting the maximality of $\mathcal{S}_1^*$. This proves the claim. □

Claim 4: *H contains a loop, $h \geq 2$ and $d_2 \geq 2$.*

Proof of Claim 4: Let $Y_1, Y_2, ..., Y_{|\mathcal{S}^*|}$ be the members of $\mathcal{S}^*$ such that $|Y_j| \geq |Y_k|$ for every $1 \leq j \leq k \leq |\mathcal{S}^*|$. Let $Y_{|\mathcal{S}^*|+i} = X_i$ for every $i \in [h - |U^*|]$, where X_i is defined in the statement of Claim 3.

We first show that $|Y_j| \geq d_j$ for every $j \in [h]$. Suppose to the contrary that there exists $j \in [h]$ with $|Y_j| < d_j$. It is clear that $|Y_j| \geq d_{|U^*|+1} \geq d_j$ when $|U^*| + 1 \leq j \leq h$. So there exists $i_j \in [r]$ such that $Y_j \in \mathcal{S}_{i_j}^*$. Since $Y_j \in \mathcal{S}_{i_j}^*$, $d - i_j + 1 = |Y_j| < d_j$, so $d - d_j + 1 < i_j$. Since $|Y_j| < d_j$, $|\bigcup_{i=1}^{d-d_j+1} U_i| = |\bigcup_{i=1}^{d-d_j+1} \mathcal{S}_i^*| \leq j - 1 < |\{u \in V(H) : \deg_H(u) \geq d - (d - d_j + 1) + 1\}|$, so $d - d_j + 1 \geq r \geq i_j$ by the definition of r, a contradiction.

So $|Y_j| \geq d_j$ for every $j \in [h]$. Hence for every $j \in [h]$, there exists $Y_j' \subseteq Y_j$ with $|Y_j'| = d_j$. Since $Y_1, Y_2, ..., Y_h$ are pairwise disjoint and $\bigcup_{j=1}^{h} Y_j$ is free with respect to $\mathcal{E}$ by Claim 3, we know $Y_1', Y_2', ..., Y_h'$ are pairwise disjoint and $\bigcup_{j=1}^{h} Y_j'$ is free with respect to $\mathcal{E}$. If H does not contain a loop, then G contains an H-immersion by Lemma 4.4, a contradiction.

So H contains a loop. Since H is not exceptional, $h \geq 2$ and $d_2 \geq 2$. □

Claim 5: *For every $v \in U^* \cup \{v_i : 1 \leq i \leq h - |U^*|\}$, there exist $Z_v \subseteq E(G)$ with $|Z_v| \leq (3hd^2)^{d+1} + dh$ and $[A_v, B_v] \in \mathcal{E} - Z_v$ of order at most one such that $v \in A_v$.*

Proof of Claim 5: Let v be a vertex in $U^* \cup \{v_i : 1 \leq i \leq h - |U^*|\}$. Let $\mathcal{S}_1 = \mathcal{S}^* \cup \{X_i : 1 \leq i \leq h - |U^*|\}$. Let $\mathcal{S}_2 = \{W \subseteq E(G) : W$ consists of two edges of G incident with $v\}$. Let $\mathcal{S}_3 = \{W \subseteq E(G) : W$ consists of two edges of G sharing at least one common end $u \in V(G) - \{v\}\}$.

For $i \in [3]$, let J_i be a set such that $\mathcal{S}_i$ can be written as $\{Y_{i,j} : j \in J_i\}$. Let $J_1^* = J_1$, $J_2^* = \emptyset$, and $J_3^* = \emptyset$. So $\bigcup_{i=1}^{3} \bigcup_{j \in J_i^*} Y_{i,j}$ is free with respect to $\mathcal{E}$. Let $k_1 = |\mathcal{S}_1|$. Let $k_2 = dh$ and let $k_3 = dh$. So $\max\{k_1, k_2, k_3\} \leq hd$. Note that for every $S \in \bigcup_{i=1}^{3} \mathcal{S}_i$, $|S| \leq \max\{d, 2\} \leq d$ since $d_2 \geq 2$. Since $\theta \geq 3(hd \cdot 3 \cdot d)^{d+1} + 3d$, by Lemma 3.3, either

(i') there exist J_1', J_2', J_3' with $J_i^* \subseteq J_i' \subseteq J_i$ and $|J_i'| = k_i$ for each $i \in [3]$ such that $\bigcup_{i=1}^{3} \bigcup_{j \in J_i'} Y_{i,j}$ is free with respect to $\mathcal{E}$, and $X_{i,j} \cap X_{i',j'} = \emptyset$ for all distinct pairs $(i, j), (i', j')$ with $1 \leq i \leq i' \leq 3$, $j \in J_i'$ and $j' \in J_{i'}'$, or (ii') there exists $Z'_v \subseteq E(G)$ with $|Z'_v| \leq (3hd^2)^{d+1}$ and $i^* \in [3]$ with $|J^*_{i^*}| < k_{i^*}$ such that for every $j \in J_{i^*}$, either $Y_{i^*,j} \cap Z'_v \neq \emptyset$ or $Y_{i^*,j}$ is not free with respect to $\mathcal{E} - Z'_v$.

Suppose that (i') holds. Let $\mathcal{S}'_2 = \{Y_{2,j} : j \in J'_2\}$ and $\mathcal{S}'_3 = \{Y_{3,j} : j \in J'_3\}$. Let $X_0 = \bigcup_{Y \in \mathcal{S}'_2 \cup \mathcal{S}'_3} Y$. Let $D_v = \mathcal{S}'_3$. Since $|J'_2| = hd$, there exists a partition $(D_u : u \in (U^* \cup \{v_i : 1 \leq i \leq h - |U^*|\}) - \{v\})$ of $\mathcal{S}'_2$ into subsets of size at least d. Note that each member of D_v consists of two edges incident with a vertex in $V(G) - \{v\}$, and for every $u \in (U^* \cup \{v_i : 1 \leq i \leq h - |U^*|\}) - \{v\}$, each member of D_u consists of two edges incident with $v \in V(G) - \{u\}$. Hence G contains an H-immersion by Lemma 4.4, a contradiction.

Therefore, (ii') holds. Since $k_1 = |J^*_1|$, $i^* \in \{2, 3\}$. Let $Z_v = Z'_v \cup \bigcup_{S \in \mathcal{S}_1} S$. So $|Z_v| \leq (3hd^2)^{d+1} + dh \leq \xi$.

If $i^* = 2$, then let $u = v$; otherwise, let u be a vertex in $V(G) - \{v\}$ such that there exists $X \in \mathcal{S}_{i^*}$ such that u is a common end of all edges in X. If there exists at most one edge of $G - Z_v$ incident with u, then there exists $[A_u, B_u] \in \mathcal{E} - Z_v$ of order at most one such that $u \in A_u$. If there exist at least two edges of $G - Z_v$ incident with u, then let W be a set of two edges of $G - Z_v$ incident with u, so $W \in \mathcal{S}_{i^*}$. Since W is disjoint from Z'_v, W is not free with respect to $\mathcal{E} - Z_v$, so there exists $W' \subseteq W$ and $[A_W, B_W] \in \mathcal{E} - (Z_v \cup W')$ of order less than $|W - W'| = 2 - |W'|$ such that every edge in $W - W'$ has every end in A_W. Since u is an end of any edge of W, $u \in A_W$. Note that $[A_W, B_W] \in \mathcal{E} - Z_v$ has order at most one.

So the claim follows if $i^* = 2$. Hence we may assume that $i^* = 3$. Define $U = \{v\}$ and $C = Z_v$. Then for every $w \in V(G) - U$, either w is incident with at most one edge in $G - Z_v$, or $w \in V(G) - \{v\}$ is a common end of all edges in X for some $X \in \mathcal{S}_3$. But in either case, there exists $[A_w, B_w] \in \mathcal{E} - Z_v$ of order at most $1 \leq d_2 - 1 = d_{|U|+1} - 1$ such that $w \in A_w$, contradicting Claim 2. $\square$

Claim 6: *There exists $Z_0 \subseteq E(G)$ with $|Z_0| \leq (2hd^2)^{d+1} + hd$ such that for every $v \in V(G) - (U^* \cup \{v_i : 1 \leq i \leq h - |U^*|\})$, there exists $[A_v, B_v] \in \mathcal{E} - Z_0$ of order at most one such that $v \in A_v$.*

Proof of Claim 6: Let $\mathcal{S}_1 = \mathcal{S}^* \cup \{X_i : 1 \leq i \leq h - |U^*|\}$. Let $\mathcal{S}_2 = \{W \subseteq E(G) : W$ consists of two edges of G sharing at least one common end $u \in V(G) - (U^* \cup \{v_i : 1 \leq i \leq h - |U^*|\})\}$. For $i \in [2]$, let J_i be a set such that $\mathcal{S}_i$ can be written as $\{Y_{i,j} : j \in J_i\}$. Let $J^*_1 = J_1$ and $J^*_2 = \emptyset$. Let $k_1 = |J^*_1|$ and $k_2 = hd$. Note that $\max\{k_1, k_2\} \leq hd$ and $|S| \leq \max\{d, 2\} = d$ for every $S \in \mathcal{S}_1 \cup \mathcal{S}_2$, since $d_2 \geq 2$. Since $\theta \geq 3(hd \cdot 2 \cdot d)^{d+1} + 3d$ and $|J^*_1| = k_1$, by Lemma 3.3, either

(i') there exist J'_1, J'_2 with $J^*_i \subseteq J'_i \subseteq J_i$ and with $|J'_i| = k_i$ for each $i \in [2]$ such that the members of $\{Y_{1,j} : j \in J'_1\} \cup \{Y_{2,j} : j \in J'_2\}$ are pairwise disjoint, and $\bigcup_{i=1}^{2} \bigcup_{j \in J'_i} Y_{i,j}$ is free with respect to $\mathcal{E}$, or

(ii') there exists $Z'_0 \subseteq E(G)$ with $|Z'_0| \leq (2hd^2)^{d+1}$ such that for every $Y_{2,j} \in \mathcal{S}_2$, either $Y_{2,j} \cap Z'_0 \neq \emptyset$, or $Y_{2,j}$ is not free with respect to $\mathcal{E} - Z'_0$.

If (i') holds, then G contains an H-immersion by Lemma 4.4. So (ii') holds. Let $Z_0 = Z_0' \cup \bigcup_{Y \in \mathcal{S}_1} Y$. Then $|Z_0| \leq (2hd^2)^{d+1} + hd$. For every $v \in V(G) - (U^* \cup \{v_i : 1 \leq i \leq h - |U^*|\})$, either v is incident with at most one edge of $G - Z_0$, or there exists $W \in \mathcal{S}_2$ consisting of two edges of $G - Z_0$ incident with v such that W is not free with respect to $\mathcal{E} - Z_0$. For the former, there exists $[A_v, B_v] \in \mathcal{E} - Z_0$ of order at most one such that $v \in A_v$ and we are done. So we may assume that there exists $W \in \mathcal{S}_2$ consisting of two edges of $G - Z_0$ incident with v such that W is not free with respect to $\mathcal{E} - Z_0$. Since W is disjoint from Z_0', W is not free with respect to $\mathcal{E} - Z_0$, so there exists $W' \subseteq W$ and $[A_W, B_W] \in \mathcal{E} - (Z_0 \cup W')$ of order less than $|W - W'| = 2 - |W'|$ such that every edge in $W - W'$ has every end in A_W. Since v is an end of any edge of W, $v \in A_W$. Note that $[A_W, B_W] \in \mathcal{E} - Z_0$ has order at most one. This proves the claim. □

Define $C = Z_0 \cup \bigcup_{v \in U^* \cup \{v_i : 1 \leq i \leq h - |U^*|\}} Z_v$. So $|C| \leq (2hd^2)^{d+1} + hd + h \cdot ((3hd^2)^{d+1} + dh) \leq \xi$. By Claims 5 and 6, for every $v \in V(G)$, there exists $[A_v, B_v] \in \mathcal{E} - C$ of order at most $1 \leq d_1 - 1$ such that $v \in A_v$. It is a contradiction to Claim 2 by choosing $U = \emptyset$. This proves the theorem. □

5. Isolating an immersion

The main result of this section is Lemma 5.6 which states that if a graph that does not contain many edge-disjoint H-immersions has an edge-tangle $\mathcal{E}$ of large order that controls a large complete graph-thorns, then one can delete a bounded number of edges to push all H-immersions in the remaining graph into the first entry of an edge-cut belonging to $\mathcal{E}$. The proof of Lemma 5.6 follows from an induction on the number of components of H. The main difficulty lies at the base case, namely the case that H is connected. This base case will be proved in Lemma 5.5.

Now we sketch the proof of Lemma 5.5. Assume that G does not contain k edge-disjoint H-immersions. Then G does not contain a large graph H' with $|V(H')| = |V(H)|$ as an immersion. So Theorem 4.6 implies that one can delete a bounded number of edges from G such that for every vertex v in the remaining graph not contained in a set U with $|U| \leq |V(H)| - 1$, there exists an edge-cut $[A_v, B_v]$ in $\mathcal{E}$ of small order such that $v \in A_v$. Hence for every H-immersion Π in the remaining graph, there exists an edge-cut $[A_\Pi, B_\Pi]$ in $\mathcal{E}$ of bounded order such that all its branch vertices are contained in $U \cup A_\Pi$, and A_Π contains at least one branch vertex.

Assume that we can further delete a set of edges of bounded size from G to either decrease $|U|$ or decrease the order of $[A_\Pi, B_\Pi]$ for all H-immersions Π. As $|U|$ is bounded, by repeatedly deleting small sets of edges a bounded number of times, at some point we will keep decreasing the order of $[A_\Pi, B_\Pi]$ for all H-immersions Π, so that eventually $[A_\Pi, B_\Pi]$ will have order 0 for all H-immersions Π. It implies that $\Pi(H)$ is contained in $G[A_\Pi]$ as H is connected and A_Π contains at least one branch vertex. So Lemma 5.5 is proved.

So it suffices to show that we can further delete a set of edges of bounded size to either decrease $|U|$ or decrease the order of $[A_\Pi, B_\Pi]$ for all H-immersions Π.

First we show that if we cannot delete a small number of edges to reduce $|U|$, then one can find a set S_u of many edges incident with u for every vertex $u \in U$, such that $S_{u'}$ and $S_{u''}$ are pairwise disjoint for distinct $u', u'' \in U$, and $\bigcup_{u \in U} S_u$ is free with respect to $\mathcal{E}$. This is the purpose of Lemma 5.1 and can be proved by an application of Menger's theorem.

We may assume that we cannot delete a small set of edges to decrease $|U|$, for otherwise we are done. So we may assume that those sets S_u's exist. Then we shall show that we can delete a bounded number of edges to reduce the order of $[A_\Pi, B_\Pi]$ for all H-immersions Π. This is the purpose of Lemma 5.2. Then Lemma 5.5 follows from repeatedly applying Lemma 5.2.

Now we sketch the proof of Lemma 5.2. For each H-immersion $\Pi = (\pi_V, \pi_E)$, since all branch vertices of Π are contained in $A_\Pi \cup U$, for every edge $e \in E(H)$ with $V(\pi_E(e)) \cap U = \emptyset$, $V(\pi_E(e)) \cap A_\Pi \neq \emptyset$. We say that $G[A_\Pi]$ "fully realizes" an edge e of H if $\pi_E(e) \subseteq G[A_\Pi]$; $G[A_\Pi]$ "partially realizes" an edge e of H if $\pi_E(e) \not\subseteq G[A_\Pi]$ but $V(\pi_E(e)) \cap A_\Pi \neq \emptyset$. Since A_Π contains at least one branch vertex, at least one edge of H is fully realized or partially realized by $G[A_\Pi]$. This leads to the notion of "shell" defined right above the statement of Lemma 5.2, which is a collection of subgraphs of H indicating what the vertices of H whose corresponding branch vertices are contained in U are, what the edges fully realized by $G[A_\Pi]$ are, and what the edges partially realized by $G[A_\Pi]$ are. Note that for each partially realized edge e of H, one can find an edge between A_Π and B_Π contained in $\pi_E(e)$.

Hence, if we can find many H-immersions $\Pi_1, \Pi_2, ...$, where $G[A_{\Pi_i}] \cap \Pi_i(H)$ are pairwise edge-disjoint, such that the union of $\bigcup_{u \in U} S_u$ and the set of edges between A_{Π_i} and B_{Π_i} over all i is free, then we can "link" those edges to create k edge-disjoint H-immersions by using Lemma 4.3 to obtain a contradiction. So it implies that we cannot find such H-immersions. Then Lemma 3.3 implies one can delete a bounded number of edges to reduce the order of $[A_\Pi, B_\Pi]$ for all H-immersions Π, so that Lemma 5.2 is proved.

Now we formally prove all results in this section.

Lemma 5.1. *For any positive integers h, w, there exists a nonnegative integer $\xi^* = \xi^*(h, w)$ such that the following holds. Let θ, ξ, p be positive integers with $\theta \geq \xi + \xi^* + 1$. Let G be a graph and $\mathcal{E}$ an edge-tangle in G of order at least θ. Assume that there exist $Y_0 \subseteq E(G)$ with $|Y_0| \leq \xi$, $U_0 \subseteq V(G)$ with $|U_0| \leq h - 1$ and a family $\mathcal{F}_0 \subseteq \mathcal{E} - Y_0$ of edge-cuts of $G - Y_0$ of order less than p. Then there exist $U \subseteq U_0$, a set Z with $Y_0 \subseteq Z \subseteq E(G)$ with $|Z| \leq |Y_0| + \xi^*$, a family $\mathcal{F} \subseteq \mathcal{E} - Z$ of edge-cuts of $G - Z$ of order less than p and a collection $\{S_u : u \in U\}$ such that the following hold.*

1. *$U_0 - U \subseteq \bigcup_{[A,B] \in \mathcal{F}} A$ and $\mathcal{F}_0 \subseteq \mathcal{F}$.*
2. *For every $u \in U$, S_u consists of w edges of $G - Z$ incident with u.*
3. *$S_u \cap S_{u'} = \emptyset$ for distinct $u, u' \in U$.*
4. *$\bigcup_{u \in U} S_u$ is free with respect to $\mathcal{E} - Z$.*

Proof. Let $a_0 = 0$, and for every positive integer i, let $a_i = a_{i-1}+2h^2w$. Define $\xi^* = a_{h-1}$.

Let r be an integer with $0 \leq r \leq h-1$ such that there exist $Y_r \subseteq E(G)$ with $Y_0 \subseteq Y_r$ and $|Y_r| \leq |Y_0| + a_r$, $U_r \subseteq U_0$ with $|U_r| \leq |U_0| - r$ and a family $\mathcal{F}_r \subseteq \mathcal{E} - Y_r$ of edge-cuts of $G - Y_r$ of order less than p with $U_0 - U_r \subseteq \bigcup_{[A,B]\in\mathcal{F}_r} A$ and $\mathcal{F}_0 \subseteq \mathcal{F}_r$. Note that such a number r exists as we can choose $r = 0$. We assume that r is as large as possible.

We shall prove that this lemma is true if we take $U = U_r$, $Z = Y_r$ and $\mathcal{F} = \mathcal{F}_r$. It suffices to prove the existence of a collection $\{S_u : u \in U_r\}$ satisfying Statements 2-4.

If $r \geq |U_0|$, then $U_r = \emptyset$, so Statements 2-4 hold. So we may assume that $r \leq |U_0|-1 \leq h-2$.

Claim 1: *There exists a collection $\{S_u : u \in U_r\}$ of pairwise disjoint sets such that for every $u \in U_r$, S_u consists of w edges of $G - Y_r$ incident with u.*

Proof of Claim 1: Let U'_r be a minimal subset of U_r such that there exists a collection $\{S_u : u \in U_r - U'_r\}$ of pairwise disjoint sets such that for every $u \in U_r - U'_r$, S_u consists of w edges incident with u of $G - Y_r$ whose every end is in $U_r - U'_r$. So for every two distinct $u, u' \in U'_r$, there exist at most $2w-1$ edges of $G - Y_r$ between u, u' by the minimality of U'_r. Let Y be the set consists of the non-loop edges whose both ends are in U'_r. Note that $|Y| \leq \binom{|U'_r|}{2}(2w-1) \leq (h-1)^2(2w-1)$. Let $G' = G - (Y_r \cup Y)$.

To prove this claim, it suffices to show that there exists a collection $\{S_u : u \in U'_r\}$ of pairwise disjoint sets such that for every $u \in U'_r$, S_u consists of w edges of G' incident with u.

Define H' to be the directed graph such that the following hold.

- $V(H')$ is the disjoint union of a set Q and a set R, where Q is a copy of U'_r and R is a copy of $V(G)$. For each $u \in U'_r$, we denote the copy of u in Q by u'; for each $v \in V(G)$, we denote the copy of v in R by v'.
- Every edge of H' is from Q to R.
- For every $u' \in Q$ and $v' \in R$ with $u \neq v$, the number of edges of H' from u' to v' equals the number of edges of G' with ends u, v.
- For every $u' \in Q$ and $v' \in R$ with $u = v$, the number of edges of H' from u' to v' equals the number of loops of G' incident with u.

