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A Discrete Model For Bike Share Inventory

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Abstract

In this paper, a discrete Markov chain model is developed to describe the inventory at a bike share station. The uniqueness of solutions is first studied. Then the model calibration is considered by investigating a constrained optimization problem. Numerical simulations involving real data are conducted to demonstrate the model effectiveness as well.

AMS Subject Classifications: 39A60, 49J21.
Keywords: Markov chain, bike share system, optimization.

1 Introduction

Bike share systems have become more and more popular around the world. The reader is referred to [3, 8–10] and the references therein for a brief review of bike share history as well as the development of bike sharing in USA. The rebalancing of bikes among bike stations is an important solution to supplement the bikes or free the docks at a station, and has been studied from different perspectives; see for example [5, 13, 14]. Clearly, an accurate prediction of station inventory, i.e. the number of bikes at a station, will significantly increase the efficiency of rebalancing. Motivated by this thought, we will develop models to describe the changes in the inventory at a bike station.

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It is notable that differential equation (DE) models based on the mean-field method have been developed to investigate the bike share problem, see for example [6,7,12,15]. Due to some technical reasons, those models considered the proportions of stations with k bikes at time t instead of the inventory at an individual station, see [15]. As a result, those works focus on the macro trend of a bike share system, which is different from our objective.

At the individual station level, Graef et al. [8] developed a type of deterministic fractional DE model to study the inventory at individual stations. Those models were further investigated in [16]. Since the outputs of the deterministic models should be understood as the average inventory, it is difficult to exactly match the model outputs with a sample path, i.e. the real data on a single day. To overcome this limitation, we will view the station inventory as a stochastic process X and model it by a Markov chain model. Due to the fact that almost all real datasets are discretely sampled, we will directly consider the discrete Markov chain models so that it is easy to calibrate the models with real data. In addition to the average inventory $E(X)$, our model will also output the standard deviation σ that allows us to calculate the confidence intervals $[E(X)-\sigma, E(X)+\sigma]$ and $[E(X)-2\sigma, E(X)+2\sigma]$. Therefore, our models may return more reasonable results than the deterministic models.

This paper is organized as follows: after this introduction, the discrete bike share inventory model is developed in Section 2. The general model calibration methodology is presented in Section 3. Numerical simulations are given in Section 4. The last section, Section 5, contains a summary and a discussion of the future work.

2 Discrete Bike Share Inventory Model

In this section, we will develop a Markov chain model for the inventory at a bike share station. Assume the maximum capacity at the station is N and the number of bikes at the station is described by a discrete stochastic process X. Hence the possible values of X are $[0,N]_{\mathbb{Z}}$, where $[a,b]_{\mathbb{Z}}$ denotes the discrete interval $\{a,\dots,b\}$ for any integers a and b with $a \le b$. Our goal is to develop a model to predict the probability when $X = x$ at time t, denoted by $p(x,t)$.

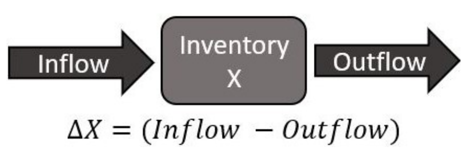

Figure 2.1: Station inventory flowchart.

Assume the station inventory X is determined by two processes, the bike return (Inflow) process and bike pickup (Outflow) process, see Figure 2.1. Then the net inventory change ΔX is the difference of the inflow and the outflow. Let t_0 be the initial time of the model and Δt be a sufficiently small time step size such that for any $t \geq t_0$, the only possible values of ΔX over $[t, t+\Delta t]$ are $\{-1, 0, 1\}$. Then all the possible changes of ΔX over $[t, t+\Delta t]$ and the associated probabilities are listed in Table 2.1.

ΔX	Probability
1	$a(x,t)\Delta t$ – the probability a bike is returned, $a(N,t) \equiv 0$;
-1	$d(x,t)\Delta t$ – the probability a bike is picked up, $d(0,t) \equiv 0$;
0	$1 - (a(x,t) + d(x,t))\Delta t$ – otherwise.

Table 2.1: Possible changes of ΔX over $[t, t+\Delta t]$.

Remark 2.1. Note that $x = N$ implies the station is full. Hence it is impossible to return a bike, i.e. $a(N, \cdot) \equiv 0$. Similarly, $x = 0$ implies the station is empty. Hence it is impossible to rent a bike, i.e. $d(0, \cdot) \equiv 0$.

