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A Family of Banach Spaces Over $\mathbb{R}^\infty$

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In T. L. Gill and W. W. Zachary, *Functional Analysis and the Feynman Operator Calculus* (Springer, New York, 2016), the topology of $\mathbb{R}^\infty$ was replaced with a new topology and denoted by $\mathbb{R}_I^\infty$. This space was then used to construct Lebesgue measure on $\mathbb{R}_I^\infty$ in a manner that is no more difficult than the same construction on $\mathbb{R}^n$. More important for us, a new class of separable Banach spaces $\mathrm{KS}^p[\mathbb{R}^n]$, $1 \leq p \leq \infty$, for the HK-integrable functions, was introduced. These spaces also contain the L^p spaces and the Schwartz space as continuous dense embeddings. This paper extends the work in T. L. Gill and W. W. Zachary, *Functional Analysis and the Feynman Operator Calculus* (Springer, New York, 2016) from $\mathrm{KS}^p[\mathbb{R}^n]$ to $\mathrm{KS}^p[\mathbb{R}_I^\infty]$.

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1. Introduction

The standard topology for $\mathbb{R}^\infty$ defines open sets to be the Cartesian product of an arbitrary finite number of open sets in $\mathbb{R}$, while the remaining infinite number are copies of $\mathbb{R}$ (cylindrical sets). This automatically makes any attempt to directly define Lebesgue measure impossible. We take the opposite approach, which defines a new topology on $\mathbb{R}^\infty$. First, we define Lebesgue measure directly on the Hilbert cube $I_0 = [-\frac{1}{2}, \frac{1}{2}]^{\aleph_0}$ by $\lambda_\infty(I_0) = 1$ and set $I_n = \prod_{i=n+1}^{\infty}[-\frac{1}{2}, \frac{1}{2}]$.

Definition 1.1. If $\mathfrak{B}[\mathbb{R}^n]$ is the Borel σ-algebra and $A, B \in \mathbb{R}^n$ are open, we define nth-order box sets in $\mathbb{R}^\infty$ by $A_n = A \times I_n, B_n = B \times I_n$, satisfying:

(1) $A_n \cup B_n = (A \cup B) \times I_n$,
(2) $A_n \cap B_n = (A \cap B) \times I_n$ and
(3) $B_n^c = B^c \times I_n$.

Definition 1.2. We define $\mathbb{R}_I^n = \mathbb{R}^n \times I_n \subset \mathbb{R}^\infty$. If T is a linear transformation on $\mathbb{R}^n$ and $A_n = A \times I_n$, we define $T^{[n]}$ on $\mathbb{R}_I^n$ by $T^{[n]}[A_n] = T[A] \times I_n$.

We define the topology on $\mathbb{R}_I^n$ via the following class of open sets:

$$\mathfrak{O}_n = \{U \times I_n : U \text{ open in } \mathbb{R}^n\}$$

and let $\mathfrak{B}[\mathbb{R}_I^n]$ be the natural Borel σ-algebra.

For any $A_n \in \mathfrak{B}[\mathbb{R}_I^n]$, we define $\lambda_\infty(A_n)$ on $\mathbb{R}_I^n$ by the product measure:

$$\lambda_\infty(A_n) = \lambda_n(A) \times \prod_{i=n+1}^{\infty} \lambda_1(I) = \lambda_n(A).$$

Theorem 1.3. *$\lambda_\infty(\cdot)$ is a translationally and rotationally invariant measure on $\mathfrak{B}[\mathbb{R}_I^n]$, which is equivalent to n-dimensional Lebesgue measure on $\mathbb{R}^n$.*

Since $\mathbb{R}_I^n \subset \mathbb{R}_I^{n+1}$, we have an increasing sequence and define

$$\hat{\mathbb{R}}_I^\infty = \lim_{n\to\infty} \mathbb{R}_I^n = \bigcup_{k=1}^{\infty} \mathbb{R}_I^k.$$

Let $\mathfrak{X}_1 = \hat{\mathbb{R}}_I^\infty$ and let τ_1 be the topology induced by the class of open sets $\mathfrak{O} \subset \mathfrak{X}_1$:

$$\mathfrak{O} = \bigcup_{n=1}^{\infty} \mathfrak{O}_n = \bigcup_{n=1}^{\infty} \{U \times I_n : U \text{ open in } \mathbb{R}^n\}.$$

Let $\mathfrak{X}_2 = \mathbb{R}^\infty \backslash \hat{\mathbb{R}}_I^\infty$ and let τ_2 be discrete topology on $\mathfrak{X}_2$ induced by the discrete metric so that, for $x, y \in \mathfrak{X}_2, x \neq y, d_2(x,y) = 1$ and for $x = y$, $d_2(x,y) = 0$.

Definition 1.4. We define $(\mathbb{R}_I^\infty, \tau)$ to be the coproduct $(\mathfrak{X}_1, \tau_1) \oplus (\mathfrak{X}_2, \tau_2)$, of $(\mathfrak{X}_1, \tau_1)$ and $(\mathfrak{X}_2, \tau_2)$, so that every open set in $(\mathbb{R}_I^\infty, \tau)$ is the disjoint union of two open sets $G_1 \bigcup G_2$, with G_1 in $(\mathfrak{X}_1, \tau_1)$ and G_2 in $(\mathfrak{X}_2, \tau_2)$. It follows that $\mathbb{R}_I^\infty = \mathbb{R}^\infty$ as sets. However, since every point in $\mathfrak{X}_2$ is open and closed in $\mathbb{R}_I^\infty$ and no point is open and closed in $\mathbb{R}^\infty$, they are not equal as topological spaces.

