Minimum Membership Covering and Hitting

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Abstract

Set cover is a well-studied problem with application in many fields. A well-known variant of this problem is the Minimum Membership Set Cover problem: Given a set of points and a set of objects, the objective is to cover all points while minimizing the maximum number of objects that contain any one point. A dual of this problem is the Minimum Membership Hitting Set problem: Given a set of points and a set of objects, the objective is to stab all of the objects while minimizing the maximum number of points that an object contains. We study both of these variants in a geometric setting with various types of geometric objects in the plane, including axis-parallel line segments, axis-parallel strips, rectangles that are anchored on a horizontal line from one side, rectangles that are stabbed by a horizontal line, and rectangles that are anchored on one of two horizontal lines (i.e., each rectangle shares its top or its bottom edge (or both) with one of the input horizontal lines). For each of these problems we either prove NP-hardness or we give a polynomial-time algorithm. In particular, we show that it is NP-complete to decide whether there exists a solution with depth exactly 1 for either the Minimum Membership Set Cover or the Minimum

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Membership Hitting Set problem. In addition, we study a generalized version of the Minimum Membership Hitting Set problem.

Keywords: Minimum membership set cover, Minimum membership hitting set, Anchored rectangles, NP-hard

1. Introduction

The set cover problem is one of the fundamental problems in computer science and combinatorial optimization. This problem, and its many variants, play an important role in modelling various problems arising in practical scenarios. One of its variants is the *Minimum Membership Set Cover (MMSC)* problem, which is defined in a geometric setting as follows.

> **Minimum Membership Set Cover (*MMSC*):** Given a point set P and a set O of objects (regions), cover all the points in P with a subset $O' \subseteq O$ of objects such that the maximum depth of a point is minimized, where the *depth of a point* $p \in P$ is the number of objects in O' that contain it. We say that O' is a *cover* of P, and we let $d(O')$ denote the maximum depth of any point $p \in P$ with respect to O'.

A related problem that is "dual" to the *MMSC* problem is the *Minimum Membership Hitting Set (MMHS)* problem, defined as follows.

> **Minimum Membership Hitting Set (*MMHS*):** Given a point set P and a set O of objects (regions) determine a subset $P' \subseteq P$ of points stabbing (intersecting) all objects O such that the maximum depth of an object is minimized, where the *depth of an object* $o \in O$ is the number of points in P' that stab it. We say that P' is a *hitting set* of O, and we let $d(P')$ denote the maximum depth of any object $o \in O$ with respect to P'.

In addition to the above two problems, we consider a *generalized* version of the *MMHS* problem, the *Generalized Minimum Membership Hitting Set (GMMHS)* problem, where, instead of a point set and an object set, we are given two sets

R ("red") and B ("blue") of objects. The objective is to stab (intersect) all of the objects in B using a subset $R' \subseteq R$ such that the maximum number of red objects in R' hitting any single object in B is minimized. The *MMHS* problem is the special case of *GMMHS* in which the red objects R are points; the generalization is that now R is not just a set of points but is a more general type of region. We prove that even a very special case of *GMMHS*is NP-hard, namely that in which the blue/red regions are horizontal/vertical line segments of unit length.

Applications and motivation: The minimum membership set cover problem is motivated by an application in interference reduction in wireless networks [1]. We are given a set of "clients", which are served by some "servers". Each server has some transmission range within which it can serve clients. If a client is within the ranges of more than one server, then the client experiences interference in the signals it receives from the multiple servers. Therefore, one seeks to choose a set of servers to serve all the clients such that the maximum interference of any client is minimum possible.

The minimum membership set cover problem with rectangles anchored on a horizontal line has an application to wireless coverage [2]. One is given a set of clients (points) in the plane. There is a base station (a point) that serves these clients. The base station uses a directional antenna to transmit beams (a circular sector with angle θ and radius r) to the clients. The goal is to choose a set of beams to serve all the clients such that the maximum interference of any client is minimum possible. In [2], the authors show that this problem in polar coordinate systems can be reduced to the minimum membership set cover problem with rectangles anchored on a horizontal line.

1.1. Previous Work

The very well studied standard set cover problem is NP-hard. A simple greedy heuristic gives a $O(\log n)$-factor approximation, and it is NP-hard to compute an approximation better than logarithmic [3]. The Minimum Membership Set Cover variant was first introduced by Kuhn et al. [1]. They showed that the problem cannot be approximated better than $O(\log n)$ and gave an algorithm achieving approximation factor $O(\log n)$. Erlebach and van Leeuwen [4] considered the geometric variant of the problem, proving that for unit squares and unit disks the problem is NP-hard and that there does not exist a polynomial-time factor 2 approximation algorithm, unless P=NP. Further, for unit squares, they provided a factor 5 approximation algorithm for the case in which the optimum objective value is bounded by a constant. Recently, Nandy et al. [5] reconsidered the same problem and gave polynomial-time algorithms for both unweighted and weighted intervals on the real line. Also recently, Narayanswami et al. [6], considered the problem of hitting a set of horizontal segments with vertical segments while minimizing the number of times a vertical segment is hit by the chosen horizontal segments. They showed that this problem is NP-hard and cannot be approximated better than factor 2. Further, if the segments are of unbounded length (i.e., they are lines), then it can be solved in polynomial time (see also [7] for this algorithm and some generalizations of this problem). In a somewhat different, but related, direction, *capacitated* geometric set cover instances have been studied, e.g., the capacitated discrete unit disk cover, in which we seek a minimum-cardinality subset of a given set of unit disks in order to cover a given set of points, with an upper bound (capacity constraint, α) on how many points can be covered by any one disk; for $\alpha \geq 3$ the problem is NP-complete, and a PTAS (polynomial-time approximation scheme) is known [8].

