Facets, weak facets, and extreme functions of the Gomory–Johnson infinite group problem*

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Abstract. We investigate three competing notions that generalize the notion of a facet of finite-dimensional polyhedra to the infinite-dimensional Gomory–Johnson model. These notions were known to coincide for continuous piecewise linear functions with rational breakpoints. We show that two of the notions, extreme functions and facets, coincide for the case of continuous piecewise linear functions, removing the hypothesis regarding rational breakpoints. We prove an if-and-only-if version of the Gomory–Johnson Facet Theorem. Finally, we separate the three notions using discontinuous examples.

1 Introduction

1.1 Facets in the finite-dimensional case

Let G be a finite index set. The space $\mathbb{R}^{(G)}$ of real-valued functions $y\colon G \to \mathbb{R}$ is isomorphic to and routinely identified with the Euclidean space $\mathbb{R}^{|G|}$. Let $\mathbb{R}^G$ denote its dual space. It is the space of functions $\alpha\colon G \to \mathbb{R}$, which we consider as linear functionals on $\mathbb{R}^{(G)}$ via the pairing $\langle \alpha, y \rangle = \sum_{r\in G} \alpha(r)y(r)$. Again it is routinely identified with the Euclidean space $\mathbb{R}^{|G|}$, and the dual pairing $\langle \alpha, y \rangle$ is the Euclidean inner product. A (closed, convex) rational polyhedron of $\mathbb{R}^{(G)}$ is the set of $y\colon G \to \mathbb{R}$ satisfying $\langle \alpha_i, y \rangle \geq \alpha_{i,0}$, where $\alpha_i \in \mathbb{Z}^G$ are integer linear

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functionals and $\alpha_{i,0} \in \mathbb{Z}$, for i ranging over another finite index set I. We refer to [22,9] for the standard notions of polyhedral geometry.

Consider an integer linear optimization problem in $\mathbb{R}^{(G)}$, i.e., the problem of minimizing a linear functional $\eta \in \mathbb{R}^G$ over a feasible set $F \subseteq \{\, y\colon G \to \mathbb{Z}_+ \,\} \subset \mathbb{R}^{(G)}_+$, or, equivalently, over the convex hull $R = \operatorname{conv} F \subset \mathbb{R}^{(G)}_+$. A *valid inequality* for R is an inequality of the form $\langle \pi, y\rangle \geq \pi_0$, where $\pi \in \mathbb{R}^G$, which holds for all $y \in R$ (equivalently, for all $y \in F$). If R is closed, it is exactly the set of all y that satisfy all valid inequalities. In the following we will restrict ourselves to the case that $R \subseteq \mathbb{R}^{(G)}_+$ is a polyhedron of blocking type [22, section 9.2], i.e., a polyhedron in $\mathbb{R}^{(G)}_+$ whose recession cone is the positive orthant. Then it suffices to consider normalized valid inequalities $\langle \pi, y\rangle \geq \pi_0$ with $\pi \geq 0$ and $\pi_0 = 1$, together with the trivial inequalities $y(r) \geq 0$.

Let $P(\pi)$ denote the set of functions $y \in F$ for which the inequality $\langle \pi, y\rangle \geq 1$ is tight, i.e., $\langle \pi, y\rangle = 1$. If $P(\pi) \neq \emptyset$, then $\langle \pi, y\rangle \geq 1$ is a *tight valid inequality.* Then R is exactly the set of all $y \geq 0$ that satisfy all tight valid inequalities. A valid inequality $\langle \pi, y\rangle \geq 1$ is called *minimal* if there is no other valid inequality $\langle \pi', y\rangle \geq 1$ where $\pi' \neq \pi$ such that $\pi' \leq \pi$ pointwise. One can show that a minimal valid inequality is tight. A valid inequality $\langle \pi, y\rangle \geq 1$ is called *facet-defining* if

$$\begin{aligned} &\text{for every valid inequality } \langle \pi', y\rangle \geq 1 \text{ such that } P(\pi) \subseteq P(\pi'),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{we have } P(\pi) = P(\pi'), \end{aligned} \tag{wF}$$

or, in other words, if the face induced by $\langle \pi, y\rangle \geq 1$ is maximal [22, section 8.4]. Because R is of blocking type, it has full affine dimension [22, section 9.2]. Hence, there is a unique minimal representation of R by a finite system of linear inequalities (up to reordering them and multiplying them by positive real numbers) which are in bijection with the facets [22, section 8.4]. Because of our normalization, this implies the following two equivalent characterizations of facet-defining inequalities of the form $\langle \pi', y\rangle \geq 1$:

$$\begin{aligned} &\text{for every valid inequality } \langle \pi', y\rangle \geq 1 \text{ such that } P(\pi) \subseteq P(\pi'),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{we have } \pi = \pi', \end{aligned} \tag{F}$$

and

$$\begin{aligned} &\text{if } \langle \pi^1, y\rangle \geq 1 \text{ and } \langle \pi^2, y\rangle \geq 1 \text{ are valid inequalities, and } \pi = \tfrac{1}{2}(\pi^1 + \pi^2)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{then } \pi = \pi^1 = \pi^2. \end{aligned} \tag{E}$$

1.2 Facets in the infinite-dimensional Gomory–Johnson model

It is perhaps not surprising that the three conditions (wF), (F), and (E) are no longer equivalent when R is a general convex set that is not polyhedral, and in particular when we change from the finite-dimensional to the infinite-dimensional setting. In the present paper, however, we consider a particular case of an infinite-dimensional model, in which this question has eluded researchers for a long time.

Let $G = \mathbb{Q}$ or $G = \mathbb{R}$ and let $\mathbb{R}^{(G)}$ now denote the space of finite-support functions $y\colon G \to \mathbb{R}$. The so-called *infinite group problem* was introduced by Gomory and Johnson in their seminal papers [13,14]. Let $F = F_f(G, \mathbb{Z}) \subseteq \mathbb{R}^{(G)}_+$ be the set of all finite-support functions $y\colon G \to \mathbb{Z}_+$ satisfying the equation

$$\sum_{r \in G} r\, y(r) \equiv f \pmod 1 \tag{1}$$

where f is a given element of $G \setminus \mathbb{Z}$. We study its convex hull $R = R_f(G, \mathbb{Z}) \subseteq \mathbb{R}^{(G)}_+$, consisting of the functions $y\colon G \to \mathbb{R}_+$ that can be written as (finite) convex combinations of elements of F, and which are therefore finite-support functions as well.

Valid inequalities for R are of the form $\langle \pi, y \rangle \geq \pi_0$, where π comes from the dual space $\mathbb{R}^G$, which is the space of all real-valued functions (without the finite-support condition). When $G = \mathbb{Q}$, then R is again of "blocking type" (see, for example, [8, section 5]), and so we again may assume $\pi \geq 0$ and $\pi_0 = 1$.

If $G = \mathbb{R}$ (the setting of the present paper), typical pathologies from the analysis of functions of a real variable come into play. By [4, Proposition 2.4], there is an infinite-dimensional subspace $\Pi^* \subset \mathbb{R}^G$ of functions π^* such that the equations $\langle \pi^*, y \rangle = 0$ are valid for R. The functions π^* are constructed using a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$, and each $\pi^* \in \Pi^*$, $\pi^* \neq 0$ has a graph whose topological closure is $\mathbb{R}^2$. Recently, Basu et al. [2, Theorem 3.5] showed that for every valid inequality $\langle \pi, y \rangle \geq \pi_0$ there exists a valid inequality $\langle \pi', y \rangle \geq \pi_0$ with $\pi' \geq 0$ such that $\pi' - \pi \in \Pi^*$. Thus, ignoring trivial inequalities with $\pi_0 \leq 0$, we may once again assume $\pi \geq 0$ and normalize to $\pi_0 = 1$. We call such functions π *valid functions.* In contrast to Gomory and Johnson [13,14], who only considered continuous functions π, this class of functions contains many interesting discontinuous functions such as the Gomory fractional cut.

(Minimal) valid functions π that satisfy the conditions (wF), (F), and (E), are called *weak facets*, *facets*, and *extreme functions*, respectively. The relation of these notions, in particular of facets and extreme functions, has remained unclear in the literature. For example, Basu et al. [1], responding to a claim by Gomory and Johnson in [15], wrote:

> The statement that extreme functions are facets appears to be quite nontrivial to prove, and to the best of our knowledge there is no proof in the literature. We therefore cautiously treat extreme functions and facets as distinct concepts, and leave their equivalence as an open question.

The survey [4, section 2.2] summarizes what was known about the relation of the three notions: Facets form a subset of the intersection of extreme functions and weak facets; see Figure 1. For the family $\mathcal{F}_1$ of continuous piecewise linear functions with rational breakpoints, [4, Proposition 2.8] and [5, Theorem 8.6] proved that (E) $\Leftrightarrow$ (F). Moreover, in this case, (wF) $\Rightarrow$ (F) can be shown easily as another consequence of [5, Theorem 8.6]. Thus (E), (F), (wF) are equivalent when π is a continuous piecewise linear function with rational breakpoints.

1.3 Contribution of this paper

A well known sufficient condition for facetness of a minimal valid function π is the Gomory–Johnson Facet Theorem. In its strong form, due to Basu–Hildebrand–Köppe–Molinaro [7], it reads:

Theorem 1.1 (Facet Theorem, strong form, [7, Lemma 34]; see also [4, Theorem 2.12]). *Suppose for every minimal valid function π', $E(\pi) \subseteq E(\pi')$ implies $\pi' = \pi$. Then π is a facet.*

(Here $E(\pi)$ is the *additivity domain* of π, defined in section 2.) We show (Theorem 4.4 below) that, in fact, **this holds as an "if and only if" statement.** The technique of the proof of this converse is not surprising, but the result is crucial for the remainder of the paper, and it closes a gap in the literature.

As we mentioned above, for the family $\mathcal{F}_1$ of continuous piecewise linear functions with rational breakpoints, Basu et al. [4, Proposition 2.8] showed that the notions of extreme functions and facets coincide. This was a consequence of Basu et al.'s finite oversampling theorem, which connects the extremality of a function $\pi \in \mathcal{F}_1$ to the extremality of its restriction in a finite group problem [3]. We **sharpen this result by removing the hypothesis regarding rational breakpoints.**

Theorem 1.2. *Let $\mathcal{F}_4$ be the family of continuous piecewise linear functions (not necessarily with rational breakpoints). Then*

$$\{\, \pi \in \mathcal{F}_4 : \pi \text{ is extreme} \,\} = \{\, \pi \in \mathcal{F}_4 : \pi \text{ is a facet} \,\}.$$

The proof relies on our new characterization of facets, as well as on a technical development on so-called *effective perturbation functions* in section 3, which is also of independent interest.

Then we investigate the notions of facets and weak facets in the case of discontinuous functions. This appears to be a first in the published literature. All papers that consider discontinuous functions only used the notion of extreme functions.

We give **three discontinuous functions that furnish the separation of the three notions** (section 6): A function ψ that is extreme, but is neither a weak facet nor a facet; a function π that is not an extreme function (nor a facet), but is a weak facet; and a function $\hat{\pi}$ that is extreme and a weak facet but is not a facet; see Figure 1. Two of these three separations are obtained by extending a rather complicated construction from the authors' paper [19]; the proofs are in part computer-assisted.

It remains an open question whether this separation can also be done using continuous (necessarily non–piecewise linear) functions. We discuss this question in the conclusions of the paper, section 7.

2 Minimal valid functions and their perturbations

Following [4], we define possibly discontinuous piecewise linear functions π on $\mathbb{R}$ as follows. Take a collection $\mathcal{P}_1$ of closed proper intervals (*one-dimensional faces*)

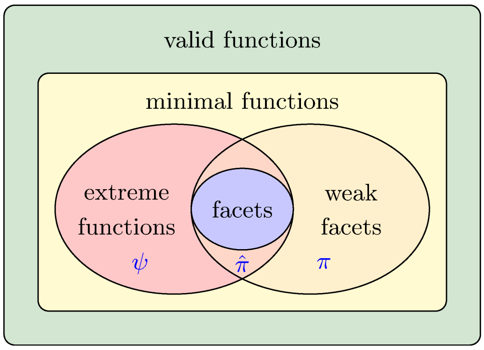

Fig. 1. Separation of the three notions in the discontinuous case

$I \subseteq \mathbb{R}$ such that $\mathbb{R} = \bigcup \mathcal{P}_1$ (*completeness*) and the intersection of any two distinct $I_1, I_2 \in \mathcal{P}_1$ is either empty or a singleton that consists of a common endpoint of I_1 and I_2 (*face-to-face property*). Let $\mathcal{P}_0$ be the set of singletons (*zero-dimensional faces, vertices*) arising as intersections $I_1 \cap I_2$ for $I_1, I_2 \in \mathcal{P}_1$. Define $\mathcal{P} = \{\emptyset\} \cup \mathcal{P}_0 \cup \mathcal{P}_1$, which we refer to as a *polyhedral complex*. We assume that it is *locally finite*, i.e., every compact interval of $\mathbb{R}$ has a nonempty intersection with only finitely many elements of $\mathcal{P}$. We call a function π *piecewise linear* over the complex $\mathcal{P}$ if it is affine linear on the relative interior of each face $I \in \mathcal{P}$. This is a nontrivial condition only for the one-dimensional faces $I = [a, b] \in \mathcal{P}_1$, for which it means that π is affine linear on the open interval (a, b). To express limits, for $x \in I$ we denote

$$\pi_I(x) = \lim_{\substack{u \to x \\ u \in \operatorname{rel\,int}(I)}} \pi(u). \tag{2}$$

We have

$$\pi(x) = \pi_I(x) \quad \text{for all } x \text{ in the relative interior of the face } I \in \mathcal{P}, \tag{3}$$

and thus π_I is the extension of the affine linear function on $\operatorname{rel\,int}(F)$ to the closed face F. When π is continuous, we have

$$\pi(x) = \pi_I(x) \quad \text{for all } x \text{ in the face } I \in \mathcal{P}. \tag{4}$$

Example 2.1. Consider the discontinuous piecewise linear function ψ shown in Figure 2, which will become important in section 6. It was constructed by Hildebrand (2013, unpublished; reported in [4]) and is available in the electronic compendium of extreme functions [20] as `hildebrand_discont_3_slope_1()`. Here $\mathcal{P}_1$ consists of the one-dimensional faces (closed proper intervals) $[0, \frac{1}{8}]$, $[\frac{1}{8}, \frac{3}{8}]$, $[\frac{3}{8}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{5}{8}]$, $[\frac{5}{8}, \frac{7}{8}]$, $[\frac{7}{8}, 1]$, and their translations by integers. $\mathcal{P}_0$ consists of the singletons corresponding to all endpoints of these intervals. For $I = [0, \frac{1}{8}] \in \mathcal{P}_1$, we have the linear function $\pi_I(x) = 6x$ for $x \in I$, and $\pi(x) = \pi_I(x)$ for $x \in \operatorname{rel\,int}(I) = (0, \frac{1}{8})$. For $I = \operatorname{rel\,int}(I) = \{\frac{1}{8}\} \in \mathcal{P}_0$, we have $\pi(\frac{1}{8}) = \pi_I(\frac{1}{8}) = \frac{1}{4}$.

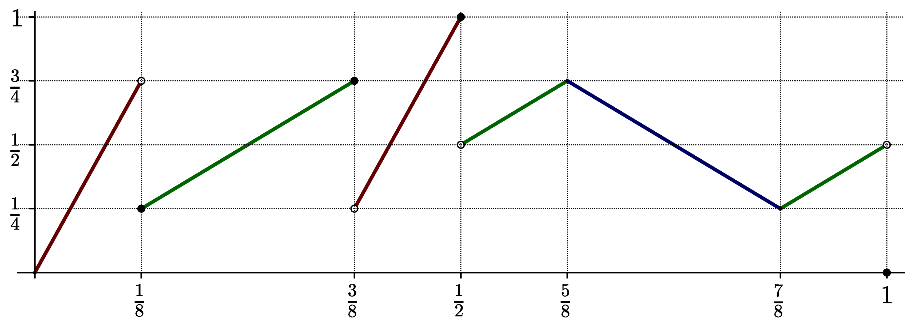

Fig. 2. The piecewise linear function ψ = hildebrand_discont_3_slope_1().

For a function $\pi\colon \mathbb{R} \to \mathbb{R}$, define the *subadditivity slack* of π as $\Delta\pi(x, y) := \pi(x) + \pi(y) - \pi(x + y)$; then π is subadditive if and only if $\Delta\pi(x, y) \geq 0$ for all $x, y \in \mathbb{R}$. Denote the *additivity domain* of π by

$$E(\pi) = \{\, (x, y) \mid \Delta\pi(x, y) = 0 \,\}.$$

By a theorem of Gomory and Johnson [13] (see [4, Theorem 2.6]), the minimal valid functions are exactly the subadditive functions $\pi\colon \mathbb{R} \to \mathbb{R}_+$ that satisfy $\pi(0) = 0$, are periodic modulo 1 and satisfy the *symmetry condition* $\pi(x) + \pi(f - x) = 1$ for all $x \in \mathbb{R}$. As a consequence, minimal valid functions are bounded between 0 and 1.

To combinatorialize the additivity domains of piecewise linear subadditive functions, we work with a two-dimensional polyhedral complex $\Delta\mathcal{P}$. It is defined as the collection of (closed) polyhedra

$$F(I, J, K) = \bigl\{\, (x, y) \in \mathbb{R} \times \mathbb{R} \bigm| x \in I,\ y \in J,\ x + y \in K \,\bigr\}$$

for $I, J, K \in \mathcal{P}$, which we refer to as the *faces* of $\Delta\mathcal{P}$. As I, J, and K can be proper intervals or singletons of $\mathcal{P}$, the nonempty faces F of $\Delta\mathcal{P}$ can be zero-, one-, or two-dimensional. Figure 3 (left) shows $\Delta\mathcal{P}$ corresponding to the function ψ of Example 2.1. Define the projections $p_1, p_2, p_3\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as $p_1(x, y) = x$, $p_2(x, y) = y$, $p_3(x, y) = x + y$.

In the continuous case, since the function π is piecewise linear over $\mathcal{P}$, we have by (4) that $\Delta\pi$ is affine linear over each face $F \in \Delta\mathcal{P}$. Let π be a minimal valid function for $R_f(\mathbb{R}, \mathbb{Z})$ that is piecewise linear over $\mathcal{P}$. Following [4], we define the *space of perturbation functions with prescribed additivities* $E = E(\pi)$

$$\bar{\Pi}^E(\mathbb{R}, \mathbb{Z}) = \left\{ \bar{\pi}\colon \mathbb{R} \to \mathbb{R} \;\middle|\; \begin{array}{rl} \bar{\pi}(0) = 0 & \\ \bar{\pi}(f) = 0 & \\ \Delta\bar{\pi}(x, y) = 0 & \text{for all } (x, y) \in E \\ \bar{\pi}(x + z) = \bar{\pi}(x) & \text{for all } x \in \mathbb{R},\ z \in \mathbb{Z} \end{array} \right\}. \qquad (5)$$

When π is discontinuous, one also needs to consider the limit points where the subadditivity slacks are approaching zero at the relative boundary of a face.

Let F be a face of $\Delta\mathcal{P}$. For $(x,y) \in F$, we denote

$$\Delta\pi_F(x,y) := \lim_{\substack{(u,v)\to(x,y)\\ (u,v)\in \operatorname{rel\,int}(F)}} \Delta\pi(u,v). \tag{6}$$

(For $(x,y) \in \operatorname{rel\,int}(F)$, we have $\Delta\pi_F(x,y) = \Delta\pi(x,y)$. In particular, for zero-dimensional faces $F = \{(x,y)\}$, we have $\operatorname{rel\,int}(F) = \{(x,y)\}$, so the only sequence considered in the limit is the constant sequence (x,y), and thus the limit is just the value $\Delta\pi(x,y)$.) Define

$$E_F(\pi) = \{\, (x,y) \in F \mid \Delta\pi_F(x,y) \text{ exists, and } \Delta\pi_F(x,y) = 0 \,\}.$$

Notice that in the above definition of $E_F(\pi)$, we include the condition that the limit denoted by $\Delta\pi_F(x,y)$ exists, so that this definition can as well be applied to functions π (and $\bar\pi$) that are not piecewise linear over $\mathcal{P}$.

We denote by $E_\bullet(\pi,\mathcal{P})$ the family of sets $E_F(\pi)$, indexed by $F \in \Delta\mathcal{P}$. Define the *space of perturbation functions with prescribed additivities and limit-additivities* $E_\bullet = E_\bullet(\pi,\mathcal{P})$

$$\bar\Pi^{E_\bullet}(\mathbb{R},\mathbb{Z}) = \left\{ \bar\pi\colon \mathbb{R}\to\mathbb{R} \;\middle|\; \begin{array}{rl} \bar\pi(0) = 0 & \\ \bar\pi(f) = 0 & \\ \Delta\bar\pi_F(x,y) = 0 & \text{for } (x,y)\in E_F(\pi),\ F\in\Delta\mathcal{P} \\ \bar\pi(x+z) = \bar\pi(x) & \text{for } x\in\mathbb{R},\ z\in\mathbb{Z} \end{array} \right\}. \tag{7}$$

Remark 2.2. Let $\bar\pi \in \bar\Pi^E(\mathbb{R},\mathbb{Z})$. The third condition of (5) is equivalent to $E(\pi) \subseteq E(\bar\pi)$. Let $\bar\pi \in \bar\Pi^{E_\bullet}(\mathbb{R},\mathbb{Z})$. The third condition of (7) is equivalent to $E_F(\pi) \subseteq E_F(\bar\pi)$ for all faces $F \in \Delta\mathcal{P}$, which is stronger than $E(\pi) \subseteq E(\bar\pi)$ in (5). Thus, in general, $\bar\Pi^{E_\bullet}(\mathbb{R},\mathbb{Z}) \subseteq \bar\Pi^{E}(\mathbb{R},\mathbb{Z})$. If π is continuous, then $\Delta\pi_F(x,y) = \Delta\pi(x,y)$ for $(x,y) \in F$. Therefore, $E(\pi) \subseteq E(\bar\pi)$ implies that $E_F(\pi) \subseteq E_F(\bar\pi)$ for all faces $F \in \Delta\mathcal{P}$, hence $\bar\Pi^{E_\bullet}(\mathbb{R},\mathbb{Z}) = \bar\Pi^{E}(\mathbb{R},\mathbb{Z})$.

3 Effective perturbation functions

Following [19], we define the vector space

$$\tilde\Pi^\pi(\mathbb{R},\mathbb{Z}) = \left\{ \tilde\pi\colon \mathbb{R}\to\mathbb{R} \mid \exists\, \epsilon > 0 \text{ s.t. } \pi^{\pm} = \pi \pm \epsilon\tilde\pi \text{ are minimal valid} \right\}, \tag{8}$$

whose elements are called *effective perturbation functions* for π. Because of [4, Lemma 2.11 (i)], a function π is extreme if and only if $\tilde\Pi^\pi(\mathbb{R},\mathbb{Z}) = \{0\}$. Note that every function $\tilde\pi \in \tilde\Pi^\pi(\mathbb{R},\mathbb{Z})$ is bounded.

It is clear that if $\tilde\pi \in \tilde\Pi^\pi(\mathbb{R},\mathbb{Z})$, then $\tilde\pi \in \bar\Pi^{E_\bullet}(\mathbb{R},\mathbb{Z})$, where $E_\bullet = E_\bullet(\pi,\mathcal{P})$; see [3, Lemma 2.7] or [19, Lemma 2.1].

The other direction does not hold in general, but requires additional hypotheses. Let $\bar\pi \in \bar\Pi^{E_\bullet}(\mathbb{R},\mathbb{Z})$. In [6, Theorem 3.13] (see also [4, Theorem 3.13]), it is proved that if π and $\bar\pi$ are continuous and $\bar\pi$ is piecewise linear, we have

$\bar{\pi} \in \tilde{\Pi}^{\pi}(\mathbb{R}, \mathbb{Z})$. (Similar arguments also appeared in the earlier literature, for example in the proof of [3, Theorem 3.2].)

We will need a more general version of this result. Consider the following definition. Given a locally finite complete polyhedral complex $\mathcal{P}$ of $\mathbb{R}$, we call a function $\bar{\pi}\colon \mathbb{R} \to \mathbb{R}$ *piecewise Lipschitz continuous* over $\mathcal{P}$, if it is Lipschitz continuous over the relative interior of each face of the complex. Under this definition, piecewise Lipschitz continuous functions can be discontinuous at the relative boundaries of the faces.

Theorem 3.1. *Let $\pi\colon \mathbb{R} \to \mathbb{R}$ be a minimal valid function that is piecewise linear over a locally finite polyhedral complex $\mathcal{P}$. Let $\bar{\pi} \in \bar{\Pi}^{E_\bullet}(\mathbb{R}, \mathbb{Z})$ be a perturbation function, where $E_\bullet = E_\bullet(\pi, \mathcal{P})$. Suppose that $\bar{\pi}$ is piecewise Lipschitz continuous over $\mathcal{P}$. Then $\bar{\pi}$ is an effective perturbation function, $\bar{\pi} \in \tilde{\Pi}^{\pi}(\mathbb{R}, \mathbb{Z})$.*

Proof. Let

$$\begin{aligned} m := \min\{\, \Delta\pi_F(x, y) \mid (x, y) \in \operatorname{vert}(\Delta\mathcal{P}),\ &F \text{ is a face of } \Delta\mathcal{P} \\ &\text{such that } (x, y) \in F \text{ and } \Delta\pi_F(x, y) \neq 0 \,\}; \end{aligned}$$

Because π is minimal, it is periodic modulo 1; thus $\bar{\pi} \in \bar{\Pi}^{E_\bullet}(\mathbb{R}, \mathbb{Z})$ implies that $\bar{\pi}$ is also periodic modulo 1. Because $\mathcal{P}$ is locally finite, only finitely many faces of it have a nonempty intersection with $[0, 1]$. Take a positive number C that is greater than the Lipschitz constant of $\bar{\pi}$ on the relative interior of each of these finitely many faces. Then because of periodicity, C is larger than the Lipschitz constant on all faces of $\mathcal{P}$. Moreover, because π is piecewise linear over $\mathcal{P}$, periodic, and nonconstant (as $\pi(0) = 0$ and $\pi(f) = 1$), all faces of $\mathcal{P}$ are bounded. Hence $\bar{\pi}$ is bounded, and therefore

$$M := \sup_{(x,y)\in\mathbb{R}^2} |\Delta\bar{\pi}(x, y)|$$

is finite. If $M = 0$, then π is additive; because it is also piecewise Lipschitz continuous and periodic, it follows that $\bar{\pi} \equiv 0$, and thus $\bar{\pi} \in \tilde{\Pi}^{\pi}(\mathbb{R}, \mathbb{Z})$ holds trivially. In the following, we assume $M > 0$. Define $\epsilon := \min\left\{\frac{m}{M}, \frac{m}{8C}\right\}$. We also have $m > 0$, since π is subadditive and $\Delta\pi$ is non-zero somewhere. Thus, $\epsilon > 0$. Let $\pi^+ = \pi + \epsilon\bar{\pi}$ and $\pi^- = \pi - \epsilon\bar{\pi}$, which we collectively refer to as $\pi^\pm$. We want to show that $\pi^\pm$ are minimal valid.

We claim that π^+ and π^- are subadditive functions. Let $(x, y) \in [0, 1]^2$. Let F be a face of $\Delta\mathcal{P}$ such that $(x, y) \in F$. We denote the limit (6) of $\pi^\pm$ by $\Delta\pi^\pm_F(x, y)$; we will show that it is nonnegative. First, assume $\Delta\pi_F(x, y) = 0$. It follows from $E_F(\pi) \subseteq E_F(\bar{\pi})$ that $\Delta\bar{\pi}_F(x, y) = 0$. Therefore, $\Delta\pi^\pm_F(x, y) = 0$. Next, assume $\Delta\pi_F(x, y) \neq 0$. Consider $S = \{\, (u, v) \in F \mid \Delta\pi_F(u, v) = 0 \,\}$, which is a closed set since $\Delta\pi_F$ is continuous over the face F.