By Table 2.1 and the idea of the Chapman–Kolmogorov equation, we derive a discrete nonhomogeneous Markov chain model defined by

$$p(0, t_{i+1}) = p(0, t_i)[1 - a(0, t_i)\Delta t] + p(1, t_i)d(1, t_i)\Delta t, \tag{2.1}$$

$$\begin{aligned} p(x, t_{i+1}) =& p(x, t_i)[1 - (a(x, t_i) + d(x, t_i))\Delta t] + p(x+1, t_i)d(x+1, t_i)\Delta t \\ &+ p(x-1, t_i)a(x-1, t_i)\Delta t, \quad x = 1, \ldots, N-1, \end{aligned} \tag{2.2}$$

$$p(N, t_{i+1}) = p(N, t_i)[1 - d(N, t_i)\Delta t] + p(N-1, t_i)a(N-1, t_i)\Delta t, \tag{2.3}$$

where $t_i = t_0 + i\Delta t$, $i = 0, \ldots, T-1$.

Assume the inventory at t_0 is x_0. Then the initial condition for Model (2.1)–(2.3) is

$$p(x, t_0) = \delta_{x,x_0}, \quad x \in [0, N]_{\mathbb{Z}} \tag{2.4}$$

with δ_{x,x_0} the Kronecker delta defined by

$$\delta_{x,x_0} := \begin{cases} 1, & x = x_0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $P(t_i) = (p(0, t_i), \ldots, p(N, t_i))^T \in \mathbb{R}^{N+1}$ be a $(N+1)$ dimensional column vector with $(\cdot)^T$ being the transpose. Then (2.1)–(2.4) can be written as the vector form

$$P(t_{i+1}) = A(t_i)P(t_i), \quad i = 0, \ldots, T-1, \tag{2.5}$$

$$P(t_0) = (p(0, t_0), \ldots, p(N, t_0))^T = (\mathbf{1}_{x_0}(0), \ldots, \mathbf{1}_{x_0}(N))^T, \tag{2.6}$$

where $A(t_i) = [\mathbf{a_{m,n}}(t_i)] \in M_{(N+1)\times(N+1)}$ is a $(N+1) \times (N+1)$ tridiagonal matrix defined by

$$\mathbf{a_{m,n}}(t_i) = \begin{cases} 1 - (a(m, t_i) + d(m, t_i))\Delta t, & m = n, \ m = 0, \ldots, N, \\ d(n, t_i)\Delta t, & n = m+1, \ m = 0 \ldots, N-1, \\ a(n, t_i)\Delta t, & n = m-1, \ m = 1, \ldots, N, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

It is clear that Model (2.5), (2.6) (or (2.1)–(2.4)) is a discrete linear time-variant system. Some useful results of Model (2.5), (2.6) are given as follows.

Theorem 2.2. *Assume* $a(x,t_i) \geq 0$ *and* $d(x,t_i) \geq 0$ *for any* $x \in [0,N]_{\mathbb{Z}}$ *and* $i \in [0,T-1]_{\mathbb{Z}}$. *Then*

(a) Model (2.5)*,* (2.6) *has a unique solution defined by*

$$P(t_i) = \left(\prod_{s=0}^{i-1} A(t_s)\right) P(t_0), \quad i = 1, \ldots, T.$$

(b) Assume that Δt *is sufficiently small such that for any* $x \in [0,N]_{\mathbb{Z}}$ *and* $i \in [0,T-1]_{\mathbb{Z}}$*,* $a(x,t_i)\Delta t \in [0,1]$*,* $d(x,t_i)\Delta t \in [0,1]$*, and* $(a(x,t_i) + d(x,t_i))\Delta t \in [0,1]$*. Let* p *be the solution of* (2.1)–(2.4)*. Then we have* $p(x,t_i) \in [0,1]$*,* $x \in [0,N]_{\mathbb{Z}}$*, and* $\sum_{x=0}^{N} p(x,t_i) = 1$*,* $i = 1, \ldots, T$.

Proof. Part (a) is well-known, see for example [11, Theorem 4.1].