Closely related to the set cover problem is the maximum coverage problem. Here, a universe set U, a collection C of subsets of U, and a positive integer k is given; the goal is to find at most k sets from C that cover a maximum number of elements from U. This problem is also NP-hard and has a $(1 - \frac{1}{e})$ factor (greedy) approximation algorithm [9]. The geometric set cover problem in $I\!R^2$ is NP-hard for several simple classes of objects, such as disks [10], squares [10], etc. However, the same problem on a real line $I\!R$ is solvable in $O(n \log n)$ time. There is a PTAS for geometric set cover instances with unit disks and unit squares as objects [11]. Another variant of the set cover problem is the *unique cover problem*: given a set P of points and a set T of objects in the plane, the objective is to find a subset $T' \subseteq T$ of objects such that the objects T' cover a maximum number of points whose depth is exactly 1. This problem is NP-hard for both unit disks and unit squares [4]. For unit disks, a 4.31-factor approximation algorithm is available for the unique cover problem [12], and for unit squares a PTAS exists [13].

Recently, Mehrabi [14] considered a variant of the set cover problem, called the *unique set cover problem*. Here also the input is a set P of points and a set T of objects in the plane; the goal is to find a subset $T' \subseteq T$ of objects such that the number of points whose depth is exactly 1 is maximized. He showed that this problem is NP-complete for unit disks and unit squares in the plane. Further, for unit squares he designed a PTAS using a mod-one transformation trick of Chan and Hu for the red-blue set cover problem [15]. Another related problem is the weighted depth problem [16, 17, 18], where the input is a set P of points and a set T of n weighted boxes; the goal is to find a point whose depth is maximum. In $\mathbb{R}^d$, this problem can be solved in time $O(n^d)$ [16].

1.2. Our Contributions: Overview

In this paper we present the following results.

Minimum Membership Set Cover (*MMSC*) problem

We give a polynomial-time algorithm for deciding if there exists a cover with depth one for the *MMSC* problem with objects that are rectangles anchored on a horizontal line. In contrast, we show that if the objects are rectangles that intersect a horizontal line (versus that are anchored, sharing a side with a horizontal line), the *MMSC* problem is NP-hard. We also prove NP-hardness for the cases of objects that are axis-parallel strips or rectangles anchored on two horizontal lines.

Minimum Membership Hitting Set (*MMHS*) problem

We give a polynomial-time algorithm for deciding if there exists a hitting set with depth one for the *MMHS* problem with objects that are rectangles anchored on a horizontal line. In contrast, we show that if the objects are rectangles that intersect a horizontal line, the *MMHS* problem is NP-hard. We also prove NP-hardness for the cases of objects that are axis-parallel strips or rectangles anchored on two horizontal lines.

Generalized Minimum Membership Hitting Set (*GMMHS*) problem We show that *GMMHS*, with object sets B, R given as unit-length horizontal/vertical line segments, is NP-hard; even deciding if a solution exists with depth one is NP-complete. We also give a 5-approximation algorithm if the optimal objective function is bounded by a constant.

It is noted that, in all of our NP-completeness proofs, we prove that it is NP-complete to decide whether there exists a solution with depth exactly 1. Since the depth is an integer, any approximation algorithm returns a solution greater than or equal to 2. Thus, each of the problems shown to be NP-complete does not have a polynomial-time algorithm with approximation factor smaller than 2 (unless P=NP).

Equivalence of MMSC and MMHS with unit disks/squares. There is a connection (equivalence) between the *MMSC* and *MMHS* problems where the input objects are either unit disks or unit squares. Consider the case of unit squares. Given an instance $C = (P, T)$ of the *MMSC* problem, with a set P of points and a set T of unit squares, we consider a "dual" instance, H, of a *MMHS* problem whose regions are specified by the set of unit squares centered on the points $p \in P$, and whose points are specified as the center points of the squares $t \in T$. We then note that determining a solution to the *MMSC* problem C is equivalent to determining a solution to the *MMHS* problem H. Thus, we conclude, by applying the results in [4, 5]: The *MMHS* problem is NP-complete with unit squares and unit disks and there exists a 5-approximation for the *MMHS* problem with unit squares where the optimal objective value is bounded by a constant.

1.3. Definitions and Notations

In the *3SAT* problem we are given a *CNF* formula ϕ with n variables $\mathcal{X}$=x_1, x_2,. . .,x_n and m clauses $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ where each clause is a disjunction of exactly 3 literals, and the objective is to decide whether there is a truth assignment to variables such that ϕ is satisfiable. This problem is known to be NP-complete [19]. In the planar version of this problem, each variable and clause represents a vertex and there is an edge between a variable vertex and a clause vertex if and only if the corresponding clause contains the corresponding literal. Finally, the resulting bipartite graph is planar. This problem is called the *Planar-3SAT* problem and Lichtenstein [20] proved that this problem is also NP-complete. Later on, Knuth and Raghunathan [21] showed that every instance of the *Planar-3SAT* problem can be represented using the following rectilinear representation. The variables are placed on a horizontal line and the clauses containing 3 legs each connecting those variables either from above or below the horizontal line such that no two clause legs intersect. This problem is called the *Rectilinear-Planar-3SAT* problem and is also NP-complete [21]. A *Positive-1-in-3SAT* problem is a *3SAT* problem, however the objective is different: here, the objective is to decide whether there is a truth assignment to the variables such that exactly one literal per clause is true. Schaefer [22] proved that this problem is NP-complete. This problem can be represented using the rectilinear representation as defined above; we refer to it as the *Rectilinear-Positive-Planar-1-in-3SAT* problem (see Figure 1). Mulzer and Rote [23] proved that it is also NP-complete.

We now define some terminology. Let $\mathcal{C}_{above} \subseteq \mathcal{C}$ be the set of clauses in a *PP1in3SAT* formula ϕ that connect to the variables from above. Similarly, let $\mathcal{C}_{below} \subseteq \mathcal{C}$ be the set of clauses that connect to the variables from below. For each variable x_i, $1 \le i \le n$, we order the clauses in $\mathcal{C}_{above}$ left to right that connect x_i. Let $C_\ell \in \mathcal{C}_{above}$ be a clause containing the three variables x_i, x_j, and x_k. Then, according to the ordering defined above, we assume that C_ℓ is the ℓ_1-, ℓ_2-, and ℓ_3-th clause for the variables x_i, x_j, and x_k, respectively. For example, the clause C_3 is a 3-rd, 1-st, and 1-st clause for the variables x_3, x_4, and x_5, respectively, in the *PP1in3SAT* instance in Figure 1. We also say that the clause C_ℓ connects to x_i by left, to x_j by middle, and to x_k by right legs.

