



Global Effect of Non-Conservative Perturbations on Homoclinic Orbits

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Received: 4 September 2019 / Accepted: 5 October 2020
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Abstract

We study the effect of time-dependent, non-conservative perturbations on the dynamics along homoclinic orbits to a normally hyperbolic invariant manifold. We assume that the unperturbed system is Hamiltonian, and the normally hyperbolic invariant manifold is parametrized via action-angle coordinates. The homoclinic excursions can be described via the scattering map, which gives the future asymptotic of an orbit as a function of its past asymptotic. We provide explicit formulas, in terms of convergent integrals, for the perturbed scattering map expressed in action-angle coordinates. We illustrate these formulas for perturbations of both uncoupled and coupled rotator-pendulum systems.

Keywords Melnikov method · Homoclinic orbits · Scattering map

Mathematics Subject Classification Primary 37J40, 37C29, 34C37;
Secondary 70H08

To the memory of Florin Diacu.

M. Gidea: Research of M.G. was partially supported by NSF Grant DMS-1814543. R. de la Llave: Research of R.L. was partially supported by NSF Grant DMS-1800241, and H2020-MCA-RISE #734577. M. Musser: Research of M.M. was partially supported by NSF Grant DMS-1814543.

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1 Introduction

1.1 Brief Description of the Main Results and Methodology

In this paper we study the effect of small, non-conservative, time-dependent perturbations on the dynamics along homoclinic orbits in Hamiltonian systems. We describe this dynamics via the scattering map, and estimate the effect of the perturbation on the scattering map. We illustrate the computation of the perturbed scattering map on two simple models: an uncoupled rotator-pendulum system, and a coupled rotator-pendulum system. However, similar computations can be obtained for more general systems.

Our approach is based on geometric methods and on Melnikov theory. The geometric framework presents the following situation. There exists a normally hyperbolic invariant manifold (NHIM) whose stable and unstable manifolds intersect. The orbits in the intersection are homoclinic orbits which are bi-asymptotic to the normally hyperbolic invariant manifold. To each transverse homoclinic intersection we can associate a scattering map, which assigns to the foot-point of the unstable fiber passing through a given homoclinic point the foot-point of the stable fiber passing through the same homoclinic point. The scattering map can also be defined in the special case when the stable and unstable manifolds coincide. In either case, the scattering map is a diffeomorphism of an open subset of the NHIM onto its image. When the system is Hamiltonian and the NHIM is a symplectic manifold, the scattering map is a symplectic map. If a small, Hamiltonian perturbation is added to the system, then the scattering map remains symplectic. This is no longer the case when a non-conservative perturbation is added to the system: the perturbed scattering map does not need to be symplectic.

In the rotator-pendulum models that we consider, the NHIM can be parametrized via action-angle coordinates, so the scattering map can be described in terms of these coordinates as well. For the uncoupled rotator-pendulum system, the unperturbed scattering map is the identity. For the coupled rotator-pendulum the unperturbed scattering map is a shift in the angle coordinates. Then we add a small, non-conservative, time-dependent perturbation. Using Melnikov theory, we compute the perturbed scattering map up to the first order with respect to the size of the perturbation. We provide expressions for the difference between the perturbed scattering map and the unperturbed one, relative to the action and angle coordinates, in terms of convergent improper integrals of the perturbation evaluated along homoclinic orbits of the unperturbed system. One important aspect in the computation is that, in the perturbed system, the action is a slow variable, while the angle is a fast variable.

Similar computations of the perturbed scattering map, when the perturbation is Hamiltonian, have been done in, e.g. [7]. The effect of the perturbation on the action component of the scattering map is relatively easy to compute directly. On the other hand, the effect on the angle component of the scattering map is more complicated to compute, since this is a fast variable. To circumvent this difficulty, the paper [7] uses the symplecticity of the scattering map to estimate indirectly the effect of the perturbation on the angle component. In our case, since we consider non-conservative

perturbations, this type of argument no longer holds. We therefore perform a direct computation of the effect of the perturbation on the angle component of the scattering map.

The rotator-pendulum models describe the behavior of systems with slow variables (action/energy) when the level sets of these conserved quantities have different topologies (or include separatrices). These situations appear in normal forms near resonances. In the uncoupled model, these slow variables do not affect each other, but in the coupled model, they do.

The coupling between the slow variables of different topologies is a real effect and cannot be eliminated by making further changes of variables. As we will see, the coupling between the actions and the energies of penduli, leads to the systems having a non-trivial scattering map.

Both the uncoupled and the coupled rotator pendulum system are fundamental examples in studying Hamiltonian instability, and in particular the Arnold diffusion problem. The main idea is that the perturbed scattering map can be used to show the existence of pseudo-orbits—generated by successive applications of the scattering map—that move $O(1)$ with respect to the perturbation inside the NHIM. Then a shadowing lemma [20] can be used to show the existence of true orbits that move $O(1)$ in a neighborhood of the NHIM. A recent survey on this topic can be found in [19].

1.2 Related Works

The Melnikov method has been developed to study the persistence of periodic orbits and of homoclinic/heteroclinic connections under periodic perturbations [31].

One well-known application of the Melnikov method is to show that degenerate homoclinic orbits in the unperturbed system yield transverse homoclinic orbits in the perturbed system, see, e.g., [9–12,21,25,33,36]. The effect of the homoclinic orbits is given in terms of certain improper integrals referred to as ‘Melnikov integrals’. In some of these papers the integrals are only conditionally convergent, and the sequence of limits of integration must be carefully chosen in order to obtain the correct dynamic meaning.

Another important application of the Melnikov method is to estimate the effect of the perturbations on the scattering map, which is associated to homoclinic excursions to a normally hyperbolic invariant manifold. In the case when the perturbation is given by a time-periodic or quasi-periodic Hamiltonian, this effect is estimated in, e.g., [4–8,13,14,20]. The effect on the scattering map of general time-dependent Hamiltonian perturbations is studied in, e.g., [17,18].

Some other papers of related interest include [1,22–24,29,30,34].

A novelty of our paper is that we study the effect on the scattering map of general time-dependent perturbations that can be non-conservative. The methodology used in some of the earlier papers, which relies on the symplectic properties of the scattering map, does not extend to the non-conservative case.

We also note that the results here are global in the sense that they apply to all homoclinics to a NHIM, while other results only apply to homoclinics to fixed points or periodic/quasi-periodic orbits.

1.3 Structure of the Paper

In Sect. 2 we provide the set-up for the problem, and describe the models that make the main focus of the subsequent results: the uncoupled and the coupled rotator-pendulum system subject to general time-dependent, non-conservative perturbations. In Sect. 3, we describe the main tools—normally hyperbolic invariant manifolds and the scattering map. In Sect. 4 we provide some master lemmas that are used in the subsequent calculations. The main results in the case of the uncoupled rotator-pendulum system are formulated and proved in Sect. 5. Theorem 5.1 gives sufficient conditions for the existence of transverse homoclinic intersections for the perturbed system. Theorem 5.4 provides estimates on the effect of the perturbation on the action-component of the scattering map. Theorem 5.6 provides estimates on the effect of the perturbation on the angle-component of the scattering map. In Sect. 5.8 we show that, when the perturbation is Hamiltonian, the formulas obtained in Theorems 5.4 and 5.6 are equivalent to the corresponding formulas in [7]. The main results on the effect of the perturbation on the scattering map in the case of the coupled rotator-pendulum system— Theorems 6.2 and 6.3—are formulated and proved in Sect. 6. Surprisingly, we obtain the same formulas as in the case of the the uncoupled rotator-pendulum system.

2 Set-Up

Consider a $\mathcal{C}^{r+1}$ -smooth manifold M of dimension $(2m)$, where $r \geq r_0$ for some suitable r_0 . Each point $z \in M$ is described via a system of local coordinates $(u, v) \in \mathbb{R}^{2m}$, i.e., $z = z(u, v)$. Assume that M is endowed with the standard symplectic form

$$\Omega = du \wedge dv = \sum_{i=1}^m du_i \wedge dv_i, \quad (2.1)$$

defined on local coordinate charts.

On M we consider a non-autonomous system of differential equations

$$\dot{z} = \mathcal{X}_\varepsilon(z; \varepsilon) = \mathcal{X}^0(z) + \varepsilon \mathcal{X}^1(z, t; \varepsilon), \quad (2.2)$$

where $\mathcal{X}^0 : M \rightarrow TM$ is a $\mathcal{C}^r$ -differentiable vector field on M , $\mathcal{X}^1 : M \times \mathbb{R} \times \mathbb{R} \rightarrow TM$ is a time-dependent, parameter dependent $\mathcal{C}^r$ -differentiable vector field on M , and $\varepsilon \in \mathbb{R}$ is a ‘smallness’ parameter, taking values in some interval $(-\varepsilon_0, \varepsilon_0)$ around 0. Moreover, we assume that $\mathcal{X}^1 = \mathcal{X}^1(z, t; \varepsilon)$ is uniformly differentiable in all variables.

The flow of (2.2) will be denoted by Φ_ε^t .

Above, the dependence of $\mathcal{X}^1(z, t; \varepsilon)$ on the time t is assumed to be of a general type, not necessarily periodic or quasi-periodic. In the particular case of a periodic perturbation, we require that t is defined mod 1, or, equivalently $t \in \mathbb{T}^1$. In the particular case of a quasi-periodic perturbation, we require that the vector field $\mathcal{X}^1$ is of the form $\mathcal{X}^1(z, \chi(t); \varepsilon)$, for $\chi : \mathbb{R} \rightarrow \mathbb{T}^k$ of the form $\chi(t) = \phi_0 + t\varpi$ for some $k \geq 2$, $\phi_0 \in \mathbb{T}^k$ and $\varpi \in \mathbb{R}^k$ a rationally independent vector, i.e., satisfying the following condition: $h \in \mathbb{Z}^k$ and $h \cdot \varpi = 0$ imply $h = 0$.

Below, we will consider some situations when the vector fields $\mathcal{X}^0, \mathcal{X}^1$ satisfy additional assumptions.

2.1 The Unperturbed System

We assume that the vector field $\mathcal{X}^0$ represents an autonomous Hamiltonian vector field, that is, $\mathcal{X}^0 = J\nabla_z H_0$ for some $\mathcal{C}^{r+1}$ -smooth Hamiltonian function $H_0 : M \rightarrow \mathbb{R}$, where J is an almost complex structure compatible with the standard symplectic form given by (2.1), and the gradient ∇ is with respect to the associated Riemannian metric¹.

Below we describe some of the geometric structures that are the subject of our study. These geometric structures are defined in Sect. 3.3.

(H0-i) There exists a $(2d)$ -dimensional manifold $\Lambda_0 \simeq D \times \mathbb{T}^d \subseteq M$ that is a normally hyperbolic invariant manifold (NHIM) for the Hamiltonian flow Φ_0^t of H_0 , where D is a closed d -dimensional ball in $\mathbb{R}^d$. The NHIM Λ_0 is symplectic with the induced symplectic form from M .

(H0-ii) The manifold Λ_0 is foliated by d -dimensional invariant tori, and is parametrized via some ‘action-angle’ coordinates characterized by the following properties. Each d -dimensional torus in the foliation corresponds to a fixed value of the action coordinate, and the angle coordinate is symplectically conjugate with the action coordinate relative to the symplectic form induced on Λ_0 . The flow Φ_0^t on each such torus is a linear flow.

(H0-iii) The unstable and stable manifolds $W^u(\Lambda_0), W^s(\Lambda_0)$ of Λ_0 coincide, i.e., $W^u(\Lambda_0) = W^s(\Lambda_0)$. We consider two cases:

- (a) For each $z \in \Lambda_0$, $W^u(z) = W^s(z)$;
- (b) For each $z \in \Lambda_0$, there exists $z' \in \Lambda_0$ such that $W^u(z) = W^s(z')$.

Condition **(H0-i)** says that there exists a NHIM for the flow, which is symplectic when endowed with the restriction to Λ_0 of the symplectic form Ω on M . Condition **(H0-ii)** says that the dynamics restricted to the NHIM is integrable, with the action being an integral of motion. As we shall see in Sects. 2.3 and 2.4, in the case of rotator-pendulum systems, the NHIM corresponds to fixing the variables of the ‘generalized’ pendulum at the hyperbolic equilibrium. The d -dimensional tori that foliate the NHIM correspond to energy level sets of the ‘generalized’ rotator. Condition **(H0-iii)** says that there exist homoclinic orbits to the NHIM which are degenerate, as they correspond to the unstable and stable manifolds of the NHIM which coincide.

¹ $g(u, v) = \omega(u, Jv)$.

As models for systems with the above properties, we consider an uncoupled and a coupled rotator-pendulum system, which are described in detail in Sects. 2.3 and 2.4.

We will show that if the perturbation $\mathcal{X}^1$ satisfies some verifiable conditions, then the unstable and stable manifolds of the perturbed NHIM intersect transversally for all $\varepsilon \neq 0$ sufficiently small, so there exist transverse homoclinic orbits to the NHIM. The goal will be to quantify the effect of the perturbation on the dynamics along homoclinic orbits. This effect will be measured in terms of the changes in the action and angle coordinates when the orbit follows a homoclinic excursion.

2.2 The Perturbation

The vector field $\mathcal{X}^1$ is a time-dependent, parameter-dependent vector field on M . In the general case we will not assume that $\mathcal{X}^1$ is Hamiltonian, so the system (2.2) can be subject to dissipation or forcing.

We will also derive results for the particular case when the perturbation $\mathcal{X}^1$ in (2.2) is Hamiltonian, that is, it is given by

$$\mathcal{X}^1(z, t; \varepsilon) = J \nabla_z H_1(z, t; \varepsilon), \quad (2.3)$$

where H_1 is a time-dependent, parameter-dependent $\mathcal{C}^{r+1}$ -smooth Hamiltonian function on M .

2.3 Model 1: The Rotator-Pendulum System

This model is described by an autonomous Hamiltonian H_0 of the form:

$$\begin{aligned} H_0(p, q, I, \theta, t) &= h_0(I) + h_1(p, q) \\ &= h_0(I) + \sum_{i=1}^n \varsigma_i \left(\frac{1}{2} p_i^2 + V_i(q_i) \right), \end{aligned} \quad (2.4)$$

with $p = (p_1, \dots, p_n) \in \mathbb{R}^n$, $q = (q_1, \dots, q_n) \in \mathbb{T}^n$, $\varsigma_i \in \{-1, +1\}$ for $i = 1, \dots, n$, $I = (I_1, \dots, I_d) \in \mathbb{R}^d$, $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{T}^d$, and $t \in \mathbb{R}$. We will also denote $z = z(p, q, I, \theta)$.