In a similar manner, if $\mathfrak{B}[\mathbb{R}_I^n]$ is the Borel σ-algebra for $\mathbb{R}_I^n$, then $\mathfrak{B}[\mathbb{R}_I^n] \subset \mathfrak{B}[\mathbb{R}_I^{n+1}]$, so we can define $\hat{\mathfrak{B}}[\mathbb{R}_I^n]$ by

$$\hat{\mathfrak{B}}[\mathbb{R}_I^\infty] = \lim_{n\to\infty} \mathfrak{B}[\mathbb{R}_I^n] = \bigcup_{k=1}^{\infty} \mathfrak{B}[\mathbb{R}_I^k].$$

Let $\mathfrak{B}[\mathbb{R}_I^\infty]$ be the smallest σ-algebra containing $\hat{\mathfrak{B}}[\mathbb{R}_I^\infty] \cup \mathcal{P}(\mathbb{R}^\infty \backslash \bigcup_{k=1}^\infty \mathbb{R}_I^k)$, where $\mathcal{P}(\cdot)$ is the power set. It is obvious that the class $\mathfrak{B}[\mathbb{R}_I^\infty]$ coincides with the Borel σ-algebra generated by the τ-topology on $\mathbb{R}_I^\infty$.

Lemma 1.5. $\hat{\mathfrak{B}}[\mathbb{R}_I^\infty] \subset \mathfrak{B}[\mathbb{R}^\infty]$

Proof. It suffices to prove that $\mathbb{R}_I^n \in \mathfrak{B}[\mathbb{R}^\infty]$ for all n. Let $n \in \mathbb{N}$ and define $O_i^{(m)}$ by

$$O_i^{(m)} = \mathbb{R}^{i-1} \times \left(-\frac{1}{2} - \frac{1}{m}, \frac{1}{2} + \frac{1}{m}\right) \times \prod_{k>i} \mathbb{R},$$

then

$$O_i = \bigcap_{m\in\mathbb{N}} O_i^{(m)} = \mathbb{R}^{i-1} \times \left[-\frac{1}{2}, \frac{1}{2}\right] \times \prod_{k>i} \mathbb{R}.$$

Finally, we have

$$\mathbb{R}_I^n = \bigcap_{i>n} O_i.$$

□

We note that O_i is a G_δ set, which is not open in $\mathbb{R}^\infty$, so that $\mathbb{R}_I^n$ is not open in $\mathbb{R}^\infty$.

1.1. *The extension of $\lambda_\infty(\cdot)$ to $\mathbb{R}_I^\infty$*

We know that $\lambda_\infty(\cdot)$ is a countably additive measure on $\mathfrak{B}(\mathbb{R}_I^n)$ for each $n \in \mathbb{N}$, but we cannot say the same for $\mathfrak{B}(\mathbb{R}_I^\infty)$. We now indicate how to provide a (constructive) extension of $\lambda_\infty(\cdot)$ to a countably additive measure on $\mathfrak{B}(\mathbb{R}_I^\infty)$. All proofs can be found in [1]. (This version is equivalent to the one first found by Yamasaki [2] in 1980.) Let $K_n = K \times I_n \in \mathbb{R}_I^n$, $K_{n_i} = K \times I_{n_i} \in \mathbb{R}_I^{n_i}$ be compact sets, with $n, n_i \in \mathbb{N}$. Define

$$\Delta_0 = \{K_n \in \mathfrak{B}(\mathbb{R}_I^n) : \lambda_\infty(K_n) < \infty\},$$

$$\Delta = \left\{ P_N = \bigcup_{i=1}^{N} K_{n_i} \text{ with } \lambda_\infty(K_{n_i} \cap K_{n_j}) = 0, i \neq j \right\}.$$

Definition 1.6. If $P_N \in \Delta$, we define

$$\lambda_\infty(P_N) = \sum_{i=1}^{N} \lambda_\infty(K_{n_i}).$$

Since $P_N \in \mathfrak{B}(\mathbb{R}_I^n)$ for some n and $\lambda_\infty(\cdot)$ is a measure on $\mathfrak{B}(\mathbb{R}_I^n)$, the next result follows:

Lemma 1.7. *Let* $P_{N_1}, P_{N_2} \in \Delta$ *then*

(1) *If* $P_{N_1} \subset P_{N_2}$, *then* $\lambda_\infty(P_{N_1}) \le \lambda_\infty(P_{N_2})$.
(2) *If* $\lambda_\infty(P_{N_1} \cap P_{N_2}) = 0$, *then* $\lambda_\infty(P_{N_1} \cup P_{N_2}) = \lambda_\infty(P_{N_2}) + \lambda_\infty(P_{N_2})$.