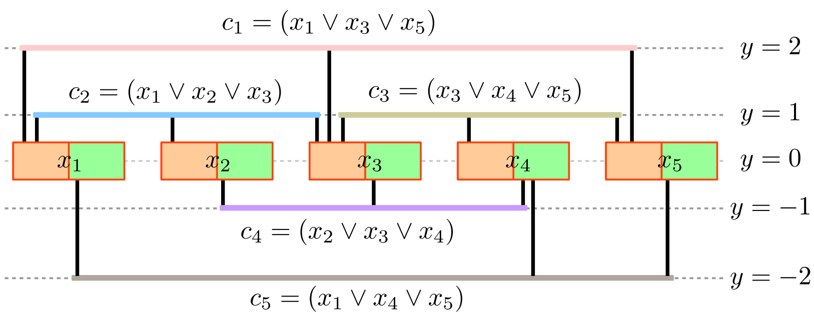


Figure 1: Representation of a *Rectilinear-Positive-Planar-1-in-3SAT* problem.

2. Minimum Membership Set Cover Problem

2.1. Rectangles Anchored on a Horizontal Line

We observe that, in polynomial time, one can decide if there exists a cover of depth one for the *MMSC* problem with rectangles anchored on a horizontal line from one side (*MMSCRAHL*). The idea is as follows. We assign a weight to each rectangle, given by the number of points it contains. Now we have an input of a set of weighted rectangles anchored on a horizontal line and a set of points. We now find, for this instance, a maximum independent set of rectangles (no two of them share a point), using the algorithm described by Chan and Grant [24] (the *pack-regions* problem). This requires polynomial time. Next, we verify in $O(1)$ time whether or not the size of the maximum independent set is equal to the number of points. If this is true, then we ensure that there is a cover of the points with depth exactly 1.

2.2. Axis-Parallel Strips

In this section we prove that the *MMSC* problem with axis-parallel strips (*MMSCS*) is NP-hard. We give a reduction from the Positive-1-in-3SAT (*P1in3SAT*) problem (see Section 1.3 for the definition). Let ϕ be a *P1in3SAT* formula. We generate an instance $Z(S, P)$ of the *MMSCS* problem from ϕ in the following way, where S is a set of strips and P is a set of points.

Variable gadget: For variable x_i, the gadget consists of one vertical strip v_i, one horizontal strip h_i, and a point p_i. The point is covered by both v_i and h_i (see Figure 2). Clearly, either v_i or h_i will cover p_i with depth one. We assume that choosing h_i makes x_i true, while choosing v_i makes x_i false.

Overall Structure: We place the variable gadgets (points) along a diagonal line. For each clause we take a vertical bounded *region*. The clause gadgets are placed sequentially one by one to the right of the variable gadgets, and each gadget is confined to its corresponding region. Between two consecutive variable horizontal strips there is an *empty space*, where we place some points corresponding to the clauses.

Clause gadget: Let $C_\ell = (x_i \vee x_j \vee x_k)$ be a clause. For this clause, we take 5 points $p_i^\ell, p_j^\ell, p_k^\ell, p_1^\ell, p_2^\ell$ and 4 vertical strips $q^\ell, r^\ell, s^\ell, t^\ell$ (see Figure 2). The points p_i^ℓ, p_j^ℓ, and p_k^ℓ are corresponding to the variables x_i, x_j and x_k respectively and are placed inside the strips h_i, h_j, and h_k respectively. The other two points p_1^ℓ and p_2^ℓ are placed in any empty space between the variable horizontal strips of x_i, x_j (i.e., between h_i and h_j) and x_j, x_k (i.e., between h_j and h_k) respectively. Points $\{p_i^\ell, p_1^\ell\}$ are contained in q^ℓ. Similarly, $\{p_1^\ell, p_j^\ell\}$, $\{p_j^\ell, p_2^\ell\}$, and $\{p_2^\ell, p_k^\ell\}$ are contained in r^ℓ, s^ℓ, and t^ℓ, respectively. These 5 points and 4 rectangles are strictly contained inside the vertical region of C_ℓ (Figure 2).

This completes the description of details of the construction, for a given instance, ϕ, of *P1in3SAT*. Finally, we note that the construction can be done in time that is polynomial in the size of the formula ϕ. We now utilize this construction to prove the following theorem.

Theorem 1. *The MMSCS problem is NP-hard.*

Proof. We prove that, ϕ is satisfiable (i.e., exactly one literal is true per clause) if and only if $Z(P, S)$ has a solution of depth one. Assume that ϕ has an assignment such that exactly one literal per clause is true. If x_i is true then select h_i; otherwise, select v_i. Now, for each clause, exactly one of $p_i^\ell, p_j^\ell, p_k^\ell$ is

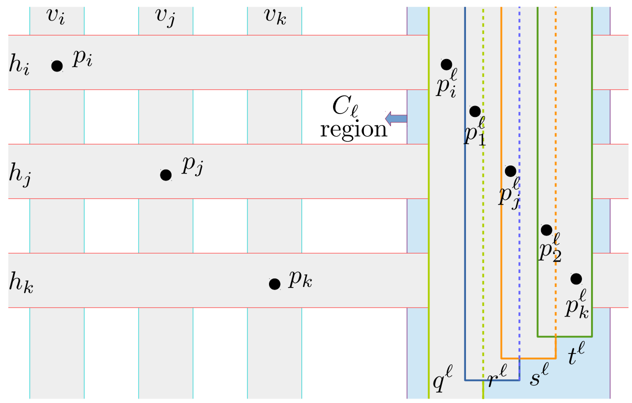


Figure 2: Gadgets of variables x_i, x_j, x_k, and clause C_ℓ and their interaction.

covered by the solution. Hence, the remaining 4 points are covered by exactly two strips with depth one.