The phase space $\mathbb{R}^{2(d+n)}$ is endowed with the symplectic form

$$\Omega = \sum_{i=1}^n dp_i \wedge dq_i + \sum_{j=1}^d dI_j \wedge d\theta_j.$$

In the above, we assume the following:

- (RP-i) The level sets of the I -variable are d -dimensional tori for all $I \in D$, for some closed d -dimensional ball $D \subset \mathbb{R}^d$.
- (RP-ii) Each potential V_i is periodic of period 1 in q_i ;

(RP-iii) Each potential V_i has a non-degenerate local maximum (in the sense of Morse), which, without loss of generality, we set at 0; that is, $V_i'(0) = 0$ and $V_i''(0) < 0$. The non-degeneracy in the sense of Morse means that, additionally, 0 is the only critical point in the level set $\{V_i(q) = V_i(0)\}$, that is, $V_i'(q_*) = 0$ and $V_i(q_*) = V_i(0)$ implies $q_* = 0$.

Condition (RP-iii) implies that each pendulum has a homoclinic orbit to $(0, 0)$.

We note that for the classical rotator, the standard assumption is that $\partial^2 h_0 / \partial I^2$ is positive definite; in our case we allow that $\partial^2 h_0 / \partial I^2$ is of indefinite sign.² For this reason we refer to h_0 as a ‘generalized’ rotator. This situation appears in several applications, such as critical inclination of satellite orbits, quasigeostrophic flows, plasma devices, and transport in magnetized plasma [3,26,28].

We note that condition (RP-iii) implies that if $(p_i(t), q_i(t))$ is a homoclinic orbit for the i -th pendulum, then $(-p_i(-t), q_i(-t))$ is also a homoclinic orbit. Hence there exist $2n$ different homoclinic orbits for (2.4), and implicitly $2n$ different scattering maps.

For the classical pendulum, the Hamiltonian is of the form $(\frac{1}{2}p_i^2 + V_i(q_i))$. In our case we allow a sign $\varsigma_i = \pm 1$ in front each pendulum, so $\partial^2 h_1 / \partial p^2$ can be of indefinite sign. This is why we refer to the terms in h_1 as ‘generalized penduli’.

In Sect. 5.1 we will show that for each closed d -dimensional ball $D \subseteq \mathbb{R}^d$, the set

$$\Lambda_0 = \{(p, q, I, \theta) \mid p = q = 0, I \in D, \theta \in \mathbb{T}^d\} \quad (2.5)$$

is a NHIM with boundary. The stable and unstable manifolds coincide, i.e., $W^u(\Lambda_0) = W^s(\Lambda_0)$, and, moreover, for each $z \in \Lambda_0$, $W^u(z) = W^s(z)$. Each point in $W^u(\Lambda_0) = W^s(\Lambda_0)$ determines a homoclinic trajectory which approaches Λ_0 in both positive and negative time.

Thus, for this model, the geometric structures described above satisfy the properties **(H0-i)**, **(H0-ii)**, **(H0-iii-a)** in Sect. 2.1.

2.4 Model 2: The Coupled Rotator-Pendulum System

This model consists of a system of generalized rotators and penduli with a coupling between the actions and the momenta, given by the autonomous Hamiltonian H_0 :

$$\begin{aligned} H_0(p, q, I, \theta, t) &= h_0(I) + h_1(p, q) + I^T M p \\ &= h_0(I) + \sum_{i=1}^n \varsigma_i \left(\frac{1}{2} p_i^2 + V_i(q_i) \right) + I^T M p, \end{aligned} \quad (2.6)$$

where $(p, q) \in \mathbb{R}^n \times \mathbb{T}^n$, $\varsigma_i \in \{-1, +1\}$ for $i = 1, \dots, n$, $(I, \theta) \in \mathbb{R}^d \times \mathbb{T}^d$, M is $d \times n$ -matrix, T denotes the transpose of a matrix, and $t \in \mathbb{R}$.

Above the vectors I and p are thought of as $(d \times 1)$ and $(n \times 1)$ matrices, respectively. As before, we denote $\mathcal{X}^0 = \nabla H_0$.

² An example is given by $h_0(I) = I^n$ with $n \geq 3$ odd, or $h_0(I) = I_1^2 - I_2^2$.

We assume that the system (2.6) satisfies the conditions (RP-i), (RP-ii), (RP-iii) from Sect. 2.3.

In Sect. 6.1 we will show that for each closed d -dimensional ball $D \subseteq \mathbb{R}^d$, the set

$$\Lambda_0 = \{(p, q, I, \theta) \mid p = -\varsigma M^T I, q = 0, I \in D, \theta \in \mathbb{T}^d\} \quad (2.7)$$

is a NHIM with boundary. The stable and unstable manifolds coincide, i.e., $W^u(\Lambda_0) = W^s(\Lambda_0)$, as in the case of the uncoupled rotator-pendulum model in Sect. 2.3. However, as we will see in Sect. 6.2, homoclinic orbits have future asymptotics that are different from the past asymptotics. That is, for each $z \in \Lambda_0$, there exists $z' \in \Lambda_0$ such that $W^u(z) = W^s(z')$, with $z \neq z'$ in general.

Thus, for this model, the geometric structures described above satisfy the properties (H0-i), (H0-ii), (H0-iii-b) in Sect. 2.1.

3 Preliminaries

3.1 Vector Fields as Differential Operators

In the sequel, we will identify vector fields with differential operators, which is a standard operation in differential geometry (see, e.g., [2]). That is, given a smooth vector field $\mathcal{X}$ and a smooth function f on the manifold M ,

$$(\mathcal{X}f)(z) = \sum_j (\mathcal{X})_j(z) (\partial_{z_j} f)(z), \quad (3.1)$$

where $z_j, j \in \{1, \dots, \dim(M)\}$, are local coordinates. Similarly, a smooth time- and parameter-dependent vector field acts as a differential operator by

$$(\mathcal{X}f)(z, t; \varepsilon) = \sum_j (\mathcal{X})_j(z, t; \varepsilon) (\partial_{z_j} f)(z). \quad (3.2)$$

If Φ^t is the flow for the vector field $\mathcal{X}$, then

$$\begin{aligned} \frac{d}{dt}(f(\Phi^t(z))) &= \nabla f(\Phi^t(z)) \cdot \frac{d}{dt}(\Phi^t(z)) = \nabla f(\Phi^t(z)) \cdot \mathcal{X}(\Phi^t(z)) \\ &= \sum_j (\mathcal{X}_j)(\Phi^t(z)) (\partial_{z_j} f)(\Phi^t(z)) = (\mathcal{X}f)(\Phi^t(z)). \end{aligned}$$

For a vector-valued function $\mathbf{F} : M \rightarrow \mathbb{R}^k$ of components $\mathbf{F} = (\mathbf{F}_i)_i$, we will denote

$$\mathcal{X}\mathbf{F} := (\mathcal{X}\mathbf{F}_i)_i.$$

3.2 Extended System

To (2.2) we associate the extended system

$$\begin{aligned}\dot{z} &= \mathcal{X}^0(z) + \varepsilon \mathcal{X}^1(z, t; \varepsilon), \\ \dot{i} &= 1,\end{aligned}\tag{3.3}$$

which is defined on the extended phase space $\tilde{M} = M \times \mathbb{R}$. We denote $\tilde{z} = (z, t) \in \tilde{M}$. The independent variable will be denoted by s from now on, and the derivative above is meant with respect to s . We will denote by $\tilde{\Phi}_\varepsilon^s$ the extended flow of (3.3). We have

$$\tilde{\Phi}_\varepsilon^s(z, t) = (\Phi_\varepsilon^s(z), t + s).$$

3.3 Normally Hyperbolic Invariant Manifolds

We briefly recall the notion of a normally hyperbolic invariant manifold [16,27].

Let M be a $\mathcal{C}^r$ -smooth manifold, Φ^t a $\mathcal{C}^r$ -flow on M . A submanifold (with or without boundary) Λ of M is a normally hyperbolic invariant manifold (NHIM) for Φ^t if it is invariant under Φ^t , and there exists a splitting of the tangent bundle of TM into sub-bundles over Λ

$$T_z M = E_z^u \oplus E_z^s \oplus T_z \Lambda, \quad \forall z \in \Lambda\tag{3.4}$$

that are invariant under $D\Phi^t$ for all $t \in \mathbb{R}$, and there exist rates

$$\lambda_- \leq \lambda_+ < \lambda_c < 0 < \mu_c < \mu_- \leq \mu_+$$

and a constant $C > 0$, such that for all $x \in \Lambda$ we have

$$\begin{aligned}C e^{t\lambda_-} \|v\| &\leq \|D\Phi^t(z)(v)\| \leq C e^{t\lambda_+} \|v\| \text{ for all } t \geq 0, \text{ if and only if } v \in E_z^s, \\ C e^{t\mu_+} \|v\| &\leq \|D\Phi^t(z)(v)\| \leq C e^{t\mu_-} \|v\| \text{ for all } t \leq 0, \text{ if and only if } v \in E_z^u, \\ C e^{|t|\lambda_c} \|v\| &\leq \|D\Phi^t(z)(v)\| \leq C e^{|t|\mu_c} \|v\| \text{ for all } t \in \mathbb{R}, \text{ if and only if } v \in T_z \Lambda.\end{aligned}\tag{3.5}$$

It is known that Λ is $\mathcal{C}^\ell$ -differentiable, with $\ell \leq r - 1$, provided that

$$\begin{aligned}\ell \mu_c + \lambda_+ &< 0, \\ \ell \lambda_c + \mu_- &> 0.\end{aligned}\tag{3.6}$$

The manifold Λ has associated unstable and stable manifolds of Λ , denoted $W^u(\Lambda)$ and $W^s(\Lambda)$, respectively, which are $\mathcal{C}^{\ell-1}$ -differentiable. They are foliated by 1-dimensional unstable and stable manifolds (fibers) of points, $W^u(z)$, $W^s(z)$, $z \in \Lambda$, respectively, which are as smooth as the flow, i.e., $\mathcal{C}^r$ -differentiable. These fibers are not invariant by the flow, but *equivariant* in the sense that

$$\begin{aligned}\Phi^t(W^u(z)) &= W^u(\Phi^t(z)), \\ \Phi^t(W^s(z)) &= W^s(\Phi^t(z)).\end{aligned}$$

The unstable and stable manifolds of Λ , $W^u(\Lambda)$ and $W^s(\Lambda)$, are tangent at Λ to

$$E_\Lambda^u = \bigcup_{z \in \Lambda} E_z^u, \text{ and } E_\Lambda^s = \bigcup_{z \in \Lambda} E_z^s,$$

respectively.

Since $W^{s,u}(\Lambda) = \bigcup_{z \in \Lambda} W^{s,u}(z)$, we can define the projections along the fibers

$$\begin{aligned}\Omega^+ : W^s(\Lambda) &\rightarrow \Lambda, \quad \Omega^+(z) = z^+ \text{ iff } z \in W^s(z^+), \\ \Omega^- : W^u(\Lambda) &\rightarrow \Lambda, \quad \Omega^-(z) = z^- \text{ iff } z \in W^u(z^-).\end{aligned}\tag{3.7}$$

The point $z^+ \in \Lambda$ is characterized by

$$d(\Phi^t(z), \Phi^t(z^+)) \leq C_z e^{t\lambda_+}, \quad \text{for all } t \geq 0.\tag{3.8}$$

and the point $z^- \in \Lambda$ by

$$d(\Phi^t(z), \Phi^t(z^-)) \leq C_z e^{t\mu_-}, \quad \text{for all } t \leq 0,\tag{3.9}$$

for some $C_z > 0$.

3.4 Scattering Map

Assume that $W^u(\Lambda)$, $W^s(\Lambda)$ have a transverse intersection along a manifold Γ satisfying:

$$\begin{aligned}T_z \Gamma &= T_x W^s(\Lambda) \cap T_x W^u(\Lambda), \text{ for all } z \in \Gamma, \\ T_x M &= T_z \Gamma \oplus T_x W^u(z^-) \oplus T_x W^s(z^+), \text{ for all } z \in \Gamma.\end{aligned}\tag{3.10}$$

Under these conditions the projection mappings $\Omega^\pm$ restricted to Γ are local diffeomorphisms. We can restrict Γ if necessary so that $\Omega^\pm$ are diffeomorphisms from Γ onto open subsets $U^\pm$ in Λ . Such a Γ will be called a *homoclinic channel*.

By definition the scattering map associated to Γ is defined as

$$\sigma : U^- \subseteq \Lambda \rightarrow U^+ \subseteq \Lambda, \quad \sigma = \Omega^+ \circ (\Omega^-|_\Gamma)^{-1}.$$

Equivalently, $\sigma(z^-) = z^+$, provided that $W^u(z^-)$ intersects $W^s(z^+)$ at a unique point $z \in \Gamma$.

If M is a symplectic manifold, Φ^t is a Hamiltonian flow on M , and $\Lambda \subseteq M$ has an induced symplectic structure, then the scattering map is symplectic. If the flow is exact Hamiltonian, the scattering map is exact symplectic. For details see [7].

3.5 The Scattering Map for the Unperturbed, Extended System

We consider the extended system from Sect. 3.2, and we express the scattering map for the unperturbed extended pendulum-rotator system in terms of the action-angle coordinates defined in Sect. 5.3.

We first note that, for the unperturbed extended system, since

$$d(\Phi_0^t(z), \Phi_0^t(z^\pm)) \rightarrow 0 \text{ as } t \rightarrow \pm\infty,$$

and $\tilde{\Phi}_0^s(z, t) = (\Phi_0^s(z), t + s)$, we have

$$d(\tilde{\Phi}_0^s(z, t), \tilde{\Phi}_0^s(z^\pm, t)) \text{ as } s \rightarrow \pm\infty.$$

This means $\tilde{z} \in W^{u,s}(\Lambda_0)$ and $\tilde{z}^\pm = \Omega^\pm(\tilde{z}) \in \Lambda_0$ have the same t -coordinate, that is, if $\tilde{z} = (z, t)$ and $\tilde{z}^\pm = (z^\pm, t^\pm)$, then $t = t^+ = t^-$.

3.6 Perturbed Normally Hyperbolic Invariant Manifolds

Since Λ_0 is a NHIM for the flow Φ_0^t of $\mathcal{X}^0$, $\tilde{\Lambda}_0 = \Lambda_0 \times \mathbb{R}$ is a NHIM for the flow $\tilde{\Phi}_0^s$ of the extended system (3.3).

Recall that $\mathcal{X}^1 = \mathcal{X}^1(z, t; \varepsilon)$ is assumed to be uniformly differentiable in all variables. The theory of normally hyperbolic invariant manifolds, [15,27,32] asserts that there exists ε_0 such that the manifold $\tilde{\Lambda}_0$ persists as a normally hyperbolic manifold $\tilde{\Lambda}_\varepsilon$, for all $|\varepsilon| < \varepsilon_0$, which is locally invariant under the flow $\tilde{\Phi}_\varepsilon^t$. The persistent NHIM $\tilde{\Lambda}_\varepsilon$ is $O(\varepsilon)$ -close in the $\mathcal{C}^\ell$ -topology to $\tilde{\Lambda}_0$, where ℓ is as in (3.6). The locally invariant manifolds are in fact invariant manifolds for an extension of the vector field, and they depend on that extension. Hence, they do not need to be unique.