Note that no two distinct vertices in U'_r are adjacent in G'. So there exists a bijection g between $E(H')$ and the set of edges of G' incident with U'_r such that for every edge e of $E(H')$, the ends of $g(e)$ are exactly the originals of ends of e. Define H to be the directed graph obtained from H' by adding two new vertices s, t and adding w edges from s to u' and $w|U'_r|$ edges from v' to t for each $u' \in Q$ and $v' \in R$.

Assume that there exist $w|U'_r|$ edge-disjoint directed paths $P_1, P_2, ..., P_{w|U'_r|}$ in H from s to t. So every edge of H incident with s belongs to $\bigcup_{i=1}^{w|U'_r|} P_i$. Hence for every $u' \in Q$, there exist w edges in $\bigcup_{i=1}^{w|U'_r|} P_i$ from u' to R. For each $u \in U'_r$, define $S_u = \{g(e) : e \in E(\bigcup_{i=1}^{w|U'_r|} P_i), e \text{ is from } u \text{ to } R\}$. Then the collection $\{S_u : u \in U'_r\}$ consists of pairwise disjoint sets, and S_u consists of w edges of G' incident with u for each $u \in U'_r$. So the claim holds.

Hence we may assume that there do not exist $w|U'_r|$ edge-disjoint directed paths in H from s to t. By Menger's Theorem, there exists $X \subseteq E(H)$ with $|X| < w|U'_r|$ such that there exists no directed path in $H - X$ from s to t. We assume that $|X|$ is minimum. Since for each $v' \in R$, there exist $w|U'_r| > |X|$ edges from v' to t, we know there exists an edge in $H - X$ from v' to t. So X does not contain any edge incident with t, for otherwise removing any edge incident with t from X does not create a directed path from s to t, contradicting the minimality of X. Let T be the subset of Q consisting of the vertices that can be reached from s by a directed path in $H - X$. So there exists no directed path in $H - X$ from T to t. Note that $T \neq \emptyset$ since there are more than $|X|$ edges incident with s. Since X does not contain any edge incident with t, there exist no directed path in $H - X$ from T to R. Let $X' = \{g(e) : e \in E(H - s) \cap X\}$. Then X' contains all the edges of G' incident with $\{u \in U'_r : u' \in T\}$.

Let $[A, B] = [\{u \in U'_r : u' \in T\}, V(G) - \{u \in U'_r : u' \in T\}]$. Define $Y_{r+1} = Y_r \cup Y \cup X'$, $U_{r+1} = U_r - \{u \in U'_r : u' \in T\}$ and $\mathcal{F}_{r+1} = \mathcal{F}_r \cup \{[A, B]\}$. Then $|Y_{r+1}| \leq |Y_r| + |Y| + |X'| \leq (|Y_0| + a_r) + (h-1)^2(2w-1) + w|U'_r| - 1 \leq (|Y_0| + a_r) + (h-1)^2(2w-1) + (w(h-1)-1) \leq |Y_0| + a_{r+1}$. And $|U_{r+1}| = |U_r| - |T| \leq |U_0| - (r+1)$. Note that $[A, B]$ is an edge-cut of $G - Y_{r+1}$ of order zero. So $\mathcal{F}_{r+1}$ is a family of edge-cuts of $G - Y_{r+1}$ of order less than p such that $U_0 - U_{r+1} = (U_0 - U_r) \cup \{u \in U'_r : u' \in T\} \subseteq \bigcup_{[A',B'] \in \mathcal{F}_{r+1}} A'$. Since there are at most $|Y \cup X'| < (h-1)^2(2w-1) + (w(h-1)-1) \leq \theta - |Y_r|$ edges of $G - Y_r$ incident with A, $[A, B] \in \mathcal{E} - Y_r$ by (E1) and (E3). So $[A, B] \in \mathcal{E} - Y_{r+1}$. This contradicts the maximality of r. □

Let $\{S_u : u \in U_r\}$ be a collection mentioned in Claim 1. To prove the claim, it suffices to prove that $\bigcup_{u \in U_r} S_u$ is free with respect to $\mathcal{E} - Y_r$.

Suppose to the contrary that $\bigcup_{u \in U_r} S_u$ is not free with respect to $\mathcal{E} - Y_r$. Then there exist $X \subseteq \bigcup_{u \in U_r} S_u$ and $[A, B] \in \mathcal{E} - (Y_r \cup X)$ of order less than $|(\bigcup_{u \in U_r} S_u) - X|$ such that every edge in $(\bigcup_{u \in U_r} S_u) - X$ has every end in A. Note that $(\bigcup_{u \in U_r} S_u) - X \neq \emptyset$ since there exists no edge-cut of order less than 0. Let X' be the union of X and the set of edges of $G - Y_r$ with one end in A and one end in B. So $|X'| \leq |X| + |\bigcup_{u \in U_r} S_u| \leq 2|\bigcup_{u \in U_r} S_u| \leq 2(h-1)w$.

Define $Y_{r+1} = Y_r \cup X'$, $U_{r+1} = U_r - A$, and $\mathcal{F}_{r+1} = \mathcal{F}_r \cup \{[A, B]\}$. So $|Y_{r+1}| \leq |Y_r| + |X'| \leq |Y_0| + a_r + 2(h-1)w \leq |Y_0| + a_{r+1}$. Note that every edge in $(\bigcup_{u \in U_r} S_u) - X \neq \emptyset$ has every end in A, so $U_r \cap A \neq \emptyset$. Hence $|U_{r+1}| \leq |U_r| - 1 \leq |U_0| - (r+1)$. Since $[A, B] \in \mathcal{E} - Y_r$, $[A, B] \in \mathcal{E} - Y_{r+1}$ is an edge-cut of $G - Y_{r+1}$ of order 0. So $\mathcal{F}_r \subseteq \mathcal{F}_{r+1}$ and $U_0 - U_{r+1} \subseteq \bigcup_{[A',B'] \in \mathcal{F}_{r+1}} A'$. This contradicts the maximality of r and proves the lemma. □

Let G be a graph and S a subgraph of G. We define S^+_G to be the graph obtained from S by attaching $\deg_G(v) - \deg_S(v)$ leaves to v, for each $v \in V(S)$. So every vertex in $V(S^+_G) - V(S)$ corresponds to an edge in $E(G) - E(S)$. Note that if e is an edge in

$E(G) - E(S)$ with both ends u, v in $V(S)$, then e contributes two leaves to S_G^+, where one is adjacent to u and one is adjacent to v. In particular, if $e \in E(G) - E(S)$ is a loop incident with a vertex v in S, then e contributes two leaves adjacent to v in S_G^+.

Let G and H be graphs, and let S, R be subgraphs of G, H, respectively. We say that S_G^+ *realizes* R_H^+ if S_G^+ contains a R_H^+-immersion (π_V, π_E) such that $\pi_V(V(R_H^+) - V(R)) \subseteq V(S_G^+) - V(S)$ and $\pi_V(V(R)) \subseteq V(S)$.

Let H be a graph. A *shell* of H is a collection of disjoint connected subgraphs of H such that every vertex of H is contained in a member of the collection. For any H-immersion $\Pi = (\pi_V, \pi_E)$ in a graph G, we denote the subgraph $\bigcup_{e \in E(H)} \pi_E(e) \cup \bigcup_{v \in V(H)} \pi_V(v)$ of G by $\Pi(H)$.

Lemma 5.2. *For every connected graph H and for positive integers k, p, ξ_0', there exist integers $\theta^* = \theta^*(H, k, p, \xi_0'), w^* = w^*(H, k, p, \xi_0'), \xi^* = \xi^*(H, k, p, \xi_0')$ such that the following holds. Assume that G is a graph that does not contain k edge-disjoint H-immersions and $\mathcal{E}$ is an edge-tangle in G of order at least θ^* controlling a K_w-thorns for some $w \geq w^*$. If there exist $U' \subseteq V(G)$ with $|U'| \leq |V(H)| - 1$, $Z_0' \subseteq E(G)$ with $|Z_0'| \leq \xi_0'$ and a family $\mathcal{F}' \subseteq \mathcal{E} - Z_0'$ of edge-cuts of $G - Z_0'$ of order less than p such that for every H-immersion $L = (\pi_V, \pi_E)$ in $G - Z_0'$, there exists $[A_L', B_L'] \in \mathcal{F}'$ such that $\pi_V(V(H)) \subseteq U' \cup A_L'$, then there exist $U \subseteq U'$, $Z^* \subseteq E(G)$ with $|Z^*| \leq \xi^*$ and a family $\mathcal{F}^* \subseteq \mathcal{E} - Z^*$ of edge-cuts of $G - Z^*$ such that either*

1. *$U \subset U'$, every member of $\mathcal{F}^*$ has order less than $|V(H)|p$, and for every H-immersion $\Pi = (\pi_V, \pi_E)$ in $G - Z^*$, there exists $[A_\Pi^*, B_\Pi^*] \in \mathcal{F}^*$ such that $\pi_V(V(H)) \subseteq U \cup A_\Pi^*$, or*
2. *every member of $\mathcal{F}^*$ has order less than $p - 1$, and for every H-immersion $\Pi = (\pi_V, \pi_E)$ in $G - Z^*$, either there exists $[A, B] \in \mathcal{E} - Z^*$ of order zero such that $\Pi(H) \subseteq G[A]$, or there exists $[A_\Pi^*, B_\Pi^*] \in \mathcal{F}^*$ such that $\pi_V(V(H)) \subseteq U \cup A_\Pi^*$.*

Proof. Let H be a connected graph with degree sequence $(d_1, d_2, ..., d_h)$, where $h = |V(H)|$. Let k, p, ξ_0' be positive integers. We define the following.

- Let $\xi_0 = \xi_0' + \xi_{5.1}$, where $\xi_{5.1}$ is the number $\xi^*(h, kd_1)$ mentioned in Lemma 5.1.
- Let $\xi_0'' = (khd_1(kh + 2)(kd_1 + p))^{kd_1 + p + 1}$.
- Define $\xi^* = \xi_0 + (2h^2 d_1)^h \xi_0''$, $w^* = 6k^2 h^2 d_1 p + \xi^*$ and $\theta^* = \xi^* + w^* + hp$.

Let G be a graph that does not contain k edge-disjoint H-immersions, and let $\mathcal{E}$ be an edge-tangle of order at least θ^* in G controlling a K_w-thorns α for some $w \geq w^*$. Assume there exist $U' \subseteq V(G)$ with $|U'| \leq |V(H)| - 1$, $Z_0' \subseteq E(G)$ with $|Z_0'| \leq \xi_0'$ and a family $\mathcal{F}' \subseteq \mathcal{E} - Z_0'$ of edge-cuts of $G - Z_0'$ of order less than p such that for every H-immersion $L = (\pi_V, \pi_E)$ in $G - Z_0'$, there exists $[A_L', B_L'] \in \mathcal{F}'$ such that $\pi_V(V(H)) \subseteq U' \cup A_L'$.

By Lemma 5.1, there exist $U \subseteq U'$, a set Z_0 with $Z_0' \subseteq Z_0 \subseteq E(G)$ with $|Z_0| \leq \xi_0$, a family $\mathcal{F} \subseteq \mathcal{E} - Z_0$ of edge-cuts of $G - Z_0$ of order less than p and a collection $\{S_u : u \in U\}$ such that the following hold.

- $U' - U \subseteq \bigcup_{[A,B] \in \mathcal{F}} A$ and $\mathcal{F}' \subseteq \mathcal{F}$.
- For every $u \in U$, S_u consists of kd_1 edges of $G - Z_0$ incident with u.
- $S_u \cap S_{u'} = \emptyset$ for distinct $u, u' \in U$.
- $\bigcup_{u \in U} S_u$ is free with respect to $\mathcal{E} - Z_0$.

We may assume that U is inclusion-wise minimal subject to the conditions above. So for every $u \in U$, there exists no $[A, B] \in \mathcal{E} - Z_0$ of order less than p such that $u \in A$, for otherwise, we may add $[A, B] \in \mathcal{F}$ and remove u from U.

For any subset Z of $E(G)$, we say an H-immersion Π in $G - Z$ is *active* (with respect to Z) if there does not exist $[A, B] \in \mathcal{E} - Z$ of order zero such that $\Pi(H) \subseteq G[A]$.

Suppose that this lemma does not hold.

Claim 1: *$G - Z_0$ contains an active H-immersion with respect to Z_0, and $U = U'$.*

Proof of Claim 1: If $G - Z_0$ contains no active H-immersion with respect to Z_0, then Statement 2 of this lemma holds by taking $Z^* = Z_0$ and $\mathcal{F}^* = \emptyset$, a contradiction. So $G - Z_0$ contains an active H-immersion with respect to Z_0.

Now we suppose that $U \subset U'$. Since $U' - U \subseteq \bigcup_{[A,B] \in \mathcal{F}} A$, for each $u \in U' - U$, there exists $[A_u, B_u] \in \mathcal{F}$ with $u \in A_u$. Since the order of $\mathcal{E} - Z_0$ is at least hp, we know for every $[A, B] \in \mathcal{F}$, $[A \cup \bigcup_{u \in U'-U} A_u, B \cap \bigcap_{u \in U'-U} B_u]$ is an edge-cut of order less than $|[A, B]| + (h - 1)p < hp$ and hence belongs to $\mathcal{E} - Z_0$ by Lemma 2.3. Since for every H-immersion $L = (\pi_V, \pi_E)$ in $G - Z_0 \subseteq G - Z_0'$, there exists $[A_L, B_L] \in \mathcal{F}' \subseteq \mathcal{F}$ such that $\pi_V(V(H)) \subseteq U' \cup A_L$, we know $\pi_V(V(H)) \subseteq U \cup A_L \cup \bigcup_{u \in U'-U} A_u$. So Statement 1 of this lemma follows if we take $Z^* = Z_0$ and $\mathcal{F}^* = \{[A \cup \bigcup_{u \in U'-U} A_u, B \cap \bigcap_{u \in U'-U} B_u] : [A, B] \in \mathcal{F}\}$, a contradiction. □

For any member S of some shell of H and any active H-immersion $L = (\pi_V, \pi_E)$ in $G - Z_0$ with respect to Z_0, we say that an edge-cut $[A, B]$ of $G - Z_0$ is *useful for L, S* if the following hold.

- $[A, B] \in \mathcal{E} - Z_0$ and the order of $[A, B]$ is less than p.
- $\pi_V(V(H)) \subseteq A \cup U$.
- $G[A]^+_G$ realizes S^+_H.
- For every vertex in A, there exists a path in $G[A] - Z_0$ from this vertex to an end of an edge between A and B.

Claim 2: *For every active H-immersion $L = (\pi_V, \pi_E)$ in $G - Z_0$ with respect to Z_0, there exist a shell $\mathcal{P}_L$ of H and $[A_L, B_L] \in \mathcal{E} - Z_0$ of order less than p such that $\{v\} \in \mathcal{P}_L$ for* *every $v \in V(H)$ with $\pi_V(v) \in U$, and $[A_L, B_L]$ is useful for L, S for every member S of $\mathcal{P}_L - \{\{v\} : \pi_V(v) \in U\}$.*

Proof of Claim 2: Let $L = (\pi_V, \pi_E)$ be an active H-immersion in $G - Z_0$ with respect to Z_0. Note that L is an H-immersion in $G - Z'_0$, so there exists $[A_L, B_L] \in \mathcal{F}' \subseteq \mathcal{F}$ such that $\pi_V(V(H)) \subseteq U' \cup A_L = U \cup A_L$. Since $\mathcal{F}' \subseteq \mathcal{E} - Z_0$ and every member of $\mathcal{F}'$ is an edge-cut $G - Z'_0$ of order less than p, we know $[A_L, B_L] \in \mathcal{E} - Z_0$ is an edge-cut of $G - Z_0$ of order less than p.

Let S' be the subgraph of H such that $V(S') = \{v \in V(H) : \pi_V(v) \in A_L\}$ and $E(S') = \{e \in E(H) : \pi_E(e) \subseteq G[A_L]\}$. Let $\mathcal{P}_L$ be the shell of H that is the union of the set $\{\{v\} : v \in V(H) - V(S')\}$ and the collection consisting of the components of S'. Since for every $u \in U$, there exists no $[A, B] \in \mathcal{E} - Z_0$ of order less than p such that $u \in A$, we know that $\{\{v\} : v \in V(H) - V(S')\} = \{\{v\} : v \in V(H), \pi_V(v) \in U\}$. Since $|U| \leq h - 1$, $\pi_V(v) \notin U$ for some $v \in V(H)$, so S' contains at least one vertex. Then $G[A_L]^+_G$ realizes S^+_H for every member S of $\mathcal{P}_L - \{\{v\} : v \in V(H), \pi_V(v) \in U\}$.

So there exists $[A_L, B_L] \in \mathcal{E} - Z_0$ satisfying the first three conditions of being useful for L, S, for every member S of $\mathcal{P}_L - \{\{v\} : v \in V(H), \pi_V(v) \in U\}$. We further choose such $[A_L, B_L]$ such that the order of $[A_L, B_L]$ is as small as possible, and subject to that, A_L is minimal. To show that $[A_L, B_L]$ is useful for L, S for every member S of $\mathcal{P}_L - \{\{v\} : v \in V(H), \pi_V(v) \in U\}$, it suffices to show that for every vertex in A_L, there exists a path in $G[A_L] - Z_0$ from this vertex to an end of an edge between A_L and B_L.

Since L is active, the order of $[A_L, B_L]$ is greater than zero. Suppose that there exists a vertex in A_L such that there exists no path in $G[A_L] - Z_0$ from this vertex to an edge between A_L and B_L. Then there exists a component C of $G[A_L] - Z_0$ such that there exists no path in $G - Z_0$ from $V(C)$ to any edge between A_L and B_L. We define $[A'_L, B'_L] = [A_L - V(C), B_L \cup V(C)]$. Since the order of $[A'_L, B'_L]$ is the same as the order of $[A_L, B_L]$, we know $[A'_L, B'_L] \in \mathcal{E} - Z_0$ by Lemma 2.3. By the minimality of A_L, $V(C)$ contains $\pi_V(v)$ for some $v \in V(H)$. Since H is connected and C is a component of $G[A_L] - Z_0$, C contains $\pi_E(E(H))$. So $[V(C), V(G) - V(C)]$ is an edge-cut of $G - Z_0$ of order zero such that C contains $\pi_E(E(H))$ and hence $G[V(C)]^+_G$ realizes H^+_H. Note that it implies that $\pi_V(V(H)) \cap U = \emptyset$, so $\{H\}$ is a shell $\mathcal{P}'$ of H with $\{v\} \in \mathcal{P}'$ for each $v \in V(H)$ with $\pi_V(v) \in U$. In addition, $[V(C), V(G) - V(C)] \in \mathcal{E} - Z_0$ by Lemma 2.3. Since the order of $[A_L, B_L]$ is greater than 0, it contradicts the minimality of the order of $[A_L, B_L]$. $\square$

For every shell $\mathcal{P}$ of H and every subset D of $\{v \in V(H) : \{v\} \in \mathcal{P}\}$ of size at most $|U|$, we define the following.

- Define $H_{\mathcal{P},D}$ to be the graph obtained from the disjoint union of k copies of H by for each $v \in D$, identifying the k copies of v into a vertex. Note that $|V(H_{\mathcal{P},D})| = k(|V(H)| - |D|) + |D|$, and for any two (not necessarily distinct) vertices in D, if there are ℓ edges of H between then, then there are $k\ell$ edges of $H_{\mathcal{P},D}$ between them.

Note that G contains no $H_{\mathcal{P},D}$-immersion, for otherwise G contains k edge-disjoint H-immersions.

- Define $\mathcal{Q}_{\mathcal{P},D}$ to be the shell of $H_{\mathcal{P},D}$ consisting of $\{v\}$, for each $v \in D$, and the members of $\mathcal{P} - \{\{v\} : v \in D\}$ in each copy of H.
- For each $S \in \mathcal{Q}_{\mathcal{P},D}$, define $\mathcal{X}_S = \{X : X$ is the set of edges between A_L and B_L, for some active H-immersion L in $G - Z_0$ and some edge-cut $[A_L, B_L]$ of $G - Z_0$ that is useful for $L, S\}$. Note that each member of $\mathcal{X}_S$ has size at most p.

Define $\mathcal{X}_0 = \{S_u : u \in U\}$. Recall that $\bigcup_{X \in \mathcal{X}_0} X$ is free with respect to $\mathcal{E} - Z_0$. Define $\mathcal{X}_E$ to be the collection of the 2-element subsets of $E(G - Z_0)$ each consisting of two edges having at least one common end. Define $\mathcal{X}_0^* = \mathcal{X}_0$, $k_0 = |U|$, $\mathcal{X}_E^* = \emptyset$, $k_E = khd_1$, $\mathcal{X}_S^* = \emptyset$ and $k_S = kh$ for each $S \in \mathcal{Q}_{\mathcal{P},D} - \{\{v\} : v \in D\}$. Note that $|\mathcal{Q}_{\mathcal{P},D}| \leq kh$.

