A two-sided Faulhaber-like formula involving Bernoulli polynomials

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Abstract

We give a new identity involving Bernoulli polynomials and combinatorial numbers. This provides, in particular, a Faulhaber-like formula for sums of the form $1^m(n-1)^m + 2^m(n-2)^m + \cdots + (n-1)^m 1^m$ for positive integers m and n.

1 Introduction

Bernoulli numbers B_k are given by the exponential generating function $z/(e^z - 1)$,

$$B_k = k![z^k]\frac{z}{e^z - 1}\,,$$

where $[z^n]f(z)$ is the n-th coefficient of the Taylor expansion of f around $z = 0$.

In the course of studying the distribution of the eigenvalues of the so-called *area operator* in loop quantum gravity [1] we were led to believe that the following identity held

$$\sum_{k=0}^{m}\binom{m}{k}\frac{B_{2m-k+1}}{2m-k+1}2^k = \frac{(-1)^m}{2}\left(1 - 2^{2m+1}\frac{\Gamma(1+m)^2}{\Gamma(2m+2)}\right)\,, \tag{1.1}$$

for $m \in \mathbb{N} \cup \{0\}$. The purpose of this short note is to prove this formula by proving a generalization of it. Particular cases of this general formula involve what we called a two-sided Faulhaber-like formula. A Faulhaber formula (also called Bernoulli's formula as Jacob Bernoulli was the first to write it) is given by

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j}.$$

Notice that in

$$\sum_{k=1}^{n-1} k^p = 1^p + 2^p + \cdots + (n-2)^p + (n-1)^p$$

there is an increasing sequence of addends given by powers of the integers. A particular and interesting case of the aforementioned generalized formula will involve instead a "two-sided" version of it:

$$\sum_{k=1}^{n-1} k^p (n-k)^p = 1^p (n-1)^p + 2^p (n-2)^p + \cdots + (n-2)^p 2^p + (n-1)^p 1^p.$$

Likewise, the Bernoulli numbers are generalized by considering the Bernoulli polynomials:

$$B_k(x) = k![z^k] \frac{z e^{xz}}{e^z - 1}.$$

2 Main theorem

The main result of the paper is the following

Theorem 2.1 *Given $N \in \mathbb{Z}$, $m \in \mathbb{N}$ and $w \in \mathbb{C}$, we have*

$$\sum_{k=0}^{m} \binom{m}{k} \frac{B_{m+k+1}\left(\frac{N-w}{2}\right)}{m+k+1} w^{m-k} \tag{2.1}$$
$$= \frac{(-1)^{m+1}}{2^{2m+1}} \left[\frac{(2w)^{2m+1}}{2(2m+1)\binom{2m}{m}} - \operatorname{sign}(N-1) \sum_{k=1}^{|N-1|} \left(w^2 - (|N-1| - 2k + 1)^2 \right)^m \right].$$

Before proceeding with the proof let us discuss some consequences of this formula

Remark 2.2 *It is possible to get a number of Faulhaber-like formulas from* (2.1). *The simplest one can be obtained by taking both w and N to be equal to a natural* *number $n \geq 2$.*

$$\sum_{k=1}^{n-1} k^m (n-k)^m = \frac{n^{2m+1}}{(2m+1)\binom{2m}{m}} + 2(-1)^m \sum_{k=0}^{m} \binom{m}{k} \frac{B_{m+k+1}}{m+k+1} n^{m-k} = \\ = \frac{n^{2m+1}}{(2m+1)\binom{2m}{m}} - 2(-1)^m \sum_{k=0}^{m} \binom{m}{k} \zeta(-m-k) n^{m-k} . \tag{2.2}$$

where we have used the well known relation between the zeta Riemann function and the Bernoulli numbers

$$\zeta(1-N) = -\frac{B_N}{N}.$$

Equation (2.2) *appears often in the literature obtained through different methods (see for instance [2, page 10]).*