If $S = \emptyset$, then $\Delta\pi_F(u, v) \geq m$ for any $(u, v) \in \operatorname{vert}(F)$. We have $\Delta\pi_F(x, y) \geq m$ by the fact that $\Delta\pi_F$ is affine over F. Hence, in this case,

$$\begin{aligned} \Delta\pi^\pm_F(x, y) &= \Delta\pi_F(x, y) \pm \epsilon\Delta\bar{\pi}_F(x, y) \\ &\geq \Delta\pi_F(x, y) - \epsilon\,|\Delta\bar{\pi}_F(x, y)| \geq m - \frac{m}{M}M \geq 0. \end{aligned}$$

Now consider the case $S \neq \emptyset$. Let d denote the Euclidean distance from (x, y) to S. Since S is a closed set, there exists a point $(x', y') \in S$ such that $(x - x')^2 + (y - y')^2 = d^2$. Let $I = p_1(F), J = p_2(F)$ and $K = p_3(F)$. Then $x, x' \in I,\ y, y' \in J$ and $x + y, x' + y' \in K$. It follows from $E_F(\pi) \subseteq E_F(\bar\pi)$ and $\Delta\pi_F(x', y') = 0$ that $\Delta\bar\pi_F(x', y') = 0$. Therefore,

$$\begin{aligned}\Delta\bar\pi_F(x, y) &= \Delta\bar\pi_F(x, y) - \Delta\bar\pi_F(x', y')\\ &= \bar\pi_I(x) - \bar\pi_I(x') + \bar\pi_J(y) - \bar\pi_J(y') + \bar\pi_K(x + y) - \bar\pi_K(x' + y'),\end{aligned}$$

where $\bar\pi_I(x) = \lim_{u\to x, u\in \operatorname{rel\,int}(I)} \bar\pi(u)$ as in (2). Since $\bar\pi$ is Lipschitz continuous over $\operatorname{rel\,int}(I), \operatorname{rel\,int}(J)$ and $\operatorname{rel\,int}(K)$, we have that

$$\begin{aligned}|\bar\pi_I(x) - \bar\pi_I(x')| &\le C\,|x - x'| \le Cd;\\ |\bar\pi_J(y) - \bar\pi_J(y')| &\le C\,|y - y'| \le Cd;\\ |\bar\pi_K(x + y) - \bar\pi_K(x' + y')| &\le C\,|x + y - x' - y'| \le 2Cd.\end{aligned}$$

Hence $|\Delta\bar\pi_F(x, y)| \le 4Cd$. Applying a geometric estimate (Lemma A.1 in Appendix A with $g = \Delta\pi_F$) shows that $\Delta\pi_F(x, y) \ge \frac{md}{2}$. Therefore, in the case where $S \neq \emptyset$,

$$\begin{aligned}\Delta\pi^{\pm}_F(x, y) &= \Delta\pi_F(x, y) \pm \epsilon\Delta\bar\pi_F(x, y)\\ &\ge \Delta\pi_F(x, y) - \epsilon\,|\Delta\bar\pi_F(x, y)| \ge \frac{md}{2} - \frac{m}{8C}(4Cd) = 0.\end{aligned}$$

We showed that $\pi^{\pm}$ are subadditive. Since $\bar\pi \in \bar\Pi^E(\mathbb{R}, \mathbb{Z})$, we have $\pi^{\pm}(0) = \pi(0) = 0$ and $\pi^{\pm}(f) = \pi(f) = 1$. The last result along with $E(\pi) \subseteq E(\bar\pi)$ imply that $\pi^+(x) + \pi^+(y) = \pi^-(x) + \pi^-(y) = 1$ if $x + y \equiv f \pmod 1$. The functions $\pi^{\pm}$ are non-negative. Indeed, suppose that $\pi^+(x) < 0$ for some $x \in \mathbb{R}$, then it follows from the subadditivity that $\pi^+(nx) \le n\pi^+(x)$ for any $n \in \mathbb{Z}_+$, which is a contradiction to the boundedness of π^+.

Thus, $\pi^{\pm}$ are minimal valid functions. We conclude that $\bar\pi \in \tilde\Pi^{\pi}(\mathbb{R}, \mathbb{Z})$. □

4 Extreme functions and facets

In this section, we discuss the relations between the notions of extreme functions and facets. We first review the definition of a facet, following [4, section 2.2.3]; cf. ibid. for a discussion of this notion in the earlier literature, in particular [15] and [11].

Let $P(\pi)$ denote the set of functions $y\colon \mathbb{R} \to \mathbb{Z}_+$ with finite support satisfying

$$\sum_{r\in\mathbb{R}} r\,y(r) \equiv f \pmod 1 \quad\text{and}\quad \sum_{r\in\mathbb{R}} \pi(r)y(r) = 1.$$

A valid function π is called a *facet* if for every valid function π' such that $P(\pi) \subseteq P(\pi')$ we have that $\pi' = \pi$. Equivalently, a valid function π is a facet if this condition holds for all such *minimal* valid functions π' [7].

Remark 4.1. In our paper we investigate the notions of facets (and weak facets) in particular for the case of discontinuous functions. This appears to be a first in the published literature. All papers that consider discontinuous functions only used the notion of extreme functions. In particular, Dey–Richard–Li–Miller [12], who were the first to consider previously known discontinuous functions as first-class members of the Gomory–Johnson hierarchy of valid functions, use extreme functions exclusively; whereas [11], which was completed by a subset of the authors in the same year, uses (weak) facets exclusively. The same is true in Dey's Ph.D. thesis [10]: The notion of extreme functions is used in chapters regarding discontinuous functions; whereas the notion of facets is used when talking about (2-row) continuous functions. Dey (2016, personal communication) remembers that at that time, he and his coauthors were aware that facets were the strongest notion and they would strive to establish facetness of valid functions whenever possible. However, in the excellent survey [21], facets are no longer mentioned and the exposition is in terms of extreme functions.

Remark 4.2. In the discontinuous case, the additivity in the limit plays a role in extreme functions, which are characterized by the non-existence of an effective perturbation function $\tilde{\pi} \not\equiv 0$. However facets (and weak facets, see the next section) are defined through $P(\pi)$, which does not capture the limiting additive behavior of π. The additivity domain $E(\pi)$, which appears in the Facet Theorem as discussed below, also does not account for additivity in the limit.

A well known sufficient condition for facetness of a minimal valid function π is the Gomory–Johnson Facet Theorem. We have stated its strong form, due to Basu–Hildebrand–Köppe–Molinaro [7], in the introduction as Theorem 1.1. In order to prove our "if and only if" version, we need the following lemma.

Lemma 4.3. *Let π and π' be minimal valid functions. Then $E(\pi) \subseteq E(\pi')$ if and only if $P(\pi) \subseteq P(\pi')$.*

Proof. The "if" direction is proven in [7, Theorem 20]; see also [4, Theorem 2.12]. We now show the "only if" direction, using the subadditivity of π. Assume that $E(\pi) \subseteq E(\pi')$. Let $y \in P(\pi)$. Let $\{r_1, r_2, \dots, r_n\}$ denote the finite support of y. By definition, the function y satisfies that $y(r_i) \in \mathbb{Z}_+$, $\sum_{i=1}^n r_i y(r_i) \equiv f \pmod 1$, and $\sum_{i=1}^n \pi(r_i) y(r_i) = 1$. Since π is a minimal valid function, we have that $1 = \sum_{i=1}^n \pi(r_i) y(r_i) \geq \pi\big(\sum_{i=1}^n r_i y(r_i)\big) = \pi(f) = 1$. Thus, each subadditivity inequality here is tight for π, and is also tight for π' since $E(\pi) \subseteq E(\pi')$. We obtain $\sum_{i=1}^n \pi'(r_i) y(r_i) = \pi'\big(\sum_{i=1}^n r_i y(r_i)\big) = \pi'(f) = 1$, which implies that $y \in P(\pi')$. Therefore, $P(\pi) \subseteq P(\pi')$. □

Theorem 4.4 (Facet Theorem, "if and only if" version). *A minimal valid function π is a facet if and only if for every minimal valid function π', $E(\pi) \subseteq E(\pi')$ implies $\pi' = \pi$.*

Proof. It follows from the Facet Theorem in the strong form (Theorem 1.1) and Lemma 4.3. □

Recall the space $\bar{\Pi}^E(\mathbb{R},\mathbb{Z})$ of perturbation functions with prescribed additivities $E = E(\pi)$ from section 2. In [4, page 25, section 3.6], the Facet Theorem is reformulated in terms of perturbation functions as follows:

> If π is not a facet, then there exists a non-zero $\bar{\pi} \in \bar{\Pi}^{E(\pi)}(\mathbb{R},\mathbb{Z})$ such that $\pi' = \pi + \bar{\pi}$ is a minimal valid function.

The authors of [4] caution that this last statement is not an "if and only if" statement. We now prove that actually the following "if and only if" version holds.

Lemma 4.5. *A minimal valid function π is a facet if and only if there is no non-zero $\bar{\pi} \in \bar{\Pi}^E(\mathbb{R},\mathbb{Z})$, where $E = E(\pi)$, such that $\pi + \bar{\pi}$ is minimal valid.*

Proof. Let π be a minimal valid function.

Assume that π is a facet. Let $\bar{\pi} \in \bar{\Pi}^E(\mathbb{R},\mathbb{Z})$ where $E = E(\pi)$ such that $\pi' = \pi + \bar{\pi}$ is minimal valid. It is clear that $E(\pi) \subseteq E(\pi')$. By Theorem 4.4, $\pi' = \pi$. Thus, $\bar{\pi} \equiv 0$.

Assume there is no non-zero $\bar{\pi} \in \bar{\Pi}^E(\mathbb{R},\mathbb{Z})$, where $E = E(\pi)$, such that $\pi + \bar{\pi}$ is minimal valid. Let π' be a minimal valid function such that $E(\pi) \subseteq E(\pi')$. Consider $\bar{\pi} = \pi' - \pi$. We have that $\bar{\pi} \in \bar{\Pi}^E(\mathbb{R},\mathbb{Z})$ and that $\pi + \bar{\pi} = \pi'$ is minimal valid. Then $\bar{\pi} \equiv 0$ by the assumption. Hence, $\pi' = \pi$. It follows from Theorem 4.4 that π is a facet. □

We will not use this lemma in the following.

Now we come to the proof of a main theorem stated in the introduction.

Proof (of Theorem 1.2). Let π be a continuous piecewise linear minimal valid function. As mentioned in [4, section 2.2.4], [7, Lemma 1.3] showed that if π is a facet, then π is extreme.

We now prove the other direction by contradiction. Suppose that π is extreme, but is not a facet. Then by Theorem 4.4, there exists a minimal valid function $\pi' \neq \pi$ such that $E(\pi) \subseteq E(\pi')$. Since π is continuous piecewise linear and $\pi(0) = \pi(1) = 0$, there exists $\delta > 0$ such that $\Delta\pi(x,y) = 0$ and $\Delta\pi(-x,-y) = 0$ for $0 \leq x, y \leq \delta$. The condition $E(\pi) \subseteq E(\pi')$ implies that $\Delta\pi'(x,y) = 0$ and $\Delta\pi'(-x,-y) = 0$ for $0 \leq x, y \leq \delta$ as well. As the function π' is bounded, it follows from the Interval Lemma (see [4, Lemma 4.1], for example) that π' is affine linear on $[0,\delta]$ and on $[-\delta,0]$. We also know that $\pi'(0) = 0$ as π' is minimal valid. Using the subadditivity, we obtain that π' is Lipschitz continuous.

Let $\bar{\pi} = \pi' - \pi$. Then $\bar{\pi} \not\equiv 0$, $\bar{\pi} \in \bar{\Pi}^E(\mathbb{R},\mathbb{Z})$ where $E = E(\pi)$, and $\bar{\pi}$ is Lipschitz continuous. Since π is continuous, we have $\bar{\Pi}^E(\mathbb{R},\mathbb{Z}) = \bar{\Pi}^{E_\bullet}(\mathbb{R},\mathbb{Z})$ by Remark 2.2. By Theorem 3.1, there exists $\epsilon > 0$ such that $\pi^{\pm} = \pi \pm \epsilon\bar{\pi}$ are distinct minimal valid functions. This contradicts the assumption that π is an extreme function.

Thus, the equality {extreme functions in $\mathcal{F}_4$} = {facets in $\mathcal{F}_4$} is proved. □

5 Weak facets

We first review the definition of a weak facet, following [4, section 2.2.3]; cf. ibid. for a discussion of this notion in the earlier literature, in particular [15] and [11]. A valid function π is called a *weak facet* if for every valid function π' such that $P(\pi) \subseteq P(\pi')$ we have $P(\pi) = P(\pi')$.

As we mentioned above, to prove that π is a facet, it suffices to consider π' that is minimal valid. The following lemma shows it is also the case for weak facets.

Lemma 5.1. (1) *Let π be a valid function. If π is a weak facet, then π is minimal valid.*
(2) *Let π be a minimal valid function. Suppose that for every minimal valid function π', we have that $P(\pi) \subseteq P(\pi')$ implies $P(\pi) = P(\pi')$. Then π is a weak facet.*
(3) *A minimal valid function π is a weak facet if and only if for every minimal valid function π', we have that $E(\pi) \subseteq E(\pi')$ implies $E(\pi) = E(\pi')$.*

Proof. (1) Suppose that π is not minimal valid. Then, by [7, Theorem 1], π is dominated by another minimal valid function π', with $\pi(x_0) > \pi'(x_0)$ at some x_0. Let $y \in P(\pi)$. We have

$$1 = \sum_{r\in\mathbb{R}} \pi(r)y(r) \geq \sum_{r\in\mathbb{R}} \pi'(r)y(r) \geq \pi'\Bigl(\sum_{r\in\mathbb{R}} r\, y(r)\Bigr) = \pi'(f) = 1.$$

Hence equality holds throughout, implying that $y \in P(\pi')$. Therefore, $P(\pi) \subseteq P(\pi')$. Now consider y with $y(x_0) = y(f - x_0) = 1$ and $y(x) = 0$ otherwise. It is easy to see that $y \in P(\pi')$, but $y \notin P(\pi)$ since $\pi(x_0) + \pi(f - x_0) > \pi'(x_0) + \pi'(f - x_0) = 1$. Therefore, $P(\pi) \subsetneq P(\pi')$, a contradiction to the weak facet assumption on π.

(2) Consider any valid function π^* (not necessarily minimal) such that $P(\pi) \subseteq P(\pi^*)$. Let π' be a minimal function that dominates π^*: $\pi' \leq \pi^*$. From the proof of (1) we know that $P(\pi^*) \subseteq P(\pi')$. Thus, $P(\pi) \subseteq P(\pi')$. By hypothesis, we have that $P(\pi) = P(\pi^*) = P(\pi')$. Therefore, π is a weak facet.

(3) Direct consequence of (2) and Lemma 4.3. □

By Theorem 1.2, for continuous piecewise linear functions, the notions of extreme functions and facets are the same. Next we discuss the relation to weak facets. We have the following theorem.

Theorem 5.2. *Let $\mathcal{F}$ be a subfamily of the family $\mathcal{F}_4$ of continuous piecewise linear functions such that*

> *existence of an effective perturbation for any minimal valid $\pi \in \mathcal{F}$*
> *implies existence of a piecewise linear effective perturbation.*

Let $\pi \in \mathcal{F}$. The following are equivalent. (E) *π is extreme,* (F) *π is a facet,* (wF) *π is a weak facet.*

Before proving the theorem, we discuss a hierarchy of known subfamilies that satisfy the hypothesis.

Remark 5.3. As shown in [3] (for a stronger statement, see [5, Theorem 8.6]), the family $\mathcal{F}_1$ of continuous piecewise linear functions with rational breakpoints is such a subfamily where existence of an effective perturbation implies existence of a piecewise linear effective perturbation.

Remark 5.4. Zhou [23, Chapter 4] introduces a completion procedure for deciding the extremality of piecewise linear functions, which is known to terminate for all functions with rational breakpoints and some functions with irrational breakpoints. Let $\mathcal{F}_2 \supset \mathcal{F}_1$ be the family of continuous piecewise linear functions with rational breakpoints for which the procedure terminates. In this case, by [23, Lemma 4.11.3, Theorems 4.11.4, 4.11.6], the space of effective perturbations has a precise description as a direct sum of a finite-dimensional space of continuous piecewise linear functions and finitely many spaces of Lipschitz functions. Because the spaces of Lipschitz functions contain nonzero piecewise linear functions, this implies that $\mathcal{F}_2$ satisfies the hypothesis of Theorem 5.2.

Remark 5.5. Hildebrand–Köppe–Zhou [16,17] consider the family $\mathcal{F}_3 \supseteq \mathcal{F}_2$ of continuous piecewise linear functions that have a *finitely presented moves closure* [16, Assumption 4.2]. For these functions, by [16, Theorems 4.14–4.16], the space of effective perturbations has a direct sum decomposition of the same type as for the family $\mathcal{F}_2$, and again this implies that the family $\mathcal{F}_3$ satisfies the hypothesis of Theorem 5.2.

Open question 5.6 *It is an open question whether the whole family $\mathcal{F}_4 \supseteq \mathcal{F}_3$ of all continuous piecewise linear functions satisfies the hypothesis of Theorem 5.2.*

Proof (of Theorem 5.2). By Theorem 1.2 and the fact that {facets} $\subseteq$ {extreme functions} $\cap$ {weak facets}, it suffices to show that {weak facets} $\subseteq$ {extreme functions}.

Assume that π is a weak facet, thus π is minimal valid by Lemma 5.1. We show that π is extreme. For the sake of contradiction, suppose that π is not extreme. By the assumption $\pi \in \mathcal{F}$, there exists a piecewise linear perturbation function $\bar\pi \not\equiv 0$ such that $\pi \pm \bar\pi$ are minimal valid functions. Furthermore, by [4, Lemma 2.11], we know that $\bar\pi$ is continuous, and $E(\pi) \subseteq E(\bar\pi)$. By taking the union of the breakpoints, we can define a common refinement, which will still be denoted by $\mathcal{P}$, of the complexes for π and for $\bar\pi$. In other words, we may assume that π and $\bar\pi$ are both continuous piecewise linear over $\mathcal{P}$. Since $\Delta\bar\pi \not\equiv 0$, we may assume without loss of generality that $\Delta\bar\pi(x, y) > 0$ for some $(x, y) \in \operatorname{vert}(\Delta\mathcal{P})$. Define

$$\epsilon = \min \left\{ \frac{\Delta\pi(x, y)}{\Delta\bar\pi(x, y)} \;\middle|\; (x, y) \in \operatorname{vert}(\Delta\mathcal{P}),\ \Delta\bar\pi(x, y) > 0 \right\}.$$

Notice that $\epsilon > 0$, since $\Delta\pi \geq 0$ and $E(\pi) \subseteq E(\bar\pi)$. Let $\pi' = \pi - \epsilon\bar\pi$. Then π' is a bounded continuous function piecewise linear over $\mathcal{P}$, such that $\pi' \neq \pi$.

The function π' is subadditive, since $\Delta\pi'(x, y) \geq 0$ for each $(x, y) \in \operatorname{vert}(\Delta\mathcal{P})$. As in the proof of Theorem 3.1, it can be shown that π' is non-negative, $\pi'(0) = 0$, $\pi'(f) = 1$, and that π' satisfies the symmetry condition. Therefore, π' is a minimal valid function. Let (u, v) be a vertex of $\Delta\mathcal{P}$ satisfying $\Delta\bar{\pi}(u, v) > 0$ and $\Delta\pi(u, v) = \epsilon\Delta\bar{\pi}(u, v)$. We know that $\Delta\pi'(u, v) = \Delta\pi(u, v) - \epsilon\Delta\bar{\pi}(u, v) = 0$, hence $(u, v) \in E(\pi')$. However, $(u, v) \notin E(\pi)$, since $\Delta\bar{\pi}(u, v) > 0$ implies that $\Delta\pi(u, v) \neq 0$. Therefore, $E(\pi) \subsetneq E(\pi')$. By Lemma 5.1(3), we have that π is not a weak facet, a contradiction. □

Remark 5.7. The theorem is stated for functions π and $\bar{\pi}$ that are piecewise over the same complex $\mathcal{P}$. This is not a restriction because if we are given two complexes $\mathcal{P}$ and $\bar{\mathcal{P}}$, then we can define a new complex, the *common refinement* of $\mathcal{P}$ and $\bar{\mathcal{P}}$, whose set of vertices is the union of those of $\mathcal{P}$ and $\bar{\mathcal{P}}$.

6 Separation of the notions in the discontinuous case

6.1 Extreme, but not a weak facet

The definitions of facets and weak facets fail to account for additivities-in-the-limit, which are a crucial feature of the extremality test for discontinuous functions. This allows us to separate the notion of extreme functions from the other two notions. Below we do this by observing that the discontinuous piecewise linear extreme function $\psi =$ hildebrand_discont_3_slope_1(), which appeared above in Example 2.1, works as a separating example.

Theorem 6.1. *The function* $\psi =$ hildebrand_discont_3_slope_1() *is a one-sided discontinuous piecewise linear function with rational breakpoints that is extreme, but is neither a weak facet nor a facet.*

Proof. The function $\psi =$ hildebrand_discont_3_slope_1() is extreme (Hildebrand, 2013, unpublished, reported in [4]). The extremality proof appears as [18, Example 7.2]; it can also be verified using the software [20].[3] The function ψ is piecewise linear on a complex $\mathcal{P}$, which is illustrated in Figure 3 (left). Consider the minimal valid function $\psi' =$ discontinuous_facets_paper_example_psi_prime() defined by

$$\psi'(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}]; \\ \psi(x) & \text{if } x \in (\frac{1}{2}, 1). \end{cases}$$

It can be considered as piecewise linear on the same complex $\mathcal{P}$. Observe that $E(\psi)$ is a strict subset of $E(\psi')$. See Figure 3 for an illustration of this inclusion. Thus, by Lemma 5.1(3), the function ψ is not a weak facet (nor a facet). □

[3] The command h = hildebrand_discont_3_slope_1(); extremality_test(h) carries out the verification.

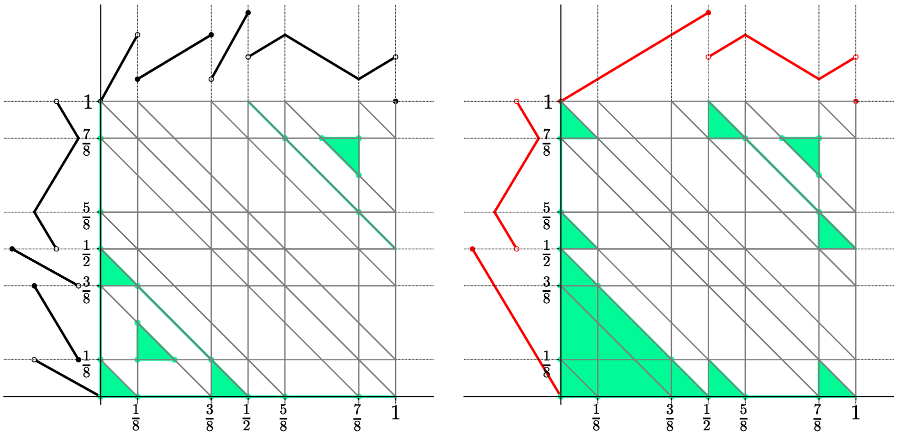

Fig. 3. Two diagrams of functions h (*graphs on the top and the left*) and polyhedral complexes $\Delta\mathcal{P}$ (*gray solid lines*) with additive domains $E(\mathsf{h})$ (*shaded in green*). (*Left, black graph*) $\mathsf{h} = \mathsf{hildebrand\_discont\_3\_slope\_1}() = \psi$. (*Right, red graph*) $\mathsf{h} = \psi'$ from the proof of Theorem 6.1.

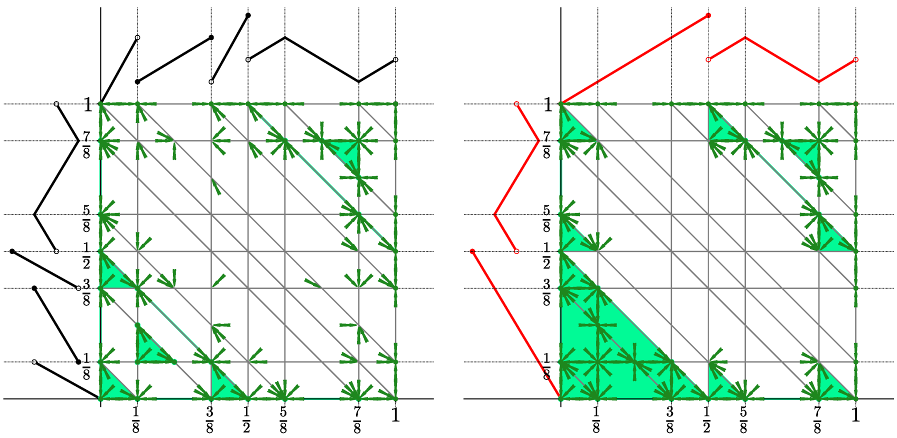

Fig. 4. Two diagrams of functions h (*graphs on the top and the left borders*) and polyhedral complexes $\Delta\mathcal{P}$ (*gray solid lines*) with additive domains $E(\mathsf{h})$ (*green triangles*) and $E_\bullet(\mathsf{h}, \mathcal{P})$ (*green arrows*). (*Left*) $\mathsf{h} = \mathsf{hildebrand\_discont\_3\_slope\_1}() = \psi$. (*Right*) $\mathsf{h} = \mathsf{discontinuous\_facets\_paper\_example\_psi\_prime}() = \psi'$ from the proof of Theorem 6.1.

We remark that there is no inclusion relation between the limit-additivities captured in the set families $E_\bullet(\psi, \mathcal{P})$ and $E_\bullet(\psi', \mathcal{P})$, as illustrated in the diagrams in Figure 4. We will explain these diagrams on an example only; see [18], where these types of diagrams for discontinuous piecewise linear functions were introduced, for a full discussion.[4] Consider $(\frac{3}{8}, \frac{3}{8})$ as a vertex of the face $F \in \Delta\mathcal{P}$ that is the triangle to the northeast of it. The limit of $\Delta\psi$ within $\operatorname{rel\,int}(F)$ to $(\frac{3}{8}, \frac{3}{8})$ is $\lim_{(x,y)\to(\frac{3}{8}^+,\frac{3}{8}^+)} \Delta\psi(x, y) = 0$, thus we have additivity in the limit. This is indicated by the green arrow from the northeast of $(\frac{3}{8}, \frac{3}{8})$. But the corresponding limit of $\Delta\psi'$ is $\lim_{(x,y)\to(\frac{3}{8}^+,\frac{3}{8}^+)} \Delta\psi'(x, y) > 0$. As a result, the perturbation $\bar\psi = \psi' - \psi$ is not an effective perturbation for ψ: For any $\epsilon > 0$, the function $\psi - \epsilon\bar\psi$ violates subadditivity near $(\frac{3}{8}, \frac{3}{8})$. In fact, $\bar\psi$ does not belong to the space $\bar\Pi^{E_\bullet}(\mathbb{R}, \mathbb{Z})$ with $E_\bullet = E_\bullet(\psi, \mathcal{P})$.

6.2 Weak facet, but not extreme

The other separations appear to require more complicated constructions. Recently in [19], the authors constructed a two-sided discontinuous piecewise linear minimal valid function, $\pi =$ kzh_minimal_has_only_crazy_perturbation_1(), which is not extreme, but which is not a convex combination of other piecewise linear minimal valid functions; see Table 1 in Appendix B for the definition and Figure 5 for a graph.

