Dual-Feasible Functions for Integer Programming and Combinatorial Optimization: Algorithms, Characterizations, and Approximations*

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Abstract. Within the framework of the superadditive duality theory of integer programming, we study two types of dual-feasible functions of a single real variable [Alves, C. et al.: *Dual-Feasible Functions for Integer Programming and Combinatorial Optimization: Basics, Extensions and Applications.* Springer (2016)]. We introduce software that automates testing piecewise linear functions for maximality and extremality, enabling a computer-based search. We build a connection to cut-generating functions in the Gomory–Johnson and related models, complete the characterization of maximal functions, and prove analogues of the Gomory–Johnson 2-slope theorem and the Basu–Hildebrand–Molinaro approximation theorem.

Keywords: Dual-feasible functions, cut-generating functions, integer programming, 2-slope theorem, computer-based search

1 Introduction

The duality theory of integer linear optimization is a multifaceted research topic that connects to cutting plane theory and the theory of value functions of parametric optimization problems. The central objects on the dual side of this theory are superadditive functionals.

* Extended abstracts with some of the results and proofs of the present paper have appeared as *Köppe, M., Wang, J.: Structure and interpretation of dual-feasible functions. Electronic Notes in Discrete Mathematics 62 (Supplement C), 153–158 (2017), LAGOS'17 IX Latin and American Algorithms, Graphs and Optimization,* https://doi.org/10.1016/j.endm.2017.10.027 and *Köppe, M., Wang, J.: Characterization and approximation of strong general dual feasible functions. In: Lee, J., Rinaldi, G., Mahjoub, A.R. (eds.) Combinatorial Optimization, ISCO 2018, Lecture Notes in Computer Science, vol. 10856, pp. 265–276. Springer International Publishing, Cham (2018),* https://doi.org/10.1007/978-3-319-96151-4_23.

** The authors gratefully acknowledge partial support from the National Science Foundation through grant DMS-1320051 (M. Köppe). A part of this work was done while the first author (M. Köppe) was visiting the Simons Institute for the Theory of Computing. It was partially supported by the DIMACS/Simons Collaboration on Bridging Continuous and Discrete Optimization through NSF grant CCF-1740425.

Dual-feasible functions. In the present paper, we make no attempt of describing this duality theory in its full generality. Rather, we focus on two simple and fundamental settings in which superadditive functionals of a single real variable appear. These settings, following the recent monograph [1], can be defined as follows. *Classical dual-feasible functions (cDFFs)* are functions $\phi\colon D \to D$ such that

$$\sum_{i\in I} x_i \leq 1 \quad \Rightarrow \quad \sum_{i\in I} \phi(x_i) \leq 1 \tag{1}$$

holds for any family $\{x_i\}_{i\in I} \subseteq D$ indexed by a finite index set I, where $D = [0, 1]$. In [1, Chapters 2 and 3], these functions are studied alongside with *general dual-feasible functions (gDFFs)*, which satisfy the same property (1) for the extended domain $D = \mathbb{R}$. (There is an equivalent definition of these functions in terms of valid inequalities for infinite-dimensional integer programming models, which we suppress until subsection 5.2.)

Dual-feasible functions provide strong dual bounds in a branch-and-bound algorithm and strong valid inequalities in a cutting-plane procedure. They have been applied to this effect in particular for combinatorial optimization problems that benefit from a column-generation approach, such as bin-packing or cutting-stock problems [28,30]. We refer to [1, section 2.5] for further pointers to the literature.

For each of the two settings, classical and general, there is a hierarchy of DFFs regarding their strength. The *maximal* DFFs are those that are pointwise non-dominated. As we will see later, they are the ones that have a characterization involving superadditivity. Among the maximal DFFs, a DFF is said to be *extreme* if it can not be written as a convex combination of two other maximal DFFs. (This is an analogue of the notion of facet-defining inequalities.)

Cut-generating functions. These features of the theory are remarkably similar to the ones in the study of the valid inequalities for the Gomory–Johnson infinite group problem [14,15], and the broader context of cut-generating functions [12,31]. Our paper is an attempt to determine the precise relation between DFFs and cut-generating functions, and to transfer recent advances in the study of the latter to the DFF setting, in the hope that they will prove useful there.

In particular, we note that cut-generating functions are closely tied to the setting of tableau cuts in a simplex-based cutting plane procedure. They play an essential role in generating valid inequalities which cut off the *current fractional basic solution*. In this way, they explain and generalize the Gomory fractional cut and Gomory mixed-integer cut, the workhorses of state-of-art integer programming solvers. The cut-generating functions in the Gomory–Johnson model [14,15] are related to Gomory's corner relaxation [13], which is obtained by relaxing the non-negativity of all basic variables in the tableau. Thus, basic integer variables are allowed to take any value in $\mathbb{Z}$ in the relaxations. (Cut-generating functions for stronger relaxations have been studied too: In the model of Yıldız and Cornuéjols [31], basic variables are constrained to some set $S \subset \mathbb{R}$ with suitable properties. We will come back to this model later.)

The close relation of cut-generating functions to a particular algorithmic framework possibly limits their applicability. Our paper can be seen as a preliminary study toward a larger goal: We would like to make the powerful results for cut-generating functions — and the rich toolsets that were used to obtain them — available for spaces of functions that arise from the study of broader algorithmic frameworks. This includes algorithms for large-scale problems based on decomposition techniques. The context in which the study of DFFs arose, combinatorial problems with Dantzig–Wolfe decomposition, is one such setting.

Contributions and structure of the paper. After a brief review of key results for DFFs in section 2, we begin the paper in section 3 by transferring and extending recent algorithmic techniques [5,20,32,16,17] developed by Basu, Hildebrand, Hong, Köppe, and Zhou for cut-generating functions in the single-row Gomory–Johnson model [6,7] to DFFs. The algorithms are implemented in the current version of the software [27]. We focus on cDFFs. Like the majority of the development in [1, Chapter 2], we consider piecewise linear functions that are allowed to be discontinuous at the breakpoints. Similar to [23], we provide an electronic compendium of the known extreme DFFs from [1]. The extremality of the functions from this library is proved in [1] by studying analytical properties of extreme DFFs. We complement this by our algorithmic techniques, leading to automatic maximality and extremality tests for cDFFs. They are based on the methods of polyhedral complexes and functional equations from [5,20] and the inverse semigroup techniques from [16,17]. On the basis of the automatic maximality and extremality test, we use a computer-based search technique based on polyhedral computation and filtering to find new extreme DFFs. Our search reveals that the classic dual-feasible functions are much richer than what is represented by the families of functions described in the literature. We hope that our software facilitates experimentation and further study.

Then, in section 4, we turn to the study of gDFFs. Here we transfer techniques used by Yıldız and Cornuéjols for the study of their previously mentioned model of cut-generating functions, which generalizes the Gomory–Johnson framework. Their results extended the characterization of minimal Gomory–Johnson cut-generating functions in terms of the so-called generalized symmetry condition. Inspired by the characterization of minimal Yıldız–Cornuéjols cut-generating functions and using similar techniques, we give a full characterization of maximal gDFFs, closing a gap in [1].

In section 5, we investigate the relation between classical and gDFFs and cut-generating functions. First, in subsection 5.1, we introduce a conversion from Gomory–Johnson functions to DFFs, which under some conditions generates maximal or extreme cDFFs and gDFFs. The Gomory–Johnson model is well-studied and the literature provides a large library of known functions. From our conversion, we obtain 2-slope extreme DFFs and maximal DFFs with arbitrary number of slopes. This work is also a possible starting point for constructing new parametric families of DFFs with special properties.

Then, in subsection 5.2, we focus on gDFFs, which allow us to build a more precise connection. GDFFs turn out to have a very close relation to a model studied by Jeroslow [21], Blair [10] and Bachem et al. [3], which we refer to by $Y_{=1}$ and a certain relaxation of this model, which we denote by $Y_{\leq 1}$. Both $Y_{=1}$ and $Y_{\leq 1}$ can be studied in the Yıldız–Cornuéjols model [31] with various sets S. GDFFs generate valid inequalities for the Yıldız–Cornuéjols model with $S = (-\infty, 0]$, and cut-generating functions generate valid inequalities for the Jeroslow model where $S = \{0\}$. These two families of functions are then connected by an operation known as "tilting."

Perhaps the most famous result of Gomory and Johnson's masterpiece [14,15] is the 2-slope theorem, showing that every continuous piecewise linear minimal valid function that has only two different slope values is an extreme function. In section 6, we show a similar result for gDFFs: Any 2-slope maximal gDFF, for which one slope value is 0, is extreme. In contrast, we show that for cDFFs, there cannot exist a 2-slope theorem. We have a counterexample, a maximal cDFF with 2 slopes and 3 "connected covered components" (a concept from the algorithmic extremality test) that is not extreme.

Finally, in section 7, we turn to the approximation theory of gDFFs. Basu et al. [9] proved that the 2-slope extreme Gomory–Johnson cut-generating functions are dense in the set of continuous minimal functions. We prove a similar approximation theorem, which indicates that almost all continuous maximal gDFFs can be approximated by extreme (2-slope) gDFFs as close as we desire. Unlike the 2-slope fill-in procedure that Basu et al. [9] used, we always use 0 as one slope value in our fill-in procedure, which is necessary since the 2-slope theorem of gDFFs requires 0 to be one slope value. It remains an open question whether maximal cDFFs can also be approximated in the same way by extreme functions.

2 Key results from the DFF literature

Definition 2.1 *A function $\phi\colon [0,1] \to [0,1]$ is called a (valid)* classical dual-feasible function (cDFF)*, if for any finite list of real numbers $x_i \in [0,1]$, $i \in I$, it holds that $\sum_{i\in I} x_i \leq 1 \Rightarrow \sum_{i\in I} \phi(x_i) \leq 1$. A function $\phi\colon \mathbb{R} \to \mathbb{R}$ is called a (valid)* general dual-feasible function (gDFF)*, if for any finite list of real numbers $x_i \in \mathbb{R}$, $i \in I$, it holds that $\sum_{i\in I} x_i \leq 1 \Rightarrow \sum_{i\in I} \phi(x_i) \leq 1$.*

Definition 2.2 *A cDFF/gDFF is* maximal *if it is not (pointwise) dominated by a distinct cDFF/gDFF. A cDFF/gDFF is* extreme *if it cannot be written as a convex combination of other two different cDFFs/gDFFs.*

In the monograph [1], the authors explored maximality of both cDFFs and gDFFs.

Theorem 2.3 (Characterization of maximal cDFFs, [1, Theorem 2.1]). *A function $\phi\colon [0,1] \to [0,1]$ is a maximal cDFF if and only if $\phi(0) = 0$, ϕ is superadditive and ϕ is symmetric in the sense $\phi(x) + \phi(1-x) = 1$ for all $x \in [0,1]$.*

Theorem 2.4 (Conditions for maximality of gDFFs, [1, Theorem 3.1]). *Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a given function. If ϕ satisfies the following conditions, then ϕ is a maximal gDFF: (i) $\phi(0) = 0$. (ii) ϕ is symmetric in the sense $\phi(x)+\phi(1-x) = 1$ for all $x \in \mathbb{R}$. (iii) ϕ is superadditive. (iv) There exists an $\epsilon > 0$ such that $\phi(x) \geq 0$ for all $x \in (0, \epsilon)$.*

If ϕ is a maximal gDFF, then ϕ satisfies conditions (i), (iii) and (iv).

Remark 2.5 *The function $\phi(x) = cx$ for $0 \leq c < 1$ is a maximal gDFF but it does not satisfy condition (ii).*

Remark 2.6 *Note that conditions (i), (iii) and (iv) guarantee that any maximal gDFF is nondecreasing and consequently nonnegative on $\mathbb{R}_+$.*

Different approaches to construct non-trivial cDFFs from "simple" functions are explained in [1], including convex combination and function composition.

Proposition 2.7 ([1, Section 2.3.1]) *If ϕ_1 and ϕ_2 are two maximal cDFFs, then $\alpha\phi_1 + (1-\alpha)\phi_2$ is also a maximal cDFF, for $0 < \alpha < 1$.*

Proposition 2.8 ([1, Proposition 2.3]) *If ϕ_1 and ϕ_2 are two maximal cDFFs, then the composed function $\phi_1(\phi_2(x))$ is also a maximal cDFF.*

Maximal gDFFs can also be obtained by extending maximal cDFFs to the domain $\mathbb{R}$. Theorem 2.9 uses quasiperiodic extensions and Theorem 2.10 uses affine functions when x is not in $[0, 1]$. Throughout the paper, we use $\{a\}$ to represent the fractional part of a.

Theorem 2.9 ([1, Proposition 3.10]). *Let ϕ be a maximal cDFF, then there exists $b_0 \geq 1$ such that for all $b > b_0$ the following function $\hat{\phi}(x)$ is a maximal gDFF.*

$$\hat{\phi}(x) = \begin{cases} b \times \lfloor x \rfloor + \phi(\{x\}) & \text{if } x \leq 1 \\ 1 - \hat{\phi}(1-x) & \text{if } x > 1 \end{cases}.$$

Theorem 2.10 ([1, Proposition 3.12]). *Let ϕ be a maximal cDFF, then there exists $b \geq 1$ such that the following function $\hat{\phi}(x)$ is a maximal gDFF.*

$$\hat{\phi}(x) = \begin{cases} bx + 1 - b & \text{if } x < 0 \\ bx & \text{if } x > 1 \\ \phi(x) & \text{if } 0 \leq x \leq 1 \end{cases}.$$

Proposition 2.11 shows that every maximal gDFF is the sum of a linear function and a bounded function. Proposition 2.12 explains the behavior of nonlinear maximal gDFFs at given points.

Proposition 2.11 ([1, Proposition 3.4]) *If $\phi\colon \mathbb{R} \to \mathbb{R}$ is a maximal gDFF and $t = \sup\{\frac{\phi(x)}{x} : x > 0\}$, then we have $\lim_{x\to\infty} \frac{\phi(x)}{x} = t \leq -\phi(-1)$, and for any $x \in \mathbb{R}$, it holds that: $tx - \max\{0, t-1\} \leq \phi(x) \leq tx$.*

Proposition 2.12 ([1, Proposition 3.5]) *If $\phi\colon \mathbb{R} \to \mathbb{R}$ is a maximal gDFF and not of the kind $\phi(x) = cx$ for $0 \le c < 1$, then $\phi(1) = 1$ and $\phi(\frac{1}{2}) = \frac{1}{2}$.*

The following proposition utilizes the fact that maximal gDFFs are superadditive and nondecreasing, which can be used to generate valid inequalities for general linear integer optimization problems.

Proposition 2.13 ([1, Proposition 5.1]) *If ϕ is a maximal gDFF and $L = \{x \in \mathbb{Z}_+^n : \sum_{j=1}^n a_{ij}x_j \le b_i, i = 1, 2, \ldots, m\}$, then for any i, $\sum_{j=1}^n \phi(a_{ij})x_j \le \phi(b_i)$ is a valid inequality for L.*

3 Automatic tests and search for classical DFFs

In this section, we restrict ourselves to piecewise linear cDFFs. We introduce the automatic maximality and extremality tests of given piecewise linear functions, and a computer-based search method which is used to find new extreme functions. Our methods are released as part of the software [27], which is written in Python, using the framework of SageMath [29], a comprehensive Python-based open source computer algebra system. In this paper, a function name shown in typewriter font refers to the `cutgeneratingfunctionology.dff` module of our SageMath program [27].

3.1 Definition of piecewise linear functions and polyhedral complexes underlying the algorithmic maximality test of classical DFFs

We begin with a definition of piecewise linear functions $\phi\colon [0, 1] \to [0, 1]$ that are allowed to be discontinuous, similar to [5, section 2.1] and [20,6]. Let $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1$. Denote by $B = \{a_0, a_1, \ldots, a_{n-1}, a_n\}$ the set of all possible *breakpoints*. The 0-dimensional faces are defined to be the singletons, $\{a_i\}$, $a_i \in B$, and the 1-dimensional faces are the closed intervals, $[a_i, a_{i+1}]$, $i = 0, \ldots, n-1$. Together they form $\mathcal{P} = \mathcal{P}_B$, a finite polyhedral complex. We call a function $\phi\colon [0, 1] \to \mathbb{R}$ *piecewise linear* over $\mathcal{P}_B$ if for each face $I \in \mathcal{P}_B$, there is an affine linear function $\phi_I : \mathbb{R} \to \mathbb{R}$, $\phi_I(x) = c_I x + b_I$ such that $\phi(x) = \phi_I(x)$ for all $x \in \operatorname{rel\,int}(I)$. Under this definition, piecewise linear functions can be discontinuous. Let $I = [a_i, a_{i+1}]$. The function ϕ can be determined on the open intervals $\operatorname{int}(I) = (a_i, a_{i+1})$ by linear interpolation of the limits $\phi(a_i^+) = \lim_{x\to a_i, x>a_i} \phi(x) = \phi_I(a_i)$ and $\phi(a_{i+1}^-) = \lim_{x\to a_{i+1}, x<a_{i+1}} \phi(x) = \phi_I(a_{i+1})$. We say the function ϕ is continuous piecewise linear over $\mathcal{P}_B$ if it is affine over each of the cells of $\mathcal{P}_B$ (thus automatically imposing continuity).