Claim 3: *For every shell $\mathcal{P}$ of H and every subset D of $\{v \in V(H) : \{v\} \in \mathcal{P}\}$ of size at most $|U|$, there exist $Z_{\mathcal{P},D} \subseteq E(G) - Z_0$ with $|Z_{\mathcal{P},D}| \leq \xi_0''$ and $S \in \mathcal{Q}_{\mathcal{P},D} - \{\{v\} : v \in D\}$ such that for every $X \in \mathcal{X}_S$, either $X \cap Z_{\mathcal{P},D} \neq \emptyset$, or X is not free with respect to $\mathcal{E} - (Z_0 \cup Z_{\mathcal{P},D})$.*

Proof of Claim 3: Let $\mathcal{P}$ be a shell of H and D a subset of $\{v \in V(H) : \{v\} \in \mathcal{P}\}$ of size at most $|U|$. Notice that $|\mathcal{X}_0^*| = k_0$. By Lemma 3.3, one of the following holds.

(i) There exist a collection $\mathcal{X}_0'$ of size k_0 with $\mathcal{X}_0^* \subseteq \mathcal{X}_0' \subseteq \mathcal{X}_0$, a collection $\mathcal{X}_E'$ of size $k_E = khd_1$ with $\mathcal{X}_E^* \subseteq \mathcal{X}_E' \subseteq \mathcal{X}_E$ and collections $\mathcal{X}_S'$ of size $k_S = kh$ with $\mathcal{X}_S^* \subseteq \mathcal{X}_S' \subseteq \mathcal{X}_S$ for each $S \in \mathcal{Q}_{\mathcal{P},D} - \{\{v\} : v \in D\}$ such that $\mathcal{X}_0' \cup \mathcal{X}_E' \cup \bigcup_{S \in \mathcal{Q}_{\mathcal{P},D} - \{\{v\}:v \in D\}} \mathcal{X}_S'$ consists of pairwise disjoint members, and the union of its members is free with respect to $\mathcal{E} - Z_0$.

(ii) There exist $Z_{\mathcal{P},D} \subseteq E(G) - Z_0$ with $|Z_{\mathcal{P},D}| \leq (khd_1(|\mathcal{Q}_{\mathcal{P},D}| - |D| + 2)(kd_1 + p))^{kd_1+p+1} \leq \xi_0''$ and $S \in \mathcal{Q}_{\mathcal{P},D} - \{\{v\} : v \in D\}$ such that for every $X \in \mathcal{X}_S$, either $X \cap Z_{\mathcal{P},D} \neq \emptyset$, or X is not free with respect to $\mathcal{E} - (Z_0 \cup Z_{\mathcal{P},D})$.

(iii) There exists $Z_{\mathcal{P},D} \subseteq E(G) - Z_0$ with $|Z_{\mathcal{P},D}| \leq (khd_1(|\mathcal{Q}_{\mathcal{P},D}| - |D| + 2)(kd_1 + p))^{kd_1+p+1} \leq \xi_0''$ such that every set of two edges of $G - (Z_0 \cup Z_{\mathcal{P},D})$ sharing at least one common end is not free with respect to $\mathcal{E} - (Z_0 \cup Z_{\mathcal{P},D})$.

Note that Statement (iii) cannot hold by Lemma 2.18 since $\theta \geq \xi_0 + \xi_0'' + 2$. To prove this claim, it suffices to show that Statement (i) does not hold.

Suppose to the contrary that Statement (i) holds. We shall derive a contradiction by showing that G contains k edge-disjoint H-immersions.

Let X be the union of the members of $\mathcal{X}_0' \cup \mathcal{X}_E' \cup \bigcup_{S \in \mathcal{Q}_{\mathcal{P},D} - \{\{v\}:v \in D_{\mathcal{P}}\}} \mathcal{X}_S'$. So X is free with respect to $\mathcal{E} - Z_0$, and $|X| \leq hkd_1 + 2hkd_1 + kh \cdot kh \cdot p \leq 4h^2k^2d_1p$. Since α is a K_w-thorns controlled by $\mathcal{E}$, there exists a $K_{w-\xi_0}$-thorns α' in $G - Z_0$ controlled by $\mathcal{E} - Z_0$. Note that $w - \xi_0 \geq w^* - \xi_0 \geq \frac{3}{2}|X|$.

Suppose that there exist $Y \subseteq X$ and an edge-cut $[A, B]$ of $G - (Z_0 \cup Y)$ of order less than $|X| - |Y|$ such that every edge in $X - Y$ is incident with vertices in A and

$A \cap V(\alpha'(t)) = \emptyset$ for some $t \in V(K_{w-\xi_0})$. By (E1), $[A, B]$ or $[B, A]$ is in $\mathcal{E} - (Z_0 \cup Y)$, and hence $[A, B]$ or $[B, A]$ is in $\mathcal{E} - Z_0$. Since X is free with respect to $\mathcal{E} - Z_0$, $[A, B] \notin \mathcal{E} - Z_0$. Since $\mathcal{E} - Z_0$ controls α', $[B, A] \notin \mathcal{E} - Z_0$, a contradiction.

Therefore, there do not exist $Y \subseteq X$ and an edge-cut $[A, B]$ of $G - (Z_0 \cup Y)$ of order less than $|X| - |Y| \leq \frac{2}{3}(w^* - \xi_0) - |Y|$ such that every edge in $X - Y$ is incident with vertices in A and $A \cap V(\alpha'(t)) = \emptyset$ for some $t \in V(K_{w-\xi_0})$.

For each $S \in \mathcal{Q}_{\mathcal{P},D} - \{\{v\} : v \in D\}$ and $X_S \in \mathcal{X}'_S$,

- define L_S to be an H-immersion $(\pi_V^{(S)}, \pi_E^{(S)})$ in $G - Z_0$ such that X_S is the set of edges between A_{L_S} and B_{L_S}, where $[A_{L_S}, B_{L_S}]$ is a useful edge-cut of $G - Z_0$ for L_S, S,
- let $(\pi_V^{(S,1)}, \pi_E^{(S,1)})$ be an S_H^+-immersion in $G[A_{L_S}]_G^+$ such that $\pi_V(V(S_H^+) - V(S)) \subseteq V(G[A_{L_S}]_G^+) - A_{L_S}$ and $\pi_V(V(S)) \subseteq A_{L_S}$, and
- let f_{X_S} be the injection from $V(S^+) - V(S)$ to X_S such that for every $x \in V(S^+) - V(S)$, $f_{X_S}(x)$ is the edge in X_S contained in $\pi_E^{(S,1)}(e)$, where e is the edge in S_H^+ incident with x.

Define ι_D to be an injection from D to U, and for every $v \in D$, define $f'_{X_{\{v\}}}$ to be an injection from the set of edges of $H_{\mathcal{P},D}$ incident with v to the set $S_{\iota_D(v)}$, and define $f_{X_{\{v\}}}$ to be the injection from $V(H_{\mathcal{P},D}[\{v\}]_{H_{\mathcal{P},D}}^+) - \{v\}$ to $S_{\iota_D(v)}$ such that $f_{X_{\{v\}}}(e) = f'_{X_{\{v\}}}(e')$ for every $e \in V(H_{\mathcal{P},D}[\{v\}])_{H_{\mathcal{P},D}}^+ - \{v\}$ and edge e' of $H_{\mathcal{P},D}[\{v\}]_{H_{\mathcal{P},D}}^+$ (so e' is an edge of $H_{\mathcal{P},D}$) incident with e. Note that for every $v \in D$, $\{v\}$ is a member of $\mathcal{Q}_{\mathcal{P},D}$. So for every $S \in \mathcal{Q}_{\mathcal{P},D}$, f_{X_S} is defined. In addition, $k_E \geq |E(H_{\mathcal{P},D})|$, so there exists an injection ι from $E(H_{\mathcal{P},D})$ to $\mathcal{X}'_E$.

For each edge e of $H_{\mathcal{P},D}$ not contained in any member of $\mathcal{Q}_{\mathcal{P},D}$, we define the following.

- Say e has one end in $S_1 \in \mathcal{Q}_{\mathcal{P},D}$ and one end in $S_2 \in \mathcal{Q}_{\mathcal{P},D}$. Note that S_1 and S_2 are not necessarily distinct, and e corresponds to a leaf e_1 in S_1^+ and a leaf e_2 in S_2^+, where $e_1 \neq e_2$ even if $S_1 = S_2$ or e is a loop. We define $W_e = \{f_{X_{S_1}}(e_1), f_{X_{S_2}}(e_2)\}$.
- Define $W'_e = \iota(e)$. Note that W'_e is a member of $\mathcal{X}'_E$.
- Define $\{W_{e,1}, W_{e,2}\}$ to be a partition of $W_e \cup W'_e$ into two sets of size two each containing exactly one element in W_e.

Let W be the union of $W_{e,1} \cup W_{e,2}$ over all edges e of $H_{\mathcal{P},D}$ not contained in any member of $\mathcal{Q}_{\mathcal{P},D}$. Let $\mathcal{W} = \{W_{e,1}, W_{e,2} : e \in E(H_{\mathcal{P},D})$ not contained in any member of $\mathcal{Q}_{\mathcal{P},D}\}$. Note that W is a subset of X and $\mathcal{W}$ is a partition of W. Let $\mathcal{R} = \{\{x\} : x \in X - W\}$, and let $\mathcal{R}^* = \mathcal{R} \cup \mathcal{W}$. Note that $\mathcal{R}$ is a partition of $X - W$ and $\mathcal{R}^*$ is a partition of X.

Recall that there do not exist $Y \subseteq X$ and an edge-cut $[A, B]$ of $G - (Z_0 \cup Y)$ of order less than $|X| - |Y| \leq \frac{2}{3}(w - \xi_0) - |Y|$ such that every edge in $X - Y$ is incident with vertices in A and $A \cap V(\alpha'(t)) = \emptyset$ for some $t \in V(K_{w-\xi_0})$. So by Lemma 4.3 (where the partition mentioned in Lemma 4.3 is taken to be $\mathcal{R}^*$), there exists a collection $\{T_x : x \in X - W\} \cup \{T_{e,i} : i \in [2], e \in E(H_{\mathcal{P},D})$ not contained in any member of $\mathcal{Q}_{\mathcal{P},D}\}$

of pairwise edge-disjoint connected subgraphs in $G - Z_0$ such that $x \in E(T_x)$ for every $x \in X - W$, and $E(T_{e,i}) \cap X = E(T_{e,i}) \cap W = W_{e,i}$ for each e and i.

Let $\mathcal{Q}'_{\mathcal{P},D} = \mathcal{Q}_{\mathcal{P},D} - \{\{v\} : v \in D\}$. Note that $\bigcup_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} X_S$ contains all edges between $\bigcup_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} A_{L_S}$ and $\bigcap_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} B_{L_S}$. Suppose there exists an edge $x \in X$ whose every end is in $\bigcup_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} A_{L_S}$. Let $Y = X - \{x\}$. Hence, $[\bigcup_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} A_{L_S}, \bigcap_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} B_{L_S}]$ is an edge-cut of $G - (Z_0 \cup Y)$ of order $0 < |X| - |Y|$ such that every edge in $X - Y$ has every end in $\bigcup_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} A_{L_S}$. Note that $[\bigcup_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} A_{L_S}, \bigcap_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} B_{L_S}] \in \mathcal{E} - (Z_0 \cup Y)$ by Lemma 2.3. So X is not free with respect to $\mathcal{E} - Z_0$, a contradiction.

Hence every edge in X has at most one end in $\bigcup_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} A_{L_S}$. In particular, since $\bigcup_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} X_S$ contains all edges between $\bigcup_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} A_{L_S}$ and $\bigcap_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} B_{L_S}$, every edge in a member of $\mathcal{X}'_E$ has every end in $\bigcap_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} B_{L_S}$. Therefore, X consists of all edges between $\bigcup_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} A_{L_S}$ and $\bigcap_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} B_{L_S}$, and some edge whose every end is in $\bigcap_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} B_{L_S}$. It follows that each subgraph $T_{e,i}$ does not contain an edge whose every end is in $\bigcup_{S \in \mathcal{Q}'_{\mathcal{P},D}, X_S \in \mathcal{X}'_S} A_{L_S}$.

Since X is free with respect to $\mathcal{E} - Z_0$ and $X_{S'} \cap X_{S''} = \emptyset$ for distinct $S', S'' \in \mathcal{Q}'_{\mathcal{P},D}$, we have $A_{L_{S'}} \cap A_{L_{S''}} = \emptyset$ for distinct $S', S'' \in \mathcal{Q}'_{\mathcal{P},D}$ by Lemma 2.17. So the subgraphs $T_{e,i}$ together with the intersection of the image of $\pi_E^{(S)}$ and $G[A_{L_S}]$, for each $S \in \mathcal{Q}'_{\mathcal{P},D}$, define a subgraph of $G - Z_0$ containing an $H_{\mathcal{P},D}$-immersion (π_V, π_E) in $G - Z_0$ with $\pi_V(v) = \iota_D(v)$ for every $v \in D$, a contradiction. This proves the claim. $\square$

Let Z^* be the union of Z_0 and the sets $Z_{\mathcal{P},D}$ over all shells $\mathcal{P}$ of H and subsets D of $\{v \in V(H) : \{v\} \in \mathcal{P}\}$ of size at most $|U|$ mentioned in Claim 3. Note that there are at most $h^h(hd_1)^h$ different shells of H, and for each shell $\mathcal{P}$ of H, there are at most 2^h different subsets of $\{v \in V(H) : \{v\} \in \mathcal{P}\}$. So $|Z^*| \leq \xi_0 + (2h^2d_1)^h\xi''_0 \leq \xi^*$.

Note that we may assume that there exists an active H-immersion in $G - Z^*$ with respect to Z^*, for otherwise Statement 2 of this lemma holds by taking $\mathcal{F}^* = \emptyset$. Note that every H-immersion in $G - Z^*$ is an immersion in $G - Z'_0$.

Claim 4: *For every active H-immersion $L = (\pi_V, \pi_E)$ in $G - Z^*$ with respect to Z^*, there exists $[A^*_L, B^*_L] \in \mathcal{E} - Z^*$ of order less than $p - 1$ such that $\pi_V(V(H)) \subseteq U \cup A^*_L$.*

Proof of Claim 4: Let $L = (\pi_V, \pi_E)$ be an active H-immersion in $G - Z^*$ with respect to Z^*. So L is an active H-immersion in $G - Z_0$ with respect to Z_0. By Claim 2, there exist a shell $\mathcal{P}_L$ of H and $[A_L, B_L] \in \mathcal{E} - Z_0$ of order less than p such that $\{v\} \in \mathcal{P}_L$ for every $v \in V(H)$ with $\pi_V(v) \in U$, and $[A_L, B_L]$ is useful for L, S for every member S of $\mathcal{P}_L - \{\{v\} : \pi_V(v) \in U\}$. Let X be the set of edges of $G - Z_0$ between A_L and B_L. So $|X| \leq p - 1$. Let $D = \{v \in V(H) : \pi_V(v) \in U\}$. So D is a subset of $\{v \in V(H) : \{v\} \in \mathcal{P}_{\mathcal{L}}\}$ of size at most $|U|$. Since $|U| \leq |V(H)| - 1$, $X \in \mathcal{X}_S$ for every member S of $\mathcal{Q}_{\mathcal{P}_L,D} - \{\{v\} : v \in D\}$.

By Claim 3, either $X \cap Z^* \neq \emptyset$ or X is not free with respect to $\mathcal{E} - Z^*$. If $X \cap Z^* \neq \emptyset$, then $[A_L, B_L]$ is an edge-cut of $G - Z^*$ of order less than its order in $G - Z_0$, so $[A_L, B_L] \in$

$\mathcal{E} - Z^*$ is an edge-cut of $G - Z^*$ of order less than $p - 1$ such that $\pi_V(V(H)) \subseteq U \cup A_L$ and we are done. So we may assume that $X \cap Z^* = \emptyset$.

Hence X is not free with respect to $\mathcal{E} - Z^*$. So there exist $Y \subseteq X$ and $[A, B] \in \mathcal{E} - (Z^* \cup Y)$ of order less than $|X| - |Y|$ such that every edge in $X - Y$ has every end in A. We assume that the order of $[A, B]$ is as small as possible, and subject to that, A is maximal.

Since $[A, B]$ is an edge-cut of $G - Z^*$ of order less than $|X| \leq p - 1$, $[A, B] \in \mathcal{E} - Z^*$. So we are done if $A_L \subseteq A$.

So we may assume that $A_L \not\subseteq A$. Let $A^* = A \cup A_L$ and $B^* = B \cap B_L$. Since every edge of $G - (Z^* \cup Y)$ between A_L, B_L is an edge in $X - Y$, it is not incident with B. So $[A^*, B^*]$ is an edge-cut of $G - (Z^* \cup Y)$ of order at most the order of $[A, B]$ with $A^* \supset A$. Hence $[A^*, B^*] \in \mathcal{E} - (Z^* \cup Y)$ by Lemma 2.3. But this contradicts the choice of $[A, B]$. This proves the claim. □

Define $\mathcal{F}^* = \{[A_L^*, B_L^*] : L$ is an active H-immersion in $G - Z^*$ with respect to $Z^*\}$, where $[A_L^*, B_L^*]$ is the edge-cut mentioned in Claim 4. Then Statement 2 of this lemma follows. □

Lemma 5.3. *For every connected graph H and for positive integers k, p, ξ, there exist integers $\theta^* = \theta^*(H, k, p, \xi), w^* = w^*(H, k, p, \xi), \xi^* = \xi^*(H, k, p, \xi), p^* = p^*(H, k, p, \xi)$ such that the following holds. Assume that G is a graph that does not contain k edge-disjoint H-immersions and $\mathcal{E}$ is an edge-tangle of G of order at least θ^* controlling a K_w-thorns for some $w \geq w^*$. If there exist $U' \subseteq V(G)$ with $|U'| \leq |V(H)| - 1$, $Z \subseteq E(G)$ with $|Z| \leq \xi$ and a family $\mathcal{F}' \subseteq \mathcal{E} - Z$ of edge-cuts of $G - Z$ of order less than p such that for every H-immersion $L = (\pi_V, \pi_E)$ in $G - Z$, there exists $[A'_L, B'_L] \in \mathcal{F}'$ such that $\pi_V(V(H)) \subseteq U' \cup A'_L$, then there exist $U \subseteq U'$, $Z^* \subseteq E(G)$ with $|Z^*| \leq \xi^*$ and a family $\mathcal{F}^* \subseteq \mathcal{E} - Z^*$ of edge-cuts of $G - Z^*$ such that either*

1. *$U \subset U'$, every member of $\mathcal{F}^*$ has order less than p^*, and for every H-immersion $\Pi = (\pi_V, \pi_E)$ in $G - Z^*$, there exists $[A_\Pi^*, B_\Pi^*] \in \mathcal{F}^*$ such that $\pi_V(V(H)) \subseteq U \cup A_\Pi^*$, or*
2. *for every H-immersion Π in $G - Z^*$, there exists $[A, B] \in \mathcal{E} - Z^*$ of order zero such that $\Pi(H) \subseteq G[A]$.*

Proof. Let H be a connected graph and k, p, ξ be positive integers. Let $h = |V(H)|$. We define the following.

- Let $\xi_0 = \xi$, $\theta_0 = 0$ and $w_0 = 0$.
- For every positive integer i with $1 \leq i \leq p$, define $\theta_i = \theta_{i-1} + \theta_{5.2}(H, k, p-i+1, \xi_{i-1})$, $w_i = w_{i-1} + w_{5.2}(H, k, p-i+1, \xi_{i-1})$, and $\xi_i = \xi_{5.2}(H, k, p-i+1, \xi_{i-1})$, where $\theta_{5.2}, w_{5.2}, \xi_{5.2}$ are the numbers θ^*, w^*, ξ^* mentioned in Lemma 5.2, respectively.
- Define $\theta^* = \sum_{i=1}^{p} \theta_i$, $w^* = \sum_{i=1}^{p} w_i$, $\xi^* = \sum_{i=1}^{p} \xi_i$ and $p^* = hp$.

Let G be a graph that does not contain k edge-disjoint H-immersions, and let $\mathcal{E}$ be an edge-tangle of G of order at least θ^* controlling a K_w-thorns for some $w \geq w^*$. Assume that there exist $U' \subseteq V(G)$ with $|U'| \leq |V(H)|-1$, $Z \subseteq E(G)$ with $|Z| \leq \xi$ and a family $\mathcal{F}' \subseteq \mathcal{E} - Z$ of edge-cuts of $G - Z$ of order less than p such that for every H-immersion $L = (\pi_V, \pi_E)$ in $G - Z$, there exists $[A'_L, B'_L] \in \mathcal{F}'$ such that $\pi_V(V(H)) \subseteq U' \cup A'_L$.

