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ON AN INTEGRAL OF *J*-BESSEL FUNCTIONS AND ITS APPLICATION TO MAHLER MEASURE

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With an appendix by J. S. Friedman†

Abstract

Cogdell *et al.* [‘Evaluating the Mahler measure of linear forms via Kronecker limit formulas on complex projective space’, *Trans. Amer. Math. Soc.* (2021), to appear] developed infinite series representations for the logarithmic Mahler measure of a complex linear form with four or more variables. We establish the case of three variables by bounding an integral with integrand involving the random walk probability density $a\int_0^\infty tJ_0(at)\prod_{m=0}^2 J_0(r_m t)\,dt$, where J_0 is the order-zero Bessel function of the first kind and a and r_m are positive real numbers. To facilitate our proof we develop an alternative description of the integral's asymptotic behaviour at its known points of divergence. As a computational aid for numerical experiments, an algorithm to calculate these series is presented in the appendix.

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1. Introduction

The Mahler measure of a multi-variable complex polynomial figures prominently in many mathematical contexts. Lehmer sought large primes by relating the growth of the Pierce numbers, $\prod_{i=1}^d(1 \pm \alpha_i^m)$, where the α_i are the roots of the polynomial, to that of the Mahler measure of the polynomial (see [13]). Shinder and Vlasenko showed that Mahler measure is related to certain L-values of modular forms (see [9]). Values of the Mahler measure have interpretations in ergodic theory [13] and also arise in the study of topological polynomial invariants [9]; its ubiquity makes its effective computation of some importance.

1.1. Calculating Mahler measure. If the arsenal of an analyst is stocked with inequalities, the stockpile of one studying Mahler measure might be rife with series

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representations. Considerable toil is involved with numerically evaluating logarithmic Mahler measure directly from its integral definition. The inefficiency of this direct method has stressed the necessity of expressing Mahler measures in terms of fast-converging infinite series, so that a truncated series gives a high-precision estimate in a timely manner [1]. Analytic conjectures on closed-form expressions relating to Mahler measures are not infrequently conceived and then sharpened as the result of extensive computations [1, 3, 13], so such formulations can be of considerable value.

Much progress in this vein has been made by Rodriguez-Villegas *et al.*, who established such expressions in terms of J-Bessel functions [12]. Borwein *et al.* [4] established series expressions for the Mahler measure of the linear forms $x_0 + x_1 + \cdots + x_n$, involving the even moments of the $(n+1)$-step densities p_{n+1}. More recently, Cogdell, Jorgenson and Smajlović have obtained a series formulation for the logarithmic Mahler measure of an arbitrary complex linear form by expressing the log-norm of a linear polynomial as an infinite series [5]. This latter investigation settled the case of four or more variables. Our aim is to establish the Cogdell–Jorgenson–Smajlović Mahler measure series representations for the unexamined case of three variables. We invoke a result due to Nicholson on three-step uniform random walks of varying but prescribed step lengths (see [15]). Also, we develop an alternative description of the associated integral's asymptotic behaviour more amenable to our proof and which provides further insight on a related integral.

1.2. Random walks. Suppose a man wanders into the complex plane, finds himself at the origin and determines to go on a ramble. He walks from his starting point for some distance r_m at angle θ_m, both chosen at whim, and does this n times successively. Curious observers wish to know the probability his distance from the origin at the conclusion of the n stretches is between r and $r + \delta r$ for some pre-determined $r, \delta > 0$. This is the well-known problem of the random walk in the plane [15]. The study of this problem began with Pearson, whose motivation was to construct an idealised system modelling the complex natural phenomenon of species migration [11]; the integrals associated to such probability densities have been called *ramble integrals* in Pearson's honour. Kluyver established the classical result that for a positive number a, the probability density p_{n+1} associated to an $(n+1)$-step walk has the Bessel integral representation

$$p_{n+1}(a) = a \int_0^\infty t J_0(at) \prod_{m=0}^{n} J_0(t)\, dt,$$

corresponding to the case where each step length is 1 [4]. Nicholson generalised this result in the case of three steps where the wanderer's step lengths need not coincide. We restate this important finding in Theorem 1.1(i), for which we now establish notation.