The manifold $\tilde{\Lambda}_\varepsilon$ can be parametrized via a $\mathcal{C}^\ell$ -diffeomorphism $\tilde{k}_\varepsilon : \tilde{\Lambda}_0 \rightarrow \tilde{\Lambda}_\varepsilon$, where $\tilde{k}_0 = \text{Id}_{\tilde{\Lambda}_0}$, and $\tilde{k}_\varepsilon$ is $O(\varepsilon)$ -close to $\tilde{k}_0$ in the $\mathcal{C}^\ell$ -smooth topology on compact sets. Through $\tilde{k}_\varepsilon$, the perturbed NHIM $\tilde{\Lambda}_\varepsilon$ can be parametrized in terms of the variables (I, θ, t) , where (I, θ) are the action-angle variables on Λ_0 .

For details, see [5].

For the perturbed NHIM $\tilde{\Lambda}_\varepsilon$, $|\varepsilon| < \varepsilon_0$, there exists an invariant splitting of the tangent bundle similar to that in (3.4), and $D\tilde{\Phi}_\varepsilon^t$ satisfies expansion/contraction relations similar to those in (3.5), for some constants $\tilde{C}$, $\tilde{\lambda}_-$, $\tilde{\lambda}_+$, $\tilde{\mu}_-$, $\tilde{\mu}_+$, $\tilde{\lambda}_c$, $\tilde{\mu}_c$. These constants are independent of ε , and can be chosen as close as desired to the unperturbed ones, that is, to C , λ_- , λ_+ , μ_- , μ_+ , λ_c , μ_c , respectively, by choosing ε_0 suitably small.

There exist unstable and stable manifolds $W^u(\tilde{\Lambda}_\varepsilon)$, $W^s(\tilde{\Lambda}_\varepsilon)$ associated to $\tilde{\Lambda}_\varepsilon$, and there exist corresponding projection maps $\Omega^- : W^u(\tilde{\Lambda}_\varepsilon) \rightarrow \tilde{\Lambda}_\varepsilon$, and $\Omega^+ : W^s(\tilde{\Lambda}_\varepsilon) \rightarrow \tilde{\Lambda}_\varepsilon$. For $\tilde{z}^+ = \Omega^+(\tilde{z})$, with $\tilde{z} \in W^s(\tilde{\Lambda}_\varepsilon)$ we have

$$d(\Phi^t(\tilde{z}), \Phi^t(\tilde{z}^+)) \leq C_{\tilde{z}} e^{t\tilde{\lambda}_+}, \quad \text{for all } t \geq 0. \quad (3.11)$$

and for $\tilde{z}^- = \Omega^-(\tilde{z})$, with $\tilde{z} \in W^u(\tilde{\Lambda}_\varepsilon)$ we have

$$d(\Phi^t(\tilde{z}), \Phi^t(\tilde{z}^-)) \leq C_{\tilde{z}} e^{t\tilde{\mu}^-}, \quad \text{for all } t \leq 0, \quad (3.12)$$

for some $\tilde{C}_{\tilde{z}} > 0$. The constant $\tilde{C}_{\tilde{z}}$ can be chosen uniformly bounded provided we restrict to $\tilde{z}$ in the local unstable and stable manifolds $W_{\text{loc}}^u(\tilde{\Lambda}_\varepsilon)$, $W_{\text{loc}}^s(\tilde{\Lambda}_\varepsilon)$. Hence we can replace $\tilde{C}_{\tilde{z}}$ by some $\tilde{C}$.

To simplify notation, from now on we will drop the symbol $\sim$ from $\tilde{C}$, $\tilde{\lambda}_-$, $\tilde{\lambda}_+$, $\tilde{\mu}_-$, $\tilde{\mu}_+$, $\tilde{\lambda}_c$, $\tilde{\mu}_c$.

4 Master Lemmas

In this section we define some abstract Melnikov-type integral operators and study their properties, which will be used in the next sections. The derivations are similar to the ones in [18].

We consider the setting of 3.6. There exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, there exists a normally hyperbolic invariant manifold $\tilde{\Lambda}_\varepsilon$ for $\tilde{\Phi}_\varepsilon^s$. For each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ there exists a homoclinic channel $\tilde{\Gamma}_\varepsilon$ (see Sect. 3.4), which depends $\mathcal{C}^\ell$ -smoothly on ε , and determines the projections $\Omega^\pm : \tilde{\Gamma}_\varepsilon \rightarrow \Omega^\pm(\tilde{\Gamma}_\varepsilon) \subseteq \tilde{\Lambda}_\varepsilon$, which are local diffeomorphisms as in (3.7). We regard $\tilde{\Phi}_\varepsilon^s$, $\tilde{\Lambda}_\varepsilon$, $\tilde{\Gamma}_\varepsilon$ as perturbations of $\tilde{\Phi}_0^s$, $\tilde{\Lambda}_0$, $\tilde{\Gamma}_0$, for $\varepsilon \neq 0$ small.

Let $\tilde{z}_\varepsilon = (z_\varepsilon, t_\varepsilon) \in \tilde{\Gamma}_\varepsilon$ be a homoclinic point for $\tilde{\Phi}_\varepsilon^s$. Because of the smooth dependence of the normally hyperbolic manifold and of its stable and unstable manifolds on the perturbation, there is a homoclinic point $\tilde{z}_0 = (z_0, t_0) \in \tilde{\Gamma}_0$ for $\tilde{\Phi}_0^s$ that is $O(\varepsilon)$ -close to $\tilde{z}_\varepsilon$, that is

$$\tilde{z}_\varepsilon = \tilde{z}_0 + O(\varepsilon). \quad (4.1)$$

There exist many unperturbed homoclinic points $\tilde{z}_0$ satisfying (4.1); we choose a point $\tilde{z}_0$ such that $t_0 = t_\varepsilon$.

Let

$$\tilde{z}_\varepsilon \in \tilde{M} \mapsto \mathbf{F}(\tilde{z}_\varepsilon) \in \mathbb{R}^k$$

be a uniformly $\mathcal{C}^{r_0}$ -smooth mapping, for $k \geq 1$ and $1 \leq r_0 \leq r$.

We define the integral operators

$$\begin{aligned} \mathcal{I}^+(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon) &= \int_0^{+\infty} \left(\mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^+)) - \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon)) \right) ds, \\ \mathcal{I}^-(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon) &= \int_{-\infty}^0 \left(\mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^-)) - \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon)) \right) ds. \end{aligned} \quad (4.2)$$

Lemma 4.1 (Master Lemma 1) *The improper integrals (4.2) are convergent. The operators $\mathcal{I}^+(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon)$ and $\mathcal{I}^-(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon)$ are linear in $\mathbf{F}$.*

Proof The linearity of the operators follows from the linearity properties of integrals.

To prove convergence, we will use that the exponential contraction along the stable (unstable) manifold in forward (backward) time, given by (3.11) and (3.12). For the stable manifold, we have

$$|\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^+) - \tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon)| < C e^{s\lambda_+}, \text{ for } s \geq 0,$$

where C is the positive constant and λ_+ is the negative contraction rate from Sect. 3.6.

Recall that $\mathbf{F}$ is uniformly $\mathcal{C}^0$ -differentiable, so it is Lipschitz with Lipschitz constant C_F . Thus,

$$\begin{aligned} |\mathfrak{J}^+(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon)| &= \left| \int_0^\infty \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^+)) - \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon)) ds \right| \\ &\leq \int_0^\infty C_F C e^{s\lambda_+} ds \\ &= -C_F C \frac{1}{\lambda_+} \end{aligned}$$

Note, the last expression is positive since $\lambda_+ < 0$. Thus the integral is bounded and therefore convergent. The proof for the convergence of $\mathfrak{J}^-(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon)$ is similar. The difference is that the limits of integration are from $-\infty$ to 0 and the contraction rate is $-\mu_- < 0$

Note that the statement of Lemma 4.1 holds if $\mathbf{F}$ is replaced by

$$\mathcal{X}_\varepsilon \mathbf{F} = (\mathcal{X}^0 + \varepsilon \mathcal{X}^1) \mathbf{F}.$$

Due to the assumptions in Sect. 2, $\mathcal{X}_\varepsilon \mathbf{F}$ is Lipschitz uniformly in ε . This fact will be used in the proof of the next lemma. $\square$

Lemma 4.2 (*Master Lemma 2*)

$$\begin{aligned} \mathbf{F}(\tilde{z}_\varepsilon^+) - \mathbf{F}(\tilde{z}_\varepsilon) &= -\mathfrak{J}^+((\mathcal{X}^0 + \varepsilon \mathcal{X}^1) \mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon), \\ \mathbf{F}(\tilde{z}_\varepsilon^-) - \mathbf{F}(\tilde{z}_\varepsilon) &= \mathfrak{J}^-((\mathcal{X}^0 + \varepsilon \mathcal{X}^1) \mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon). \end{aligned} \quad (4.3)$$

Proof To prove this lemma, we will begin by computing the derivative of the i^{th} component $\mathbf{F}_i$ of $\mathbf{F}$ along the perturbed flow. For $\tilde{z}$ a point in $\tilde{M}$, using (3.2) we have

$$\begin{aligned} \frac{d}{ds} \mathbf{F}_i(\tilde{\Phi}_\varepsilon^s(\tilde{z})) &= \nabla \mathbf{F}_i(\tilde{\Phi}_\varepsilon^s(\tilde{z})) \cdot \frac{d}{ds} \tilde{\Phi}_\varepsilon^s(\tilde{z}) \\ &= \mathcal{X}^0 \mathbf{F}_i(\tilde{\Phi}_\varepsilon^s(\tilde{z})) + \varepsilon \mathcal{X}^1 \mathbf{F}_i(\tilde{\Phi}_\varepsilon^s(\tilde{z})). \end{aligned}$$

With the above result, we can now compute the difference in (4.3). Note that we define a vector field, $\mathcal{X}$, acting on a vector valued function, $\mathbf{F}$, as

$$\mathcal{X} \mathbf{F} = (\mathcal{X} \mathbf{F}_i)_i.$$

We have

$$\begin{aligned} \mathbf{F}(\tilde{z}_\varepsilon^+) - \mathbf{F}(\tilde{z}_\varepsilon) &= \mathbf{F}(\tilde{\Phi}_\varepsilon^T(\tilde{z}_\varepsilon^+)) - \mathbf{F}(\tilde{\Phi}_\varepsilon^T(\tilde{z}_\varepsilon)) \\ &\quad - \int_0^T \frac{d}{ds} \left(\mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^+)) - \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon)) \right) ds. \end{aligned}$$

Letting T approach infinity, the first difference vanishes because the homoclinic point $\tilde{z}_\varepsilon$ and its foot point $\tilde{z}_\varepsilon^+$ approach each other. We then can rewrite the integral using the expression for the derivative of $\mathbf{F}$ along the flow:

$$\begin{aligned} & - \int_0^\infty \left((\mathcal{X}^0 + \varepsilon \mathcal{X}^1) \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^+)) - (\mathcal{X}^0 + \varepsilon \mathcal{X}^1) \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon)) \right) ds \\ &= -\mathfrak{J}^+((\mathcal{X}^0 + \varepsilon \mathcal{X}^1) \mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon). \end{aligned}$$

The convergence of the above integral is due to Lemma 4.1 applied to $\mathcal{X}_\varepsilon \mathbf{F} = (\mathcal{X}^0 + \varepsilon \mathcal{X}^1) \mathbf{F}$.

The proof for $\mathfrak{J}^-((\mathcal{X}^0 + \varepsilon \mathcal{X}^1) \mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon)$ is similar. The main difference is that the limits of integration are from $-\infty$ to 0. $\square$

Lemma 4.3 (Master Lemma 3)

$$\begin{aligned} \mathfrak{J}^+(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon) &= \mathfrak{J}^+(\mathbf{F}, \tilde{\Phi}_0^s, \tilde{z}_0) + O(\varepsilon^\varrho), \\ \mathfrak{J}^-(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon) &= \mathfrak{J}^-(\mathbf{F}, \tilde{\Phi}_0^s, \tilde{z}_0) + O(\varepsilon^\varrho), \end{aligned} \tag{4.4}$$

for $0 < \varrho < 1$. When $\mathbf{F}$ is replaced by $\mathcal{X}_\varepsilon \mathbf{F} = (\mathcal{X}^0 + \varepsilon \mathcal{X}^1) \mathbf{F}$ in the above, the integrals on the right-hand side are evaluated with $\mathcal{X}^1 = \mathcal{X}^1(\cdot; 0)$.

Proof To prove this lemma, we will use both the Gronwall inequality from the Appendix A and the Lipschitz property of $\mathbf{F}$. The Gronwall inequality (A.6) gives

$$\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^+) = \tilde{\Phi}_0^s(\tilde{z}_0^+) + O(\varepsilon^{\rho_1}),$$

and

$$\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon) = \tilde{\Phi}_0^s(\tilde{z}_0) + O(\varepsilon^{\rho_1}),$$

where $0 < \rho_1 < 1$. Note that these equalities hold on an interval of time $0 < t < k \ln\left(\frac{1}{\varepsilon}\right)$, for $k \leq \frac{1-\rho}{C_0}$, where C_0 is the Lipschitz constant of $\mathcal{X}^0$; see Appendix A.

Before using the results from Gronwall, we will split the integrals into two parts:

$$\begin{aligned} \mathfrak{J}^+(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon) - \mathfrak{J}^+(\mathbf{F}, \tilde{\Phi}_0^s, \tilde{z}_0) &= \int_0^\infty \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^+)) - \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon)) ds \\ &\quad - \int_0^\infty \mathbf{F}(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \mathbf{F}(\tilde{\Phi}_0^s(\tilde{z}_0)) ds. \end{aligned}$$

Each integral can be written as

$$\begin{aligned} & \int_0^T \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^+)) - \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon)) ds + \int_T^\infty \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^+)) - \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon)) ds, \\ & \int_0^T \mathbf{F}(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \mathbf{F}(\tilde{\Phi}_0^s(\tilde{z}_0)) ds + \int_T^\infty \mathbf{F}(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \mathbf{F}(\tilde{\Phi}_0^s(\tilde{z}_0)) ds. \end{aligned}$$

Examining the second term of the first integral, we have

$$\begin{aligned} \left| \int_T^\infty \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^+)) - \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon)) ds \right| & \leq \int_T^\infty C_{\mathbf{F}} C e^{s\lambda_+} ds \\ & = C \frac{1}{|\lambda_+|} e^{T\lambda_+} \end{aligned}$$

where $\mathbf{C} = C_{\mathbf{F}} C$.

Now if we let $T = k \ln\left(\frac{1}{\varepsilon}\right)$, then the integral is bounded by

$$C \frac{1}{|\lambda_+|} \varepsilon^{k|\lambda_+|}.$$

More importantly, we have shown that the integral is of the order $O(\varepsilon^{\rho_2})$ with $\rho_2 = k|\lambda_+|$.