Let $Z_0 = Z$ and $\mathcal{F}_0 = \mathcal{F}'$. Let r be an integer with $0 \leq r \leq p-1$ such that there exist a set $Z_r \subseteq E(G)$ with $|Z_r| \leq \xi_r$ and a family $\mathcal{F}_r \subseteq \mathcal{E} - Z_r$ of edge-cuts of $G - Z_r$ of order less than $p - r$ such that for every H-immersion $\Pi = (\pi_V, \pi_E)$ in $G - Z_r$, there exists $[A, B] \in \mathcal{F}_r$ such that $\pi_V(V(H)) \subseteq U' \cup A$. Note that such an integer r exists as $r = 0$ is a candidate. We assume that r is as large as possible.

Applying Lemma 5.2 by taking $(H, k, p, \xi'_0, U', Z'_0, \mathcal{F}') = (H, k, p-r, \xi_r, U', Z_r, \mathcal{F}_r)$, there exist $U \subseteq U'$, $Z^* \subseteq E(G)$ with $|Z^*| \leq \xi_{r+1}$ and a family $\mathcal{F}^* \subseteq \mathcal{E} - Z^*$ of edge-cuts of $G - Z^*$ such that either

(i) $U \subset U'$, every member of $\mathcal{F}^*$ has order less than $h\cdot(p-r)$, and for every H-immersion $\Pi = (\pi_V, \pi_E)$ in $G - Z^*$, there exists $[A, B] \in \mathcal{F}^*$ such that $\pi_V(V(H)) \subseteq U \cup A$, or
(ii) every member of $\mathcal{F}^*$ has order less than $p - r - 1$ and for every H-immersion $\Pi = (\pi_V, \pi_E)$ in $G - Z^*$, either there exists $[A, B] \in \mathcal{E} - Z^*$ of order zero such that $\Pi(H) \subseteq G[A]$, or there exists $[A^*_\Pi, B^*_\Pi] \in \mathcal{F}^*$ such that $\pi_V(V(H)) \subseteq U \cup A^*_\Pi$.

If (i) holds, then since $\xi^* \geq \xi_{r+1}$ and $p^* \geq h(p-r)$, Statement 1 of this lemma holds. So we may assume that (ii) holds.

Assume that $r = p-1$. Since there exists no edge-cut of order less than zero, $\mathcal{F}^* = \emptyset$. So for every H-immersion Π in $G - Z^*$, there exists $[A, B] \in \mathcal{E} - Z^*$ of order zero such that $\Pi(H) \subseteq G[A]$. Hence Statement 2 of this lemma holds.

So we may assume that $r \leq p-2$. Define $Z_{r+1} = Z^*$, and define $\mathcal{F}_{r+1} = \mathcal{F}^* \cup \{[A, B] \in \mathcal{E} - Z^* : [A, B]$ is an edge-cut of $G - Z^*$ of order zero$\}$. Since $r \leq p-2$, every member of $\mathcal{F}_{r+1}$ has order less than $p - r - 1$. This contradicts the maximality of r and completes the proof. □

Lemma 5.4. *For every connected graph H on at least two vertices and for every positive integer k, there exist integers $\theta = \theta(H, k), w = w(H, k), \xi = \xi(H, k)$ with $\theta > w + \xi$ such that the following holds. If G is a graph that does not contain k edge-disjoint H-immersions and $\mathcal{E}$ is an edge-tangle in G of order at least θ controlling a $K_{w'}$-thorns for some $w' \geq w$, then there exist $Z \subseteq E(G)$ with $|Z| \leq \xi$ and $[A, B] \in \mathcal{E} - Z$ of order zero such that $G[A]$ contains all H-immersions in $G - Z$.*

Proof. Let H be a connected graph with degree sequence $(d_1, d_2, ..., d_h)$, where $h = |V(H)| \geq 2$. Let k be a positive integer. We define the following.

- Let $\theta_0 = \theta_{4.6}(kd_1, h)$ and let $\xi_0 = \xi_{4.6}(kd_1, h)$, where $\theta_{4.6}$ and $\xi_{4.6}$ are the numbers θ and ξ mentioned in Theorem 4.6, respectively.

- Let $p_0 = kd_1h$ and $w_0 = 3kd_1h$.
- For every positive integer i, let $\theta_i = \theta_{i-1} + \theta_{5.3}(H, k, p_{i-1}, \xi_{i-1})$, $w_i = w_{5.3}(H, k, p_{i-1}, \xi_{i-1})$, $\xi_i = \xi_{5.3}(H, k, p_{i-1}, \xi_{i-1})$ and $p_i = p_{5.3}(H, k, p_{i-1}, \xi_{i-1})$, where $\theta_{5.3}, w_{5.3}, \xi_{5.3}$ and $p_{5.3}$ are the numbers θ^*, w^*, ξ^* and p^* mentioned in Lemma 5.3, respectively.
- Define $\xi = \sum_{i=0}^{h} \xi_i$, $w = w_h + \xi$ and $\theta = \theta_h + w + \xi + \sum_{i=0}^{h} p_i$.

Let G be a graph that does not contain k edge-disjoint H-immersions, and let $\mathcal{E}$ be an edge-tangle of order at least θ in G controlling a $K_{w'}$-thorns α for some $w' \geq w$. We may assume that $k \geq 2$, for otherwise the lemma holds by choosing $Z = \emptyset$.

Define H_k to be the graph obtained from H by duplicating each edge k times. Note that H_k is a graph on h vertices with maximum degree kd_1. Since $h \geq 2$ and H is connected, $d_2 \geq 1$. So H_k contains at least two vertices of degree at least $k \geq 2$ and hence is not an exceptional graph. Since G does not contain k edge-disjoint H-immersions, G does not contain an H_k-immersion. By Theorem 4.6, there exist $Z_0 \subseteq E(G)$ with $|Z_0| \leq \xi_0$, $U_0 \subseteq V(G)$ with $|U_0| \leq h-1$ and a family $\mathcal{F}'_0 \subseteq \mathcal{E} - Z_0$ of edge-cuts such that for each $v \in V(G) - U_0$, there exists $[A_v, B_v] \in \mathcal{F}'_0$ of order less than kd_1 with $v \in A_v$.

For every H-immersion $L=(\pi_V,\pi_E)$ in $G-Z_0$, define $[A^*_L,B^*_L]=[\bigcup_{v\in V(H),\pi_V(v)\notin U_0} A_{\pi_V(v)}, \bigcap_{v\in V(H),\pi_V(v)\notin U_0} B_{\pi_V(v)}]$. Note that each $[A^*_L, B^*_L]$ has order less than kd_1h and hence belongs to $\mathcal{E} - Z_0$ by Lemma 2.3. Define $\mathcal{F}_0 = \{[A^*_L, B^*_L] : L$ is an H-immersion in $G - Z_0\}$ to be a collection of edge-cuts of $G - Z_0$. Note that for every H-immersion $L = (\pi_V, \pi_E)$ in $G - Z_0$, $\pi_V(V(H)) \subseteq U_0 \cup A^*_L$.

Let r be an integer with $0 \leq r \leq h-1$ such that there exist $U_r \subseteq V(G)$ with $|U_r| \leq h-1-r$, $Z_r \subseteq E(G)$ with $|Z_r| \leq \xi_r$ and a family $\mathcal{F}_r \subseteq \mathcal{E} - Z_r$ of edge-cuts of $G - Z_r$ of order less than p_r such that for every H-immersion $L = (\pi_V, \pi_E)$ in $G - Z_r$, there exists $[A'_L, B'_L] \in \mathcal{F}_r$ such that $\pi_V(V(H)) \subseteq U_r \cup A'_L$. Note that such an integer r exists since $r = 0$ is a candidate. We assume that r is as large as possible.

Apply Lemma 5.3 by taking $(H, k, p, \xi, U', Z, \mathcal{F}') = (H, k, p_r, \xi_r, U_r, Z_r, \mathcal{F}_r)$, there exist $U \subseteq U_r$, $Z^* \subseteq E(G)$ with $|Z^*| \leq \xi_{r+1}$ and a family $\mathcal{F}^* \subseteq \mathcal{E} - Z^*$ of edge-cuts of $G - Z^*$ such that either

(i) $U \subset U_r$, every member of $\mathcal{F}^*$ has order less than p_{r+1}, and for every H-immersion $\Pi = (\pi_V, \pi_E)$ in $G - Z^*$, there exists $[A^*_\Pi, B^*_\Pi] \in \mathcal{F}^*$ such that $\pi_V(V(H)) \subseteq U \cup A^*_\Pi$, or
(ii) for every H-immersion Π in $G - Z^*$, there exists $[A^*_\Pi, B^*_\Pi] \in \mathcal{E} - Z^*$ of order zero such that $\Pi(H) \subseteq G[A^*_\Pi]$.

We first suppose that (i) holds. If $r = h-1$, then $U_r = \emptyset$, so (i) does not hold, a contradiction. So $r \leq h-2$. But it is a contradiction to the maximality of r by defining $U_{r+1} = U$, $Z_{r+1} = Z^*$ and $\mathcal{F}_{r+1} = \mathcal{F}^*$.

Hence (i) does not hold. So (ii) holds. Let $[A, B] = [\bigcup_L A^*_L, \bigcap_L B^*_L]$, where the union and intersection are over all H-immersions L in $G - Z^*$ and each $[A^*_L, B^*_L]$ is the member of $\mathcal{E} - Z^*$ mentioned in (ii). Note that $[A, B]$ has order zero. So $[A, B] \in \mathcal{E} - Z^*$ by

Lemma 2.3. Note that for every H-immersion Π in $G - Z^*$, $\Pi(H) \subseteq G[A]$. Then this lemma follows by taking $Z = Z^*$. □

Now we drop the requirement of the number of vertices of H from Lemma 5.4.

Lemma 5.5. *For every connected graph H and for every positive integer k, there exist integers $\theta = \theta(H,k), w = w(H,k), \xi = \xi(H,k)$ with $\theta > w + \xi$ such that the following holds. If G is a graph that does not contain k edge-disjoint H-immersions and $\mathcal{E}$ is an edge-tangle in G of order at least θ controlling a $K_{w'}$-thorns for some $w' \geq w$, then there exist $Z \subseteq E(G)$ with $|Z| \leq \xi$ and $[A,B] \in \mathcal{E} - Z$ of order zero such that $G[A]$ contains all H-immersions in $G - Z$.*

Proof. Let H be a connected graph and let k be a positive integer. By Lemma 5.4, we may assume $|V(H)| = 1$. Note that we may assume $|E(H)| \geq 1$, for otherwise every graph on at least one vertex contains arbitrarily many edge-disjoint H-immersions. Hence H is a one-vertex graph with at least one loop. Let H' be the graph obtained from H by subdividing one edge of H once.

Define $\xi = \xi_{5.4}(H',k) + (k-1)(k|E(H)|-1)$, $w = w_{5.4}(H',k)$ and $\theta = \theta_{5.4}(H',k) + w + \xi$, where $\xi_{5.4}, w_{5.4}, \theta_{5.4}$ are the numbers ξ, w, θ mentioned in Lemma 5.4, respectively.

Let G be a graph that does not contain k edge-disjoint H-immersions and $\mathcal{E}$ an edge-tangle in G of order at least θ controlling a $K_{w'}$-thorns for some $w' \geq w$. Since G does not contain k edge-disjoint H-immersions, no vertex of G is incident with at least $k|E(H)|$ loops, and there are at most $k-1$ vertices of G incident with at least $|E(H)|$ loops. Hence there exists $Z_0 \subseteq E(G)$ with $|Z_0| \leq (k-1)(k|E(H)|-1)$ such that no vertex in $G - Z_0$ is incident with at least $|E(H)|$ loops in $G - Z_0$.

Since G does not contain k edge-disjoint H-immersions, G does not contain k edge-disjoint H'-immersions. By Lemma 5.4, there exist $Z' \subseteq E(G)$ with $|Z'| \leq \xi_{5.4}(H',k)$ and $[A,B] \in \mathcal{E} - Z'$ of order zero such that $G[A]$ contains all H'-immersions in $G - Z'$. Let $Z = Z_0 \cup Z'$. So $|Z| \leq \xi$ and $[A,B] \in \mathcal{E} - Z$ is an edge-cut of $G - Z$ of order zero.

Suppose that there exists an H-immersion Π in $G - Z$ such that $\Pi(H) \not\subseteq G[A]$. Since H is connected, $\Pi(H) \subseteq G[B]$. So $\Pi(H)$ does not contain an H'-immersion. Hence $\Pi(H)$ consists of one vertex and $|E(H)|$ loops. But no vertex in $G - Z$ is incident with at least $|E(H)|$ loops in $G - Z$, a contradiction. This proves the lemma. □

The following is the main result of this section, which says that the connectivity of the graph H in Lemma 5.5 can be dropped.

Lemma 5.6. *For every graph H and for every positive integer k, there exist integers $\theta = \theta(H,k), w = w(H,k), \xi = \xi(H,k)$ with $\theta > w + \xi$ such that the following holds. If G is a graph that does not contain k edge-disjoint H-immersions and $\mathcal{E}$ is an edge-tangle in G of order at least θ controlling a $K_{w'}$-thorns for some $w' \geq w$, then there exist $Z \subseteq E(G)$ with $|Z| \leq \xi$ and $[A,B] \in \mathcal{E} - Z$ of order zero in $G - Z$ such that $G[B] - Z$ contains no H-immersion.*

Proof. Let H be a graph and let k be a positive integer. Let p be the number of components of H. We shall prove this lemma by induction on p. When $p = 1$, this lemma holds by taking θ, w, ξ to be the numbers $\theta_{5.5}(H, k), w_{5.5}(H, k), \xi_{5.5}(H, k)$, where $\theta_{5.5}, w_{5.5}, \xi_{5.5}$ are the integers θ, w, ξ mentioned in Lemma 5.5. So we may assume that $p \geq 2$ and the lemma holds for every graph with less than p components.

Define $\mathcal{F}_1$ to be the set of graphs that can be obtained from H by adding an edge between different components. Define $\mathcal{F}_2$ to be the set of graphs that can be obtained from H by subdividing an edge and adding an edge between this new vertex and another component of H. Define $\mathcal{F}_3$ to be the set of graphs that can be obtained from H by subdividing two edges in different components and either adding an edge between those two new vertices or identifying the two new vertices. Define $\mathcal{F}_4$ to be the set of graphs that can be obtained from H by subdividing an edge and identify this new vertex with a vertex in another component. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$. Note that $|\mathcal{F}| \leq |V(H)|^2 + |E(H)||V(H)| + 2|E(H)|^2 + |E(H)||V(H)| \leq 5(|V(H)|^2 + |E(H)|^2)$. Since every graph in $\mathcal{F}$ contains less than p components, by the induction hypothesis, for every graph $F \in \mathcal{F}$, there exist integers $\theta(F, k), w(F, k), \xi(F, k)$ such that the lemma holds.

Define $w(H, k) = \sum_{F \in \mathcal{F}} w(F, k)$, $\xi(H, k) = \sum_{F \in \mathcal{F}} \xi(F, k)$, and $\theta(H, k) = w(H, k) + \xi(H, k) + \sum_{F \in \mathcal{F}} \theta(F, k)$. We shall prove that the numbers $\theta(H, k), w(H, k)$ and $\xi(H, k)$ satisfy the lemma. Let $\theta = \theta(H, k)$, $w = w(H, k)$ and $\xi = \xi(H, k)$.

Let G be a graph that does not contain k edge-disjoint H-immersions and $\mathcal{E}$ an edge-tangle in G of order at least θ controlling a $K_{w'}$-thorns for some $w' \geq w$. Note that for every $F \in \mathcal{F}$ and every $Z \subseteq E(G)$, any subgraph of $G - Z$ containing an F-immersion contains an H-immersion. So for every $F \in \mathcal{F}$, G does not contain k edge-disjoint F-immersions. By the induction hypothesis, for every $F \in \mathcal{F}$, there exist $Z_F \in E(G)$ with $|Z_F| \leq \xi(F, k)$ and $[A_F, B_F] \in \mathcal{E} - Z_F$ of order zero in $G - Z_F$ such that $G[B_F] - Z_F$ contains no F-immersion. Define $Z = \bigcup_{F \in \mathcal{F}} Z_F$ and $[C, D] = [\bigcup_{F \in \mathcal{F}} A_F, \bigcap_{F \in \mathcal{F}} B_F]$. So $[C, D]$ has order zero in $G - Z$ and $G[D] - Z$ contains no F-immersion for each $F \in \mathcal{F}$. Since $[C, D]$ has order zero, $[C, D] \in \mathcal{E} - Z$ by Lemma 2.3. Define $[A, B]$ to be the edge-cut of $G - Z$ of order zero such that $[A, B] \in \mathcal{E} - Z$ and $C \subseteq A$, and subject to those, A is maximal. Then the maximality of A implies that $G[B] - Z$ is connected by Lemma 2.3.

Suppose that $G[B] - Z$ contains an H-immersion. Then for each $i \in [p]$, $G[B] - Z$ contains an H_i-immersion $\Pi_i = (\pi_V^{(i)}, \pi_E^{(i)})$, where H_i is the i-th component of H, such that the images of $\pi_V^{(1)}, ..., \pi_V^{(p)}$ are pairwise disjoint and the images of $\pi_E^{(1)}, ..., \pi_E^{(p)}$ are pairwise edge-disjoint. If there exist distinct $i, j \in [p]$ such that $V(\Pi_i(H_i)) \cap V(\Pi_j(H_j)) \neq \emptyset$, then $G[B] - Z$ contains an F'-immersion for some $F' \in \mathcal{F}_3 \cup \mathcal{F}_4 \subseteq \mathcal{F}$, a contradiction. Since $G[B] - Z$ is connected, there exist distinct $i, j \in [p]$ and a path P in $G[B] - Z$ of length at least one from $V(\Pi_i(H_i))$ to $V(\Pi_j(H_j))$ internally disjoint from $\bigcup_{\ell=1}^{p} V(\Pi_\ell(H_\ell))$. But it implies that $G[B] - Z$ contains an F'-immersion for some $F' \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \subseteq \mathcal{F}$, a contradiction. Therefore, $G[B] - Z$ contains no H-immersion. $\square$

6. Edge-tangles in 4-edge-connected graphs

A $m \times n$ *grid* is the graph with vertex-set $\{1, 2, ..., n\} \times \{1, 2, ..., m\}$ and two vertices $(x, y), (x', y')$ are adjacent if and only if $|x - x'| + |y - y'| = 1$. For every $i \in [m]$, the *i-th row* of a $m \times n$ grid is the subgraph induced by $\{(x, i) : x \in [n]\}$. For every $j \in [n]$, the *j-th column* of a $m \times n$ grid is the subgraph induced by $\{(j, y) : y \in [m]\}$.

For every positive integer r, the *diagonal vertices* of the $r \times 2r$ wall are the vertices $\{(2i - 1, i) : 1 \leq i \leq r\}$.

Lemma 6.1 *([1, Theorem (1.5)]). For every $g > 1$, there exists $b \geq 0$ such that the following holds. Let (π_V, π_E) be a wall-subdivision in a graph G, and let S be a subset of the image of π_V of the diagonal vertices of the wall such that for every pair of distinct vertices x, y in S, G contains four edge-disjoint paths from x to y. If $|S| \geq b$, then there exists a $g \times g$ grid-immersion (π'_V, π'_E) in G such that the image of π'_V is contained in S.*

In fact, in [1], Chudnovsky et al. proved that the grid-immersion (π'_V, π'_E) mentioned in Lemma 6.1 is a "strong immersion." We omit the definition of strong immersions as we do not need this notion in the rest of the paper. But we remark that every H-strong immersion is an H-immersion. On the other hand, the following lemma shows that if we do not require (π'_V, π'_E) to be a strong immersion, we can strengthen Lemma 6.1 by showing that the mentioned wall-subdivision (π_V, π_E) can be replaced by a wall-immersion.

Lemma 6.2. *For every $g > 1$, there exists $b \geq 0$ such that the following holds. Let (π_V, π_E) be a wall-immersion in a graph G, and let S be a subset of the image of π_V of the diagonal vertices of the wall such that for every pair of distinct vertices x, y in S, G contains four edge-disjoint paths from x to y. If $|S| \geq b$, then there exists a $g \times g$ grid-immersion (π'_V, π'_E) in G such that the image of π'_V is contained in S.*

Proof. Let g be an integer with $g > 1$. Define b to be the number b mentioned in Lemma 6.1.

Let G be a graph and let W be a wall such that (π_V, π_E) is a W-immersion in G. Let S be a subset of the image of π_V of the diagonal vertices of W. Assume that $|S| \geq b$, and for every pair of distinct vertices x, y in S, G contains four edge-disjoint paths from x to y.