Let $K(k) := \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-1/2}\, d\theta$ be the complete elliptic integral of the first kind, $r_0, r_1, r_2 > 0$ be the step lengths of a random walk and order $r_0 \geq r_1 \geq r_2$ without loss of generality. Let $a > 0$ and define a_1, a_2, a_3, a_4 by ordering the set $\{a, r_0, r_1, r_2\}$ so that $\{a_1 \geq a_2 \geq a_3 \geq a_4\}$. In the case where $a_1 \leq a_2 + a_3 + a_4$, set

$$\Delta^2 := \tfrac{1}{16}(r_0 + r_1 + r_2 - a)(a + r_1 + r_2 - r_0)(a + r_0 + r_2 - r_1)(a + r_0 + r_1 - r_2) \geq 0.$$

1.3. Logarithmic Mahler measure. The Mahler measure $M(P)$ of an $(n+1)$-variable complex polynomial P is defined by

$$M(P) = \exp\Big(\frac{1}{(2\pi)^{n+1}} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \log(|P(e^{i\theta_0}, e^{i\theta_1}, \ldots, e^{i\theta_n})|)\, d\theta_0\, d\theta_1 \ldots d\theta_n\Big).$$

The logarithmic Mahler measure is defined as $m(P) := \log M(P)$. Let

$$P_D(Z_0, \ldots, Z_n) := W_0 Z_0 + W_1 Z_1 + \cdots + W_n Z_n$$

be a linear form in $n+1$ complex variables and $D := (W_0, \ldots, W_n)$ be its tuple of coefficients. Let $d(D) = |W_0| + \cdots + |W_n|$ and $c(D) := \sqrt{(n+1)(|W_0|^2 + \cdots + |W_n|^2)}$.

1.4. Our main results. The primary implement to establish the series representation for the Mahler measure is the following result.

THEOREM 1.1. *Define $I(a) := \int_0^\infty tJ_0(at) \prod_{m=0}^{2} J_0(r_m t)\, dt$ and let S be the set given by $S := \{r_0 + r_1 - r_2, r_0 - r_1 + r_2, -(r_0 - r_1 - r_2)\}$, requiring the elements to be strictly positive. Set $S^* := S \cup \{0, r_0 + r_1 + r_2\}$ or $S \cup \{r_0 - r_1 - r_2, r_0 + r_1 + r_2\}$, according as we have $r_0 - r_1 - r_2 <$ or ≥ 0.*

(i) *For any $a > 0$, the integral $I(a)$ is finite unless $a \in S$, differentiable unless $a \in S^*$ and has closed form*

$$I(a) = \begin{cases} 0 & \text{if } a_1 > a_2 + a_3 + a_4, \\ \dfrac{1}{\pi^2 \Delta} K\Big(\dfrac{\sqrt{a r_0 r_1 r_2}}{\Delta}\Big) & \text{if } \Delta^2 > a r_0 r_1 r_2, \\ \dfrac{1}{\pi^2 \sqrt{a r_0 r_1 r_2}} K\Big(\dfrac{\Delta}{\sqrt{a r_0 r_1 r_2}}\Big) & \text{if } \Delta^2 < a r_0 r_1 r_2. \end{cases}$$

(ii) *For $b \in S$, the integral $I(a)$ diverges at $a = b$ with $I(a) = O(\log|a - b|)$ as $a \to b$.*

Before continuing, we pause to examine the features of various densities for a three-step walk, which are of some analytic interest. We write $p_3(a; r_0, r_1, r_2)$ for the density corresponding to the ramble with step-length tuple (r_0, r_1, r_2). The density exhibits logarithmic singularities at points which vary according to the step-length combination and is differentiable between these points. The integral $I(a)$ vanishes to the left of $r_0 - r_1 - r_2$ and to the right of $r_0 + r_1 + r_2$, as here $a_1 > a_2 + a_3 + a_4$. The rambler's prospect of concluding their travel at distance from the origin within the sum of the three steps taken, or inside the distance $r_0 - r_1 - r_2$, is certain and hopeless, respectively, so has probability 1 and zero in these intervals. Since $p_3(a; r_0, r_1, r_2) = aI(a)$ is the derivative with respect to a of this probability [5], $I(a)$ must be zero in these intervals. See Figure 1, where $p_3(a; 5, 4, 3)$ illustrates Kluyver's example of the integral 'defining distinct analytic functions in different intervals' [8].