Unlike Gomory–Johnson cut-generating functions, which may be discontinuous at 0 on both sides, a classical maximal DFF is always continuous at 0 from the right and at 1 from the left.

Lemma 3.1 *Any piecewise linear maximal cDFF is continuous at 0 from the right and continuous at 1 from the left.*

Proof. Consider ϕ to be a piecewise linear maximal cDFF, and $\phi(x) = sx + b$ on the first open interval (a_0, a_1). Note that the maximality of ϕ implies that $\phi(0) = 0$. Choose $x = y = \frac{a_1}{3}$, and based on superadditivity, we have

$$\phi(x) + \phi(y) \leq \phi(x + y) \Rightarrow sx + b + sy + b \leq s(x + y) + b \Rightarrow b \leq 0.$$

Since b is also the right limit at 0, so b is nonnegative. Therefore, $b = 0$, which implies ϕ is continuous at 0 from the right. By the symmetry condition, ϕ is continuous at 1 from the left. □

Similar to [5,6,7,8,16], we introduce the function $\nabla\phi\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $\nabla\phi(x, y) = \phi(x + y) - \phi(x) - \phi(y)$. The function $\nabla\phi$ measures the slack in the superadditivity condition. Observe that the piecewise linearity of ϕ induces piecewise linearity of $\nabla\phi$. In order to express the domains of linearity of $\nabla\phi(x, y)$, and thus domains of additivity and strict superadditivity, we introduce the two-dimensional polyhedral complex $\Delta\mathcal{P} = \Delta\mathcal{P}_B$. The faces F of the complex are defined as follows. Let $I, J, K \in \mathcal{P}_B$, so each of I, J, K is either a breakpoint of ϕ or a closed interval delimited by two consecutive breakpoints. Then $F = F(I, J, K) = \{\, (x, y) \in \mathbb{R} \times \mathbb{R} : x \in I,\, y \in J,\, x + y \in K \,\}$. The projections $p_1, p_2, p_3 \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are defined as $p_1(x, y) = x$, $p_2(x, y) = y$, $p_3(x, y) = x + y$. Let $F \in \Delta\mathcal{P}$ and let $(u, v) \in F$. We define

$$\nabla\phi_F(u, v) = \lim_{\substack{(x,y)\to(u,v)\\ (x,y)\in \operatorname{rel\,int}(F)}} \nabla\phi(x, y),$$

which allows us to conveniently express limits to boundary points of F, in particular to vertices of F, along paths within $\operatorname{rel\,int}(F)$. It is clear that $\nabla\phi_F(u, v)$ is affine over F, and $\nabla\phi(u, v) = \nabla\phi_F(u, v)$ for all $(u, v) \in \operatorname{rel\,int}(F)$. We will use $\operatorname{vert}(F)$ to denote the set of vertices of the face F.

Let ϕ be a piecewise linear maximal DFF. We now define the *additive faces* of the two-dimensional polyhedral complex $\Delta\mathcal{P}$ of ϕ. When ϕ is continuous, we say that a face $F \in \Delta\mathcal{P}$ is additive if $\nabla\phi = 0$ over all F. Notice that $\nabla\phi$ is affine over F, the condition is equivalent to $\nabla\phi(u, v) = 0$ for any $(u, v) \in \operatorname{vert}(F)$. When ϕ is discontinuous, following [19], we say that a face $F \in \Delta\mathcal{P}$ is additive if F is contained in a face $F' \in \Delta\mathcal{P}$ such that $\nabla\phi_{F'}(x, y) = 0$ for any $(x, y) \in F$. Since $\nabla\phi$ is affine in the relative interiors of each face of $\Delta\mathcal{P}$, the last condition is equivalent to $\nabla\phi_{F'}(u, v) = 0$ for any $(u, v) \in \operatorname{vert}(F)$.

3.2 Maximality test

We introduce an efficient method to check the maximality of a given piecewise linear function. The code `maximality_test(`ϕ`)` implements a fully automatic test whether a given function ϕ is maximal, by using the information that is described in $\Delta\mathcal{P}$.

Based on Theorem 2.3, we need to first check that the range of the function stays in $[0, 1]$ and $\phi(0) = 0$. Since we assume the function is piecewise linear with finitely many breakpoints, only function values and left and right limits at the

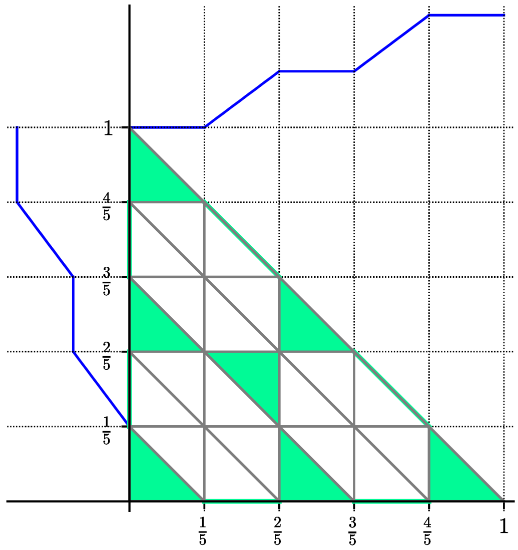

Fig. 1: Maximal cDFF $\phi_{BJ,1}(x;C) = \frac{\lfloor Cx \rfloor + \max(0, \frac{\{Cx\}-\{C\}}{1-\{C\}})}{\lfloor C \rfloor}$ for $C = \frac{5}{2}$.

breakpoints need to be checked. Similarly, the symmetry condition only needs to be checked on all breakpoints including the left and right limits at each breakpoint. In regards to the superadditivity, it suffices to check $\nabla\phi(u,v) \geq 0$ for any $(u,v) \in \text{vert}(F)$, including the limit values $\nabla\phi_F(u,v)$ when ϕ is discontinuous.

As for the diagrams of $\Delta\mathcal{P}$, we start with a triangle complex $I = J = K = [0,1]$, and then refine I, J, K based on the set of breakpoints, namely B. In practice, the code `plot_2d_diagram_dff(`ϕ`)` will show vertices where superadditivity or symmetry condition is violated (marked red). It also paints additive faces green, including 1-dimensional and 0-dimensional additive faces, which are additive edges and vertices not contained in any higher dimensional additive faces.

Figure 1 is an example of the $\Delta\mathcal{P}$ of a maximal cDFF. We also plot the function on the upper and left borders. There is no vertex where superadditivity or symmetry condition is violated, so the function is maximal.

3.3 Extremality test

Our automatic extremality test, `extremality_test_dff`, builds upon the techniques of the grid-free extremality test for the Gomory–Johnson setting, which is described in [32, Chapter 4] and [20,18,16] and the forthcoming paper [17], and implemented in [27].

In this subsection, we provide the technical results that allow us to adapt these techniques to cDFFs. First there is a simple necessary condition for piecewise linear extreme cDFFs.

Lemma 3.2 *Let ϕ be a piecewise linear extreme cDFF. If ϕ is strictly increasing, then $\phi(x) = x$. In other words, there is no strictly increasing piecewise linear extreme cDFF except for $\phi(x) = x$.*

Proof. We know ϕ is continuous at 0 from the right. Suppose $\phi(x) = sx$, $x \in [0, a_1)$ and $s > 0$, since ϕ is not strictly increasing if $s = 0$. We claim that s is the smallest slope value of ϕ. Suppose otherwise $\phi(x) = s'x + t$, $x \in [r, r + \epsilon]$ with $s' < s$ and $\epsilon < a_1$. In order to satisfy the superadditivity, we have $\phi(r + \epsilon) \geq \phi(\epsilon) + \phi(r)$, which can be reduced to $s' \geq s$. The contradiction indicates that s is the smallest slope value. We have $s \leq 1$ since $\phi(1) = 1$. Similarly if $s = 1$, then $\phi(x) = x$.

Next, we can assume $0 < s < 1$. Define a function:

$$\phi_1(x) = \frac{\phi(x) - sx}{1 - s}.$$

It is not hard to show $\phi_1(x) = 0$ for $x \in [0, a_1)$, and $\phi_1(1) = 1$. The function ϕ_1 is superadditive because it is obtained by subtracting a linear function from a superadditive function. These two together guarantee that ϕ_1 stays in the range $[0, 1]$. The function ϕ_1 satisfies the symmetry condition due to the following equation:

$$\phi_1(x) + \phi_1(1 - x) = \frac{\phi(x) + \phi(1 - x) - sx - s(1 - x)}{1 - s} = 1.$$

Therefore, ϕ_1 is also a maximal cDFF. Moreover, $\phi(x) = sx + (1 - s)\phi_1(x)$ implies ϕ is not extreme, since it can be expressed as a convex combination of two different maximal cDFFs: x and ϕ_1. □

Next we give the definition of effective perturbation functions.

Definition 3.3 *Let ϕ be a maximal cDFF. Then a function $\tilde{\phi}\colon [0, 1] \to \mathbb{R}$ is called an* effective perturbation function *of ϕ, if there exists $\epsilon > 0$, such that $\phi + \epsilon\tilde{\phi}$ and $\phi - \epsilon\tilde{\phi}$ are both maximal cDFFs.*

From the definition above, the zero function is always an effective perturbation function, and we call it the trivial effective perturbation function. There exists a nontrivial effective perturbation function of ϕ if and only if ϕ is not extreme.

Effective perturbations of a DFF ϕ have a close relation to the functions ϕ in regards to continuity and additivity.

Lemma 3.4 *Let ϕ be a piecewise linear maximal cDFF. If ϕ is continuous on a proper interval $I \subseteq [0, 1]$, then for any perturbation function $\tilde{\phi}$, we have that $\tilde{\phi}$ is Lipschitz continuous on the interval I. Furthermore, $\tilde{\phi}$ is continuous at all points at which ϕ is continuous.*

Proof. We know ϕ is continuous at 0 from the right. Let $\tilde{\phi}$ be an effective perturbation function. Since ϕ is piecewise linear, there exists a nonnegative s, such that $\phi(x) = sx$ on the first interval $[0, a_1)$. Let $I = J = K = [0, a_1]$, and let $F = F(I, J, K)$. Then for any $x \in I$, $y \in J$, $x + y \in K$, $\nabla\phi_F(x, y) = s(x + y) - sx - sy = 0$. Thus, F is a two-dimensional additive face of ΔP. From the Interval Lemma, we know that there exists $\tilde{s}$, such that $\tilde{\phi}(x) = \tilde{s}x$, when $x \in [0, a_1)$. Since $\tilde{\phi}$ is an effective perturbation function, there exists $\epsilon > 0$, such that $\phi^+ = \phi + \epsilon\tilde{\phi}$ and $\phi^- = \phi - \epsilon\tilde{\phi}$ are both maximal cDFFs. We know that ϕ^+ and ϕ^- have slope $s^+ = s + \epsilon\tilde{s} \geq 0$ and $s^- = s - \epsilon\tilde{s} \geq 0$ respectively.

Let $I \subseteq [0, 1]$ be a proper interval where ϕ is continuous. Since ϕ is piecewise linear, there exists a positive constant C such that $|\phi(x) - \phi(y)| \leq C|x - y|$, for any $x, y \in I$. We can simply choose C to be the largest absolute value of the slopes of ϕ. Assume $x \geq y$ and $x - y < a_1$, from the superadditivity of ϕ^+ and ϕ^-, $\phi^+(x) \geq \phi^+(y) + \phi^+(x - y) = \phi^+(y) + s^+(x - y)$ and $\phi^-(x) \geq \phi^-(y) + \phi^-(x - y) = \phi^-(y) + s^-(x - y)$. It follows that $-(C + s^-)(x - y) \leq \epsilon(\tilde{\phi}(x) - \tilde{\phi}(y)) \leq (C + s^+)(x - y)$. Therefore, $|\tilde{\phi}(x) - \tilde{\phi}(y)| \leq \tilde{C}|x - y|$, where $\tilde{C} = \frac{1}{\epsilon}\max(C + s^-, C + s^+)$. Hence, $\tilde{\phi}$ is Lipschitz continuous on the interval I. □

We remark that, in contrast to the Gomory–Johnson setting, Lemma 3.4 holds without further hypotheses, and so the subtle issues regarding two-sided discontinuous functions explored in [26] do not arise for our cDFFs.

For the following lemma, recall from subsection 3.1 the notation $\nabla\tilde{\phi}_F(x, y)$ to denote the limit within the face F of the two-dimensional complex.

Lemma 3.5 *Let ϕ be a piecewise linear maximal cDFF. For any effective perturbation function $\tilde{\phi}$, we have that $\tilde{\phi}$ satisfies additivity where ϕ satisfies additivity. This also holds true in the limit: If $F \in \Delta\mathcal{P}$, $(x, y) \in F$, and $\nabla\phi_F(x, y) = 0$, then $\nabla\tilde{\phi}_F(x, y) = 0$.*

Proof. Since $\tilde{\phi}$ is an effective perturbation function, there exists $\epsilon > 0$, such that $\phi^+ = \phi + \epsilon\tilde{\phi}$ and $\phi^- = \phi - \epsilon\tilde{\phi}$ are both maximal cDFFs. If ϕ satisfies additivity at (x, y), we have $\phi(x) + \phi(y) = \phi(x + y)$. Applying superadditivity of ϕ^+ and ϕ^- at (x, y), we get $\tilde{\phi}(x) + \tilde{\phi}(y) = \tilde{\phi}(x + y)$. Likewise, if the limit $\nabla\phi_F(x, y)$ is zero, then the superadditivity of ϕ^+ and ϕ^- implies that the limit $\nabla\phi_F(x, y)$ exists and is zero. □

From the continuity (Lemma 3.4) and additivity (Lemma 3.5), our algorithm deduces further properties of every effective perturbation function $\tilde{\phi}$. One tool is the famous Gomory–Johnson Interval Lemma; we include a version of it below.

Lemma 3.6 (Interval Lemma) *[6, Lemma 4.1] Let $a_1 < a_2$ and $b_1 < b_2$. Consider the intervals $A = [a_1, a_2]$, $B = [b_1, b_2]$, and $A + B = [a_1 + b_1, a_2 + b_2]$. Let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, and $h : A + B \to \mathbb{R}$ be bounded functions on A, B and $A + B$, respectively. If $f(a) + g(b) = h(a + b)$ for all $a \in A$ and $b \in B$, then f, g, and h are affine functions with identical slopes in the intervals A, B, and $A + B$, respectively.*

Using this lemma and additional techniques from [32, Chapter 4], [18,16], our algorithm constructs a list of pairwise disjoint *connected covered components* $C_1, \ldots, C_k$, each of which is a finite union of open intervals with the following property. If $\tilde{\pi}$ is any effective perturbation function and C_i is one of the connected covered components, then the restrictions of $\tilde{\pi}$ to the intervals of C_i are affine functions with identical slopes. Like in the Gomory–Johnson case, we can prove the finiteness of this construction when the breakpoints of ϕ are rational.

If there is some *uncovered interval*, i.e., an open interval of $[0,1] \setminus \bigcup_i C_i$, then a nontrivial "equivariant" perturbation is guaranteed to exist, and hence `extremality_test_dff` returns `False`. (Our algorithm can actually construct such a perturbation using the method of inverse semigroups of restricted translations and reflections from [32, Chapter 4], [18,16]; we suppress all details.)

On the other hand, if the domain $[0,1]$ is covered by the closures of $C_1, \ldots, C_k$, then any effective perturbation function is guaranteed to be piecewise linear. The existence of a nontrivial effective perturbation function depends on whether a finite dimensional linear system has a nontrivial solution.

We start with the continuous case. Suppose ϕ is a continuous piecewise linear maximal cDFF, thus any effective perturbation function $\tilde{\phi}$ must be continuous. If the domain $[0,1]$ is covered by the closures of $C_1, \ldots, C_k$, then $\tilde{\phi}$ has the same slope value on each C_i and we denote the slope value by s_i. Note that the effective perturbation function $\tilde{\phi}$ is uniquely determined by the set of slope values $\{s_1, \ldots, s_k\}$. Specifically, there exists a vector-valued linear function $g\colon [0,1] \to \mathbb{R}^k$ so that $\tilde{\phi}(x) = g(x) \cdot (s_1, \ldots, s_k)$. The ith coordinate of $g(x)$ represents the total length of the connected component C_i contained in the interval $[0,x]$, i.e. $g(x) = (|C_1 \cap [0,x]|, \ldots, |C_k \cap [0,x]|)$.