Since W is simple, $\pi_E(e)$ does not contain any loop of G for every $e \in E(W)$. In addition, for every pair of distinct vertices x, y in S, any path from x to y does not contain any loop. So we may assume that G is loopless by deleting all loops of G.

Let G' be the graph obtained from G by subdividing every edge once. Let H be the graph obtained from $L(G')$ by for each $v \in V(G) \subseteq V(G')$, adding a vertex u_v adjacent to every vertex in $\text{cl}(v)$ in G'. Then it is clear that H admits a W-subdivision (π''_V, π''_E)

such that $\pi''_V(x) = u_{\pi_V(x)}$ for every $x \in V(W)$. In particular, $\pi''_V(\pi_V^{-1}(S)) = \{u_s : s \in S\}$. Note that since G is loopless, for every edge e of G with ends x, y, there exists an edge in H with one end in $V(\text{cl}(x))$ and one end in $V(\text{cl}(y))$, and we also denote this edge in H as e. Since every wall does not contain a loop, the image of π''_E of each edge of W is path in H.

For every pair of distinct vertices x, y of S, there exist four edge-disjoint paths P_1, P_2, P_3, P_4 in G from x to y, so there exist four paths Q_1, Q_2, Q_3, Q_4 in H from u_x to u_y such that $E(Q_i)$ contains $E(P_i)$ for $1 \leq i \leq 4$. If we choose those paths $Q_1, ..., Q_4$ such that the sum of their length is minimum, then $Q_1, ..., Q_4$ are pairwise edge-disjoint. Therefore, by Lemma 6.1, there exists a $g \times g$ grid-immersion (π'''_V, π'''_E) in H such that the image of π'''_V is contained in $\{u_s : s \in S\}$.

Note that if for every $v \in V(G)$, we identify the vertices in $\{u_v\} \cup \text{cl}(v)$ into a vertex and delete all resulting loops, then we obtain G. By the same procedure, we know there exists a $g \times g$ wall-immersion (π^*_V, π^*_E) in G such that the image of π^*_V is $\{s \in S : u_s$ is in the image of $\pi'''_V\}$. This proves the lemma. □

Recall that every large wall has a natural edge-tangle in it by Lemma 2.13, and every immersion induces an edge-tangle by Lemma 2.10. The following lemma shows that every graph with no edge-cut of order three but with an edge-tangle induced by an immersion of a large wall has an edge-tangle controlling a large complete graph-thorns.

Lemma 6.3. *For any positive integers θ and t, there exists a positive integer $w = w(\theta, t)$ with $w > \theta$ such that the following holds. If G is a graph with no edge-cut of order three, and $\mathcal{E}$ is an edge-tangle in G of order w induced by a $2w \times 4w$ wall-immersion and the natural tangle of order w in the $2w \times 4w$ wall, then there exists an edge-tangle $\mathcal{E}' \subseteq \mathcal{E}$ of order at least θ in G controlling a K_t-thorns.*

Proof. Let θ and t be positive integers. Let $\theta' = \theta + t$. Let b be the number mentioned in Lemma 6.2 by taking $g = 4\theta'$. Define $w = b + 2$. Note that $w \geq (4\theta')^2 + 2 > \theta' + 2 > \theta + 2$.

Denote the $2w \times 4w$ wall by W and denote the $4\theta' \times 4\theta'$ grid by R. Let S be the set of diagonal vertices of W not contained in the first and the last column of W. So $|S| \geq b$. Let G be a graph with no edge-cut of order three, and let $\mathcal{E}$ be an edge-tangle in G of order w induced by a W-immersion (π_V, π_E) in G and the natural edge-tangle in W of order w. By Lemma 2.14, for every edge-cut $[A, B]$ of G of order less than w, $[A, B] \in \mathcal{E}$ if and only if B contains the image of π_V of all vertices of a column of W.

Claim 1: *For any two vertices x, y in $\pi_V(S)$, there exist four edge-disjoint paths in G between x and y.*

Proof of Claim 1: Let $x, y \in \pi_V(S)$. So there exist $x', y' \in S$ such that $x = \pi_V(x')$ and $y = \pi_V(y')$. Since x' and y' are diagonal vertices in W not belong to the first and last column of W, there exist three edge-disjoint paths in W from x' to y'. Hence there exist three edge-disjoint paths in G from x to y. Suppose that there do not exist four edge-disjoint paths in G from x to y. Then there exists an edge-cut $[A, B]$ of G of order at most three such that $x \in A$ and $y \in B$. Since there are three edge-disjoint paths in G from x to y, the order of $[A, B]$ is exactly three, contradiction that G has no edge-cut of order three. □

By Lemma 6.2 and Claim 1, G admits an R-immersion (π'_V, π'_E) such that $\pi'_V(V(R))$ is contained in $\pi_V(S)$. Define $\mathcal{E}'$ to be the collection of all edge-cuts $[A, B]$ of G of order less than θ' such that B contains the image of π'_V of all vertices of a row of R.

A wall that is a subgraph of R is *canonical* if every its row is a subgraph of a row of R and every its column is a subgraph of the union of two consecutive columns of R. Note that for every canonical $2\theta' \times 4\theta'$ wall W' and for every $[A, B] \in \mathcal{E}'$, B contains the image of π'_V of all vertices of a row of W', so B intersects the image of π'_V of vertices in at least θ' columns of W'.

Since R contains a canonical $2\theta' \times 4\theta'$ wall W^* as a subgraph, for every $[A, B] \in \mathcal{E}'$, B intersects the image of π'_V of vertices in at least θ' columns of W^*. By Lemma 2.14, $\mathcal{E}'$ is the edge-tangle in G of order $\theta' \geq \theta$ induced by an W^*-immersion and the nature edge-tangle in W^* of order θ'.

For every i with $1 \leq i \leq t \leq \theta'$, define $\alpha(v_i)$ to be the union of the image of π'_E of the edges in the i-th column and the edges in the i-th row of R, where we write $V(K_t) = \{v_j : 1 \leq j \leq t\}$. So α is a K_t-thorns.

We claim that $\mathcal{E}'$ controls α. Suppose to the contrary that there exist $[A, B] \in \mathcal{E}'$ with order less than t and $v \in V(K_t)$ such that $V(\alpha(v)) \cap B = \emptyset$. Since $[A, B] \in \mathcal{E}'$, B contains the image of π'_V of all vertices of a row of R. Since $\alpha(v)$ intersects the image of π'_V of each row, $B \cap V(\alpha(v)) \neq \emptyset$, a contradiction. Hence $\mathcal{E}'$ controls a K_t-thorns α.

It suffices to prove that $\mathcal{E}' \subseteq \mathcal{E}$ to complete the proof. Let $[A, B] \in \mathcal{E}'$. So the order of $[A, B]$ is less than θ'. Since $\pi'_V(V(R)) \subseteq \pi_V(S)$ and B contains the image of π'_V of all vertices of a row of R, we know B contains at least θ' vertices in $\pi_V(S)$. Since different vertices in S belong to different columns of W, B intersects the image of π_V of vertices in at least θ' columns of W. Since $\theta' < w$, $[A, B] \in \mathcal{E}$ by Lemma 2.14. This proves that $\mathcal{E}' \subseteq \mathcal{E}$. □

The following theorem is the main result of this section.

Theorem 6.4. *For any positive integers k and θ with $\theta > k$, there exists a positive integer $w = w(k, \theta)$ such that if G is a graph with no edge-cut of order three, and $\mathcal{E}$ is an edge-tangle in G of order at least w, then $\mathcal{E}_\theta$ controls a K_k-thorns, where $\mathcal{E}_\theta$ is the edge-tangle in G of order θ such that $\mathcal{E}_\theta \subseteq \mathcal{E}$.*

Proof. Let k and θ be positive integers with $\theta > k$. Note that $\theta \geq 2$. Let $w_1 = w_{6.3}(\theta, k)$, where $w_{6.3}$ is the integer w mentioned in Lemma 6.3. Note that $w_1 > \theta \geq 2$ by Lemma 6.3. Define $w = w_{2.16}(w_1, k)$, where $w_{2.16}$ is the integer w mentioned in Lemma 2.16.

For every integer t and for every edge-tangle $\mathcal{E}$ in a graph of order at least t, let $\mathcal{E}_t$ be the edge-tangle in the same graph of order t such that $\mathcal{E}_t \subseteq \mathcal{E}$.

Let G be a graph with no edge-cut of order three, and let $\mathcal{E}$ be an edge-tangle in G of order at least w. By Lemma 2.16, either

(i) there exists $v \in V(G)$ incident with at least k edges in G such that $v \in B$ for every $[A, B] \in \mathcal{E}_{w_1}$, or
(ii) $\mathcal{E}_{w_1}$ is induced by a $2w_1 \times 4w_1$ wall-immersion and the natural edge-tangle of order w_1 in the $2w_1 \times 4w_1$ wall.

We first assume that (i) holds. Define α to be a K_k-thorns such that $\alpha(h)$ is an edge of G incident with v for each $h \in V(K_k)$. We shall prove that $\mathcal{E}_\theta$ controls α. Let $[A, B] \in \mathcal{E}_\theta$ with order less than k. Since $\theta < w_1$, $[A, B] \in \mathcal{E}_{w_1}$. Hence, $v \in B \cap V(\alpha(h))$ for every $h \in V(K_k)$. Therefore, $\mathcal{E}_\theta$ controls a K_k-thorns.

So we may assume that (ii) holds. That is, $\mathcal{E}_{w_1}$ is induced by a $2w_1 \times 4w_1$ wall-immersion and the natural edge-tangle of order w_1 in the $2w_1 \times 4w_1$ wall. By Lemma 6.3, there exists an edge-tangle $\mathcal{E}' \subseteq \mathcal{E}_{w_1}$ of order at least θ in G controlling a K_k-thorns. Therefore, $\mathcal{E}_\theta = \mathcal{E}'_\theta$ controls a K_k-thorns. □

7. Erdős-Pósa property

We say that a graph G is *nearly 3-cut free* if $|V(G)| \geq 2$, G is connected and for every edge-cut of G of order three, the edges between A and B are parallel with the same ends.

Lemma 7.1. *If G is either a nearly 3-cut free graph or a graph with $|V(G)| = 1$, then there exist a tree T and a partition $\{X_t : t \in V(T)\}$ of $V(G)$ such that the following hold.*

1. *For every $t \in V(T)$, either $|X_t| = 1$ or $G[X_t]$ does not have an edge-cut of order three.*
2. *If there is an edge of G with one end in X_{t_1} and one end in X_{t_2} for some distinct $t_1, t_2 \in V(T)$, then t_1 is adjacent to t_2 in T.*
3. *For every edge t_1t_2 of T, there are exactly three edges with one end in X_{t_1} and one end in X_{t_2}, and those edges are parallel with the same ends.*

Proof. We prove this lemma by induction on $|V(G)|$. If either G does not have an edge-cut of order three or G has only one vertex, then we are done by taking the tree on one vertex and the partition of $V(G)$ with one part. This proves the base case and we may assume that $|V(G)| \geq 2$ and the lemma holds for every nearly 3-cut free graph on less than $|V(G)|$ vertices. And we may assume that there exists an edge-cut $[A, B]$ of G of order three. Since G is nearly 3-cut free with $|V(G)| \geq 2$, the edges between A and B are three parallel edges with the same ends u, v, say $u \in A$ and $v \in B$.

Suppose that $|A| \geq 2$ and $G[A]$ is not nearly 3-cut free. Then there exists an edge-cut $[A', B']$ of $G[A]$ of order zero or three such that either there is no edge between A' and

B', or the edges between A', B' are not parallel with the same ends. By symmetry, we may assume that $u \in B'$. So $[A', B' \cup B]$ is an edge-cut of G such that the edges between $A', B' \cup B$ are the edges between A', B'. Since G is nearly 3-cut free, there is at least one edge between $A', B' \cup B$, and the edges between $A', B' \cup B$ are parallel with the same ends. Therefore, there is at least one edge between A', B', and the edges between A', B' are parallel with the same ends, a contradiction.

Hence either $|A| = 1$ or $G[A]$ is nearly 3-cut free. Similarly, either $|B| = 1$ or $G[B]$ is nearly 3-cut free. By the induction hypothesis, there exist trees T_A, T_B, a partition $\{Y_t : t \in V(T_A)\}$ of A and a partition $\{Z_t : t \in V(T_B)\}$ of B satisfying the three properties mentioned in the lemma. Let $t_u \in V(T_A)$ and $t_v \in V(T_B)$ be the vertices such that $u \in X_{t_u}$ and $v \in X_{t_v}$. Define T to be the tree obtained from the union of T_A and T_B by adding the edge $t_u t_v$. For every $t \in V(T)$, define $X_t = Y_t$ if $t \in V(T_A)$, and $X_t = Z_t$ if $t \in V(T_B)$. Then T and the partition $\{X_t : t \in V(T)\}$ of $V(G)$ satisfy the three properties mentioned in the lemma. $\square$

Now we are ready to address the Erdős-Pósa property. The purpose of Lemma 7.3 is to deal with the main difficulty of the proof of Theorem 1.1. Lemma 7.3 implies Theorem 1.1 for the case when H has no isolated vertices and G is nearly 3-cut free.

We give the intuition of the statement of Lemma 7.3 and sketch its proof. We shall prove that given a nearly 3-cut free graph G, if G does not contain k edge-disjoint H-immersions, then we can hit all H-immersions in G by a set of edges with bounded size. We assume that H is connected in the proof sketch, as the case that H is disconnected follows from a relatively easier argument by (more or less) induction on the number of components of H. We shall prove it by induction on k, and assume that G does not contain k edge-disjoint H-immersions. Note that as long as there exist an edge-cut $[A, B]$ of G, a hitting set of H-immersions of $G[A]$ and a hitting set of H-immersions of $G[B]$, we can obtain a hitting set of H-immersions of G by taking the union of those two hitting sets together with all edges between A, B, since H is connected. So there is a win if there exists an edge-cut $[A, B]$ of G of small order such that each $G[A]$ and $G[B]$ has a hitting set of small size. If both $G[A]$ and $G[B]$ contain H-immersions, then each of $G[A]$ and $G[B]$ does not contain $k-1$ edge-disjoint H-immersions, so we expect to obtain hitting sets of H-immersions in $G[A]$ and $G[B]$ by induction on k. However, the induction does not apply, as $G[A]$ and $G[B]$ might not be nearly 3-cut free. So instead of considering $G[A] = G - B$ and $G[B] = G - A$, we consider the graph G_A obtained from G by contracting B and the graph G_B obtained from G by contracting A. Note that G_A and G_B are nearly 3-cut free. But contracting a subset of $V(G)$ can create more H-immersions. So we should treat those new vertices obtained by contractions as special vertices. This is the purpose of the set S and function γ stated in Lemma 7.3. It can be helpful (though not completely true) to think that each vertex v in S corresponds to a subset of vertices that induces a subgraph that contains $\gamma(v)$ edge-disjoint H-immersions. This setting allows us to apply induction on $k - \sum_{v \in S} \gamma(v)$ for G_A and G_B. In addition, each of those special vertices is obtained by contracting one side of an edge-cut, so its degree equals the order of the edge-cut. As we will only contract the sides of edge-cuts of bounded order, the degree of those special vertices in S is bounded. Since H has no isolated vertices, each H-immersion in G intersecting S must intersect an edge incident with a vertex in S. Hence if we can hit all H-immersions in $G - S$ by a set of edges of bounded size, then we can hit all H-immersions in G by a set of edges of bounded size by further including all edges incident with S, as long as $|S|$ is bounded. Indeed, $|S|$ is bounded by $\sum_{v \in S} \gamma(v)$.

Further intuition and proof sketch of Lemma 7.3 will be stated after we prove the following easy lemma which is the base case of Lemma 7.3.

Lemma 7.2. *For every connected graph H that has exactly one edge and every function $g : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \to \mathbb{N} \cup \{0\}$, there exists a function $f : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \to \mathbb{N} \cup \{0\}$ such that for every graph G, every positive integer k, every $S \subseteq V(G)$ and every function $\gamma : S \to \mathbb{N}$, if S does not contain any vertex of degree at least $g(k, \sum_{v \in S} \gamma(v))$, then either $G - S$ contains $k - \sum_{v \in S} \gamma(v)$ edge-disjoint H-immersions, or there exists $Z \subseteq E(G)$ with $|Z| \leq f(k, \sum_{v \in S} \gamma(v))$ such that $G - Z$ does not contain an H-immersion.*

Proof. Let H be a connected graph that has exactly one edge, and let $g : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \to \mathbb{N} \cup \{0\}$ be a function. Note that H is either K_2 or the one-vertex graph with one loop. It was shown in [2, Chapter 9, Exercise 6] that there exists a function $h : \mathbb{N} \to \mathbb{N}$ such that for every simple graph G, either G contains k edge-disjoint cycles, or there exists $Z \subseteq E(G)$ with $|Z| \leq h(k)$ such that $G - Z$ has no cycle. Define f to be the function such that $f(x, y) = h(x + yg(x, y)) + 2(x + yg(x, y))^2$ for any $x \in \mathbb{N}$ and $y \in \mathbb{N} \cup \{0\}$.

Let G be a graph, k a positive integer, S a subset of $V(G)$ and $\gamma : S \to \mathbb{N}$ a function such that S does not contain any vertex of degree at least $g(k, \sum_{v \in S} \gamma(v))$. Let $d = \sum_{v \in S} \gamma(v)$.

Since every vertex of S has degree less than $g(k, d)$, if G contains $k + |S|g(k, d)$ edge-disjoint H-immersions, then $G - S$ contains $k \geq k - d$ edge-disjoint H-immersions. Note that $|S| \leq d$ since $\gamma(v) \geq 1$ for every $v \in S$. So to prove this lemma, it suffices to prove that either G contains $k + dg(k, d)$ edge-disjoint H-immersions, or there exists $Z \subseteq E(G)$ with $|Z| \leq f(k, d)$ such that $G - Z$ does not contain an H-immersion.

We first assume that $H = K_2$. If G contains at least $k + dg(k, d)$ non-loop edges, then G contains $k + dg(k, d)$ edge-disjoint H-immersions; if G has less than $k + dg(k, d)$ non-loop edges, there exists $Z \subseteq E(G)$ with $|Z| \leq k + dg(k, d) \leq f(k, d)$ such that $G - Z$ has no non-loop edge and has no H-immersion. So this lemma holds if $H = K_2$.

Now we assume that H is the one-vertex graph with one loop. We assume that G does not contain $k + dg(k, d)$ edge-disjoint H-immersions and show that there exists a set $Z \subseteq E(G)$ with $|Z| \leq f(k, d)$ such that $G - Z$ has no H-immersion. So G does not contain $k + dg(k, d)$ loops, and there do not exist two distinct vertices such that there are $2(k + dg(k, d))$ parallel edges between them. In addition, G does not contain $k + dg(k, d)$ distinct pairs of distinct vertices of G such that there are at least two edges between each pair. Hence G has at most $k + dg(k, d) - 1$ loops, no pair of distinct vertices of G has at least $2(k+dg(k,d))$ edges between them, and there are at most $k+dg(k,d)-1$ pairs of distinct vertices having parallel edges between them. So there exists $Z_1 \subseteq E(G)$ with $|Z_1| \leq k+dg(k,d)-1+(k+dg(k,d)-1)(2(k+dg(k,d))-1) \leq 2(k+dg(k,d))^2$ such that $G-Z_1$ is simple. Since G does not contain $k+dg(k,d)$ edge-disjoint H-immersions, $G-Z_1$ is a simple graph that does not contain $k+dg(k,d)$ edge-disjoint cycles. By [2, Chapter 9, Exercise 6], there exists $Z_2 \subseteq E(G-Z_1)$ with $|Z_2| \leq h(k+dg(k,d))$ such that $G-(Z_1 \cup Z_2)$ is a simple graph with no cycle and hence has no H-immersion. This proves the lemma since $|Z_1 \cup Z_2| \leq h(k+dg(k,d))+2(k+dg(k,d))^2 \leq f(k,d)$. □

Now we continue the intuition and proof sketch of Lemma 7.3. Recall that we aim to prove that given a nearly 3-cut free graph G and a connected graph H, if $G-S$ does not contain $k-\sum_{v\in S}\gamma(v)$ edge-disjoint H-immersions, then we can hit all H-immersions in G by a set of edges with bounded size, where S is a special set of vertices whose size is bounded by $\sum_{v\in S}\gamma(v)$. Also recall that our setting for the set S of special vertices allows us to apply induction on G_A and G_B whenever we have an edge-cut $[A,B]$ of small order such that both $G[A]$ and $G[B]$ contain H-immersions, and the degree of the vertices in S can be bounded if we only work on edge-cuts of bounded order. So now we may assume that there exists no edge-cut $[A,B]$ of G of small order such that each $G[A]$ and $G[B]$ contains an H-immersion. This will allow us to define an edge-tangle in G of large order (see Claims 3-5), by simply seeing which side of each edge-cut contains an H-immersion. Note that the order of the edge-tangle is related to the degree of the vertices in S and the order of the edge-cuts that we can work with. For a technical reason, we need this number to be depend on $|S|$ (or more precisely, $\sum_{v\in S}\gamma(v)$). And that is the reason why we consider the function g in Lemma 7.3 to indicate the degree condition of the vertices of S. For another technical reason, we want this function g growing sufficiently quickly, and that is the motivation of the notion of "H-legal" functions defined below. If G has no edge-cut of order three, then we know this edge-tangle controls a K_w-thorns for some large w by Theorem 6.4, and hence we can obtain a hitting set by Lemma 5.6. So we may assume that G is nearly 3-cut free but has an edge-cut of order three. Hence we can decompose G into pieces with no edge-cut of order three in a tree-like fashion by Lemma 7.1. Claims 6-8 tell us that we can use the tree to reduce the problem to a piece of G with no edge-cut of order three and hence complete the proof.