Part (b) can be proven by induction. By (2.1)–(2.4), it is easy to verify $p(x,t_1) \geq 0$, $x \in [0,N]_{\mathbb{Z}}$, and $\sum_{x=0}^{N} p(x,t_1) = \sum_{x=0}^{N} p(x,t_0) = 1$. Assume for any $i \in [0,T-1]_{\mathbb{Z}}$, $p(x,t_i) \geq 0$, $x \in [0,N]_{\mathbb{Z}}$, and $\sum_{x=0}^{N} p(x,t_i) = 1$. Then by (2.1)–(2.3), $p(x,t_{i+1}) \geq 0$, $x \in [0,N]_{\mathbb{Z}}$, and $\sum_{x=0}^{N} p(x,t_{i+1}) = \sum_{x=0}^{N} p(x,t_i) = 1$. □

Remark 2.3. Theorem 2.2 ensures that when the step size Δt is sufficiently small, the solution of Model (2.5), (2.6) will always be a probability mass function defined on $[0,N]_{\mathbb{Z}}$. Therefore, Model (2.5), (2.6) defines a discrete nonhomogeneous Markov chain model.

3 Model Calibration

It is obvious that inflow and outflow probability functions a and d in Table 2.1 play a crucial role in Model (2.5), (2.6). In this section, we will explore various forms of a and d and examine the model performance by calibrating the model with real data.

Throughout this section, we assume that the inflow probability function a is proportional to the normalized average inflow function $\tilde{a}(t)$ and the outflow probability function d is proportional to the normalized average outflow function $\tilde{d}(t)$. Both $\tilde{a}$ and $\tilde{d}$ can be aggregated from historical data published online; see for example [1, 2]. Therefore, we let a and d be defined by

$$a(x,t;\Theta) = \begin{cases} u_1(t;\Theta)\tilde{a}(t), & x = 0,\dots,N-1, \\ 0, & x = N, \end{cases} \tag{3.1}$$

and

$$d(x,t;\Theta) = \begin{cases} u_2(t;\Theta)\tilde{d}(t), & x = 0,\dots,N-1, \\ 0, & x = 0, \end{cases} \tag{3.2}$$

where u_1 and u_2 are two non-negative functions determined by both time t and a parameter vector $\Theta = (\theta_1,\dots,\theta_r) \in \mathbb{R}^r$. In the sequel, we will use $a(\cdot;\Theta)$, $d(\cdot;\Theta)$ etc. to emphasize that the functions rely on Θ. For instance,

$$P(t_{i+1}) = A(t_i;\Theta)P(t_i), \quad i = 0,\dots,T-1. \tag{3.3}$$

We will calibrate Model (3.3), (2.6) with real data by adjusting Θ. We always assume that the station inventory is observed at t_0 and t_T. Additionally, let K be a finite subset of $[1, T-1]_{\mathbb{Z}}$ that the station inventory are observed at $\{t_i \mid i \in K\}$. Let $\{Y(t_i) \mid i \in K \cup \{t_0, t_T\}\}$ be the average inventory observation. Define $L : \mathbb{R}^r \to \mathbb{R}$ by

$$\begin{aligned} L(\Theta) = \sum_{i \in K\cup\{t_0\}} \alpha_i \left(\sum_{x=0}^{N} xp(x,t_i) - Y(t_i)\right)^2 + \alpha_T \left(\sum_{x=0}^{N} xp(x,t_T) - Y(t_T)\right)^2 \\ + \alpha_{T+1} f(\Theta), \end{aligned} \tag{3.4}$$

where $\{\alpha_i\}$ are preset positive weights and $f : \mathbb{R}^r \to \mathbb{R}$ is a continuous and coercive function, i.e.

$$\lim_{\|\Theta\|\to\infty} f(\Theta) = \infty. \tag{3.5}$$

Then the model calibration can be conducted by finding a $\Theta \in \mathbb{R}^r$ that minimizes the objective function $L(\Theta)$ subject to the constraint (3.3), (2.6).

Remark 3.1. It is clear that all the norms in $\mathbb{R}^r$ are equivalent since $\mathbb{R}^r$ is a finite dimensional space. Without loss of generality, the Euclidean norm is chosen in (3.5). Other norms can be used as well.

We first consider the existence of the minimum point of L.