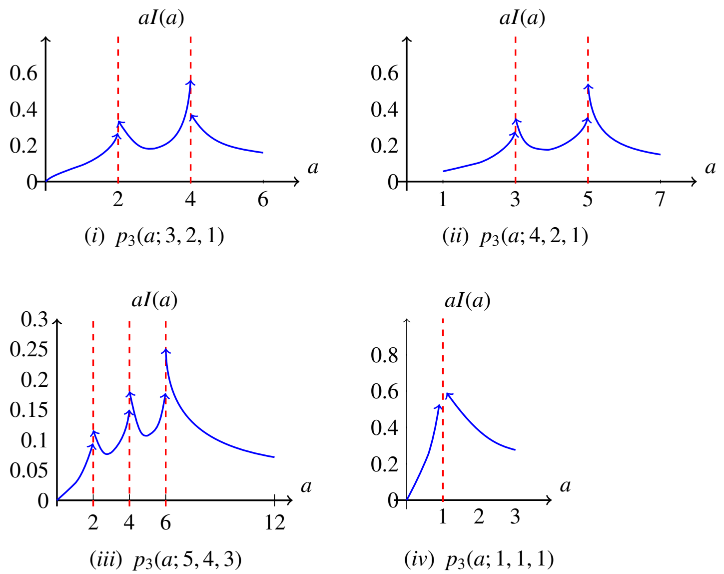

FIGURE 1. Various ramble integrals.

COROLLARY 1.2. *With notation as in Section 1.3, let*

$$a(n,k,D) = \sum_{l_0+\cdots+l_n=k, l_m\ge 0} \binom{k}{l_0, l_1, \ldots, l_n}^2 |W_0|^{l_0} \ldots |W_n|^{l_n}, \quad \binom{k}{l_0, \ldots, l_n} = \frac{k!}{l_0! \ldots l_n!}.$$

Then, for $n = 2$, the logarithmic Mahler measure $m(P_D)$ of the linear polynomial P_D is given by

$$m(P_D) = \log c(D) - \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k a(n,k,D)}{c(D)^{2k}}. \tag{1.1}$$

COROLLARY 1.3. *Let $H_0 := 0$ and $H_l := \sum_{j=1}^{l} 1/j$, $l \in \mathbb{N}_+$, be the harmonic numbers and, for any integer $l \ge 0$, define*

$$S_D(l) := \sum_{j=1}^{\infty} \frac{2j+l}{j(j+l)} \sum_{k=0}^{j} \binom{j+l+k-1}{k} \binom{j}{k} \frac{(-1)^k a(n,k,D)}{c(D)^{2k}}.$$

(i) *For $n = 2$ and all $l \ge 0$ with $D \ne r(1, 1, \ldots, 1)$ for some $r \ne 0$,*

$$m(P_D) = \log c(D) - \tfrac{1}{2} H_l - \tfrac{1}{2} S_D(l). \tag{1.2}$$

(ii) *Additionally, if $l \in \{0, 1\}$, then (1.2) holds for any D.*

By taking these results together with those of the paper [5], the Mahler measure series in (1.1) and (1.2) are valid for arbitrary linear polynomials of three or more variables ($n \geq 2$). Mahler measure calculation in the two-variable case ($n = 1$) is met in standard complex analysis texts using Jensen's formula (see [10, page 345].

The proof of Corollaries 1.2 and 1.3 yields error bounds for the truncated series given in (1.1) and (1.2). We denote the constant arising from bounding $I(a)$ as A_D and $|S|$ for the size of the singularity set S and obtain the following results.