Definition 1.8. If $G \subset \mathbb{R}_I^\infty$ is any open set, we define

$$\lambda_\infty(G) = \lim_{N\to\infty} \sup\{\lambda_\infty(P_N) : P_N \in \Delta, P_N \subset G\}.$$

Theorem 1.9. *If* $\mathfrak{O}$ *is the class of open sets in* $\mathfrak{B}(\mathbb{R}_I^\infty)$, *we have*

(1) $\lambda_\infty(\mathbb{R}_I^\infty) = \infty$.
(2) *If* $G_1, G_2 \in \mathfrak{O}, G_1 \subset G_2$, *then* $\lambda_\infty(G_1) \le \lambda_\infty(G_2)$.
(3) *If* $\{G_k\} \subset \mathfrak{O}$, *then*

$$\lambda_\infty\left(\bigcup_{k=1}^{\infty} G_k\right) \le \sum_{k=1}^{\infty} \lambda_\infty(G_k).$$

(4) *If the* G_k *are disjoint, then*

$$\lambda_\infty\left(\bigcup_{k=1}^{\infty} G_k\right) = \sum_{k=1}^{\infty} \lambda_\infty(G_k).$$

If F is an arbitrary compact set in $\mathfrak{B}(\mathbb{R}_I^\infty)$, we define

$$\lambda_\infty(F) = \inf\{\lambda_\infty(G) : F \subset G, G \text{ open}\}. \quad (1)$$

Remark 1.10. At this point, we see the power of $\mathfrak{B}(\mathbb{R}_I^\infty)$. Unlike $\mathfrak{B}(\mathbb{R}^\infty)$, Eq. (1) is well defined for $\mathfrak{B}(\mathbb{R}_I^\infty)$ because it has a sufficient number of open sets of finite measure.

1.1.1. *Bounded outer measure*

Definition 1.11. Let A be an arbitrary set in $\mathbb{R}_I^\infty$.

(1) The outer measure (on $\mathbb{R}_I^\infty$) is defined by

$$\lambda_\infty^*(A) = \inf\{\lambda_\infty(G) : A \subset G, G \text{ open}\}.$$

We let $\mathfrak{L}_0$ be the class of all A with $\lambda_\infty^*(A) < \infty$.

(2) If $A \in \mathfrak{L}_0$, we define the inner measure of A by

$$\lambda_\infty,(*)(A) = \sup\{\lambda_\infty(F) : F \subset A, F \text{ compact}\}.$$

(3) We say that A is a bounded measurable set if $\lambda_\infty^*(A) = \lambda_\infty,(*)(A)$, and define the measure of A, $\lambda_\infty(A)$, by $\lambda_\infty(A) = \lambda_\infty^*(A)$.

The following theorem characterizes the properties of Lebesgue measure on $\mathbb{R}_I^\infty$ (see [1]).

Theorem 1.12. *The measure space* $(\mathbb{R}_I^\infty, \mathfrak{B}[\mathbb{R}_I^\infty], \lambda_\infty)$ *has the following properties:*

(1) $\lambda_\infty(\mathfrak{X}_2) = 0$.
(2) *For every* $A \in \mathfrak{L}[\mathbb{R}_I^\infty]$ *(Lebesgue sets) and* $\varepsilon > 0$, *there exist a compact set* $F \subset A$ *and an open set* $G \supset A$ *such that* $\lambda_\infty(G\backslash F) < \varepsilon$, *so that* $\lambda_\infty(\cdot)$ *is regular.*
(3) *There exists a family of compact sets* $\{A_n\} \subset \mathfrak{B}[\mathbb{R}_I^\infty]$, *with* $\lambda_\infty[A_n] < \infty$ *and a set* N *with* $\lambda_\infty[N] = 0$, *such that* $\mathbb{R}_I^\infty = \bigcup_{n=1}^\infty A_n \cup N$ *(i.e.,* $\lambda_\infty(\cdot)$ *is* σ*-finite).*
(4) *For* $A \in \mathfrak{B}[\mathbb{R}_I^\infty]$, $\lambda_\infty(A - x) = \lambda_\infty(A)$ *if and only if* $x \in \ell_1$.

1.2. *Measurable functions*

In this section, we discuss measurable functions on $\mathbb{R}_I^\infty$. Let $x = (x_1, x_2, x_3, \dots) \in \mathbb{R}_I^\infty$. Fixing n with $I_n = \prod_{k=n+1}^\infty [-\frac{1}{2}, \frac{1}{2}]$, we set $h_n(\hat{x}) = \chi_{I_n}(\hat{x})$, where $\hat{x} = (x_i)_{i=n+1}^\infty$.

Definition 1.13. Let $\mathcal{M}^n$ represent the class of Lebesgue measurable functions on $\mathbb{R}^n$. If $x \in \mathbb{R}_I^\infty$ and $f^n \in \mathcal{M}^n$, let $\bar{x} = (x_i)_{i=1}^n$ and define an essentially tame measurable function of order n (or e_n-tame) on $\mathbb{R}_I^\infty$ by $f(x) = f^n(\bar{x}) \otimes h_n(\hat{x})$. We let

$$\mathcal{M}_I^n = \{f(x) : f(x) = f^n(\bar{x}) \otimes h_n(\hat{x}), x \in \mathbb{R}_I^\infty\}$$

be the class of all e_n-tame functions.