In the general case where ϕ may be discontinuous, the effective perturbation function $\tilde{\phi}$ may also be discontinuous. If $\tilde{\phi}$ may be discontinuous at x, then there exist jumps $h^- = \tilde{\phi}(x) - \tilde{\phi}(x^-)$ and $h^+ = \tilde{\phi}(x^+) - \tilde{\phi}(x)$, where h^-, h^+ represent the left and right discontinuity at x respectively. Note that h^-, h^+ could also be 0 representing (left/right) continuity at point x. It is not hard to see there are only finitely many points where discontinuity may occur, since discontinuity can only occur at the endpoints of connected components $C_1, \ldots, C_k$. The effective perturbation function $\tilde{\phi}$ is uniquely determined by the set of slope values $\{s_1, \ldots, s_k\}$ and potential jumps $\{h_1, \ldots, h_m\}$. The general form of an effective perturbation function $\tilde{\phi}$ can then be expressed using a vector-valued linear function $g\colon [0,1] \to \mathbb{R}^{k+m}$, slope variables and jump variables so that

$$\tilde{\phi}(x) = g(x) \cdot (s_1, \ldots, s_k, h_1, \ldots, h_m). \tag{2}$$

The last m coordinates of $g(x)$ are binaries indicating whether those potential jumps are contained in the interval $[0,x]$. Observe that the function g is determined only by the original function ϕ.

The next step is to find all constraints that $\tilde{\phi}(x)$ needs to satisfy and solve a linear system of $(s_1, \ldots, s_k, h_1, \ldots, h_m)$. If there is only the trivial solution, then `extremality_test_dff` returns `True`. If one nontrivial function $\tilde{\phi}(x)$ is found, then `extremality_test_dff` returns `False`. We use the following proposition. Recall from subsection 3.1 the notation $\tilde{\phi}(a^-)$ to represent the left limit to a.

Proposition 3.7 *Let ϕ be a piecewise linear maximal cDFF. Assume $\phi(a_1^-) = 0$ where a_1 is the breakpoint of ϕ next to 0, and assume ϕ has no uncovered interval. Let $\hat{B} = B \cup \bigcup_i \partial C_i$ be the union of breakpoints of ϕ and endpoints of the intervals of covered components, and $\hat{\mathcal{P}}$ be the new complex based on $\hat{B}$. The functions $\tilde{\phi}\colon [0,1] \to \mathbb{R}$ defined by (2), using the slope variables and jump variables given by all covered components of ϕ, are piecewise linear over $\hat{\mathcal{P}}$. Construct a linear system of equations for $\tilde{\phi}$:*

$$\nabla\tilde{\phi}_F(x,y) = 0 \qquad \text{for all } F \in \Delta\hat{P},\ (x,y) \in \operatorname{vert}(F) \text{ such that } \nabla\phi_F(x,y) = 0, \tag{3a}$$

$$\tilde{\phi}(1) = \tilde{\phi}(a_1^-) = 0. \tag{3b}$$

If there is only the trivial solution $\tilde{\phi}(x) = 0$, then ϕ is extreme. If there is some nontrivial solution $\tilde{\phi}$, then there exists $\epsilon > 0$ such that $\phi + \epsilon\tilde{\phi}$ and $\phi - \epsilon\tilde{\phi}$ are both maximal; thus $\tilde{\phi}$ is a nontrivial effective perturbation and ϕ is not extreme.

Proof. Note that if $\tilde{\phi}$ is an effective perturbation function, then it must satisfy the linear system (3) because of Lemma 3.5. By assumption, $[0,1]$ is covered by the closures of $C_1, \dots, C_k$, where each C_i is a connected covered component. We know that $\tilde{\phi}$ is affine linear on each C_i with the same slope. Observe that if the linear system has only the trivial solution, then the only effective perturbation function is the zero function, thus ϕ is extreme.

Suppose there is a nonzero solution $\tilde{\phi}$ to the linear system (3); it is by definition a piecewise linear function on $[0,1]$ with possible discontinuities at the breakpoints. Let

$$\begin{aligned} \delta &= \min\{\, \nabla\phi_F(x,y) \quad : F \in \Delta\hat{\mathcal{P}},\ (x,y) \in \operatorname{vert}(F),\ \nabla\phi_F(x,y) > 0 \,\}, \\ \sigma &= \max\{\, |\nabla\tilde{\phi}_F(x,y)| : F \in \Delta\hat{\mathcal{P}},\ (x,y) \in \operatorname{vert}(F),\ \nabla\phi_F(x,y) > 0 \,\}. \end{aligned}$$

Note the minimum and maximum are over a finite set. Choose $\epsilon = \frac{\delta}{\max(\sigma,1)} > 0$; then we claim that $\phi + \epsilon\tilde{\phi}$ and $\phi - \epsilon\tilde{\phi}$ are both superadditive. Let $F \in \Delta\hat{P}$ and $(x,y) \in \operatorname{vert}(F)$. We compute $\nabla(\phi + \epsilon\tilde{\phi})_F(x,y) = \nabla\phi_F(x,y) + \epsilon\nabla\tilde{\phi}_F(x,y)$. If $\nabla\phi_F(x,y) = 0$, then by (3), also $\nabla\tilde{\phi}_F(x,y) = 0$. Otherwise $\nabla\phi_F(x,y) > 0$, and then $\nabla\phi_F(x,y) + \epsilon\nabla\tilde{\phi}_F(x,y) \geq \delta - \epsilon\sigma \geq 0$. Thus, $\phi \pm \epsilon\tilde{\phi}$ are both superadditive.

Consider ϕ and $\tilde{\phi}$ on the interval $[0, a_1)$. The function ϕ is the zero function on $[0, a_1)$ since $\phi(0) = \phi(0^+) = \phi(a_1^-) = 0$. Note that $(0, a_1)$ belongs to some covered component, i.e., $(0, a_1) \subset C_i$ for some i. Then $\tilde{\phi}$ is also a linear function on $(0, a_1)$. Due to $\tilde{\phi}(0) = \tilde{\phi}(0^+) = \tilde{\phi}(a_1^-) = 0$, we know that $\tilde{\phi}$ is also the zero function on $[0, a_1)$. Then $\phi + \epsilon\tilde{\phi}$ and $\phi - \epsilon\tilde{\phi}$ are both nonnegative on $[0, a_1)$, so they are monotone increasing on $[0,1]$ by superadditivity. Since $(\phi \pm \epsilon\tilde{\phi})(1) = \phi(1) \pm \epsilon\tilde{\phi}(1) = 1$ and the functions are monotone increasing, $\phi \pm \epsilon\tilde{\phi}$ both stay in the range of $[0,1]$.

The symmetry condition of $\phi + \epsilon\tilde{\phi}$ and $\phi - \epsilon\tilde{\phi}$ is implied by the symmetry condition of ϕ. Indeed, for every face $F = F(I,J,K) \in \Delta\hat{P}$ such that $K = \{1\}$ and every vertex $(x, 1-x) \in \operatorname{vert}(F)$, we have $\nabla\phi_F(x, 1-x) = 0$. Then from the linear system (3), we also have $\nabla\tilde{\phi}_F(x,1-x)=0$, and thus $\nabla(\phi\pm\epsilon\tilde{\phi})_F(x,1-x) = \nabla\phi_F(x,1-x)\pm\epsilon\nabla\tilde{\phi}_F(x,1-x)=0$.

Therefore, both $\phi+\epsilon\tilde{\phi}$ and $\phi-\epsilon\tilde{\phi}$ are maximal cDFFs, thus ϕ is not extreme. □

3.4 Computer-based search

In this subsection, we discuss how computer-based search can help in finding extreme cDFFs. Most known cDFFs in the monograph [1] have a similar structure: continuous cDFFs are 2-slope functions, and discontinuous cDFFs have slope 0 in every affine linear piece.

We transfer a computer-based search technique from [25] for Gomory–Johnson functions to cDFFs. Our goal is to find piecewise linear extreme cDFFs with rational breakpoints, which have fixed common denominator $q\in\mathbb{N}$. The strategy is to discretize the interval $[0,1]$ and consider discrete functions on $B_q := \frac{1}{q}\mathbb{Z}\cap[0,1]$, or, equivalently, vectors in $\mathbb{R}^{q+1}$ whose components are the function values on the grid B_q. In this space, we define a polytope by inequalities from the characterization of maximality. Extreme points of the polytope can be found by vertex enumeration tools. Recent advances in polyhedral computation (Normaliz, version 3.2.0; see [11]) allow us to reach $q=31$ in under a minute of CPU time. Candidates for extreme cDFFs ϕ are obtained by interpolating values on $\frac{1}{q}\mathbb{Z}\cap[0,1]$ from each extreme point (discrete function). Then we use our extremality test to filter out the non-extreme functions.

Based on a detailed computational study regarding the performance of vertex enumeration codes in [25], we consider two libraries, the Parma Polyhedra Library (PPL) and Normaliz. Both are convenient to use within the software SageMath [29].

We now introduce some notation, which will allow us to make precise statements that also include the discontinuous case.

Definition 3.8 *We use B_q to denote the set $\{0,\frac{1}{q},\frac{2}{q},\dots,\frac{q-1}{q},1\}$. Denote $\Phi_C(q)$ to be the set of all maximal continuous piecewise linear cDFFs with breakpoints in B_q, and $\Phi_D(q)$ to be the set of all maximal possibly discontinuous piecewise linear cDFFs with breakpoints in B_q.*

Theorem 3.9. *Both $\Phi_C(q)$ and $\Phi_D(q)$ are linearly isomorphic to finite dimensional convex polytopes $\Phi'_C(q)\subset\mathbb{R}^{q+1}$ and $\Phi'_D(q)\subset\mathbb{R}^{3q-1}$, respectively, if q is fixed.*

Proof. Continuous case. Note that any maximal cDFF $\phi\in\Phi_C(q)$ is uniquely determined by the values at the breakpoints. So we just need to consider discrete functions on B_q that are the restrictions $\phi|_{B_q}$ of ϕ to B_q. Since ϕ is maximal, $\phi|_{B_q}$ should also satisfy superadditivity and the symmetry condition.

For each possible breakpoint $\frac{i}{q}$, we introduce a variable v_i to be the value $\phi(\frac{i}{q})$. Considering inequalities from superadditivity, the symmetry condition and $0\le v_i\le 1$, $v_0=0$, we get a polytope in $q+1$ dimensional space, because there are only finitely many inequalities and each variable is bounded. We denote this polytope by $\Phi'_C(q)$.

It is not hard to prove the convex combination of two maximal continuous piecewise linear cDFFs with breakpoints in B_q is also in $\Phi_C(q)$.

We can get ϕ back by interpolating $\phi|_{B_q}$. Therefore $\Phi_C(q)$ is linearly isomorphic to $\Phi'_C(q)$, a finite dimensional convex polytope.

Discontinuous case. Consider the linear map from $\Phi_D(q)$ to $\mathbb{R}^{3q-1}$ given by

$$\phi \mapsto \Big(\phi(0),\, \phi(\tfrac{1}{q}^-),\, \phi(\tfrac{1}{q}),\, \phi(\tfrac{1}{q}^+),\, \ldots,\, \phi(\tfrac{q-1}{q}^-),\, \phi(\tfrac{q-1}{q}),\, \phi(\tfrac{q-1}{q}^+),\, \phi(1)\Big),$$

where $\phi(a^-)$ and $\phi(a^+)$ again represent the left and right limits to a respectively. Denote the image by $\Phi'_D(q)$. This map is invertible by interpolating linearly between the given limit values of ϕ near the breakpoints. Moreover, we know from the maximality test (subsection 3.2) that it suffices to test the limits $\nabla\phi_F(x,y)$ to the vertices of the complex $\Delta\mathcal{P}$ within faces $F \ni (x,y)$ of $\Delta\mathcal{P}$. Each vertex (x,y) is contained in at most 12 faces F. Each of the limits can be expressed as a linear combination of values and limits of ϕ at the breakpoints, such as $\phi(x^+)+\phi(y^-)-\phi((x+y)^-)$, $\phi(x^-)+\phi(y)-\phi((x+y)^-)$, etc. Therefore, $\Phi'_D(q)$ is a polytope. □

A function ϕ that we obtain from the interpolation of the discrete function values of a vertex of $\Phi'_C(q)$, or the interpolation of the function values and limits of a vertex of $\Phi'_D(q)$, is only a candidate of extreme functions. We need to use the extremality test described in subsection 3.3 to pick those actual extreme cDFFs. The following theorem provides an easier verification for extremality: if ϕ has no uncovered interval, then we can claim we find an extreme cDFF.

Theorem 3.10. *Let ϕ be a function from interpolating values of some extreme point of the polytope $\Phi'_C(q)$ or $\Phi'_D(q)$. Then ϕ is extreme if and only if there is no uncovered interval.*

Proof. Here we only give the proof for continuous case, and the proof for discontinuous case is similar.

Suppose ϕ is obtained by in interpolating the discrete function $\phi|_{B_q}$, which is an extreme point of the polytope $\Phi'_C(q)$, and $\tilde{\phi}$ is an effective perturbation function.

If there is an uncovered interval, by subsection 3.3, there exists an effective "equivariant" perturbation function, and the function is not extreme.

If there is no uncovered interval for ϕ, then the interval $[0,1]$ is covered by the closures of $C_1,\ldots,C_k$, where each C_i is a connected covered component. Since every breakpoint of ϕ is in the form of $\frac{i}{q}$, the endpoints of C_i are also in the form of $\frac{i}{q}$. We know ϕ and $\tilde{\phi}$ are affine linear on each C_i with the same slope by the Interval Lemma, and continuity of ϕ implies continuity of $\tilde{\phi}$. Therefore, we know $\tilde{\phi}$ is also a continuous function with breakpoints in B_q, which means

$\phi + \epsilon\tilde{\phi}$ and $\phi + \epsilon\tilde{\phi}$ both have the same property. The maximality of $\phi + \epsilon\tilde{\phi}$ and $\phi + \epsilon\tilde{\phi}$ implies their restrictions to B_q are also in the polytope $\Phi'_C(q)$, and

$$\phi|_{B_q} = \frac{(\phi + \epsilon\tilde{\phi})|_{B_q} + (\phi - \epsilon\tilde{\phi})|_{B_q}}{2}.$$

Since $\phi|_{B_q}$ is an extreme point of the polytope $\Phi'_C(q)$, then $\phi|_{B_q} = (\phi + \epsilon\tilde{\phi})|_{B_q} = (\phi - \epsilon\tilde{\phi})|_{B_q}$, which implies $\phi = \phi + \epsilon\tilde{\phi} = \phi - \epsilon\tilde{\phi}$. Therefore, ϕ is extreme. □

Table 1 shows the results and the computation time for computing all vertices of $\Phi'_C(q)$ for different values of q. We then use Theorem 3.10 to filter out those non-extreme functions which have uncovered intervals. As we can see in the table, the actual extreme cDFFs are much fewer than the vertices of the polytope $\Phi'_C(q)$. PPL is faster when q is small and Normaliz performs well when q is relatively large. We can observe that the time cost increases dramatically as q gets large. Similar to [25], we can apply the preprocessing program "redund" provided by lrslib (version 5.08), which removes redundant inequalities using Linear Programming. However, in contrast to the computation in [25], removing redundancy from the system does not improve the efficiency. Instead, for relatively large q, the time cost after preprocessing is a little more than that of before preprocessing for both PPL and Normaliz.

For example, for $q = 31$, among 91761 functions interpolated from extreme points, there are 1208 extreme cDFFs, most of which do not belong to known families.

4 Characterization of maximal general DFFs

Alves et al. [1] provided several sufficient conditions and necessary conditions of maximal gDFFs in Theorem 2.4, but they do not match precisely. Inspired by the characterization of minimal cut-generating functions in the Yıldız–Cornuéjols model [31], we complete the characterization of maximal gDFFs.

Proposition 4.1 *A function $\phi\colon \mathbb{R} \to \mathbb{R}$ is a maximal gDFF if and only if the following conditions hold:*

(i) $\phi(0) = 0$.
(ii) ϕ *is superadditive.*
(iii) $\phi(x) \geq 0$ *for all* $x \in \mathbb{R}_+$.
(iv) $\phi(x) = \inf_k\{\frac{1}{k}(1 - \phi(1 - kx)) : k \in \mathbb{Z}_{++}\}$ *for all* $x \in \mathbb{R}$.

Proof. Suppose ϕ is a maximal gDFF, then conditions $(i), (ii), (iii)$ hold by Theorem 2.4, which also implies that ϕ is monotone increasing. For any $x \in \mathbb{R}$ and $k \in \mathbb{Z}_+$, $kx+(1-kx) = 1 \Rightarrow k\phi(x)+\phi(1-kx) \leq 1$. So $\phi(x) \leq \frac{1}{k}(1-\phi(1-kx))$ for any positive integer k, then $\phi(x) \leq \inf_k\{\frac{1}{k}(1 - \phi(1 - kx)) : k \in \mathbb{Z}_+\}$.