A similar argument shows that

$$\int_T^\infty \mathbf{F}(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \mathbf{F}(\tilde{\Phi}_0^s(\tilde{z}_0)) ds$$

is of the order $O(\varepsilon^{\rho_2})$.

Returning to the integrals from 0 to T , we have

$$\begin{aligned} & \int_0^T \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^+)) - \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon)) ds - \int_0^T \mathbf{F}(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \mathbf{F}(\tilde{\Phi}_0^s(\tilde{z}_0)) ds \\ & = \int_0^T \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^+)) - \mathbf{F}(\tilde{\Phi}_0^s(\tilde{z}_0^+)) ds - \int_0^T \mathbf{F}(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon)) - \mathbf{F}(\tilde{\Phi}_0^s(\tilde{z}_0)) ds. \end{aligned}$$

Now we can apply the Gronwall inequality (A.6) as well as the Lipschitz property of $\mathbf{F}$. This show that the difference of the integrals is bounded by

$$\int_0^T C_{\mathbf{F}} K \varepsilon^{\rho_1} ds,$$

for some $K > 0$ and $0 < \rho_1 < 1$.

Thus, each integral is of the order

$$O\left(\varepsilon^{\rho_1} \ln\left(\frac{1}{\varepsilon}\right)\right) = O(\varepsilon^{\rho_3}),$$

for some $\rho_3 < \rho_1 < 1$.

Finally, let $\rho = \min\{\rho_2, \rho_3\}$. Returning to the original expression, we have

$$|\mathfrak{J}^+(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, \tilde{z}_\varepsilon) - \mathfrak{J}^+(\mathbf{F}, \tilde{\Phi}_0^s, \tilde{z}_0)| = O(\varepsilon^\rho).$$

□

Lemma 4.4 (Master Lemma 4) *If $\|\mathbf{F}\|_{\mathcal{C}^1}$ is $O(\varepsilon)$ then*

$$\begin{aligned}\mathfrak{J}^+(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, z_\varepsilon) &= \mathfrak{J}^+(\mathbf{F}, \tilde{\Phi}_0^s, z_0) + O(\varepsilon^{1+\rho}), \\ \mathfrak{J}^-(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, z_\varepsilon) &= \mathfrak{J}^-(\mathbf{F}, \tilde{\Phi}_0^s, z_0) + O(\varepsilon^{1+\rho}),\end{aligned}\tag{4.5}$$

for $0 < \rho < 1$. The integrals on the right-hand side are evaluated with $\mathcal{X}^1 = \mathcal{X}^1(\cdot; 0)$.

Proof By the mean value theorem, we have

$$|\mathbf{F}(z_1) - \mathbf{F}(z_2)| \leq C_{\mathbf{F}}|z_1 - z_2|,$$

where $C_{\mathbf{F}} = \|D\mathbf{F}\|_{\mathcal{C}^0}$ is the Lipschitz constant of $\mathbf{F}$. The existence of the Lipschitz constant $C_{\mathbf{F}} < \infty$ follows from the assumption that $\mathbf{F}$ is bounded together with its derivatives. Now, by the hypothesis, $C_{\mathbf{F}} = O(\varepsilon)$. The proof is now similar to the proof of Lemma 4.3. Essentially, the Lipschitz constant of $\mathbf{F}$, $C_{\mathbf{F}}$, is replaced with a quantity of order $O(\varepsilon)$. Thus,

$$|\mathfrak{J}^+(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, z_\varepsilon) - \mathfrak{J}^+(\mathbf{F}, \tilde{\Phi}_0^s, z_0)| = O(\varepsilon)O(\varepsilon^{\rho_1}) + O(\varepsilon)O(\varepsilon^{\rho_2}).$$

Finally, let $\rho = \min\{\rho_1, \rho_2\}$.

The proof for $|\mathfrak{J}^-(\mathbf{F}, \tilde{\Phi}_\varepsilon^s, z_\varepsilon) - \mathfrak{J}^-(\mathbf{F}, \tilde{\Phi}_0^s, z_0)|$ follows similarly. □

Remark 4.5 We note that in Lemma 4.4 the assumption that $\|\mathbf{F}\|_{\mathcal{C}^1} = O(\varepsilon)$ can be replaced by the weaker assumption that $\|D\mathbf{F}\|_{\mathcal{C}^0} = O(\varepsilon)$. We also note that the Lemmas 4.1, 4.2, 4.3, 4.4 are valid when $\mathbf{F}$ is replaced by $\mathcal{X}_\varepsilon \mathbf{F} = (\mathcal{X}^0 + \varepsilon \mathcal{X}^1)\mathbf{F}$. In the sequel these lemmas will be applied in both situations.

5 Scattering Map for the Perturbed, Uncoupled Rotator-Pendulum System

5.1 Normally Hyperbolic Invariant Manifold for the Unperturbed Rotator-Pendulum System

Consider the unperturbed rotator-pendulum system described in Sect. 2.3.

The point $(0, 0)$ is a hyperbolic fixed point for each pendulum, the characteristic exponents are $\mu_i = (-V_i''(0))^{1/2} > 0$, $\lambda_i = -(-V_i''(0))^{1/2} = -\mu_i < 0$, the corresponding unstable and stable eigenvectors are $v_i^u = (-(-V_i''(0))^{1/2}, 1)$ and

$v_i^s = ((-V_i''(0))^{1/2}, 1)$, and the corresponding unstable and stable eigenspaces are $E_i^u = \text{Span}(v_i^u)$ and $E_i^s = \text{Span}(v_i^s)$, for $i = 1, \dots, n$.

Define

$$\begin{aligned}\lambda_- &= \min_i \lambda_i, \quad \lambda_+ = \max_i \lambda_i, \\ \mu_- &= \min_i \mu_i = -\lambda_+, \quad \mu_+ = \max_i \mu_i = -\lambda_- \\ \lambda_c &= -\mu_c,\end{aligned}\tag{5.1}$$

and choose $0 < \mu_c < \mu_-$ be some arbitrarily small positive number, and $\lambda_c = -\mu_c$. We have the following spectrum gap

$$\lambda_- \leq \lambda_+ < \lambda_c < 0 < \mu_c < \mu_- \leq \mu_+.$$

Also, define

$$\begin{aligned}E_z^u &= \oplus_{i=1, \dots, n} E_i^u, \\ E_z^s &= \oplus_{i=1, \dots, n} E_i^s.\end{aligned}\tag{5.2}$$

It immediately follows that for each closed d -dimensional ball $D \subseteq \mathbb{R}^d$, the set

$$\Lambda_0 = \{(p, q, I, \theta) \mid p = q = 0, I \in D, \theta \in \mathbb{T}^d\}\tag{5.3}$$

is a NHIM with boundary, where the rates $\lambda_-, \lambda_+, \mu_-, \mu_+, \lambda_c$, and μ_c from Sect. 3.3 are the ones defined by (5.1), and the unstable and stable spaces E_z^u and E_z^s at $z \in \Lambda_0$ are the ones given by (5.2).

5.2 The Scattering Map for the Unperturbed, Extended, Uncoupled Pendulum-Rotator System

Since we have $W^s(\tilde{\Lambda}_0) = W^u(\tilde{\Lambda}_0)$ and for each $\tilde{z} \in \tilde{\Lambda}_0$, $W^s(\tilde{z}) = W^u(\tilde{z})$, the corresponding scattering map $\tilde{\sigma}_0$ is the identity map wherever it is defined. Thus, $\tilde{\sigma}_0(\tilde{z}^-) = \tilde{z}^+$ implies $\tilde{z}^- = \tilde{z}^+$, or, equivalently

$$\tilde{\sigma}_0(I, \theta, t) = (I, \theta, t).\tag{5.4}$$

5.3 Coordinate System for the Unperturbed, Uncoupled, Rotator-Pendulum System

For each generalized pendulum, we define a new local coordinate system (y_i, x_i) as follows.

The coordinate y_i is chosen to be equal to the energy of the i -th generalized pendulum, i.e.,

$$y_i = \varsigma_i(p_i^2/2 + V_i(q_i)),$$

and is defined in a whole neighborhood of one of its separatrices.

The coordinate x_i is defined by

$$dx_i = \frac{ds_i}{\|\nabla y_i\|},$$

where $ds_i = (dp_i^2 + dq_i^2)^{1/2} = (p_i'(t)^2 + q_i'(t)^2)^{1/2} dt_i$ is the arc length element along the energy level, and t_i is the time along the energy level of the i -th generalized pendulum. Since $(p_i', q_i') = (-V_i'(q_i), p_i)$ we have $\|(p_i', q_i')\| = \|\nabla y_i\|$, therefore $dx_i = dt_i$. That is, the coordinate x_i equals to the time t_i it takes the solution $(p_i(t), q_i(t))$ to go from some initial point (p_i^0, q_i^0) to (p_i, q_i) . The value q_i^0 can be chosen uniformly for all energy levels, and p_i^0 is implicitly given by the energy condition.

A direct computation shows that

$$dx_i = \frac{ds}{\|y_i\|} = \frac{-V_i'(q_i)dp_i + p_i dq_i}{p_i^2 + V_i'(q_i)^2},$$

hence

$$dy_i \wedge dx_i = (p_i dp_i + V_i'(q_i) dq_i) \wedge \left(\frac{-V_i'(q_i) dp_i + p_i dq_i}{p_i^2 + V_i'(q_i)^2} \right) = dp_i \wedge dq_i.$$

The coordinate system (y_i, x_i) constructed above cannot be defined in a neighborhood of the separatrix that contains the hyperbolic equilibrium point, since this is a critical point of the energy function. We define this coordinate system only in some neighborhood $\mathcal{N}_i$ of a segment of the separatrix.

Then, if we let $\mathcal{N} = \Pi_{i=1}^n \mathcal{N}_i$, we obtain the symplectic coordinates (y, x, I, θ) on $\mathcal{N}$. In these coordinates the Hamiltonian H_0 is given by

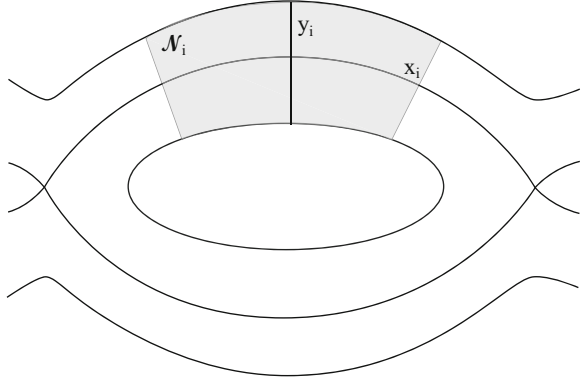
$$H_0(y, x, I, \theta) = h_0(I) + h_1(y) = h_0(I) + \sum_{i=1}^n y_i, \text{ for } (y, x, I, \theta) \in \mathcal{N}. \quad (5.5)$$

We use this coordinate system in Sect. 5.5 to measure the splitting of the stable and unstable manifolds in the perturbed system. However, the fact that the coordinate system is symplectic is not at all essential, and will not be used in the sequel.

5.4 Evolution Equations

Consider the coordinate system (y, x, I, θ) defined in Sect. 5.3. We will identify the vector fields $\mathcal{X}^0$ and $\mathcal{X}^1$ with derivative operators acting on functions, as described in Sect. 3.1.

Fig. 1 Coordinate system (y, x) in a neighborhood of a segment of the separatrix



Since $\mathcal{X}^0 = J\nabla H_0$ is a Hamiltonian vector field, using the Poisson bracket $\{\cdot, \cdot\}$, we have

$$\begin{aligned}\mathcal{X}^0 y &= \{y, H_0\} = \{y, h_0(I) + h_1(y, x)\} = \{y, h_1(y, x)\} = -\frac{\partial h_1}{\partial x}, \\ \mathcal{X}^0 x &= \{x, H_0\} = \{x, h_0(I) + h_1(y, x)\} = \{x, h_1(y, x)\} = \frac{\partial h_1}{\partial y}, \\ \mathcal{X}^0 I &= \{I, H_0\} = \{I, h_0(I) + h_1(y, x)\} = -\frac{\partial h_0}{\partial \theta} = 0, \\ \mathcal{X}^0 \theta &= \{\theta, H_0\} = \{\theta, h_0(I) + h_1(y, x)\} = \frac{\partial h_0}{\partial I} = \omega(I).\end{aligned}$$

When $\mathcal{X}^1 = J\nabla H_1$ is a Hamiltonian vector field, similarly we have

$$\begin{aligned}\mathcal{X}^1 y &= \{y, H_1\} = -\frac{\partial H_1}{\partial x}, \\ \mathcal{X}^1 x &= \{x, H_1\} = \frac{\partial H_1}{\partial y}, \\ \mathcal{X}^1 I &= \{I, H_1\} = -\frac{\partial H_1}{\partial \theta}, \\ \mathcal{X}^1 \theta &= \{\theta, H_1\} = \frac{\partial H_1}{\partial I}.\end{aligned}$$

Using the above formulas, we provide below the evolution equations of the coordinates (y, x, I, θ) , expressing the time-derivative of each coordinate along a solution of the perturbed system. We include the expression for the general case, as well as for the special case when the perturbation $\mathcal{X}^1$ is Hamiltonian.

$$\begin{aligned}\dot{y} &= \mathcal{X}^0 y + \varepsilon \mathcal{X}^1 y = -\frac{\partial H_0}{\partial x} + \varepsilon \mathcal{X}^1 y \\ &= -\frac{\partial h_1}{\partial x} - \varepsilon \frac{\partial H_1}{\partial x}.\end{aligned}\tag{5.6}$$

$$\begin{aligned}\dot{x} &= \mathcal{X}^0 x + \varepsilon \mathcal{X}^1 x = \frac{\partial H_0}{\partial y} + \varepsilon \mathcal{X}^1 x \\ &= \frac{\partial h_1}{\partial y} + \varepsilon \frac{\partial H_1}{\partial y}.\end{aligned}\tag{5.7}$$

$$\begin{aligned}\dot{I} &= \mathcal{X}^0 I + \varepsilon \mathcal{X}^1 I = -\frac{\partial H_0}{\partial \theta} + \varepsilon \mathcal{X}^1 I \\ &= -\varepsilon \frac{\partial H_1}{\partial \theta}.\end{aligned}\tag{5.8}$$

$$\begin{aligned}\dot{\theta} &= \mathcal{X}^0 \theta + \varepsilon \mathcal{X}^1 \theta = \frac{\partial H_0}{\partial I} + \varepsilon \mathcal{X}^1 \theta \\ &= \frac{\partial h_0}{\partial I} + \varepsilon \frac{\partial H_1}{\partial I}.\end{aligned}\tag{5.9}$$

Note that the evolution equations for the x - and y -coordinate from above are only valid for $(y, x, I, \theta) \in \mathcal{N}$ from Sect. 5.3.