COROLLARY 1.4. *Let $E_1(N; n, D)$ be the right-hand side of (1.1). Then*

$$|m(P_D) - E_1(N; n, D)| \leq |S| \frac{\sqrt[4]{2\pi} A_D c(D)^2}{3\sqrt[4]{N^3}}.$$

COROLLARY 1.5. *Let $E_2(N; n, D)$ be the right-hand side of (1.2). For $l = 1$,*

$$|m(P_D) - E_2(N; n, D)| \leq |S| \frac{3\sqrt[4]{2} A_D c(D)^2}{\sqrt{\pi}\sqrt{N}}.$$

COROLLARY 1.6. *Let $A(D, l) = 6\sqrt{2}(1 - d(D)^2/c(D)^2)^{-(l-1)/2}$. For $l \geq 2$,*

$$|m(P_D) - E_2(N; n, D)| \leq |S| \frac{3\sqrt{2} c(D)^2 A_D A(D, l)}{2\sqrt{N}}.$$

1.5. Finer truncation bounds and Mahler measure estimates. One may refine these truncation bounds experimentally by utilising the algorithm presented in the appendix to compute a truncated series at some N and then comparing the result to known values. All computations in this section employ (1.2) with $l = 1$. By suitably modifying the given code, an experimental bound for $|m(P_D) - E_1(N; n, D)|$ may be similarly obtained.

Consider $m(x_0 + x_1 + x_2)$ for which high-precision estimates are available [1]. Computing for values of N up to 200, we observe that $|m(P_D) - E_2(N; n, D)| \cdot \sqrt{N} \leq C$ for $C \approx 3.8 \times 10^{-2}$, so one might estimate the error bound as simply $C/\sqrt{N}$, eliminating A_D and the other constant terms altogether. For an arbitrary linear polynomial, we have recourse to an identity of Cassaigne and Maillot (see [9]), which relates Mahler measure to the Bloch–Wigner dilogarithm function and the usual logarithm. Let $r_m := |W_m|$ be the lengths of the coefficients $\{W_0, W_1, W_2\}$ of P_D and $r_0 \geq r_1 \geq r_2$ without loss of generality. We have

$$\pi m(P_D) = \begin{cases} \gamma_0 \log r_0 + \gamma_1 \log r_1 + \gamma_2 \log r_2 + \mathcal{D}\left(\dfrac{r_2}{r_1} e^{i\gamma_0}\right) & \text{triangle case,} \\ \pi \log r_0 & \text{nontriangle case,} \end{cases}$$

where the triangle case means $\{r_0, r_1, r_2\}$ can form the sides of a triangle, the nontriangle case is its negation and γ_m is the angle opposite the side r_m.

The Bloch–Wigner dilogarithm is $\mathcal{D}(\alpha) := \Im\mathfrak{m}(Li_2(\alpha)) + \arg(1 - \alpha) \cdot \log|\alpha|$ for $\alpha \in \mathbb{C}\backslash[1, \infty)$, where Li_2 denotes the analytic continuation of the usual dilogarithm to $\mathbb{C}\backslash[1, \infty)$ [16]. In Table 1, we present approximations of Mahler measures, corresponding dilogarithms computed therefrom and estimates for the constant C. The logarithms are computed independently. We do not certify the correctness of the digits,

TABLE 1. Mahler measures.

D	$m(p_D)$	$\log r_0$	$\alpha = (r_2/r_1)e^{i\gamma_0}$	$\mathcal{D}(\alpha)$	C
$(3, 2, 1)$	1.0986	1.0986	–	–	0.028
$(4, 2, 1)$	1.3862	1.3862	–	–	0.064
(e^2, e, e)	2.0000	2	–	–	0.080
$(1, 1, 1)$	0.3203	–	$e^{\pi i/3}$	1.0149	0.038
$(\sqrt{2}, 1, 1)$	0.4648	–	$e^{\pi i/2}$	0.9159	0.027
$(1.732, 1, 1)$	0.5815	–	$e^{2\pi i/3}$	0.6766	0.027
$(1.8478, 1, 1)$	0.6272	–	$e^{3\pi i/4}$	0.5238	0.034
$(1.932, 1, 1)$	0.6624	–	$e^{5\pi i/6}$	0.3569	0.035

but note that they coincide with known logarithm and dilogarithm values to at least four digits. One may also obtain an *analytic* refinement of the error bounds via numerical integration using the closed form of $I(a)$, but the estimate is unsurprisingly much cruder. Nevertheless by employing this method one may conclude, for example, that $|m(P_D) - E_2(N; n, D)| \le C/\sqrt{N}$ for $C \approx 2.324$, where $D = (1, 1, 1)$.

1.6. Organisation of the paper. In Section 2 we include relevant facts from the literature. In Section 3 we establish our main results and, finally, in Appendix A, Friedman presents an algorithm to compute the terms $a(n, k, D)$ and $S_D(l)$ as an aid to Mahler measure numerical evaluations.