If there exists x_0 such that $\phi(x_0) < \inf_k\{\frac{1}{k}(1 - \phi(1 - kx_0))\colon k \in \mathbb{Z}_+\}$, then define a function ϕ_1 which takes value $\inf_k\{\frac{1}{k}(1 - \phi(1 - kx_0))\colon k \in \mathbb{Z}_+\}$ at x_0

Table 1: Search for extreme cDFFs and efficiency of vertex enumeration codes (continuous case)

Polytope $\Phi_C(q)$						Running times (s)	
q	dim	inequalities		vertices	extreme DFF	PPL	Normaliz
		original	minimized				
2	0	4	3	1	1	0.00006	0.002
3	1	5	5	2	1	0.00009	0.006
5	2	9	7	3	2	0.00014	0.007
7	3	15	10	5	3	0.0002	0.007
9	4	23	14	9	3	0.0004	0.008
11	5	33	18	14	7	0.0006	0.010
13	6	45	23	25	8	0.001	0.012
15	7	59	29	66	14	0.003	0.018
17	8	75	35	94	22	0.005	0.025
19	9	93	42	221	32	0.010	0.042
21	10	113	50	677	55	0.036	0.105
23	11	135	58	1360	105	0.110	0.226
25	12	159	67	3898	189	0.526	0.725
27	13	185	77	12279	291	5.1	2.991
29	14	213	87	28877	626	41	9.285
31	15	243	98	91761	1208	595	35.461

and $\phi(x)$ if $x \neq x_0$. We claim that ϕ_1 is a gDFF which dominates ϕ. Given a function $y\colon \mathbb{R} \to \mathbb{Z}_+$, with finite support satisfying $\sum_{x\in\mathbb{R}} x\, y(x) \leq 1$. We have $\sum_{x\in\mathbb{R}} \phi_1(x)\, y(x) = \phi_1(x_0)\, y(x_0) + \sum_{x\neq x_0} \phi(x)\, y(x)$. If $y(x_0) = 0$, then it is clear that $\sum_{x\in\mathbb{R}} \phi_1(x)\, y(x) \leq 1$. Let $y(x_0) \in \mathbb{Z}_+$, then $\phi_1(x_0) \leq \frac{1}{y(x_0)}(1 - \phi(1 - y(x_0)\, x_0))$ by definition of ϕ_1, then $\phi_1(x_0)\, y(x_0) + \phi(1 - y(x_0)\, x_0) \leq 1$. Since ϕ is superadditive and monotone increasing, we get $\sum_{x\neq x_0} \phi(x)\, y(x) \leq \phi(\sum_{x\neq x_0} x\, y(x)) \leq \phi(1 - y(x_0)\, x_0)$. From the two inequalities we conclude that ϕ_1 is a gDFF and dominates ϕ, which contradicts the maximality of ϕ. So the condition (iv) holds.

Suppose there is a function $\phi\colon \mathbb{R} \to \mathbb{R}$ satisfying all four conditions. Choose $x = 1$ and $k = 1$, we can get $\phi(1) \leq 1$ from (iv). Together with conditions $(i), (ii), (iii)$, it guarantees that ϕ is a gDFF by the definition of gDFFs. Assume that there is a gDFF ϕ_1 dominating ϕ and there exists x_0 such that $\phi_1(x_0) > \phi(x_0) = \inf_k\{\frac{1}{k}(1 - \phi(1 - kx_0))\colon k \in \mathbb{Z}_+\}$. So there exists some $k \in \mathbb{Z}_+$ such that

$$\begin{aligned} &\phi_1(x_0) > \frac{1}{k}(1 - \phi(1 - kx_0)) \\ \Leftrightarrow\ &k\phi_1(x_0) + \phi(1 - kx_0) > 1 \\ \Rightarrow\ &k\phi_1(x_0) + \phi_1(1 - kx_0) > 1. \end{aligned}$$

The last step contradicts the fact that ϕ_1 is a gDFF. Therefore, ϕ is maximal. □

Parallel to the restricted minimal and strongly minimal functions in the Yıldız–Cornuéjols model [31], "restricted maximal" and "strongly maximal" gDFFs are defined by strengthening the notion of maximality.

Definition 4.2 *We say that a gDFF ϕ is* implied via scaling *by a gDFF ϕ_1, if $\beta\phi_1 \geq \phi$ for some $0 \leq \beta \leq 1$. We call a gDFF $\phi\colon \mathbb{R} \to \mathbb{R}$* restricted maximal *if ϕ is not implied via scaling by a distinct gDFF ϕ_1. We say that a gDFF ϕ is* implied *by a gDFF ϕ_1, if $\phi(x) \leq \beta\phi_1(x) + \alpha x$ for some $0 \leq \alpha, \beta \leq 1$ and $\alpha + \beta \leq 1$. We call a gDFF $\phi\colon \mathbb{R} \to \mathbb{R}$* strongly maximal *if ϕ is not implied by a distinct gDFF ϕ_1.*

Note that restricted maximal gDFFs are maximal and strongly maximal gDFFs are restricted maximal. Based on the definition of strong maximality, $\phi(x) = x$ is implied by the zero function, so ϕ is not strongly maximal, though it is extreme. We include the characterizations of restricted maximal and strongly maximal gDFFs here, which only involve the standard symmetry condition instead of the generalized symmetry condition.

Theorem 4.3. *A function $\phi\colon \mathbb{R} \to \mathbb{R}$ is a restricted maximal gDFF if and only if the following conditions hold:*

(i) $\phi(0) = 0$.
(ii) ϕ *is superadditive.*
(iii) $\phi(x) \geq 0$ *for all* $x \in \mathbb{R}_+$.

(iv) $\phi(x) + \phi(1-x) = 1$ *for all* $x \in \mathbb{R}$.

Proof. It is easy to show that ϕ is valid and restricted maximal if ϕ satisfies conditions $(i - iv)$. Suppose ϕ is a restricted maximal gDFF, then we only need to prove condition (iv), since restricted maximality implies maximality.

Suppose there exists some x such that $\phi(x) + \phi(1-x) < 1$. By the characterization of maximality, $\phi(x) = \inf_k\{\frac{1}{k}(1 - \phi(1-kx))\colon k \in \mathbb{Z}_+\}$.

Case 1: Suppose there exists some $k \in \mathbb{N}$ such that $\phi(x) = \frac{1}{k}(1 - \phi(1-kx))$. By superadditivity $k\phi(x) = 1 - \phi(1-kx) = 1 - \phi(1-x-(k-1)x) \geq 1 - \phi(1-x) + \phi((k-1)x) \geq 1 - \phi(1-x) + (k-1)\,\phi(x)$, which implies $\phi(x) + \phi(1-x) \geq 1$, in contradiction to the assumption above.

Case 2: Suppose otherwise $\phi(x) < \frac{1}{k}(1 - \phi(1-kx))$ for any positive integer k. Therefore, for any $\epsilon > 0$, there exists a corresponding $k_\epsilon \in \mathbb{N}$, such that

$$\phi(x) < \frac{1}{k_\epsilon}(1 - \phi(1 - k_\epsilon x)) < \phi(x) + \epsilon.$$

Then $\phi(k_\epsilon x) \leq 1 - \phi(1 - k_\epsilon x) < k_\epsilon \phi(x) + k_\epsilon \epsilon$, or equivalently $\frac{\phi(k_\epsilon x)}{k_\epsilon} < \phi(x) + \epsilon$. Since ϕ is superadditive, $\phi(x) \leq \frac{\phi(k_\epsilon x)}{k_\epsilon}$. Let ϵ go to 0 in the inequality $\phi(x) \leq \frac{\phi(k_\epsilon x)}{k_\epsilon} < \phi(x) + \epsilon$, and we have $\lim_{\epsilon \to 0} \frac{\phi(k_\epsilon x)}{k_\epsilon} = \phi(x)$. It is easy to see that $\lim_{\epsilon \to 0} k_\epsilon = +\infty$.

Next, we will show that $\phi(kx) = k\phi(x)$ for any positive integer k. Suppose $\bar{k}$ is the smallest integer such that $\frac{\phi(\bar{k}x)}{\bar{k}} = \phi(x) + \delta$ for some $\delta > 0$. Then for any $i \geq \bar{k}$, there exist $\lambda_i, r_i \in \mathbb{Z}_+$, such that $i = \lambda_i \bar{k} + r_i$, $0 \leq r_i < \bar{k}$. Then

$$\begin{aligned}\phi(ix) &= \phi(\lambda_i \bar{k} x + r_i x) \geq \lambda_i \phi(\bar{k}x) + \phi(r_i x)\\ &\geq \lambda_i \bar{k}\phi(x) + \lambda_i \bar{k}\delta + r_i \phi(x) = i\phi(x) + (i - r_i)\delta.\end{aligned}$$

Therefore $\frac{\phi(ix)}{i} \geq \phi(x) + \delta - \frac{r_i}{i}\delta$ for any $i \geq \bar{k}$. Since r_i is bounded, $\frac{\phi(ix)}{i} \geq \phi(x) + \frac{\delta}{2}$ for any $i \geq 2\bar{k}$, which contradicts $\lim_{\epsilon \to 0} \frac{\phi(k_\epsilon x)}{k_\epsilon} = \phi(x)$. We have $\phi(kx) = k\phi(x)$ for any positive integer k. From Proposition 2.12 we know $\phi(1) = 1$, and we have

$$\begin{aligned}k\phi(x) = \phi(kx) &\geq (k-1)\phi(1) + \phi(1 - k(1-x))\\ \Leftrightarrow 1 - \phi(x) &\leq \frac{1 - \phi(1 - k(1-x))}{k}\\ \Rightarrow 1 - \phi(x) &\leq \inf_k \frac{1 - \phi(1 - k(1-x))}{k} = \phi(1-x).\end{aligned}$$

The above inequality contradicts our original assumption.

In both cases, we have a contradiction if $\phi(x) + \phi(1-x) < 1$. Therefore, $\phi(x) + \phi(1-x) = 1$, which completes the proof. □

Theorem 4.4. *A function* $\phi\colon \mathbb{R} \to \mathbb{R}$ *is a strongly maximal gDFF if and only if* ϕ *is a restricted maximal gDFF and* $\lim_{\epsilon \to 0^+} \frac{\phi(\epsilon)}{\epsilon} = 0$.

Proof. Suppose ϕ is strongly maximal, we only need to show $\lim_{\epsilon\to 0^+}\frac{\phi(\epsilon)}{\epsilon} = 0$ since strong maximality implies restricted maximality. We first show that $\liminf_{\epsilon\to 0^+}\frac{\phi(\epsilon)}{\epsilon} = 0$. It is clear that $\liminf_{\epsilon\to 0^+}\frac{\phi(\epsilon)}{\epsilon} \geq 0$ since ϕ is restricted maximal. Assume $\liminf_{\epsilon\to 0^+}\frac{\phi(\epsilon)}{\epsilon} = s > 0$, then there exist $\delta > 0$ and $s' < s$ (small enough) such that $\phi(x) \geq s'x$ for $x \in [0, \delta]$. Define a new function $\phi_1(x) = \frac{\phi(x)-s'x}{1-s'}$, and ϕ is implied by ϕ_1. Note that ϕ_1 is a restricted maximal gDFF. The strong maximality of ϕ implies $\phi_1(x) = \phi(x) = x$. Therefore, $\phi(x) = x$ is not strongly maximal. This contradiction implies that $\liminf_{\epsilon\to 0^+}\frac{\phi(\epsilon)}{\epsilon} = 0$.

Next we show that $\lim_{\epsilon\to 0^+}\frac{\phi(\epsilon)}{\epsilon} = 0$. Suppose on the contrary there exists some positive s such that $\limsup_{\epsilon\to 0^+}\frac{\phi(\epsilon)}{\epsilon} = 3s > 0$. There exist two positive and decreasing sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ approaching 0, such that $\phi(x_n) > 2sx_n$ and $\phi(y_n) < sy_n$. Fix y_1 and choose $0 < x_n < y_1$ and $k \in \mathbb{Z}_{++}$ such that $y_1 \geq kx_n \geq \frac{y_1}{2}$. Since ϕ is superadditive and nondecreasing, $\phi(y_1) \geq \phi(kx_n) \geq k\phi(x_n) > 2ksx_n \geq sy_1$, which contradicts the choice of y_1. Then we have $\limsup_{\epsilon\to 0^+}\frac{\phi(\epsilon)}{\epsilon} = \liminf_{\epsilon\to 0^+}\frac{\phi(\epsilon)}{\epsilon} = 0$, and $\lim_{\epsilon\to 0^+}\frac{\phi(\epsilon)}{\epsilon} = 0$ for a strongly maximal gDFF ϕ.

On the other hand, we assume ϕ is restricted maximal and $\lim_{\epsilon\to 0^+}\frac{\phi(\epsilon)}{\epsilon} = 0$. Suppose ϕ is implied by a gDFF ϕ_1 meaning $\phi(x) \leq \beta\phi_1(x) + \alpha x$ and $\beta, \alpha \geq 0, \beta + \alpha \leq 1$. Let $x = 1$, then $1 \leq \beta\phi_1(1) + \alpha \leq \beta + \alpha \leq 1$. We know that $\beta = 1 - \alpha$. Note that $\beta\phi_1(x) + \alpha x$ is also a gDFF (a convex combination of two gDFFs ϕ_1 and x), then $\phi(x) = (1-\alpha)\phi_1(x) + \alpha x$ due to the maximality of ϕ. Divide by x from the above equation and take the lim inf as $x \to 0^+$, we can conclude $\alpha = 0$. So ϕ is strongly maximal. □

Remark 4.5 *Let ϕ be a maximal gDFF that is not linear. By Proposition 2.12 we know that $\phi(1) = 1$. If ϕ is implied via scaling by a gDFF ϕ_1, or equivalently $\beta\phi_1 \geq \phi$ for some $0 \leq \beta \leq 1$, then $\beta\phi_1(1) \geq \phi(1) = 1$. Since $\beta \leq 1$ and $\phi_1(1) \leq 1$, we have $\beta = 1$ and ϕ is dominated by ϕ_1. The maximality of ϕ implies $\phi = \phi_1$, so ϕ is restricted maximal. Therefore, we have a simpler version of the characterization of maximal gDFFs.*

Theorem 4.6. *A function $\phi\colon \mathbb{R} \to \mathbb{R}$ is a maximal gDFF if and only if $\phi(x) = sx$ for some $0 \leq s < 1$ or ϕ is a restricted maximal gDFF, i.e., ϕ satisfies the following conditions:*

(i) $\phi(0) = 0$.
(ii) ϕ *is superadditive.*
(iii) $\phi(x) \geq 0$ *for all* $x \in \mathbb{R}_+$.
(iv) $\phi(x) + \phi(1-x) = 1$ *or* $\phi(x) = sx$, $0 \leq s < 1$.

We use Zorn's lemma to show that maximal, restricted maximal and strongly maximal gDFFs exist, and they are potentially stronger than just valid gDFFs. The proof is analogous to the proof of [31, Theorem 1, Proposition 6, Theorem 9].

Theorem 4.7. *(i) Every gDFF is dominated by a maximal gDFF.*

(ii) Every gDFF is implied via scaling by a restricted maximal gDFF.
(iii) Every gDFF is implied by a strongly maximal gDFF.

Proof. Part (i). If the gDFF ϕ is already maximal, then it is dominated by itself. We assume ϕ is not maximal. Define a set $A = \{\text{ valid gDFF } \hat{\phi} : \hat{\phi}(x) \geq \phi(x) \text{ for } x \in \mathbb{R}\,\}$. We consider $(A, \leq)$ as a partially ordered set, where the partial order $\phi_1 \leq \phi_2$ is imposed by the pointwise inequality $\phi_1(x) \leq \phi_2(x)$ for all $x \in \mathbb{R}$. Consider a chain C and a function $\phi_C(x) = \sup_{\phi' \in C} \phi'(x)$. We claim ϕ_C is an upper bound of the chain C and it is contained in A.

First we prove ϕ_C is a well-defined function. For any fixed $r_0 \in \mathbb{R}$, based on the definition of gDFF, we know that for any $\phi' \in C$ it holds that $\phi'(r_0) + \phi(-r_0) \leq \phi'(r_0) + \phi'(-r_0) \leq 0$. Note that $\phi(-r_0)$ is a fixed constant and it forces that $\sup_{\phi' \in C} \phi'(r_0) < \infty$. So we know that $\phi_C(x) = \sup_{\phi' \in C} \phi'(x) < \infty$ for any $x \in \mathbb{R}$.

Next, we prove ϕ_C is a valid gDFF and dominates ϕ. It is clear that $\phi_C \geq \phi$, so we only need to show ϕ_C is a valid gDFF. Suppose on the contrary ϕ_C is not valid, then there exist $(x_i)_{i=1}^m$ such that $\sum_{i=1}^m x_i \leq 1$ and $\sum_{i=1}^m \phi_C(x_i) = 1 + \epsilon$ for some $\epsilon > 0$. Since there are only finite number of x_i, we can choose a function $\phi' \in C$ such that $\phi_C(x_i) < \phi'(x_i) + \frac{\epsilon}{m}$ for $i = 1, 2, \ldots, m$. Then $1 + \epsilon = \sum_{i=1}^m \phi_C(x_i) < \sum_{i=1}^m (\phi'(x_i) + \frac{\epsilon}{m}) \leq 1 + \epsilon$. The last step is due to the fact that ϕ' is a valid gDFF. From the contradiction we know that ϕ_C is a valid gDFF.