On the other hand, assume that there is a cover of the points with depth one. Now, for each variable gadget, to cover p_i we need one of the two strips h_i or v_i. We set variable x_i to be true if h_i is in the solution; otherwise, we set x_i to be false. Now consider any clause C_ℓ. Since the depth of the solution (indeed a cover of all points) is one, exactly one of $p_i^\ell, p_j^\ell, p_k^\ell$ corresponding to C_ℓ is covered by a variable horizontal strip. We set this variable to be true. Hence, exactly one literal per clause is true in ϕ. □

Corollary 1. *The MMSC problem with rectangles each anchored on one of two orthogonal lines (MMSCRATOL) is NP-hard. (Consider a vertical line $x = -M$ and a horizontal line $y = -M$, for M sufficiently large; then very tall or very wide rectangles anchored on these lines are axis-parallel strips.)*

2.3. Rectangles Intersecting a Horizontal Line

In this section we prove that the *MMSC* problem with rectangles intersecting a horizontal line (*MMSCRIHL*) is NP-hard. The reduction is from the *PP1in3SAT* problem [23]. From an arbitrary instance ϕ of the *PP1in3SAT* problem, we construct an instance Z of the *MMSCRIHL* problem, where the rectangles in Z intersect a horizontal line L.

Variable gadget: The gadget for the variable x_i consists of $12m$ rectangles $\{1, 2, \ldots, 12m\}$ and $12m - 1$ points $\{p_1, p_2, \ldots, p_{12m-1}\}$ (see Figure 3(a)), where m is the number of clauses in ϕ. The points are along the top edges of the rectangles. The 1-st and the $12m$-th rectangles contain the points p_1 and p_{12m-1}, respectively, and the j-th rectangle contains the p_{j-1}-th and p_j-th points, for $2 \le j \le 12m - 1$. We note that the first $6m$ rectangles $\{1, 2, \ldots, 6m\}$ are responsible for the clauses in $\mathcal{C}_{above}$, whereas the next $6m$ rectangles $\{6m+1, 6m+2, \ldots, 12m\}$ are responsible for the clauses in $\mathcal{C}_{below}$. All of the rectangles are intersecting a horizontal line L. Now, in order to cover all of the points while minimizing the depth, we have only two distinct optimal solutions: either all even-numbered or all odd-numbered rectangles with depth exactly one. This gives the truth value of the variable x_i.

Clause gadget: We first modify the *PP1in3SAT* problem in the following way. Note that the variables of ϕ are placed on a horizontal line ($y = 0$). We move the variables vertically up such that they are placed on a horizontal line $y = m + 1$ (above the y-values of all the clauses in $\mathcal{C}_{above}$) (see Figure 4). The clauses in $\mathcal{C}_{above}$ are placed above L and below the line $y = m+1$ while connecting the same set of variables as before. Note that these clauses now connect the variables from below. On the contrary, the clauses in $\mathcal{C}_{below}$ are placed below L and still connect to the same set of variables from below.

Let us now consider the set $\mathcal{C}_{above}$ of clauses. Notice that, in the definition of the *PP1in3SAT* problem these clauses can be ordered in increasing y-direction (see Figure 1). Here we reverse the order of the clauses (see Figure 3(b)). Now for each clause $C \in \mathcal{C}_{above}$ we take a *rectangular box* whose top boundary is the segment of C in the modified construction. The bottom boundary of the box touches the line L. Each box has a thin strip along the top edge of that box, called the *tape* of that clause. Similarly, we reverse the order of the clauses in $\mathcal{C}_{below}$ and for each clause C we take a box whose bottom boundary is the segment of C in the modified construction. The top boundary of the box touches the line L. Now here the tape is along the bottom boundary of each box.

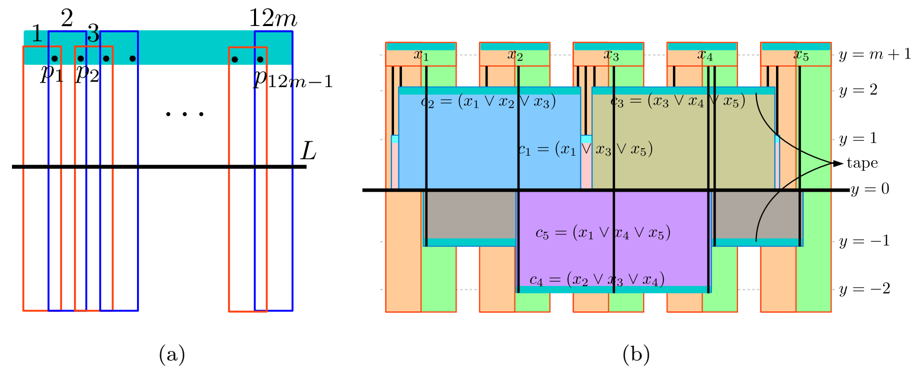


Figure 3: (a) A variable gadget. (b) Position of the clause gadgets.

Let $C_\ell = (x_i \vee x_j \vee x_k)$ be a clause in $\mathcal{C}_{above}$. We say that x_i is a *left*, x_j is a *middle*, and x_k is a *right* variable for C_ℓ. We take 5 points; point p_i^ℓ corresponding to x_i, points $p_j^\ell, q_j^\ell, r_j^\ell$ corresponding to x_j, and point p_k^ℓ corresponding to x_k; and 4 rectangles $s_1^\ell, s_2^\ell, s_3^\ell, s_4^\ell$. The rectangle s_1^ℓ covers the points $\{p_i^\ell, p_j^\ell\}$, s_2^ℓ covers the points $\{p_i^\ell, q_j^\ell\}$, s_3^ℓ covers the points $\{p_j^\ell, p_k^\ell\}$, and s_4^ℓ covers the points $\{r_j^\ell, p_k^\ell\}$ (see Figure 4). The rectangles are placed inside the box and the points are placed inside the tape of C_ℓ.