2. Background

2.1. *J*-Bessel functions. Recall that $J_0(t)$ is a solution to Bessel's differential equation [7] and hence continuous. Poisson's formal expansion of $J_0(t)$ [15, page 194] for large arguments (that is, $|t| \ge 45$ [7]) is given by

$$J_0(t) = \sqrt{\frac{2}{\pi t}}\left[\cos\left(t - \frac{\pi}{4}\right)P_0(t) + \sin\left(t - \frac{\pi}{4}\right)Q_0(t)\right]. \tag{2.1}$$

We use this expansion for $t \ge 1$ without loss of generality. Stieltjes discovered useful estimates for the series $P_0(t)$ and $Q_0(t)$ in a finite number of terms and we shall utilise the approximations [15, page 208],

$$P_0(t) = 1 - \theta_1 \frac{9}{128t^2} \quad \text{and} \quad Q_0(t) = -\frac{1}{8t} + \theta_2 \frac{225}{3072} \cdot \frac{1}{t^3}, \tag{2.2}$$

where $0 < \theta_1, \theta_2 < 1$. By [14, Theorem 7.31.2], J_0 is bounded. In particular,

$$|J_0(c(D)vt)| \le \sqrt{\frac{2}{\pi c(D)v}} \quad \text{for all } t \ge 1. \tag{2.3}$$

2.2. Integral evaluations involving J-Bessel functions. Here we summarise integral evaluations from [6, 6.699-1 and 6.699-2, page 731] for the cases $\lambda = -\frac{1}{2}$ and $\nu = 0$:

$$\int_0^\infty t^{-1/2} J_0(at)\sin(bt)\,dt = 2^{1/2}a^{-3/2}bF\Big(\frac{3}{4},\frac{3}{4};\frac{3}{2};\Big(\frac{b}{a}\Big)^2\Big) \quad \text{for } 0 < b < a, \tag{2.4}$$

$$\int_0^\infty t^{-1/2} J_0(at)\sin(bt)\,dt = b^{-1/2}\frac{\sqrt{2\pi}}{2}F\Big(\frac{3}{4},\frac{1}{4};1;\Big(\frac{a}{b}\Big)^2\Big) \quad \text{for } 0 < a < b, \tag{2.5}$$

$$\int_0^\infty t^{-1/2} J_0(at)\cos(bt)\,dt = \frac{2^{-1/2}a^{-1/2}\Gamma(1/4)}{\Gamma(3/4)}F\Big(\frac{1}{4},\frac{1}{4};\frac{1}{2};\Big(\frac{b}{a}\Big)^2\Big) \quad \text{for } 0 < b < a, \tag{2.6}$$

$$\int_0^\infty t^{-1/2} J_0(at)\cos(bt)\,dt = \frac{b^{-1/2}\sqrt{2\pi}}{2}F\Big(\frac{1}{4},\frac{3}{4};1;\Big(\frac{a}{b}\Big)^2\Big) \quad \text{for } 0 < a < b, \tag{2.7}$$

where Γ denotes the Gamma function and F denotes the Gaussian hypergeometric series. Note that the evaluations are finite for the given arguments of the respective functions.

2.3. The Ramanujan asymptotic formula for the Gaussian hypergeometric series. We characterise the behaviour of the above integrals as a approaches b, for which we examine the asymptotic behaviour of the hypergeometric series $F(\alpha,\beta;\alpha+\beta;z)$. Let $B(\alpha,\beta)$ denote the Euler Beta function and define

$$R := R(\alpha,\beta) = -\psi(\alpha) - \psi(\beta) - 2\gamma_{EM}, \quad \psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)},$$

where γ_{EM} denotes the Euler–Mascheroni constant. As $a \to b$, the argument z of F in the evaluations in Section 2.2 satisfies $0 < z < 1$ and approaches 1. The Ramanujan asymptotic formula [2, page 96] gives

$$F(\alpha,\beta;\alpha+\beta;z) = \frac{1}{B(\alpha,\beta)}[R - \log(1-z) + O((1-z)\log(1-z))].$$