This function has 40 breakpoints $0 = x_0 < x_1 < \cdots < x_{39} < x_{40} = 1$ within $[0, 1]$. It has two *special intervals* $(l, u) = (x_{17}, x_{18})$ and $(f - u, f - l) = (x_{19}, x_{20})$, where $f = x_{37} = \frac{4}{5}$, $l = \frac{219}{800}$, $u = \frac{269}{800}$, on which every nonzero perturbation is *microperiodic*, namely invariant under the action of the dense additive group $T = \langle t_1, t_2\rangle_{\mathbb{Z}}$, where $t_1 = a_1 - a_0 = x_{10} - x_6 = \frac{77}{7752}\sqrt{2}$ and $t_2 = a_2 - a_0 = x_{13} - x_6 = \frac{77}{2584}$. Below we prove that it furnishes another separation.

Theorem 6.2. *The function $\pi =$* kzh_minimal_has_only_crazy_perturbation_1() *is a two-sided discontinuous piecewise linear function (with some irrational breakpoints) that is not extreme (nor a facet), but is a weak facet.*

Proof. By [19, Theorem 5.1], we know that the function π is minimal valid, but is not extreme. Let π' be a minimal valid function such that $E(\pi) \subseteq E(\pi')$. We want to show that $E(\pi) = E(\pi')$. Consider $\bar\pi = \pi' - \pi$, which is a bounded $\mathbb{Z}$-periodic function satisfying that $E(\pi) \subseteq E(\bar\pi)$. As a difference of minimal valid functions, it satisfies the symmetry condition

$$\bar\pi(x) + \bar\pi(y) = 0 \quad \text{for } x, y \in \mathbb{R} \text{ such that } x + y = f \tag{9}$$

as well as the conditions

$$\bar\pi(0) = \bar\pi(\tfrac{f}{2}) = \bar\pi(f) = \bar\pi(\tfrac{1+f}{2}) = \bar\pi(1) = 0. \tag{10}$$

[4] The graphs in Figure 3 can be reproduced with the command plot_2d_diagram_additive_domain_sans_limits(h), those in Figure 4 with the command plot_2d_diagram_additive_domain_sans_limits(h) + plot_2d_diagram_with_cones(h).

We reuse parts of the proof of [19, Theorem 5.1, Part (ii)], applying it to the perturbation $\bar{\pi}$.

First, as in the proof of [19, Theorem 5.1, Part (ii)], we prove the following claim:

(o) The function $\bar{\pi}$ is piecewise linear on $\mathcal{P}$ outside of the special intervals, with unknown slopes $\bar{c}_1, \bar{c}_3 \in \mathbb{R}$ on all intervals where π has slopes c_1 and c_3, respectively.

See Table 1 for a list of the intervals. We reuse the computer-assisted proof in [19, Appendix C] to prove Claim (o). Because $E(\pi) \subseteq E(\bar{\pi})$, if a two-dimensional face $F \in \Delta\mathcal{P}$ satisfies

$$\Delta\pi(x, y) = 0 \quad \text{for } (x, y) \in \operatorname{rel\,int}(F), \tag{11}$$

then we also have

$$\Delta\bar{\pi}(x, y) = 0 \quad \text{for } (x, y) \in \operatorname{rel\,int}(F). \tag{12}$$

Table 2 shows a list of faces $F \in \Delta\mathcal{P}$ with this propety. Our proof repeatedly applies the Gomory–Johnson Interval Lemma in the form of [4, Theorem 4.3] to these faces. (This version of the theorem only requires the boundedness of the function $\bar{\pi}$; this is contrast to the proof in [19]. The latter uses a version that is stated for effective perturbations only.) By the theorem, $\bar{\pi}$ is affine linear with the same slope on the open intervals $\operatorname{int}(p_i(F))$ for $i = 1, 2, 3$.

Then the proof considers the edges $F \in \Delta\mathcal{P}$ that satisfy (11) shown in Table 3. Let $\{i, j\} \subset \{1, 2, 3\}$ such that $p_i(F)$ and $p_j(F)$ are proper intervals. Let $L \subseteq F$ be a line segment such that $\bar{\pi}$ is affine linear on $p_i(L)$. Then by (12), $\bar{\pi}$ is also affine linear with the same slope on $p_j(L)$. (There is another difference to the proof in [19]: property (11) is more specific than the hypothesis of [19, Theorem 3.3]. The latter only requires limit-additivities $\Delta\pi_{F'}(x, y) = 0$ for $(x, y) \in \operatorname{rel\,int}(F)$ where $F' \supseteq F$ is an enclosing face. This distinction is crucial because we have no control over the limit-additivities of $\bar{\pi}$.)

Next we establish the following stronger claim:

(i) We have $\bar{\pi}(x) = 0$ for $x \notin (l, u) \cup (f - u, f - l)$.

Our proof is again similar to the one of [19, Theorem 5.1, Part (ii)], but in contrast to that, we consider only the restriction of $\bar{\pi}$ to

$$[0, l] \cup [u, f - u] \cup [f - l, 1],$$

where the function is piecewise linear by (o). The restricted function is determined by a finite system of parameters as follows: two slope parameters $\bar{c}_1$ and $\bar{c}_3$, 19 parameters that determine the function value $\bar{\pi}(x_i)$ at each breakpoint, and 18 parameters that determine the midpoint function value $\bar{\pi}(\frac{x_i + x_{i+1}}{2})$ on each interval of $\mathcal{P}$ except for the special intervals. (Here we used the symmetry condition (9), as well as the conditions (10) to reduce the number of parameters.) We set up a finite linear system of equations that expresses the additivity relations (12) for faces F that satisfy (11). We do this by writing equations $\Delta\bar\pi_F(x,y)=0$ for $(x,y)\in\operatorname{vert}(F)$ for these faces F. The system has full rank; a regular 39×39 subsystem is shown in Tables 4 and 5. Therefore the unique solution of the system is 0, and Claim (i) is proved.

Next, we show that

(ii) $\bar\pi$ is constant on each coset $\bar x+T\in\mathbb{R}/T$ on the special interval (l,u), and likewise on the special interval $(f-u,f-l)$.

The function $\bar\pi$ satisfies the additivity relations (12) from the faces $F(\{a_i\},[l,u],[f-u,f-l])$ for $i=0,1,2$, where $a_0=x_6$, $a_1=x_{10}=a_0+t_1=\frac{77}{7752}\sqrt{2}+\frac{19}{100}$, and $a_2=x_{13}=a_0+t_2$. These faces appear in Table 6; see also Figure 5. Let $\bar x$ be an arbitrary real number. Then there exists a point $\hat x\in(l,u)$ such that $\bar x-\hat x\in T$ and $\hat x\pm t_i\in(l,u)$. Then $\bar\pi$ and $\hat x$ satisfy the hypothesis of [19, Lemma B.1]. Writing $\bar x-\hat x=\lambda_1t_1+\lambda_2t_2$ for some $\lambda_1,\lambda_2\in\mathbb{Z}$, the lemma gives $\bar\pi(\bar x)-\bar\pi(\hat x)=\sum_{i=1}^{2}\lambda_i(\bar\pi(a_i)-\bar\pi(a_0))$. Using $\bar\pi(a_i)=0$, as the points a_i lie outside of the special intervals, we obtain $\bar\pi(\bar x)=\bar\pi(\hat x)$. Using the symmetry relation given by (12) for the face $F([l,u],[f-u,f-l],\{f\})$, we obtain that $\bar\pi$ is constant on the set $(f-u,f-l)\cap(f-\bar x+T)$ as well.

Next, using the face $F([l,u],[l,u],\{l+u\})$, which satisfies (11), and $\bar\pi(l+u)=0$ from (i) because $l+u$ lies outside the special intervals, we obtain that

(iii) $\bar\pi(x)+\bar\pi(y)=0$ for $x,y\in(l,u)$ such that $x+y=l+u$.

Together with (ii), we obtain that

(iv) $\bar\pi(x)+\bar\pi(y)=0$ for $x,y\in(l,u)$ such that $x+y\in(l+u)+T$.

We now show that $\bar\pi$ also satisfies the following condition:

(v) $|\bar\pi(x)|\le s$ for all $x\in(l,u)\cup(f-u,f-l)$,

where

$$s=\pi(x_{39}^-)+\pi(1+l-x_{39})-\pi(l)=\tfrac{19}{23998}. \tag{13}$$

Indeed, by (iii) and (9), it suffices to show that for any $x\in(l,u)$, we have $\bar\pi(x)\ge -s$. Suppose, for the sake of contradiction, that there is $\bar x\in(l,u)$ such that $\bar\pi(\bar x)<-s$. Since the group T is dense in $\mathbb{R}$, we can find $x\in(l,u)$ such that $x\in\bar x+T$ and x is arbitrarily close to $1+l-x_{39}$. We choose x so that $\delta=x-(1+l-x_{39})\in(0,\frac{-s-\bar\pi(\bar x)}{c_2-c_3})$, where c_2 and c_3 denote the slope of π on the pieces (l,u) and $(0,x_1)$, respectively. See Table 1 for the concrete values of the parameters. Let $y=1+l-x$. Then $y=x_{39}-\delta$. It follows from (i) that $\bar\pi(y)=0$ and $\bar\pi(x+y)=\bar\pi(l)=0$. Now consider $\Delta\pi'(x,y)=\pi'(x)+\pi'(y)-\pi'(x+y)$, where

$$\begin{aligned}\pi'(x)&=\bar\pi(x)+\pi(x)=\bar\pi(x)+\pi(1+l-x_{39})+\delta c_2;\\ \pi'(y)&=\pi(y)\qquad\quad\;=\pi(x_{39}^-)-\delta c_3;\\ \pi'(x+y)&=\pi(x+y)\qquad\;=\pi(l).\end{aligned}$$

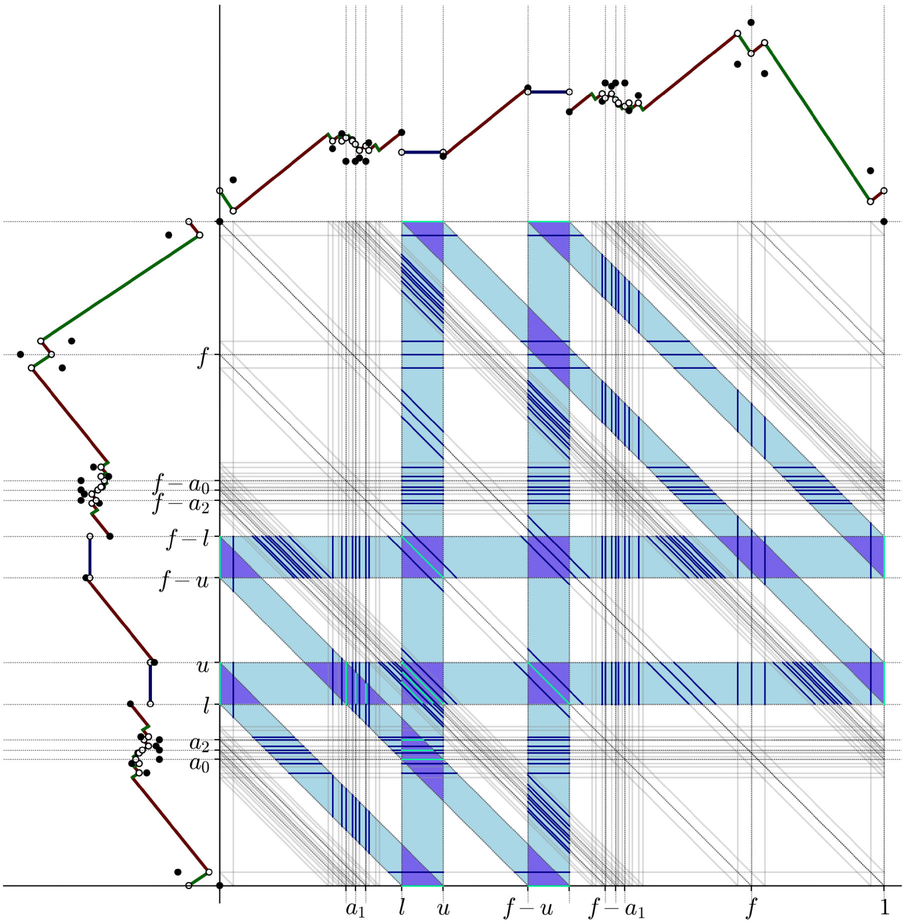

Fig. 5. Diagram of the polyhedral complex $\Delta\mathcal{P}$ of the function π = kzh_minimal_has_only_crazy_perturbation_1() (shown at the top and left borders), where two-dimensional faces F are color-coded according to the values n_F: $n_F = 0$ (*white*), $n_F = 1$ (*light blue*), $n_F = 2$ (*medium lavender blue*). One-dimensional faces F with $n_F > 0$ are shown in (a) *light green* if $\Delta\pi_F(u, v) = 0$, (b) *dark blue* if $\Delta\pi_F(u, v) \geq n_F \cdot s$ for $(u, v) \in \operatorname{vert}(F)$.

Since $x - \bar{x} \in T$, the condition (ii) implies that $\bar{\pi}(x) = \bar{\pi}(\bar{x})$. We have

$$\begin{aligned}\Delta\pi'(x, y) &= \bar{\pi}(\bar{x}) + [\pi(1 + l - x_{39}) + \pi(x_{39}^-) - \pi(l)] + \delta(c_2 - c_3)\\ &= \bar{\pi}(\bar{x}) + s + \delta(c_2 - c_3) < 0,\end{aligned}$$

a contradiction to the subadditivity of π'. Therefore, $\bar{\pi}$ satisfies condition (v).

Let F be a face of $\Delta\mathcal{P}$. Denote by $n_F \in \{0, 1, 2\}$ the number of projections $p_i(\operatorname{rel\,int}(F))$ for $i = 1, 2, 3$ that intersect with $(l, u) \cup (f - u, f - l)$. See Figure 5;

note that there is no face F with $n_F = 3$. It follows from the conditions (i) and (v) that

$$|\Delta\bar{\pi}(x,y)| \leq n_F \cdot s \quad \text{for any } (x,y) \in \operatorname{rel\,int}(F).$$

Moreover, we make the following claim.

(vi) If $F \in \Delta\mathcal{P}$ has $n_F \neq 0$, then either
  (a) $\Delta\pi_F(u,v) = 0$ for all $(u,v) \in \operatorname{vert}(F)$, or
  (b) $\Delta\pi_F(u,v) \geq n_F \cdot s$ for all $(u,v) \in \operatorname{vert}(F)$, and the inequality is strict for at least one vertex.

We have verified this claim computationally (using exact computations) in our software, by enumerating all faces F of $\Delta\mathcal{P}$ with $n_F > 0$.[5] We provide the relevant data of the function in Appendix B (Tables 6 and 7) for the reader's reference and for archival purposes.

Next, we show the following simple corollary of Claim (vi):

(vii) If $F \in \Delta\mathcal{P}$ has $n_F \neq 0$, then either
  (a) $\Delta\pi(x,y) = 0$ for all $(x,y) \in \operatorname{rel\,int}(F)$, or
  (b) $\Delta\pi(x,y) > n_F \cdot s$ for all $(x,y) \in \operatorname{rel\,int}(F)$.

To prove this, assume that $n_F \neq 0$. Since $\Delta\pi_F$ is affine linear on F, $\Delta\pi(x,y)$ for $(x,y) \in \operatorname{rel\,int}(F)$ is a strict convex combination of $\{\, \Delta\pi_F(u,v) \mid (u,v) \in \operatorname{vert}(F)\,\}$. As at least one of the inequalities $\Delta\pi_F(u,v) \geq n_F \cdot s$ is strict, (b) follows.

Finally, we prove the following claim:

(viii) For $(x,y) \in \mathbb{R}^2$ such that $\Delta\pi(x,y) > 0$, we have $\Delta\pi'(x,y) > 0$.

To prove this, consider the (unique) face $F \in \Delta\mathcal{P}$ such that $(x,y) \in \operatorname{rel\,int}(F)$. If $n_F = 0$, then $\Delta\bar{\pi}(x,y) = 0$, and hence $\Delta\pi'(x,y) = \Delta\pi(x,y) > 0$. Otherwise, because we have $\Delta\pi(x,y) > 0$ by assumption, the above case (b) applies, and hence $\Delta\pi'(x,y) = \Delta\pi(x,y) + \Delta\bar{\pi}(x,y) > 0$ holds when $n_F \neq 0$ as well.

We obtain that $E(\pi') \subseteq E(\pi)$. This, together with the assumption $E(\pi) \subseteq E(\pi')$, implies that $E(\pi) = E(\pi')$. We conclude, by Lemma 5.1(3), that π is a weak facet. □

6.3 Extreme and weak facet, but not a facet

For the remaining separation, we construct an extreme function $\hat{\pi}$ as follows. In [19, Theorem 5.1], the authors showed that $\pi =$ kzh_minimal_has_only_crazy_perturbation_1() admits an effective locally microperiodic perturbation that is supported on the cosets $l + T$, $u + T$ of the group T on the special interval (l, u)

[5] The enumeration is done by the function generate_faces_with_projections_intersecting. A fully automatic verification is carried out by the command kzh_minimal_has_only_crazy_perturbation_1_check_subadditivity_slacks().

and, equivariantly, on the cosets $f - l + T$, $f - u + T$ on the special interval $(f - u, l - u)$.

We perturb the function π instead on infinitely many (almost all) cosets of the group T on the two special intervals as follows. Consider the involution (point reflection) $\rho_{l+u} \colon x \mapsto l + u - x = f - a_0 - x$, which has the unique fixed point $\frac{l+u}{2}$. Because $\rho_{l+u}(x + t) = \rho_{l+u}(x) - t$ for $x \in \mathbb{R}$ and $t \in T$, the involution can be considered as a map from the quotient $\mathbb{R}/T$ (whose elements are the cosets of T) to itself. The set of fixed points of $\mathbb{R}/T$ under this map is

$$C = \left\{ \tfrac{l+u}{2} + T,\ \tfrac{l+u-t_1}{2} + T,\ \tfrac{l+u-t_2}{2} + T,\ \tfrac{l+u-(t_1+t_2)}{2} + T \right\}.$$

The remaining elements of $\mathbb{R}/T$ are paired by the involution into two-element orbits $\{x, \rho_{l+u}(x)\}$. Fix a choice function c^+ that maps each of the two-element sets $\{x, \rho_{l+u}(x)\} \subset \mathbb{R}/T$ to one of its two elements. (We remark that the existence of such a choice function does *not* depend on the axiom of choice because the sets in question are finite.) Then define

$$C^+ = \left\{ c^+(\{x, \rho_{l+u}(x)\}) \in \mathbb{R}/T \;\middle|\; x \in \mathbb{R}/T,\ x \notin C \right\}.$$

Using these sets, we define for every $x \in [0, 1]$

$$\hat\pi(x) = \begin{cases} \pi(x) & \text{if } x \notin (l,u) \cup (f-u, f-l), \text{ or} \\ & \text{if } x \in (l,u) \text{ such that } x + T \in C, \text{ or} \\ & \text{if } x \in (f-u, f-l) \text{ such that } f - x + T \in C; \\ \pi(x) + s & \text{if } x \in (l,u) \text{ such that } x + T \in C^+, \text{ or} \\ & \text{if } x \in (f-u, f-l) \text{ such that } f - x + T \in C^+; \\ \pi(x) - s & \text{otherwise,} \end{cases} \tag{14}$$

where s is the constant defined in (13) in the proof of Theorem 6.2. We extend this function to $\mathbb{R}$ by setting $\hat\pi(x + z) = \hat\pi(x)$ for $x \in \mathbb{R}$ and $z \in \mathbb{Z}$.

Theorem 6.3. *The function* $\hat\pi$ = kzh_extreme_and_weak_facet_but_not_facet()[6] *defined in* (14) *is a two-sided discontinuous, non–piecewise linear function that is extreme and a weak facet, but is not a facet.*

Proof. Let $\bar\pi = \hat\pi - \pi$. By definition of $\hat\pi$, the function $\bar\pi$ is periodic modulo 1. Moreover, it satisfies the symmetry condition (9), the conditions (10), as well as the conditions (i) to (v) in the proof of Theorem 6.2. We claim that $\hat\pi$ is subadditive. To this end, recall the notation n_F from the proof of Theorem 6.2.

For all faces $F \in \Delta\mathcal{P}$ with $n_F = 0$, because $\hat\pi$ equals π outside of the special intervals, we have $\Delta\bar\pi_F(x, y) = 0$ for $(x, y) \in \operatorname{rel\,int}(F)$, and thus $\Delta\hat\pi_F(x, y) = \Delta\pi_F(x, y)$ for $(x, y) \in \operatorname{rel\,int}(F)$.

Next, consider the faces F with $n_F > 0$. By Claim (vii) from Theorem 6.2, we either have (a) $\Delta\pi_F(x, y) = 0$ for all $(x, y) \in \operatorname{rel\,int}(F)$, or (b) $\Delta\pi_F(x, y) > n_F \cdot s$ for all $(x, y) \in \operatorname{rel\,int}(F)$.

[6] The authors thank Jiawei Wang for his help with implementing this function in the software.

From Table 6 (see also Figure 5), we see that these faces satisfying (a) are exactly the following (up to replacing $F(I,J,K)$ by $F(J,I,K)$ and up to $\mathbb{Z}$-periodicity):

(1) $F = F([l,u],\{a_i\},[f-u,f-l])$ for $i = 1,2,3$. Denoting $t_0 = 0$ for convenience, we have $a_i = a_0 + t_i$, where $t_i \in T$. Fix i and let $(x,a_i) \in \operatorname{rel\,int}(F)$, so $x \in (l,u)$ and $x + a_i \in (f-u, f-l)$. Because $\bar\pi(a_i) = 0$, as a_i lies outside of the special intervals, and $(l,u) \ni f - (x + a_i) = f - a_0 - x - t_i = \rho_{l+u}(x) - t_i \in \rho_{l+u}(x) + T$, we have $\Delta\bar\pi(x,a_i) = \bar\pi(x) + \bar\pi(a_i) - \bar\pi(x+a_i) = \bar\pi(x) + \bar\pi(f-(x+a_i)) = \bar\pi(x) + \bar\pi(\rho_{l+u}(x)) = 0$.

(2) $F = F([l,u],[l,u],\{f-a_i\})$ for $i = 1,2,3$. Again fix i and let $(x,y) \in \operatorname{rel\,int}(F)$, so $x + y = f - a_i = f - a_0 - t_i$ and $x,y \in (l,u)$. Then $\Delta\bar\pi(x,y) = \bar\pi(x) + \bar\pi((f-a_0) - x - t_i) - \bar\pi(f-a_i) = \bar\pi(x) + \bar\pi(\rho_{l+u}(x)) = 0$.

(3) $F = F([l,u],[f-u,f-l],\{f\})$. Then, by the symmetry condition (9), we have $\Delta\bar\pi(x,y) = 0$.

(4) $F(\{0\},[l,u],[l,u])$ and $F(\{0\},[f-u,f-l],[f-u,f-l])$. Here $\Delta\bar\pi(x,y) = 0$ trivially.

Again, we conclude that $\Delta\hat\pi_F(x,y) = \Delta\pi_F(x,y)$ for $(x,y) \in \operatorname{rel\,int}(F)$.

Finally, consider the faces F with $n_F > 0$ that satisfy (b), i.e., $\Delta\pi_F(x,y) > n_F \cdot s$ for $(x,y) \in \operatorname{rel\,int}(F)$. Because $|\Delta\bar\pi_F(x,y)| \le n_F \cdot s$, we have $\Delta\hat\pi_F(x,y) > 0$ for $(x,y) \in \operatorname{rel\,int}(F)$.

Hence, $\hat\pi$ is subadditive as claimed, and therefore a minimal valid function, and in fact $E(\hat\pi) = E(\pi)$.

Let π' be a minimal valid function such that $E(\hat\pi) \subseteq E(\pi')$. Then, as shown in the proof of Theorem 6.2, we have $E(\hat\pi) = E(\pi')$. It follows from Lemma 5.1(3) that $\hat\pi$ is a weak facet. However, the function $\hat\pi$ is not a facet, since $E(\hat\pi) = E(\pi)$ but $\hat\pi \neq \pi$. Next, we show that $\hat\pi$ is an extreme function.

Suppose that $\hat\pi$ can be written as $\hat\pi = \frac{1}{2}(\pi^1 + \pi^2)$, where π^1, π^2 are minimal valid functions. Then $E(\hat\pi) \subseteq E(\pi^1)$ and $E(\hat\pi) \subseteq E(\pi^2)$. Let $\bar\pi^1 = \pi^1 - \pi$ and $\bar\pi^2 = \pi^2 - \pi$. We have that $E(\pi) \subseteq E(\bar\pi^1)$ and $E(\pi) \subseteq E(\bar\pi^2)$. Hence, as shown in the proof of Theorem 6.2, $\bar\pi^1$ and $\bar\pi^2$ satisfy the symmetry condition (9) and the conditions (i) to (v). We will show that $\bar\pi^1 = \bar\pi^2$.

For $x \notin (l,u) \cup (f-u, f-l)$, we have $\bar\pi^i(x) = 0$ $(i = 1,2)$ by condition (i). It remains to prove that $\bar\pi^1(x) = \bar\pi^2(x)$ for $x \in (l,u) \cup (f-u,f-l)$. By the symmetry condition (9), it suffices to consider $x \in (l,u)$. We distinguish three cases. If $x+T \in C$, then condition (iv) implies $\bar\pi^i(x) = 0$ $(i = 1,2)$. If $x+T \in C^+$, then $\bar\pi(x) = s$ by definition. Notice that $\bar\pi^1 + \bar\pi^2 = \pi^1 + \pi^2 - 2\pi = 2\hat\pi - 2\pi = 2\bar\pi$, and that $\bar\pi^i(x) \le s$ $(i = 1,2)$ by condition (v). We have $\bar\pi^i(x) = s$ $(i = 1,2)$ in this case. If $x + T \notin C$ and $x + T \notin C^+$, then $\bar\pi(x) = -s$, and hence $\bar\pi^i(x) = -s$ $(i = 1,2)$. Therefore, $\bar\pi^1 = \bar\pi^2$ and $\pi^1 = \pi^2$, which proves that the function $\hat\pi$ is extreme. □

7 Conclusion

As a conclusion to our paper, we discuss the three notions relative to subspaces of functions.

Gomory and Johnson introduced the notion of facets in [15] in a setting in which valid functions, by definition, are continuous functions. Following the discussion in [1], a continuous valid function π is defined to be a *facet in the sense of Gomory–Johnson* if $P(\pi) \subseteq P(\pi')$ implies $\pi' = \pi$ for every *continuous* valid function π'. As remarked in [1], every continuous facet is also a facet in the sense of Gomory–Johnson. We have a partial converse as follows.

Corollary 7.1. *Every continuous piecewise linear function (not necessarily with rational breakpoints) that is a facet in the sense of Gomory–Johnson is also a facet.*

Proof. Let π be a continuous piecewise linear minimal valid function that is not a facet. Then π is not an extreme function. Thus $\pi = \frac{1}{2}(\pi^1 + \pi^2)$ with some minimal valid functions $\pi^1, \pi^2 \neq \pi$, which are Lipschitz continuous by [4, Lemma 2.11 (iv)] and satisfy $E(\pi) \subseteq E(\pi^i)$ by [4, Lemma 2.11 (ii)]. Setting $\pi' = \pi^1$, it follows from Lemma 4.3 that $P(\pi) \subseteq P(\pi')$. Therefore π is not a facet in the sense of Gomory–Johnson. □

Open question 7.2 *Is every facet in the sense of Gomory–Johnson a facet?*

An approach to resolve this question in the negative would be to construct a continuous non–piecewise linear minimal valid function π such that there exists a minimal valid function $\pi' \neq \pi$ with $P(\pi) \subseteq P(\pi')$ (equivalently, $E(\pi) \subseteq E(\pi')$) that is discontinuous, and all such functions π' are discontinuous. Note that the differences $\bar{\pi} = \pi' - \pi$ cannot be effective perturbations for π, because all effective perturbations of a continuous function π are Lipschitz continuous by [4, Lemma 2.11 (iv)].