We have shown that every chain in the set A has a upper bound in A. By Zorn's lemma, we know there is a maximal element in the set A, which is the desired maximal gDFF.

Part (ii). By (i) we only need to show every maximal gDFF ϕ is implied via scaling by a restricted maximal gDFF. Based on Theorem 4.6, ϕ is either restricted maximal or a linear function. If ϕ is restricted maximal, then it is implied via scaling by itself. If ϕ is a linear function, then it is implied via scaling by $\phi'(x) = x$.

Part (iii). Suppose ϕ is a linear function, and $\phi(x) = s_0 x$ where $0 \leq s_0 \leq 1$. Observe that ϕ is implied by any strongly maximal gDFF ϕ_1, since we have $s_0 x \leq 0 \times \phi_1(x) + s_0 x$.

Now we assume that ϕ is nonlinear. By (ii) we only need to show every restricted maximal gDFF ϕ is implied by a strongly maximal gDFF. If ϕ is already strongly maximal, then it is implied by itself. Suppose ϕ is not strongly maximal.

First we claim that $\lim_{\epsilon \to 0^+} \frac{\phi(\epsilon)}{\epsilon}$ exists. The proof of the claim follows the proof of Theorem 4.4 so we omit it here. Since ϕ is not strongly maximal, $\lim_{\epsilon \to 0^+} \frac{\phi(\epsilon)}{\epsilon} > 0$ by Theorem 4.4. If $\lim_{\epsilon \to 0^+} \frac{\phi(\epsilon)}{\epsilon} = 1$, then ϕ is the linear functions $\phi(x) = x$. We can assume that $1 > \lim_{\epsilon \to 0^+} \frac{\phi(\epsilon)}{\epsilon} = s > 0$. Define a new function $\phi_1(x) = \frac{\phi - sx}{1-s}$ and we want to show ϕ_1 is a strongly maximal gDFF. Note that $\phi_1(0) = 0$, ϕ_1 is superadditive, $\phi_1(x) + \phi_1(1-x) = 1$ and $\lim_{\epsilon \to 0^+} \frac{\phi_1(\epsilon)}{\epsilon} = 0$. We only need to prove $\phi_1(x)$ is nonnegative if x is nonnegative and near 0. Suppose on the contrary there exist $r_0 > 0$ and $\epsilon > 0$ such that

$\phi(r_0) = sr_0 - \epsilon$. There also exists a positive and decreasing sequence $(x_n)_{n=1}^{\infty}$ approaching 0 and satisfying $\frac{\phi(x_n)}{x_n} > s - \frac{\epsilon}{2r_0}$. Choose x_n small enough and $k \in \mathbb{Z}_{++}$ such that $r_0 \geq kx_n \geq r_0 - \frac{\epsilon}{2s}$. Since ϕ is superadditive and nondecreasing, we have

$$sr_0 - \epsilon = \phi(r_0) \geq \phi(kx_n) \geq k\phi(x_n) > ksx_n - \frac{k\epsilon x_n}{2r_0} \geq sr_0 - \frac{\epsilon}{2} - \frac{\epsilon}{2} = sr_0 - \epsilon.$$

The above contradiction implies that $\phi(x) \geq sx$ for positive x near 0. Therefore ϕ_1 is strongly maximal and ϕ is implied by ϕ_1. □

5 Relation to cut-generating functions

5.1 Relation to Gomory–Johnson functions

In this section, we relate both cDFFs and gDFFs to the Gomory–Johnson cut-generating functions. In fact, new DFFs, especially extreme ones, can be discovered by converting Gomory–Johnson functions to DFFs. We first introduce the Gomory–Johnson cut-generating functions; details can be found in [6,7]. Consider the single-row Gomory–Johnson model, which takes the following form:

$$x + \sum_{r \in \mathbb{R}} r\, y(r) = b, \quad b \notin \mathbb{Z}, b > 0, \tag{4}$$

$$x \in \mathbb{Z},\ y : \mathbb{R} \to \mathbb{Z}_+, \text{ and } y \text{ has finite support.}$$

Let $\pi \colon \mathbb{R} \to \mathbb{R}$ be a nonnegative function. Then by definition π is a valid Gomory–Johnson function if $\sum_{r \in \mathbb{R}} \pi(r)\, y(r) \geq 1$ holds for any feasible solution (x, y). Minimal (pointwise non-dominating) functions are characterized by subadditivity and several other properties.

As maximal cDFFs and gDFFs are superadditive and minimal Gomory–Johnson functions are subadditive, underlying the conversion is that subtracting subadditive functions from linear functions gives superadditive functions; but the details are more complicated.

Theorem 5.1. *Let π be a minimal piecewise linear Gomory–Johnson function corresponding to a row of the form (4) with the right hand side b. Assume π is continuous at 0 from the right. Then there exists $\delta > 0$, such that for all $0 < \lambda < \delta$, the function $\phi_\lambda \colon \mathbb{R} \to \mathbb{R}$, defined by $\phi_\lambda(x) = \frac{bx - \lambda\pi(bx)}{b - \lambda}$, is a maximal gDFF and its restriction $\phi_\lambda|_{[0,1]}$ is a maximal cDFF. These functions have the following properties.*

(i) *π has k different slopes if and only if ϕ_λ has k different slopes. If $b > 1$, then π has k different slopes if and only if $\phi_\lambda|_{[0,1]}$ has k different slopes.*
(ii) *The gDFF ϕ_λ is extreme if π is also continuous with only 2 slope values where its positive slope s satisfies $sb > 1$ and $\lambda = \frac{1}{s}$. The cDFF $\phi_\lambda|_{[0,1]}$ is extreme if π and λ satisfy the previous conditions and $b > 3$.*

Proof. First we prove ϕ_λ is a maximal gDFF if λ is small enough. As a minimal valid Gomory–Johnson function, π is $\mathbb{Z}$-periodic, $\pi(0) = 0$, π is subadditive and $\pi(x) + \pi(b - x) = 1$ for all $x \in \mathbb{R}$ [6]. Note that ϕ_λ is defined on $\mathbb{R}$, since π is $\mathbb{Z}$-periodic and defined on $\mathbb{R}$. It is not hard to check $\phi_\lambda(0) = 0$. Since ϕ_λ is obtained by subtracting a subadditive function from a linear function, it is superadditive.

The symmetry condition of ϕ_λ is due to the following equation:

$$\begin{aligned}\phi_\lambda(x) + \phi_\lambda(1-x) &= \frac{bx - \lambda\pi(bx)}{b-\lambda} + \frac{b(1-x) - \lambda\pi(b(1-x))}{b-\lambda} \\ &= \frac{b - \lambda(\pi(bx) + \pi(b(1-x)))}{b-\lambda} = 1.\end{aligned}$$

The last step is from the symmetry condition of π and $\pi(b) = 1$. Since π is piecewise linear and continuous at 0 from the right. Let s be the largest slope of π, then the largest slope of $\pi(bx)$ is bs. Choose $\delta = \frac{1}{s}$, then if $\lambda < \delta$, the slope of bx is always no smaller than the slope of $\lambda\pi(bx)$. There exists an $\epsilon > 0$ such that $\phi_\lambda(x) \geq 0$ for all $x \in (0, \epsilon)$. Therefore, ϕ_λ is a maximal gDFF by Theorem 2.4 and $\phi_\lambda|_{[0,1]}$ is a maximal cDFF by Theorem 2.3.

Part (i). Suppose π has slope s on the interval (a_i, a_{i+1}), then by calculation $\phi_\lambda(x) = \frac{bx-\lambda\pi(bx)}{b-\lambda}$ has slope $s' = \frac{b(1-\lambda s)}{b-\lambda}$ on the interval $(\frac{a_i}{b}, \frac{a_{i+1}}{b})$. So if π has slope s_1, s_2 on interval (a_i, a_{i+1}) and (a_j, a_{j+1}) respectively, and ϕ_λ has slope s_1', s_2' on interval $(\frac{a_i}{b}, \frac{a_{i+1}}{b})$ and $(\frac{a_j}{b}, \frac{a_{j+1}}{b})$ respectively, then $s_1 = s_2$ if and only if $s_1' = s_2'$. From the above fact we can conclude π has k different slopes if and only if ϕ_λ has k different slopes.

Since π is $\mathbb{Z}$-periodic, ϕ_λ is quasiperiodic with period $\frac{1}{b}$. If $b > 1$, the interval $[0, 1]$ contains a whole period, which has pieces with all different slope values. So π has k different slopes if and only if $\phi_\lambda|_{[0,1]}$ has k different slopes.

Part (ii). If $sb > 1$ and $\lambda = \frac{1}{s}$, then it is not hard to show ϕ_λ is also continuous piecewise linear with only 2-slope values, and $\phi_\lambda(x) = 0$ for $x \in [0, \frac{\epsilon}{b}]$, i,e., one slope value is 0. From the above results, we know ϕ_λ is a maximal gDFFs.

We use the idea of the extremality test in subsection 3.3. Since π is extreme from the Gomory–Johnson 2-Slope Theorem [14], all intervals are covered and there are 2 covered components. Suppose $(x, y, x+y)$ is an additive vertex, which means $\pi(x) + \pi(y) = \pi(x + y)$. From arithmetic computation, $(\frac{x}{b}, \frac{y}{b}, \frac{x+y}{b})$ is an additive vertex, i.e., $\phi_\lambda(\frac{x}{b}) + \phi_\lambda(\frac{y}{b}) = \phi_\lambda(\frac{x+y}{b})$. So the additive faces for ϕ_λ are just a scaling of those for π. In regards to ϕ_λ, all intervals are covered and there are only 2 covered components, and $\phi_\lambda(1) = 1$ and $\phi_\lambda(x) = 0$ for $x \in [0, \frac{\epsilon}{b}]$ guarantee that the interval $[0, 1]$ contains the 2 covered components.

Assume $\phi_\lambda = \frac{\phi_1+\phi_2}{2}$, where ϕ_1 and ϕ_2 are maximal gDFFs. By Theorem 2.4 and definition, $\phi_1(x) = \phi_2(x) = 0$ for $x \in [0, \frac{\epsilon}{b}]$ and $\phi_1(1) = \phi_2(1) = 1$. The functions ϕ_1 and ϕ_2 satisfy the additivity where ϕ_λ satisfies the additivity, otherwise one of ϕ_1 and ϕ_2 violates the superadditivity. So the additive faces of ϕ_λ are still additive faces of ϕ_1 and ϕ_2. By the Interval Lemma [6] and values at point $\frac{\epsilon}{b}$ and 1, we can show ϕ_1 and ϕ_2 both have 2 covered components and these covered components are the same as those of ϕ_λ. Thus ϕ_1 and ϕ_2 are both continuous 2-slope functions and one slope value is 0, due to nondecreasing condition. Suppose the 2 covered components within $[0,1]$ are C_1 and C_2, where C_1 and C_2 are disjoint unions of closed intervals. We assume ϕ_1 and ϕ_2 have slope 0 on C_1 and slope s_1 and s_2 on C_2 respectively. The condition $\phi_1(1) = \phi_2(1) = 1$ implies that $0 \times |C_1| + s_1 \times |C_2| = 1$ and $0 \times |C_1| + s_2 \times |C_2| = 1$, where $|C_1|$ and $|C_2|$ denote the measure of C_1 and C_2. So we have $s_1 = s_2$. All these properties guarantee that ϕ_1 and ϕ_2 are equal to each other, therefore ϕ_λ is extreme.

We assume $b > 3$. If all intervals are covered for the restriction $\phi_\lambda|_{[0,1]}$, then we can use the same arguments to show $\phi_\lambda|_{[0,1]}$ is extreme. So we only need to show all intervals are covered by additive faces in the triangular region: $R = \{(x, y) \colon x, y, x + y \in [0, 1]\}$. Maximality of $\phi_\lambda|_{[0,1]}$, especially the symmetry condition, implies that if $(x, y, x+y)$ is an additive vertex, so is $(1-x-y, y, 1-x)$. The fact implies that the covered components are symmetric about $x = \frac{1}{2}$, i.e., x is covered $\Leftrightarrow$ $1 - x$ is covered and they are in the same covered components. From the scaling of additive faces of π, the additive faces of $\phi_\lambda|_{[0,1]}$ contained in the square $[0, \frac{1}{b}]^2$ cover the interval $[0, \frac{1}{b}]$, and the additive faces of $\phi_\lambda|_{[0,1]}$ contained in the square $[\frac{1}{b}, \frac{2}{b}] \times [0, \frac{1}{b}]$ cover the interval $[\frac{1}{b}, \frac{2}{b}]$. Similarly, we can use additive faces contained in $\lceil \frac{b}{2} \rceil = \lceil \frac{1}{2} / \frac{1}{b} \rceil$ such whole squares to cover the interval $[0, \frac{1}{2}]$. The condition $b > 3$ guarantees that those $\lceil \frac{b}{2} \rceil$ whole squares are contained in the region R. Together with the symmetry of covered components, we can conclude all intervals are covered, thus $\phi_\lambda|_{[0,1]}$ is extreme.

This concludes the proof of the theorem. □

Remark 5.2 *(1) A construction of k-slope extreme Gomory–Johnson functions has been found for any arbitrary $k \in \mathbb{N}$ [4]. Therefore, there exist maximal gDFFs and cDFFs with an arbitrary number of slopes.*

(2) The restriction $\phi_\lambda|_{[0,1]}$ is not always extreme as a cDFF even if ϕ_λ is extreme as a gDFF. See an example in Remark 6.5.

(3) Note that ϕ_λ is quasiperiodic since π is $\mathbb{Z}$-periodic. However, not all maximal gDFFs are quasiperiodic (See Theorem 2.10). Therefore, the conversion is not surjective.

5.2 Relation to Yıldız and Cornuéjols cut-generating functions

In this subsection, we focus on gDFFs since they have the extended domain $\mathbb{R}$. We define an infinite dimensional space Y called "the space of nonbasic variables" as $Y = \{y : y \colon \mathbb{R} \to \mathbb{Z}_+ \text{ and } y \text{ has finite support}\}$, and we refer to the zero function as the origin of Y. In this section, we study valid inequalities of certain subsets of the space Y and connect gDFFs to a particular family of cut-generating functions.

In the paper of Yıldız and Cornuéjols [31], the authors considered the following generalization of the Gomory–Johnson model:

$$x = f + \sum_{r \in \mathbb{R}} r\, y(r), \tag{5}$$

$$x \in S,\ y : \mathbb{R} \to \mathbb{Z}_+, \text{ and } y \text{ has finite support,}$$

where S can be any nonempty subset of $\mathbb{R}$. A function $\pi\colon \mathbb{R} \to \mathbb{R}$ is called a *valid cut-generating function* if the inequality $\sum_{r\in\mathbb{R}} \pi(r)\, y(r) \geq 1$ holds for all feasible solutions (x, y) to (5). In order to ensure that such cut-generating functions exist, they only consider the case $f \notin S$. Otherwise, if $f \in S$, then $(x, y) = (f, 0)$ is a feasible solution and there is no function π which can make the inequality $\sum_{r\in\mathbb{R}} \pi(r)\, y(r) \geq 1$ valid. Note that $y \in Y$ for any feasible solution (x, y) to (5), and all valid inequalities in the form of $\sum_{r\in\mathbb{R}} \pi(r)\, y(r) \geq 1$ to (5) are inequalities which separate the origin of Y.

We consider two different but related models in the form of (5). Let $f = -1$, $S = \{0\}$, and the feasible region $Y_{=1} = \{y : \sum_{r\in\mathbb{R}} r\, y(r) = 1,\ y\colon \mathbb{R} \to \mathbb{Z}_+ \text{ and } y \text{ has finite support}\}$. Let $f = -1$, $S = (-\infty, 0]$, and the feasible region $Y_{\leq 1} = \{y : \sum_{r\in\mathbb{R}} r\, y(r) \leq 1,\ y\colon \mathbb{R} \to \mathbb{Z}_+ \text{ and } y \text{ has finite support}\}$. It is immediate to check that the latter model is the relaxation of the former. Therefore $Y_{=1} \subsetneq Y_{\leq 1}$ and any valid inequality for $Y_{\leq 1}$ is also valid for $Y_{=1}$.

Jeroslow [21], Blair [10] and Bachem et al. [3] studied minimal valid inequalities of the set $Y_{=b} = \{y : \sum_{r\in\mathbb{R}} r\, y(r) = b,\ y\colon \mathbb{R} \to \mathbb{Z}_+ \text{ and } y \text{ has finite support}\}$. Note that $Y_{=b}$ is the set of feasible solutions to (5) for $S = \{0\}$, $f = -b$. The notion "minimality" they used is in fact the restricted minimality in the Yıldız–Cornuéjols model. In this section, we use the terminology introduced by Yıldız and Cornuéjols. Jeroslow [21] showed that finite-valued subadditive (restricted minimal) functions are sufficient to generate all necessary valid inequalities of $Y_{=b}$ for bounded mixed integer programs. Kılınç-Karzan and Yang [22] discussed whether finite-valued functions are sufficient to generate all necessary inequalities for the convex hull description of disjunctive sets. Interested readers are referred to [22] for more details on the sufficiency question. Blair [10] extended Jeroslow's result to rational mixed integer programs. Bachem et al. [3] characterized restricted minimal cut-generating functions under some continuity assumptions, and showed that restricted minimal functions satisfy the symmetry condition.