3. Proof of the main results

PROOF OF THEOREM 1.1. (i) The convergence behaviour and closed form for $I(a)$ is a reformulation of Nicholson's result (see [15, page 414]). To examine differentiability, let b_1 and b_2 be two consecutive points in S^*, $a \in (b_1, b_2)$ and

$$k := \min\Big\{\frac{\sqrt{ar_0r_1r_2}}{\Delta}, \frac{\Delta}{\sqrt{ar_0r_1r_2}}\Big\} \in [0,1).$$

Define $C(a)$ to be the relevant coefficient of $K(k)$, that is, $C(a) := 1/\pi^2\Delta$ for $k = \sqrt{ar_0r_1r_2}/\Delta$ and $C(a) := 1/\pi^2\sqrt{ar_0r_1r_2}$ otherwise. Note that $C(a)$ and $K(k)$ are indeed well-defined functions of a on this interval, by the continuity of k as a function of a and the fact that $I(a)$ diverges if and only if $a \in S$. Both $C(a)$ and the argument k are differentiable functions of a on (b_1, b_2) and the elliptic integral $K(k)$ is differentiable for $k = f(a) \in (0, 1)$, so $I(a) = C(a) \cdot K(f(a))$ is differentiable at a. For $a \in (0, r_0 - r_1 - r_2)$ or $(r_0 + r_1 + r_2, \infty)$, $I(a)$ is continually zero and so differentiable. It is clear that (two-sided) differentiability fails at the points of S^*.

(ii) Since $tJ_0(at)\prod_{m=0}^{2} J_0(r_m t)$ is integrable on [0, 1], we consider the integral on the interval $[1, \infty)$. By applying Poisson's formal expansion (2.1), Szegö's bound for J_0 (2.3), Stieltjes' estimates (2.2) for the auxiliary functions $P_0(t)$ and $Q_0(t)$, standard inequalities and elementary trigonometric identities,

$$\int_1^\infty tJ_0(at)\prod_{m=0}^{2} J_0(r_m t)\,dt = \sum_{i=1}^{4}\Big(\alpha_i\int_1^\infty t^{-1/2}J_0(at)\cos(a_i t)\,dt + \beta_i\int_1^\infty t^{-1/2}J_0(at)\sin(a_i t)\,dt\Big) + \int_1^\infty B(t)\,dt,$$

where the α_i and β_i are nonzero constants satisfying $\alpha_1 = -\beta_1$ and $\alpha_4 = -\beta_4$, the a_i are constants lying in the set $\{r_0 \pm r_1 \pm r_2\}$ and the function $B(t) \in L^1([1, \infty))$. Next apply the closed evaluations (2.4)–(2.7) for the individual integrals and then invoke the Ramanujan asymptotic formula for the hypergeometric series F. By (i), $I(a)$ converges at $a = r_0 + r_1 + r_2$ and $a = r_0 - r_1 - r_2 > 0$ and we obtain $I(a) = O(\log|a - b|)$ for $a \to b \in S$, as claimed. □

REMARK 3.1. From the above analysis, we can obtain some additional information concerning the behaviour of integrals of the form $\int_1^\infty t^{-1/2}J_0(at)(\cos(at) - \sin(at))\,dt$ for $a > 0$. Although $\int_1^\infty t^{-1/2}J_0(at)\cos(at)\,dt$ and $\int_1^\infty t^{-1/2}J_0(at)\sin(at)\,dt$ diverge individually, $\int_1^\infty t^{-1/2}J_0(at)(\cos(at) - \sin(at))\,dt$ must be finite.

PROOF OF COROLLARY 1.2. Armed with Theorem 1.1, we are now ready to establish Corollary 1.2. By [5, Equation (46)],

$$\left|2m(P_D) - 2\log c(D) + \sum_{j=1}^{N}\frac{1}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{(-1)^k a(n,k,D)}{c(D)^{2k}}\right| \le \sum_{j=N+1}^{\infty}\frac{1}{j}I_{D_1},$$

where

$$I_{D_1} := \left|c(D)^2\int_0^{d(D)/c(D)}(1 - v^2)^j v\Big(\int_0^\infty tJ_0(c(D)vt)\prod_{m=0}^{2}J_0(r_m t)\,dt\Big)dv\right|$$