Theorem 3.2. *Let L be defined by* (3.4)*. Assume functions a and d defined by* (3.1) *and* (3.2) *are continuous with respect to Θ, and f is a continuous and coercive function. Then there exists a $\Theta^* \in \mathbb{R}^r$ that minimizes $L(\Theta)$ subject to the constraint* (3.3)*,* (2.6)*.*

Proof. Let $\{\Theta_n\} \subset \mathbb{R}^r$ be a minimizing sequence of L, i.e.

$$\lim_{n\to\infty} L(\Theta_n) = \inf_{\Theta\in\mathbb{R}^r} L(\Theta).$$

Since f is coercive and

$$\sum_{i\in K\cup\{t_0\}} \alpha_i \left(\sum_{x=0}^{N} xp(x,t_i) - Y(t_i)\right)^2 + \alpha_T \left(\sum_{x=0}^{N} xp(x,t_T) - Y(t_T)\right)^2 \geq 0,$$

L is coercive as well. Hence $\{\Theta_n\}$ is bounded. Thus there exist $\Theta^* \in \mathbb{R}^r$ and a subsequence $\{\Theta_{n'}\} \subset \{\Theta_n\}$ with $\Theta_{n'} \to \Theta^*$ as $n' \to \infty$. Therefore,

$$\lim_{n'\to\infty} L(\Theta_{n'}) = L(\Theta^*) = \inf_{\Theta\in\mathbb{R}^r} L(\Theta).$$

The proof is complete. □

Remark 3.3. (a) The involvement of the coercive function f in (3.4) serves two purposes: (i) it is needed in the proof of Theorem 3.2 to guarantee the existence of the minimum point; and (ii) it is used to control certain properties, e.g. the magnitudes, of the inflow and outflow probabilities.

(b) In practice, the coefficient α_{T+1} in (3.4) may be much smaller than α_i, $i = 0, \ldots, \alpha_T$.

The necessary conditions of the minimum point can also be studied by the idea given in [4, Section 2.2].

Theorem 3.4. *Let $\mathbf{a}_{x,y}(t;\Theta)$ be defined by* (2.7)*, $(x,y) \in [0,N]_{\mathbb{Z}} \times [0,N]_{\mathbb{Z}}$. Assume the functions f and $\mathbf{a}_{x,y}$ are differentiable with respect to θ_k, $k = 1,\ldots,r$. Let $\Theta \in \mathbb{R}^r$ be a minimum point of L defined by* (3.4) *subject to* (3.3)*,* (2.6)*, $P(t_i) = (p(0,t_i),\ldots,p(N,t_i))^T$ be the solution of* (3.3)*,* (2.6) *respect to Θ, and*

$$\{\lambda(t_i) = (\lambda_0(t_i),\ldots,\lambda_N(t_i)) \in \mathbb{R}^{N+1} \mid i = 1,\ldots,T\}$$

be a multiplier sequence. Then P, λ, and Θ must simultaneously satisfy (3.3)*,* (2.6)*,*

$$\lambda_x(t_T) = 2x\alpha_T \left(\sum_{x=0}^{N} xp(x,t_T) - Y(t_T)\right), \tag{3.6}$$

$$\lambda_x(t_i) = \sum_{j=0}^{N} \lambda_j(t_{i+1})\mathbf{a}_{j,x}(t_i;\Theta) + 2\alpha_i x \left[\sum_{x=0}^{N} xp(x,t_i) - Y(t_i)\right], \tag{3.7}$$

$$x = 0,\ldots,N,\ i = 1,\ldots,\ T-1,$$

and

$$\alpha_{T+1}\frac{\partial}{\partial\theta_k} f(\Theta) + \sum_{i=0}^{T-1}\sum_{x=0}^{N}\sum_{y=0}^{N} \lambda_x(t_{i+1})p(y,t_i)\frac{\partial}{\partial\theta_k}\mathbf{a}_{x,y}(t_i;\Theta) = 0, \quad k = 1,\ldots,r.$$

Proof. This result is proven by the Lagrange multiplier method. Let Φ be defined by

$$\Phi(p,\lambda,\Theta) = \sum_{i\in K\cup\{t_0\}} \alpha_i \left(\sum_{x=0}^{N} xp(x,t_i) - Y(t_i)\right)^2 + \alpha_T \left(\sum_{x=0}^{N} xp(x,t_T) - Y(t_T)\right)^2$$
$$+ \alpha_{T+1} f(\Theta) + \sum_{i=0}^{T-1}\sum_{x=0}^{N} \lambda_x(t_{i+1}) \left[\sum_{y=0}^{N} \mathbf{a}_{x,y}(t_i;\Theta)p(y,t_i) - p(x,t_{i+1})\right].$$

Then the necessary conditions can be obtained by setting the partial derivatives of Φ with respect to all the components of p, λ, and Θ to 0. We omit the details. $\square$

4 Numerical Simulation

In this section, we will explore three types of functions $\{u_1, u_2\}$ and apply the methodology developed in Section 3 to calibrate Model (3.3), (2.6).