Yıldız–Cornuéjols cut-generating functions provide valid inequalities which separate the origin, but clearly there exist other types of valid inequalities. If we let $f \in S$, then there does not exist a valid inequality separating the origin, but we can still consider those which do not separate the origin.

In terms of the relaxation $Y_{\leq 1}$, gDFFs can generate the valid inequalities in the form of $\sum_{r\in\mathbb{R}} \phi(r)\, y(r) \leq 1$, and such inequalities do not separate the origin. Note that there is no valid inequality separating the origin since $0 \in Y_{\leq 1}$. The gDFFs can also be viewed as valid functions in the following pure integer linear programming model:

$$1 \geq \sum_{r\in\mathbb{R}} r\, y(r), \tag{6}$$

$$y : \mathbb{R} \to \mathbb{Z}_+, \text{ and } y \text{ has finite support.}$$

Notice that ϕ is a valid gDFF if the inequality $\sum_{r\in\mathbb{R}} \phi(r)\, y(r) \leq 1$ holds for all feasible solutions y to (6).

Cut-generating functions provide valid inequalities which separate the origin for $Y_{=1}$, but such inequalities are not valid for $Y_{\leq 1}$. In terms of inequalities that do not separate the origin, any inequality in the form of $\sum_{r\in\mathbb{R}} \phi(r)\, y(r) \leq 1$ generated by some gDFF ϕ is valid for $Y_{\leq 1}$ and hence valid for $Y_{=1}$, since the model of $Y_{\leq 1}$ is the relaxation of that of $Y_{=1}$. Clearly, there also exist valid inequalities which do not separate the origin for $Y_{=1}$ but are not valid for $Y_{\leq 1}$.

Yıldız and Cornuéjols [31] introduced the notions of minimal, restricted minimal and strongly minimal cut-generating functions. We consider the cut-generating functions to the model (5) when $f = -1$, $S = \{0\}$, and we restate the definitions of minimality of such cut-generating functions. A valid cut-generating function π is called *minimal* if it does not dominate another valid cut-generating function π'. A cut-generating function π' implies a cut-generating function π via scaling if there exists $\beta \geq 1$ such that $\pi \geq \beta\pi'$. A valid cut-generating function π is *restricted minimal* if there is no other cut-generating function π' implying π via scaling. A cut-generating function π' implies a cut-generating function π if there exist α, β, and $\beta \geq 0, \alpha + \beta \geq 1$ such that $\pi(x) \geq \beta\pi'(x) + \alpha x$. A valid cut-generating function π is *strongly minimal* if there is no other cut-generating function π' implying π. Yıldız and Cornuéjols also characterized minimal and restricted minimal functions without additional assumptions. As for the strong minimality and extremality, they mainly focused on the case where $f \in \overline{\mathrm{conv}(S)}$ and $\overline{\mathrm{conv}(S)}$ is full-dimensional. We instead discuss the strong minimality and extremality when $f = -1$, $S = \{0\}$ in Remark 5.6.

We show that gDFFs are closely related to cut-generating functions for $Y_{=1}$. The main idea is that valid inequalities generated by cut-generating functions for $Y_{=1}$ can be lifted to valid inequalities generated by gDFFs for the relaxation $Y_{\leq 1}$. The procedure involves adding a multiple of the defining equality $\sum_{r\in\mathbb{R}} r\, y(r) = 1$ to a valid inequality, which is called "tilting" by Aráoz et al. [2].

We include the characterizations of minimal and restricted minimal cut-generating functions for $Y_{=1}$ below. Bachem et al. had the same characterization [3, Theorem] as Theorem 5.4 under continuity assumptions at the origin.

Theorem 5.3 ([31, Theorem 2]). *A function $\pi\colon \mathbb{R} \to \mathbb{R}$ is a minimal cut-generating function for $Y_{=1}$ if and only if $\pi(0) = 0$, π is subadditive, and $\pi(x) = \sup_k\{\frac{1}{k}(1 - \pi(1 - kx)) : k \in \mathbb{N}\}$.*

Theorem 5.4 ([31, Proposition 5]). *A function $\pi\colon \mathbb{R} \to \mathbb{R}$ is a restricted minimal cut-generating function for $Y_{=1}$ if and only if π is minimal and $\pi(1) = 1$.*

The following theorem describes the conversion between gDFFs and cut-generating functions for $Y_{=1}$. Unlike Gomory–Johnson cut-generating functions, Yıldız–Cornuéjols cut-generating functions can be converted to gDFFs and the other way around.

Theorem 5.5. *Given a valid/maximal/restricted maximal gDFF ϕ, then for every $0 < \lambda < 1$, the following function is a valid/minimal/restricted minimal cut-generating function for $Y_{=1}$:*

$$\pi_\lambda(x) = \frac{x - (1 - \lambda)\, \phi(x)}{\lambda}.$$

Given a valid/minimal/restricted minimal cut-generating function π for $Y_{=1}$, which is Lipschitz continuous at $x = 0$, then there exists $\delta > 0$ such that for all $0 < \lambda < \delta$ the following function is a valid/maximal/restricted maximal gDFF:

$$\phi_\lambda(x) = \frac{x - \lambda\,\pi(x)}{1-\lambda}, \quad 0 < \lambda < 1.$$

Proof. Part (i). The proof of valid functions.

We want to show that π_λ is a a valid cut-generating function for $Y_{=1}$. Suppose there is a function $y : \mathbb{R} \to \mathbb{Z}_+$, y has finite support, and $\sum_{r\in\mathbb{R}} r\, y(r) = 1$. We want to show that for $\lambda \in (0, 1)$:

$$\begin{aligned}
&\sum_{r\in\mathbb{R}} \pi_\lambda(r)\, y(r) \geq 1 \\
\Leftrightarrow\ &\sum_{r\in\mathbb{R}} \frac{r - (1-\lambda)\,\phi(r)}{\lambda}\, y(r) \geq 1 \\
\Leftrightarrow\ &\sum_{r\in\mathbb{R}} (r - (1-\lambda)\,\phi(r))\, y(r) \geq \lambda \\
\Leftrightarrow\ &\sum_{r\in\mathbb{R}} r\, y(r) - (1-\lambda) \sum_{r\in\mathbb{R}} \phi(r)\, y(r) \geq \lambda \\
\Leftrightarrow\ &\sum_{r\in\mathbb{R}} \phi(r)\, y(r) \leq 1.
\end{aligned}$$

The last step is derived from $\sum_{r\in\mathbb{R}} r\, y(r) = 1$ and ϕ is a gDFF.

On the other hand, the Lipschitz continuity of π at 0 guarantees that $\phi_\lambda(x) \geq 0$ for $x \geq 0$ if λ is small enough. Then the proof for validity of ϕ_λ is analogous to the proof above.

Part (ii). The proof of maximal/minimal functions.

As stated in Theorem 5.3, π is minimal if and only if $\pi(0) = 0$, π is subadditive and $\pi(x) = \sup_k\{\frac{1}{k}(1 - \pi(1 - kx)) : k \in \mathbb{N}\}$, which is called the generalized symmetry condition. If $\pi_\lambda(x) = \frac{x-(1-\lambda)\,\phi(x)}{\lambda}$, then $\pi_\lambda(0) = 0$ and π_λ is subadditive.

$$
\begin{aligned}
&\sup_k\{\frac{1}{k}(1-\pi_\lambda(1-kx)) : k \in \mathbb{Z}_+\}\\
=\ &\sup_k\{\frac{1}{k}(1-\frac{1-kx-(1-\lambda)\,\phi(1-kx)}{\lambda}) : k \in \mathbb{Z}_+\}\\
=\ &\sup_k\{\frac{kx-(1-\lambda)(1-\phi(1-kx))}{k\lambda} : k \in \mathbb{Z}_+\}\\
=\ &\sup_k\{\frac{x}{\lambda}-\frac{1-\lambda}{\lambda}\frac{1}{k}(1-\phi(1-kx)) : k \in \mathbb{Z}_+\}\\
=\ &\frac{x}{\lambda}-\frac{1-\lambda}{\lambda}\inf_k\{\frac{1}{k}(1-\phi(1-kx)) : k \in \mathbb{Z}_+\}\\
=\ &\frac{x}{\lambda}-\frac{1-\lambda}{\lambda}\phi(x)\\
=\ &\pi_\lambda(x).
\end{aligned}
$$

Therefore, π_λ is minimal.

On the other hand, given a minimal cut-generating function π, let $\phi_\lambda(x) = \frac{x-\lambda\,\phi(x)}{1-\lambda}$, then it is easy to see the superadditivity and $\phi_\lambda(0) = 0$. The generalized symmetry can be proven similarly. The Lipschitz continuity of π at 0 implies that $\phi_\lambda(x) \geq 0$ for any $x \geq 0$ if λ is chosen properly.

Part (iii). The proof of restricted maximal/minimal functions.

As stated in Theorem 5.4, π is restricted minimal if and only if $\pi(0) = 0$, π is subadditive and $\pi(x) = \sup_k\{\frac{1}{k}(1-\pi(1-kx)) : k \in \mathbb{Z}_+\}$, and $\pi(1) = 1$. Given a restricted maximal gDFF ϕ, we have $\phi(1) = 1$, which implies $\pi_\lambda(1) = 1$.

On the other hand, a restricted minimal π satisfying $\pi(1) = 1$, then $\phi_\lambda(1) = 1$. Based on the maximality of ϕ_λ, we know ϕ_λ is restricted maximal. □

Remark 5.6 *We discuss the distinctions between these two families of functions.*

- *(i) It is not hard to prove that extreme gDFFs are always maximal. However, unlike cut-generating functions for $Y_{=1}$, extreme gDFFs are not always restricted maximal. For instance, $\phi(x) = 0$ is an extreme gDFF but not restricted maximal.*
- *(ii) By applying the proof of [31, Proposition 28], we can show that no strongly minimal cut-generating function for $Y_{=1}$ exists. However, there exist strongly maximal gDFFs by Theorem 4.7. Moreover, we can use the same conversion formula in Theorem 5.5 to convert a restricted minimal cut-generating function to a strongly maximal gDFF (see Theorem 5.7 below). In fact, it suffices to choose a proper λ such that $\lim_{\epsilon\to 0^+}\frac{\phi_\lambda(\epsilon)}{\epsilon} = 0$ by the characterization of strongly maximal gDFFs (Theorem 4.4).*
- *(iii) There is no extreme piecewise linear cut-generating function π for $Y_{=1}$ which is Lipschitz continuous at $x = 0$, except for $\pi(x) = x$. If π is such an extreme function, then for any λ small enough, we claim that ϕ_λ is an*

extreme gDFF. Suppose $\phi_\lambda = \frac{1}{2}\phi^1 + \frac{1}{2}\phi^2$ *and let* $\pi^1_\lambda, \pi^2_\lambda$ *be the corresponding cut-generating functions of* ϕ^1, ϕ^2 *by Theorem 5.5. Note that* $\pi = \frac{1}{2}(\pi^1_\lambda + \pi^2_\lambda)$*, which implies* $\pi = \pi^1_\lambda = \pi^2_\lambda$ *and* $\phi_\lambda = \phi^1_\lambda = \phi^2_\lambda$*. Thus* ϕ_λ *is extreme. By Lemma 6.2 and the extremality of* ϕ_λ*, we know* $\phi_\lambda(x) = x$ *or there exists* $\epsilon > 0$*, such that* $\phi_\lambda(x) = 0$ *for* $x \in [0, \epsilon)$*. If* $\phi_\lambda(x) = x$*, then* $\pi(x) = x$*. Otherwise,* $\lim_{x\to 0^+} \frac{\phi_\lambda(x)}{x} = 0$ *for any small enough* λ*. The equation*

$$0 = \lim_{x\to 0^+} \frac{\phi_\lambda(x)}{x} = \lim_{x\to 0^+} \frac{x - \lambda\pi(x)}{(1-\lambda)x} = \frac{1 - \lambda \lim_{x\to 0^+} \frac{\pi(x)}{x}}{1-\lambda}$$

implies $\lim_{x\to 0^+} \frac{\pi(x)}{x} = \frac{1}{\lambda}$ *for any small enough* λ*, which is not possible. Therefore,* π *cannot be extreme except for* $\pi(x) = x$*.*

Theorem 5.7. *Given a non-linear restricted minimal cut-generating function* π *for* $Y_{=1}$*, which is Lipschitz continuous at* 0*, then there exists* $\lambda > 0$ *such that the following function is a strongly maximal gDFF:*

$$\phi_\lambda(x) = \frac{x - \lambda\,\pi(x)}{1-\lambda}.$$

6 Two-slope theorem for general DFFs

In this section, we prove a 2-slope theorem for extreme gDFFs, in the spirit of the 2-slope theorem of Gomory and Johnson [14,15]. First we introduce the lemma showing that extreme gDFFs have certain structures. Similar to Lemma 3.1 and Lemma 3.2, by studying the superadditivity of maximal gDFFs, it is not hard to prove the following lemma.

Lemma 6.1 *Piecewise linear maximal gDFFs are continuous at* 0 *from the right.*

Lemma 6.2 *Let* ϕ *be a piecewise linear extreme gDFF.*

(i) If ϕ *is strictly increasing, then* $\phi(x) = x$*.*
(ii) If ϕ *is not strictly increasing, then there exists* $\epsilon > 0$*, such that* $\phi(x) = 0$ *for* $x \in [0, \epsilon)$*.*

Proof. Similar to the proof of Lemma 3.2, we can assume $\phi(x) = sx$, $x \in [0, a_1)$. If $s = 1$, then $\phi(x) = x$. If $s = 0$, then ϕ is not strictly increasing therefore (ii) holds.

Next, we assume $0 < s < 1$. Define a function:

$$\phi_1(x) = \frac{\phi(x) - sx}{1-s}.$$

Clearly $\phi_1(x) = 0$ for $x \in [0, a_1)$. The function ϕ_1 is superadditive because it is obtained by subtracting a linear function from a superadditive function. We have that

$$\begin{aligned}
\phi_1(x) &= \frac{\phi(x) - sx}{1-s} \\
&= \frac{1}{1-s}[\inf_k\{\frac{1}{k}(1-\phi(1-kx)) : k \in \mathbb{Z}_+\} - sx] \\
&= \frac{1}{1-s}[\inf_k\{\frac{1}{k}(1-[(1-s)\phi_1(1-kx) + s(1-kx)]) : k \in \mathbb{Z}_+\} - sx] \\
&= \frac{1}{1-s}[\inf_k\{\frac{1}{k}[(1-s) + skx - (1-s)\phi_1(1-kx)] : k \in \mathbb{Z}_+\} - sx] \\
&= \frac{1}{1-s}\inf_k\{\frac{1}{k}[(1-s) - (1-s)\phi_1(1-kx)] : k \in \mathbb{Z}_+\} \\
&= \inf_k\{\frac{1}{k}(1-\phi_1(1-kx)) : k \in \mathbb{Z}_+\}.
\end{aligned}$$

The above equation shows that ϕ_1 satisfies the generalized symmetry condition in Proposition 4.1. Therefore, ϕ_1 is also a maximal gDFF. The condition $\phi(x) = sx + (1-s)\phi_1(x)$ implies ϕ is not extreme, since it can be expressed as a convex combination of two different maximal gDFFs: x and ϕ_1. □

From Lemma 6.2, we know 0 must be one slope value of a piecewise linear extreme gDFF ϕ, except for $\phi(x) = x$. Next, we introduce the 2-slope theorem for extreme gDFFs. The proof of the following two-slope theorem follows closely that of the Gomory–Johnson's two-slope theorem.

Theorem 6.3 (Two-Slope Theorem for gDFFs). *Let ϕ be a continuous piecewise linear strongly maximal gDFF with only 2 slope values, then ϕ is extreme.*

Proof. Since ϕ is strongly maximal with 2 slope values, we know one slope value must be 0 by Theorem 4.4. Suppose $\phi = \frac{1}{2}(\phi_1+\phi_2)$, where ϕ_1, ϕ_2 are two maximal gDFFs. From Proposition 2.12, we know $\phi(1) = 1$, which implies $\phi_1(1) = \phi_2(1) = 1$. Let s be the other slope value of ϕ. Due to superadditivity of ϕ, s is the limiting slope of ϕ at 0^- and 0 is the limiting slope of ϕ at 0^+. More precisely, there exist $\epsilon, \delta > 0$ such that $\phi(x) = sx$ for $x \in [-\epsilon, 0]$ and $\phi(x) = 0$ for $x \in [0, \delta]$. We want to show ϕ_1, ϕ_2 have slope 0 where ϕ has slope 0, and ϕ_1, ϕ_2 have slope s where ϕ has slope s.