5.5 Existence of Transverse Homoclinic Connections

Consider the coordinate system $z = (y, x, I, \theta)$ on $\mathcal{N}$, defined in Sect. 5.3. Relative to this coordinate system, in the unperturbed case, $W^u(\tilde{\Lambda}_0) = W^s(\tilde{\Lambda}_0)$ are locally given by $y = 0$.

A point $\tilde{z}_0 \in W^u(\tilde{\Lambda}_0) = W^s(\tilde{\Lambda}_0)$ can be written in terms of the extended coordinates (y, x, I, θ, t) as

$$\tilde{z}_0 = (0, x_0, I, \theta, t),$$

and in terms of the original coordinates (p, q, I, θ, t) as

$$\tilde{z}_0 = (p_0, q_0, I, \theta, t).$$

Applying the flow to this point yields

$$\tilde{\Phi}_0^s(\tilde{z}_0) = (p(s), q(s), I, \theta + \omega(I)s, t + s).$$

We cannot express the effect of the flow $\tilde{\Phi}_0^s$ on this point in terms of the (y, x, I, θ, t) -coordinates for all times s , since the latter coordinates are only defined in some neighborhood $\mathcal{N}$.

A point $\tilde{z}_0^* \in \tilde{\Lambda}_0$ can be written in terms of the (p, q, I, θ, t) -coordinates as

$$\tilde{z}_0^* = (0, 0, I, \theta, t),$$

and the effect of the flow to this point yields

$$\tilde{\Phi}_0^s(\tilde{z}_0^*) = (0, 0, I, \theta + \omega(I)s, t + s).$$

Note that the (y, x, I, θ, t) -coordinates are not defined on $\tilde{\Lambda}_0$.

In the perturbed case, for $\varepsilon \neq 0$ small, we can locally describe both the stable and unstable manifolds as graphs of $\mathcal{C}^{\ell-1}$ -smooth functions $y_\varepsilon^s, y_\varepsilon^u$, over (x_0, I, θ, t) given by

$$\begin{aligned} y_\varepsilon^s &= y_\varepsilon^s(x_0, I, \theta, t), \\ y_\varepsilon^u &= y_\varepsilon^u(x_0, I, \theta, t), \end{aligned} \quad (5.10)$$

respectively, for $(0, x_0, I, \theta) \in \mathcal{N}$.

The result below gives sufficient conditions for the existence of a transverse homoclinic intersection of $W^s(\tilde{\Lambda}_\varepsilon)$ and $W^u(\tilde{\Lambda}_\varepsilon)$. The proof is essentially the same as for Proposition 2.6. in [18], except that the latter is under the assumption that the perturbation is Hamiltonian. Therefore we will omit the proof.

Theorem 5.1 *For $(0, x_0, I, \theta, t) \in \mathcal{N}$, the difference between $y_\varepsilon^s(x_0, I, \theta, t)$ and $y_\varepsilon^u(x_0, I, \theta, t)$ is given by*

$$\begin{aligned} y_\varepsilon^u - y_\varepsilon^s &= -\varepsilon \int_{-\infty}^{+\infty} \left(\mathcal{X}^1 y(0, 0, I, \theta + \omega(I)s, t + s) \right. \\ &\quad \left. - \mathcal{X}^1 y(p(s), q(s), I, \theta + \omega(I)s, t + s) \right) ds \\ &\quad + O\left(\varepsilon^{1+\rho}\right) \\ &= -\varepsilon \int_{-\infty}^{+\infty} \left(\{y, H_1\}(0, 0, I, \theta + \omega(I)s, t + s) \right. \\ &\quad \left. - \{y, H_1\}(p(s), q(s), I, \theta + \omega(I)s, t + s) \right) ds \\ &\quad + O\left(\varepsilon^{1+\rho}\right). \end{aligned} \quad (5.11)$$

The second formula corresponds to the case when the perturbation is Hamiltonian.

If $x^ = x^*(I, \theta, t)$ is a non-degenerate zero of the mapping*

$$\begin{aligned} x_0 \in \mathbb{R}^n \mapsto & - \int_{-\infty}^{+\infty} \left((\mathcal{X}^1 y)(0, 0, I, \theta + \omega(I)s, t + s) \right. \\ & \left. - (\mathcal{X}^1 y)(p(s), q(s), I, \theta + \omega(I)s, t + s) \right) ds \in \mathbb{R}^n, \end{aligned} \quad (5.12)$$

then there exists $\varepsilon_0 > 0$ sufficiently small such that for all $0 < |\varepsilon| < \varepsilon_0$ $W^s(\tilde{\Lambda}_\varepsilon)$ and $W^u(\tilde{\Lambda}_\varepsilon)$ have a transverse homoclinic intersection which can be parametrized as

$$y_\varepsilon^u(x^*(I, \theta, t), I, \theta, t) = y_\varepsilon^s(x^*(I, \theta, t), I, \theta, t),$$

for (I, θ, t) in some open set in $\mathbb{R}^d \times \mathbb{T}^d \times \mathbb{T}^1$.

Remark 5.2 In (5.11) and (5.12) we use both coordinate systems (y, x, I, θ, t) and (p, q, I, θ, t) . We use the first coordinate system to measure the splitting between

the stable and unstable manifolds, and the second coordinate system to integrate the perturbation along the separatrix orbit as well as along the NHIM. Note that, while the (y, x, I, θ) -coordinates are defined only locally on $\mathcal{N}$, the function $y = (y_i)_{i=1, \dots, n}$ where $y_i = \varsigma_i(p_i^2/2 + V_i(q_i))$ is defined everywhere.

Remark 5.3 In the case when both the system and the perturbation are Hamiltonian, it is shown in [18] that the corresponding condition (5.12) holds for a $\mathcal{C}^1$ -open and $\mathcal{C}^\infty$ -dense set of perturbation H_1 . In particular, condition (5.12) is generic.

When the perturbation is non-conservative, it is possible that $W^s(\tilde{\Lambda}_\varepsilon)$ and $W^u(\tilde{\Lambda}_\varepsilon)$ do not intersect for any $\varepsilon \neq 0$, even though for $\varepsilon = 0$ we have $W^s(\tilde{\Lambda}_0) = W^u(\tilde{\Lambda}_0)$. That is, a non-conservative perturbation can destroy the homoclinic intersection. The condition (5.12) that guarantees the existence of such an intersection is non-generic. The next sections are under the assumption that $W^s(\tilde{\Lambda}_\varepsilon)$ and $W^u(\tilde{\Lambda}_\varepsilon)$ intersect transversally for $0 < |\varepsilon| < \varepsilon_0$.

5.6 Change in Action by the Scattering Map

Assume:

- $\tilde{z}_\varepsilon$ is a homoclinic point for the perturbed system, i.e., $\tilde{z}_\varepsilon \in W^s(\tilde{\Lambda}_\varepsilon) \cap W^u(\tilde{\Lambda}_\varepsilon)$,
- $\tilde{z}_\varepsilon^\pm = \Omega^\pm(\tilde{z}_\varepsilon) \in \tilde{\Lambda}_\varepsilon$,
- $\tilde{z}_0$ is a homoclinic point for the unperturbed system, i.e., $\tilde{z}_0 \in W^s(\tilde{\Lambda}_0) \cap W^u(\tilde{\Lambda}_0)$, corresponding to $\tilde{z}_\varepsilon$ via (4.1), and
- $\tilde{z}_0^\pm = \Omega^\pm(\tilde{z}_0) \in \tilde{\Lambda}_0$.

The existence of the homoclinic point $\tilde{z}_\varepsilon$ is guaranteed provided that the conditions from Theorem 5.1 are met.

Under the above assumptions, we have $\tilde{\sigma}_\varepsilon(\tilde{z}_\varepsilon^-) = \tilde{z}_\varepsilon^+$, and $\tilde{\sigma}_0(\tilde{z}_0^-) = \tilde{z}_0^+$. We recall here that for the unperturbed system, the scattering map is the identity $\tilde{\sigma}_0 = \text{Id}$, hence, in terms of action-angle coordinates (I, θ) , $I(\tilde{z}_0^-) = I(\tilde{z}_0^+)$, and $\theta(\tilde{z}_0^-) = \theta(\tilde{z}_0^+)$.

The result below describes the relation between $\tilde{\sigma}_\varepsilon$ and $\tilde{\sigma}_0$ in terms of the action coordinate I .

Theorem 5.4 *The change in action I by the scattering map $\tilde{\sigma}_\varepsilon$ is given by:*

$$\begin{aligned} I(\tilde{z}_\varepsilon^+) - I(\tilde{z}_\varepsilon^-) &= -\varepsilon \int_{-\infty}^{+\infty} \left(\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0)) \right) ds \\ &\quad + O(\varepsilon^{1+\rho}) \\ &= -\varepsilon \int_{-\infty}^{+\infty} \left(\{I, H_1\}(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \{I, H_1\}(\tilde{\Phi}_0^s(\tilde{z}_0)) \right) ds \\ &\quad + O(\varepsilon^{1+\rho}). \end{aligned} \quad (5.13)$$

where $\tilde{z}_0^+ = \tilde{z}_0^- = \tilde{z}_0^\pm$, and $0 < \rho < 1$.

The second formula corresponds to the case when the perturbation is Hamiltonian. The integrals on the right-hand side are evaluated with $\mathcal{X}^1 = \mathcal{X}^1(\cdot; 0)$ and $H_1 = H_1(\cdot; 0)$, respectively.

Proof The key observation is that the unperturbed Hamiltonian H_0 for the rotator-pendulum system does not depend on θ , hence I is a slow variable, as it can be seen from (5.8).

The proof follows immediately from Lemma 5.5 below, by making $\varsigma = 0$, and subtracting (5.15) from (5.14). We also use the fact that for the unperturbed pendulum-rotator system the foot points of the stable fiber and of the unstable fiber through the same point $\tilde{z}_0$ coincide, i.e., $\tilde{z}_0^- = \tilde{z}_0^+$ which we denote $\tilde{z}_0^\pm$. $\square$

Lemma 5.5 *For any $\varsigma \in \mathbb{R}$ we have*

$$\begin{aligned} & I(\tilde{\Phi}_\varepsilon^\varsigma(\tilde{z}_\varepsilon^+)) - I(\tilde{\Phi}_\varepsilon^\varsigma(\tilde{z}_\varepsilon)) \\ &= -\varepsilon \int_0^{+\infty} \left(\mathcal{X}^1 I(\tilde{\Phi}_0^{s+\varsigma}(\tilde{z}_0^+)) - \mathcal{X}^1 I(\tilde{\Phi}_0^{s+\varsigma}(\tilde{z}_0)) \right) ds + O(\varepsilon^{1+\rho}), \\ &= -\varepsilon \int_0^{+\infty} \left(\{I, H_1\}(\tilde{\Phi}_0^{s+\varsigma}(\tilde{z}_0^+)) - \{I, H_1\}(\tilde{\Phi}_0^{s+\varsigma}(\tilde{z}_0)) \right) ds + O(\varepsilon^{1+\rho}), \end{aligned} \quad (5.14)$$

and

$$\begin{aligned} & I(\tilde{\Phi}_\varepsilon^\varsigma(\tilde{z}_\varepsilon^-)) - I(\tilde{\Phi}_\varepsilon^\varsigma(\tilde{z}_\varepsilon)) \\ &= +\varepsilon \int_{-\infty}^0 \left(\mathcal{X}^1 I(\tilde{\Phi}_0^{s+\varsigma}(\tilde{z}_0^-)) - \mathcal{X}^1 I(\tilde{\Phi}_0^{s+\varsigma}(\tilde{z}_0)) \right) ds + O(\varepsilon^{1+\rho}) \\ &= +\varepsilon \int_{-\infty}^0 \left(\{I, H_1\}(\tilde{\Phi}_0^{s+\varsigma}(\tilde{z}_0^-)) - \{I, H_1\}(\tilde{\Phi}_0^{s+\varsigma}(\tilde{z}_0)) \right) ds + O(\varepsilon^{1+\rho}). \end{aligned} \quad (5.15)$$

where $0 < \rho < 1$.

The second formula in each equation corresponds to the case when the perturbation is Hamiltonian.

Proof We will only prove (5.14) as (5.15) follows similarly.

We first apply Lemmas 4.1 and 4.2 for $\mathbf{F} = (\mathcal{X}^0 + \varepsilon \mathcal{X}^1)I$ and $\mathbf{F} = I$, respectively, obtaining

$$\begin{aligned} I(\tilde{\Phi}_\varepsilon^\varsigma(\tilde{z}_\varepsilon^+)) - I(\tilde{\Phi}_\varepsilon^\varsigma(\tilde{z}_\varepsilon)) &= - \int_0^{+\infty} \left((\mathcal{X}^0 I + \varepsilon \mathcal{X}^1 I)(\tilde{\Phi}_\varepsilon^{s+\varsigma}(\tilde{z}_\varepsilon^+)) \right. \\ &\quad \left. - (\mathcal{X}^0 I + \varepsilon \mathcal{X}^1 I)(\tilde{\Phi}_\varepsilon^{s+\varsigma}(\tilde{z}_\varepsilon)) \right) ds. \end{aligned}$$

Using (5.8), since $\mathcal{X}^0 I = \frac{\partial H_0}{\partial \theta} = 0$, we obtain

$$\begin{aligned} I(\tilde{\Phi}_\varepsilon^\varsigma(\tilde{z}_\varepsilon^+)) - I(\tilde{\Phi}_\varepsilon^\varsigma(\tilde{z}_\varepsilon)) &= -\varepsilon \int_0^{+\infty} \left((\mathcal{X}^1 I)(\tilde{\Phi}_\varepsilon^{s+\varsigma}(\tilde{z}_\varepsilon^+)) \right. \\ &\quad \left. - (\mathcal{X}^1 I)(\tilde{\Phi}_\varepsilon^{s+\varsigma}(\tilde{z}_\varepsilon)) \right) ds. \end{aligned}$$

Using Lemma 4.3 for $\mathbf{F} = \mathcal{X}^1 I$, we can replace the perturbed flow by the unperturbed flow by making an error of order $O(\varepsilon^\varrho)$, yielding

$$I(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon^+)) - I(\tilde{\Phi}_\varepsilon^s(\tilde{z}_\varepsilon)) = -\varepsilon \int_0^{+\infty} \left((\mathcal{X}^1 I)(\tilde{\Phi}_0^{s+\varepsilon}(\tilde{z}_0^+)) - (\mathcal{X}^1 I)(\tilde{\Phi}_0^{s+\varepsilon}(\tilde{z}_0)) \right) ds + O(\varepsilon^{1+\varrho}).$$

Finally, we note that in the pendulum-rotator system the foot-points of the stable fiber and of the unstable fiber through the same homoclinic point $\tilde{z}_0$ coincide, i.e., $\tilde{z}_0^- = \tilde{z}_0^+ = \tilde{z}_0^\pm$.