Definition 1.14. A function $f : \mathbb{R}_I^\infty \to \mathbb{R}$ is said to be measurable and we write $f \in \mathcal{M}_I$, if there is a sequence $\{f_n \in \mathcal{M}_I^n\}$ of e_n-tame functions, such that $\lim_{n\to\infty} f_n(x) = f(x)$ λ_∞-(a.e).

The existence of functions satisfying Definition 1.14 is not obvious, so we have [3].

Theorem 1.15. (Existence) *Suppose that* $f : \mathbb{R}_I^\infty \to (-\infty, \infty)$ *and* $f^{-1}(A) \in \mathfrak{B}[\mathbb{R}_I^\infty]$ *for all* $A \in \mathfrak{B}[\mathbb{R}]$. *Then there exists a family of functions* $\{f_n\}$, $f_n \in \mathcal{M}_I^n$, *such that* $f_n(x) \to f(x)$, λ_∞*-(a.e).*

Remark 1.16. From Theorem 1.12(1), we see that any set A, of nonzero measure is concentrated in $\mathfrak{X}_1$ (i.e., $\lambda_\infty(A) = \lambda_\infty(A \cap \mathfrak{X}_1)$). It also follows that the essential support of the limit function $f(x)$ in Definition 1.14 (i.e., $\{x | f(x) \neq 0\}$) is concentrated in $\mathbb{R}_I^N$, for some N.

1.3. *Integration theory on* $\mathbb{R}_I^\infty$

In this section, we provide a constructive theory of integration on $\mathbb{R}_I^\infty$ using the known properties of

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integration on $\mathbb{R}_I^n$. This approach has the advantage that all the standard theorems for Lebesgue measure apply. (The proofs are the same as for integration on $\mathbb{R}^n$.) Let $L^1[\mathbb{R}_I^n]$ be the class of integrable functions on $\mathbb{R}_I^n$. Since $L^1[\mathbb{R}_I^n] \subset L^1[\mathbb{R}_I^{n+1}]$, we define $L^1[\hat{\mathbb{R}}_I \infty] = \bigcup_{n=1}^{\infty} L^1[\mathbb{R}_I^n]$.

Definition 1.17. We say that a measurable function $f \in L^1[\mathbb{R}_I^\infty]$, if there is a Cauchy-sequence $\{f_n\} \subset L^1[\hat{\mathbb{R}}_I^\infty]$, with $f_n \in L^1[\mathbb{R}_I^n]$ and $\lim_{n\to\infty} f_n(x) = f(x)$, λ_∞-(a.e).

Theorem 1.18. $L^1[\hat{\mathbb{R}}_I^\infty] = L^1[\mathbb{R}_I^\infty]$.

Proof. We know that $L^1[\hat{\mathbb{R}}_I^\infty] \supset L^1[\mathbb{R}_I^n]$ for all n so it suffices to prove that $L^1[\hat{\mathbb{R}}_I^\infty]$ is closed. Let f be a limit point of $L^1[\hat{\mathbb{R}}_I^\infty]$ ($f \in L^1[\mathbb{R}_I^\infty]$). If $f = 0$, we are done, so assume $f \neq 0$. From our remarks above, we know that if A_f is the support of f, then $\lambda_\infty(A_f) = \lambda_\infty(A_f \cap \mathfrak{X}_1)$. Thus, $A_f \cap \mathfrak{X}_1 \subset \mathbb{R}_I^N$ for some N. This means that there is a function $f' \in L^1[\mathbb{R}_I^{N+1}]$ with $\lambda_\infty(\{\mathbf{x} : f(\mathbf{x}) \neq f'(\mathbf{x})\}) = 0$. It follows that $f(\mathbf{x}) = f'(\mathbf{x})$-(a.e). Recalling that $L^1[\mathbb{R}_I^n]$ is a set of equivalence classes, we see that $L^1[\hat{\mathbb{R}}_I^\infty] = L^1[\mathbb{R}_I^\infty]$. □

Definition 1.19. If $f \in L^1[\mathbb{R}_I^\infty]$, we define the integral of f by [4]

$$\int_{\mathbb{R}_I^\infty} f(x)d\lambda_\infty(x) = \lim_{n\to\infty} \int_{\mathbb{R}_I^\infty} f_n(x)d\lambda_\infty(x),$$

where $\{f_n\} \subset L^1[\mathbb{R}_I^\infty]$ is any Cauchy-sequence converging to $f(x)$-(a.e).

Theorem 1.20. *Let $f \in L^1[\mathbb{R}_I^\infty]$, $M > 0$, $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions with $f_n \to f$ (a.e) and let $\{g_n\}_{n=1}^\infty$ is a sequence of functions in $L^1[\mathbb{R}_I^\infty]$ converging to $g \in L^1[\mathbb{R}_I^\infty]$ (a,e): then we have the following*:

(1) $L^1[\mathbb{R}_I^\infty]$ *is linear and* $\int_{\mathbb{R}_I^\infty} |f(x)|d\lambda_\infty(x) < \infty$.