Variable and clause interaction: We now describe the placement of the clause rectangles and points with respect to the variable rectangles. Let $1, 2, \ldots$ be the left to right order the clauses in $\mathcal{C}_{above}$ that connect to the variable x_i. In this order, assume that C_ℓ be the ℓ_1-, ℓ_2-, and ℓ_3-th clause for the variables x_i, x_j, and x_k respectively. Then we do the following.

- ⇝ Since x_i is a left variable in C_ℓ, place the point p_i^ℓ inside the $(6\ell_1 - 2)$-th rectangle of the gadget of x_i.
- ⇝ Since x_j is a middle variable in C_ℓ, place the point p_j^ℓ inside the $(6\ell_2 - 2)$-th rectangle of the gadget of x_j. Also place the point q_j^ℓ and r_j^ℓ inside the $(6k - 3)$-th and $(6k - 1)$-th rectangles of the gadget of x_j.
- ⇝ Since x_k is a right variable in C_ℓ, place the point p_k^ℓ inside the $(6\ell_3 - 2)$-th

rectangle of the gadget of x_k.

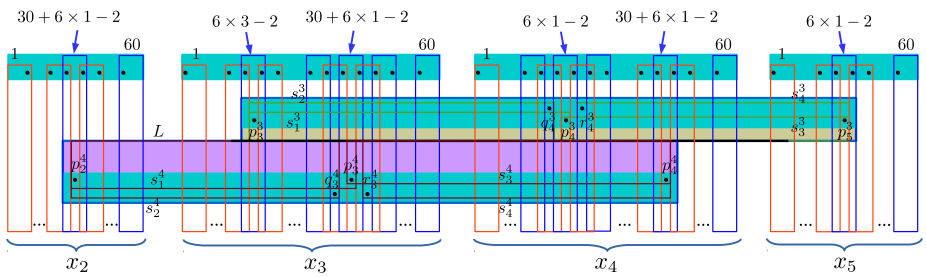


Figure 4: Interaction between the variable and clause gadgets. We demonstrate the interaction of C_3 and C_4 with the variables in the *P1in3SAT* instance in Figure 1.

A similar construction can be made for the clauses in $\mathcal{C}_{below}$, but using the last $6m$ rectangles in the variable gadgets. See Figure 4 for the construction described above. We now prove the following theorem.

Theorem 2. *The MMSCRIHL problem is NP-hard.*

Proof. We prove that exactly one literal is true in every clause of ϕ if and only if the *MMSCRIHL* problem has a cover of depth 1. Assume that there is an assignment to the variables of ϕ that satisfies exactly one literal per clause. For a variable x_i, if it is true, then select the even indexed rectangles; otherwise, select the odd indexed rectangles from the gadget of x_i. Let us consider a clause $C_\ell = (x_i \vee x_j \vee x_k)$. Since exactly one literal per clause is true, exactly one of p_i^ℓ or p_j^ℓ, or p_k^ℓ is covered by a variable rectangle. Clearly, the remaining points in the clause gadget are covered by the clause rectangles with depth one.

In the reverse direction, assume that the *MMSCRIHL* problem has a cover of depth 1. To cover the points in a variable gadget and in order to make their depth 1, there are only two possibilities to select the rectangles. We set the variable x_i to be true if all even indexed rectangles are selected from the gadget of x_i; otherwise, set x_i to be false. Now consider a clause $C_\ell = (x_i \vee x_j \vee x_k)$. Now in C_ℓ, if more than one literal is true then the depth of a point in the gadget of C_ℓ will be more than 1. If the clause is not satisfiable then also either at least one point is not covered or there will be a point whose depth will be more than one. The only possibility is exactly one literal per clause is true. Hence, the theorem. □

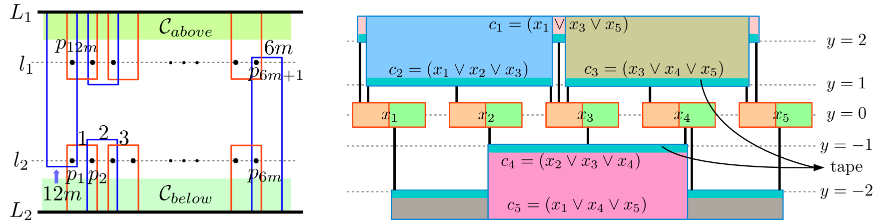
2.4. Rectangles Anchored on Two Horizontal Lines

In this section we prove that the *MMSC* problem with rectangles anchored on two horizontal lines (*MMSCRATHL*) is NP-hard. We give a reduction from the *PP1in3SAT* problem [23].

Variable gadget: To construct the variable gadget (see Figure 5(a)) of x_i, we first take two parallel lines L_1 and L_2. We consider $12m$ points on two imaginary horizontal lines l_1 and l_2 in between L_1 and L_2 where each of l_1 and l_2 contains $6m$ points. We also consider $12m$ rectangles $1, 2, \ldots, 12m$. The rectangles $1, 2, \ldots, 6m$ are anchored on the line L_2 and the remaining rectangles are anchored on the line L_1. The i-th rectangle covers exactly two points p_i and p_{i+1}, for $1 \le i \le 12m - 1$ and the rectangle $12m$ covers the points p_{12m} and p_1. Now in order to cover all the points while minimizing the depth, we have only two different optimal solutions; either all even numbered or all odd numbered rectangles with depth exactly 1. This gives the truth value of the variable x_i.

Clause gadget: We first consider the set $\mathcal{C}_{below}$ of clauses in ϕ. These clauses can be ordered in decreasing y-direction (see Figure 1). Now for each clause $C \in \mathcal{C}_{below}$ we take a *rectangular box* whose top boundary is the segment of C. The bottom boundary of the box touches the line L_2. Each box has a thin strip along the top edge of that box, called the *tape* of that clause. Similarly, we construct the boxes and tapes for the clauses for $\mathcal{C}_{above}$. See Figure 5(b).