Remark 2.3 *For $N = 1$, Equation* (2.1) *gives the beautiful expression (equivalent to equation (1.17) of [3]) valid for any $w \in \mathbb{C}$,*

$$\sum_{k=0}^{m} \binom{m}{k} \frac{B_{m+k+1}\left(\frac{1-w}{2}\right)}{m+k+1} w^{m-k} = \frac{(-1)^{m+1}}{2} \frac{w^{2m+1}}{(2m+1)\binom{2m}{m}} . \tag{2.3}$$

Remark 2.4 *Sums involving*

$$\frac{B_{\beta m+k+1}}{\beta m+k+1} = -\zeta(-k-\beta m)$$

with integer $\beta \geq 2$ can also be studied although a more complicated approach is needed involving complex analysis and combinatorial identities. Nonetheless, the results are not as neat as (2.1) *and each case has to be studied separately.*

Remark 2.5 *It is also possible to generalize* (2.1) *for fractional values of N but, again, no systematic approach has been found. One such expression is when $w = N = 1/2$*

$$(-1)^{m+1} \sum_{k=0}^{m} \binom{m}{k} \frac{B_{m+k+1}}{m+k+1} 2^k = \frac{1}{2^{m+2}(2m+1)\binom{2m}{m}} + \frac{1}{2^{3m+2}} \sum_{k=0}^{m} (-1)^k \binom{m}{k} E_{2k}$$

where the E_n are the Euler numbers [4, entry A122045].

Proof of Theorem 2.1
The result is a consequence, on one hand, of the following easy-to-prove formula for the Bernoulli polynomials

$$\begin{aligned} B_n(x+r) - B_n(x) = n\,\mathrm{sign}(r) \Bigg(&\sum_{k=1}^{|r|-1} (x + k\,\mathrm{sign}(r) - 1)^{n-1} \\ &+ \frac{1+\mathrm{sign}(r)}{2}(x+r-1)^{n-1} + \frac{1-\mathrm{sign}(r)}{2}(x-1)^{n-1} \Bigg)\,, \end{aligned} \tag{2.4}$$

valid for $r \in \mathbb{Z}$ and $x \in \mathbb{C}$, which is a direct consequence of

$$B_n(x+1) - B_n(x) = nx^{n-1}\,,$$

and, on the other hand, of the remarkable identity obtained by Sun (equation (1.14) of [3])

$$\begin{aligned} &(-1)^k \sum_{j=0}^{k} \binom{k}{j} x^{k-j} \frac{B_{\ell+j+1}(y)}{\ell+j+1} + (-1)^\ell \sum_{j=0}^{\ell} \binom{\ell}{j} x^{\ell-j} \frac{B_{k+j+1}(z)}{k+j+1} \\ &= \frac{(-x)^{k+\ell+1}}{(k+\ell+1)\binom{k+\ell}{k}} \end{aligned} \tag{2.5}$$

where $k\,,\ell \in \mathbb{N}$ and $x+y+z=1$.

Taking now $x=w$, $y=(N-w)/2$, $z = 1-(N+w)/2$ and $k=\ell=m \in \mathbb{N}$ in (2.5) we obtain

$$\begin{aligned} &(-1)^m \sum_{j=0}^{m} \binom{m}{j} w^{m-j} \frac{B_{m+k+1}\left(\frac{N-w}{2}\right)}{m+j+1} \\ &= \frac{(-w)^{2m+1}}{(2m+1)\binom{2m}{m}} + (-1)^{m+1} \sum_{j=0}^{m} \binom{m}{j} w^{m-j} \frac{B_{m+j+1}\left(1-\frac{N+w}{2}\right)}{m+j+1}\,. \end{aligned}$$

Using now equation (2.4) to rewrite the last term in terms of $B_{m+j+1}\left(\frac{N-w}{2}\right)$, we finally obtain (2.1). ■

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