(2) *If* $|f_n| \leq M$, *then*

$$\int_{\mathbb{R}_I^\infty} f(x)d\lambda_\infty(x) = \lim_{n\to\infty} \int_{\mathbb{R}_I^\infty} f_n(x)d\lambda_\infty(x).$$

(3) *If* $|f_n| \leq g(x)$ (a,e), *then*

$$\int_{\mathbb{R}_I^\infty} f(x)d\lambda_\infty(x) = \lim_{n\to\infty} \int_{\mathbb{R}_I^\infty} f_n(x)d\lambda_\infty(x).$$

(4) *If* $|f_n| \leq g_n(x)$ (a,e), *then*

$$\int_{\mathbb{R}_I^\infty} f(x)d\lambda_\infty(x) = \lim_{n\to\infty} \int_{\mathbb{R}_I^\infty} f_n(x)d\lambda_\infty(x).$$

(5) *If the sequence* $\{f_n\}_{n=1}^\infty$ *is non-negative and increasing then*

$$\int_{\mathbb{R}_I^\infty} f(x)d\lambda_\infty(x) = \lim_{n\to\infty} \int_{\mathbb{R}_I^\infty} f_n(x)d\lambda_\infty(x).$$

(6) *If the sequence* $\{f_n\}_{n=1}^\infty$ *is non-negative then*

$$\int_{\mathbb{R}_I^\infty} f(x)d\lambda_\infty(x) \leq \liminf \int_{\mathbb{R}_I^\infty} f_n(x)d\lambda_\infty(x).$$

2. The Kuelbs–Steadman Spaces $KS^p[\mathbb{R}_I^\infty]$

Since the work of Henstock [5] and Kurzweil [6], the most important finitely additive measure on $\mathbb{R}^n$ is the one generated by the Henstock–Kurzweil integral (HK-integral). It generalizes the Lebesgue, Bochner and Pettis integrals and it is equivalent to the Denjoy and Perron integrals. Moreover, it is much easier to learn and understand compared to these and the Lebesgue integral. It also provides useful variants of the same theorems that have made the Lebesgue integral so important. The most important factor preventing the widespread use of the HK-integral in mathematics, engineering and physics has been the lack of a Banach space structure comparable to the L^p spaces for the Lebesgue integral.

The possibility for change in this condition began indirectly in 1965, when Gross [7] proved that every separable Banach space contains a separable Hilbert space as a continuous dense embedding. This work was a generalization of Wiener's theory, which used the (densely embedded Hilbert) Sobolev space $\mathbb{H}_0^1[0,1] \subset \mathbb{C}_0[0,1]$. Then, in 1970, Kuelbs [8], generalized Gross' theorem to include the Hilbert space rigging $\mathbb{H}_0^1[0,1] \subset \mathbb{C}_0[0,1] \subset L^2[0,1]$. A general version of Gross–Kuelbs theorem can be stated as follows.

Theorem 2.1. *Let $\mathcal{B}$ be a separable Banach space. Then, separable Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and a positive trace class operator T_{12} defined on $\mathcal{H}_2$ exist such that $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ all as continuous dense embeddings, with $(T_{12}^{1/2}u, T_{12}^{1/2}v)_1 = (u,v)_2$ and $(T_{12}^{-1/2}u, T_{12}^{-1/2}v)_2 = (u,v)_1$.*

This work in relationship to the HK-integral first appeared in the dissertation of Steadman at Howard University in 1988 (see [4]). To understand the connection, we need to see the proof of the $\mathcal{H}_2$ part of the Gross–Kuelbs theorem.

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Lemma 2.2. *If $\mathcal{B}$ is a separable Banach space, there exists a separable Hilbert space $\mathcal{H} \supset \mathcal{B}$ as a continuous dense embedding.*

Proof. Let $\{e_k\}$ be a countable dense sequence on the unit ball of $\mathcal{B}$, and let $\{e_k^*\}$ be any fixed set of corresponding duality mappings (i.e., for each k, $e_k^* \in \mathcal{B}^*$ and $e_k^*(e_k) = \langle e_k, e_k^* \rangle = \|e_k\|_{\mathcal{B}}^2 = \|e_k^*\|_{\mathcal{B}^*}^2 = 1$). For each k, let $t_k = \frac{1}{2^k}$, and define (u, v) as follows:

$$(u, v) = \sum_{k=1}^{\infty} t_k e_k^*(u)\bar{e}_k^*(v) = \sum_{k=1}^{\infty} \frac{1}{2^k} e_k^*(u)\bar{e}_k^*(v).$$