In the case of the Hamiltonian perturbation, we only need to substitute $\mathcal{X}^1 I = \{I, H_1\}$. $\square$

5.7 Change in Angle by the Scattering Map

Under the same assumptions as at the beginning of Sect. 5.6, below we provide a result that describes the relation between $\tilde{\sigma}_\varepsilon$ and $\tilde{\sigma}_0$ in terms of the angle coordinate θ .

Theorem 5.6 *The change in angle θ by the scattering map $\tilde{\sigma}_\varepsilon$ is given by:*

$$\begin{aligned} & \theta(\tilde{z}_\varepsilon^+) - \theta(\tilde{z}_\varepsilon^-) \\ &= -\varepsilon \int_{-\infty}^{+\infty} \mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0)) ds \\ & \quad + \varepsilon \int_0^{+\infty} (\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0))) ds \cdot \left(\frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \\ & \quad + O(\varepsilon^{1+\varrho}) \\ &= -\varepsilon \int_{-\infty}^{+\infty} \{\theta, H_1\}(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \{\theta, H_1\}(\tilde{\Phi}_0^s(\tilde{z}_0)) ds \\ & \quad + \varepsilon \int_0^{+\infty} (\{I, H_1\}(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \{I, H_1\}(\tilde{\Phi}_0^s(\tilde{z}_0))) ds \cdot \left(\frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \\ & \quad + O(\varepsilon^{1+\varrho}). \end{aligned} \tag{5.16}$$

where $\tilde{z}_0^+ = \tilde{z}_0^- = \tilde{z}_0^\pm$, $I_0 = I(\tilde{z}^\pm)$, and $0 < \varrho < 1$. In the second term on the right-hand side the integral is thought of as a $1 \times d$ matrix, and $\frac{\partial^2 h_0}{\partial I^2}(I_0)$ as a $d \times d$ matrix. Also $\{\theta, H_1\}$, $\{I, h_1\}$ are $1 \times d$ matrices.

The second formula corresponds to the case when the perturbation is Hamiltonian. The integrals on the right-hand side are evaluated with $\mathcal{X}^1 = \mathcal{X}^1(\cdot; 0)$ and $H_1 = H_1(\cdot; 0)$, respectively.

Proof Unlike in Theorem 5.4, where I is a slow variable, θ is a fast variable, as it can be seen from (5.9). However, we will show that the differences

$$\theta(\tilde{z}_\varepsilon^+) - \theta(\tilde{z}_\varepsilon)$$

and

$$\theta(\tilde{z}_\varepsilon^-) - \theta(\tilde{z}_\varepsilon)$$

are slow quantities. Then, taking the difference,

$$\theta(\tilde{z}_\varepsilon^+) - \theta(\tilde{z}_\varepsilon^-)$$

is $O(\varepsilon)$.

We begin with $\theta(\tilde{z}_\varepsilon^+) - \theta(\tilde{z}_\varepsilon)$. Using Lemmas 4.1 and 4.2 for $\mathbf{F} = (\mathcal{X}^0 + \varepsilon\mathcal{X}^1)\theta$ and $\mathbf{F} = \theta$, respectively, we obtain

$$\begin{aligned} \theta(\tilde{z}_\varepsilon^+) - \theta(\tilde{z}_\varepsilon) &= - \int_0^{+\infty} \left((\mathcal{X}^0\theta + \varepsilon\mathcal{X}^1\theta)(\tilde{\Phi}_\varepsilon^\zeta(\tilde{z}_\varepsilon^+)) \right. \\ &\quad \left. - (\mathcal{X}^0\theta + \varepsilon\mathcal{X}^1\theta)(\tilde{\Phi}_\varepsilon^\zeta(\tilde{z}_\varepsilon)) \right) d\zeta. \end{aligned} \quad (5.17)$$

From (5.9) we have

$$\mathcal{X}^0\theta + \varepsilon\mathcal{X}^1\theta = \frac{\partial h_0}{\partial I} + \varepsilon\mathcal{X}^1(\theta),$$

and (5.17) becomes

$$\begin{aligned} &- \int_0^{+\infty} \left(\frac{\partial h_0}{\partial I}(\tilde{\Phi}_\varepsilon^\zeta(\tilde{z}_\varepsilon^+)) - \frac{\partial h_0}{\partial I}(\tilde{\Phi}_\varepsilon^\zeta(\tilde{z}_\varepsilon)) \right) d\zeta \\ &- \varepsilon \int_0^{+\infty} \left((\mathcal{X}^1\theta)(\tilde{\Phi}_\varepsilon^\zeta(\tilde{z}_\varepsilon^+)) - (\mathcal{X}^1\theta)(\tilde{\Phi}_\varepsilon^\zeta(\tilde{z}_\varepsilon)) \right) d\zeta. \end{aligned} \quad (5.18)$$

The second integral in (5.18) has a factor of ε , so we will focus on the first integral. Recall that $\frac{\partial h_0}{\partial I}$ depends only on I . So the first integral in (5.18) can be written as

$$- \int_0^{+\infty} \left(\frac{\partial h_0}{\partial I}(I(\tilde{\Phi}_\varepsilon^\zeta(\tilde{z}_\varepsilon^+))) - \frac{\partial h_0}{\partial I}(I(\tilde{\Phi}_\varepsilon^\zeta(\tilde{z}_\varepsilon))) \right) d\zeta.$$

Let us first consider the case when h_0 is of one-degree-of-freedom, i.e. $I \in \mathbb{R}$. We can use the integral version of the Mean Value Theorem to rewrite the integral. Recall,

$$f(x + \Delta x) - f(x) = \Delta x \int_0^1 f'(x + t\Delta x) dt$$

Using

$$\begin{aligned} f &= \frac{\partial h_0}{\partial I} \\ x + \Delta x &= I(\tilde{\Phi}_\varepsilon^\zeta(\tilde{z}_\varepsilon^+)) \end{aligned}$$

$$x = I \left(\tilde{\Phi}_\varepsilon^\zeta (\tilde{z}_\varepsilon) \right)$$

the integral becomes

$$- \int_0^{+\infty} \left((I_\varepsilon^{\zeta,+} - I_\varepsilon^\zeta) \int_0^1 \frac{\partial^2 h_0}{\partial I^2} (I_\varepsilon^\zeta + t(I_\varepsilon^{\zeta,+} - I_\varepsilon^\zeta)) dt \right) d\zeta$$

where we denote $I_\varepsilon^{\zeta,+} = I \left(\tilde{\Phi}_\varepsilon^\zeta (\tilde{z}_\varepsilon^+) \right)$ and $I_\varepsilon^\zeta = I \left(\tilde{\Phi}_\varepsilon^\zeta (\tilde{z}_\varepsilon) \right)$.

We use Gronwall's inequality as in Lemma A.2 to rewrite the inside integral of the second partial derivative as

$$\int_0^1 \frac{\partial^2 h_0}{\partial I^2} (I_\varepsilon^\zeta + t(I_\varepsilon^{\zeta,+} - I_\varepsilon^\zeta)) dt = \int_0^1 \frac{\partial^2 h_0}{\partial I^2} (I_0^\zeta + t(I_0^{\zeta,+} - I_0^\zeta)) dt + O(\varepsilon^\varrho).$$

Now $I_0^{\zeta,+} = I_0^\zeta = I_0$ because I is constant along the unperturbed flow, hence the above integral equals

$$\frac{\partial^2 h_0}{\partial I^2} (I_0) + O(\varepsilon^\varrho).$$

We now apply Lemma 5.5 to rewrite $I_\varepsilon^{\zeta,+} - \tilde{I}_\varepsilon^\zeta$, so the integral becomes

$$\begin{aligned} & \varepsilon \int_0^{+\infty} \int_0^{+\infty} (\mathcal{X}^1 I(\tilde{\Phi}_0^{s+\zeta}(\tilde{z}_0^+)) - \mathcal{X}^1 I(\tilde{\Phi}_0^{s+\zeta}(\tilde{z}_0))) d\varsigma ds \cdot \left(\frac{\partial^2 h_0}{\partial I^2} (I_0) \right) \\ & + O(\varepsilon^{1+\varrho}) \end{aligned} \quad (5.19)$$

This integral has a factor of ε , and the remaining term is $O(\varepsilon^{1+\varrho})$, thus $\theta(\tilde{z}_\varepsilon^+) - \theta(\tilde{z}_\varepsilon)$ is a slow quantity.

Denote by $\mathcal{J}$ the antiderivative of

$$s \mapsto (\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0)))$$

which approaches 0 as $s \rightarrow \pm\infty$; we recall here that $\tilde{z}_0^+ = \tilde{z}_0^- = \tilde{z}_0^\pm$. We have

$$\begin{aligned} \mathcal{J}(s) &= - \int_s^{+\infty} (\mathcal{X}^1 I(\tilde{\Phi}_0^v(\tilde{z}_0^+)) - \mathcal{X}^1 I(\tilde{\Phi}_0^v(\tilde{z}_0))) dv \\ &= \int_{-\infty}^s (\mathcal{X}^1 I(\tilde{\Phi}_0^v(\tilde{z}_0^-)) - \mathcal{X}^1 I(\tilde{\Phi}_0^v(\tilde{z}_0))) dv. \end{aligned} \quad (5.20)$$

Making the change of variable $v = s + \varsigma$ the integral in (5.19) becomes

$$\int_0^{+\infty} \int_s^{+\infty} (\mathcal{X}^1 I(\tilde{\Phi}_0^v(\tilde{z}_0^+)) - \mathcal{X}^1 I(\tilde{\Phi}_0^v(\tilde{z}_0))) dv ds = - \int_0^{+\infty} \mathcal{J}(s) ds. \quad (5.21)$$

Using Integration by Parts we obtain

$$\begin{aligned} - \int_0^{+\infty} \mathcal{J}(s) ds &= -s \mathcal{J}(s) \Big|_0^{+\infty} + \int_0^{+\infty} (\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0))) s ds \\ &= \int_0^{+\infty} (\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0))) s ds. \end{aligned} \quad (5.22)$$

In the above, the quantity $s \mathcal{J}(s)$ obviously equals to 0 at $s = 0$, and equals to 0 when $s \rightarrow \infty$ since, by l'Hopital Rule

$$\lim_{s \rightarrow \infty} \frac{\mathcal{J}(s)}{s^{-1}} = \lim_{s \rightarrow \infty} - \frac{(\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0)))}{s^{-2}} = 0,$$

since $(\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0)))$ approaches 0 at exponential rate.

Applying Lemma 4.3 to the second integral in (5.18), and combining with the above we obtain

$$\begin{aligned} &\theta(\tilde{z}_\varepsilon^+) - \theta(\tilde{z}_\varepsilon) \\ &= +\varepsilon \int_0^{+\infty} (\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0))) s ds \cdot \left(\frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \\ &\quad - \varepsilon \int_0^{+\infty} \mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0)) ds + O(\varepsilon^{1+q}). \end{aligned} \quad (5.23)$$

Similarly, for $\theta(\tilde{z}_\varepsilon^-) - \theta(\tilde{z}_\varepsilon)$ we obtain an expression as a sum of two integrals

$$\begin{aligned} &\theta(\tilde{z}_\varepsilon^-) - \theta(\tilde{z}_\varepsilon) \\ &= +\varepsilon \int_{-\infty}^0 \mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0)) ds \\ &\quad - \varepsilon \int_{-\infty}^0 (\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0))) s ds \cdot \left(\frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \\ &\quad + O(\varepsilon^{1+q}). \end{aligned} \quad (5.24)$$

In the case when $d = 1$, recalling that $\tilde{z}_0^+ = \tilde{z}_0^- = \tilde{z}_0^\pm$, we conclude that

$$\begin{aligned} &\theta(\tilde{z}_\varepsilon^+) - \theta(\tilde{z}_\varepsilon^-) \\ &= -\varepsilon \int_{-\infty}^{+\infty} \mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0)) ds \\ &\quad + \varepsilon \int_0^{+\infty} (\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0))) s ds \cdot \left(\frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \\ &\quad + O(\varepsilon^{1+q}) \end{aligned}$$

In the case where $I \in \mathbb{R}^d$, we can use the vectorial version of the Mean Value Theorem. For $f : \mathbb{R}^d \rightarrow \mathbb{R}$, we have

$$f(\mathbf{x} + t \Delta \mathbf{x}) - f(\mathbf{x}) = \left\langle \Delta \mathbf{x}, \int_0^1 \nabla f(\mathbf{x} + t \Delta \mathbf{x}) dt \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^d$.

Setting

$$\begin{aligned} f &= \frac{\partial h_0}{\partial I_j} \\ \mathbf{x} + \mathbf{1x} &= I_\varepsilon^{\zeta, +} \\ \mathbf{x} &= I_\varepsilon^\zeta \end{aligned}$$

and proceeding as before, the first integral that appears in the computation of $\theta_j(\tilde{z}_\varepsilon^+) - \theta_j(\tilde{z}_\varepsilon)$ becomes

$$\begin{aligned} & - \int_0^{+\infty} \left(\frac{\partial h_0}{\partial I_j}(I(\tilde{\Phi}_\varepsilon^\zeta(\tilde{z}_\varepsilon^+))) - \frac{\partial h_0}{\partial I_j}(I(\tilde{\Phi}_\varepsilon^\zeta(\tilde{z}_\varepsilon))) \right) d\zeta \\ &= +\varepsilon \int_0^{+\infty} \int_0^{+\infty} \left\langle \mathcal{X}^1 I(\tilde{\Phi}_0^{s+\zeta}(\tilde{z}_0)) - \mathcal{X}^1 I(\tilde{\Phi}_0^{s+\zeta}(\tilde{z}_0)), \frac{\partial^2 h_0}{\partial I \partial I_j}(I_0) \right\rangle d\zeta ds \\ & \quad + O(\varepsilon^{1+\varrho}), \\ &= -\varepsilon \int_0^{+\infty} \left\langle \mathcal{J}(s), \frac{\partial^2 h_0}{\partial I \partial I_j}(I_0) \right\rangle ds + O(\varepsilon^{1+\varrho}), \end{aligned}$$

where we now denote by $\mathcal{J}(s)$ the vector-valued function whose component $\mathcal{J}_i(s)$ represents the antiderivative of

$$s \mapsto (\mathcal{X}^1 I_i(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I_i(\tilde{\Phi}_0^s(\tilde{z}_0)))$$

which approaches 0 as $s \rightarrow \pm\infty$, for $i = 1, \dots, d$.

Using Integration by Parts the last expression can be written as

$$+\varepsilon \int_0^{+\infty} \left\langle \mathcal{X}^1 I(\Phi_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\Phi_0^s(\tilde{z}_0)), \frac{\partial^2 h_0}{\partial I \partial I_j}(I_0) \right\rangle s ds + O(\varepsilon^{1+\varrho}).$$

The second integral that appears in the computation of $\theta_j(\tilde{z}_\varepsilon^+) - \theta_j(\tilde{z}_\varepsilon)$ has the same form as in the 1-dimensional case $d = 1$.