4.1 Data Description and Processing

Bike station data from Capital Bikeshare Bike [1] in Washington DC are used in our experiments. We collected data from a bike station between 10 August 2019 and 1 December 2019. The original data was roughly sampled at 1 record every minute, but at times there could be a delay in the data recording. To reduce any issue with data processing, the data were bucketed in half hour buckets and the average values (the return counts, rental counts, and bikes counts at each bike station) of that time period were taken. To ensure consistency, the inventory, return counts (inflow), and the pickup counts (outflow) are trimmed such that the start and end times are consistent. Then the average inventory $\{Y(t_i)\}$, normalized average inflow $\{\tilde{a}(t_i)\}$, and normalized average outflow $\{\tilde{d}(t_i)\}$ are calculated, $i = 0, \ldots, 47$.

4.2 Choice of Inflow/Outflow Probability Functions

We will consider three types of functions: polynomials, step functions, and piecewise linear functions.

(1) Polynomials of degree 6. In this case, $\Theta = (\theta_1, \ldots, \theta_{14}) \in \mathbb{R}^{14}$. Let u_1, u_2, and $f(\Theta)$ be defined by

$$u_1(t;\Theta) = \sum_{l=0}^{6} \theta_{l+1} t^l, \quad u_2(t;\Theta) = \sum_{l=0}^{6} \theta_{l+8} t^l,$$

and

$$f(\Theta) = \int_{t_0}^{t_{47}} \left[(u_1(s;\Theta) - \eta_1)^2 + (u_2(s;\Theta) - \eta_2)^2\right] ds.$$

(2) Step functions. In this case, $\Theta = (\theta_1, \dots, \theta_{94}) \in \mathbb{R}^{94}$. Let u_1, u_2, and $f(\Theta)$ be defined by

$$u_1(t;\Theta) = \begin{cases} \theta_1, & t_0 \le t < t_1, \\ \theta_2, & t_1 \le t < t_2, \\ \cdots & \cdots \\ \theta_{47}, & t_{46} \le t \le t_{47}, \end{cases} \qquad u_2(t;\Theta) = \begin{cases} \theta_{48}, & t_0 \le t < t_1, \\ \theta_{49}, & t_1 \le t < t_2, \\ \cdots & \cdots \\ \theta_{94}, & t_{46} \le t \le t_{47}, \end{cases}$$

and $f(\Theta) = \sum_{k=1}^{94} \theta_k^2$.

(3) Piecewise linear functions. In this case, $\Theta = (\theta_1, \dots, \theta_{96}) \in \mathbb{R}^{96}$. Let u_1, u_2, and $f(\Theta)$ be defined by

$$u_1(t;\Theta) = \theta_{i+1} + \frac{\theta_{i+2} - \theta_{i+1}}{t_{i+1} - t_i}(t - t_i), \quad t \in [t_i, t_{i+1}],\ i = 0, \dots, 46,$$
$$u_2(t;\Theta) = \theta_{i+49} + \frac{\theta_{i+50} - \theta_{i+49}}{t_{i+1} - t_i}(t - t_i), \quad t \in [t_i, t_{i+1}],\ i = 0, \dots, 46,$$

and $f(\Theta) = \sum_{k=1}^{96} \theta_k^2$.

Then the model (3.3), (2.6) is calibrated by solving the constrained optimization problem (3.4), (3.3), (2.6) with respect to these three types of functions. Based on the calibrated model solutions, the mean

$$E(X(t_i)) = \sum_{x=0}^{N} x p(x, t_i)$$

and the standard deviation

$$\sigma(t_i) = \sqrt{\sum_{x=0}^{N} x^2 p(x, t_i) - \left(\sum_{x=0}^{N} x p(x, t_i) \right)^2}$$

are calculated, $i = 1, \dots, T$. Then the average historical inventory data and $E(X)$ are given in Figures 4.1–4.3. To reflect the stochastic characteristics of our model, two regions formed by $E(X) \pm \sigma$ and $E(X) \pm 2\sigma$ respectively are plotted in the figures as well.