It is clear that (u, v) is an inner product on $\mathcal{B}$. Let $\mathcal{H}$ be the completion of $\mathcal{B}$ with respect to this inner product. It is clear that $\mathcal{B}$ is dense in $\mathcal{H}$, and

$$\|u\|_{\mathcal{H}}^2 = \sum_{k=1}^{\infty} t_k |e_k^*(u)|^2 \le \sup_k |e_k^*(u)|^2 = \|u\|_{\mathcal{B}}^2,$$

so the embedding is continuous. □

Now, note that, if $\mathcal{B}$ is $L^1[\mathbb{R}^n]$,

$$|e_k^*(u)|^2 = \left| \int_{\mathbb{R}^n} e_k^*(\mathbf{x}) u(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|^2, \tag{2}$$

where $e_k^*(\mathbf{x}) \in L^\infty[\mathbb{R}^n]$. It is clear that the Hilbert space $\mathcal{H}$, will contain some non-absolutely integrable functions, but we cannot say which ones will or will not be in there. This gave Steadman the needed hint for her Hilbert space design. Fix n and let $\mathbb{Q}_I^n$ be the set $\{\mathbf{x} \in \mathbb{R}_I^n\}$ such that the first n coordinates $(x_1, x_2, \ldots, x_n)$ are rational. Since this is a countable dense set in $\mathbb{R}_I^n$, we can arrange it as $\mathbb{Q}_I^n = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\}$. For each l and i, let $\mathbf{B}_l(\mathbf{x}_i)$ be the closed cube centered at $\mathbf{x}_i$, with sides parallel to the coordinate axes and edge $e_l = \frac{1}{2^{l-1}\sqrt{n}}, l \in \mathbb{N}$. Now, choose the natural order which maps $\mathbb{N} \times \mathbb{N}$ bijectively to $\mathbb{N}$, and let $\{\mathbf{B}_k, k \in \mathbb{N}\}$ be the resulting set of (all) closed cubes $\{\mathbf{B}_l(\mathbf{x}_i) | (l, i) \in \mathbb{N} \times \mathbb{N}\}$ centered at a point in $\mathbb{Q}_I^n$. Let $\mathcal{E}_k(\mathbf{x})$ be the characteristic function of $\mathbf{B}_k$, so that $\mathcal{E}_k(\mathbf{x})$ is in $L^p[\mathbb{R}_I^n] \cap L^\infty[\mathbb{R}_I^n]$ for $1 \le p < \infty$. Define $F_k(\cdot)$ on $L^1[\mathbb{R}_I^n]$ by

$$F_k(f) = \int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_\infty(\mathbf{x}). \tag{3}$$

Since $\mathbf{B}_k$ is a cube with sides parallel to the coordinate axes and $\mathcal{E}_k(\mathbf{x})$ is the characteristic function of $\mathbf{B}_k$, $F_k(\cdot)$ is well defined for all HK-integrable functions. It is also a bounded linear functional on $L^p[\mathbb{R}_I^n]$ for each k, with $\|F_k\|_\infty \le 1$ and, if $F_k(f) = 0$ for all k, $f = 0$, so that $\{F_k\}$ is fundamental on $L^p[\mathbb{R}_I^n]$ for $1 \le p \le \infty$. Fix $t_k > 0$ such that $\sum_{k=1}^{\infty} t_k = 1$ and define an inner product $(\cdot)$ on $L^1[\mathbb{R}_I^n]$ by

$$(f, g) = \sum_{k=1}^{\infty} t_k F_k(f) \bar{F}_k(g). \tag{4}$$

The completion of $L^1[\mathbb{R}_I^n]$ in this inner product is the Kuelbs–Steadman space, $\mathrm{KS}^2[\mathbb{R}_I^n]$. To see directly that $\mathrm{KS}^2[\mathbb{R}_I^n]$ contains the HK-integrable functions, let f be HK-integrable, then

$$\begin{aligned}
\|f\|_{\mathrm{KS}^2}^2 &= \sum_{k=1}^{\infty} t_k |F_k(f)|^2 \\
&\le \sup_k |F_k(f)|^2 \\
&= \sup_k \left| \int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_n(\mathbf{x}) \right|^2 \\
&< \infty,
\end{aligned}$$

so $f \in \mathrm{KS}^2[\mathbb{R}_I^n]$.

Theorem 2.3. *For each p, $1 \le p \le \infty$, $\mathrm{KS}^2[\mathbb{R}_I^n] \supset L^p[\mathbb{R}_I^n]$ as a continuous dense subspace.*

Proof. By construction, $\mathrm{KS}^2[\mathbb{R}_I^n]$ contains $L^1[\mathbb{R}_I^n]$ densely, so we need to only show that $\mathrm{KS}^2[\mathbb{R}_I^n] \supset L^q[\mathbb{R}_I^n]$ for $q \ne 1$. If $f \in L^q[\mathbb{R}_I^n]$ and $q < \infty$, we have

$$\begin{aligned}
\|f\|_{\mathrm{KS}^2} &= \left[\sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_\infty(\mathbf{x}) \right|^{\frac{2q}{q}} \right]^{1/2} \\
&\le \left[\sum_{k=1}^{\infty} t_k \left(\int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x}) |f(\mathbf{x})|^q d\lambda_\infty(\mathbf{x}) \right)^{\frac{2}{q}} \right]^{1/2} \\
&\le \sup_k \left(\int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x}) |f(\mathbf{x})|^q d\lambda_\infty(\mathbf{x}) \right)^{\frac{1}{q}} \\
&\le \|f\|_q.
\end{aligned}$$