Basu et al. [2] highlight the subspace of Lipschitz continuous functions. All minimal valid functions that are liftable to cut-generating function pairs for the mixed integer problem belong to this space [2, Remark 2.7]. Define a *facet in the sense of Lipschitz* to be a Lipschitz continuous function such that $P(\pi) \subseteq P(\pi')$ implies $\pi' = \pi$ for every Lipschitz continuous valid function π'. Thus we can ask:

Open question 7.3 *Is every facet in the sense of Lipschitz a facet?*

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A Auxiliary result

In the proof of Theorem 3.1, we need the following elementary geometric estimate.

Lemma A.1. *Let $F \subset [0,1]^2$ be a convex polygon with vertex set* $\operatorname{vert}(F)$*, and let $g\colon F \to \mathbb{R}$ be an affine linear function. Suppose that for each $v \in \operatorname{vert}(F)$, either $g(v) = 0$ or $g(v) \geq m$ for some $m > 0$. Let $S = \{\, x \in F \mid g(x) = 0 \,\}$, and assume that S is nonempty. Then $g(x) \geq m\, d(x, S)/2$ for any $x \in F$, where $d(x, S)$ denotes the Euclidean distance from x to S.*

Proof. Let $x \in F$ be arbitrary. We may write

$$x = \sum_{v \in \operatorname{vert}(F)} \alpha_v v$$

for some $\alpha_v \in [0,1]$ with $\sum_v \alpha_v = 1$.

Since S is a closed set, for each $v \in \operatorname{vert}(F)$, there exists $s_v \in S$ such that $d(v, S) = d(v, s_v)$. Let $s^* = \sum_{v \in \operatorname{vert}(F)} \alpha_v s_v$. We have that $s^* \in S$ since the set S is convex. Thus,

$$\begin{aligned} d(x, S) &\leq d(x, s^*) && \text{(by definition)} \\ &= d\Big(\sum_v \alpha_v v, \sum_v \alpha_v s_v\Big) \\ &\leq \sum_v \alpha_v d(v, s_v) && \text{(by the triangle inequality)} \\ &= \sum_v \alpha_v d(v, S). \end{aligned}$$

For those $v \in \operatorname{vert}(F)$ with $g(v) = 0$, we have $v \in S$ by definition and thus $d(v, S) = 0$. Therefore,

$$d(x, S) \leq \sum_{\substack{v \in \operatorname{vert}(F) \\ g(v) \geq m}} \alpha_v d(v, S) \leq 2 \sum_{\substack{v \in \operatorname{vert}(F) \\ g(v) \geq m}} \alpha_v.$$

Using the affine linearity of g, it thus follows that

$$g(x) = \sum_{v \in \operatorname{vert}(F)} \alpha_v g(v) = \sum_{\substack{v \in \operatorname{vert}(F) \\ g(v) = 0}} \alpha_v g(v) + \sum_{\substack{v \in \operatorname{vert}(F) \\ g(v) \geq m}} \alpha_v g(v) \geq \frac{m\, d(x, S)}{2}.$$

□

B Data of the function $\pi =$ kzh_minimal_has_only_crazy_perturbation_1()

The following pages provide tables with data of the piecewise linear function $\pi =$ kzh_minimal_has_only_crazy_perturbation_1() of Theorem 6.2.

Table 1 defines the function by listing the breakpoints x_i and the values and the left and right limits at the breakpoints. (A version of this table has previously appeared in [19].)

Tables 2 and 3 list the faces $F = F(I, J, K)$ of the complex $\Delta\mathcal{P}$ that we use for proving piecewise linearity of $\bar{\pi}$ outside of the special intervals, i.e., Claim (o) in the proof of Theorem 6.2. In all tables, the faces are listed by lexicographically increasing triples (I, J, K); and of the two equivalent faces $F(I, J, K)$ and $F(J, I, K)$, we only show the lexicographically smaller one.

Table 4 shows a list of faces F that satisfy $\Delta\bar{\pi}_F(x, y) = 0$ for all $(x, y) \in \operatorname{rel\,int}(F)$. This property can be verified by inspecting the provided list of vertices of each face. A selection of one vertex (u, v) for each listed face F, listed first in the table, suffices to form a full-rank homogeneous linear system of equations $\Delta\bar{\pi}_F(u, v) = 0$. We obtained the selection of faces and their vertices by Gaussian elimination. The full-rank system, shown in Table 5, proves that $\bar{\pi}$ is 0 outside of the special intervals, Claim (i) in the proof of Theorem 6.2.

Finally, Tables 6 and 7 list the faces F whose projections $p_i(\operatorname{rel\,int}(F))$, $i = 1, 2, 3$, overlap with the special intervals ($n_F > 0$). They are relevant for verifying Claims (ii), (iii), and (vi) in the proof of Theorem 6.2. For each face F, we list the values of the subadditivity slack $\Delta\pi_F(u, v)$ for all vertices (u, v) of F in nondecreasing order from left to right. If there is an enclosing face $F' \supset F$ with $\Delta\pi_{F'}(u, v) = \Delta\pi_F(u, v)$ for all vertices (u, v) of F because of one-sided continuity, then we suppress F in the table. All numbers have been rounded to 3 decimals for presentation. Claims (ii) and (iii) use faces with $\Delta\pi_F(x, y) = 0$ for $(x, y) \in F$. To verify Claim (v), note that if $\Delta\pi_F(u, v) = 0$ for one vertex of F, then $\Delta\pi_F(u, v) = 0$ for all vertices of F. Next, note that for all other faces F with $n_F > 0$, the inequality $\Delta\pi_F(u, v) \geq n_F \cdot s$ (where $s \approx 0.001$) is satisfied and tight for at most one vertex (u, v) of each face. These vertices are marked by the word "(tight)" in the tables; we have $n_F = 1$ for each of these faces. All remaining subadditivity slacks $\Delta\pi_F(u, v)$ for vertices $(u, v) \in \operatorname{vert}(F)$ exceed $0.003 \geq 3 \cdot s$.

Table 1. The piecewise linear function $\pi =$ kzh_minimal_has_only_crazy_perturbation_1(), defined by its values and limits at the breakpoints. If a limit is omitted, it equals the value.

i	x_i	$\pi(x_i^-) = \pi_{[x_{i-1},x_i]}(x_i)$	$\pi(x_i)$	$\pi(x_i^+) = \pi_{[x_i,x_{i+1}]}(x_i)$	slope
0	0	$\frac{101}{650}$	0	$\frac{101}{650}$	$c_3 = -5$
1	$\frac{101}{5000}$	$\frac{707}{13000}$	$\frac{2727}{13000}$	$\frac{707}{13000}$	$c_1 = \frac{35}{13}$
2	$\frac{60153}{369200}$		$\frac{421071}{959920}$		$c_3 = -5$
3	$\frac{849}{5000}$	$\frac{4851099}{11999000}$	$-\frac{1925}{71994}\sqrt{2} + \frac{4851099}{11999000}$	$\frac{4851099}{11999000}$	$c_1 = \frac{35}{13}$
4	$\frac{1925}{298129}\sqrt{2} + \frac{849}{5000}$		$\frac{67375}{3875677}\sqrt{2} + \frac{4851099}{11999000}$		$c_3 = -5$
5	$\frac{77}{7752}\sqrt{2} + \frac{849}{5000}$	$\frac{385}{93016248}\sqrt{2} + \frac{4851099}{11999000}$	$\frac{2695}{100776}\sqrt{2} + \frac{4851099}{11999000}$	$\frac{385}{93016248}\sqrt{2} + \frac{4851099}{11999000}$	$c_1 = \frac{35}{13}$
6	$a_0 = \frac{19}{100}$	$-\frac{1925}{71994}\sqrt{2} + \frac{275183}{599950}$	$\frac{18196}{59995}$	$-\frac{1925}{71994}\sqrt{2} + \frac{275183}{599950}$	$c_1 = \frac{35}{13}$
7	$\frac{77}{22152}\sqrt{2} + \frac{281986521}{1490645000}$		$-\frac{385}{22152}\sqrt{2} + \frac{10467633}{22933000}$		$c_3 = -5$
8	$\frac{40294}{201875}$	$\frac{848837}{2099500}$	$\frac{795836841}{1937838500}$	$\frac{848837}{2099500}$	$c_1 = \frac{35}{13}$
9	$\frac{36999}{184600}$		$\frac{975607}{2399800}$		$c_3 = -5$
10	$a_1 = \frac{77}{7752}\sqrt{2} + \frac{19}{100}$	$-\frac{385}{7752}\sqrt{2} + \frac{275183}{599950}$	$\frac{385}{93016248}\sqrt{2} + \frac{18196}{59995}$	$-\frac{385}{7752}\sqrt{2} + \frac{275183}{599950}$	$c_3 = -5$
11	$\frac{1051}{5000}$	$\frac{4291761}{11999000}$	$-\frac{1925}{71994}\sqrt{2} + \frac{4291761}{11999000}$	$\frac{4291761}{11999000}$	$c_1 = \frac{35}{13}$
12	$\frac{1925}{298129}\sqrt{2} + \frac{1051}{5000}$		$\frac{67375}{3875677}\sqrt{2} + \frac{4291761}{11999000}$		$c_3 = -5$
13	$a_2 = \frac{14199}{64600}$	$\frac{192500}{3875677}\sqrt{2} + \frac{240046061}{775135400}$	$\frac{50943}{167960}$	$\frac{192500}{3875677}\sqrt{2} + \frac{240046061}{775135400}$	$c_3 = -5$
14	$\frac{77}{7752}\sqrt{2} + \frac{1051}{5000}$	$\frac{385}{93016248}\sqrt{2} + \frac{4291761}{11999000}$	$\frac{2695}{100776}\sqrt{2} + \frac{4291761}{11999000}$	$\frac{385}{93016248}\sqrt{2} + \frac{4291761}{11999000}$	$c_1 = \frac{35}{13}$
15	$\frac{77}{22152}\sqrt{2} + \frac{342208579}{1490645000}$		$-\frac{385}{22152}\sqrt{2} + \frac{122181831}{298129000}$		$c_3 = -5$
16	$\frac{193799}{807500}$		$\frac{187742}{524875}$		$c_1 = \frac{35}{13}$
17	$l = A = \frac{219}{800}$		$\frac{933}{2080}$	$\frac{51443}{147680}$	$c_2 = \frac{5}{11999}$
18	$u = A_0 = \frac{269}{800}$	$\frac{668809}{1919840}$	$\frac{683}{2080}$		$c_1 = \frac{35}{13}$
19	$f - u = \frac{371}{800}$		$\frac{1397}{2080}$	$\frac{1251031}{1919840}$	$c_2 = \frac{5}{11999}$
20	$f - l = \frac{421}{800}$	$\frac{96237}{147680}$	$\frac{1147}{2080}$		$c_1 = \frac{35}{13}$

i	x_i	$\pi(x_i^-) = \pi_{[x_{i-1},x_i]}(x_i)$	$\pi(x_i)$	$\pi(x_i^+) = \pi_{[x_i,x_{i+1}]}(x_i)$	slope
21	$\frac{452201}{807500}$		$\frac{337133}{524875}$		$c_1 = \frac{35}{13}$
22	$-\frac{77}{22152}\sqrt{2} + \frac{850307421}{1490645000}$		$\frac{385}{22152}\sqrt{2} + \frac{175947169}{298129000}$		$c_3 = -5$
23	$-\frac{77}{7752}\sqrt{2} + \frac{2949}{5000}$	$-\frac{385}{93016248}\sqrt{2} + \frac{7707239}{11999000}$	$-\frac{2695}{100776}\sqrt{2} + \frac{7707239}{11999000}$	$-\frac{385}{93016248}\sqrt{2} + \frac{7707239}{11999000}$	$c_1 = \frac{35}{13}$
24	$f - a_2 = \frac{37481}{64600}$	$-\frac{192500}{3875677}\sqrt{2} + \frac{535089339}{775135400}$	$\frac{117017}{167960}$	$-\frac{192500}{3875677}\sqrt{2} + \frac{535089339}{775135400}$	$c_3 = -5$
25	$-\frac{1925}{298129}\sqrt{2} + \frac{2949}{5000}$		$-\frac{67375}{3875677}\sqrt{2} + \frac{7707239}{11999000}$		$c_3 = -5$
26	$\frac{2949}{5000}$	$\frac{7707239}{11999000}$	$\frac{1925}{71994}\sqrt{2} + \frac{7707239}{11999000}$	$\frac{7707239}{11999000}$	$c_1 = \frac{35}{13}$
27	$f - a_1 = -\frac{77}{7752}\sqrt{2} + \frac{61}{100}$	$\frac{385}{7752}\sqrt{2} + \frac{324767}{599950}$	$-\frac{385}{93016248}\sqrt{2} + \frac{41799}{59995}$	$\frac{385}{7752}\sqrt{2} + \frac{324767}{599950}$	$c_3 = -5$
28	$\frac{110681}{184600}$		$\frac{1424193}{2399800}$		$c_3 = -5$
29	$\frac{121206}{201875}$	$\frac{1250663}{2099500}$	$\frac{1142001659}{1937838500}$	$\frac{1250663}{2099500}$	$c_1 = \frac{35}{13}$
30	$-\frac{77}{22152}\sqrt{2} + \frac{910529479}{1490645000}$		$\frac{385}{22152}\sqrt{2} + \frac{12465367}{22933000}$		$c_3 = -5$
31	$f - a_0 = l + u = \frac{61}{100}$	$\frac{1925}{71994}\sqrt{2} + \frac{324767}{599950}$	$\frac{41799}{59995}$	$\frac{1925}{71994}\sqrt{2} + \frac{324767}{599950}$	$c_1 = \frac{35}{13}$
32	$-\frac{77}{7752}\sqrt{2} + \frac{3151}{5000}$	$-\frac{385}{93016248}\sqrt{2} + \frac{7147901}{11999000}$	$-\frac{2695}{100776}\sqrt{2} + \frac{7147901}{11999000}$	$-\frac{385}{93016248}\sqrt{2} + \frac{7147901}{11999000}$	$c_1 = \frac{35}{13}$
33	$-\frac{1925}{298129}\sqrt{2} + \frac{3151}{5000}$		$-\frac{67375}{3875677}\sqrt{2} + \frac{7147901}{11999000}$		$c_3 = -5$
34	$\frac{3151}{5000}$	$\frac{7147901}{11999000}$	$\frac{1925}{71994}\sqrt{2} + \frac{7147901}{11999000}$	$\frac{7147901}{11999000}$	$c_1 = \frac{35}{13}$
35	$\frac{235207}{369200}$		$\frac{538849}{959920}$		$c_3 = -5$
36	$\frac{3899}{5000}$	$\frac{12293}{13000}$	$\frac{10273}{13000}$	$\frac{12293}{13000}$	$c_1 = \frac{35}{13}$
37	$f = \frac{4}{5}$	$\frac{549}{650}$	1	$\frac{549}{650}$	$c_3 = -5$
38	$\frac{4101}{5000}$	$\frac{899}{1000}$	$\frac{9667}{13000}$	$\frac{899}{1000}$	$c_1 = \frac{35}{13}$
39	$\frac{4899}{5000}$	$\frac{101}{1000}$	$\frac{3333}{13000}$	$\frac{101}{1000}$	$c_3 = -5$
40	1	$\frac{101}{650}$	0	$\frac{101}{650}$	$c_1 = \frac{35}{13}$

Table 2: Two-dimensional faces F with additivity on $\operatorname{int}(F)$ for proving piecewise linearity outside of the special intervals. All intervals I, J, K are closed and elements of the complex $\mathcal{P}$; notation $\langle a, b\rangle$: endpoints are not reached by the projection of the face; (a, b): function π is one-sided discontinuous at the endpoints from within the interval; $[a, b]$: function π is one-sided continuous at the endpoints from within the interval.

Face $F = F(I, J, K)$				vertices of F								
I	J	K	slope	u	v	$u+v$	u	v	$u+v$	u	v	$u+v$
$\langle x_1, x_2\rangle$	$\langle x_1, x_2\rangle$	$\langle x_1, x_2]$	c_1	${x_1}^+$	${x_1}^+$	$2{x_1}^+$	${x_1}^+$	$-x_1{+}{x_2}^-$	${x_2}^-$	$-x_1{+}{x_2}^-$	${x_1}^+$	${x_2}^-$
$\langle x_1, x_2\rangle$	$[u, f{-}u\rangle$	$\langle u, f{-}u]$	c_1	${x_1}^+$	u^+	$u{+}{x_1}^+$	${x_1}^+$	$f{-}u{-}{x_1}^-$	$f{-}u^-$	$f{-}2u^-$	u^+	$f{-}u^-$
$\langle x_1, x_2\rangle$	$[x_{35}, x_{36}\rangle$	$\langle x_{35}, x_{36})$	c_1	${x_1}^+$	${x_{35}}^+$	$x_1{+}{x_{35}}^+$	${x_1}^+$	$-x_1{+}{x_{36}}^-$	${x_{36}}^-$	$-x_{35}{+}{x_{36}}^-$	${x_{35}}^+$	${x_{36}}^-$
$\langle x_{38}, x_{39})$	$\langle x_{38}, x_{39})$	$(x_{38}{+}1, x_{39}{+}1\rangle$	c_3	$x_{38}{-}x_{39}{+}1^+$	${x_{39}}^-$	$x_{38}{+}1^+$	${x_{39}}^-$	$x_{38}{-}x_{39}{+}1^+$	$x_{38}{+}1^+$	${x_{39}}^-$	${x_{39}}^-$	$2{x_{39}}^-$

Table 3: One-dimensional faces F with additivity on $\operatorname{rel\,int}(F)$ for proving piecewise linearity outside of the special intervals. All intervals I, J, K are closed and elements of the complex $\mathcal{P}$; notation $\langle a,b\rangle$: endpoints are not reached by the projection of the face; (a,b): function π is one-sided discontinuous at the endpoints from within the interval; $[a,b]$: function π is one-sided continuous at the endpoints from within the interval.

Face $F = F(I,J,K)$				vertices of F					
I	J	K	slope	u	v	$u+v$	u	v	$u+v$
$\langle 0,x_1\rangle$	$\{a_0\}$	$[x_9,a_1)$	c_3	$-a_0+x_9{}^+$	a_0	$x_9{}^+$	$-a_0+a_1{}^-$	a_0	$a_1{}^-$
$\langle 0,x_1)$	$\{a_0\}$	(a_1,x_{11})	c_3	$-a_0+a_1{}^+$	a_0	$a_1{}^+$	$x_1{}^-$	a_0	$a_0+x_1{}^-$
$\langle 0,x_1\rangle$	$\{a_1\}$	$[x_{12},a_2)$	c_3	$-a_1+x_{12}{}^+$	a_1	$x_{12}{}^+$	$-a_1+a_2{}^-$	a_1	$a_2{}^-$
$\langle 0,x_1)$	$\{a_1\}$	(a_2,x_{14})	c_3	$-a_1+a_2{}^+$	a_1	$a_2{}^+$	$x_1{}^-$	a_1	$a_1+x_1{}^-$
$\langle 0,x_1)$	$\{a_2\}$	$[x_{15},x_{16}]$	c_3	$-a_2+x_{15}{}^+$	a_2	$x_{15}{}^+$	$x_1{}^-$	a_2	$a_2+x_1{}^-$
$(0,x_1)$	$\{x_{36}\}$	(x_{36},f)	c_3	0^+	x_{36}	$x_{36}{}^+$	$x_1{}^-$	x_{36}	$x_1+x_{36}{}^-$
$(0,x_1)$	$\{x_{38}\}$	$(x_{38},x_{39}\rangle$	c_3	0^+	x_{38}	$x_{38}{}^+$	$x_1{}^-$	x_{38}	$x_1+x_{38}{}^-$
$(x_1,x_2\rangle$	$\{x_3\}$	$(a_0,x_7]$	c_1	$x_1{}^+$	x_3	$x_1+x_3{}^+$	$-x_3+x_7{}^-$	x_3	$x_7{}^-$
$(x_1,x_2\rangle$	$\{a_0\}$	$(x_{11},x_{12}]$	c_1	$x_1{}^+$	a_0	$a_0+x_1{}^+$	$-a_0+x_{12}{}^-$	a_0	$x_{12}{}^-$
$(x_1,x_2\rangle$	$\{a_1\}$	$(x_{14},x_{15}]$	c_1	$x_1{}^+$	a_1	$a_1+x_1{}^+$	$-a_1+x_{15}{}^-$	a_1	$x_{15}{}^-$
$(x_1,x_2\rangle$	$\{a_2\}$	$[x_{16},l]$	c_1	$x_1{}^+$	a_2	$a_2+x_1{}^+$	$-a_2+l^-$	a_2	l^-
$(x_1,x_2\rangle$	$\{x_{36}\}$	(f,x_{38})	c_1	$x_1{}^+$	x_{36}	$x_1+x_{36}{}^+$	$-x_{36}+x_{38}{}^-$	x_{36}	$x_{38}{}^-$
$\{x_3\}$	$[x_{30},f-a_0)$	$\langle x_{35},x_{36})$	c_1	x_3	$x_{30}{}^+$	$x_3+x_{30}{}^+$	x_3	$f-a_0{}^-$	$f-a_0+x_3{}^-$
$\{a_0\}$	$[x_{25},x_{26})$	$\langle x_{35},x_{36})$	c_1	a_0	$x_{25}{}^+$	$a_0+x_{25}{}^+$	a_0	$x_{26}{}^-$	$a_0+x_{26}{}^-$
$\{a_0\}$	$(x_{26},f-a_1)$	$(x_{36},f\rangle$	c_3	a_0	$x_{26}{}^+$	$a_0+x_{26}{}^+$	a_0	$f-a_1{}^-$	$f+a_0-a_1{}^-$
$\{a_0\}$	$(f-a_1,x_{28}]$	$\langle x_{36},f\rangle$	c_3	a_0	$f-a_1{}^+$	$f+a_0-a_1{}^+$	a_0	$x_{28}{}^-$	$a_0+x_{28}{}^-$
$\{a_0\}$	$[x_{33},x_{34})$	$\langle f,x_{38})$	c_1	a_0	$x_{33}{}^+$	$a_0+x_{33}{}^+$	a_0	$x_{34}{}^-$	$a_0+x_{34}{}^-$
$\{a_0\}$	$(x_{34},x_{35}]$	$(x_{38},x_{39}\rangle$	c_3	a_0	$x_{34}{}^+$	$a_0+x_{34}{}^+$	a_0	$x_{35}{}^-$	$a_0+x_{35}{}^-$
$\{a_0\}$	$\langle x_{38},x_{39})$	$[x_2+1,x_3+1)$	c_3	a_0	$-a_0+x_2+1^+$	x_2+1^+	a_0	$x_{39}{}^-$	$a_0+x_{39}{}^-$

Table 3: One-dimensional faces F with additivity on $\operatorname{rel\,int}(F)$ for proving piecewise linearity outside of the special intervals (ctd.)

Face $F = F(I, J, K)$				vertices of F					
I	J	K	slope	u	v	$u+v$	u	v	$u+v$
$\{a_0\}$	$(x_{39}, 1\rangle$	$(x_3{+}1, x_4{+}1]$	c_1	a_0	${x_{39}}^+$	$a_0{+}{x_{39}}^+$	a_0	$-a_0{+}x_4{+}1^-$	$x_4{+}1^-$
$\{a_1\}$	$[x_{22}, x_{23})$	$\langle x_{35}, x_{36})$	c_1	a_1	${x_{22}}^+$	$a_1{+}{x_{22}}^+$	a_1	${x_{23}}^-$	$a_1{+}{x_{23}}^-$
$\{a_1\}$	$(x_{23}, f{-}a_2)$	$(x_{36}, f\rangle$	c_3	a_1	${x_{23}}^+$	$a_1{+}{x_{23}}^+$	a_1	$f{-}{a_2}^-$	$f{+}a_1{-}{a_2}^-$
$\{a_1\}$	$(f{-}a_2, x_{25}]$	$\langle x_{36}, f\rangle$	c_3	a_1	$f{-}{a_2}^+$	$f{+}a_1{-}{a_2}^+$	a_1	${x_{25}}^-$	$a_1{+}{x_{25}}^-$
$\{a_1\}$	$(f{-}a_0, x_{32})$	$\langle f, x_{38})$	c_1	a_1	$f{-}{a_0}^+$	$f{-}a_0{+}{a_1}^+$	a_1	${x_{32}}^-$	$a_1{+}{x_{32}}^-$
$\{a_1\}$	$(x_{32}, x_{33}]$	$(x_{38}, x_{39}\rangle$	c_3	a_1	${x_{32}}^+$	$a_1{+}{x_{32}}^+$	a_1	${x_{33}}^-$	$a_1{+}{x_{33}}^-$
$\{a_1\}$	$\langle x_{38}, x_{39})$	$[x_4{+}1, x_5{+}1)$	c_3	a_1	$-a_1{+}x_4{+}1^+$	$x_4{+}1^+$	a_1	${x_{39}}^-$	$a_1{+}{x_{39}}^-$
$\{a_1\}$	$(x_{39}, 1\rangle$	$(x_5{+}1, a_0{+}1)$	c_1	a_1	${x_{39}}^+$	$a_1{+}{x_{39}}^+$	a_1	$a_0{-}a_1{+}1^-$	$a_0{+}1^-$
$\{a_2\}$	$[f{-}l, x_{21}]$	$\langle x_{35}, x_{36})$	c_1	a_2	$f{-}l^+$	$f{+}a_2{-}l^+$	a_2	${x_{21}}^-$	$a_2{+}{x_{21}}^-$
$\{a_2\}$	$[x_{21}, x_{22}]$	$(x_{36}, f\rangle$	c_3	a_2	${x_{21}}^+$	$a_2{+}{x_{21}}^+$	a_2	${x_{22}}^-$	$a_2{+}{x_{22}}^-$
$\{a_2\}$	$[x_{28}, x_{29})$	$\langle f, x_{38})$	c_1	a_2	${x_{28}}^+$	$a_2{+}{x_{28}}^+$	a_2	${x_{29}}^-$	$a_2{+}{x_{29}}^-$
$\{a_2\}$	$(x_{29}, x_{30}]$	$(x_{38}, x_{39}\rangle$	c_3	a_2	${x_{29}}^+$	$a_2{+}{x_{29}}^+$	a_2	${x_{30}}^-$	$a_2{+}{x_{30}}^-$
$\{a_2\}$	$\langle x_{38}, x_{39})$	$[x_7{+}1, x_8{+}1)$	c_3	a_2	$-a_2{+}x_7{+}1^+$	$x_7{+}1^+$	a_2	${x_{39}}^-$	$a_2{+}{x_{39}}^-$
$\{a_2\}$	$(x_{39}, 1\rangle$	$(x_8{+}1, x_9{+}1]$	c_1	a_2	${x_{39}}^+$	$a_2{+}{x_{39}}^+$	a_2	$-a_2{+}x_9{+}1^-$	$x_9{+}1^-$
$\{x_{36}\}$	$(x_{39}, 1)$	$\langle x_{35}{+}1, x_{36}{+}1)$	c_1	x_{36}	${x_{39}}^+$	$x_{36}{+}{x_{39}}^+$	x_{36}	1^-	$x_{36}{+}1^-$

Table 4: Faces F with additivity on $\operatorname{rel\,int}(F)$, one vertex of each providing an equation $\Delta\bar{\pi}_F(u,v) = 0$, to form a full-rank homogeneous linear system in the proof of Theorem 6.2. All intervals I, J, K are closed and elements of the complex $\mathcal{P}$; notation $\langle a,b\rangle$: endpoints are not reached by the projection of the face; (a,b): function π is one-sided discontinuous at the endpoints from within the interval; $[a,b]$: function π is one-sided continuous at the endpoints from within the interval.