Case 1: Suppose $[a, b]$ is a closed interval where ϕ has slope value 0. Choose $\delta' = \min(\delta, \frac{b-a}{2}) > 0$. Let $I = [0, \delta']$, $J = [a, b-\delta']$, $K = [a, b]$, then I, J, K are three non-empty and proper intervals. Clearly $\phi(x) + \phi(y) = \phi(x+y)$ for $x \in I, y \in J$. Since ϕ_1, ϕ_2 are also superadditive, they satisfy the equality where ϕ satisfy the equality. In other words, $\phi_i(x) + \phi_i(y) = \phi_i(x+y)$ for $x \in I$, $y \in J$, $i = 1, 2$. By the Interval Lemma, ϕ_1 is affine over $[a, b]$ and $[0, \delta']$ with the same slope value l_1. Similarly, ϕ_2 is affine over $[a, b]$ and $[0, \delta']$ with the same slope value l_2. It is clear that $l_1 = l_2 = 0$ since ϕ_1, ϕ_2 are increasing and $0 = \frac{1}{2}(l_1 + l_2)$.

Case 2: Suppose $[c, d]$ is a closed interval where ϕ has slope value s. Choose $\epsilon' = \min(\epsilon, \frac{d-c}{2})$. Let $I = [-\epsilon', 0]$, $J = [c+\epsilon', d]$, $K = [c, d]$, it is clear that

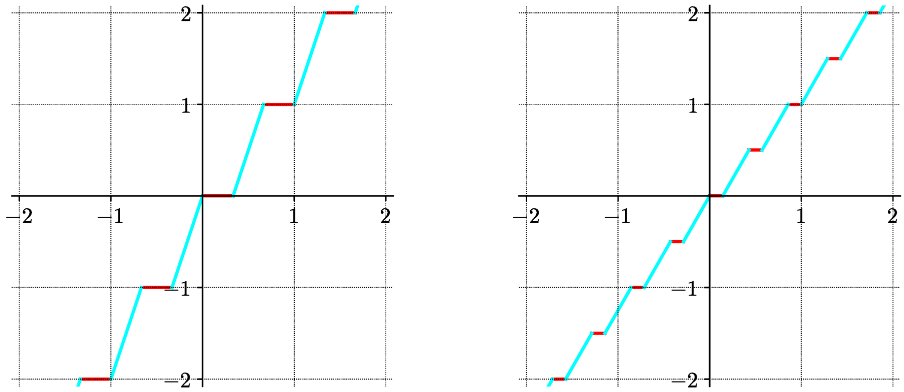

Fig. 2: Graphs of $\phi_{BJ,1}$ [1, Example 3.1] for $C = 3/2$ (left) and $C = 7/3$ (right).

$\phi(x) + \phi(y) = \phi(x + y)$ for $x \in I, y \in J$. Similarly we can prove that ϕ_i is affine over $[c, d]$ and $[-\epsilon', 0]$ with the same slope value s_i $(i = 1, 2)$.

Consider the interval $[0 = a_0, a_1, \ldots, a_n = 1]$, where ϕ has slope 0 over $[a_k, a_{k+1}]$ with k even and slope s over $[a_k, a_{k+1}]$ with k odd. Then ϕ_i have slope 0 over $[a_k, a_{k+1}]$ with k even and slope s_i over $[a_k, a_{k+1}]$ with k odd. Let L_0 and L_s be the total length of intervals where ϕ has slope 0 and s, respectively. Then $s \cdot L_s + 0 \cdot L_0 = 1$. It is possible that ϕ_i has jumps at breakpoints a_k, but it can only jump up since ϕ_i is increasing. Suppose $h_i \geq 0$ are the total jumps of ϕ_i at discontinuous points. From $\phi_i(1) = 1$ we can obtain the following equation:

$$s_i \cdot L_s + 0 \cdot L_0 + h_i = 1 \quad (i = 1, 2).$$

Note that $s = \frac{1}{2}(s_1 + s_2)$ and $s \cdot L_s + 0 \cdot L_0 = 1$. So $s_1 = s_2 = s$ and $h_1 = h_2 = 0$ which implies ϕ_1, ϕ_2 are continuous and $\phi_1 = \phi_2 = \phi$. Thus, ϕ is extreme. □

Remark 6.4 *Alves et al. [1] showed the following functions by Burdet and Johnson with one parameter $C \geq 1$ are maximal gDFFs, where $\{a\}$ represents the fractional part of a:*

$$\phi_{BJ,1}(x; C) = \frac{\lfloor Cx \rfloor + \max(0, \frac{\{Cx\} - \{C\}}{1 - \{C\}})}{\lfloor C \rfloor}.$$

We can prove that they are extreme. If $C \in \mathbb{N}$, then $\phi_{BJ,1}(x) = x$. If $C \notin \mathbb{N}$, $\phi_{BJ,1}$ is a continuous 2-slope maximal gDFF with one slope value 0, therefore it is extreme by Theorem 6.3. Figure 2 shows two examples of $\phi_{BJ,1}$ and they are constructed by the Python function `phi_bj_1_gdff`.

Remark 6.5 *However, the analogous result does not hold for cDFFs. In other words, the restriction $\phi|_{[0,1]}$ is not always extreme as a cDFF even if ϕ is extreme as a gDFF. In fact, $\phi_{BJ,1}(x; C)|_{[0,1]}$ is not extreme as a cDFF for $1 < C < 2$,* *though it is a continuous 2-slope maximal cDFF with one slope value 0. We found an interesting counterexample by computer-based search; it is shown in Figure 3 and Figure 4.*

7 Restricted maximal general DFFs are almost extreme

In the previous section, we have shown that any continuous 2-slope strongly maximal gDFF is extreme. In this section, we prove that extreme gDFFs are dense in the set of continuous restricted maximal gDFFs. Equivalently, for any given continuous restricted maximal gDFF ϕ, there exists an extreme gDFF ϕ_{ext} which approximates ϕ as close as desired (with the infinity norm). The idea of the proof is inspired by the approximation theorem of Gomory–Johnson functions [9]. We first introduce the main theorem in this section. The approximation[1] is implemented for piecewise linear functions with finitely many pieces.

Theorem 7.1 (Approximation Theorem). *Let ϕ be a continuous restricted maximal gDFF, then for any $\epsilon > 0$, there exists an extreme gDFF ϕ_{ext} such that $\|\phi - \phi_{\text{ext}}\|_\infty < \epsilon$.*

Remark 7.2 *The result cannot be extended to maximal gDFF. Note that $\phi(x) = sx$ is maximal but not extreme for $0 < s < 1$. Any non-trivial extreme gDFF ϕ' satisfies $\phi'(1) = 1$, and $\phi'(1) - \phi(1) = 1 - s > 0$ and $1 - s$ is a fixed positive constant. Therefore, $\phi(x) = sx$ cannot be arbitrarily approximated by an extreme gDFF.*

We briefly explain the structure of the proof. First we approximate a continuous restricted maximal gDFF ϕ by a piecewise linear maximal gDFF ϕ_{pwl}. Next, we perturb ϕ_{pwl} such that the new maximal gDFF ϕ_{loose} satisfies $\nabla\phi_{\text{loose}}(x, y) > \gamma > 0$ for "most" $(x, y) \in \mathbb{R}^2$. After applying the 2-slope fill-in procedure to ϕ_{loose}, we get a superadditive 2-slope function $\phi_{\text{fill-in}}$, which is not symmetric anymore. Finally, we symmetrize $\phi_{\text{fill-in}}$ to get the desired ϕ_{ext}.

By studying the superadditivity of maximal gDFFs near the origin, it is not hard to prove Lemma 7.3. By choosing a large enough $q \in \mathbb{N}$ and interpolating the function over $\frac{1}{q}\mathbb{Z}$ we can obtain Lemma 7.4.

Lemma 7.3 *Any continuous restricted maximal gDFF ϕ is uniformly continuous.*

Proof. Since ϕ is continuous at 0 and nondecreasing, for any $\epsilon > 0$, there exists $\delta > 0$ such that $-\delta < t \leq 0$ implies $-\epsilon < \phi(t) \leq 0$. For any x, y with $-\delta < x - y < 0$, by superadditivity we have $0 \geq \phi(x) - \phi(y) \geq \phi(x - y) > -\epsilon$. So ϕ is uniformly continuous. □

Lemma 7.4 *Let ϕ be a continuous restricted maximal gDFF, then for any $\epsilon > 0$, there exists a piecewise linear continuous restricted maximal gDFF ϕ_{pwl}, such that $\|\phi - \phi_{\text{pwl}}\|_\infty < \frac{\epsilon}{3}$.*

[1] See the constructor `two_slope_approximation_gdff_linear`.

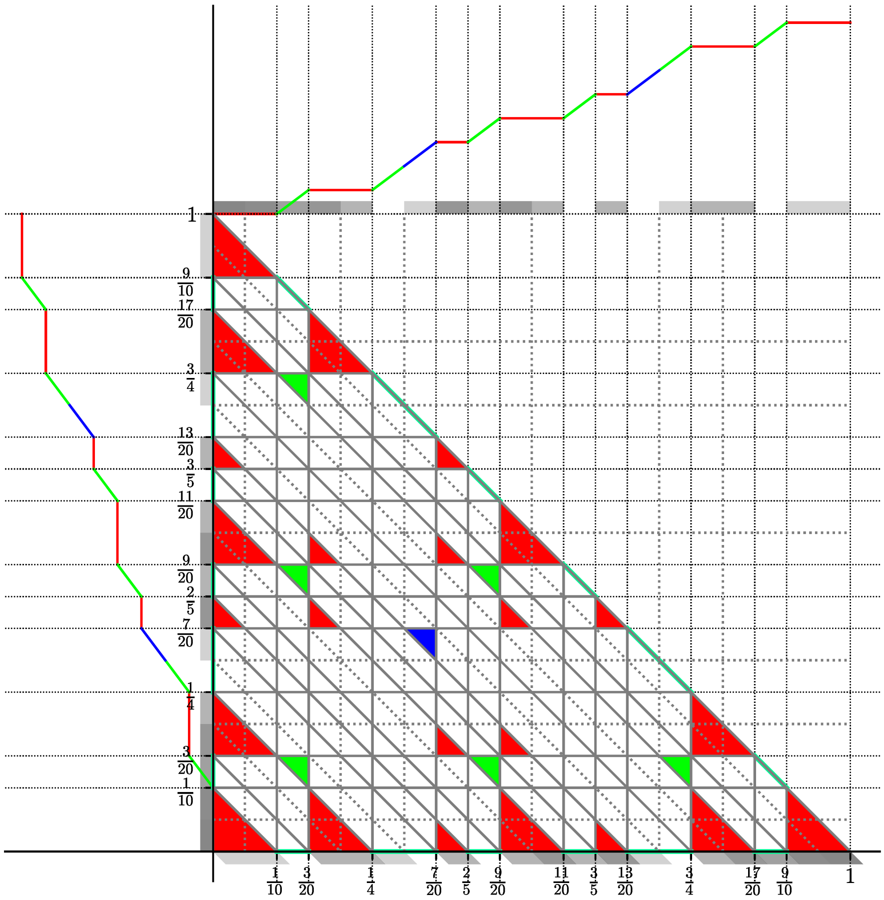

Fig. 3: Continuous 2-slope maximal non-extreme cDFF `w_2slope_3covered_nonextreme` with 3 connected covered components. We use 3 different colors to color additive faces to represent 3 different covered components. The colors on the function are consistent with the colors of additive faces. We plot the function on the left and upper border. The shadows represent covered components from the projections of additive faces in 3 directions.

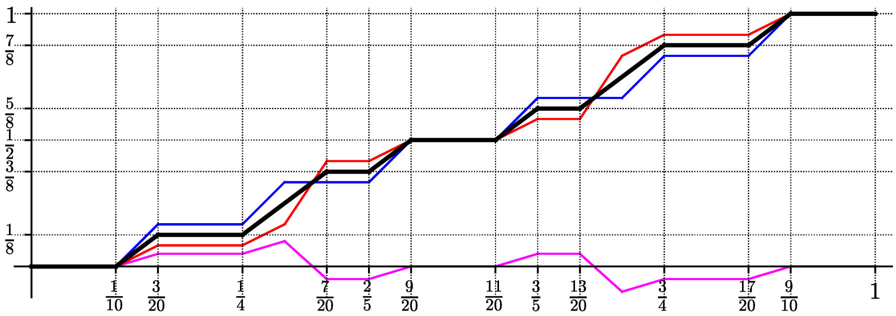

Fig. 4: Continuous 2-slope maximal non-extreme cDFF `w_2slope_3covered_nonextreme` (in black), an effective perturbation function $\tilde{\pi}$ (magenta), and functions $\phi^{\pm} = \phi \pm \epsilon\tilde{\phi}$ (blue and red).

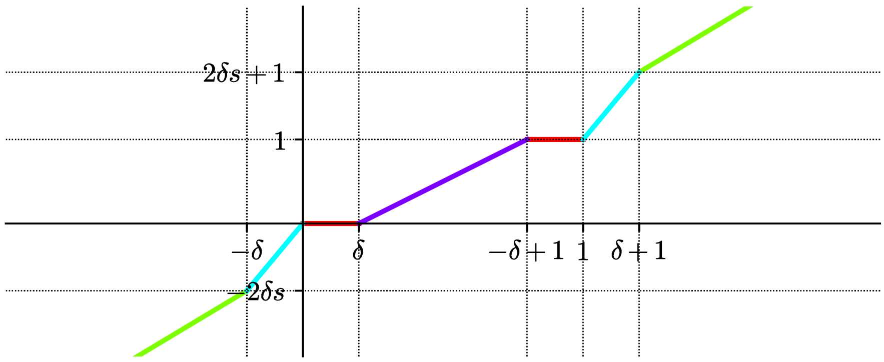

Fig. 5: The graph of $\phi_{s,\delta}$ for $s = 2$ and $\delta = \frac{1}{5}$

Next, we introduce a parametric family of restricted maximal gDFFs $\phi_{s,\delta}$ which will be used to perturb ϕ_{pwl}. Define

$$\phi_{s,\delta}(x) = \begin{cases} sx - s\delta & \text{if } x < -\delta \\ 2sx & \text{if } -\delta \leq x < 0 \\ 0 & \text{if } 0 \leq x < \delta \\ \frac{1}{1-2\delta}x - \frac{\delta}{1-2\delta} & \text{if } \delta \leq x < 1-\delta \\ 1 & 1-\delta \leq x < 1 \\ 2sx - 2s + 1 & 1 \leq x < 1+\delta \\ sx - s + 1 + s\delta & x \geq 1+\delta \end{cases}.$$

The function $\phi_{s,\delta}$ is a continuous piecewise linear function, which has breakpoints: $-\delta, 0, \delta, 1-\delta, 1, 1+\delta$ and slope values: $s, 2s, 0, \frac{1}{1-s\delta}, 0, 2s, s$ in each affine piece. Figure 5 is the graph of one $\phi_{s,\delta}$ function constructed by the Python function `phi_s_delta`.

Let $E_\delta = \{(x, y) \in \mathbb{R}^2 : -\delta < x < \delta \text{ or } -\delta < y < \delta \text{ or } 1-\delta < x+y < 1+\delta\}$.

Lemma 7.5 *The function $\phi_{s,\delta}$ is a continuous restricted maximal gDFF and $\nabla\phi_{s,\delta}(x, y) \geq \delta$ for $(x, y) \notin E_\delta$, if $s > 1$ and $0 < \delta < \min\{\frac{s-1}{2s}, \frac{1}{3}\}$.*

Proof. Verifying the above properties of $\phi_{s,\delta}$ is a routine computation by analyzing the superadditivity slack at every vertex in the two-dimensional polyhedral complex (cf. subsection 3.1) of $\phi_{s,\delta}$. See Appendix A for more details. □

Lemma 7.6 *Let ϕ_{pwl} be a piecewise linear continuous restricted maximal gDFF, then for any $\epsilon > 0$, there exists a piecewise linear continuous restricted maximal gDFF ϕ_{loose} satisfying: (i) $\|\phi_{\text{loose}} - \phi_{\text{pwl}}\|_\infty < \frac{\epsilon}{3}$; (ii) there exist $\delta, \gamma > 0$ such that $\nabla\phi_{\text{loose}}(x, y) \geq \gamma$ for (x, y) not in E_δ.*

Proof. By Proposition 2.11, let $t = \lim_{x\to\infty} \frac{\phi_{\text{pwl}}(x)}{x}$, then $tx - t + 1 \leq \phi_{\text{pwl}}(x) \leq tx$. We can assume $t > 1$, otherwise ϕ_{pwl} is the identity function and the result is trivial. Choose $s = t$ and δ small enough such that $0 < \delta < \min\{\frac{s-1}{2s}, \frac{1}{3}, \frac{1}{q}\}$, where q is the denominator of breakpoints of ϕ_{pwl} in previous lemma. We know that the limiting slope of maximal gDFF $\phi_{t,\delta}$ is also t and $tx - t + 1 \leq \phi_{t,\delta}(x) \leq tx$, which implies $\|\phi_{t,\delta} - \phi_{\text{pwl}}\|_\infty \leq t - 1$.