Hence, $f \in \mathrm{KS}^2[\mathbb{R}_I^n]$. For $q = \infty$, first note that $\mathrm{vol}(\mathbf{B}_k)^2 \le [\frac{1}{\sqrt{n}}]^{2n} \le 1$, so we have

$$\begin{aligned}
|f|_{\mathrm{KS}^2} &= \left[\sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_\infty(\mathbf{x}) \right|^2 \right]^{1/2} \\
&\le \left[\left[\sum_{k=1}^{\infty} t_k [\mathrm{vol}(\mathbf{B}_k)]^2 \right] [\operatorname{ess\,sup} |f|]^2 \right]^{1/2} \\
&\le \|f\|_\infty.
\end{aligned}$$

Thus, $f \in \mathrm{KS}^2[\mathbb{R}_I^n]$, and $L^\infty[\mathbb{R}_I^n] \subset \mathrm{KS}^2[\mathbb{R}_I^n]$. □

Before proceeding to additional discussion, we construct the $\mathrm{KS}^p[\mathbb{R}_I^n]$ spaces. Define

$$\|f\|_{\mathrm{KS}^p} = \left[\sum_{k=1}^{\infty} t_k |F_k(f)|^p\right]^{1/p}$$
$$\|f\|_{\mathrm{KS}^\infty} = \sup_k |F_k(f)|.$$

It is easy to see that $\|\cdot\|_{\mathrm{KS}^p}$ defines a norm on L^p. If KS^p is the completion of L^p with respect to this norm, we have the following theorem.

Theorem 2.4. *For each q, $1 \le q \le \infty$, $\mathrm{KS}^p[\mathbb{R}_I^n] \supset L^q[\mathbb{R}_I^n]$ as a dense continuous embedding.*

Proof. As in the previous theorem, by construction $\mathrm{KS}^p[\mathbb{R}_I^n]$ contains $L^p[\mathbb{R}_I^n]$ densely, so we need to only show that $\mathrm{KS}^p[\mathbb{R}_I^n] \supset L^q[\mathbb{R}_I^n]$ for $q \neq p$. First, suppose that $p < \infty$. If $f \in L^q[\mathbb{R}_I^n]$ and $q < \infty$, we have

$$\begin{aligned}
\|f\|_{\mathrm{KS}^p} &= \left[\sum_{k=1}^{\infty} t_k \left|\int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_\infty(\mathbf{x})\right|^{\frac{qp}{q}}\right]^{1/p} \\
&\le \left[\sum_{k=1}^{\infty} t_k \left(\int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x}) |f(\mathbf{x})|^q d\lambda_\infty(\mathbf{x})\right)^{\frac{p}{q}}\right]^{1/p} \\
&\le \sup_k \left(\int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x}) |f(\mathbf{x})|^q d\lambda_\infty(\mathbf{x})\right)^{\frac{1}{q}} \\
&\le \|f\|_q.
\end{aligned}$$

Hence, $f \in \mathrm{KS}^p[\mathbb{R}_I^n]$. For $q = \infty$, we have

$$\begin{aligned}
\|f\|_{\mathrm{KS}^p} &= \left[\sum_{k=1}^{\infty} t_k \left|\int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_\infty(\mathbf{x})\right|^{p}\right]^{1/p} \\
&\le \left[\left[\sum_{k=1}^{\infty} t_k [\mathrm{vol}(\mathbf{B}_k)]^p\right] [\operatorname{ess\,sup} |f|]^p\right]^{1/p} \\
&\le M\|f\|_\infty.
\end{aligned}$$

Thus $f \in \mathrm{KS}^p[\mathbb{R}_I^n]$, and $L^\infty[\mathbb{R}_I^n] \subset \mathrm{KS}^p[\mathbb{R}_I^n]$. The case $p = \infty$ is obvious. □

Theorem 2.5. *For KS^p, $1 \le p \le \infty$, we have*

(1) *If $f, g \in \mathrm{KS}^p$, then $\|f + g\|_{\mathrm{KS}^p} \le \|f\|_{\mathrm{KS}^p} + \|g\|_{\mathrm{KS}^p}$ (Minkowski inequality).*
(2) *If K is a weakly compact subset of L^p, it is a compact subset of KS^p.*
(3) *If $1 < p < \infty$, then KS^p is uniformly convex.*
(4) *If $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, then the dual space of KS^p is KS^q.*
(5) *$\mathrm{KS}^\infty \subset \mathrm{KS}^p$, for $1 \le p < \infty$.*

Proof. The proof of (1) follows from the classical case for sums. The proof of (2) follows from the fact that if $\{f_m\}$ is any weakly convergent sequence in K with limit f, then

$$\int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x})[f_m(\mathbf{x}) - f(\mathbf{x})] d\lambda_\infty(\mathbf{x}) \to 0$$

for each k. It follows that $\{f_m\}$ converges strongly to f in KS^p.

The proof of (3) follows from a modification of the proof of the Clarkson inequalities for l^p norms.