The placement of the clause points and rectangles is similar to the placement of the clause points and rectangles described in Section 2.3. The clause structure is exactly the same as in Section 2.3. For a clause $C_\ell = (x_i \vee x_j \vee x_k)$ in $\mathcal{C}_{below}$ with x_i, x_j, and x_k as *left*, *middle*, and *right* variable, we take 5 points; point p_i^ℓ corresponding to x_i, points $p_j^\ell, q_j^\ell, r_j^\ell$ corresponding to x_j, and point p_k^ℓ corresponding to x_k; and 4 rectangles $s_1^\ell, s_2^\ell, s_3^\ell, s_4^\ell$. The rectangle s_1^ℓ cover the points $\{p_i^\ell, p_j^\ell\}$, s_2^ℓ cover the points $\{p_i^\ell, q_j^\ell\}$, s_3^ℓ cover the points $\{p_j^\ell, p_k^\ell\}$, and



(a) (b)

Figure 5: (a) A variable gadget. The clause gadgets are placed inside the shaded regions that are anchored on either L_1 or L_2. (b) Position of the clause gadgets.

s_4^ℓ cover the points $\{r_j^\ell, p_k^\ell\}$. The rectangles are placed inside the box and the points are placed inside the tape of C_ℓ.

Variable and clause interaction: Observe that, the way the clauses in $\mathcal{C}_{above}$ are connected to the variables in Section 2.3 (Figure 3(b)), here the same way the clauses in $\mathcal{C}_{below}$ are connected to the variables. Therefore, the interaction between the variables and the clauses is similar to that in Section 2.3, but now here we consider a clause $C \in \mathcal{C}_{below}$ instead of a clause $C \in \mathcal{C}_{above}$. As in the proof of Theorem 2, we conclude:

Theorem 3. *The MMSCRATHL problem is NP-hard.*

3. Minimum Membership Hitting Set Problem

3.1. Rectangles Anchored on a Horizontal Line

We observe that in polynomial time one can decide if there exists a hitting set of depth one for the *MMHS* problem with rectangles anchored on a horizontal line from one side (*MMHSRAHL*). The idea is similar to that applied in Section 2.1. Here we assign a weight to each point, given by the number of rectangles it stabs. Now we have an input of a set of rectangles anchored on a horizontal line and a set of weighted points. We find, for this instance, a maximum weight set of points (no two of them share a rectangle), using the algorithm described by Chan and Grant [24] (the *pack-points* problem); this takes polynomial time.

Next, we verify in $O(1)$ time whether or not the size of the solution is equal to the number of rectangles. If this is true, then we know that there is a hitting set for the instance with depth exactly 1.

3.2. Axis-Parallel Strips

We prove that the *MMHS* problem with axis-parallel strips (*MMHSS*) is NP-hard using a reduction from the *P1in3SAT* problem. We generate an instance $Z(S,P)$ of the *MMHSS* problem from ϕ, an instance of the *P1in3SAT* problem.

The gadget for a variable x_i includes $2m-1$ horizontal strips $\{1,2,\ldots,2m-1\}$ and $2m$ points $\{p_1,p_2,\ldots,p_{2m}\}$. The j-th strip contains the points p_j and p_{j+1}, for $1 \leq j \leq 2m-1$ (see Figure 6(a)). The points are on a vertical line. However, we move some of the points to the right to some clause gadgets at later stage. It is observed that there are exactly two different sets of points, either all even indexed or all odd indexed, which stab all the strips with depth exactly 1. We stack the variable gadgets vertically from top to bottom.

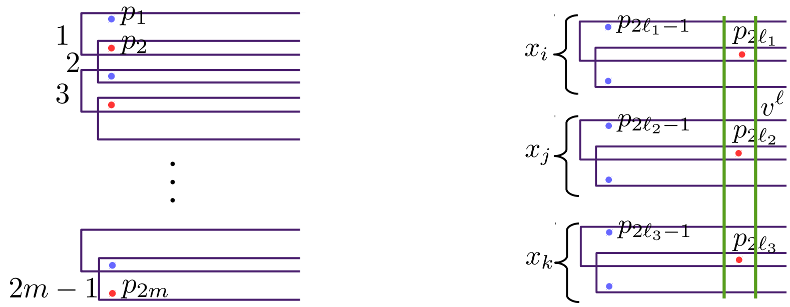
The gadget for a clause C_ℓ is a vertical strip v^ℓ. The clause gadgets are placed one after another to the right of the points corresponding to the variable gadgets.

For each variable, we order the clauses that contains it. Let C_ℓ be a clause that contains x_i, x_j, x_k, then according to this ordering we say that C_ℓ is a ℓ_1-th, ℓ_2-th, and ℓ_3-th clause for x_i, x_j, and x_k respectively. Now for the clause C_ℓ we move the three points $p_{2\ell_1}$, $p_{2\ell_2}$, and $p_{2\ell_3}$ in the horizontal orientation from the gadgets of x_i, x_j, and x_k respectively to inside v^ℓ.

Clearly, the number of strips and points are polynomial with respect to the number of variables and clauses in ϕ. Hence the construction can be done in polynomial time. We now prove the following theorem.

Theorem 4. *The MMHSS problem is NP-hard.*

Proof. We prove that exactly one literal is true in each clause of ϕ if and only if Z has a hitting set with depth exactly 1. For variable x_i, we choose even indexed points if x_i is true, else choose odd indexed points. This clearly stabs



(a) (b)

Figure 6: (a) Variable gadget. (b) Clause gadget and its interaction with variable gadgets.

all variable strips with depth 1. Since exactly one literal is true in each clause of ϕ, exactly one point will stab a clause strip. On the other hand assume that there is a hitting set of points with depth exactly 1. Now stabbing all the variable strips with depth 1 requires either all even or all odd indexed points. So we set x_i to be true if even indexed points are selected; otherwise, set x_i to be false. Since the depth of the hitting set is 1, exactly one point in a clause strip is selected. □

3.3. Rectangles Intersecting a Horizontal Line

In this section we show that the *MMHS* problem with rectangles intersecting a horizontal line (*MMHSRIHL*) is NP-hard. Here we give a reduction from the *PP1in3SAT* problem.