Recall that an isolated vertex in a graph is a vertex of degree zero. For a graph H with no isolated vertices, we say that a function g is *H-legal* if g is a function from $\mathbb{N} \times (\mathbb{N} \cup \{0\})$ to $\mathbb{N} \cup \{0\}$ satisfying that

- $g(x,y) \geq g(x,y')+2y$ and $g(x,y) \geq g(x',y)$ for every $x,x' \in \mathbb{N}$, $y,y' \in \mathbb{N}\cup\{0\}$ with $x \geq x'$ and $y > y'$, and
- for any positive integers m and n,

$g(m,n)$

$$\geq \max_{H'} \{ w_{6.4}\big(w_{5.6}(H', m + (n-1) \cdot g(m, n-1)), \theta_{5.6}(H', m + (n-1) \cdot g(m, n-1))\big) + 3m|V(H')|d_{H'} \},$$

where the maximum is over all graphs H' with no isolated vertices and with $|E(H')| \leq |E(H)|$, and $w_{6.4}$ is the integer w mentioned in Theorem 6.4, and $\theta_{5.6}, w_{5.6}$ are the integers θ, w mentioned in Lemma 5.6, respectively, and $d_{H'}$ is the maximum degree of H'.

Note that if g is H-legal for some graph H with no isolated vertices, then g is H''-legal for any graph H'' with no isolated vertices with $|E(H'')| \leq |E(H)|$. And it is easy to see that H-legal functions exist for any graph H with no isolated vertices.

Lemma 7.3. *For every graph H with no isolated vertices, there exists an H-legal function $g^* : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \to \mathbb{N} \cup \{0\}$ such that for every H-legal function $g : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \to \mathbb{N} \cup \{0\}$ with $g \geq g^*$, there exists a function $f : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \to \mathbb{N} \cup \{0\}$ such that for every nearly 3-cut free graph G, every positive integer k, every $S \subseteq V(G)$ and every function $\gamma : S \to \mathbb{N}$, if S does not contain any vertex of degree at least $g(k, \sum_{v \in S} \gamma(v))$, then either $G - S$ contains $k - \sum_{v \in S} \gamma(v)$ edge-disjoint H-immersions, or there exists $Z \subseteq E(G)$ with $|Z| \leq f(k, \sum_{v \in S} \gamma(v))$ such that $G - Z$ does not contain an H-immersion.*

Proof. Let H be a graph with no isolated vertices. Denote $|V(H)|$ by h and the maximum degree of H by d. Since H has no isolated vertices, $d \geq 1$.

We shall prove this lemma by induction on $|E(H)|$. If H contains only one edge, then H is connected since H has no isolated vertices, so the lemma holds by Lemma 7.2 by choosing g^* to be any H-legal function. This proves the base case of the induction. We assume that this lemma is true for every graph H' without isolated vertices with $|E(H')| < |E(H)|$ and denote the corresponding function g^* and the corresponding function f (when some H'-legal function g with $g \geq g^*$ is given) by $g^*_{H'}$ and $f_{H',g}$, respectively.

We define the following.

- Define $g^* : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \to \mathbb{N} \cup \{0\}$ such that the following hold.
  - For every positive integer m, define $g^*(m, 0) = w_{6.4}(w_{5.6}(H, m), \theta_{5.6}(H, m)) + 3mhd + \sum_{H'} g^*_{H'}(m, 0)$, where $w_{6.4}$ is the integer w mentioned in Theorem 6.4, and $\theta_{5.6}, w_{5.6}$ are the integers θ, w mentioned in Lemma 5.6, respectively, and the last sum is over all graphs H' with no isolated vertices and with less edges than H.
  - For every positive integers m, n, define $g^*(m, n) = w_{6.4}(w_{5.6}(H, m + (n-1) \cdot g^*(m, n-1)), \theta_{5.6}(H, m + (n-1) \cdot g^*(m, n-1))) + 3mhd + g^*(m, n-1) + 2n + \theta_{5.6}(H, m + (n-1) \cdot g^*(m, n-1)) + \sum_{H'} g^*_{H'}(m, n)$, where the last sum is over all graphs H' with no isolated vertices and with less edges than H.

Note that $\theta_{5.6}(H,t) > w_{5.6}(H,t)$ for any positive integer t by Lemma 5.6. Clearly, g^* is H-legal.

- For every H-legal function g with $g \geq g^*$, we define the following.
  - Let $f'_{H,g} : \mathbb{N} \to \mathbb{N} \cup \{0\}$ be the function such that $f'_{H,g}(k) = \sum_{H'} \sum_{i=0}^{k} f_{H',g}(k,i)$ for every positive integer k, where the first sum is taken over all graphs H' with no isolated vertices having less edges than H. Note that there are only finitely many such graphs H', as H' has no isolated vertices.
  - Define $f : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \to \mathbb{N} \cup \{0\}$ to be the function satisfying the following.
    * $f(m,n) = 0$ for every integers m, n with $0 < m \leq n$.
    * $f(m,n) = 2f(m,n+1) + (m+n+2)g(m,n+1) + (2^{h+1}-4)f'_{H,g}(mg(m,n+2g(m,n+1))) + mhd$ for every integers m, n with $m > n \geq 0$.

    Note that f depends on g, but we do not add subscript g to describe f for simplicity of notations.

We shall prove that the functions g^* and f defined above satisfy the conclusion of this lemma for the graph H. That is, we shall prove that for every H-legal function g with $g \geq g^*$, the function f satisfies the property that for every nearly 3-cut free graph G, every positive integer k, every set $S \subseteq V(G)$ and every function $\gamma : S \to \mathbb{N}$ such that S does not contain any vertex of degree at least $g(k, \sum_{v \in S} \gamma(v))$, either $G - S$ contains $k - \sum_{v \in S} \gamma(v)$ edge-disjoint H-immersions, or there exists $Z \subseteq E(G)$ with $|Z| \leq f(k, \sum_{v \in S} \gamma(v))$ such that $G - Z$ does not contain an H-immersion.

We do induction on $k - \sum_{v \in S} \gamma(v)$. Suppose to the contrary that there exists a tuple (g, G, k, S, γ) such that the following hold.

(i) g is an H-legal function with $g \geq g^*$, G is a nearly 3-cut free graph, k is a positive integer, S is a subset of $V(G)$, and $\gamma : S \to \mathbb{N}$ is a function such that S does not contain any vertex of degree at least $g(k, \sum_{v \in S} \gamma(v))$.

(ii) $G - S$ does not contain $k - \sum_{v \in S} \gamma(v)$ edge-disjoint H-immersions, but there does not exist $Z \subseteq E(G)$ with $|Z| \leq f(k, \sum_{v \in S} \gamma(v))$ such that $G - Z$ has no H-immersion.

(iii) Subject to (i) and (ii), $k - \sum_{v \in S} \gamma(v)$ is minimum.

Note that (ii) implies that $k - \sum_{v \in S} \gamma(v) \geq 1$, so the minimum mentioned in (iii) exists.

In the rest of the proof, we denote $\sum_{v \in S} \gamma(v)$ by $\bar{r}$.

For every edge-cut $[A, B]$ of G with $A \neq \emptyset \neq B$, define G_A (and G_B, respectively) to be the graphs obtained from G by identifying B (and A, respectively) into one new vertex v_B (and v_A, respectively), and deleting all resulting loops. Define $S_A = (S \cap A) \cup \{v_B\}$ and $S_B = (S \cap B) \cup \{v_A\}$. Note that the degree of v_B in G_A and the degree of v_A in G_B are the order of $[A, B]$.

Claim 1: *For every edge-cut $[A, B]$ of G with $A \neq \emptyset \neq B$, G_A and G_B are nearly 3-cut free.*

Proof of Claim 1: Since $A \neq \emptyset \neq B$, G_A and G_B contain at least two vertices. Since G is nearly 3-cut free, G is connected, so G_A and G_B are connected. If G_A is not nearly 3-cut free, then there exists an edge-cut $[X, Y]$ of G_A with $v_B \in Y$ of order three such that the edges between X and Y are not parallel edges with the same ends. But then $[X, (Y - \{v_B\}) \cup B]$ is an edge-cut of G of order three such that the edges in between are not parallel edges with the same ends, a contradiction. So G_A is nearly 3-cut free. Similarly, G_B is nearly 3-cut free. $\square$

Claim 2: *Let θ be a positive integer. If there exist $W \subseteq V(G)$, an edge-cut $[A, B]$ of G of order less than θ and a set $Z_0 \subseteq E(G)$ containing all edges between A and B such that $G[A] - (W \cup Z_0)$ and $G[B] - (W \cup Z_0)$ do not contain H-immersions, then there exists Z with $Z_0 \subseteq Z \subseteq E(G)$ and $|Z| \leq |Z_0| + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + \theta))$ such that $G - (W \cup Z)$ has no H-immersion.*

Proof of Claim 2: If $A = \emptyset$, then $G = G[B]$, so we are done by taking $Z = Z_0$. Similarly, we are done if $B = \emptyset$. So we may assume that $A \neq \emptyset \neq B$ and hence G_A and G_B contain at least two vertices and are nearly 3-cut free by Claim 1.

If H is connected, then every H-immersion in $G - Z_0$ must be in $G[A]$ or $G[B]$, as Z_0 contains all edges between A and B. So we are done by taking $Z = Z_0$.

Now we assume that H is not connected. Let $H_1, H_2, ..., H_p$ be the components of H, where $p \geq 2$. For every set I with $\emptyset \subset I \subset [p]$, define Q_I to be the disjoint union of H_i over all $i \in I$.

Since H has no isolated vertices, every H-immersion in G_A (or G_B, respectively) intersecting S_A (or S_B, respectively) must intersect an edge incident with a vertex in S_A (or S_B, respectively). Since every vertex in $S_A - \{v_B\}$ has degree in G_A at most $g(k, \bar{r}) - 1$ in G_A and v_B has degree in G_A at most $\theta - 1$, for every I with $\emptyset \subset I \subset [p]$ and any nonnegative integer k', if G_A contains at least $k' + (|S_A| - 1)(g(k, \bar{r}) - 1) + \theta - 1$ edge-disjoint Q_I-immersions, then $G[A] - S_A$ contains k' edge-disjoint Q_I-immersions. Similarly, for every I with $\emptyset \subset I \subset [p]$ and any nonnegative integer k', if G_B contains at least $k' + (|S_B| - 1)(g(k, \bar{r}) - 1) + \theta - 1$ edge-disjoint Q_I-immersions, then $G[B] - S_B$ contains k' edge-disjoint Q_I-immersions.

Note that for every nonnegative integer k' and set I with $\emptyset \subset I \subset [p]$, if $G[A] - S_A$ contains k' edge-disjoint Q_I-immersions and $G[B] - S_B$ contains k' edge-disjoint $Q_{[p]-I}$-immersions, then $G - S$ contains k' edge-disjoint H-immersions. Hence, since $G - S$ has no $k - \bar{r}$ edge-disjoint H-immersions, for every I with $\emptyset \subset I \subset [p]$, either G_A does not contain $(k - \bar{r}) + (|S_A| - 1)(g(k, \bar{r}) - 1) + \theta - 1$ edge-disjoint Q_I-immersions, or G_B does not contain $(k - \bar{r}) + (|S_B| - 1)(g(k, \bar{r}) - 1) + \theta - 1$ edge-disjoint $Q_{[p]-I}$-immersions.

As $k - \bar{r} \geq 1$, $\max\{|S_A|, |S_B|\} \leq |S| + 1 \leq \bar{r} + 1 \leq k$. Hence, for every I with $\emptyset \subset I \subset [p]$, either G_A does not contain $(k - 1)g(k, \bar{r}) - \bar{r} + \theta$ edge-disjoint Q_I-immersions, or G_B does not contain $(k - 1)g(k, \bar{r}) - \bar{r} + \theta$ edge-disjoint $Q_{[p]-I}$-immersions.

Define $\gamma_A : S_A \to \mathbb{N}$ to be the function such that $\gamma_A(v_B) = \theta + \sum_{v \in S \cap B} \gamma(v)$ and $\gamma_A(x) = \gamma(x)$ for every $x \in S \cap A$. Define $\gamma_B : S_B \to \mathbb{N}$ to be the function such that $\gamma_B(v_A) = \theta + \sum_{v \in S \cap A} \gamma(v)$ and $\gamma_B(x) = \gamma(x)$ for every $x \in S \cap B$.

Note that $\sum_{v \in S_A} \gamma_A(v) = \theta + \bar{r}$ and $\sum_{v \in S_B} \gamma_B(v) = \theta + \bar{r}$. So $g(k, \sum_{v \in S_A} \gamma_A(v)) = g(k, \theta + \bar{r}) \geq g(k, \bar{r}) + 2\theta$ since g is H-legal. Similarly, $g(k, \sum_{v \in S_B} \gamma_B(v)) \geq g(k, \bar{r}) + 2\theta$.

Let $k_A = kg(k, \sum_{v \in S_A} \gamma_A(v))$ and let $k_B = kg(k, \sum_{v \in S_B} \gamma_B(v))$. Hence, if I is a set with $\emptyset \subset I \subset [p]$ such that $G_A - S_A$ does not contain $(k-1)g(k, \bar{r}) - \bar{r} + \theta$ edge-disjoint Q_I-immersions, then $G_A - S_A$ does not contain $((k-1)g(k, \bar{r}) - \bar{r} + \theta + \sum_{v \in S_A} \gamma_A(v)) - \sum_{v \in S_A} \gamma_A(v) = ((k-1)g(k, \bar{r}) + 2\theta) - \sum_{v \in S_A} \gamma_A(v) \leq kg(k, \sum_{v \in S_A} \gamma_A(v)) - \sum_{v \in S_A} \gamma_A(v) = k_A - \sum_{v \in S_A} \gamma_A(v)$ edge-disjoint Q_I-immersions. Similarly, if I is a set with $\emptyset \subset I \subset [p]$ such that $G_B - S_B$ does not contain $(k-1)g(k, \bar{r}) - \bar{r} + \theta$ edge-disjoint $Q_{[p]-I}$-immersions, then $G_B - S_B$ does not contain $k_B - \sum_{v \in S_B} \gamma_B(v)$ edge-disjoint $Q_{[p]-I}$-immersions.

Therefore, for every I with $\emptyset \subset I \subset [p]$, either $G_A - S_A$ does not contain $k_A - \sum_{v \in S_A} \gamma_A(v)$ edge-disjoint Q_I-immersions, or G_B does not contain $k_B - \sum_{v \in S_B} \gamma_B(v)$ edge-disjoint $Q_{[p]-I}$-immersions.

Note that every vertex in S_A has degree in G_A less than $g(k, \bar{r}) + \theta \leq g(k, \sum_{v \in S_A} \gamma_A(v)) \leq g(k_A, \sum_{v \in S_A} \gamma_A(v))$, since g is H-legal. Similarly, every vertex in S_B has degree in G_B less than $g(k, \bar{r}) + \theta \leq g(k_B, \sum_{v \in S_B} \gamma_B(v))$.

Recall that G_A and G_B are nearly 3-cut free graphs. For every I with $\emptyset \subset I \subset [p]$, Q_I and $Q_{[p]-I}$ are graphs with no isolated vertices and with less edges than H, g is Q_I-legal and $Q_{[p]-I}$-legal, and $g \geq g^* \geq g^*_{Q_I} + g^*_{Q_{[p]-I}}$, so by the induction hypothesis, either there exists $Z_{A,I} \subseteq E(G_A)$ with $|Z_{A,I}| \leq f_{Q_I,g}(k_A, \sum_{v \in S_A} \gamma_A(v))$ such that $G_A - Z_{A,I}$ does not contain an Q_I-immersion, or there exists $Z_{B,I} \subseteq E(G_B)$ with $|Z_{B,I}| \leq f_{Q_{[p]-I},g}(k_B, \sum_{v \in S_B} \gamma_B(v))$ such that $G_B - Z_{B,I}$ does not contain an $Q_{[p]-I}$-immersion.

Note that $k_A = kg(k, \sum_{v \in S_A} \gamma_A(v)) \geq \sum_{v \in S_A} \gamma_A(v)$ and $k_B = kg(k, \sum_{v \in S_B} \gamma_B(v)) \geq \sum_{v \in S_B} \gamma_B(v)$ since g is H-legal. Therefore, for every I with $\emptyset \subset I \subset [p]$, there exists $Z_I \subseteq E(G)$ with $|Z_I| \leq f_{Q_I,g}(k_A, \sum_{v \in S_A} \gamma_A(v)) + f_{Q_{[p]-I},g}(k_B, \sum_{v \in S_B} \gamma_B(v)) \leq \sum_{i=0}^{k_A} f_{Q_I,g}(k_A, i) + \sum_{i=0}^{k_B} f_{Q_{[p]-I},g}(k_B, i) \leq f'_{H,g}(kg(k, \bar{r} + \theta))$ such that either $G_A - Z_I$ has no Q_I-immersion or $G_B - Z_I$ has no $Q_{[p]-I}$-immersion.

Define $Z = Z_0 \cup \bigcup_{\emptyset \subset I \subset [p]} Z_I$. Note that $|Z| \leq |Z_0| + (2^p - 2) \cdot f'_{H,g}(kg(k, \bar{r} + \theta)) \leq |Z_0| + (2^h - 2) f'_{H,g}(kg(k, \bar{r} + \theta))$, since $p \leq h$.

Suppose that $G - (W \cup Z)$ contains an H-immersion. Since Z contains all edges between A and B, and $G[A] - (W \cup Z_0)$ and $G[B] - (W \cup Z_0)$ do not contain H-immersions, there exists I with $\emptyset \subset I \subset [p]$ such that $G[A] - Z$ contains a Q_I-immersion and $G[B] - Z$ contains a $Q_{[p]-I}$-immersion, contradicting the existence of Z_I. This proves the claim. □

Claim 3: *There exists no edge-cut $[A, B]$ of G of order less than $g(k, 1 + \bar{r})$ such that $G[A] - S$ contains an H-immersion and $G[B] - S$ contains an H-immersion.*

Proof of Claim 3: Suppose to the contrary that there exists an edge-cut $[A, B]$ of G of order less than $g(k, 1 + \bar{r})$ such that $G[A] - S$ contains an H-immersion and $G[B] - S$ contains an H-immersion. Note that $\deg_{G_A}(v_B) < g(k, 1+\bar{r})$ and $\deg_{G_B}(v_A) < g(k, 1+\bar{r})$. Since both $G[A]$ and $G[B]$ contain H-immersions, $A \neq \emptyset \neq B$, so G_A and G_B are nearly 3-cut free by Claim 1.

Since $G[B] - S$ contains an H-immersion, $G_A - S_A = G[A] - S$ does not contain $k - \bar{r} - 1$ edge-disjoint H-immersions, for otherwise $G - S$ contains $k - \bar{r}$ edge-disjoint H-immersions, contradicting (ii). Define $\gamma_A : S_A \to \mathbb{N}$ to be the function such that $\gamma_A(v_B) = 1 + \sum_{v \in S \cap B} \gamma(v)$, and $\gamma_A(x) = \gamma(x)$ for every $x \in S \cap A$. So $G_A - S_A$ does not contain $k - \bar{r} - 1 = k - \sum_{v \in S_A} \gamma_A(v)$ edge-disjoint H-immersions. Furthermore, every vertex in S_A has degree in G_A less than $\max\{g(k, \bar{r}), g(k, 1 + \bar{r})\} = g(k, 1 + \bar{r}) = g(k, \sum_{v \in S_A} \gamma_A(v))$ since g is H-legal. Hence the tuple $(g, G_A, k, S_A, \gamma_A)$ satisfies (i). Since $k - \sum_{v \in S_A} \gamma_A(v) = k - 1 - \bar{r} < k - \bar{r}$, by (iii), $(g, G_A, k, S_A, \gamma_A)$ does not satisfy (ii). So there exists $Z_A \subseteq E(G_A)$ with $|Z_A| \leq f(k, \sum_{v \in S_A} \gamma_A(v)) = f(k, \bar{r} + 1)$ such that $G_A - Z_A$ does not contain an H-immersion.