Thus, for the vector $\theta(\tilde{z}_\varepsilon^+) - \theta(\tilde{z}_\varepsilon)$ we obtain

$$\begin{aligned} & \theta(\tilde{z}_\varepsilon^+) - \theta(\tilde{z}_\varepsilon) \\ &= +\varepsilon \int_0^{+\infty} (\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0))) ds \cdot \left(\frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \\ & \quad - \varepsilon \int_0^{+\infty} (\mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0))) ds + O(\varepsilon^{1+\varrho}), \end{aligned} \quad (5.25)$$

where in the first expression on the right-hand side the integral is thought of as a $1 \times d$ vector, and $\frac{\partial^2 h_0}{\partial I^2}(I_0)$ as a $d \times d$ matrix.

Computing $\theta_j(\tilde{z}_\varepsilon^-) - \theta_j(\tilde{z}_\varepsilon)$ in a similar fashion and combining with the above we conclude

$$\begin{aligned} & \theta_j(\tilde{z}_\varepsilon^+) - \theta_j(\tilde{z}_\varepsilon^-) \\ &= +\varepsilon \int_{-\infty}^{+\infty} (\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0))) ds \cdot \left(\frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \\ & \quad - \varepsilon \int_{-\infty}^{+\infty} (\mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0))) ds + O(\varepsilon^{1+\varrho}). \end{aligned}$$

□

5.8 Comparison with Similar Results

Consider the special case when the perturbation $\mathcal{X}^1$ is Hamiltonian and time-periodic in t , i.e., $\mathcal{X}^1 = J\nabla H_1$ for some $H_1 = H_1(z, t)$, with $t \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Then the scattering map is exact symplectic and depends smoothly on the parameter ε , so it can be computed perturbatively. More precisely, the scattering map, in terms of a local system of coordinates (I, θ, t) on $\tilde{\Lambda}_\varepsilon$, can be expanded in powers of ε as follows:

$$\tilde{\sigma}_\varepsilon = \tilde{\sigma}_0 + \varepsilon J\nabla S \circ \tilde{\sigma}_0 + O(\varepsilon^2), \quad (5.26)$$

where S_0 is a $\mathcal{C}^\ell$ -smooth Hamiltonian function defined on some open subset of $\tilde{\Lambda}_\varepsilon$. Hence $J\nabla S_0$ represents a Hamiltonian vector field on $\tilde{\Lambda}_\varepsilon$. See [7]. Formula (5.27) is no longer true in the case of perturbations that are not Hamiltonian.

In the case of the uncoupled pendulum-rotator system, since $\tilde{\sigma}_0 = \text{Id}$, we have

$$\tilde{\sigma}_\varepsilon = \text{Id} + \varepsilon J\nabla S + O(\varepsilon^2), \quad (5.27)$$

and the Hamiltonian function S that generates the scattering map can be computed explicitly as follows. Let

$$\begin{aligned} p^0(\tau + t\bar{1}) &= (p_1^0(\tau_1 + t), \dots, p_n^0(\tau_n + t)), \\ q^0(\tau + t\bar{1}) &= (q_1^0(\tau_1 + t), \dots, q_n^0(\tau_n + t)), \end{aligned} \quad (5.28)$$

be a parametrization of the system of separatrices of the penduli, where $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n$ and $\bar{1} = (1, \dots, 1) \in \mathbb{R}^n$.

Define

$$L(\tau, I, \theta, t) = - \int_{-\infty}^{+\infty} \left(H_1(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1}), I, \theta + \omega(I)s, t + s) - H_1(0, 0, I, \theta + \omega(I)s, t + s) \right) ds \quad (5.29)$$

Assume that the map

$$\tau \in \mathbb{R}^n \mapsto L(\tau, I, \theta, t) \in \mathbb{R} \quad (5.30)$$

has a non-degenerate critical point τ^* , which is locally given, by the implicit function theorem, by

$$\tau^* = \tau^*(I, \theta, t). \quad (5.31)$$

Hence

$$\frac{\partial L}{\partial \tau}(\tau^*(I, \theta, t), I, \theta, t) = 0. \quad (5.32)$$

The function L defined in (5.29) is referred to as the Melnikov potential, measures the global effect of the perturbation on homoclinic orbits of the unperturbed system. The non-degenerate critical points of (5.30) yield the existence of transverse homoclinic orbits for the perturbed system. These non-degenerate critical points are equivalent to the non-degenerate zeroes of the mapping (5.12) in Theorem 5.1. See, e.g., [18].

Then define the auxiliary function $\mathcal{L}$ by

$$\mathcal{L}(I, \theta, t) = L(\tau^*(I, \theta, t), I, \theta, t). \quad (5.33)$$

It is not difficult to show that $\mathcal{L}$ satisfies the following relation for all $\sigma \in \mathbb{R}$:

$$\mathcal{L}(I, \theta, t) = \mathcal{L}(I, \theta - \omega(I)\sigma, t - \sigma). \quad (5.34)$$

In particular, for $\sigma = t$, we have $\mathcal{L}(I, \theta, t) = \mathcal{L}(I, \theta - \omega(I)t, 0)$. If we denote by $\mathcal{L}^*$ the function defined by

$$\mathcal{L}^*(I, \bar{\theta}) = \mathcal{L}(I, \theta - \omega(I)t, 0), \text{ for } \bar{\theta} = \theta - \omega(I)t, \quad (5.35)$$

then

$$\mathcal{L}(I, \theta, t) = \mathcal{L}^*(I, \bar{\theta}), \text{ for } \bar{\theta} = \theta - \omega(I)t. \quad (5.36)$$

This says that the function $\mathcal{L}$, while nominally depending on two vector variables I , θ and one real variable t , depends in fact only on two vector variables I and $\bar{\theta}$.

Through computations similar to those in the proofs of Theorems 5.4 and 5.6, one obtains (see [7]) that the Hamiltonian function S that generates the scattering map is given by

$$S(I, \theta, t) = -L(I, \theta, t). \quad (5.37)$$

For $\tilde{\sigma}_\varepsilon(I^-, \theta^-, t^-) = (I^+, \theta^+, t^+)$, from (5.27) we obtain

$$I^+ - I^- = \varepsilon \frac{\partial \mathcal{L}}{\partial \theta}(I, \theta, t) + O(\varepsilon^{1+q}), \quad (5.38)$$

$$\theta^+ - \theta^- = -\varepsilon \frac{\partial \mathcal{L}}{\partial I}(I, \theta, t) + O(\varepsilon^{1+q}), \quad (5.39)$$

$$t^+ - t^- = 0. \quad (5.40)$$

From (5.33) and (5.32)

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial I}(I, \theta, t) &= \frac{\partial L}{\partial \tau}(\tau^*(I, \theta, t), I, \theta, t) \frac{\partial \tau^*}{\partial I}(I, \theta, t) + \frac{\partial L}{\partial I}(\tau^*(I, \theta, t), I, \theta, t) \\ &= \frac{\partial L}{\partial I}(\tau^*(I, \theta, t), I, \theta, t), \\ \frac{\partial \mathcal{L}}{\partial \theta}(I, \theta, t) &= \frac{\partial L}{\partial \tau}(\tau^*(I, \theta, t), I, \theta, t) \frac{\partial \tau^*}{\partial \theta}(I, \theta, t) + \frac{\partial L}{\partial I}(\tau^*(I, \theta, t), I, \theta, t) \\ &= \frac{\partial L}{\partial \theta}(\tau^*(I, \theta, t), I, \theta, t). \end{aligned} \quad (5.41)$$

From (5.29), and using the fact that $\mathcal{X}^1 I = \{I, H_1\} = -\frac{\partial H_1}{\partial \theta}$, we obtain:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \theta}(I, \theta, t) &= - \int_{-\infty}^{+\infty} \left(\frac{\partial H_1}{\partial \theta}(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1}), I, \theta + \omega(I)s, t + s) \right. \\ &\quad \left. - \frac{\partial H_1}{\partial \theta}(0, 0, I, \theta + \omega(I)s, t + s) \right) ds \\ &= - \int_{-\infty}^{+\infty} (\{I, H_1\}(0, 0, I, \theta + \omega(I)s, t + s) \\ &\quad - \{I, H_1\}(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1}), I, \theta + \omega(I)s, t + s)) ds. \end{aligned} \quad (5.42)$$

Above, note that the point $(0, 0, I, \theta + \omega(I)s, t + s)$ corresponds to $\tilde{z}_0^\pm$, and the point $(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1}), I, \theta + \omega(I)s, t + s)$ corresponds to $\tilde{z}_0$ in Sect. 5.6. Thus, the formula for the change in the action by the scattering map in (5.38) is the same as the one given in Theorem 5.4.

From (5.29), and using that $\mathcal{X}^1\theta = \{\theta, H_1\} = \frac{\partial H_1}{\partial I}$, $\mathcal{X}^1 I = \{I, H_1\} = -\frac{\partial H_1}{\partial \theta}$, we obtain:

$$\begin{aligned}
& \frac{\partial \mathcal{L}}{\partial I}(I, \theta, t) \\
&= - \int_{-\infty}^{+\infty} \left(\frac{\partial H_1}{\partial I}(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1}), I, \theta + \omega(I)s, t + s) \right. \\
&\quad \left. - \frac{\partial H_1}{\partial I}(0, 0, I, \theta + \omega(I)s, t + s) \right) ds \\
&\quad - \int_{-\infty}^{+\infty} \left(\frac{\partial H_1}{\partial \theta}(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1}), I, \theta + \omega(I)s, t + s) \right. \\
&\quad \left. - \{I, H_1\}(0, 0, I, \theta + \omega(I)s, t + s) \right) (D_I \omega(I)s) ds \\
&= \int_{-\infty}^{+\infty} (\{\theta, H_1\}(0, 0, I, \theta + \omega(I)s, t + s) \\
&\quad - \{\theta, H_1\}(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1}), I, \theta + \omega(I)s, t + s)) ds \\
&\quad - \int_{-\infty}^{+\infty} (I, H_1\}(0, 0, I, \theta + \omega(I)s, t + s) \\
&\quad - \{I, H_1\}(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1}), I, \theta + \omega(I)s, t + s)) (D_I \omega(I)s) ds.
\end{aligned} \tag{5.43}$$

Since $D\omega(I) = \frac{\partial^2 h_0}{\partial I^2}(I)$, and noting that it is independent of the variable of integration, so it can be moved outside of the integral, the formula for the change in the angle by the scattering map in (5.39) is the same as the one given in Theorem 5.6.

6 Scattering Map for the Perturbed, Coupled Rotator-Pendulum System

6.1 Normally Hyperbolic Invariant Manifold for the Unperturbed Coupled Rotator-Pendulum System

The equations of motion for (2.6) are given by

$$\begin{aligned}
\dot{p} &= -\frac{\partial h_1}{\partial q}, \\
\dot{q} &= \frac{\partial h_1}{\partial p} + M^T I, \\
\dot{I} &= -\frac{\partial h_0}{\partial \theta} = 0, \\
\dot{\theta} &= \frac{\partial h_0}{\partial I} + Mp = \omega(I) + Mp.
\end{aligned} \tag{6.1}$$

Each component I_j , $j = 1, \dots, d$, is a first integral of the system, hence the vector I is conserved along trajectories.

Let

$$K = h_1(p, q) + I^T M p = \sum_{i=1}^n \varsigma_i \left(\frac{p_i^2}{2} + V(q_i) \right) + I^T M p. \quad (6.2)$$

Denote by ς the $(n \times n)$ -diagonal matrix whose i -th diagonal entry is equal to the sign ς_i in front of the i -th pendulum in (6.2). It follows that for each fixed I , the point $(p, q) = (-\varsigma M^T I, 0)$ is a hyperbolic fixed point for the (p, q) -system.

For each I fixed, the function

$$K_i = \varsigma_i \left(\frac{p_i^2}{2} + V(q_i) \right) + I^T M_{*i} p_i, \text{ for } i = 1, \dots, n, \quad (6.3)$$

where M_{*i} denotes the i -th column of M , represents a Hamiltonian of pendulum-type for the (p_i, q_i) -system, and is a first integral of the system. The energy level of the hyperbolic fixed point of the (p_i, q_i) -system is

$$\begin{aligned} K_i(-\varsigma_i M_{*i}^T I, 0) &= \frac{1}{2} \varsigma_i M_{*i}^T I \cdot M_{*i}^T I - \varsigma_i M_{*i}^T I \cdot M_{*i}^T I \\ &= -\frac{1}{2} \varsigma_i M_{*i}^T I \cdot M_{*i}^T I. \end{aligned}$$

The corresponding separatrix satisfies the equation

$$\varsigma_i \left(\frac{1}{2} (p_i + \varsigma_i M_{*i}^T I)^2 + V(q_i) \right) = 0. \quad (6.4)$$

The eigenvalues and the corresponding eigenvectors at the hyperbolic fixed point $(p_i, q_i) = (-\varsigma_i M_{*i}^T I, 0)$ are the same as for the uncoupled rotator-pendulum system in Sect. 5.1, and it similarly follows that

$$\Lambda_0 = \{(p, q, I, \theta) \mid p = -\varsigma M^T I, q = 0, I \in D, \theta \in \mathbb{T}^d\} \quad (6.5)$$

is a NHIM with boundary for the flow Φ_0^t of (2.6).

The stable and unstable manifolds of Λ_0 coincide, i.e., $W^u(\Lambda_0) = W^s(\Lambda_0)$, as in the case of the uncoupled rotator-pendulum model in Sect. 2.3. The corresponding homoclinic manifolds satisfy the implicit equations (6.4) for $i = 1, \dots, n$. However, homoclinic orbits have, in general, future asymptotics different from the past asymptotics. That is, for each $z \in W^u(z^-) = W^s(z^+)$, we have that $d(\Phi_0^t(z), \Phi_0^t(z^\pm)) \rightarrow 0$ as $t \rightarrow \pm\infty$, where z^- and z^+ do not need be the same point.

6.2 The Scattering Map for the Unperturbed, Extended, Coupled Pendulum-Rotator System

The equations (6.1) imply that, along a homoclinic orbit we have $\dot{\theta} = \omega(I) + Mp$, while along an orbit in Λ_0 we have $\dot{\theta} = \omega(I) - M_{\mathcal{S}} M^T I$. The difference in the rate of change of θ along homoclinic orbits and along asymptotic orbits in the NHIM makes the corresponding scattering map no longer equal to the identity map, but equal to a shift in the angle θ .