The variable gadget is similar to the variable gadget defined in Section 3.2, but now there are $4m-1$ strips, $\{1, 2, \ldots, 4m-1\}$, and $4m$ points, $\{p_1, p_2, \ldots, p_{4m}\}$ instead of $2m-1$ strips and $2m$ points. These strips are now vertical and they are bounded above and below so that they become rectangles. Further, they are intersecting a horizontal line L. Recall that the j-th strip contains the points p_j and p_{j+1}, for $1 \le j \le 4m-1$. It is now clear that there are exactly two different sets of points, $P_1^i = \{p_1, p_3, \ldots, p_{4m-1}\}$ and $P_1^i = \{p_2, p_4, \ldots, p_{4m}\}$, that stab all the rectangles such that the depth of the solution is exactly 1.

The clause gadget is similar to that in Section 2.3, but now, for each clause, the rectangular box of Section 2.3 is itself a rectangle, and each rectangle has a tape (a rectangle corresponding to a clause in $\mathcal{C}_{above}$ has a tape along its top boundary, and a rectangle corresponding to a clause in $\mathcal{C}_{below}$ has a tape along its bottom boundary). We now use a similar process as in Section 3.2 to shift (vertically) points from the variable gadgets to the tapes of the clause rectangles.

Let $C_\ell = (x_i \vee x_j \vee x_k)$ be a clause in C_{above}. As in Section 2.3, assume that x_i is a left, x_j is a middle, and x_k is a right variable for C_ℓ. Also let C_ℓ be the ℓ_1-, ℓ_2-, and ℓ_3-th clause for x_i, x_j, and x_k respectively.

We now move the three points $p_{2\ell_1}$, $p_{2\ell_2}$, and $p_{2\ell_3}$ in the vertical orientation from the gadgets of x_i, x_j, and x_k respectively to inside the tape of the clause C_ℓ. A similar construction can be done for the the clauses in $\mathcal{C}_{below}$, however, the points $\{p_{2m+1}, p_{2m+2}, \ldots, p_{4m}\}$ are responsible for these clauses and shifted vertically to the tapes of the clause rectangles accordingly.

Clearly the construction is made in polynomial time in terms of the size of the formula. Since no two tapes contain points corresponding to two different clauses, as in the proof of Theorem 4, we conclude the following theorem.

Theorem 5. *The MMHSRIHL problem is NP-hard.*

3.4. Rectangles Anchored on Two Horizontal Lines

We show that the *MMHSRATHL* problem is NP-hard. Here, also we give a reduction from the *PP1in3SAT* problem.

The variable gadget is identical to the variable gadget in Section 2.4; however, here we take $4m + 8$ points $\{p_1, p_2, \ldots, p_{4m+8}\}$ on two imaginary horizontal lines l_1 and l_2, with $2m + 4$ points each. We also take $4m + 8$ rectangles $\{1, 2, \ldots, 4m+8\}$ such that $2m+4$ rectangles $\{1, 2, \ldots, 2m+4\}$ are anchored on L_2 and the remaining $2m+4$ rectangles $\{2m+5, 2m+6, \ldots, 4m+8\}$ are anchored on the line L_1. Clearly, there are two optimal sets of points, $P_1^i = \{p_1, p_3, \ldots, p_{4m+7}\}$ and $P_1^i = \{p_2, p_4, \ldots, p_{4m+8}\}$, stabbing the rectangles with depth exactly 1.

The clause gadgets are also identical to the clause gadgets in Section 2.4 with the difference that each rectangular box is a rectangle itself that contains a tape (rectangles corresponding to clauses in $\mathcal{C}_{below}$ have a tape along their top boundaries, and rectangles corresponding to clauses in $\mathcal{C}_{above}$ have a tape along their bottom boundaries).

The variable clause interaction is made by vertically shifting some points from the variable gadgets to this tape (similar to Section 3.3). Observe that in Section 3.3 we describe the interaction for the clauses in $\mathcal{C}_{above}$. Here we consider the clauses in $\mathcal{C}_{below}$ due to the similar reason as in Section 2.4.

Similar to the proof of Theorem 4 and Theorem 5, we conclude with the following theorem.

Theorem 6. *The MMHSRATHL problem is NP-hard.*

4. The *GMMHS* problem of Stabbing Horizontal Unit Segments with Vertical Unit Segments

NP-hardness: We prove that the *GMMHS* problem of stabbing horizontal unit segments by vertical unit segments (*GMMHSUSeg*) is NP-hard. The reduction is from the *PP1in3SAT* problem.

Variable gadget: Each variable gadget consists of a *variable chain* and at most $2m$ *clause chains*, each corresponding to a clause leg that connects to a variable.

Variable chain: Each variable chain consists of $8m + 2$ unit horizontal segments $\{h_1, h_2, \dots, h_{8m+2}\}$ positioned like a rectangular fashion (see Figure 7). The segments $\{h_1, h_2, \dots, h_{4m}\}$ are on a horizontal line and are responsible for connecting the clause chains to the variable chain from above. Similarly, the segments $\{h_{4m+2}, h_{4m+3}, \dots, h_{8m+1}\}$ are on another horizontal line and are responsible for connecting the clause chains to the variable chain from below.

Clause chains: Let C_ℓ be a clause in $\mathcal{C}_{above}$ that connects the variables x_i, x_j, and x_k through left, middle, and right legs respectively. Then for a left or middle, or right leg, we construct a *left* or *middle*, or *right* chain respectively. The

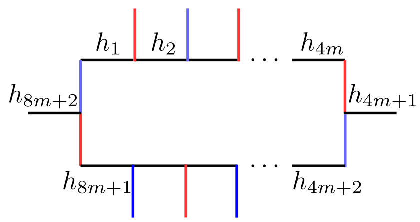


Figure 7: A variable gadget.

left and middle chains are depicted in Figure 8(a) and Figure 8(b) respectively. The right chain is similar to the left chain but flipped vertically.