Similarly, there exists $Z_B \subseteq E(G_B)$ with $|Z_B| \leq f(k, \bar{r} + 1)$ such that $G_B - Z_B$ does not contain an H-immersion. Note that every edge of G_A incident with v_B is an edge between A and B. So Z_A is a subset of $E(G)$. Similarly, Z_B is a subset of $E(G)$.

Let Z' be the set of edges of G with one end in A and one end in B. Define $Z_0 = Z_A \cup Z_B \cup Z'$. Note that $|Z_0| \leq 2f(k, \bar{r} + 1) + g(k, 1 + \bar{r})$.

Since $G[A] - Z_0$ is a subgraph of $G_A - Z_A$ and $G[B] - Z_0$ is a subgraph of $G_B - Z_B$, we know that $G[A] - Z_0$ and $G[B] - Z_0$ do not contain H-immersions. Note that Z_0 contains all edges between A and B. Applying Claim 2 by taking $\theta = g(k, 1 + \bar{r})$ and $W = \emptyset$, we know that there exists Z with $Z_0 \subseteq Z \subseteq E(G)$ and $|Z| \leq |Z_0| + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1 + \bar{r}))) \leq f(k, \bar{r})$ such that $G - Z$ has no H-immersion. Hence (g, G, k, S, γ) does not satisfy (ii), a contradiction. □

Claim 4: *For every edge-cut $[A, B]$ of G of order less than $g(k, 1 + \bar{r})$, exactly one of $G[A] - S$ or $G[B] - S$ contains an H-immersion.*

Proof of Claim 4: Suppose to the contrary that this claim does not hold. So there exists an edge-cut $[A, B]$ of G of order less than $g(k, 1 + \bar{r})$ such that $G[A] - S$ and $G[B] - S$ do not contain H-immersions by Claim 3. Applying Claim 2 by taking $\theta = g(k, 1 + \bar{r})$, $W = S$ and Z_0 to be the set of the edges between A and B, we obtain $Z \subseteq E(G)$ with $|Z| \leq g(k, 1 + \bar{r}) + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1 + \bar{r})))$ such that $G - (S \cup Z)$ has no H-immersion.

Let Z' be the union of Z and the set of edges incident with vertices in S. Since $|S| \leq \bar{r} \leq k - 1$ and every vertex in S has degree less than $g(k, \bar{r})$, $|Z'| \leq |Z| + (k - 1)(g(k, \bar{r}) - 1) \leq kg(k, 1 + \bar{r}) + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1 + \bar{r}))) \leq f(k, \bar{r})$. Since H has no isolated vertices, $G - Z'$ has no H-immersion, contradicting (ii). □

Define $\mathcal{E}$ to be the collection of edge-cuts of G such that $[A, B] \in \mathcal{E}$ if and only if $[A, B]$ has order less than $g(k, 1 + \bar{r})$ and $G[B] - S$ contains an H-immersion. Note that Claim 4 implies that $G[A] - S$ does not contain an H-immersion for every $[A, B] \in \mathcal{E}$.

Claim 5: *$\mathcal{E}$ is an edge-tangle in G of order $g(k, 1 + \bar{r})$.*

Proof of Claim 5: Claim 4 implies that $\mathcal{E}$ satisfies (E1).

Suppose that $\mathcal{E}$ does not satisfy (E2). So there exist edge-cuts $[A_1, B_1], [A_2, B_2], [A_3, B_3] \in \mathcal{E}$ with $B_1 \cap B_2 \cap B_3 = \emptyset$. Hence $\{A_1, B_1 \cap A_2, B_1 \cap B_2 \cap A_3\}$ is a partition of $V(G)$. Let $[C_1, D_1] = [A_1 \cup (B_1 \cap A_2), B_1 \cap B_2 \cap A_3]$. Note that $G[C_1] - (A_1 \cup S) \subseteq G[B_1 \cap A_2] - S \subseteq G[A_2] - S$. Since $[A_2, B_2] \in \mathcal{E}$, $G[C_1] - (A_1 \cup S)$ does not contain an H-immersion. Since $[A_3, B_3] \in \mathcal{E}$, $G[D_1] - (A_1 \cup S) \subseteq G[A_3] - (A_1 \cup S)$ does not contain an H-immersion. Note that every edge between C_1, D_1 is either between A_1, B_1 or between A_2, B_2. So the order of $[C_1, D_1]$ is less than $2g(k, 1 + \bar{r})$. Applying Claim 2 by taking $\theta = 2g(k, 1 + \bar{r})$, $W = A_1 \cup S$, $[A, B] = [C_1, D_1]$ and Z_0 to be the set of all edges between C_1 and D_1, there exists $Z_1^* \subseteq E(G)$ with $|Z_1^*| \leq 2g(k, 1 + \bar{r}) + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + 2g(k, 1 + \bar{r})))$ such that $G - (A_1 \cup S \cup Z_1^*)$ has no H-immersion. Hence $G[B_1] - (S \cup Z_1^*) = G - (A_1 \cup S \cup Z_1^*)$ does not contain an H-immersion. Since $[A_1, B_1] \in \mathcal{E}$, $G[A_1] - (S \cup Z_1^*)$ does not contain an H-immersion. Applying Claim 2 by taking $\theta = g(k, 1 + \bar{r})$, $W = S$, $[A, B] = [A_1, B_1]$ and Z_0 to be the union of Z_1^* and the set of all edges between A_1, B_1, there exists $Z_2^* \subseteq E(G)$ with $|Z_2^*| \leq (|Z_1^*| + g(k, 1 + \bar{r})) + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1 + \bar{r}))) \leq 3g(k, 1 + \bar{r}) + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + 2g(k, 1 + \bar{r}))) + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1 + \bar{r}))) \leq 3g(k, 1 + \bar{r}) + 2(2^h - 2)f'_{H,g}(kg(k, \bar{r} + 2g(k, 1 + \bar{r})))$ such that $G - (S \cup Z_2^*)$ does not contain an H-immersion. Let Z_3^* be the union of Z_2^* and the set of all edges of G incident with S. Since H has no isolated vertices, $G - Z_3^*$ does not contain an H-immersion. Note that $|Z_3^*| \leq |Z_2^*| + |S|(g(k, \bar{r}) - 1) \leq (\bar{r} + 3)g(k, 1 + \bar{r}) + 2(2^h - 2)f'_{H,g}(kg(k, \bar{r} + 2g(k, 1 + \bar{r}))) \leq f(k, \bar{r})$. It contradicts (ii). So $\mathcal{E}$ satisfies (E2).

Finally, suppose that there exists $[A, B] \in \mathcal{E}$ such that there are less than $g(k, 1 + \bar{r})$ edges incident with B, then $G[B] - (E(G[B]) \cup S)$ has no H-immersion. Since $[A, B] \in \mathcal{E}$, $G[A] - (E(G[B]) \cup S) = G[A] - S$ has no H-immersion. Applying Claim 2 by taking $\theta = g(k, 1 + \bar{r})$, $W = S$, and Z_0 to be the union of $E(G[B])$ and the set of edges between A, B, we know there exists $Z \subseteq E(G)$ with $|Z| \leq 2g(k, 1 + \bar{r}) + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1 + \bar{r})))$ such that $G - (Z \cup S)$ has no H-immersion. Let Z^* be the union of Z and the set of all edges of G incident with S. Then $G - Z^*$ does not contain an H-immersion. But $|Z^*| \leq 2g(k, 1 + \bar{r}) + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1 + \bar{r}))) + \bar{r}(g(k, \bar{r}) - 1) \leq (\bar{r} + 2)g(k, 1 + \bar{r}) + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1 + \bar{r}))) \leq f(k, \bar{r})$, contradicting (ii). Hence $\mathcal{E}$ satisfies (E3). □

Let T be the tree and $\mathcal{P} = \{X_t : t \in V(T)\}$ the partition of $V(G)$ satisfying Lemma 7.1. For each $t \in (T)$, we call X_t the bag at t. For each edge $e \in E(T)$, there exists an edge-cut $[A_e, B_e]$ of G such that each A_e and B_e is the union of the bags of the vertices in a component of $T - e$. So $[A_e, B_e]$ has order at most three and the edges between A_e and B_e are the parallel edges with the same ends. Since $\mathcal{E}$ is an edge-tangle of order greater than three, $[A_e, B_e] \in \mathcal{E}$ or $[B_e, A_e] \in \mathcal{E}$ but not both. If $[A_e, B_e] \in \mathcal{E}$, then we direct e such that B_e contains the bag of the head of e; otherwise, we direct e in the opposite direction. Hence, we obtain an orientation of T and there exists a vertex t^* of T of out-degree zero.

Claim 6: *There exist a set R of loops of $G[X_{t^*}]$ with $|R| \leq (k-1)hd$ and a set $U \subseteq E(T)$ with $|U| \leq (k-1)hd$ such that every edge in U is incident with t^*, and for every H-immersion $\Pi = (\pi_V, \pi_E)$ in G, one of the following holds.*

- $\Pi(H)$ *intersects* S.
- $\Pi(H)$ *contains a non-loop edge of* $G[X_{t^*}]$.
- $\Pi(H)$ *contains an edge in* R.
- $V(\Pi(H)) \cap X_{t^*} \neq \emptyset$ *and there exists* $e \in U$ *such that* $V(\Pi(H)) \cap A_e \neq \emptyset$.

Proof of Claim 6: Recall that for any edge e of T, the edges between $[A_e, B_e]$ are parallel edges. So for every H-immersion $\Pi = (\pi_V, \pi_E)$ in G and every edge x of H, since $\pi_E(x)$ is a path or a cycle, there are at most two edges e of T incident with t^* such that $V(\pi_E(x)) \cap A_e \neq \emptyset$. Therefore, for every H-immersion $\Pi = (\pi_V, \pi_E)$ in G, there exists a set W_Π of edges of T incident with t^* with $|W_\Pi| \leq 2|E(H)| \leq hd$ such that $\Pi(H) \cap G[\bigcap_{e \in W_\Pi} B_e] - X_{t^*} = \emptyset$, and for every $e \in W_\Pi$, $V(\Pi(H)) \cap A_e \neq \emptyset$. In addition, for every H-immersion $\Pi = (\pi_V, \pi_E)$ in G, $\Pi(H)$ contains a loop e' of G only if there exists a loop e of H such that $\pi_E(e) = e'$. So for every H-immersion $\Pi = (\pi_V, \pi_E)$ in G, there exists a set R_Π of loops of G incident with X_{t^*} with $|R_\Pi| \leq |E(H)| \leq hd$ such that R_Π consists of the loops of G incident with X_{t^*} contained in $\Pi(H)$.

Let $\mathcal{C}$ be a maximal collection of H-immersions in $G - S$ such that

- for every member Π of $\mathcal{C}$, $\Pi(H)$ does not contain any non-loop edge of $G[X_{t^*}]$, and
- for distinct members Π_1, Π_2 of $\mathcal{C}$, $R_{\Pi_1} \cap R_{\Pi_2} = \emptyset$ and $W_{\Pi_1} \cap W_{\Pi_2} = \emptyset$.

Note that members of $\mathcal{C}$ are pairwise edge-disjoint H-immersions in $G - S$. So $|\mathcal{C}| < k - \bar{r} \leq k$.

Define $R = \bigcup_{\Pi \in \mathcal{C}} R_\Pi$ and define $U = \bigcup_{\Pi \in \mathcal{C}} W_\Pi$. Hence $|R| \leq (k-1)hd$ and $|U| \leq (k-1)hd$.

Let Π be an H-immersion in G. We may assume that Π does not satisfy the first two conclusions of this claim, for otherwise we are done. So Π is an H-immersion in $G - S$.

Suppose that $V(\Pi(H)) \cap X_{t^*} = \emptyset$. Since $\Pi(H) \cap G[\bigcap_{e \in W_\Pi} B_e] - X_{t^*} = \emptyset$, $\Pi(H) \subseteq G[\bigcup_{e \in W_\Pi} A_e]$. Hence Π is an H-immersion in $G[\bigcup_{e \in W_\Pi} A_e] - S$. So $[\bigcup_{e \in W_\Pi} A_e, \bigcap_{e \in W_\Pi} B_e] \notin \mathcal{E}$ by the definition of $\mathcal{E}$. Since $|W_\Pi| \leq hd$, the order of $[\bigcup_{e \in W_\Pi} A_e, \bigcap_{e \in W_\Pi} B_e]$ is at most $3hd$ which is less than the order of $\mathcal{E}$. Since $[A_e, B_e] \in \mathcal{E}$ for each $e \in W_\Pi$, $[\bigcup_{e \in W_\Pi} A_e, \bigcap_{e \in W_\Pi} B_e] \in \mathcal{E}$ by Claim 5 and Lemma 2.3, a contradiction.

Hence $V(\Pi(H)) \cap X_{t^*} \neq \emptyset$. Suppose $R_\Pi = \emptyset$ and $W_\Pi = \emptyset$. Since $W_\Pi = \emptyset$, $\Pi(H) \subseteq G[X_{t^*}]$. Since $R_\Pi = \emptyset$ and $\Pi(H) \subseteq G[X_{t^*}]$, $\Pi(H)$ does not contain any loop of G. Since H has no isolated vertex, $\Pi(H)$ contains a non-loop edge of $G[X_{t^*}]$, so Π satisfies the second conclusion of this claim, a contraction.

Hence either $R_\Pi \neq \emptyset$ or $W_\Pi \neq \emptyset$. If $\Pi \in \mathcal{C}$ and $R_\Pi \neq \emptyset$, then Π satisfies the third conclusion of this claim. If $\Pi \in \mathcal{C}$ and $W_\Pi \neq \emptyset$, then Π satisfies the fourth conclusion of this claim. So we may assume that $\Pi \notin \mathcal{C}$. By the maximality of $\mathcal{C}$, there exists $\Pi' \in \mathcal{C}$

such that either $R_\Pi \cap R_{\Pi'} \neq \emptyset$ or $W_\Pi \cap W_{\Pi'} \neq \emptyset$. If $R_\Pi \cap R_{\Pi'} \neq \emptyset$, then $\Pi(H)$ contains an edge in R, so Π satisfies the third conclusion of this claim. So we may assume that $W_\Pi \cap W_{\Pi'} \neq \emptyset$. Then there exists $e \in W_{\Pi'} \subseteq U$ such that $V(\Pi(H)) \cap A_e \neq \emptyset$. So Π satisfies the fourth conclusion of this claim. □

Claim 7: *For every $[A, B] \in \mathcal{E}$ of order less than $g(k, 1 + \bar{r}) - 3(k-1)hd$, $G[B \cap X_{t^*}]$ contains at least $f(k, \bar{r}) - kg(k, 1 + \bar{r}) - (k-1)hd - (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1 + \bar{r})))$ non-loop edges.*

Proof of Claim 7: Let $m = f(k, \bar{r}) - kg(k, 1 + \bar{r}) - (k-1)hd - (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1 + \bar{r})))$. Suppose to the contrary that there exists $[A, B] \in \mathcal{E}$ of order less than $g(k, 1 + \bar{r}) - 3(k-1)hd$ such that $G[B \cap X_{t^*}]$ contains less than m non-loop edges.

Let U be the set of edges of T incident with t^* and R the set of loops of $G[X_{t^*}]$ mentioned in Claim 6. Note that $|U| \leq (k-1)hd$ and $|R| \leq (k-1)hd$. Let $A' = A \cup \bigcup_{e \in U} A_e$ and $B' = B \cap \bigcap_{e \in U} B_e$. Note that the order of $[A', B']$ is at most $|[A, B]| + 3(k-1)hd < g(k, 1 + \bar{r})$. By Lemma 2.3, $[A', B'] \in \mathcal{E}$. By Claim 6, for every H-immersion $\Pi = (\pi_V, \pi_E)$ in $G - S$, one of the following holds.

- $\Pi(H)$ contains a non-loop edge of $G[X_{t^*}]$ or a loop in R.
- $V(\Pi(H)) \cap A' \neq \emptyset$.

Let Z_0 be the set consisting of all non-loop edges in $G[B \cap X_{t^*}]$ and all edges of G between A' and B'. Let $Z = Z_0 \cup R$. Then $G[B'] - (S \cup Z)$ has no H-immersion. In addition, by the definition of $\mathcal{E}$, $G[A'] - S$ has no H-immersion. Applying Claim 2 by taking $(\theta, W, [A, B], Z_0) = (g(k, 1 + \bar{r}), S, [A', B'], Z)$, there exists $Z' \subseteq E(G)$ with $|Z'| \leq |Z| + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1 + \bar{r}))) \leq (m + g(k, 1 + \bar{r}) + (k-1)hd) + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1 + \bar{r}))) \leq f(k, \bar{r}) - (k-1)g(k, \bar{r})$ such that $G - (S \cup Z')$ has no H-immersion.

Let Z_S be the set of edges incident with vertices S. So $|Z_S| \leq (k-1)g(k, \bar{r})$. Let $Z^* = Z' \cup Z_S$. Then $|Z^*| \leq f(k, \bar{r})$ and $G - Z^*$ has no H-immersion, contradicting (ii). □

Since $f(k, \bar{r}) - kg(k, 1 + \bar{r}) - (k-1)hd - (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1 + \bar{r}))) \geq 2$, Claim 7 implies that X_{t^*} contains at least two vertices and hence $G[X_{t^*}]$ does not have an edge-cut of order three.

For every vertex v in X_{t^*}, define $Q_v = \{u \in V(G) - X_{t^*}$: every path in G from u to X_{t^*} contains $v\}$. Note that Q_v is empty if $N_G(v) \subseteq X_{t^*}$. Furthermore, since G is connected, by Lemma 7.1, for every $u \in V(G) - X_{t^*}$, there exists a unique $v \in X_{t^*}$ such that $u \in Q_v$.

Define $\mathcal{E}'$ to be the set of edge-cuts $[A', B']$ of $G[X_{t^*}]$ of order less than $g(k, 1 + \bar{r}) - 3(k-1)hd$ such that $[A', B'] \in \mathcal{E}'$ if and only if $[A' \cup \bigcup_{v \in A'} Q_v, B' \cup \bigcup_{v \in B'} Q_v] \in \mathcal{E}$.

Claim 8: *$\mathcal{E}'$ is an edge-tangle of order $g(k, 1 + \bar{r}) - 3(k-1)hd$ in $G[X_{t^*}]$.*

Proof of Claim 8: For every edge-cut $[A', B']$ of $G[X_{t^*}]$, the order of $[A' \cup \bigcup_{v \in A'} Q_v, B' \cup \bigcup_{v \in B'} Q_v]$ equals the order of $[A', B']$. Since $\mathcal{E}$ is an edge-tangle, either $[A' \cup \bigcup_{v \in A'} Q_v, B' \cup \bigcup_{v \in B'} Q_v] \in \mathcal{E}$ or $[B' \cup \bigcup_{v \in B'} Q_v, A' \cup \bigcup_{v \in A'} Q_v] \in \mathcal{E}$. So $\mathcal{E}'$ satisfies (E1).

If there exist $[A'_i, B'_i] \in \mathcal{E}'$ for $i \in [3]$ such that $A'_1 \cup A'_2 \cup A'_3 = X_{t^*}$, then $[A'_i \cup \bigcup_{v \in A'_i} Q_v, B'_i \cup \bigcup_{v \in B'_i} Q_v] \in \mathcal{E}$ for $i \in [3]$, but $A'_1 \cup A'_2 \cup A'_3 \cup \bigcup_{v \in A'_1 \cup A'_2 \cup A'_3} Q_v = V(G)$, a contradiction. So $\mathcal{E}'$ satisfies (E2).

In addition, for every edge-cut $[A', B'] \in \mathcal{E}'$, the number of edges of $G[X_{t^*}]$ incident with B' is at least $|E(G[B'])| = |E(G[(B' \cup \bigcup_{v \in B'} Q_v) \cap X_{t^*}])| \geq f(k, \bar{r}) - kg(k, 1+\bar{r}) - (k-1)hd - (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1+\bar{r}))) \geq g(k, 1+\bar{r}) - 3(k-1)hd$ by Claim 7. So $\mathcal{E}'$ satisfies (E3). $\square$

By Claim 8, $\mathcal{E}'$ is an edge-tangle of order $g(k, 1+\bar{r}) - 3(k-1)hd \geq w_{6.4}(w_{5.6}(H, k + \bar{r} \cdot g(k, \bar{r})), \theta_{5.6}(H, k + \bar{r} \cdot g(k, \bar{r})))$ in $G[X_{t^*}]$, where the last inequality follows from the assumption that g is H-legal.

Define $\mathcal{E}_k$ and $\mathcal{E}'_k$ to be the subsets of $\mathcal{E}$ and $\mathcal{E}'$ consisting of edge-cuts of order less than $\theta_{5.6}(H, k+\bar{r} \cdot g(k, \bar{r}))$, respectively. So $\mathcal{E}_k$ and $\mathcal{E}'_k$ are edge-tangles of order $\theta_{5.6}(H, k+\bar{r} \cdot g(k, \bar{r}))$ in G and $G[X_{t^*}]$, respectively. Let $w_k = w_{5.6}(H, k + \bar{r} \cdot g(k, \bar{r}))$.