Face $F = F(I,J,K)$			selected vertex			other vertices of F					
I	J	K	u	v	$u+v$	u	v	$u+v$	u	v	$u+v$
$\langle 0,x_1\rangle$	$\{a_0\}$	$\{x_8\}$	$-a_0+x_8$	a_0	x_8						
$\langle 0,x_1\rangle$	$\{a_0\}$	$[x_9,a_1)$	$-a_0+x_9{}^+$	a_0	$x_9{}^+$	$-a_0+a_1{}^-$	$a_0{}^-$	$a_1{}^-$			
$\langle 0,x_1\rangle$	$\{a_0\}$	(a_1,x_{11})	$x_1{}^-$	a_0	$a_0+x_1{}^-$	$-a_0+a_1{}^+$	$a_0{}^-$	$a_1{}^+$			
$\langle 0,x_1\rangle$	$\{a_1\}$	$[x_{12},a_2)$	$-a_1+x_{12}{}^+$	a_1	$x_{12}{}^+$	$-a_1+a_2{}^-$	$a_1{}^-$	$a_2{}^-$			
$\langle 0,x_1\rangle$	$\{a_1\}$	(a_2,x_{14})	$x_1{}^-$	a_1	$a_1+x_1{}^-$	$-a_1+a_2{}^+$	$a_1{}^-$	$a_2{}^+$			
$\langle 0,x_1\rangle$	$\{a_2\}$	$\{x_{15}\}$	$-a_2+x_{15}$	a_2	x_{15}						
$\langle 0,x_1\rangle$	$\{a_2\}$	$[x_{15},x_{16}]$	$x_1{}^-$	a_2	$a_2+x_1{}^-$	$-a_2+x_{15}{}^+$	$a_2{}^-$	$x_{15}{}^+$			
$(0,x_1)$	$\{x_{36}\}$	(x_{36},f)	0^+	x_{36}	$x_{36}{}^+$	$x_1{}^-$	$x_{36}{}^-$	$x_1+x_{36}{}^-$			
$(0,x_1)$	$\{x_{38}\}$	$(x_{38},x_{39}\rangle$	0^+	x_{38}	$x_{38}{}^+$	$x_1{}^-$	$x_{38}{}^-$	$x_1+x_{38}{}^-$			
$(x_1,x_2\rangle$	$(x_1,x_2\rangle$	$\langle x_1,x_2]$	$x_1{}^+$	$x_1{}^+$	$2x_1{}^+$	$x_1{}^+$	$-x_1+x_2{}^-$	$x_2{}^-$	$-x_1+x_2{}^-$	$x_1{}^+$	$x_2{}^-$
$(x_1,x_2\rangle$	$\{x_3\}$	$(a_0,x_7]$	$x_1{}^+$	x_3	$x_1+x_3{}^+$	$-x_3+x_7{}^-$	$x_3{}^-$	$x_7{}^-$			
$(x_1,x_2\rangle$	$\{a_0\}$	$(x_{11},x_{12}]$	$x_1{}^+$	a_0	$a_0+x_1{}^+$	$-a_0+x_{12}{}^-$	$a_0{}^-$	$x_{12}{}^-$			
$\langle x_1,x_2\rangle$	$\{a_0\}$	$\{x_{12}\}$	$-a_0+x_{12}$	a_0	x_{12}						
$(x_1,x_2\rangle$	$\{a_1\}$	$(x_{14},x_{15}]$	$x_1{}^+$	a_1	$a_1+x_1{}^+$	$-a_1+x_{15}{}^-$	$a_1{}^-$	$x_{15}{}^-$			
$(x_1,x_2\rangle$	$\{x_{11}\}$	$\langle x_{14},x_{15}]$	$x_1{}^+$	x_{11}	$x_1+x_{11}{}^+$	$-x_{11}+x_{15}{}^-$	$x_{11}{}^-$	$x_{15}{}^-$			
$(x_1,x_2\rangle$	$\{a_2\}$	$[x_{16},l]$	$x_1{}^+$	a_2	$a_2+x_1{}^+$	$-a_2+l^-$	$a_2{}^-$	l^-			
$(x_1,x_2\rangle$	$\{x_{16}\}$	$\langle x_{16},l]$	$x_1{}^+$	x_{16}	$x_1+x_{16}{}^+$	$l-x_{16}{}^-$	$x_{16}{}^-$	l^-			
$(x_1,x_2\rangle$	$\{u\}$	$\langle u,f-u]$	$x_1{}^+$	u	$u+x_1{}^+$	$f-2u^-$	u^-	$f-u^-$			

Table 4: Faces F with additivity on $\operatorname{rel\,int}(F)$, one vertex of each providing an equation $\Delta\bar{\pi}_F(u,v)=0$ (ctd.)

Face $F=F(I,J,K)$			selected vertex			other vertices of F					
I	J	K	u	v	$u+v$	u	v	$u+v$	u	v	$u+v$
$\langle x_1,x_2\rangle$	$\{f-l\}$	$\langle f-l,x_{21}]$	${x_1}^+$	$f-l$	${f-l+x_1}^+$	${-f+l+x_{21}}^-$	${f-l}^-$	${x_{21}}^-$			
$\langle x_1,x_2\rangle$	$\{x_{23}\}$	$\{f-a_0\}$	$f-a_0-x_{23}$	x_{23}	$f-a_0$						
$\langle x_1,x_2\rangle$	$\{x_{35}\}$	$\langle x_{35},x_{36})$	${x_1}^+$	x_{35}	${x_1+x_{35}}^+$	${-x_{35}+x_{36}}^-$	${x_{35}}^-$	${x_{36}}^-$			
$\langle x_1,x_2\rangle$	$\{x_{36}\}$	(f,x_{38})	${x_1}^+$	x_{36}	${x_1+x_{36}}^+$	${-x_{36}+x_{38}}^-$	${x_{36}}^-$	${x_{38}}^-$			
$\{a_0\}$	$\{x_{32}\}$	$\langle f,x_{38}\rangle$	a_0	x_{32}	a_0+x_{32}						
$\{a_0\}$	$[x_{33},x_{34})$	$\langle f,x_{38})$	a_0	${x_{33}}^+$	${a_0+x_{33}}^+$	${a_0}^-$	${x_{34}}^-$	${a_0+x_{34}}^-$			
$\{a_0\}$	$(x_{34},x_{35}]$	$(x_{38},x_{39}\rangle$	a_0	${x_{34}}^+$	${a_0+x_{34}}^+$	${a_0}^-$	${x_{35}}^-$	${a_0+x_{35}}^-$			
$\{a_1\}$	$[x_{30},f-a_0)$	$\langle f,x_{38}\rangle$	a_1	${x_{30}}^+$	${a_1+x_{30}}^+$	${a_1}^-$	${f-a_0}^-$	${f-a_0+a_1}^-$			
$\{a_1\}$	$(f-a_0,x_{32})$	$\langle f,x_{38})$	a_1	${f-a_0}^+$	${f-a_0+a_1}^+$	${a_1}^-$	${x_{32}}^-$	${a_1+x_{32}}^-$			
$\{a_1\}$	$(x_{32},x_{33}]$	$(x_{38},x_{39}\rangle$	a_1	${x_{32}}^+$	${a_1+x_{32}}^+$	${a_1}^-$	${x_{33}}^-$	${a_1+x_{33}}^-$			
$\{a_1\}$	$\langle x_{38},x_{39}\rangle$	$\{x_4+1\}$	a_1	$-a_1+x_4+1$	x_4+1						
$\{x_{11}\}$	$\{x_{22}\}$	$\langle x_{35},x_{36}\rangle$	x_{11}	x_{22}	$x_{11}+x_{22}$						
$\{a_2\}$	$[x_{16},l\rangle$	$\langle u,f-u]$	a_2	${x_{16}}^+$	${a_2+x_{16}}^+$	${a_2}^-$	${f-a_2-u}^-$	${f-u}^-$			
$\{a_2\}$	$\{x_{28}\}$	$\langle f,x_{38}\rangle$	a_2	x_{28}	a_2+x_{28}						
$\{a_2\}$	$[x_{28},x_{29})$	$\langle f,x_{38})$	a_2	${x_{28}}^+$	${a_2+x_{28}}^+$	${a_2}^-$	${x_{29}}^-$	${a_2+x_{29}}^-$			
$\{a_2\}$	$(x_{29},x_{30}]$	$(x_{38},x_{39}\rangle$	a_2	${x_{29}}^+$	${a_2+x_{29}}^+$	${a_2}^-$	${x_{30}}^-$	${a_2+x_{30}}^-$			
$\{x_{30}\}$	$\langle x_{39},1\rangle$	$\{f-a_1+1\}$	x_{30}	$f-a_1-x_{30}+1$	$f-a_1+1$						
$\{x_{33}\}$	$\langle x_{39},1\rangle$	$\{f-a_0+1\}$	x_{33}	$f-a_0-x_{33}+1$	$f-a_0+1$						
$\{x_{35}\}$	$\langle x_{38},x_{39}\rangle$	$\{f-a_0+1\}$	x_{35}	$f-a_0-x_{35}+1$	$f-a_0+1$						
$\{x_{38}\}$	$(x_{39},1)$	$(f+1,x_{38}+1)$	x_{38}	${x_{39}}^+$	${x_{38}+x_{39}}^+$	${x_{38}}^-$	1^-	${x_{38}+1}^-$			
$\langle x_{38},x_{39})$	$\langle x_{38},x_{39})$	$(x_{38}+1,x_{39}+1\rangle$	${x_{39}}^-$	${x_{39}}^-$	${2x_{39}}^-$	${x_{38}-x_{39}+1}^+$	${x_{39}}^-$	${x_{38}+1}^+$	${x_{39}}^-$	${x_{38}-x_{39}+1}^+$	${x_{38}+1}^+$

Table 5: Homogeneous linear system for determining the restriction of $\bar{\pi}$ to the non-special intervals in the proof of Theorem 6.2. The variables are the values of $\bar{\pi}$ at breakpoints (•) and at midpoints of intervals between breakpoints (−), and the slopes $\bar{c}_1$, $\bar{c}_3$ of $\bar{\pi}$ on the non-special intervals. Matrix coefficients are abbreviated as $+ = 1$, $- = -1$, and $\cdot = 0$.

Eqn. $\Delta\bar{\pi}_F(u,v)=0$			Coefficients of $\bar{\pi}(x_i)$ and $\bar{\pi}(\frac{x_i+x_{i+1}}{2})$		Coefficients of slopes	
u	v	$u+v$	$-\bullet^{x_1}-\bullet-\bullet-\bullet-\bullet-\bullet^{a_0}-\bullet-\bullet-\bullet-\bullet^{a_1}-\bullet-\bullet-\bullet^{a_2}-\bullet-\bullet-\bullet-\bullet^{l}\bullet^{u}$	$\wr\ -\bullet^{x_{38}}$	$\bar{c}_1$	$\bar{c}_3$
x_{36}	0^+	x_{36}^+	2 − ·	· ·	·	·
a_0	$-a_0+x_8$	x_8	+ · · · · · · · · · · + · · · − · · · · · · · · · · · · · · · · · · ·	· ·	·	$\frac{1}{2}(-2a_0-x_1+2x_8)$
a_0	$-a_0+x_9^+$	x_9^+	+ · · · · · · · · · · + · · · · · · − · · · · · · · · · · · · · · · ·	· ·	·	$\frac{1}{2}(-2a_0+a_1-x_1+x_9)$
a_0	x_1^-	$a_0+x_1^-$	+ · · · · · · · · · · + · · · · · · · · − · · · · · · · · · · · · · ·	· ·	·	$\frac{1}{2}(-a_0+a_1)$
a_1	$-a_1+x_{12}^+$	x_{12}^+	+ · · · · · · · · · · · · · · · · · · + · · · · − · · · · · · · · · ·	· ·	·	$\frac{1}{2}(-2a_1+a_2-x_1+x_{12})$
a_1	x_1^-	$a_1+x_1^-$	+ · · · · · · · · · · · · · · · · · · + · · · · · · − · · · · · · · ·	· ·	·	$\frac{1}{2}(-a_1+a_2)$
a_2	$-a_2+x_{15}$	x_{15}	+ · + · · · − · · · · ·	· ·	·	$\frac{1}{2}(-2a_2-x_1+2x_{15})$
a_2	x_1^-	$a_2+x_1^-$	+ · + · · · · − · · · ·	· ·	·	$\frac{1}{2}(-a_2+x_{15})$
x_{38}	0^+	x_{38}^+	+ ·	· +	·	$\frac{1}{2}(-a_1-x_1-x_{32}+x_{39})$
x_{36}	x_1^+	$x_1+x_{36}^+$	· − + ·	− ·	$\frac{1}{2}(-x_2-x_{36}+x_{38})$	·
x_{35}	x_1^+	$x_1+x_{35}^+$	· · 2 − ·	· ·	$\frac{1}{2}(-x_1-x_2-x_{35}+x_{36})$	·
x_3	x_1^+	$x_1+x_3^+$	· · + · · + · · · · · · − ·	· ·	$\frac{1}{2}(-x_2-x_3+x_7)$	·
a_0	x_1^+	$a_0+x_1^+$	· · + · · · · · · · · + · · · · · · · · · · − · · · · · · · · · · · ·	· ·	$\frac{1}{2}(-a_0+x_{12}-x_2)$	·
a_0	$-a_0+x_{12}$	x_{12}	· · + · · · · · · · · + · · · · · · · · · · · − · · · · · · · · · · ·	· ·	$\frac{1}{2}(-2a_0-x_1+2x_{12}-x_2)$	·
x_{23}	$f-a_0-x_{23}$	$f-a_0$	· · + · · · · · · · · + · · · · · · · · · · · · · · · − · · · · · · ·	· ·	$\frac{1}{2}(2f-2a_0-x_1-x_2-2x_{23})$	·
a_1	x_1^+	$a_1+x_1^+$	· · + · · · · · · · · · · · · · · · · + · · · · · · · · − · · · · · ·	· ·	$\frac{1}{2}(-a_1+x_{15}-x_2)$	·
x_{11}	x_1^+	$x_1+x_{11}^+$	· · + · · · · · · · · · · · · · · · · · · + · · · · · · − · · · · · ·	· ·	$\frac{1}{2}(a_1-2x_{11}+x_{15}-x_2)$	·
x_{11}	x_{22}	$x_{11}+x_{22}$	· · + · · · · · · · · · · · · · · · · · · + · · · · · · · − · · · · ·	· ·	$\frac{1}{2}(-2x_{11}-2x_{22}+x_{35}+x_{36})$	·
a_2	x_1^+	$a_2+x_1^+$	· · + · + · · · · · · − · ·	· ·	$\frac{1}{2}(-a_2+l-x_2)$	·

Table 5: Homogeneous linear system for determining the restriction of $\bar{\pi}$ to the non-special intervals in the proof of Theorem 6.2 (ctd.)

Eqn. $\Delta\bar{\pi}_F(u,v)=0$			Coefficients of $\bar{\pi}(x_i)$ and $\bar{\pi}(\frac{x_i+x_{i+1}}{2})$	Coefficients of slopes	
u	v	$u+v$	$-\overset{x_1}{\bullet}-\bullet-\bullet-\bullet-\bullet-\overset{a_0}{\bullet}-\bullet-\bullet-\bullet-\overset{a_1}{\bullet}-\bullet-\bullet-\overset{a_2}{\bullet}-\bullet-\bullet-\bullet-\overset{l}{\bullet}\overset{u}{\bullet}\wr-\overset{x_{38}}{\bullet}$	$\bar{c}_1$	$\bar{c}_3$
x_{16}	${x_1}^+$	$x_1+{x_{16}}^+$	··+····························+−·· ··	$\frac{1}{2}(a_2+l-2x_{16}-x_2)$	·
$f-l$	${x_1}^+$	$f-l+{x_1}^+$	··+·····························+−· ··	$\frac{1}{2}(-f-a_2+l-x_1-x_2+x_{36})$	·
u	${x_1}^+$	$u+{x_1}^+$	··+·······························+ ··	$\frac{1}{2}(f-2u-x_1-x_2)$	·
${x_1}^+$	${x_1}^+$	$2{x_1}^+$	··+································ ··	$\frac{1}{2}(-x_1-x_2)$	·
x_{35}	$f-a_0-x_{35}+1$	$f-a_0+1$	···−·······+······················· ··	·	$\frac{1}{2}(2f-2a_0-a_1-x_{32}-2x_{35}-x_{39}+2)$
a_0	${x_{34}}^+$	$a_0+{x_{34}}^+$	····−······+······················· ··	·	$\frac{1}{2}(-a_1-x_{32}+x_{34}-x_{35}+x_{39})$
a_0	${x_{33}}^+$	$a_0+{x_{33}}^+$	······−····+······················· −·	$\frac{1}{2}(-a_0+x_1-x_{33}+x_{36})$	·
x_{33}	$f-a_0-x_{33}+1$	$f-a_0+1$	·······−···+······················· −·	$\frac{1}{2}(2f-2a_0-2x_{33}-x_{39}+1)$	·
a_1	$-a_1+x_4+1$	x_4+1	·······−···········+··············· ··	·	$\frac{1}{2}(-3a_1-x_{32}-x_{39}+2x_4+2)$
a_1	${x_{32}}^+$	$a_1+{x_{32}}^+$	········−··········+··············· ··	·	$\frac{1}{2}(-a_1-x_{33}+x_{39})$
a_0	x_{32}	a_0+x_{32}	·········−·+······················· −·	$\frac{1}{2}(-2a_0+x_1-2x_{32}+x_{36}+x_{38})$	·
a_1	$f-{a_0}^+$	$f-a_0+{a_1}^+$	··········−········+··············· −·	$\frac{1}{2}(-f+a_0-a_1+x_1+x_{36})$	·
a_1	${x_{30}}^+$	$a_1+{x_{30}}^+$	············−······+··············· −·	$\frac{1}{2}(-2a_1+x_1+x_3-x_{30}+x_{38})$	·
x_{30}	$f-a_1-x_{30}+1$	$f-a_1+1$	·············−·····+··············· −·	$\frac{1}{2}(2f-2a_1-2x_{30}-x_{39}+1)$	·
a_2	${x_{29}}^+$	$a_2+{x_{29}}^+$	··············−··········+········· ··	·	$\frac{1}{2}(-a_1+x_{29}-x_{30}-x_{32}+x_{39})$
a_2	${x_{28}}^+$	$a_2+{x_{28}}^+$	················−········+········· −·	$\frac{1}{2}(-a_2+x_1-x_{28}+x_{36})$	·
a_2	x_{28}	a_2+x_{28}	·················−·······+········· −·	$\frac{1}{2}(-2a_2+x_1-2x_{28}+x_{36}+x_{38})$	·
a_2	${x_{16}}^+$	$a_2+{x_{16}}^+$	·························+······+·· ··	$\frac{1}{2}(f-a_2-l+x_1-2x_{16})$	·
x_{38}	${x_{39}}^+$	$x_{38}+{x_{39}}^+$	··································· 2 −	·	·
${x_{39}}^-$	${x_{39}}^-$	$2{x_{39}}^-$	··································· ··	·	$\frac{1}{2}(-a_1-x_{32}-x_{39}+2)$

Table 6: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 1$ and $n_F > 0$. All intervals I, J, K are closed and elements of the complex $\mathcal{P}$; notation $\langle a, b\rangle$: endpoints are not reached by the projection of the face; (a, b): function π is one-sided discontinuous at the endpoints from within the interval; $[a, b]$: function π is one-sided continuous at the endpoints from within the interval. An asterisk marks the special intervals.

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$\{0\}$	$(l,u)^*$	$(l,u)^*$	2	0	0 0
$\{0\}$	$(f-u,f-l)^*$	$(f-u,f-l)^*$	2	0	0 0
$\{x_1\}$	$\langle x_{16}, l]$	$(l,u\rangle^*$	1	$-c_2x + (c_1-c_2)y - \frac{1280651}{2999750}$	0.256 0.310
$\{x_1\}$	$(l,u\rangle^*$	$\langle l,u\rangle^*$	2	$-c_2x + \frac{2727}{13000}$	0.210 0.210
$\{x_1\}$	$\langle l,u)^*$	$[u, f-u\rangle$	1	$-c_1x - (c_1-c_2)y + \frac{1702237}{1499875}$	0.175 0.230
$\{x_1\}$	$\langle u, f-u]$	$(f-u, f-l\rangle^*$	1	$-c_2x + (c_1-c_2)y - \frac{1527763}{1499875}$	0.175 0.230
$\{x_1\}$	$(f-u, f-l\rangle^*$	$\langle f-u, f-l\rangle^*$	2	$-c_2x + \frac{2727}{13000}$	0.210 0.210
$\{x_1\}$	$\langle f-u, f-l)^*$	$[f-l, x_{21}\rangle$	1	$-c_1x - (c_1-c_2)y + \frac{5179349}{2999750}$	0.256 0.310
$\langle x_1, x_2]$	$\{x_3\}$	$(l,u\rangle^*$	1	$(c_1-c_2)x - c_2y - \frac{1925}{71994}\sqrt{2} + \frac{12937}{230750}$	0.298 0.457
$\langle x_1, x_2\rangle$	$\{x_5\}$	$(l,u\rangle^*$	1	$(c_1-c_2)x - c_2y + \frac{2695}{100776}\sqrt{2} + \frac{12937}{230750}$	0.336 0.504
$\langle x_1, x_2\rangle$	$\{a_0\}$	$(l,u\rangle^*$	1	$(c_1-c_2)x - c_2y - \frac{1659}{36920}$	0.180 0.349
$\langle x_1, x_2\rangle$	$\{x_8\}$	$(l,u\rangle^*$	1	$(c_1-c_2)x - c_2y + \frac{18619889}{298129000}$	0.262 0.430
$\langle x_1, x_2\rangle$	$\{a_1\}$	$(l,u\rangle^*$	1	$(c_1-c_2)x - c_2y + \frac{385}{93016248}\sqrt{2} - \frac{1659}{36920}$	0.143 0.311
$\langle x_1, x_2\rangle$	$\{x_{11}\}$	$(l,u\rangle^*$	1	$(c_1-c_2)x - c_2y - \frac{1925}{71994}\sqrt{2} + \frac{4361}{461500}$	0.143 0.311
$\langle x_1, x_2\rangle$	$\{a_2\}$	$(l,u\rangle^*$	1	$(c_1-c_2)x - c_2y - \frac{870527}{19378385}$	0.100 0.268
$\langle x_1, x_2\rangle$	$\{x_{14}\}$	$(l,u\rangle^*$	1	$(c_1-c_2)x - c_2y + \frac{2695}{100776}\sqrt{2} + \frac{4361}{461500}$	0.180 0.349
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$\{x_{23}\}$	1	$c_1x + c_2y + \frac{2695}{100776}\sqrt{2} + \frac{54693}{5999500}$	0.180 0.349
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$\{f-a_2\}$	1	$c_1x + c_2y - \frac{876987}{19378385}$	0.100 0.268
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$\{x_{26}\}$	1	$c_1x + c_2y - \frac{1925}{71994}\sqrt{2} + \frac{54693}{5999500}$	0.143 0.311
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$\{f-a_1\}$	1	$c_1x + c_2y + \frac{385}{93016248}\sqrt{2} - \frac{21727}{479960}$	0.143 0.311

Table 6: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 1$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$	
$\langle x_1, x_2\rangle$	$(f{-}u, f{-}l)^*$	$\{x_{29}\}$	1	$c_1x + c_2y + \frac{240766557}{3875677000}$	0.262 0.430	
$\langle x_1, x_2\rangle$	$(f{-}u, f{-}l)^*$	$\{f{-}a_0\}$	1	$c_1x + c_2y - \frac{21727}{479960}$	0.180 0.349	
$\langle x_1, x_2\rangle$	$(f{-}u, f{-}l)^*$	$\{x_{32}\}$	1	$c_1x + c_2y + \frac{2695}{100776}\sqrt{2} + \frac{167181}{2999750}$	0.336 0.504	
$\langle x_1, x_2]$	$\langle f{-}u, f{-}l)^*$	$\{x_{34}\}$	1	$c_1x + c_2y - \frac{1925}{71994}\sqrt{2} + \frac{167181}{2999750}$	0.298 0.457	
$[x_2, x_3\rangle$	$\{x_3\}$	$\langle l, u)^*$	1	$-(c_2{-}c_3)x - c_2y - \frac{1925}{71994}\sqrt{2} + \frac{7855487}{5999500}$	0.439 0.457	
$[x_2, x_3\rangle$	$(f{-}u, f{-}l\rangle^*$	$\{x_{34}\}$	1	$c_3x + c_2y - \frac{1925}{71994}\sqrt{2} + \frac{7853487}{5999500}$	0.439 0.457	
$\{x_3\}$	$(l, u\rangle^*$	$\langle u, f{-}u]$	1	$-c_1x - (c_1{-}c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{7975987}{5999500}$	0.043 0.098	
$\{x_3\}$	$\langle l, u)^*$	$(f{-}u, f{-}l\rangle^*$	2	$-c_2x - \frac{1925}{71994}\sqrt{2} + \frac{1212849}{11999000}$	0.063 0.063	
$\{x_3\}$	$[u, f{-}u\rangle$	$\langle f{-}u, f{-}l)^*$	1	$-c_2x + (c_1{-}c_2)y - \frac{1925}{71994}\sqrt{2} - \frac{4944013}{5999500}$	0.043 0.098	
$\{x_3\}$	$(f{-}u, f{-}l\rangle^*$	$\langle x_{34}, x_{35}]$	1	$-c_3x + (c_2{-}c_3)y - \frac{1925}{71994}\sqrt{2} - \frac{16144513}{5999500}$	0.439 0.457	
$\{x_3\}$	$\langle f{-}u, f{-}l)^*$	$[x_{35}, x_{36}\rangle$	1	$-c_1x - (c_1{-}c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{6628181}{2999750}$	0.298 0.457	
$\{x_5\}$	$(l, u\rangle^*$	$\langle u, f{-}u]$	1	$-c_1x - (c_1{-}c_2)y + \frac{2695}{100776}\sqrt{2} + \frac{7975987}{5999500}$	0.119 0.135	
$\{x_5\}$	$\langle l, u)^*$	$(f{-}u, f{-}l\rangle^*$	2	$-c_2x + \frac{2695}{100776}\sqrt{2} + \frac{1212849}{11999000}$	0.139 0.139	
$\{x_5\}$	$[u, f{-}u\rangle$	$\langle f{-}u, f{-}l)^*$	1	$-c_2x + (c_1{-}c_2)y + \frac{2695}{100776}\sqrt{2} - \frac{4944013}{5999500}$	0.119 0.135	
$\{x_5\}$	$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-c_1x - (c_1{-}c_2)y + \frac{2695}{100776}\sqrt{2} + \frac{6628181}{2999750}$	0.336 0.504	
$\{a_0\}$	$(l, u)^*$	$(f{-}u, f{-}l)^*$	2	0	0 0	(ii)
$\{a_0\}$	$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-c_1x - (c_1{-}c_2)y + \frac{1012033}{479960}$	0.180 0.349	
$\{x_8\}$	$\langle x_{16}, l]$	$(f{-}u, f{-}l\rangle^*$	1	$-c_2x + (c_1{-}c_2)y - \frac{2051079943}{3875677000}$	0.182 0.208	
$\{x_8\}$	$(l, u)^*$	$\langle f{-}u, f{-}l)^*$	2	$-c_2x + \frac{104129733}{968919250}$	0.107 0.107	
$\{x_8\}$	$\langle l, u)^*$	$[f{-}l, x_{21}\rangle$	1	$-c_1x - (c_1{-}c_2)y + \frac{6295240057}{3875677000}$	0.182 0.208	
$\{x_8\}$	$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-c_1x - (c_1{-}c_2)y + \frac{8588378557}{3875677000}$	0.262 0.430	
$\{a_1\}$	$\langle x_{16}, l]$	$(f{-}u, f{-}l\rangle^*$	1	$-c_2x + (c_1{-}c_2)y + \frac{385}{93016248}\sqrt{2} - \frac{305547}{479960}$	0.062 0.100	
$\{a_1\}$	$(l, u)^*$	$\langle f{-}u, f{-}l)^*$	2	0	0 0	(ii)