with $v \in (0, 1]$ and $r_m := |W_m|$ for each m from 0 to 2. It suffices to derive a suitable bound for I_{D_1}. Set $a := c(D)v$. Then a lies in $(0, c(D)]$ and $b \le d(D) \le c(D)$ for $b \in S$ by construction and the ℓ^1– ℓ^2 norm inequality. Set $c_b := b/c(D) \in (0, 1]$. We have

$$I_{D_1} \le c(D)^2 \int_0^1 \left|(1-v^2)^j v \sum_{b\in S} \log|v-c_b| \Big(\frac{\int_0^\infty tJ_0(c(D)vt) \prod_{m=0}^2 J_0(r_m t)\,dt}{\sum_{b\in S} \log|v-c_b|} \Big) \right| dv,$$
$$\le A_D\, c(D)^2 \int_0^1 \left|(1-v^2)^j v \sum_{b\in S} \log|v-c_b|\right| dv$$

by Theorem 1.1 for some $A_D > 0$. We show that for each j and for any $b \in S$,

$$\int_0^1 \left|(1-v^2)^j v \log|v-c_b|\right| dv \le \frac{\tilde{A}}{j^{3/4}}$$

for some $\tilde{A} > 0$, which yields the result. Note that both $\log|v-c_b|$ and $(1-v^2)^j v$ are in $L^2([0,1])$ and by a change of variables the square of the norm of $(1-v^2)^j v$ is

$$\int_0^1 (1-v^2)^{2j} v^2\,dv = \frac{1}{2}\int_0^1 (1-u)^{2j} u^{1/2}\,du.$$

Utilising [6, Section 3.196.3] with $a = 0, b = 1, \mu = 3/2$ and $\nu = 2j + 1$ and applying the Cauchy–Schwarz inequality,

$$\int_0^1 \left|(1-v^2)^j v \log|v-c_b|\right| dv \le \sqrt{\frac{\Gamma(3/2)}{2\cdot(2j)^{3/2}}}\cdot \tilde{A}_1 = \frac{\tilde{A}}{j^{3/4}},$$

where $\tilde{A}_1$ denotes the L^2 norm of $\log|v-c_b|$ and $\tilde{A} = \sqrt{\Gamma(3/2)/2^{5/2}}\cdot\tilde{A}_1 > 0$, as claimed. □

PROOF OF COROLLARY 1.3. Considering (1.2) for $l = 1$ and [5, Equations (53), (54)],

$$|m(P_D) - E_2(N; n, D))| \le \frac{C}{\sqrt{N}} c(D)^2 \int_0^1 (1-v^2)^{-1/4} v^{1/2} \int_0^\infty tJ_0(c(D)vt) \prod_{m=0}^2 J_0(r_m t)\,dt\,dv, \tag{3.1}$$

where $E_2(N; n, D)$ is the right-hand side of (1.2) with $C = 2\sqrt[4]{2}/\sqrt{\pi}$. For $l \ge 2$, one must assume that $D \ne r(1,1,1)$ and [5, Equations (56) and (57)] yield

$$|m(P_D) - E_2(N; n, D)| \le \frac{1}{2}\sum_{j=N+1}^\infty \frac{2j+l}{j(j+l)} I_{D_2},$$

where

$$I_{D_2} = \frac{c(D)^2 A(D,l)}{\sqrt{2j+l}} \int_0^1 (1-v^2)^{-1/4} v^{1/2} \Big(\int_0^\infty tJ_0(c(D)vt) \prod_{m=0}^2 J_0(r_m t)\,dt \Big) dv, \tag{3.2}$$

noting that $A(D,l)$ is a constant (see Corollary 1.6) as a consequence of the assumption $D \ne r(1,1,1)$. In both of these cases it suffices to show that the (coincident) integrals in the right-hand sides of (3.1) and (3.2) converge. For $l \geq 1$, this integral is equal to

$$\int_0^1 (1-v^2)^{-1/4} v^{1/2} \sum_{b\in S} \log|v-c_b| \Big(\frac{\int_0^\infty tJ_0(c(D)vt) \prod_{m=0}^2 J_0(r_m t)\,dt}{\sum_{b\in S} \log|v-c_b|} \Big) dv$$
$$\leq A_D \Big| \int_0^1 (1-v^2)^{-1/4} v^{1/2} \sum_{b\in S} \log|v-c_b|\,dv \Big|$$

for some $A_D > 0$ by Theorem 1.1. By the Cauchy–Schwarz inequality, for each $b \in S$, the integral converges, yielding the claim for $l \geq 1$. The case $l = 0$ follows from the case $l = 1$ and a manipulation of the inner sum in [5, Equation (8)]. □