Remark 4.1. By Figures 4.1–4.3, it is clear that the model with piecewise linear functions has the best performance. However, the polynomials only have 14 variables to determine while the piecewise linear functions have 96 variables to determine. It takes much less time to solve the optimization problem (3.4), (3.3), (2.6) with polynomials. Therefore, the model with polynomials may be used either as an ad-hoc model to estimate the trend of the inventory with lower requirement on the accuracy, or to generate the initial guess of models with other types of $\{u_1, u_2\}$.

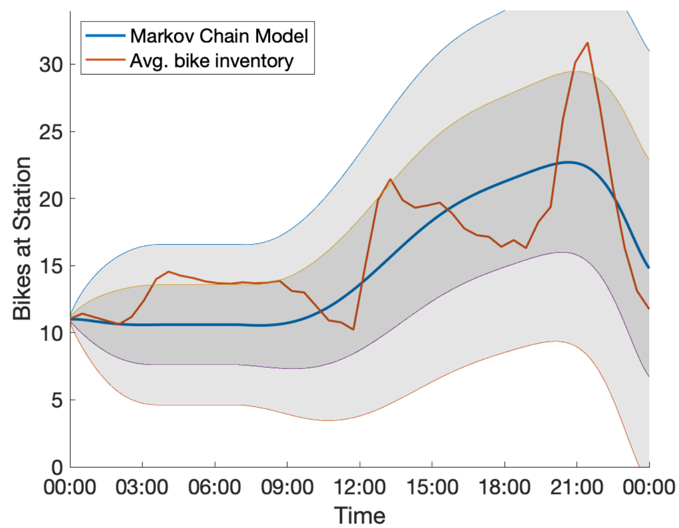

Figure 4.1: Model calibration using polynomials.

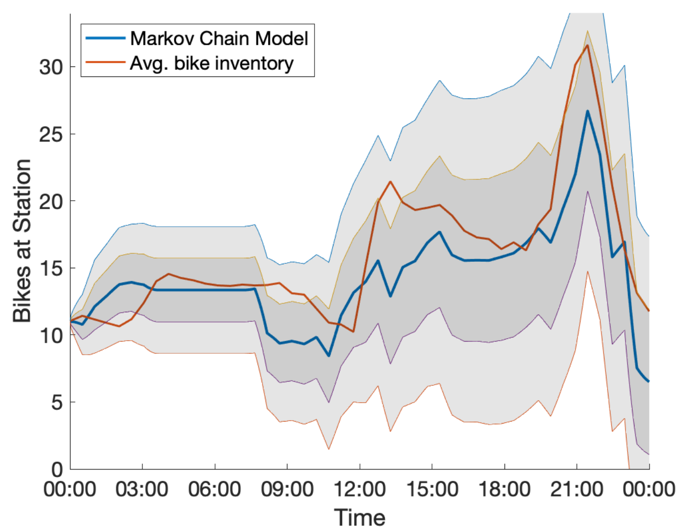

Figure 4.2: Model calibration using step functions.

4.3 Model Application

To demonstrate the application of the calibrated model, a numerical simulation is also carried out when the initial inventory is assumed to be 10 bikes at 6 am. The prediction results are given in Figures 4.4 and 4.5. The probability distributions of inventory at 3 particular time (8:00, 10:00, 12:00) are given in Figure 4.6. By Figure 4.6, there is a bigger probability that the station inventory will be 0 at 10:00. Therefore, a rebalancing

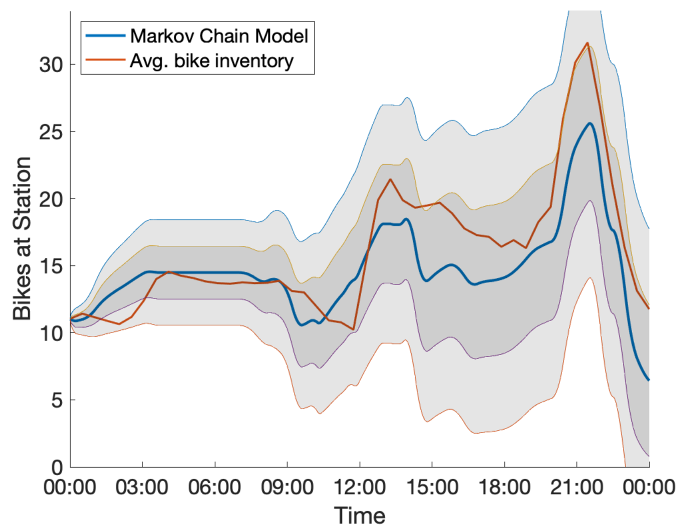

Figure 4.3: Model calibration using piecewise linear functions.

could be planned in advance as a proactive solution.