Table 6: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 1$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$	
$\{a_1\}$	$\langle l,u)^*$	$[f-l, x_{21}\rangle$	1	$-c_1x-(c_1-c_2)y+\frac{385}{93016248}\sqrt{2}+\frac{728053}{479960}$	0.062 0.100	
$\{a_1\}$	$(f-u,f-l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-c_1x-(c_1-c_2)y+\frac{385}{93016248}\sqrt{2}+\frac{1012033}{479960}$	0.143 0.311	
$\{x_{11}\}$	$\langle x_{16}, l]$	$(f-u,f-l\rangle^*$	1	$-c_2x+(c_1-c_2)y-\frac{1925}{71994}\sqrt{2}-\frac{3493057}{5999500}$	0.062 0.117	
$\{x_{11}\}$	$(l,u\rangle^*$	$\langle f-u,f-l)^*$	2	$-c_2x-\frac{1925}{71994}\sqrt{2}+\frac{653511}{11999000}$	0.017 0.017	
$\{x_{11}\}$	$\langle l,u)^*$	$[f-l, x_{21}\rangle$	1	$-c_1x-(c_1-c_2)y-\frac{1925}{71994}\sqrt{2}+\frac{9426943}{5999500}$	0.062 0.117	
$\{x_{11}\}$	$(f-u,f-l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-c_1x-(c_1-c_2)y-\frac{1925}{71994}\sqrt{2}+\frac{12976693}{5999500}$	0.143 0.311	
$\{a_2\}$	$\langle x_{16}, l]$	$(f-u,f-l\rangle^*$	1	$-c_2x+(c_1-c_2)y-\frac{24672439}{38756770}$	0.020 0.100	
$\{a_2\}$	$(l,u\rangle^*$	$\langle f-u,f-l)^*$	2	0	0 0	(ii)
$\{a_2\}$	$\langle l,u)^*$	$[f-l, x_{21}\rangle$	1	$-c_1x-(c_1-c_2)y+\frac{58790761}{38756770}$	0.020 0.100	
$\{a_2\}$	$(f-u,f-l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-c_1x-(c_1-c_2)y+\frac{40861073}{19378385}$	0.100 0.268	
$\{x_{14}\}$	$\langle x_{15}, x_{16}]$	$(f-u,f-l\rangle^*$	1	$-c_2x-(c_2-c_3)y+\frac{2695}{100776}\sqrt{2}+\frac{2449272129}{1937838500}$	0.102 0.104	
$\{x_{14}\}$	$[x_{16}, l]$	$\langle f-u,f-l\rangle^*$	1	$-c_2x+(c_1-c_2)y+\frac{2695}{100776}\sqrt{2}-\frac{3493057}{5999500}$	0.102 0.192	
$\{x_{14}\}$	$(l,u\rangle^*$	$\langle f-u,f-l)^*$	2	$-c_2x+\frac{2695}{100776}\sqrt{2}+\frac{653511}{11999000}$	0.092 0.092	
$\{x_{14}\}$	$\langle l,u)^*$	$[f-l, x_{21}]$	1	$-c_1x-(c_1-c_2)y+\frac{2695}{100776}\sqrt{2}+\frac{9426943}{5999500}$	0.102 0.192	
$\{x_{14}\}$	$\langle l,u)^*$	$[x_{21}, x_{22}\rangle$	1	$-c_3x+(c_2-c_3)y+\frac{2695}{100776}\sqrt{2}-\frac{5302727871}{1937838500}$	0.102 0.104	
$\{x_{14}\}$	$(f-u,f-l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-c_1x-(c_1-c_2)y+\frac{2695}{100776}\sqrt{2}+\frac{12976693}{5999500}$	0.180 0.349	
$\langle x_{15}, x_{16}]$	$\langle l,u)^*$	$\{x_{23}\}$	1	$c_3x+c_2y+\frac{2695}{100776}\sqrt{2}+\frac{2448626129}{1937838500}$	0.102 0.104	
$[x_{16}, l]$	$\langle l,u)^*$	$\{x_{23}\}$	1	$c_1x+c_2y+\frac{2695}{100776}\sqrt{2}-\frac{3495057}{5999500}$	0.102 0.192	
$\langle x_{16}, l]$	$\langle l,u)^*$	$\{f-a_2\}$	1	$c_1x+c_2y-\frac{24685359}{38756770}$	0.020 0.100	
$\langle x_{16}, l]$	$\langle l,u)^*$	$\{x_{26}\}$	1	$c_1x+c_2y-\frac{1925}{71994}\sqrt{2}-\frac{3495057}{5999500}$	0.062 0.117	
$\langle x_{16}, l]$	$\langle l,u)^*$	$\{f-a_1\}$	1	$c_1x+c_2y+\frac{385}{93016248}\sqrt{2}-\frac{305707}{479960}$	0.062 0.100	
$\langle x_{16}, l]$	$\langle l,u)^*$	$\{x_{29}\}$	1	$c_1x+c_2y-\frac{2052371943}{3875677000}$	0.182 0.208	
$\langle x_{16}, l]$	$\langle f-u,f-l)^*$	$\{x_{36}\}$	1	$c_1x+c_2y-\frac{1281651}{2999750}$	0.256 0.310	

Table 6: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 1$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$	
$(l,u\rangle^*$	$(l,u\rangle^*$	$\{x_{23}\}$	2	$c_2x + c_2y + \frac{2695}{100776}\sqrt{2} + \frac{649511}{11999000}$	0.092 0.092	
$(l,u\rangle^*$	$(l,u\rangle^*$	$\{f-a_2\}$	2	0	0 0	
$(l,u\rangle^*$	$(l,u\rangle^*$	$\{x_{26}\}$	2	$c_2x + c_2y - \frac{1925}{71994}\sqrt{2} + \frac{649511}{11999000}$	0.017 0.017	
$(l,u\rangle^*$	$(l,u\rangle^*$	$\{f-a_1\}$	2	0	0 0	
$(l,u\rangle^*$	$(l,u\rangle^*$	$\{x_{29}\}$	2	$c_2x + c_2y + \frac{103806733}{968919250}$	0.107 0.107	
$(l,u)^*$	$(l,u)^*$	$\{f-a_0\}$	2	0	0 0	(iii)
$\langle l,u)^*$	$\langle l,u)^*$	$\{x_{32}\}$	2	$c_2x + c_2y + \frac{2695}{100776}\sqrt{2} + \frac{1208849}{11999000}$	0.139 0.139	
$\langle l,u)^*$	$\langle l,u)^*$	$\{x_{34}\}$	2	$c_2x + c_2y - \frac{1925}{71994}\sqrt{2} + \frac{1208849}{11999000}$	0.063 0.063	
$(l,u\rangle^*$	$[u,f-u\rangle$	$\{x_{32}\}$	1	$c_2x + c_1y + \frac{2695}{100776}\sqrt{2} - \frac{4946013}{5999500}$	0.119 0.135	
$(l,u\rangle^*$	$[u,f-u\rangle$	$\{x_{34}\}$	1	$c_2x + c_1y - \frac{1925}{71994}\sqrt{2} - \frac{4946013}{5999500}$	0.043 0.098	
$\langle l,u)^*$	$\langle u,f-u]$	$\{x_{36}\}$	1	$c_2x + c_1y - \frac{1528263}{1499875}$	0.175 0.230	
$(l,u\rangle^*$	$(f-u,f-l\rangle^*$	$\{x_{36}\}$	2	$c_2x + c_2y + \frac{2513021}{11999000}$	0.210 0.210	
$(l,u\rangle^*$	$(f-u,f-l\rangle^*$	$\{x_{36}\}$	2	$c_2x + c_2y + \frac{2513021}{11999000}$	0.210 0.210	
$(l,u)^*$	$(f-u,f-l)^*$	$\{f\}$	2	0	0 0	(symm.)
$(l,u)^*$	$(f-u,f-l)^*$	$\{f\}$	2	0	0 0	(symm.)
$\langle l,u)^*$	$\langle f-u,f-l)^*$	$\{x_{38}\}$	2	$c_2x + c_2y + \frac{3072359}{11999000}$	0.256 0.256	
$\langle l,u)^*$	$\langle f-u,f-l)^*$	$\{x_{38}\}$	2	$c_2x + c_2y + \frac{3072359}{11999000}$	0.256 0.256	
$(l,u\rangle^*$	$[f-l,x_{21}\rangle$	$\{x_{38}\}$	1	$c_2x + c_1y - \frac{1891002}{1499875}$	0.156 0.211	
$(l,u)^*$	$\{x_{23}\}$	$\langle x_{38},x_{39}\rangle$	1	$(c_2-c_3)x - c_3y - \frac{2695}{100776}\sqrt{2} - \frac{1850361}{461500}$	0.200 0.513	
$(l,u)^*$	$\{f-a_2\}$	$\langle x_{38},x_{39}\rangle$	1	$(c_2-c_3)x - c_3y - \frac{76643013}{19378385}$	0.315 0.627	
$(l,u)^*$	$\{x_{26}\}$	$\langle x_{38},x_{39}\rangle$	1	$(c_2-c_3)x - c_3y + \frac{1925}{71994}\sqrt{2} - \frac{1850361}{461500}$	0.346 0.659	
$(l,u)^*$	$\{f-a_1\}$	$\langle x_{38},x_{39}\rangle$	1	$(c_2-c_3)x - c_3y - \frac{385}{93016248}\sqrt{2} - \frac{146021}{36920}$	0.394 0.706	
$(l,u)^*$	$\{x_{29}\}$	$\langle x_{38},x_{39}\rangle$	1	$(c_2-c_3)x - c_3y - \frac{1211135889}{298129000}$	0.308 0.621	

Table 6: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 1$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$(l,u)^*$	$\{f{-}a_0\}$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x - c_3y - \frac{146021}{36920}$	0.464 0.776
$(l,u)^*$	$\{x_{32}\}$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x - c_3y - \frac{2695}{100776}\sqrt{2} - \frac{935937}{230750}$	0.356 0.668
$(l,u)^*$	$\{x_{34}\}$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x - c_3y + \frac{1925}{71994}\sqrt{2} - \frac{935937}{230750}$	0.502 0.814
$(l,u)^*$	$\langle x_{35}, x_{36}\rangle$	$\{x_{39}\}$	1	$c_2x + c_1y - \frac{1592873}{1499875}$	0.671 0.839
$(l,u)^*$	$\langle x_{35}, x_{36}\rangle$	$\{1\}$	1	$c_2x + c_1y - \frac{77333}{95992}$	0.982 1.150
$(l,u)^*$	$\langle x_{35}, x_{36}\rangle$	$\{x_1{+}1\}$	1	$c_2x + c_1y - \frac{6091823}{5999500}$	0.826 0.994
$(l,u)^*$	$\{x_{36}\}$	$\langle x_1{+}1, x_2{+}1\rangle$	1	$-(c_1-c_2)x - c_1y + c_1 + \frac{6830177}{5999500}$	0.826 0.994
$(l,u)^*$	$\{f\}$	$\langle x_1{+}1, x_2{+}1\rangle$	1	$-(c_1-c_2)x - c_1y + c_1 + \frac{129419}{95992}$	0.982 1.150
$(l,u)^*$	$\{x_{38}\}$	$\langle x_1{+}1, x_2{+}1\rangle$	1	$-(c_1-c_2)x - c_1y + c_1 + \frac{1637627}{1499875}$	0.671 0.839
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$\{x_3{+}1\}$	1	$c_2x + c_3y + \frac{1925}{71994}\sqrt{2} + \frac{1140813}{230750}$	0.502 0.814
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$\{x_5{+}1\}$	1	$c_2x + c_3y - \frac{2695}{100776}\sqrt{2} + \frac{1140813}{230750}$	0.356 0.668
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$\{a_0{+}1\}$	1	$c_2x + c_3y + \frac{186259}{36920}$	0.464 0.776
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$\{x_8{+}1\}$	1	$c_2x + c_3y + \frac{1472025111}{298129000}$	0.308 0.621
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$\{a_1{+}1\}$	1	$c_2x + c_3y - \frac{385}{93016248}\sqrt{2} + \frac{186259}{36920}$	0.394 0.706
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$\{x_{11}{+}1\}$	1	$c_2x + c_3y + \frac{1925}{71994}\sqrt{2} + \frac{2303139}{461500}$	0.346 0.659
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$\{a_2{+}1\}$	1	$c_2x + c_3y + \frac{97762452}{19378385}$	0.315 0.627
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$\{x_{14}{+}1\}$	1	$c_2x + c_3y - \frac{2695}{100776}\sqrt{2} + \frac{2303139}{461500}$	0.200 0.513
$(l,u)^*$	$\{x_{39}\}$	$\langle x_{16}{+}1, l{+}1]$	1	$-(c_1-c_2)x - c_1y + c_1 + \frac{1339498}{1499875}$	0.156 0.211
$\langle l,u)^*$	$\{x_{39}\}$	$(l{+}1, u{+}1\rangle^*$	2	$-c_2y + c_2 + \frac{3333}{13000}$	0.256 0.256
$(l,u)^*$	$\{1\}$	$(l{+}1, u{+}1\rangle^*$	2	0	0 0
$[u, f{-}u\rangle$	$(f{-}u, f{-}l\rangle^*$	$\{x_{38}\}$	1	$c_1x + c_2y - \frac{2007129}{2999750}$	0.236 0.291
$\langle u, f{-}u]$	$\langle f{-}u, f{-}l)^*$	$\{x_{39}\}$	1	$c_1x + c_2y - \frac{1091117}{5999500}$	1.039 1.067
$\langle u, f{-}u]$	$\{x_{38}\}$	$(l{+}1, u{+}1\rangle^*$	1	$(c_1-c_2)x - c_2y + c_2 - \frac{1089117}{5999500}$	1.039 1.067

Table 6: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 1$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$[u, f-u\rangle$	$\{x_{39}\}$	$\langle l+1, u+1)^*$	1	$(c_1-c_2)x - c_2 y + c_2 - \frac{2006129}{2999750}$	0.236 0.291
$(f-u, f-l\rangle^*$	$(f-u, f-l\rangle^*$	$\{x_{39}\}$	2	$c_2 x + c_2 y + \frac{12556891}{11999000}$	1.047 1.047
$\langle f-u, f-l)^*$	$\langle f-u, f-l)^*$	$\{1\}$	2	$c_2 x + c_2 y + \frac{62533}{47996}$	1.303 1.303
$\langle f-u, f-l)^*$	$\langle f-u, f-l)^*$	$\{x_1+1\}$	2	$c_2 x + c_2 y + \frac{13116229}{11999000}$	1.094 1.094
$(f-u, f-l\rangle^*$	$[f-l, x_{21}\rangle$	$\{1\}$	1	$c_2 x + c_1 y - \frac{20537}{95992}$	1.203 1.230
$(f-u, f-l\rangle^*$	$[f-l, x_{21}\rangle$	$\{x_1+1\}$	1	$c_2 x + c_1 y - \frac{2542073}{5999500}$	0.993 1.075
$(f-u, f-l)^*$	$\{x_{23}\}$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x - c_1 y + c_1 - \frac{2695}{100776}\sqrt{2} + \frac{1940483}{1499875}$	0.982 1.150
$(f-u, f-l)^*$	$\{f-a_2\}$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x - c_1 y + c_1 + \frac{104498743}{77513540}$	1.062 1.230
$(f-u, f-l)^*$	$\{x_{26}\}$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x - c_1 y + c_1 + \frac{1925}{71994}\sqrt{2} + \frac{1940483}{1499875}$	1.019 1.188
$(f-u, f-l)^*$	$\{f-a_1\}$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x - c_1 y + c_1 - \frac{385}{93016248}\sqrt{2} + \frac{647057}{479960}$	1.019 1.188
$(f-u, f-l)^*$	$\{x_{29}\}$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x - c_1 y + c_1 + \frac{4808773193}{3875677000}$	0.900 1.068
$(f-u, f-l)^*$	$\{f-a_0\}$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x - c_1 y + c_1 + \frac{647057}{479960}$	0.982 1.150
$(f-u, f-l)^*$	$\{x_{32}\}$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x - c_1 y + c_1 - \frac{2695}{100776}\sqrt{2} + \frac{7482263}{5999500}$	0.826 0.994
$(f-u, f-l)^*$	$\{x_{34}\}$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x - c_1 y + c_1 + \frac{1925}{71994}\sqrt{2} + \frac{7482263}{5999500}$	0.864 1.032
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$\{x_3+1\}$	1	$c_2 x + c_1 y + \frac{1925}{71994}\sqrt{2} - \frac{5439737}{5999500}$	0.864 1.032
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$\{x_5+1\}$	1	$c_2 x + c_1 y - \frac{2695}{100776}\sqrt{2} - \frac{5439737}{5999500}$	0.826 0.994
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$\{a_0+1\}$	1	$c_2 x + c_1 y - \frac{386703}{479960}$	0.982 1.150
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$\{x_8+1\}$	1	$c_2 x + c_1 y - \frac{3538838807}{3875677000}$	0.900 1.068
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$\{a_1+1\}$	1	$c_2 x + c_1 y - \frac{385}{93016248}\sqrt{2} - \frac{386703}{479960}$	1.019 1.188
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$\{x_{11}+1\}$	1	$c_2 x + c_1 y + \frac{1925}{71994}\sqrt{2} - \frac{1290017}{1499875}$	1.019 1.188
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$\{a_2+1\}$	1	$c_2 x + c_1 y - \frac{62453497}{77513540}$	1.062 1.230
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$\{x_{14}+1\}$	1	$c_2 x + c_1 y - \frac{2695}{100776}\sqrt{2} - \frac{1290017}{1499875}$	0.982 1.150
$(f-u, f-l\rangle^*$	$\{x_{36}\}$	$\langle x_{16}+1, l+1]$	1	$-(c_1-c_2)x - c_1 y + c_1 + \frac{10379927}{5999500}$	0.993 1.075

Table 6: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 1$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$\langle f{-}u, f{-}l\rangle^*$	$\{x_{36}\}$	$\langle l{+}1, u{+}1\rangle^*$	2	$-c_2 y + c_2 + \frac{13120229}{11999000}$	1.094 1.094
$\langle f{-}u, f{-}l\rangle^*$	$\{f\}$	$\langle x_{16}{+}1, l{+}1]$	1	$-(c_1{-}c_2)x - c_1 y + c_1 + \frac{186215}{95992}$	1.203 1.230
$\langle f{-}u, f{-}l\rangle^*$	$\{f\}$	$\langle l{+}1, u{+}1\rangle^*$	2	$-c_2 y + c_2 + \frac{62549}{47996}$	1.303 1.303
$\langle f{-}u, f{-}l\rangle^*$	$\{x_{38}\}$	$\langle l{+}1, u{+}1\rangle^*$	2	$-c_2 y + c_2 + \frac{12560891}{11999000}$	1.047 1.047
$\langle f{-}u, f{-}l\rangle^*$	$\{x_{38}\}$	$[u{+}1, f{-}u{+}1\rangle$	1	$-(c_1{-}c_2)x - c_1 y + c_1 + \frac{11830883}{5999500}$	1.039 1.067
$\langle f{-}u, f{-}l\rangle^*$	$\{x_{39}\}$	$\langle u{+}1, f{-}u{+}1]$	1	$-(c_1{-}c_2)x - c_1 y + c_1 + \frac{4453871}{2999750}$	0.236 0.291
$\langle f{-}u, f{-}l\rangle^*$	$\{x_{39}\}$	$(f{-}u{+}1, f{-}l{+}1\rangle^*$	2	$-c_2 y + c_2 + \frac{3333}{13000}$	0.256 0.256
$\langle f{-}u, f{-}l\rangle^*$	$\{1\}$	$(f{-}u{+}1, f{-}l{+}1)^*$	2	0	0 0
$[f{-}l, x_{21}\rangle$	$\{x_{36}\}$	$\langle l{+}1, u{+}1\rangle^*$	1	$(c_1{-}c_2)x - c_2 y + c_2 - \frac{2540073}{5999500}$	0.993 1.075
$[f{-}l, x_{21}\rangle$	$\{f\}$	$\langle l{+}1, u{+}1)^*$	1	$(c_1{-}c_2)x - c_2 y + c_2 - \frac{20505}{95992}$	1.203 1.230
$[f{-}l, x_{21}\rangle$	$\{x_{39}\}$	$\langle f{-}u{+}1, f{-}l{+}1)^*$	1	$(c_1{-}c_2)x - c_2 y + c_2 - \frac{1890502}{1499875}$	0.156 0.211
$\{x_{23}\}$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-c_2 x + (c_1{-}c_2)y + c_2 - \frac{2695}{100776}\sqrt{2} - \frac{1289517}{1499875}$	0.982 1.150
$\{x_{23}\}$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-c_2 x - (c_2{-}c_3)y + c_2 - \frac{2695}{100776}\sqrt{2} + \frac{29942807}{5999500}$	0.200 0.513
$\{f{-}a_2\}$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-c_2 x + (c_1{-}c_2)y + c_2 - \frac{62427657}{77513540}$	1.062 1.230
$\{f{-}a_2\}$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-c_2 x - (c_2{-}c_3)y + c_2 + \frac{97768912}{19378385}$	0.315 0.627
$\{x_{26}\}$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-c_2 x + (c_1{-}c_2)y + c_2 + \frac{1925}{71994}\sqrt{2} - \frac{1289517}{1499875}$	1.019 1.188
$\{x_{26}\}$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-c_2 x - (c_2{-}c_3)y + c_2 + \frac{1925}{71994}\sqrt{2} + \frac{29942807}{5999500}$	0.346 0.659
$\{f{-}a_1\}$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-c_2 x + (c_1{-}c_2)y + c_2 - \frac{385}{93016248}\sqrt{2} - \frac{386543}{479960}$	1.019 1.188
$\{f{-}a_1\}$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-c_2 x - (c_2{-}c_3)y + c_2 - \frac{385}{93016248}\sqrt{2} + \frac{2421527}{479960}$	0.394 0.706
$\{x_{29}\}$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-c_2 x + (c_1{-}c_2)y + c_2 - \frac{3537546807}{3875677000}$	0.900 1.068
$\{x_{29}\}$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-c_2 x - (c_2{-}c_3)y + c_2 + \frac{19137618443}{3875677000}$	0.308 0.621
$\{f{-}a_0\}$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-c_2 x + (c_1{-}c_2)y + c_2 - \frac{386543}{479960}$	0.982 1.150
$\{f{-}a_0\}$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-c_2 x - (c_2{-}c_3)y + c_2 + \frac{2421527}{479960}$	0.464 0.776

Table 6: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 1$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$\{x_{32}\}$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-c_2x + (c_1-c_2)y + c_2 - \frac{2695}{100776}\sqrt{2} - \frac{5437737}{5999500}$	0.826 0.994
$\{x_{32}\}$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-c_2x - (c_2-c_3)y + c_2 - \frac{2695}{100776}\sqrt{2} + \frac{14831569}{2999750}$	0.356 0.668
$\{x_{34}\}$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-c_2x + (c_1-c_2)y + c_2 + \frac{1925}{71994}\sqrt{2} - \frac{5437737}{5999500}$	0.864 1.032
$\{x_{34}\}$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-c_2x - (c_2-c_3)y + c_2 + \frac{1925}{71994}\sqrt{2} + \frac{14831569}{2999750}$	0.502 0.814
$\langle x_{35}, x_{36}\rangle$	$\{x_{36}\}$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$(c_1-c_2)x - c_2y + c_2 - \frac{6089823}{5999500}$	0.826 0.994
$\langle x_{35}, x_{36}\rangle$	$\{f\}$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$(c_1-c_2)x - c_2y + c_2 - \frac{77301}{95992}$	0.982 1.150
$\langle x_{35}, x_{36}\rangle$	$\{x_{38}\}$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$(c_1-c_2)x - c_2y + c_2 - \frac{1592373}{1499875}$	0.671 0.839

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$. All intervals I, J, K are closed and elements of the complex $\mathcal{P}$; notation $\langle a, b\rangle$: endpoints are not reached by the projection of the face; (a, b): function π is one-sided discontinuous at the endpoints from within the interval; $[a, b]$: function π is one-sided continuous at the endpoints from within the interval. An asterisk marks the special intervals.

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$(0, x_1)$	$\langle x_{16}, l]$	$(l, u\rangle^*$	1	$-(c_2-c_3)x + (c_1-c_2)y - \frac{1155033}{2399800}$	0.100 0.155 0.256
$(0, x_1)$	$(l, u)^*$	$(l, u)^*$	2	$-(c_2-c_3)x + \frac{101}{650}$	0.054 0.054 0.155 0.155
$(0, x_1)$	$\langle l, u)^*$	$[u, f{-}u\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{2593067}{2399800}$	0.020 0.074 0.175
$(0, x_1)$	$\langle u, f{-}u]$	$(f{-}u, f{-}l\rangle^*$	1	$-(c_2-c_3)x + (c_1-c_2)y - \frac{2574933}{2399800}$	0.020 0.074 0.175
$(0, x_1)$	$(f{-}u, f{-}l)^*$	$(f{-}u, f{-}l)^*$	2	$-(c_2-c_3)x + \frac{101}{650}$	0.054 0.054 0.155 0.155
$(0, x_1)$	$\langle f{-}u, f{-}l)^*$	$[f{-}l, x_{21}\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{4012967}{2399800}$	0.100 0.155 0.256
$\langle x_1, x_2]$	$\langle x_1, x_2]$	$(l, u\rangle^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{33427}{95992}$	0.389 0.389 0.529
$\langle x_1, x_2]$	$[x_2, x_3)$	$(l, u\rangle^*$	1	$(c_1-c_2)x - (c_2-c_3)y + \frac{6683}{7384}$	0.336 0.389 0.495 0.529
$\langle x_1, x_2]$	$(x_3, x_4]$	$(l, u)^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{74041}{184600}$	0.336 0.336 0.495 0.504 0.504

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$\langle x_1, x_2\rangle$	$[x_4, x_5)$	$(l, u)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} + \frac{6683}{7384}$	0.298 0.336 0.466 0.504
$\langle x_1, x_2\rangle$	(x_5, a_0)	$(l, u)^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{1925}{71994}\sqrt{2} - \frac{74041}{184600}$	0.298 0.298 0.466 0.466
$\langle x_1, x_2\rangle$	$(a_0, x_7]$	$(l, u)^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{1925}{71994}\sqrt{2} - \frac{74041}{184600}$	0.298 0.298 0.466 0.466
$\langle x_1, x_2\rangle$	$[x_7, x_8)$	$(l, u)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + \frac{32681917}{31005416}$	0.256 0.298 0.424 0.466
$\langle x_1, x_2\rangle$	$(x_8, x_9]$	$(l, u)^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{1155033}{2399800}$	0.256 0.256 0.424 0.424
$\langle x_1, x_2\rangle$	$[x_9, a_1)$	$(l, u)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + \frac{195759}{184600}$	0.228 0.256 0.396 0.424
$\langle x_1, x_2\rangle$	(a_1, x_{11})	$(l, u)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + \frac{195759}{184600}$	0.180 0.228 0.349 0.396
$\langle x_1, x_2\rangle$	$(x_{11}, x_{12}]$	$(l, u)^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{4109}{7384}$	0.180 0.180 0.349 0.349
$\langle x_1, x_2\rangle$	$[x_{12}, a_2)$	$(l, u)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} + \frac{195759}{184600}$	0.177 0.180 0.345 0.349
$\langle x_1, x_2\rangle$	(a_2, x_{14})	$(l, u)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} + \frac{195759}{184600}$	0.143 0.177 0.311 0.345
$\langle x_1, x_2\rangle$	$(x_{14}, x_{15}]$	$(l, u)^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{1925}{71994}\sqrt{2} - \frac{4109}{7384}$	0.143 0.143 0.311 0.311
$\langle x_1, x_2\rangle$	$[x_{15}, x_{16}]$	$(l, u)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + \frac{937492041}{775135400}$	0.100 0.143 0.268 0.311
$\langle x_1, x_2\rangle$	$[x_{16}, l]$	$(l, u)^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{61117}{95992}$	0.100 0.100 0.155 0.268 0.268
$\langle x_1, x_2\rangle$	$\langle l, u\rangle^*$	$\langle l, u)^*$	2	$(c_1-c_2)x$	0.054 0.054 0.168
$\langle x_1, x_2]$	$(l, u)^*$	$[u, f-u]$	1	$-(c_1-c_2)y + \frac{88807}{95992}$	0.020 0.020 0.074 0.115 0.188 0.188
$\langle x_1, x_2]$	$\langle l, u)^*$	$(f-u, f-l\rangle^*$	2	$(c_1-c_2)x - \frac{14553}{47996}$	0.040 0.135 0.135
$\langle x_1, x_2]$	$[u, f-u]$	$(f-u, f-l\rangle^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{117913}{95992}$	0.020 0.020 0.074 0.115 0.188 0.188
$\langle x_1, x_2\rangle$	$(f-u, f-l\rangle^*$	$\langle f-u, f-l)^*$	2	$(c_1-c_2)x$	0.054 0.054 0.168
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$[f-l, x_{21}]$	1	$-(c_1-c_2)y + \frac{145603}{95992}$	0.100 0.100 0.155 0.268 0.268
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$[x_{21}, x_{22}]$	1	$(c_1-c_3)x + (c_2-c_3)y - \frac{2163307959}{775135400}$	0.100 0.143 0.268 0.311
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$[x_{22}, x_{23})$	1	$-(c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{153303}{95992}$	0.143 0.143 0.311 0.311
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$(x_{23}, f-a_2)$	1	$(c_1-c_3)x + (c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} - \frac{7055133}{2399800}$	0.143 0.177 0.311 0.345
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$(f-a_2, x_{25}]$	1	$(c_1-c_3)x + (c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} - \frac{7055133}{2399800}$	0.177 0.180 0.345 0.349