To prove (4), let

$$\ell_k^p(g) = \|g\|_{\mathrm{KS}^p}^{2-p} \left|\int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x}) g(\mathbf{x}) d\lambda_\infty(\mathbf{x})\right|^{p-2}$$

and observe that, for $p \neq 2$, $1 < p < \infty$, the linear functional

$$L_g(f) = \sum_{k=1}^{\infty} t_k \ell_k^p(g) F_k(g) \bar{F}_k(f)$$

is a duality map on KS^q for each $g \in \mathrm{KS}^p$ and that KS^p is reflexive from (3). To prove (5), note that $f \in \mathrm{KS}^\infty$ implies that $|\int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_\infty(\mathbf{x})|$ is uniformly bounded for all k. It follows that $|\int_{\mathbb{R}_I^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\lambda_\infty(\mathbf{x})|^p$ is uniformly bounded for each p, $1 \le p < \infty$. It is now clear from the definition of KS^∞ that

$$\|f\|_{\mathrm{KS}^p} = \left[\sum_{k=1}^{\infty} t_k |F_k(f)|^p\right]^{1/p} \le \|f\|_{\mathrm{KS}^\infty} < \infty.$$

□

Theorem 2.6. *For each p, $1 \le p \le \infty$, the test functions $\mathcal{D} \subset \mathrm{KS}^p(\mathbb{R}_I^n)$ as a continuous embedding.*

Proof. Since $\mathrm{KS}^\infty(\mathbb{R}_I^n)$ is continuously embedded in $\mathrm{KS}^p(\mathbb{R}_I^n)$, $1 \le q < \infty$, it suffices to prove the result for $\mathrm{KS}^\infty(\mathbb{R}_I^n)$. Suppose that $\phi_j \to \phi$ in $\mathcal{D}[\mathbb{R}_I^n]$, so that there exists a compact set $K \subset \mathbb{R}_I^n$, containing the support of $\phi_j - \phi$ and $D^\alpha \phi_j$ converges to $D^\alpha \phi$ uniformly on K for every multi-index α. Let $L = \{l \in \mathbb{N} : \text{the support of } \mathcal{E}_l \subset K\}$, then

$$\begin{aligned}
&\lim_{j\to\infty} \|D^\alpha \phi - D^\alpha \phi_j\|_{\mathrm{KS}^\infty} \\
&\quad = \lim_{j\to\infty} \sup_l |F_l(D^\alpha \phi - D^\alpha \phi_j)| \\
&\quad \le \mathrm{vol}(\mathbf{B}_l) \lim_{j\to\infty} \sup_{\mathbf{x}\in K} |D^\alpha \phi(\mathbf{x}) - D^\alpha \phi_j(\mathbf{x})| \\
&\quad \le \lim_{j\to\infty} \sup_{\mathbf{x}\in K} |D^\alpha \phi(\mathbf{x}) - D^\alpha \phi_j(\mathbf{x})| = 0.
\end{aligned}$$

It follows that $\mathcal{D}[\mathbb{R}_I^n] \subset \mathrm{KS}^p[\mathbb{R}_I^n]$ as a continuous embedding, for $1 \le p \le \infty$. Thus, by the Hahn–Banach theorem, we see that the Schwartz distributions, $\mathcal{D}'[\mathbb{R}_I^n] \subset [\mathrm{KS}^p(\mathbb{R}_I^n)]'$, for $1 \le p \le \infty$. □

2.1. *The family* $\mathrm{KS}^p[\mathbb{R}_I^\infty]$

We can now construct the spaces $\mathrm{KS}^p[\mathbb{R}_I^\infty]$, $1 \le p \le \infty$, using the same approach that led to $L^1[\mathbb{R}_I^\infty]$. Since $\mathrm{KS}^p[\mathbb{R}_I^n] \subset \mathrm{KS}^p[\mathbb{R}_I^{n+1}]$, we define $\mathrm{KS}^p[\hat{\mathbb{R}}_I^\infty] = \bigcup_{n=1}^\infty \mathrm{KS}^p[\mathbb{R}_I^n]$.

Definition 2.7. We say that a measurable function $f \in \mathrm{KS}^p[\mathbb{R}_I^\infty]$, if there is a Cauchy-sequence $\{f_n\} \subset \mathrm{KS}^p[\hat{\mathbb{R}}_I^\infty]$, with $f_n \in \mathrm{KS}^p[\mathbb{R}_I^n]$ and $\lim_{n\to\infty} f_n(x) = f(x)$, λ_∞-(a.e).

The same proof as Theorem 1.18 shows that functions in $\mathrm{KS}^p[\hat{\mathbb{R}}_I^\infty]$ differ from functions in its closure $\mathrm{KS}^p[\mathbb{R}_I^\infty]$ by sets of measure zero.

Theorem 2.8. $\mathrm{KS}^p[\hat{\mathbb{R}}_I^\infty] = \mathrm{KS}^p[\mathbb{R}_I^\infty]$.

Definition 2.9. If $f \in \mathrm{KS}^p[\mathbb{R}_I^\infty]$, we define the integral of f by

$$\int_{\mathbb{R}_I^\infty} f(x)d\lambda_\infty(x) = \lim_{n\to\infty} \int_{\mathbb{R}_I^\infty} f_n(x)d\lambda_\infty(x),$$

where $f_n \in \mathrm{KS}^p[\mathbb{R}_I^n]$ is any Cauchy sequence converging to $f(x)$.

Theorem 2.10. *If* $f \in \mathrm{KS}^p[\mathbb{R}_I^\infty]$, *then the above integral exists and all theorems that are true for* $f \in \mathrm{KS}^p[\mathbb{R}_I^n]$ *also hold for* $f \in \mathrm{KS}^p[\mathbb{R}_I^\infty]$.

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