Let us consider a clause $C \in \mathcal{C}_{above}$ that is a ℓ-th clause for the variable x_i. In the variable chain of x_i, we shift the $h_{4\ell-2}$-th segment slightly left and the $h_{4\ell-1}$-th segment slightly right (see Figure 8(c)). Place the chain for C above these two segments such that h' and $h_{4\ell-2}$ are stabbed by a vertical segment and h'' and $h_{4\ell-1}$ are stabbed by another vertical segment. Note that for each variable at most $2m$ chains are connected with its variable chain, at most m from either above or below. The variable chain and at most $2m$ left, middle, or right chains together form a big circular like arrangements of segments, called *big-cycle*. Note that, this big-cycle contains an even number of both horizontal and vertical segments and along the cycle at most 2 consecutive horizontal segments are stabbed by a vertical segment. We now have the following observation.

Observation 1. *For each variable gadget, there are two optimal solutions, either all red or all blue vertical segments each of size half of the total number of vertical segments present in a big-cycle.*

Clause gadget: Let $C_\ell \in \mathcal{C}_{above}$ be a clause that contains x_i, x_j, and x_k. The gadget for C_ℓ is a single horizontal segment h^ℓ. The position of h^ℓ with respect to the three chains corresponding to x_i, x_j, and x_k is shown in Figure 8(d).

This completes the construction. Note that this construction can be done in polynomial time with respect to the number of the variables and clauses in ϕ.

Theorem 7. *The GMMHSUSeg problem is NP-hard.*

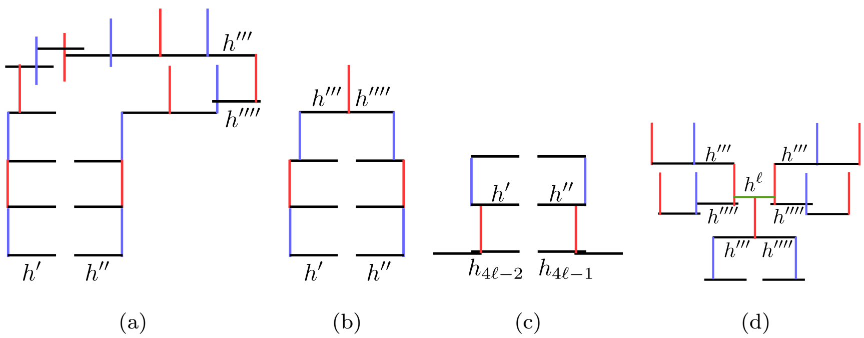


Figure 8: (a) A left chain. (b) A middle chain. (c) Attaching a clause chain to a variable chain. (d) Clause gadget and connection with the three variable gadgets.

Proof. We prove that exactly one literal per clause in ϕ is true if and only if the *GMMHSUSeg* problem has a solution with depth exactly 1. Assume that there exists a truth assignment of the variables of ϕ such that exactly one literal per clause in ϕ is true. Now consider a variable, say x_i. If x_i is true, we select all red vertical segments; otherwise, we select all blue vertical segments. Clearly the depth of any segment corresponding to any variable is exactly 1. Since in any clause, say C_ℓ exactly one literal is true, the segment h^ℓ is stabbed by exactly one vertical red segment corresponding to the true literal. Hence, the depth of the solution is exactly 1.

On the other hand, assume that there exists a solution to the *GMMH-SUSeg* problem with depth exactly 1. To stab all the horizontal segments in a variable gadget requires either all red or all blue vertical segments in order to keep the depth 1. Therefore, we set variable x_i to be true if all red segments are selected in the solution; otherwise, we set x_i to be true. Now we show that this assignment makes exactly one literal per clause of ϕ true. Consider a clause C_ℓ. Since the depth of the solution is exactly 1, exactly one of the three red segments corresponding to the three variables of C_ℓ that stab h^ℓ is selected in the solution. That means we set only that variable of C_ℓ to be true whose corresponding red segment stabs h^ℓ. This completes the proof. □

4.0.1. Approximation for the GMMHSUSeg problem:

First we convert this problem to the *MMHS* problem with unit squares. Let H and V be given sets of unit horizontal and vertical segments. For each horizontal segment $h \in H$, take a unit square $t_h \in T$ such that the bottom boundary of t_h coincides with h and for each vertical segment $v \in V$, take the top endpoint, $p_v \in P$ of v. Clearly, finding a set $V' \subseteq V$ that stabs all the horizontal segments in H while minimizing the number of times a segment in H is stabbed by segments in V' is equivalent to finding a set of points $P' \subseteq P$ that stabs all the unit squares in T while minimizing the number of points in P' that is contained in a unit square in T. (We remark that this reduction shows that the *MMHS* problem with unit squares is NP-hard, from our Theorem 7 (the NP-hardness of the *GMMHSUSeg* problem), giving an alternative (to [4]) proof of this fact.)

Since for unit squares the *MMHS* and *MMSC* problems are dual to each other, the above reduction, together with the approximation algorithm given in [4], yields the following result.

Theorem 8. *There exists a* 5*-approximation for the GMMHSUSeg problem when the optimal objective value is bounded by a constant.*

5. Conclusion

In this paper we considered the Minimum Membership Set Cover (*MMSC*) and Minimum Membership Hitting Set (*MMHS*) problems. We considered various classes of geometric objects, including axis-parallel strips, rectangles anchored on a horizontal line, rectangles anchored on two parallel horizontal lines, and rectangles intersected by a horizontal line. For the *MMSC* and *MMHS* problems with rectangles anchored on a horizontal line, we showed that the existence of solutions with depth exactly one can be solved in polynomial time. A natural open question is to design polynomial-time algorithms or proving NP-hardness for these problems (*MMSC* and *MMHS*) in this anchored rectangle setting. The *MMSC* and *MMHS* problems with other classes of geometric objects mentioned above are NP-hard. We also considered a generalized version of the Minimum Membership Hitting Set problem, the Generalized Minimum Membership Hitting Set problem (*GMMHS*) on axis-parallel unit segments, and proved that it is NP-hard. This problem admits a 5-approximation when the optimal objective value is bounded by a constant. Designing an approximation algorithm without any constraint or improving the factor are open questions.

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