Since $G[X_{t^*}]$ does not have an edge-cut of order three, by Theorem 6.4, $\mathcal{E}'_k$ controls a K_{w_k}-thorns α in $G[X_{t^*}]$. Since α is in $G[X_{t^*}] \subseteq G$, $\mathcal{E}_k$ controls α.

Since $G - S$ does not contain $k - \bar{r}$ edge-disjoint H-immersions and every vertex in S has degree in G at most $g(k, \bar{r}) - 1$, G does not contain $k - \bar{r} + |S|(g(k, \bar{r}) - 1) \leq k - \bar{r} + \bar{r} \cdot g(k, \bar{r}) \leq k + \bar{r} \cdot g(k, \bar{r})$ edge-disjoint H-immersions. Since $\mathcal{E}_k$ is an edge-tangle in G of order $\theta_{5.6}(H, k + \bar{r} \cdot g(k, \bar{r}))$ controlling a K_{w_k}-thorns, by Lemma 5.6, there exist $Z^* \subseteq E(G)$ with $|Z^*| \leq \xi_k$ and $[A, B] \in \mathcal{E}_k - Z^* \subseteq \mathcal{E} - Z^*$ of order zero such that $G[B] - Z^*$ has no H-immersion, where $\xi_k = \xi_{5.6}(H, k + \bar{r} \cdot g(k, \bar{r}))$. Note that $\xi_k < \theta_{5.6}(H, k + \bar{r} \cdot g(k, \bar{r})) \leq g(k, 1+\bar{r})$ by Lemma 5.6 and the assumption that g is H-legal.

In addition, $G[A] - S$ does not contain an H-immersion since $[A, B] \in \mathcal{E}$. So every H-immersion in $G[A]$ intersects an edge in Z_S, where Z_S is the set of edges of G incident with S. Hence $G[A] - (Z^* \cup Z_S)$ and $G[B] - (Z^* \cup Z_S)$ do not contain H-immersions. Since $[A, B] \in \mathcal{E}$, $[A, B]$ is an edge-cut of G of order less than $g(k, 1+\bar{r})$. By Claim 2, there exists Z^{**} with $Z^{**} \subseteq E(G)$ and $|Z^{**}| \leq (|Z^*| + |Z_S| + g(k, 1+\bar{r})) + (2^h - 2)f'_{H,g}(kg(k, \bar{r} + g(k, 1+\bar{r}))) \leq (2g(k, 1+\bar{r}) + (k-1)g(k, \bar{r})) + (2^h - 2)f'_H(kg(k, \bar{r} + g(k, 1+\bar{r}))) \leq f(k, \bar{r})$ such that $G - Z^{**}$ has no H-immersion, contradicting (ii). This completes the proof. $\square$

Now we drop the requirement of having no isolated vertices from Lemma 7.3. Lemma 7.4 proves Theorem 1.1 for the case that G has only one component, as every 4-edge-connected graph is nearly 3-cut-free.

Lemma 7.4. *For every graph H, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every nearly 3-cut free graph G and every positive integer k, either G contains k edge-disjoint*

H-immersions, or there exists $Z \subseteq E(G)$ with $|Z| \leq f(k)$ such that $G - Z$ contains no H-immersion.

Proof. Let H be a graph. Note that when H has no edge, every graph G contains at least one H-immersion if and only if G contains at least $|V(H)|$ vertices if and only if G contains arbitrarily many edge-disjoint H-immersions. So this theorem holds if H has no edge.

Hence we may assume that H contains at least one edge. Let H' be the graph obtained from H by deleting all isolated vertices. So H' is a graph with at least one edge and with no isolated vertices. For every positive integer k, define $f(k)$ to be the number $f_{7.3}(k, 0)$, where $f_{7.3}$ is the function mentioned in Theorem 7.3 by taking H to be H' and further taking $g = g^*$. We apply Theorem 7.3 by further taking $S = \emptyset$ and γ to be a function with empty domain, we know that for every nearly 3-cut free graph G and positive integer k, either G contains k edge-disjoint H'-immersions, or there exists $Z \subseteq E(G)$ with $|Z| \leq f(k)$ such that $G - Z$ does not contain an H'-immersion.

We shall prove that f is a function satisfying the conclusion of this lemma.

Let G be a nearly 3-cut free graph. If $|V(G)| < |V(H)|$, then clearly G does not contain an H-immersion, and we are done by choosing $Z = \emptyset$. So we may assume that $|V(G)| \geq |V(H)|$. Hence, for every $W \subseteq E(G)$ and every H'-immersion (π'_V, π'_E) of $G - W$, we can extend π'_V to an injection π_V with domain $V(H)$ by further mapping isolated vertices of H to some vertices of $G - \pi'_V(V(H'))$ such that (π_V, π'_E) is an H-immersion in $G - W$ with $E(\pi'_E(E(H'))) = E(\pi'_E(E(H)))$. Therefore, for every $W \subseteq E(G)$ and every integer k, $G - W$ contains k edge-disjoint H-immersions if and only if $G - W$ contains k edge-disjoint H'-immersions.

Now let k be a positive integer. If G does not contain k edge-disjoint H-immersions, then G does not contain k edge-disjoint H'-immersions, so there exists $Z \subseteq E(G)$ with $|Z| \leq f_{7.3}(k, 0) = f(k)$ such that $G - Z$ has no H'-immersion by Theorem 7.3. But it implies that $G - Z$ has no H-immersions. This proves the lemma. □

Theorem 1.1 is an immediate corollary of the following theorem.

Theorem 7.5. *For every graph H, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G whose every component is nearly 3-cut free and for every positive integer k, either G contains k edge-disjoint H-immersions, or there exists $Z \subseteq E(G)$ with $|Z| \leq f(k)$ such that $G - Z$ contains no H-immersion.*

Proof. Let H be a graph, and let c be the number of components of H. We define the following.

- For every graph R, define f_R to be the function f mentioned in Lemma 7.4 by taking $H = R$.
- For every $i \in [c]$, let $\mathcal{F}_i$ be the set of graphs consisting of i components of H.

- For any positive integers $m \geq 2$ and n, let $f_1(n) = (n-1) \cdot \max\{f_R(n) : R \in \mathcal{F}_1\}$ and let $f_m(n) = m^{nm} f_{m-1}(n) + (nm-1) \cdot \max\{f_R(n) : R \in \mathcal{F}_m\}$.
- Define $f : \mathbb{N} \to \mathbb{N}$ to be the function such that $f(x) = f_c(x)$ for every $x \in \mathbb{N}$.

We claim that for every $m \in [c]$, for every graph $W \in \mathcal{F}_m$, for every positive integer k and for every graph G whose every component is nearly 3-cut free, either G contains k edge-disjoint W-immersions, or there exists $Z \subseteq E(G)$ with $|Z| \leq f_m(k)$ such that $G - Z$ does not contain a W-immersion. Note that this claim implies this theorem as $H \in \mathcal{F}_c$. We shall prove this claim by induction on m.

Let $m \in [c]$, $W \in \mathcal{F}_m$, k a positive integer and G a graph whose every component is nearly 3-cut free. We assume that G does not contain k edge-disjoint W-immersions. It suffices to show that there exists $Z \subseteq E(G)$ with $|Z| \leq f_m(k)$ such that $G - Z$ has no W-immersion.

We first assume that $m = 1$. Let $G_1, G_2, ..., G_p$ be the components of G containing a W-immersion, and let k_i be the maximum number of edge-disjoint W-immersions in G_i for each $i \in [p]$. If $\sum_{i=1}^{p} k_i \geq k$, then G contains k edge-disjoint W-immersions, a contradiction. So $\sum_{i=1}^{p} k_i < k$. In particular, $p < k$. By Lemma 7.4, for every $i \in [p]$, there exists $Z_i \subseteq E(G_i)$ with $|Z_i| \leq f_W(k_i + 1)$ such that $G_i - Z_i$ has no W-immersion. Since $m = 1$, W is connected, so $G - Z$ has no W-immersion, where $Z = \bigcup_{i=1}^{p} Z_i \subseteq E(G)$. Note that $|Z| \leq \sum_{i=1}^{p} f_W(k_i + 1) \leq (k-1) f_W(k) \leq f_1(k)$. This proves the base case of the induction.

So we may assume that $m \geq 2$ and our claim holds for every smaller m.

Note that W has m components. Let $W_1, W_2, ..., W_m$ be the components of W. For every $i \in [m]$, define S_i to be the set of components of G containing an W_i-immersion. If $|S_i| \geq km$ for every $i \in [m]$, then G contains km components $G_1, ..., G_{km}$ of G such that $G_{(i-1)k+j} \in S_i$ for each $i \in [m]$ and each $j \in [k]$, so G contains k edge-disjoint W-immersions, a contradiction. Therefore, there exists $t \in [m]$ such that $|S_t| < km$.

Define L to be the disjoint union of the components of G in S_t, and define $R = G - V(L)$. Note that R has no W_t-immersion by the definition of S_t. Since $m \geq 2$, by the induction hypothesis, if L does not contain k edge-disjoint W_t-immersions, then there exists $Z_t \subseteq E(L)$ with $|Z_t| \leq f_1(k) \leq f_m(k)$ such that $L - Z_t$ has no W_t-immersion, so $G - Z_t$ has no W_t-immersion (since W_t is connected) and hence has no W-immersion.

So we may assume that L contains k edge-disjoint W_t-immersions. Hence R does not contain k edge-disjoint $(W - V(W_t))$-immersions, for otherwise G contains k edge-disjoint W-immersions. Note that $W - V(W_t) \in \mathcal{F}_{m-1}$. By the induction hypothesis, there exists $Z_R \subseteq E(R)$ with $|Z_R| \leq f_{m-1}(k)$ such that $R - Z_R$ has no $(W - V(W_t))$-immersion. So $R - Z_R$ has no W-immersion. In addition, for each component C of L, C is nearly 3-cut free and has no k edge-disjoint W-immersions, so there exists $Z_C \subseteq E(C)$ with $|Z_C| \leq f_W(k)$ such that $C - Z_C$ has no W-immersion.

Define $Z_0 = Z_R \cup \bigcup_C Z_C$, where the second union is taken over all components C of L. Note that the number of components of L equals $|S_t| \leq km - 1$. Therefore,

$|Z_0| \le f_{m-1}(k) + (km-1)f_W(k)$, and $R - Z_0$ and $C - Z_0$ do not contain a W-immersion for every component C of L.

Let ℓ be the number of components of L. Define $Q_0 = R$, and for every $i \in [\ell]$, define Q_i to be the i-th component of L. Note that $Q_i - Z_0$ has no W-immersion for every i with $0 \le i \le \ell$.

We say that $(P_0, P_1, ..., P_\ell)$ is a $(\ell+1)$-*partition* of $[m]$ if $P_0, P_1, ..., P_\ell$ are pairwise disjoint (possibly empty) proper subsets of $[m]$ with $\bigcup_{i=0}^{\ell} P_i = [m]$. Since G has no k edge-disjoint W-immersions, for every $(\ell+1)$-partition $\mathcal{P} = (P_0, ..., P_\ell)$ of $[m]$, there exists j with $0 \le j \le \ell$ such that Q_j does not contain k edge-disjoint $(\bigcup_{i \in P_j} W_i)$-immersions, so there exists $Z_\mathcal{P} \subseteq E(Q_j)$ with $|Z_\mathcal{P}| \le f_{|P_j|}(k) \le f_{m-1}(k)$ such that $Q_j - Z_\mathcal{P}$ has no $(\bigcup_{i \in P_j} W_i)$-immersions.

Define Z^* to be the union of Z_0 and $Z_\mathcal{P}$ over all $(\ell+1)$-partitions $\mathcal{P}$ of $[m]$. Since $\ell = |S_t| \le km - 1$ and there are at most $m^{\ell+1} - 1$ different $(\ell+1)$-partitions of $[m]$, $|Z^*| \le |Z_0| + (m^{\ell+1}-1)f_{m-1}(k) \le f_{m-1}(k) + (km-1)f_W(k) + (m^{km}-1)f_{m-1}(k) \le f_m(k)$.

To prove the theorem, it suffices to prove that $G - Z^*$ has no W-immersion. Suppose to the contrary that $G - Z^*$ contains a W-immersion. Since $Q_i - Z^*$ has no W-immersion for every $0 \le i \le \ell$, there exists a $(\ell+1)$-partition $\mathcal{P} = (P_0, P_1, ...P_\ell)$ of $[m]$ such that for every j with $0 \le j \le \ell$, $Q_j - Z^*$ contains a $(\bigcup_{i \in P_j} W_i)$-immersion. However, it contradicts the definition of $Z_\mathcal{P}$. This completes the proof. □

We remark that Kakimura and Kawarabayashi [8] proved that for every integer t, there exists a function f such that for every 3-minimal-cut free graph G and integer k, either G contains k edge-disjoint K_t-immersions, or there exists $Z \subseteq E(G)$ with $|Z| \le f(k)$ such that $G - Z$ has no K_t-immersion, where a graph is *3-minimal-cut free* if it is connected and it cannot be made disconnected by deleting exactly three edges while it remains connected by deleting at most two of those three edges. This result is a simple corollary of Theorem 7.5 when $t \ge 3$ (and the case $t \le 2$ is easy). Let G be a 3-minimal-cut free graph, and let G' be the graph obtained from G by deleting all cut-edges and loops and then deleting all resulting isolated vertices. Note that every component of G' is 2-edge-connected and 3-minimal-cut free. If a 2-edge-connected and 3-minimal-cut free graph has an edge-cut $[A, B]$ of order three, then one can delete at most two edges between A and B to make the graph disconnected, but it implies that some edge in $[A, B]$ is a cut-edge of the original graph, contradicting that it is 2-edge-connected. Hence every component of G' does not contain an edge-cut of order three and hence is nearly 3-cut free. In addition, when $t \ge 3$, the optimal solutions for packing and covering K_t-immersions in G are the same as the optimal solutions for packing and covering K_t-immersions in G'. Hence Theorem 7.5 implies the result in [8].

Now we prove Theorem 1.2. The following is a restatement.

Theorem 7.6. *For every loopless graph H, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every positive integer k and every graph G, either there exists k H-half-integral immersions $(\pi_V^{(1)}, \pi_E^{(1)}), ..., (\pi_V^{(k)}, \pi_E^{(k)})\}$ in G such that for each edge e of G, there exist* *at most two distinct pairs* (i, e') *with* $1 \leq i \leq k$ *and* $e' \in E(H)$ *such that* $e \in \pi_E^{(i)}(e')$*, or there exists* $Z \subseteq E(G)$ *with* $|Z| \leq f(k)$ *such that* $G - Z$ *contains no* H*-half-integral immersion.*

Proof. Define f to be the function f mentioned in Theorem 7.5 by taking $H = H$.

Let k be a positive integer. Let G be a graph, and let G' be the graph obtained from G by duplicating each edge. Note that every edge-cut of G' has even order. If $[A, B]$ is an edge-cut of a component of G' of order between one and three, then it has order two and the two edges between A and B are parallel edges with the same ends. So every component of G' is nearly 3-cut free. By Theorem 7.5, either G' contains k edge-disjoint H-immersions, or there exists $Z' \subseteq E(G')$ with $|Z'| \leq f(k)$ such that $G' - Z'$ does not contain an H-immersion.

Note that since H is loopless, for every H-immersion (π_V, π_E) in G' and $e \in E(H)$, $\pi_E(e)$ is a path in G', so there exists no $e' \in E(G)$ such that $\pi_E(e)$ contains e' and its copy in G'. So for every H-immersion (π'_V, π'_E) in G', there exists an H-half-integral immersion (π_V, π_E) such that $\pi_V(v) = \pi'_V(v)$ for every $v \in V(H)$, and for every $e \in E(H)$, $\pi_E(e)$ consists of the edges z of G such that some copy of z belongs to $E(\pi'_E(e))$. Similarly, if G' contains k edge-disjoint H-immersions, then G contains k H-half-integral immersions $(\pi_V^{(1)}, \pi_E^{(1)}), ..., (\pi_V^{(k)}, \pi_E^{(k)})$ such that for each edge e of G, there exist at most two distinct pairs (i, e') with $1 \leq i \leq k$ and $e' \in E(H)$ such that $e \in \pi_E^{(i)}(e')$, so we are done.

So we may assume that there exists $Z' \subseteq E(G')$ with $|Z'| \leq f(k)$ such that $G' - Z'$ has no H-immersion. Then $G - Z$ has no H-half-integral immersion, where $Z \subseteq E(G)$ is the set consisting of the edges of G having a copy in Z'. Note that $|Z| \leq |Z'|$. This completes the proof. □

Acknowledgments

Theorem 4.6 is based on part of an unpublished manuscript of the author in 2013. The author thanks Professor Robin Thomas and Professor Paul Wollan for some inspiring discussion that led to that manuscript. Lemma 3.2 was proved by the author when he worked on [12], and its proof was included in [12] when the first version of this paper was submitted. The author thanks Professor Luke Postle and Professor Paul Wollan for agreeing for moving the proof of Lemma 3.2 into this paper to meet a request of anonymous reviewers of this paper. The author thanks Professor Reinhard Diestel for pointing out [3–5] and thanks anonymous reviewers for careful reading and suggestions.

References

[1] M. Chudnovsky, Z. Dvořák, T. Klimošová, P. Seymour, Immersion in four-edge-connected graphs, J. Comb. Theory, Ser. B 116 (2016) 208–218.

[2] R. Diestel, Graph Theory, 5th edition, Graduate Texts in Mathematics, vol. 173, Springer-Verlag, 2016.

[3] R. Diestel, F. Hundertmark, S. Lemanczyk, Profiles of separations: in graphs, matroids, and beyond, Combinatorica 39 (2019) 37–75.

[4] R. Diestel, S. Oum, Tangle-tree duality in abstract separation systems, Adv. Math. 377 (2021) 107470.
[5] R. Diestel, S. Oum, Tangle-tree duality: in graphs, matroids and beyond, Combinatorica 39 (2019) 879–910.
[6] P. Erdős, L. Pósa, On independent circuits contained in a graph, Can. J. Math. 17 (1965) 347–352.
[7] A. Giannopoulou, O. Kwon, J.-F. Raymond, D.M. Thilikos, Packing and covering immersion-expansions of planar sub-cubic graphs, Eur. J. Comb. 65 (2017) 154–167.
[8] N. Kakimura, K. Kawarabayashi, The Erdős-Pósa property for edge-disjoint immersions in 4-edge-connected graphs, J. Comb. Theory, Ser. B 131 (2018) 138–169.
[9] K. Kawarabayashi, Y. Kobayashi, Edge-disjoint odd cycles in 4-edge-connected graphs, J. Comb. Theory, Ser. B 119 (2016) 12–27.
[10] C.-H. Liu, A global decomposition theorem for excluding immersions in graphs with no edge-cut of order three, arXiv:2006.15694, 2020.
[11] C.-H. Liu, Packing topological minors half-integrally, arXiv:1707.07221, 2017.
[12] C.-H. Liu, L. Postle, P. Wollan, Erdős-Pósa property for topological minors, manuscript, 2015.
[13] D. Marx, P. Wollan, Immersions in highly edge connected graphs, SIAM J. Discrete Math. 28 (2014) 503–520.
[14] J.-F. Raymond, I. Sau, D.M. Thilikos, An edge variant of the Erdős-Pósa property, Discrete Math. 339 (2016) 2027–2035.
[15] J.-F. Raymond, D.M. Thilikos, Recent techniques and results on the Erdős-Pósa property, Discrete Appl. Math. 231 (2017) 25–43.
[16] N. Robertson, P.D. Seymour, Graph minors. V. Excluding a planar graph, J. Comb. Theory, Ser. B 41 (1986) 92–114.
[17] N. Robertson, P.D. Seymour, Graph minors. X. Obstructions to tree-decomposition, J. Comb. Theory, Ser. B 52 (1991) 153–190.
[18] N. Robertson, P.D. Seymour, Graph minors. XIII. The disjoint paths problem, J. Comb. Theory, Ser. B 63 (1995) 65–110.
[19] N. Robertson, P.D. Seymour, Graph minors. XXIII. The Nash-Williams immersion conjecture, J. Comb. Theory, Ser. B 100 (2010) 181–205.
[20] N. Robertson, P.D. Seymour, R. Thomas, Quickly excluding a planar graph, J. Comb. Theory, Ser. B 62 (1994) 323–348.
[21] P.D. Seymour, R. Thomas, Call routing and the ratcatcher, Combinatorica 14 (1994) 217–241.
[22] P. Wollan, The structure of graphs not admitting a fixed immersion, J. Comb. Theory, Ser. B 110 (2015) 47–66.