Define $\phi_{\text{loose}} = (1 - \frac{\epsilon}{3(t-1)})\,\phi_{\text{pwl}} + \frac{\epsilon}{3(t-1)}\,\phi_{t,\delta}$. It is immediate to check ϕ_{loose} is restricted maximal, and $\|\phi_{\text{loose}} - \phi_{\text{pwl}}\|_\infty < \frac{\epsilon}{3}$ is due to $\|\phi_{t,\delta} - \phi_{\text{pwl}}\|_\infty \leq t - 1$. Based on the property of $\phi_{t,\delta}$, $\nabla\phi_{\text{loose}}(x, y) = (1 - \frac{\epsilon}{3(t-1)})\nabla\phi_{\text{pwl}}(x, y) + \frac{\epsilon}{3(t-1)}\nabla\phi_{t,\delta}(x, y) \geq \frac{\epsilon}{3(t-1)}\nabla\phi_{t,\delta}(x, y) \geq \gamma = \frac{\epsilon\delta}{3(t-1)}$ for (x, y) not in E_δ. □

Lemma 7.7 *Given a piecewise linear continuous restricted maximal gDFF ϕ_{loose} satisfying properties in previous lemma, there exists an extreme gDFF ϕ_{ext} such that $\|\phi_{\text{loose}} - \phi_{\text{ext}}\|_\infty < \frac{\epsilon}{3}$.*

Proof. Let $s^+ \geq 0$ be the largest slope of ϕ_{loose} and $\phi_{\text{loose}}(x) = s^+x$ for $x \in [-\delta, 0]$ where δ is chosen from previous lemma. Choose $q' \in \mathbb{N}_+$ such that $\frac{1}{q'}s^+ < \min\{\frac{\epsilon}{3}, \frac{\gamma}{3} = \frac{\epsilon\delta}{9(t-1)}\}$ and the breakpoints of ϕ_{loose} and $\frac{1}{2}$ are contained in $U = \frac{1}{q'}\mathbb{Z}$. Note that we can always choose a rational δ to ensure that the last step is feasible. Define a function $g\colon \mathbb{R} \to \mathbb{R}$ and a 2-slope function $\phi_{\text{fill-in}}\colon [0, 1] \to [0, 1]$:

$$g(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ s^+x & \text{if } x < 0 \end{cases},$$

$$\phi_{\text{fill-in}}(x) = \max_{u\in U}\{\phi_{\text{loose}}(u) + g(x - u)\}.$$

We claim that $\phi_{\text{fill-in}}$ is a continuous 2-slope superadditive function and $\phi_{\text{fill-in}} \leq \phi_{\text{loose}}$, $\phi_{\text{fill-in}}|_U = \phi|_U$. The proof is similar to that of [14, Theorem 3.3]. $|\phi_{\text{fill-in}}(x) - \phi_{\text{loose}}(x)| \leq \frac{1}{q}s^+ < \frac{\epsilon}{3}$ implies that $\|\phi_{\text{loose}} - \phi_{\text{fill-in}}\| < \frac{1}{q'}s^+ < \frac{\epsilon}{3}$. However, $\phi_{\text{fill-in}}$

does not necessarily satisfy the symmetry condition. If we symmetrize it and define the following function:

$$\phi_{\text{ext}}(x) = \begin{cases} \phi_{\text{fill-in}}(x) & \text{if } x \leq \frac{1}{2} \\ 1 - \phi_{\text{fill-in}}(1-x) & \text{if } x > \frac{1}{2} \end{cases}.$$

We claim that ϕ_{ext} is the desired function. It is immediate to check $\phi_{\text{ext}}(0) = 0$, ϕ_{ext} is a 2-slope continuous function and it automatically satisfies the symmetry condition. Since we use slope 0 and s^+ to do the fill-in procedure, the limiting slope of ϕ_{ext} at 0^+ is 0. Notice that $\|\phi_{\text{loose}} - \phi_{\text{ext}}\| = \|\phi_{\text{loose}} - \phi_{\text{fill-in}}\| < \frac{1}{q'}s^+ < \frac{\epsilon}{3}$ because they both satisfy the symmetry condition. So we only need to prove ϕ_{ext} is superadditive.

Case 1: If (x, y) is not in E_δ, $\nabla\phi_{\text{ext}}(x, y) \geq \nabla\phi_{\text{loose}}(x, y) - \frac{\epsilon\delta}{9(t-1)} - \frac{\epsilon\delta}{9(t-1)} - \frac{\epsilon\delta}{9(t-1)} \geq \frac{\epsilon\delta}{3(t-1)} - \frac{\epsilon\delta}{3(t-1)} = 0$.

Case 2: If $0 \leq x \leq \delta$, there are also three sub cases.

(i) If $y, x + y \leq \frac{1}{2}$, then $\nabla\phi_{\text{ext}}(x, y) = \nabla\phi_{\text{fill-in}}(x, y) \geq 0$.

(ii) If $y \leq \frac{1}{2}$ and $x + y > \frac{1}{2}$, then $\nabla\phi_{\text{ext}}(x, y) = 1 - \phi_{\text{fill-in}}(1 - x - y) - \phi_{\text{fill-in}}(x) - \phi_{\text{fill-in}}(y) \geq 1 - \phi_{\text{loose}}(1 - x - y) - \phi_{\text{loose}}(x) - \phi_{\text{loose}}(y) \geq 0$. Here we use the fact that $\phi_{\text{loose}} \geq \phi_{\text{fill-in}}$ and ϕ_{loose} is a maximal gDFF.

(iii) If $y, x + y > \frac{1}{2}$, then $\nabla\phi_{\text{ext}}(x, y) = (1 - \phi_{\text{fill-in}}(1 - x - y)) - \phi_{\text{fill-in}}(x) - (1 - \phi_{\text{fill-in}}(1 - y)) = \phi_{\text{fill-in}}(1 - y) - \phi_{\text{fill-in}}(1 - x - y) - \phi_{\text{fill-in}}(x) \geq 0$ due to superadditivity of $\phi_{\text{fill-in}}$.

Case 3: If $0 > x \geq -\delta$, based on the choice of δ and s^+, we know $\phi_{\text{ext}}(x) = s^+x$ for $0 > x \geq -\delta$. For any $y \in \mathbb{R}$, $\phi_{\text{ext}}(x + y) - \phi_{\text{ext}}(y) \geq s^+x = \phi_{\text{ext}}(x)$ since ϕ_{ext} is a 2-slope function and s^+ is the larger slope.

Similarly we can prove $\nabla\phi_{\text{ext}}(x, y) \geq 0$ if $-\delta \leq y \leq \delta$.

Case 4: If $1-\delta \leq x+y \leq 1+\delta$, let $\beta = 1-x-y$ and $-\delta \leq \beta \leq \delta$, so by case 2 and 3, $\phi_{\text{ext}}(\beta)+\phi_{\text{ext}}(x) \leq \phi_{\text{ext}}(\beta+x)$. Then we have $\phi_{\text{ext}}(x+y) = \phi_{\text{ext}}(1-\beta) = 1 - \phi_{\text{ext}}(\beta) = 1 - \phi_{\text{ext}}(\beta) + \phi_{\text{ext}}(x) - \phi_{\text{ext}}(x) \geq 1 - \phi_{\text{ext}}(\beta + x) + \phi_{\text{ext}}(x) = 1 - \phi_{\text{ext}}(1 - y) + \phi_{\text{ext}}(x) = \phi_{\text{ext}}(y) + \phi_{\text{ext}}(x)$.

We have shown that ϕ_{ext} is superadditive, then it is a continuous 2-slope strongly maximal gDFF. By the Two-Slope Theorem (Theorem 6.3), ϕ_{ext} is extreme. □

Combine the previous lemmas, and we have $\|\phi - \phi_{\text{ext}}\|_\infty \leq \|\phi - \phi_{\text{pwl}}\|_\infty + \|\phi_{\text{pwl}} - \phi_{\text{loose}}\|_\infty + \|\phi_{\text{loose}} - \phi_{\text{ext}}\|_\infty < 3 \times \frac{\epsilon}{3} = \epsilon$. We can conclude the Approximation Theorem. Observe that we always use 0 as one slope value in the fill-in procedure. It is due to the fact (Lemma 6.2) that almost all extreme gDFFs have 0 as the limiting slope at 0^+.

A Appendix: Verification of the properties of $\phi_{s,\delta}$

In this appendix, we provide some details of the proof of Lemma 7.5 regarding the properties of the function $\phi_{s,\delta}$. We explain why it suffices to check the superadditive slack at finitely many vertices in the two-dimensional polyhedral complex.

First we generalize the definition of the two-dimensional polyhedral complex to piecewise linear functions with unbounded domain. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a piecewise linear function with finitely many pieces with breakpoints $x_1 < x_2 < \cdots < x_n$. To express the domains of linearity of $\nabla\phi(x, y)$, and thus domains of additivity and strict superadditivity, we introduce the two-dimensional polyhedral complex $\Delta\mathcal{P}$, similar to the definition in subsection 3.1 for cDFFs. The faces F of the complex are defined as follows. Let $I, J, K \in \mathcal{P}$, so each of I, J, K is either a breakpoint of ϕ or a closed interval delimited by two consecutive breakpoints including $\pm\infty$. Then $F = F(I, J, K) = \{\, (x, y) \in \mathbb{R} \times \mathbb{R} : x \in I,\ y \in J,\ x + y \in K \,\}$. Let $F \in \Delta\mathcal{P}$ and observe that the piecewise linearity of ϕ induces piecewise linearity of $\nabla\phi$.

Lemma A.1 *Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a continuous piecewise linear function with finitely many pieces with breakpoints $x_1 < x_2 < \cdots < 0 < \cdots < x_n$ and ϕ has the same slope s on $(-\infty, x_1]$ and $[x_n, \infty)$. Consider a one-dimensional unbounded face F where one of I, J, K is a finite breakpoint and the other two are unbounded closed intervals, $(-\infty, x_1]$ or $[x_n, \infty)$. Then $\nabla\phi(x, y)$ is a constant along the face F.*

Proof. We only provide the proof for one case, the proofs for other cases are similar.

Suppose $I = \{x_i\}$, $J = K = [x_n, \infty)$. The vertex of F is $(x, y) = (x_i, x_n)$ if $x_i \geq 0$ and $(x, y) = (x_i, x_n - x_i)$ if $x_i < 0$. If $x_i \geq 0$, we claim that $\nabla\phi(x, y) = \nabla\phi(x_i, x_n)$ for $(x, y) \in F$. We have

$$\nabla\phi(x, y) = \phi(x_i + y) - \phi(x_i) - \phi(y) = \phi(x_i + x_n) - \phi(x_i) - \phi(x_n) = \nabla\phi(x_i, x_n).$$

The second step in the above equation is due to ϕ is affine on $[x_n, \infty)$ and $x_i + x_n, y \geq x_n$.

If $x_i < 0$, we claim that $\nabla\phi(x, y) = \nabla\phi(x_i, x_n - x_i)$ for $(x, y) \in F$. We can deduce that

$$\begin{aligned}
\nabla\phi(x, y) &= \phi(x_i + y) - \phi(x_i) - \phi(y) \\
&= (\phi(x_n) + s(x_i + y - x_n)) - \phi(x_i) - (\phi(x_n - x_i) + s(x_i + y - x_n)) \\
&= \phi(x_n) - \phi(x_i) - \phi(x_n - x_i) = \nabla\phi(x_i, x_n - x_i).
\end{aligned}$$

The second step in the above equation is due to ϕ has slope s on $[x_n, \infty)$ and $x_n - x_i, x_i + y \geq x_n$.

Case 2: Suppose $K = \{x_i\}$, $I = [x_n, \infty)$ and $J = (-\infty, x_1]$. The vertex of F is $(x, y) = (x_n, x_i - x_n)$ if $x_i \leq x_1 + x_n$ and $(x, y) = (x_i - x_1, x_1)$ if $x_i > x_1 + x_n$.

If $x_i \leq x_1 + x_n$, we claim that $\nabla\phi(x, y) = \nabla\phi(x_n, x_i - x_n)$ for $(x, y) \in F$. Similarly we have

$$\begin{aligned}
\nabla\phi(x, y) &= \phi(x_i) - \phi(x) - \phi(x_i - x) \\
&= \phi(x_i) - (\phi(x_n) + s(x - x_n)) - (\phi(x_i - x_n) - s(x - x_n)) \\
&= \phi(x_i) - \phi(x_n) - \phi(x_i - x_n) = \nabla\phi(x_n, x_i - x_n).
\end{aligned}$$

The second step in the above equation is due to ϕ has the same slope s on $(-\infty, x_1]$ and $[x_n, \infty)$ and $x \geq x_n$, $y = x_i - x \leq x_i - x_n \leq x_1$.

If $x_i > x_1 + x_n$, we claim that $\nabla\phi(x, y) = \nabla\phi(x_i - x_1, x_1)$ for $(x, y) \in F$, by the following equation:

$$\begin{aligned}\nabla\phi(x,y) &= \phi(x_i) - \phi(x_i - y) - \phi(y) \\ &= \phi(x_i) - (\phi(x_i - x_1) + s(x_1 - y)) - (\phi(x_1) - s(x_1 - y)) \\ &= \phi(x_i) - \phi(x_i - x_1) - \phi(x_1) = \nabla\phi(x_i - x_1, x_1).\end{aligned}$$

The second step in the above equation is due to ϕ has the same slope s on $(-\infty, x_1]$ and $[x_n, \infty)$ and $y \geq x_1$, $x = x_i - y \geq x_i - x_1 \geq x_n$.

Therefore, $\nabla\phi(x, y)$ is a constant for (x, y) in any fixed one-dimensional unbounded face. □

By using the piecewise linearity of $\nabla\phi$, we can prove the following lemma. Thus, it suffices to check the superadditivity slack at finitely many vertices in the two-dimensional polyhedral complex to prove the desired properties of $\phi_{s,\delta}$, i.e. $\nabla\phi_{s,\delta}(x, y) \geq \delta$ for $(x, y) \notin E_\delta$, if $s > 1$ and $0 < \delta < \min\{\frac{s-1}{2s}, \frac{1}{3}\}$.

Lemma A.2 *Define the two-dimensional polyhedral complex $\Delta\mathcal{P}$ of the function $\phi_{s,\delta}$. If $\nabla\phi_{s,\delta}(x, y) \geq \delta$ for any zero-dimensional face $(x, y) \notin E_\delta$, then $\nabla\phi_{s,\delta}(x, y) \geq \delta$ for $(x, y) \notin E_\delta$.*

Proof. Observe that $\mathbb{R}^2 - E_\delta$ is the union of finite two-dimensional faces. So we only need to show $\nabla\phi_{s,\delta}(x, y) \geq \delta$ for $(x, y) \notin E_\delta$ and (x, y) in some two-dimensional face F.

If F is bounded, then $\nabla\phi_{s,\delta}(x, y) \geq \delta$ since the inequality holds for vertices of F and $\nabla\phi$ is affine over F.

Suppose that F is unbounded and is enclosed by some bounded and some unbounded one-dimensional faces. For those bounded one-dimensional faces, $\nabla\phi_{s,\delta}(x, y) \geq \delta$ holds since the inequality holds for vertices. For any unbounded one-dimensional face F', by Lemma A.1, the $\nabla\phi$ is constant and equals to the value at the vertex of F'. We have showed that $\nabla\phi_{s,\delta}(x, y) \geq \delta$ holds for any (x, y) in the enclosing one-dimensional faces, then the inequality holds for $(x, y) \in F$ due to the piecewise linearity of $\nabla\phi$. □

Remark A.3 *In the software [27], we define a parametric family of functions $\phi_{s,\delta}$ with two variables s and δ. From the definition of $\phi_{s,\delta}$, it is clear that $\phi_{s,\delta}$ satisfies the symmetry condition. Although $\phi_{s,\delta}$ is defined in the unbounded domain $\mathbb{R}$, $\nabla\phi$ only depends on the values at the vertices of $\Delta\mathcal{P}$ which is a bounded and finite set, based on the above lemma. In order to show the superadditivity and $\nabla\phi_{s,\delta}(x, y) \geq \delta$ for $(x, y) \notin E_\delta$, only $\nabla\phi_{s,\delta}$ at all vertices of $\Delta\mathcal{P}$ needs to be checked. The Python function `phi_s_delta_is_superadditive_almost_strict` verifies the claim for given numerical values of s and δ that satisfy the hypotheses of Lemma 7.5. Using the method of parametric metaprogramming introduced in [24], the documentation tests of the Python function `phi_s_delta_check_claim` verify the claim for the full parametric family, providing an automatic proof of Lemma 7.5.*

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