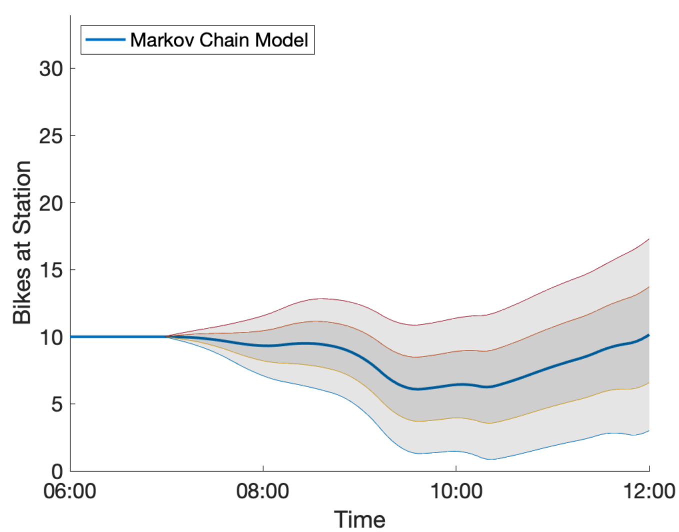

Figure 4.4: Predicted average inventory and confidence intervals using piecewise linear functions $\{u_1, u_2\}$.

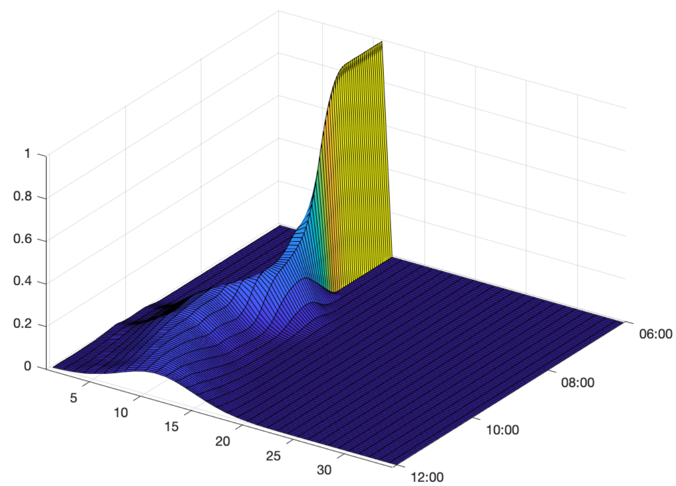

Figure 4.5: Solution of the model using piecewise linear functions $\{u_1, u_2\}$.

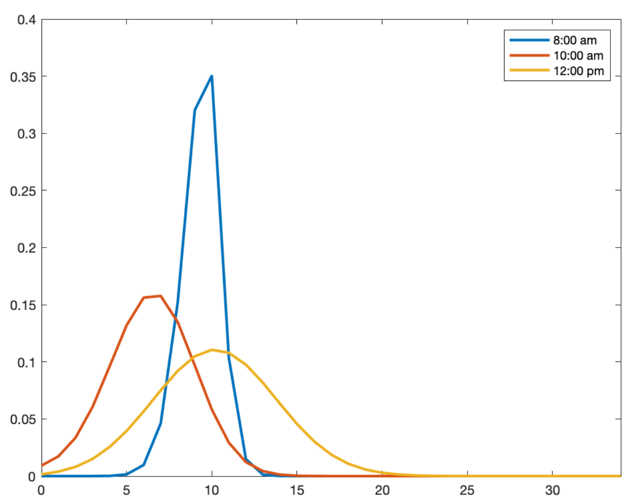

Figure 4.6: Probability distribution of inventory at 8:00, 10:00, and 12:00.

5 Conclusion and Discussion

In this paper, a discrete nonhomogeneous Markov chain model is developed to describe the inventory at a bike share station. The uniqueness of the solutions and a constrained optimization problem raised from the model calibration process are studied. Then numerical simulations are carried out to demonstrate the effectiveness of the model.

Numerical simulations show that the model performance may be improved by choosing different types of functions $\{u_1, u_2\}$. This observation motivates us to explore the feasibility of using non-traditional functions $\{u_1, u_2\}$, e.g. neural networks. Theorem 3.2 further provides guidance to develop non-standard loss function for the neural networks. This will be studied in a future paper.

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