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$[x_{25}, x_{26})$	1	$-(c_1-c_2)y + \frac{153303}{95992}$	0.180 0.180 0.349 0.349
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$(x_{26}, f-a_1)$	1	$(c_1-c_3)x + (c_2-c_3)y - \frac{7055133}{2399800}$	0.180 0.228 0.349 0.396
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$(f-a_1, x_{28}]$	1	$(c_1-c_3)x + (c_2-c_3)y - \frac{7055133}{2399800}$	0.228 0.256 0.396 0.424
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$[x_{28}, x_{29})$	1	$-(c_1-c_2)y + \frac{4012967}{2399800}$	0.256 0.256 0.424 0.424
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$(x_{29}, x_{30}]$	1	$(c_1-c_3)x + (c_2-c_3)y - \frac{91350083}{31005416}$	0.256 0.298 0.424 0.466
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$[x_{30}, f-a_0)$	1	$-(c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{4205467}{2399800}$	0.298 0.298 0.466 0.466
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$(f-a_0, x_{32})$	1	$-(c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{4205467}{2399800}$	0.298 0.298 0.466 0.466
$\langle x_1, x_2\rangle$	$(f-u, f-l)^*$	$(x_{32}, x_{33}]$	1	$(c_1-c_3)x + (c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} - \frac{297121}{95992}$	0.298 0.336 0.466 0.504
$\langle x_1, x_2]$	$(f-u, f-l)^*$	$[x_{33}, x_{34})$	1	$-(c_1-c_2)y + \frac{4205467}{2399800}$	0.336 0.336 0.495 0.504 0.504
$\langle x_1, x_2]$	$\langle f-u, f-l)^*$	$(x_{34}, x_{35}]$	1	$(c_1-c_3)x + (c_2-c_3)y - \frac{297121}{95992}$	0.336 0.389 0.495 0.529
$\langle x_1, x_2]$	$\langle f-u, f-l)^*$	$[x_{35}, x_{36}\rangle$	1	$-(c_1-c_2)y + \frac{173293}{95992}$	0.389 0.389 0.529
$[x_2, x_3)$	$[x_2, x_3)$	$\langle l, u)^*$	1	$-(c_2-c_3)x - (c_2-c_3)y + \frac{207185}{95992}$	0.477 0.477 0.495 0.495 0.529
$[x_2, x_3)$	$(x_3, x_4\rangle$	$\langle l, u)^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + \frac{2045117}{2399800}$	0.477 0.495 0.504
$[x_2, x_3)$	$(l, u\rangle^*$	$\langle u, f-u]$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{209113}{95992}$	0.081 0.115 0.135 0.188
$[x_2, x_3)$	$\langle l, u)^*$	$(f-u, f-l\rangle^*$	2	$-(c_2-c_3)x + \frac{11400}{11999}$	0.101 0.101 0.135 0.135
$[x_2, x_3)$	$[u, f-u\rangle$	$\langle f-u, f-l)^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + \frac{2393}{95992}$	0.081 0.115 0.135 0.188
$[x_2, x_3)$	$(f-u, f-l\rangle^*$	$\langle x_{33}, x_{34})$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{7213117}{2399800}$	0.477 0.495 0.504
$[x_2, x_3)$	$(f-u, f-l\rangle^*$	$(x_{34}, x_{35}]$	1	$(c_2-c_3)y - \frac{176815}{95992}$	0.477 0.477 0.495 0.495 0.529
$[x_2, x_3)$	$\langle f-u, f-l)^*$	$[x_{35}, x_{36}\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{293599}{95992}$	0.336 0.389 0.495 0.529
$(x_3, x_4]$	$(l, u\rangle^*$	$\langle u, f-u]$	1	$-(c_1-c_2)y + \frac{2093317}{2399800}$	0.081 0.106 0.135 0.135
$(x_3, x_4]$	$\langle l, u)^*$	$(f-u, f-l\rangle^*$	2	$(c_1-c_2)x - \frac{213627}{599950}$	0.101 0.101 0.126 0.126
$(x_3, x_4]$	$[u, f-u\rangle$	$\langle f-u, f-l)^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{3074683}{2399800}$	0.081 0.106 0.135 0.135
$(x_3, x_4\rangle$	$(f-u, f-l\rangle^*$	$\langle x_{34}, x_{35}]$	1	$(c_1-c_3)x + (c_2-c_3)y - \frac{7554883}{2399800}$	0.477 0.495 0.504

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$(x_3, x_4]$	$(f-u, f-l)^*$	$[x_{35}, x_{36}\rangle$	1	$-(c_1-c_2)y + \frac{4205467}{2399800}$	0.336 0.336 0.495 0.504 0.504
$[x_4, x_5)$	$(l, u\rangle^*$	$\langle u, f-u]$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{192500}{3875677}\sqrt{2} + \frac{209113}{95992}$	0.081 0.098 0.106 0.135
$[x_4, x_5)$	$\langle l, u\rangle^*$	$(f-u, f-l\rangle^*$	2	$-(c_2-c_3)x + \frac{192500}{3875677}\sqrt{2} + \frac{11400}{11999}$	0.101 0.101 0.126 0.126
$[x_4, x_5)$	$[u, f-u\rangle$	$\langle f-u, f-l\rangle^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + \frac{192500}{3875677}\sqrt{2} + \frac{2393}{95992}$	0.081 0.098 0.106 0.135
$[x_4, x_5)$	$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{192500}{3875677}\sqrt{2} + \frac{293599}{95992}$	0.298 0.336 0.466 0.504
(x_5, a_0)	$(l, u\rangle^*$	$\langle u, f-u]$	1	$-(c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{2093317}{2399800}$	0.081 0.098 0.098
(x_5, a_0)	$(l, u\rangle^*$	$(f-u, f-l\rangle^*$	2	$(c_1-c_2)x - \frac{1925}{71994}\sqrt{2} - \frac{213627}{599950}$	0.101 0.101 0.118 0.118
(x_5, a_0)	$[u, f-u\rangle$	$\langle f-u, f-l\rangle^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{1925}{71994}\sqrt{2} - \frac{3074683}{2399800}$	0.081 0.098 0.098
(x_5, a_0)	$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-(c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{4205467}{2399800}$	0.298 0.298 0.466 0.466
$(a_0, x_7]$	$\langle x_{16}, l]$	$(f-u, f-l\rangle^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{1925}{71994}\sqrt{2} - \frac{2382433}{2399800}$	0.218 0.218 0.229
$(a_0, x_7]$	$(l, u\rangle^*$	$(f-u, f-l\rangle^*$	2	$(c_1-c_2)x - \frac{1925}{71994}\sqrt{2} - \frac{213627}{599950}$	0.118 0.118 0.129 0.129
$(a_0, x_7]$	$\langle l, u\rangle^*$	$[f-l, x_{21}\rangle$	1	$-(c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{2785567}{2399800}$	0.218 0.218 0.229
$(a_0, x_7]$	$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-(c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{4205467}{2399800}$	0.298 0.298 0.466 0.466
$[x_7, x_8)$	$\langle x_{16}, l]$	$(f-u, f-l\rangle^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + \frac{14336809}{31005416}$	0.175 0.201 0.218 0.229
$[x_7, x_8)$	$(l, u\rangle^*$	$\langle f-u, f-l\rangle^*$	2	$-(c_2-c_3)x + \frac{4259700}{3875677}$	0.101 0.101 0.129 0.129
$[x_7, x_8)$	$\langle l, u\rangle^*$	$[f-l, x_{21}\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{81107369}{31005416}$	0.175 0.201 0.218 0.229
$[x_7, x_8)$	$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{99452477}{31005416}$	0.256 0.298 0.424 0.466
$(x_8, x_9]$	$\langle x_{16}, l]$	$(f-u, f-l\rangle^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{2574933}{2399800}$	0.175 0.175 0.201 0.203
$(x_8, x_9]$	$(l, u\rangle^*$	$\langle f-u, f-l\rangle^*$	2	$(c_1-c_2)x - \frac{130876}{299975}$	0.101 0.101 0.103 0.103
$(x_8, x_9]$	$\langle l, u\rangle^*$	$[f-l, x_{21}\rangle$	1	$-(c_1-c_2)y + \frac{2593067}{2399800}$	0.175 0.175 0.201 0.203
$(x_8, x_9]$	$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-(c_1-c_2)y + \frac{4012967}{2399800}$	0.256 0.256 0.424 0.424
$[x_9, a_1)$	$\langle x_{16}, l]$	$(f-u, f-l\rangle^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + \frac{1124967}{2399800}$	0.148 0.175 0.185 0.203
$[x_9, a_1)$	$(l, u\rangle^*$	$\langle f-u, f-l\rangle^*$	2	$-(c_2-c_3)x + \frac{663223}{599950}$	0.085 0.085 0.103 0.103

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$[x_9, a_1)$	$\langle l, u\rangle^*$	$[f{-}l, x_{21}\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{6292967}{2399800}$	0.148 0.175 0.185 0.203
$[x_9, a_1)$	$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{7712867}{2399800}$	0.228 0.256 0.396 0.424
(a_1, x_{11})	$\langle x_{16}, l]$	$(f{-}u, f{-}l\rangle^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + \frac{1124967}{2399800}$	0.100 0.148 0.155 0.185
(a_1, x_{11})	$(l, u\rangle^*$	$\langle f{-}u, f{-}l\rangle^*$	2	$-(c_2-c_3)x + \frac{663223}{599950}$	0.054 0.054 0.085 0.085
(a_1, x_{11})	$\langle l, u\rangle^*$	$[f{-}l, x_{21}\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{6292967}{2399800}$	0.100 0.148 0.155 0.185
(a_1, x_{11})	$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{7712867}{2399800}$	0.180 0.228 0.349 0.396
$(x_{11}, x_{12}]$	$\langle x_{16}, l]$	$(f{-}u, f{-}l\rangle^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{110213}{95992}$	0.100 0.100 0.155 0.179
$(x_{11}, x_{12}]$	$(l, u\rangle^*$	$\langle f{-}u, f{-}l\rangle^*$	2	$(c_1-c_2)x - \frac{6137}{11999}$	0.054 0.054 0.079 0.079
$(x_{11}, x_{12}]$	$\langle l, u\rangle^*$	$[f{-}l, x_{21}\rangle$	1	$-(c_1-c_2)y + \frac{96507}{95992}$	0.100 0.100 0.155 0.179
$(x_{11}, x_{12}]$	$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-(c_1-c_2)y + \frac{153303}{95992}$	0.180 0.180 0.349 0.349
$[x_{12}, a_2)$	$\langle x_{16}, l]$	$(f{-}u, f{-}l\rangle^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + \frac{192500}{3875677}\sqrt{2} + \frac{1124967}{2399800}$	0.097 0.100 0.177 0.179
$[x_{12}, a_2)$	$(l, u\rangle^*$	$\langle f{-}u, f{-}l\rangle^*$	2	$-(c_2-c_3)x + \frac{192500}{3875677}\sqrt{2} + \frac{663223}{599950}$	0.077 0.077 0.079 0.079
$[x_{12}, a_2)$	$\langle l, u\rangle^*$	$[f{-}l, x_{21}\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{192500}{3875677}\sqrt{2} + \frac{6292967}{2399800}$	0.097 0.100 0.177 0.179
$[x_{12}, a_2)$	$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{192500}{3875677}\sqrt{2} + \frac{7712867}{2399800}$	0.177 0.180 0.345 0.349
$\langle a_2, x_{14})$	$\langle x_{15}, x_{16}]$	$(f{-}u, f{-}l\rangle^*$	1	$-(c_2-c_3)x - (c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} + \frac{1794376157}{775135400}$	0.064 0.066 0.066
(a_2, x_{14})	$[x_{16}, l]$	$(f{-}u, f{-}l\rangle^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + \frac{192500}{3875677}\sqrt{2} + \frac{1124967}{2399800}$	0.064 0.066 0.097 0.155 0.177
(a_2, x_{14})	$(l, u\rangle^*$	$\langle f{-}u, f{-}l\rangle^*$	2	$-(c_2-c_3)x + \frac{192500}{3875677}\sqrt{2} + \frac{663223}{599950}$	0.054 0.054 0.077 0.077
(a_2, x_{14})	$\langle l, u\rangle^*$	$[f{-}l, x_{21}]$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{192500}{3875677}\sqrt{2} + \frac{6292967}{2399800}$	0.064 0.066 0.097 0.155 0.177
$\langle a_2, x_{14})$	$\langle l, u\rangle^*$	$[x_{21}, x_{22}\rangle$	1	$(c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} - \frac{1306423843}{775135400}$	0.064 0.066 0.066
(a_2, x_{14})	$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{192500}{3875677}\sqrt{2} + \frac{7712867}{2399800}$	0.143 0.177 0.311 0.345
$\langle x_{14}, x_{15}]$	$\langle x_{14}, x_{15}]$	$(f{-}u, f{-}l\rangle^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{1925}{35997}\sqrt{2} - \frac{102513}{95992}$	0.105 0.105 0.119
$(x_{14}, x_{15}]$	$[x_{15}, x_{16}]$	$(f{-}u, f{-}l\rangle^*$	1	$(c_1-c_2)x - (c_2-c_3)y - \frac{1925}{71994}\sqrt{2} + \frac{541041841}{775135400}$	0.064 0.066 0.091 0.105 0.119
$(x_{14}, x_{15}]$	$[x_{16}, l]$	$\langle f{-}u, f{-}l\rangle^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{1925}{71994}\sqrt{2} - \frac{110213}{95992}$	0.064 0.091 0.155 0.182

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \mathrm{vert}(F)$
$(x_{14}, x_{15}]$	$(l, u\rangle^*$	$\langle f{-}u, f{-}l)^*$	2	$(c_1-c_2)x - \frac{1925}{71994}\sqrt{2} - \frac{6137}{11999}$	0.054 0.054 0.082 0.082
$(x_{14}, x_{15}]$	$\langle l, u\rangle^*$	$[f{-}l, x_{21}]$	1	$-(c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{96507}{95992}$	0.064 0.091 0.155 0.182
$(x_{14}, x_{15}]$	$\langle l, u)^*$	$[x_{21}, x_{22}]$	1	$(c_1-c_3)x + (c_2-c_3)y - \frac{1925}{71994}\sqrt{2} - \frac{2559758159}{775135400}$	0.064 0.066 0.091 0.105 0.119
$\langle x_{14}, x_{15}]$	$\langle l, u)^*$	$[x_{22}, x_{23}\rangle$	1	$-(c_1-c_2)y - \frac{1925}{35997}\sqrt{2} + \frac{104207}{95992}$	0.105 0.105 0.119
$(x_{14}, x_{15}]$	$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-(c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{153303}{95992}$	0.143 0.143 0.311 0.311
$[x_{15}, x_{16}]$	$[x_{15}, x_{16}]$	$\langle f{-}u, f{-}l\rangle^*$	1	$-(c_2-c_3)x - (c_2-c_3)y + \frac{1909876157}{775135400}$	0.064 0.091 0.091 0.119
$[x_{15}, x_{16}]$	$[x_{16}, l]$	$\langle f{-}u, f{-}l\rangle^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + \frac{478864341}{775135400}$	0.064 0.091 0.155 0.182
$[x_{15}, x_{16}]$	$(l, u\rangle^*$	$\langle f{-}u, f{-}l)^*$	2	$-(c_2-c_3)x + \frac{243096029}{193783850}$	0.054 0.054 0.082 0.082
$[x_{15}, x_{16}]$	$\langle l, u\rangle^*$	$[f{-}l, x_{21}]$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{2148128341}{775135400}$	0.064 0.091 0.155 0.182
$[x_{15}, x_{16}]$	$\langle l, u\rangle^*$	$[x_{21}, x_{22}]$	1	$(c_2-c_3)y - \frac{1190923843}{775135400}$	0.064 0.091 0.091 0.119
$[x_{15}, x_{16}]$	$\langle l, u)^*$	$[x_{22}, x_{23})$	1	$-(c_1-c_3)x - (c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{2210305841}{775135400}$	0.064 0.066 0.091 0.105 0.119
$\langle x_{15}, x_{16}]$	$\langle l, u)^*$	$(x_{23}, f{-}a_2\rangle$	1	$(c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} - \frac{1306423843}{775135400}$	0.064 0.066 0.066
$[x_{15}, x_{16}]$	$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	1	$-(c_1-c_3)x - (c_1-c_2)y + \frac{2606756041}{775135400}$	0.100 0.143 0.268 0.311
$[x_{16}, l]$	$[x_{16}, l]$	$\langle f{-}u, f{-}l)^*$	1	$(c_1-c_2)x + (c_1-c_2)y - \frac{117913}{95992}$	0.064 0.155 0.155 0.188 0.188
$[x_{16}, l\rangle$	$(l, u\rangle^*$	$\langle f{-}u, f{-}l)^*$	2	$(c_1-c_2)x - \frac{14199}{23998}$	0.054 0.054 0.088
$[x_{16}, l]$	$(l, u\rangle^*$	$[f{-}l, x_{21}]$	1	$-(c_1-c_2)y + \frac{88807}{95992}$	0.064 0.155 0.155 0.188 0.188
$[x_{16}, l]$	$\langle l, u\rangle^*$	$[x_{21}, x_{22}]$	1	$(c_1-c_3)x + (c_2-c_3)y - \frac{2621935659}{775135400}$	0.064 0.091 0.155 0.182
$[x_{16}, l]$	$\langle l, u\rangle^*$	$[x_{22}, x_{23})$	1	$-(c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{96507}{95992}$	0.064 0.091 0.155 0.182
$[x_{16}, l]$	$\langle l, u)^*$	$(x_{23}, f{-}a_2)$	1	$(c_1-c_3)x + (c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} - \frac{8475033}{2399800}$	0.064 0.066 0.097 0.155 0.177
$\langle x_{16}, l]$	$\langle l, u)^*$	$(f{-}a_2, x_{25}]$	1	$(c_1-c_3)x + (c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} - \frac{8475033}{2399800}$	0.097 0.100 0.177 0.179
$\langle x_{16}, l]$	$\langle l, u)^*$	$[x_{25}, x_{26})$	1	$-(c_1-c_2)y + \frac{96507}{95992}$	0.100 0.100 0.155 0.179
$\langle x_{16}, l]$	$\langle l, u)^*$	$(x_{26}, f{-}a_1)$	1	$(c_1-c_3)x + (c_2-c_3)y - \frac{8475033}{2399800}$	0.100 0.148 0.155 0.185
$\langle x_{16}, l]$	$\langle l, u)^*$	$(f{-}a_1, x_{28}]$	1	$(c_1-c_3)x + (c_2-c_3)y - \frac{8475033}{2399800}$	0.148 0.175 0.185 0.203

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$\langle x_{16}, l]$	$\langle l, u\rangle^*$	$[x_{28}, x_{29})$	1	$-(c_1-c_2)y + \frac{2593067}{2399800}$	0.175 0.175 0.201 0.203
$\langle x_{16}, l]$	$\langle l, u\rangle^*$	$(x_{29}, x_{30}]$	1	$(c_1-c_3)x + (c_2-c_3)y - \frac{109695191}{31005416}$	0.175 0.201 0.218 0.229
$\langle x_{16}, l]$	$\langle l, u\rangle^*$	$[x_{30}, f-a_0)$	1	$-(c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{2785567}{2399800}$	0.218 0.218 0.229
$[x_{16}, l]$	$(f-u, f-l)^*$	$\langle x_{35}, x_{36})$	1	$-(c_1-c_2)y + \frac{145603}{95992}$	0.100 0.100 0.155 0.268 0.268
$\langle x_{16}, l]$	$\langle f-u, f-l)^*$	(x_{36}, f)	1	$(c_1-c_3)x + (c_2-c_3)y - \frac{10755033}{2399800}$	0.100 0.155 0.256
$(l, u\rangle^*$	$(l, u\rangle^*$	$\langle f-l, x_{21}]$	2	$-(c_1-c_2)x - (c_1-c_2)y + \frac{37481}{23998}$	0.054 0.054 0.088
$(l, u\rangle^*$	$(l, u\rangle^*$	$[x_{21}, x_{22}]$	2	$(c_2-c_3)x + (c_2-c_3)y - \frac{532103971}{193783850}$	0.054 0.054 0.082 0.082
$(l, u\rangle^*$	$(l, u\rangle^*$	$[x_{22}, x_{23})$	2	$-(c_1-c_2)x - (c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{19703}{11999}$	0.054 0.054 0.082 0.082
$(l, u\rangle^*$	$(l, u\rangle^*$	$(x_{23}, f-a_2)$	2	$(c_2-c_3)x + (c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} - \frac{1736777}{599950}$	0.054 0.054 0.077 0.077
$(l, u\rangle^*$	$(l, u\rangle^*$	$(f-a_2, x_{25}]$	2	$(c_2-c_3)x + (c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} - \frac{1736777}{599950}$	0.077 0.077 0.079 0.079
$(l, u\rangle^*$	$(l, u\rangle^*$	$[x_{25}, x_{26})$	2	$-(c_1-c_2)x - (c_1-c_2)y + \frac{19703}{11999}$	0.054 0.054 0.079 0.079
$(l, u\rangle^*$	$(l, u\rangle^*$	$(x_{26}, f-a_1)$	2	$(c_2-c_3)x + (c_2-c_3)y - \frac{1736777}{599950}$	0.054 0.054 0.085 0.085
$(l, u\rangle^*$	$(l, u\rangle^*$	$(f-a_1, x_{28}]$	2	$(c_2-c_3)x + (c_2-c_3)y - \frac{1736777}{599950}$	0.085 0.085 0.103 0.103
$(l, u\rangle^*$	$(l, u\rangle^*$	$[x_{28}, x_{29})$	2	$-(c_1-c_2)x - (c_1-c_2)y + \frac{515124}{299975}$	0.101 0.101 0.103 0.103
$(l, u\rangle^*$	$(l, u\rangle^*$	$(x_{29}, x_{30}]$	2	$(c_2-c_3)x + (c_2-c_3)y - \frac{11244300}{3875677}$	0.101 0.101 0.129 0.129
$(l, u\rangle^*$	$(l, u\rangle^*$	$[x_{30}, f-a_0)$	2	$-(c_1-c_2)x - (c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{1078373}{599950}$	0.118 0.118 0.129 0.129
$(l, u\rangle^*$	$(l, u\rangle^*$	$(f-a_0, x_{32})$	2	$-(c_1-c_2)x - (c_1-c_2)y - \frac{1925}{71994}\sqrt{2} + \frac{1078373}{599950}$	0.101 0.101 0.118 0.118
$\langle l, u\rangle^*$	$\langle l, u\rangle^*$	$(x_{32}, x_{33}]$	2	$(c_2-c_3)x + (c_2-c_3)y + \frac{192500}{3875677}\sqrt{2} - \frac{36600}{11999}$	0.101 0.101 0.126 0.126
$\langle l, u\rangle^*$	$\langle l, u\rangle^*$	$[x_{33}, x_{34})$	2	$-(c_1-c_2)x - (c_1-c_2)y + \frac{1078373}{599950}$	0.101 0.101 0.126 0.126
$\langle l, u\rangle^*$	$\langle l, u\rangle^*$	$(x_{34}, x_{35}]$	2	$(c_2-c_3)x + (c_2-c_3)y - \frac{36600}{11999}$	0.101 0.101 0.135 0.135
$\langle l, u\rangle^*$	$\langle l, u\rangle^*$	$[x_{35}, x_{36}\rangle$	2	$-(c_1-c_2)x - (c_1-c_2)y + \frac{88807}{47996}$	0.040 0.135 0.135
$(l, u\rangle^*$	$[u, f-u\rangle$	$(f-a_0, x_{32})$	1	$-(c_1-c_2)x - \frac{1925}{71994}\sqrt{2} + \frac{2093317}{2399800}$	0.081 0.098 0.098
$(l, u\rangle^*$	$[u, f-u\rangle$	$(x_{32}, x_{33}]$	1	$(c_2-c_3)x + (c_1-c_3)y + \frac{192500}{3875677}\sqrt{2} - \frac{381607}{95992}$	0.081 0.098 0.106 0.135

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$	
$(l,u\rangle^*$	$[u, f-u\rangle$	$[x_{33}, x_{34})$	1	$-(c_1-c_2)x + \frac{2093317}{2399800}$	0.081 0.106 0.135 0.135	
$(l,u\rangle^*$	$[u, f-u\rangle$	$(x_{34}, x_{35}]$	1	$(c_2-c_3)x + (c_1-c_3)y - \frac{381607}{95992}$	0.081 0.115 0.135 0.188	
$(l,u\rangle^*$	$[u, f-u]$	$[x_{35}, x_{36})$	1	$-(c_1-c_2)x + \frac{88807}{95992}$	0.020 0.020 0.074 0.115 0.188 0.188	
$\langle l,u\rangle^*$	$\langle u, f-u]$	(x_{36}, f)	1	$(c_2-c_3)x + (c_1-c_3)y - \frac{12174933}{2399800}$	0.020 0.074 0.175	
$(l,u\rangle^*$	$(f-u, f-l\rangle^*$	$\langle x_{35}, x_{36})$	2	$-(c_1-c_2)x - (c_1-c_2)y + \frac{25840}{11999}$	0.054 0.054 0.168	
$(l,u\rangle^*$	$(f-u, f-l\rangle^*$	$\langle x_{35}, x_{36})$	2	$-(c_1-c_2)x - (c_1-c_2)y + \frac{25840}{11999}$	0.054 0.054 0.168	
$(l,u\rangle^*$	$(f-u, f-l)^*$	(x_{36}, f)	2	$(c_2-c_3)x + (c_2-c_3)y - \frac{2306777}{599950}$	0.054 0.054 0.155 0.155	
$(l,u\rangle^*$	$(f-u, f-l)^*$	(x_{36}, f)	2	$(c_2-c_3)x + (c_2-c_3)y - \frac{2306777}{599950}$	0.054 0.054 0.155 0.155	
$(l,u\rangle^*$	$(f-u, f-l)^*$	(f, x_{38})	2	$-(c_1-c_2)x - (c_1-c_2)y + \frac{1385223}{599950}$	0.101 0.101 0.155 0.155	
$(l,u\rangle^*$	$(f-u, f-l)^*$	(f, x_{38})	2	$-(c_1-c_2)x - (c_1-c_2)y + \frac{1385223}{599950}$	0.101 0.101 0.155 0.155	
$\langle l,u\rangle^*$	$\langle f-u, f-l)^*$	$(x_{38}, x_{39}\rangle$	2	$(c_2-c_3)x + (c_2-c_3)y - \frac{48000}{11999}$	0.101 0.101 0.313	
$\langle l,u\rangle^*$	$\langle f-u, f-l)^*$	$(x_{38}, x_{39}\rangle$	2	$(c_2-c_3)x + (c_2-c_3)y - \frac{48000}{11999}$	0.101 0.101 0.313	
$(l,u\rangle^*$	$[f-l, x_{21}\rangle$	(f, x_{38})	1	$-(c_1-c_2)x + \frac{1900817}{2399800}$	0.001 0.055 0.055	(tight)
$(l,u\rangle^*$	$[f-l, x_{21}]$	$(x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x + (c_1-c_3)y - \frac{529603}{95992}$	0.001 0.055 0.159 0.212 0.472	(tight)
$(l,u\rangle^*$	$[x_{21}, x_{22}]$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x - \frac{937492041}{775135400}$	0.159 0.159 0.472 0.472	
$(l,u\rangle^*$	$[x_{22}, x_{23})$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x + (c_1-c_3)y + \frac{1925}{71994}\sqrt{2} - \frac{41331}{7384}$	0.159 0.238 0.472 0.551	
$(l,u\rangle^*$	$(x_{23}, f-a_2)$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x - \frac{192500}{3875677}\sqrt{2} - \frac{195759}{184600}$	0.238 0.238 0.551 0.551	
$(l,u\rangle^*$	$(f-a_2, x_{25}]$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x - \frac{192500}{3875677}\sqrt{2} - \frac{195759}{184600}$	0.238 0.238 0.551 0.551	
$(l,u\rangle^*$	$[x_{25}, x_{26})$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x + (c_1-c_3)y - \frac{41331}{7384}$	0.238 0.308 0.551 0.621	
$(l,u\rangle^*$	$(x_{26}, f-a_1)$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x - \frac{195759}{184600}$	0.308 0.308 0.621 0.621	
$(l,u\rangle^*$	$(f-a_1, x_{28}]$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x - \frac{195759}{184600}$	0.308 0.308 0.621 0.621	
$(l,u\rangle^*$	$[x_{28}, x_{29})$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x + (c_1-c_3)y - \frac{13612967}{2399800}$	0.308 0.315 0.621 0.627	
$(l,u\rangle^*$	$(x_{29}, x_{30}]$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x - \frac{32681917}{31005416}$	0.315 0.315 0.627 0.627	