Proposition 6.1 *Let Γ be a homoclinic manifold and $z \in \Gamma$ be a homoclinic point. The unperturbed scattering map $\sigma_0 : \Lambda_0 \rightarrow \Lambda_0$ is given by $\sigma_0(z^-) = z^+$, with $I(z^-) = I(z^+)$ and*

$$\theta(z^+) - \theta(z^-) = \int_{-\infty}^{+\infty} \left(M_{\mathcal{S}} M^T I(z^{\pm}) + Mp(\Phi_0^s(z)) \right) ds, \quad (6.6)$$

where we denote $I(z^{\pm}) = I(z^+) = I(z^-)$. Component-wise we have

$$\theta_i(z^+) - \theta_i(z^-) = \int_{-\infty}^{+\infty} \left(M_{i*} M^T I(z^{\pm}) + M_{i*} p(\Phi_0^s(z)) \right) ds, \quad (6.7)$$

for $i = 1, \dots, d$, where M_{i*} denotes the i -th row of the matrix M .

Thus

$$\sigma_0(I^-, \theta^-) = (I^+, \theta^+) = (I^-, \theta^- + \Delta(I)),$$

where the phase shift $\Delta(I)$ is given by the right hand side of (6.6), which depends only on I and the choice of homoclinic manifold.

Proof By Lemmas 4.1 and 4.2 we have

$$\begin{aligned} \theta(z^+) - \theta(z) &= - \int_0^{+\infty} \left((\mathcal{X}^0 \theta)(\Phi_0^s(z^+)) - (\mathcal{X}^0 \theta)(\Phi_0^s(z)) \right) ds \\ &= - \int_0^{+\infty} \left((\omega(I) + Mp)(\Phi_0^s(z^+)) - (\omega(I) + Mp)(\Phi_0^s(z)) \right) ds \\ &= - \int_0^{+\infty} \left(Mp(\Phi_0^s(z^+)) - Mp(\Phi_0^s(z)) \right) ds \\ &= \int_0^{+\infty} \left(M_{\mathcal{S}} M^T I(\Phi_0^s(z^+)) + Mp(\Phi_0^s(z)) \right) ds \\ &= \int_0^{+\infty} \left(M_{\mathcal{S}} M^T I(z^+) + Mp(\Phi_0^s(z)) \right) ds. \end{aligned} \quad (6.8)$$

Above we have used that that on Λ_0 we have $p = -\mathcal{S} M^T I$, and that I is a first integral, so $\omega(I(\Phi_0^s(z^+))) = \omega(I(\Phi_0^s(z)))$ and $M_{\mathcal{S}} M^T I(\Phi_0^s(z^+)) = M_{\mathcal{S}} M^T I(z^+)$. $\square$

Similarly we obtain

$$\theta(z^-) - \theta(z) = - \int_{-\infty}^0 \left(M \varsigma M^T I(z^-) + M p(\Phi_0^s(z)) \right) ds. \quad (6.9)$$

Subtracting (6.9) from (6.8), and denoting $I(z^\pm) = I(z^+) = I(z^-)$, we obtain (6.6).

Note that the phase-shift $\Delta(I)$ depends only on the action level set I and on the underlying homoclinic manifold containing the homoclinic point z . When $M = 0$ the Hamiltonian (6.1) is the same as (2.4), and $\Delta(I) = 0$. This is consistent with the fact that for (2.4) the unperturbed scattering map is the identity.

6.3 Coordinate System for the Unperturbed, Coupled, Rotator-Pendulum System

We define a new system of local coordinates (y, x, I, θ) similar to that in Sect. 6.3. Set

$$y_i = \frac{1}{2}(p_i + \varsigma_i M_{*i}^T I)^2 + V(q_i), \text{ for } i = 1, \dots, n. \quad (6.10)$$

That is, y_i is the energy of K_i , with the energy level of the separatrix set to 0. From (6.4) it follows that $y = 0$ for points (p, q, I, θ) on $W^s(\Lambda_0) = W^u(\Lambda_0)$.

Let x_i be given by

$$dx_i = \frac{ds_i}{\|\nabla_{p_i, q_i} y_i\|}, \quad (6.11)$$

where $ds_i = (dp_i^2 + dq_i^2)^{1/2} = (p_i'(t)^2 + q_i'(t)^2)^{1/2} dt_i$ is the arc-length element along the energy level, and t_i is the time along the separatrix of the i -th generalized pendulum. Since $(p_i', q_i') = (-V_i'(q_i), p_i + \varsigma_i M_{*i}^T I)$ we have $\|(p_i', q_i')\| = \|\nabla_{p_i, q_i} y_i\|$, therefore $dx_i = dt_i$. Thus x_i represents the time along the i -th separatrix.

A direct computation shows that

$$dy_i = (p_i + \varsigma_i M_{*i}^T I)(dp_i + \varsigma_i M_{*i}^T dI) + V'(q_i)dq_i$$

and

$$dx_i = \frac{-V_i'(q_i)dp_i + (p_i + \varsigma_i M_{*i}^T I)dq_i}{(p_i + \varsigma_i M_{*i}^T I)^2 + V_i'(q_i)^2}.$$

Note that $dy_i \wedge dx_i \neq dp_i \wedge dq_i$, so the corresponding coordinate change $(p, q, I, \theta) \mapsto (y, x, I, \theta)$ is not symplectic in general. Nevertheless we can still use the resulting coordinate system (y, x, I, θ) to obtain the existence of transverse homoclinic connections for the perturbed system; see Sect. 6.4.

6.4 Existence of Transverse Homoclinic Connections

Let (y, x, I, θ) be the coordinate system defined in Sect. 6.3. As in Sect. 5.5, for the perturbed system with $\varepsilon \neq 0$ small, we can locally describe both the stable and unstable manifolds as graphs of $\mathcal{C}^{\ell-1}$ -smooth functions $y_\varepsilon^s, y_\varepsilon^u$, over (x_0, I, θ, t) given by (5.10). Then Theorem 5.1 holds, the proof being identical to that for the system (2.4).

6.5 Change in Action by the Scattering Map

Using the same notation as in Sect. 5.6 we immediately obtain:

Theorem 6.2 *The change in action I by the scattering map $\tilde{\sigma}_\varepsilon$ is given by:*

$$\begin{aligned} I(\tilde{z}_\varepsilon^+) - I(\tilde{z}_\varepsilon^-) &= -\varepsilon \int_{-\infty}^{+\infty} \left(\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0)) \right) ds \\ &\quad + O(\varepsilon^{1+\rho}) \\ &= -\varepsilon \int_{-\infty}^{+\infty} \left(\{I, H_1\}(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \{I, H_1\}(\tilde{\Phi}_0^s(\tilde{z}_0)) \right) ds \\ &\quad + O(\varepsilon^{1+\rho}). \end{aligned} \tag{6.12}$$

where $\tilde{z}_0^+ = \tilde{z}_0^- = \tilde{z}_0^\pm$, and $0 < \rho < 1$.

Proof The key observation is that, for the Hamiltonian (6.1), we have

$$\mathcal{X}_\varepsilon I = -\frac{\partial H_0}{\partial \theta} + \varepsilon \mathcal{X}^1 I = \varepsilon \mathcal{X}^1 I,$$

exactly as in the case of the Hamiltonian (2.4), so the argument is identical to that in the proof of Theorem 5.4. $\square$

6.6 Change in Angle by the Scattering Map

Similarly, we obtain the relation between $\tilde{\sigma}_\varepsilon$ and $\tilde{\sigma}_0$ in terms of the angle coordinate θ .

Theorem 6.3 *The change in angle θ by the scattering map $\tilde{\sigma}_\varepsilon$ is given by:*

$$\begin{aligned}
 & \theta(\tilde{z}_\varepsilon^+) - \theta(\tilde{z}_\varepsilon^-) \\
 &= \theta(\tilde{z}_0^+) - \theta(\tilde{z}_0^-) \\
 &\quad - \varepsilon \int_{-\infty}^{+\infty} \mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \mathcal{X}^1 \theta(\tilde{\Phi}_0^s(\tilde{z}_0^-)) ds \\
 &\quad + \varepsilon \int_0^{+\infty} (\mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \mathcal{X}^1 I(\tilde{\Phi}_0^s(\tilde{z}_0^-))) s ds \cdot \left(\frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \\
 &\quad + O(\varepsilon^{1+\varrho}) \\
 &= -\varepsilon \int_{-\infty}^{+\infty} \{\theta, H_1\}(\tilde{\Phi}_0^s(\tilde{z}_0^+)) - \{\theta, H_1\}(\tilde{\Phi}_0^s(\tilde{z}_0^-)) ds \\
 &\quad + \varepsilon \int_0^{+\infty} (\{I, H_1\}(\tilde{\Phi}_0^s(\tilde{z}_0^\pm)) - \{I, H_1\}(\tilde{\Phi}_0^s(\tilde{z}_0^-))) s ds \cdot \left(\frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \\
 &\quad + O(\varepsilon^{1+\varrho}). \tag{6.13}
 \end{aligned}$$

where $\Delta(I)$ is given by Proposition 6.1, $\tilde{z}_0^+ = \tilde{z}_0^- = \tilde{z}_0^\pm$, $I_0 = I(\tilde{z}^\pm)$, and $0 < \varrho < 1$.

Proof The key observation is that, for the Hamiltonian (6.1), we have

$$\mathcal{X}_\varepsilon \theta = \frac{\partial h_0}{\partial I} + Mp + \varepsilon \mathcal{X}^1 \theta,$$

which is the same as in the case of the Hamiltonian (2.4), plus an extra term Mp . Proceeding as in the proof of Theorem 5.6, and noting that $p(\tilde{\Phi}_0^s(\tilde{z}_0)) = p(\Phi_0^s(z_0))$, the extra term leads to the additional integral

$$\int_{-\infty}^{\infty} \left(M_{\mathcal{S}} M^T I(z^\pm) + Mp(\Phi_0^s(z_0)) \right) ds$$

which is the quantity $\Delta(I) = \theta(z_0^+) - \theta(z_0^-)$ from Proposition 6.1. This extra term appears in the formula (6.13). $\square$

Appendix A. Gronwall's Inequality

In this section we apply Gronwall's Inequality to estimate the error between the solution of an unperturbed system and the solution of the perturbed system, over a time of logarithmic order with respect to the size of the perturbation.

Theorem A.1 (Gronwall's Inequality) *Given a continuous real valued function $\phi \geq 0$, and constants $\delta_0, \delta_1 \geq 0, \delta_2 > 0$, if*

$$\phi(t) \leq \delta_0 + \delta_1(t - t_0) + \delta_2 \int_{t_0}^t \phi(s) ds \tag{A.1}$$

then

$$\phi(t) \leq \left(\delta_0 + \frac{\delta_1}{\delta_2} \right) e^{\delta_2(t-t_0)} - \frac{\delta_1}{\delta_2}. \quad (\text{A.2})$$

For a reference, see, e.g., [35].

Lemma A.2 *Consider the following differential equations:*

$$\dot{z}(t) = \mathcal{X}^0(z, t) \quad (\text{A.3})$$

$$\dot{z}(t) = \mathcal{X}^0(z, t) + \varepsilon \mathcal{X}^1(z, t; \varepsilon) \quad (\text{A.4})$$

Assume that $\mathcal{X}^0$ is Lipschitz continuous in the variable z , C_0 is the Lipschitz constant of $\mathcal{X}^0$, and $\mathcal{X}^1$ is bounded with $\|\mathcal{X}^1\|_{\mathcal{C}^0} \leq C_1$, for some $C_0, C_1 > 0$. Let z_0 be a solution of the equation (A.3) and z_ε be a solution of the equation (A.4) such that

$$\|z_0(t_0) - z_\varepsilon(t_0)\| < c\varepsilon. \quad (\text{A.5})$$

Then, for $0 < \varrho_0 < 1$, $0 < k \leq \frac{1-\varrho_0}{C_0}$, and $K = c + \frac{C_1}{C_0}$, we have

$$\|z_0(t) - z_\varepsilon(t)\| < K\varepsilon^{\varrho_0}, \text{ for } 0 \leq t - t_0 \leq k \ln(1/\varepsilon). \quad (\text{A.6})$$

Proof For z_0 and z_ε solutions of (A.3) and (A.4), respectively, we have

$$z_0(t) = z_0(t_0) + \int_{t_0}^t \mathcal{X}^0(z_0(s), s) ds, \quad (\text{A.7})$$

$$z_\varepsilon(t) = z_\varepsilon(t_0) + \int_{t_0}^t \mathcal{X}^0(z_\varepsilon(s), s) ds + \varepsilon \int_{t_0}^t \mathcal{X}^1(z_\varepsilon(s), s; \varepsilon) ds. \quad (\text{A.8})$$

Subtracting, we obtain

$$\begin{aligned} \|z_\varepsilon(t) - z_0(t)\| &\leq \|z_\varepsilon(t_0) - z_0(t_0)\| + \int_{t_0}^t \|\mathcal{X}^0(z_\varepsilon(s), s) - \mathcal{X}^0(z_0(s), s)\| ds \\ &\quad + \varepsilon \int_{t_0}^t \|\mathcal{X}^1(z_\varepsilon(s), s; \varepsilon)\| ds. \end{aligned} \quad (\text{A.9})$$

Using (A.5) for the first term on the right-hand side, the Lipschitz condition on $\mathcal{X}^0$ for the second, and the boundedness of $\mathcal{X}^1$ for the third we obtain:

$$\begin{aligned} \|z_\varepsilon(t) - z_0(t)\| &\leq c\varepsilon + C_0 \int_{t_0}^t \|z_\varepsilon(s) - z_0(s)\| ds \\ &\quad + \varepsilon C_1(t - t_0). \end{aligned} \quad (\text{A.10})$$

Applying the Gronwall inequality for $\delta_0 = c$, $\delta_1 = \varepsilon C_1$, and $\delta_2 = C_0$, and recalling that $K = c + \frac{C_1}{C_0}$ we obtain

$$\begin{aligned} \|z_\varepsilon(t) - z_0(t)\| &\leq \varepsilon \left(c + \frac{C_1}{C_0} \right) e^{C_0(t-t_0)} - \varepsilon \frac{C_1}{C_0} \\ &\leq \varepsilon K e^{C_0(t-t_0)}. \end{aligned} \quad (\text{A.11})$$

If we let $0 \leq t - t_0 \leq k \ln(1/\varepsilon)$ we obtain

$$\begin{aligned} \|z_\varepsilon(t) - z_0(t)\| &\leq \varepsilon \left(c + \frac{C_1}{C_0} \right) e^{C_0(t-t_0)} - \varepsilon \frac{C_1}{C_0} \\ &\leq \varepsilon K e^{C_0 k \ln(1/\varepsilon)} \\ &= \varepsilon K \left(\frac{1}{\varepsilon} \right)^{C_0 k}. \end{aligned} \quad (\text{A.12})$$

Since $k \leq \frac{1-\varrho}{C_0}$ we conclude

$$\|z_\varepsilon(t) - z_0(t)\| \leq \varepsilon K \left(\frac{1}{\varepsilon} \right)^{1-\varrho} = K \varepsilon^\varrho. \quad (\text{A.13})$$

□

We note that, with the above argument, for a time of logarithmic order with respect to the size of the perturbation, we can only obtain an error of order $O(\varepsilon^\varrho)$ with $0 < \rho < 1$, but we cannot obtain an error of order $O(\varepsilon)$.

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