Appendix A. Numerical evaluations by Joshua Friedman

A.1. Introduction. The goal of this appendix is to compute the terms $a(n, k, D)$ and $S_D(l)$ (defined in Corollaries 1.2 and 1.3) for the case of $n = 2$ using high-precision computation software. The first step towards efficient computation is to compute the multinomial in terms of a product of binomials

$$\binom{k}{l_0, l_1, \ldots, l_n} = \binom{l_0}{l_0}\binom{l_0+l_1}{l_1}\cdots\binom{l_0+l_1+\cdots+l_n}{l_n},$$

where $l_0 + \cdots + l_n = k$. The second step is to compute all the $a(n, k, D)$ terms together, that is, for all values of k up to some pre-set maximum (in our code the constant M). We use a triple for the loop and compute all possible sums of three indices:

```
for r in 0:M
 for s in 0:M
  for t in 0:M
   k = r+s+t
```

and, each time a particular k-value appears, we add it to the running sum representing $a(n, k, D)$.

A.2. Technical details and results. Table 2 gives the first four digits of output from our algorithm. It was implemented in the language Julia using the arbitrary precision data types of BigInt and BigFloat, with a precision of 512 bits and a max of $k \leq 200$. Each line in the table took approximately 13 seconds on a single core of an Intel CPU (2.6 GHz i7) Note that we do not certify correctness of the digits.

A.3. Julia implementation of the algorithm. Note that because Julia indexes arrays starting from one rather than zero, we had to code $a(n, k, D)$ as $a[k + 1]$.

TABLE 2. Output of the algorithm.

D	l	$S_D(l)$
(1, 1, −1)	1	0.5511
(1, 1, −1)	2	0.0511
(1, 1, −1)	3	–0.28
(1, 2, 1)	1	0.5040
(1, 2, 1)	2	0.0039
(1, 2, 1)	3	–0.329
(4, 1, 1)	1	0.2164
(4, 1, 1)	2	–0.2836
(4, 1, 1)	3	–0.6169

```
#!/usr/bin/julia
const M = 200
const n = 2
const wr = BigFloat(1/2)
const ws = BigFloat(1/2)
const wt = BigFloat(1/2)
const Wr = wr^2
const Ws = ws^2
const Wt = wt^2
const C_D = (n+1)*(wr^2+ws^2+wt^2)
const l = 2
setprecision(512)

#multinomial code from https://github.com/JuliaMath/Combinatorics.jl
#We implement the multinomial as product of binomials
function multinomial(k...)
 s = 0
 result = 1
 @inbounds for i in k
  s += i
  result *= binomial(s, i)
 end
 result
end

#main function to compute the a(n,k,D) and S_D(l) terms
function f1()
 a = zeros(BigFloat,M+1)
 for r in 0:M
  for s in 0:M
   for t in 0:M
    k = r+s+t
    if k <= M
     a[k+1] += Wr^(r)*Ws^(s)*Wt^(t)
```

```
      *(multinomial(BigInt(r),BigInt(s),BigInt(t)))^2
    end
   end
  end
 end

#print the first 10 a(n,k,D)
 print("M equals ",M, " printing first 10 ",'\n' )
 for k in 0:10
  print(k,": " , a[k+1], '\n')
 end

#compute S_D(l)
 S = BigFloat(0)
 T = BigFloat(0)
 for j in 1:M
  T = BigFloat(0)
  for k in 0:j
   T += binomial(BigInt(j+l+k-1),BigInt(k))
        *binomial(BigInt(j),BigInt(k))*(-1)^k*a[k+1]/C_D^(k)
  end
 S+= BigFloat(2*j+l)/BigFloat(j*(j+l))*T
 print("l= ",l, " j= ",j, ",", "W= ",wr, ',', ws,',', wt, ",  " ,S,'\n')
 end
end
@time f1()
```

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