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$(l,u)^*$	$[x_{30}, f{-}a_0)$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x + (c_1-c_3)y + \frac{1925}{71994}\sqrt{2} - \frac{1061959}{184600}$	0.315 0.346 0.627 0.659
$(l,u)^*$	$(f{-}a_0, x_{32})$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x + (c_1-c_3)y + \frac{1925}{71994}\sqrt{2} - \frac{1061959}{184600}$	0.346 0.394 0.659 0.706
$(l,u)^*$	$(x_{32}, x_{33}]$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x - \frac{192500}{3875677}\sqrt{2} - \frac{6683}{7384}$	0.394 0.394 0.706 0.706
$(l,u)^*$	$[x_{33}, x_{34})$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x + (c_1-c_3)y - \frac{1061959}{184600}$	0.394 0.464 0.706 0.776
$(l,u)^*$	$(x_{34}, x_{35}]$	$\langle x_{38}, x_{39}\rangle$	1	$(c_2-c_3)x - \frac{6683}{7384}$	0.464 0.464 0.776 0.776
$(l,u)^*$	$[x_{35}, x_{36}\rangle$	$\langle x_{38}, x_{39})$	1	$(c_2-c_3)x + (c_1-c_3)y - \frac{557293}{95992}$	0.464 0.776 0.826 0.994
$(l,u)^*$	$\langle x_{35}, x_{36}\rangle$	$(x_{39}, 1)$	1	$-(c_1-c_2)x + \frac{4154783}{2399800}$	0.826 0.826 0.994 0.994
$(l,u)^*$	$\langle x_{35}, x_{36}\rangle$	$(1, x_1{+}1)$	1	$(c_2-c_3)x + (c_1-c_3)y + c_3 - \frac{2306217}{2399800}$	0.826 0.982 0.994 1.150
$(l,u)^*$	$\langle x_{35}, x_{36})$	$(x_1{+}1, x_2{+}1\rangle$	1	$-(c_1-c_2)x + c_1 - \frac{77333}{95992}$	0.982 0.982 1.150 1.150
$(l,u)^*$	(x_{36}, f)	$\langle x_1{+}1, x_2{+}1\rangle$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{12461783}{2399800}$	0.826 0.982 0.994 1.150
$(l,u)^*$	(f, x_{38})	$\langle x_1{+}1, x_2{+}1\rangle$	1	$-(c_1-c_2)x + c_1 - \frac{2306217}{2399800}$	0.826 0.826 0.994 0.994
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$\langle x_1{+}1, x_2{+}1]$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{513387}{95992}$	0.464 0.776 0.826 0.994
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$[x_2{+}1, x_3{+}1)$	1	$(c_2-c_3)x + c_3 + \frac{30237}{7384}$	0.464 0.464 0.776 0.776
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$(x_3{+}1, x_4{+}1]$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{997041}{184600}$	0.394 0.464 0.706 0.776
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$[x_4{+}1, x_5{+}1)$	1	$(c_2-c_3)x + c_3 - \frac{192500}{3875677}\sqrt{2} + \frac{30237}{7384}$	0.394 0.394 0.706 0.706
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$(x_5{+}1, a_0{+}1\rangle$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{1925}{71994}\sqrt{2} + \frac{997041}{184600}$	0.346 0.394 0.659 0.706
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$(a_0{+}1, x_7{+}1]$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{1925}{71994}\sqrt{2} + \frac{997041}{184600}$	0.315 0.346 0.627 0.659
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$[x_7{+}1, x_8{+}1)$	1	$(c_2-c_3)x + c_3 + \frac{122345163}{31005416}$	0.315 0.315 0.627 0.627
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$(x_8{+}1, x_9{+}1]$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{13154033}{2399800}$	0.308 0.315 0.621 0.627
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$[x_9{+}1, a_1{+}1)$	1	$(c_2-c_3)x + c_3 + \frac{727241}{184600}$	0.308 0.308 0.621 0.621
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$(a_1{+}1, x_{11}{+}1)$	1	$(c_2-c_3)x + c_3 + \frac{727241}{184600}$	0.308 0.308 0.621 0.621
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$(x_{11}{+}1, x_{12}{+}1]$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{41029}{7384}$	0.238 0.308 0.551 0.621
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$[x_{12}{+}1, a_2{+}1)$	1	$(c_2-c_3)x + c_3 - \frac{192500}{3875677}\sqrt{2} + \frac{727241}{184600}$	0.238 0.238 0.551 0.551

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \mathrm{vert}(F)$	
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$(a_2{+}1, x_{14}{+}1)$	1	$(c_2-c_3)x + c_3 - \frac{192500}{3875677}\sqrt{2} + \frac{727241}{184600}$	0.238 0.238 0.551 0.551	
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$(x_{14}{+}1, x_{15}{+}1]$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{1925}{71994}\sqrt{2} + \frac{41029}{7384}$	0.159 0.238 0.472 0.551	
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$[x_{15}{+}1, x_{16}{+}1]$	1	$(c_2-c_3)x + c_3 + \frac{2938184959}{775135400}$	0.159 0.159 0.472 0.472	
$(l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$[x_{16}{+}1, l{+}1]$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{541077}{95992}$	0.001 0.055 0.159 0.212 0.472	(tight)
$\langle l,u)^*$	$\langle x_{38}, x_{39}\rangle$	$(l{+}1, u{+}1\rangle^*$	2	$-(c_2-c_3)y + c_2 + 5$	0.101 0.101 0.313	
$(l,u\rangle^*$	$(x_{39}, 1)$	$\langle x_{16}{+}1, l{+}1]$	1	$-(c_1-c_2)x + c_1 - \frac{4560183}{2399800}$	0.001 0.055 0.055	(tight)
$(l,u)^*$	$(x_{39}, 1)$	$(l{+}1, u{+}1)^*$	2	$(c_1-c_2)y + c_2 - \frac{1649}{650}$	0.101 0.101 0.155 0.155	
$[u, f{-}u\rangle$	$(f{-}u, f{-}l)^*$	(f, x_{38})	1	$-(c_1-c_2)y + \frac{3320717}{2399800}$	0.081 0.135 0.135	
$[u, f{-}u]$	$(f{-}u, f{-}l)^*$	(x_{38}, x_{39})	1	$(c_1-c_3)x + (c_2-c_3)y - \frac{472807}{95992}$	0.081 0.135 0.293 0.961 1.195 1.222	
$\langle u, f{-}u]$	$\langle f{-}u, f{-}l)^*$	$(x_{39}, 1\rangle$	1	$-(c_1-c_2)y + \frac{6266933}{2399800}$	1.195 1.195 1.222	
$\langle u, f{-}u]$	$\langle f, x_{38})$	$(l{+}1, u{+}1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 - \frac{5362067}{2399800}$	1.195 1.195 1.222	
$[u, f{-}u]$	(x_{38}, x_{39})	$(l{+}1, u{+}1)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + c_2 + \frac{391153}{95992}$	0.081 0.135 0.293 0.961 1.195 1.222	
$[u, f{-}u\rangle$	$(x_{39}, 1)$	$\langle l{+}1, u{+}1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 - \frac{8308283}{2399800}$	0.081 0.135 0.135	
$(f{-}u, f{-}l\rangle^*$	$(f{-}u, f{-}l\rangle^*$	$\langle x_{38}, x_{39})$	2	$(c_2-c_3)x + (c_2-c_3)y - \frac{177447}{47996}$	0.941 1.202 1.202	
$(f{-}u, f{-}l)^*$	$(f{-}u, f{-}l)^*$	$(x_{39}, 1)$	2	$-(c_1-c_2)x - (c_1-c_2)y + \frac{4607379}{1199900}$	1.148 1.148 1.175 1.175 1.202 1.202	
$\langle f{-}u, f{-}l)^*$	$\langle f{-}u, f{-}l)^*$	$(1, x_1{+}1)$	2	$(c_2-c_3)x + (c_2-c_3)y + c_3 + \frac{1376879}{1199900}$	1.148 1.148 1.249 1.249	
$\langle f{-}u, f{-}l)^*$	$\langle f{-}u, f{-}l)^*$	$(x_1{+}1, x_2{+}1\rangle$	2	$-(c_1-c_2)x - (c_1-c_2)y + c_1 + \frac{62533}{47996}$	1.162 1.249 1.249	
$(f{-}u, f{-}l\rangle^*$	$[f{-}l, x_{21}\rangle$	$\langle x_{39}, 1)$	1	$-(c_1-c_2)x + \frac{5574683}{2399800}$	1.048 1.075 1.075	
$(f{-}u, f{-}l\rangle^*$	$[f{-}l, x_{21}\rangle$	$(1, x_1{+}1)$	1	$(c_2-c_3)x + (c_1-c_3)y + c_3 - \frac{886317}{2399800}$	1.048 1.075 1.149 1.230	
$(f{-}u, f{-}l)^*$	$[f{-}l, x_{21}]$	$(x_1{+}1, x_2{+}1\rangle$	1	$-(c_1-c_2)x + c_1 - \frac{20537}{95992}$	1.062 1.062 1.149 1.230 1.230	
$(f{-}u, f{-}l)^*$	$[x_{21}, x_{22}]$	$\langle x_1{+}1, x_2{+}1\rangle$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{3173215909}{775135400}$	1.019 1.062 1.188 1.230	
$(f{-}u, f{-}l)^*$	$[x_{22}, x_{23})$	$\langle x_1{+}1, x_2{+}1\rangle$	1	$-(c_1-c_2)x + c_1 + \frac{1925}{71994}\sqrt{2} - \frac{28237}{95992}$	1.019 1.019 1.188 1.188	
$(f{-}u, f{-}l)^*$	$(x_{23}, f{-}a_2)$	$\langle x_1{+}1, x_2{+}1\rangle$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 - \frac{192500}{3875677}\sqrt{2} + \frac{10181783}{2399800}$	0.985 1.019 1.153 1.188	

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$(f-u, f-l)^*$	$(f-a_2, x_{25}]$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 - \frac{192500}{3875677}\sqrt{2} + \frac{10181783}{2399800}$	0.982 0.985 1.150 1.153
$(f-u, f-l)^*$	$[x_{25}, x_{26})$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x + c_1 - \frac{28237}{95992}$	0.982 0.982 1.150 1.150
$(f-u, f-l)^*$	$(x_{26}, f-a_1)$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{10181783}{2399800}$	0.934 0.982 1.102 1.150
$(f-u, f-l)^*$	$(f-a_1, x_{28}]$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{10181783}{2399800}$	0.906 0.934 1.075 1.102
$(f-u, f-l)^*$	$[x_{28}, x_{29})$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x + c_1 - \frac{886317}{2399800}$	0.906 0.906 1.075 1.075
$(f-u, f-l)^*$	$(x_{29}, x_{30}]$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{131746401}{31005416}$	0.864 0.906 1.032 1.075
$(f-u, f-l)^*$	$[x_{30}, f-a_0)$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x + c_1 + \frac{1925}{71994}\sqrt{2} - \frac{1078817}{2399800}$	0.864 0.864 1.032 1.032
$(f-u, f-l)^*$	$(f-a_0, x_{32})$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x + c_1 + \frac{1925}{71994}\sqrt{2} - \frac{1078817}{2399800}$	0.864 0.864 1.032 1.032
$(f-u, f-l)^*$	$(x_{32}, x_{33}]$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 - \frac{192500}{3875677}\sqrt{2} + \frac{422187}{95992}$	0.826 0.864 0.994 1.032
$(f-u, f-l)^*$	$[x_{33}, x_{34})$	$\langle x_1+1, x_2+1\rangle$	1	$-(c_1-c_2)x + c_1 - \frac{1078817}{2399800}$	0.826 0.826 0.994 0.994
$(f-u, f-l)^*$	$(x_{34}, x_{35}]$	$\langle x_1+1, x_2+1]$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{422187}{95992}$	0.774 0.776 0.826 0.942 0.994
$\langle f-u, f-l)^*$	$\langle x_{34}, x_{35}]$	$[x_2+1, x_3+1\rangle$	1	$(c_2-c_3)x + c_3 + \frac{301881}{95992}$	0.774 0.776 0.776
$(f-u, f-l\rangle^*$	$[x_{35}, x_{36}\rangle$	$\langle x_1+1, x_2+1]$	1	$-(c_1-c_2)x + c_1 - \frac{48227}{95992}$	0.774 0.942 0.942
$(f-u, f-l)^*$	$[x_{35}, x_{36}\rangle$	$[x_2+1, x_3+1)$	1	$(c_2-c_3)x + (c_1-c_3)y + c_3 - \frac{168533}{95992}$	0.774 0.776 0.826 0.942 0.994
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$(x_3+1, x_4+1]$	1	$-(c_1-c_2)x + c_1 - \frac{1078817}{2399800}$	0.826 0.826 0.994 0.994
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$[x_4+1, x_5+1)$	1	$(c_2-c_3)x + (c_1-c_3)y + c_3 - \frac{192500}{3875677}\sqrt{2} - \frac{168533}{95992}$	0.826 0.864 0.994 1.032
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	(x_5+1, a_0+1)	1	$-(c_1-c_2)x + c_1 + \frac{1925}{71994}\sqrt{2} - \frac{1078817}{2399800}$	0.864 0.864 1.032 1.032
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$(a_0+1, x_7+1]$	1	$-(c_1-c_2)x + c_1 + \frac{1925}{71994}\sqrt{2} - \frac{1078817}{2399800}$	0.864 0.864 1.032 1.032
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$[x_7+1, x_8+1)$	1	$(c_2-c_3)x + (c_1-c_3)y + c_3 - \frac{59056159}{31005416}$	0.864 0.906 1.032 1.075
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$(x_8+1, x_9+1]$	1	$-(c_1-c_2)x + c_1 - \frac{886317}{2399800}$	0.906 0.906 1.075 1.075
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$[x_9+1, a_1+1)$	1	$(c_2-c_3)x + (c_1-c_3)y + c_3 - \frac{4586217}{2399800}$	0.906 0.934 1.075 1.102
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$(a_1+1, x_{11}+1)$	1	$(c_2-c_3)x + (c_1-c_3)y + c_3 - \frac{4586217}{2399800}$	0.934 0.982 1.102 1.150
$(f-u, f-l)^*$	$\langle x_{35}, x_{36}\rangle$	$(x_{11}+1, x_{12}+1]$	1	$-(c_1-c_2)x + c_1 - \frac{28237}{95992}$	0.982 0.982 1.150 1.150

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$	
$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	$[x_{12}{+}1, a_2{+}1)$	1	$(c_2-c_3)x + (c_1-c_3)y + c_3 - \frac{192500}{3875677}\sqrt{2} - \frac{4586217}{2399800}$	0.982 0.985 1.150 1.153	
$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	$(a_2{+}1, x_{14}{+}1)$	1	$(c_2-c_3)x + (c_1-c_3)y + c_3 - \frac{192500}{3875677}\sqrt{2} - \frac{4586217}{2399800}$	0.985 1.019 1.153 1.188	
$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	$(x_{14}{+}1, x_{15}{+}1]$	1	$-(c_1-c_2)x + c_1 + \frac{1925}{71994}\sqrt{2} - \frac{28237}{95992}$	1.019 1.019 1.188 1.188	
$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36}\rangle$	$[x_{15}{+}1, x_{16}{+}1]$	1	$(c_2-c_3)x + (c_1-c_3)y + c_3 - \frac{1596848091}{775135400}$	1.019 1.062 1.188 1.230	
$(f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36})$	$[x_{16}{+}1, l{+}1]$	1	$-(c_1-c_2)x + c_1 - \frac{20537}{95992}$	1.062 1.062 1.149 1.230 1.230	
$\langle f{-}u, f{-}l)^*$	$\langle x_{35}, x_{36})$	$(l{+}1, u{+}1\rangle^*$	2	$(c_1-c_2)y + c_2 - \frac{40827}{47996}$	1.162 1.249 1.249	
$(f{-}u, f{-}l\rangle^*$	(x_{36}, f)	$\langle x_{16}{+}1, l{+}1]$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{13881683}{2399800}$	1.048 1.075 1.149 1.230	
$\langle f{-}u, f{-}l)^*$	(x_{36}, f)	$(l{+}1, u{+}1\rangle^*$	2	$-(c_2-c_3)y + c_2 + \frac{6176879}{1199900}$	1.148 1.148 1.249 1.249	
$(f{-}u, f{-}l\rangle^*$	$(f, x_{38}\rangle$	$\langle x_{16}{+}1, l{+}1]$	1	$-(c_1-c_2)x + c_1 - \frac{886317}{2399800}$	1.048 1.075 1.075	
$(f{-}u, f{-}l)^*$	(f, x_{38})	$(l{+}1, u{+}1)^*$	2	$(c_1-c_2)y + c_2 - \frac{1207121}{1199900}$	1.148 1.148 1.175 1.175 1.202 1.202	
$\langle f{-}u, f{-}l)^*$	$\langle f, x_{38})$	$[u{+}1, f{-}u{+}1\rangle$	1	$-(c_1-c_2)x + c_1 - \frac{194067}{2399800}$	1.195 1.195 1.222	
$(f{-}u, f{-}l\rangle^*$	$(x_{38}, x_{39}\rangle$	$\langle l{+}1, u{+}1)^*$	2	$-(c_2-c_3)y + c_2 + \frac{254533}{47996}$	0.941 1.202 1.202	
$(f{-}u, f{-}l)^*$	(x_{38}, x_{39})	$[u{+}1, f{-}u{+}1]$	1	$-(c_1-c_2)x - (c_1-c_3)y + c_1 + \frac{597873}{95992}$	0.081 0.135 0.293 0.961 1.195 1.222	
$\langle f{-}u, f{-}l)^*$	$\langle x_{38}, x_{39})$	$(f{-}u{+}1, f{-}l{+}1\rangle^*$	2	$-(c_2-c_3)y + c_2 + 5$	0.101 0.101 0.313	
$(f{-}u, f{-}l\rangle^*$	$(x_{39}, 1)$	$\langle u{+}1, f{-}u{+}1]$	1	$-(c_1-c_2)x + c_1 - \frac{3140283}{2399800}$	0.081 0.135 0.135	
$(f{-}u, f{-}l)^*$	$(x_{39}, 1)$	$(f{-}u{+}1, f{-}l{+}1)^*$	2	$(c_1-c_2)y + c_2 - \frac{1649}{650}$	0.101 0.101 0.155 0.155	
$[f{-}l, x_{21}]$	$\langle x_{35}, x_{36})$	$(l{+}1, u{+}1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 - \frac{227257}{95992}$	1.062 1.062 1.149 1.230 1.230	
$[f{-}l, x_{21}\rangle$	(x_{36}, f)	$\langle l{+}1, u{+}1)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + c_2 + \frac{8713683}{2399800}$	1.048 1.075 1.149 1.230	
$[f{-}l, x_{21}\rangle$	$(f, x_{38}\rangle$	$\langle l{+}1, u{+}1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 - \frac{6054317}{2399800}$	1.048 1.075 1.075	
$[f{-}l, x_{21}]$	$\langle x_{38}, x_{39})$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + c_2 + \frac{334357}{95992}$	0.001 0.055 0.159 0.212 0.472	(tight)
$[f{-}l, x_{21}\rangle$	$(x_{39}, 1)$	$\langle f{-}u{+}1, f{-}l{+}1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 - \frac{9728183}{2399800}$	0.001 0.055 0.055	(tight)
$[x_{21}, x_{22}]$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + c_2 + \frac{1503951909}{775135400}$	1.019 1.062 1.188 1.230	
$[x_{21}, x_{22}]$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-(c_2-c_3)x - (c_2-c_3)y + c_2 + \frac{6038984959}{775135400}$	0.159 0.159 0.472 0.472	

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$[x_{22}, x_{23})$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 + \frac{1925}{71994}\sqrt{2} - \frac{234957}{95992}$	1.019 1.019 1.188 1.188
$[x_{22}, x_{23})$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + c_2 + \frac{1925}{71994}\sqrt{2} + \frac{326657}{95992}$	0.159 0.238 0.472 0.551
$(x_{23}, f{-}a_2)$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + c_2 - \frac{192500}{3875677}\sqrt{2} + \frac{5013783}{2399800}$	0.985 1.019 1.153 1.188
$(x_{23}, f{-}a_2)$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-(c_2-c_3)x - (c_2-c_3)y + c_2 - \frac{192500}{3875677}\sqrt{2} + \frac{19054133}{2399800}$	0.238 0.238 0.551 0.551
$(f{-}a_2, x_{25}]$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + c_2 - \frac{192500}{3875677}\sqrt{2} + \frac{5013783}{2399800}$	0.982 0.985 1.150 1.153
$(f{-}a_2, x_{25}]$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-(c_2-c_3)x - (c_2-c_3)y + c_2 - \frac{192500}{3875677}\sqrt{2} + \frac{19054133}{2399800}$	0.238 0.238 0.551 0.551
$[x_{25}, x_{26})$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 - \frac{234957}{95992}$	0.982 0.982 1.150 1.150
$[x_{25}, x_{26})$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + c_2 + \frac{326657}{95992}$	0.238 0.308 0.551 0.621
$(x_{26}, f{-}a_1)$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + c_2 + \frac{5013783}{2399800}$	0.934 0.982 1.102 1.150
$(x_{26}, f{-}a_1)$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-(c_2-c_3)x - (c_2-c_3)y + c_2 + \frac{19054133}{2399800}$	0.308 0.308 0.621 0.621
$(f{-}a_1, x_{28}]$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + c_2 + \frac{5013783}{2399800}$	0.906 0.934 1.075 1.102
$(f{-}a_1, x_{28}]$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-(c_2-c_3)x - (c_2-c_3)y + c_2 + \frac{19054133}{2399800}$	0.308 0.308 0.621 0.621
$[x_{28}, x_{29})$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 - \frac{6054317}{2399800}$	0.906 0.906 1.075 1.075
$[x_{28}, x_{29})$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + c_2 + \frac{7986033}{2399800}$	0.308 0.315 0.621 0.627
$(x_{29}, x_{30}]$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + c_2 + \frac{64975841}{31005416}$	0.864 0.906 1.032 1.075
$(x_{29}, x_{30}]$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-(c_2-c_3)x - (c_2-c_3)y + c_2 + \frac{246377163}{31005416}$	0.315 0.315 0.627 0.627
$[x_{30}, f{-}a_0)$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 + \frac{1925}{71994}\sqrt{2} - \frac{6246817}{2399800}$	0.864 0.864 1.032 1.032
$[x_{30}, f{-}a_0)$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + c_2 + \frac{1925}{71994}\sqrt{2} + \frac{7793533}{2399800}$	0.315 0.346 0.627 0.659
$(f{-}a_0, x_{32})$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 + \frac{1925}{71994}\sqrt{2} - \frac{6246817}{2399800}$	0.864 0.864 1.032 1.032
$(f{-}a_0, x_{32})$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + c_2 + \frac{1925}{71994}\sqrt{2} + \frac{7793533}{2399800}$	0.346 0.394 0.659 0.706
$(x_{32}, x_{33}]$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + c_2 - \frac{192500}{3875677}\sqrt{2} + \frac{215467}{95992}$	0.826 0.864 0.994 1.032
$(x_{32}, x_{33}]$	$\langle x_{38}, x_{39}\rangle$	$(f{-}u{+}1, f{-}l{+}1)^*$	1	$-(c_2-c_3)x - (c_2-c_3)y + c_2 - \frac{192500}{3875677}\sqrt{2} + \frac{777081}{95992}$	0.394 0.394 0.706 0.706
$[x_{33}, x_{34})$	$\langle x_{35}, x_{36}\rangle$	$(l{+}1, u{+}1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 - \frac{6246817}{2399800}$	0.826 0.826 0.994 0.994

Table 7: Subadditivity slacks $\Delta\pi_F$ for $\dim F = 2$ and $n_F > 0$ (ctd.)

I	J	K	n_F	$\Delta\pi_F(x,y),\ (x,y) \in F = F(I,J,K)$	$\Delta\pi_F(u,v),\ (u,v) \in \operatorname{vert}(F)$
$[x_{33}, x_{34})$	$\langle x_{38}, x_{39}\rangle$	$(f-u+1, f-l+1)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + c_2 + \frac{7793533}{2399800}$	0.394 0.464 0.706 0.776
$\langle x_{34}, x_{35}]$	$\langle x_{34}, x_{35}]$	$(l+1, u+1\rangle^*$	1	$-(c_2-c_3)x - (c_2-c_3)y + c_2 + \frac{685881}{95992}$	0.774 0.776 0.776
$(x_{34}, x_{35}]$	$[x_{35}, x_{36}\rangle$	$(l+1, u+1)^*$	1	$-(c_2-c_3)x + (c_1-c_2)y + c_2 + \frac{215467}{95992}$	0.774 0.776 0.826 0.942 0.994
$(x_{34}, x_{35}]$	$\langle x_{38}, x_{39}\rangle$	$(f-u+1, f-l+1)^*$	1	$-(c_2-c_3)x - (c_2-c_3)y + c_2 + \frac{777081}{95992}$	0.464 0.464 0.776 0.776
$[x_{35}, x_{36}\rangle$	$[x_{35}, x_{36}\rangle$	$\langle l+1, u+1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 - \frac{254947}{95992}$	0.774 0.942 0.942
$\langle x_{35}, x_{36})$	$\langle x_{35}, x_{36})$	$(f-u+1, f-l+1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 - \frac{284053}{95992}$	0.982 0.982 1.150 1.150
$\langle x_{35}, x_{36}\rangle$	(x_{36}, f)	$(f-u+1, f-l+1)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + c_2 + \frac{7293783}{2399800}$	0.826 0.982 0.994 1.150
$\langle x_{35}, x_{36}\rangle$	(f, x_{38})	$(f-u+1, f-l+1)^*$	1	$(c_1-c_2)x + (c_1-c_2)y + c_2 - \frac{7474217}{2399800}$	0.826 0.826 0.994 0.994
$[x_{35}, x_{36}\rangle$	$(x_{38}, x_{39}\rangle$	$(f-u+1, f-l+1)^*$	1	$(c_1-c_2)x - (c_2-c_3)y + c_2 + \frac{306667}{95992}$	0.464 0.776 0.826 0.994