FOURIER RESTRICTION ABOVE RECTANGLES

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Abstract. In this article, we study the problem of obtaining Lebesgue space inequalities for the Fourier restriction operator associated to rectangular pieces of the paraboloid and perturbations thereof. We state a conjecture for the dependence of the operator norms in these inequalities on the sidelengths of the rectangles, prove that this conjecture follows from (a slight reformulation of the) restriction conjecture for elliptic hypersurfaces, and prove that, if valid, the conjecture is essentially sharp. Such questions arise naturally in the study of restriction inequalities for degenerate hypersurfaces; we demonstrate this connection by using our positive results to prove new restriction inequalities for a class of hypersurfaces having some additive structure.

1. Introduction

Recent work [2] establishing bounds for restriction operators associated to higher order surfaces on which the curvature may vanish at some points naturally gives rise to the study of the restriction operator $\mathcal{R}_d^\ell$ associated to the rectangular piece of the paraboloid,

$$\{(|\xi|^2, \xi) : \xi \in Q^\ell\}, \qquad Q^\ell := \prod_{j=1}^d (-l_j, l_j), \qquad \ell = (l_1, \dots, l_d) \in (0,\infty]^d,$$

and perturbations thereof.

In this article, we consider the problems of establishing finiteness and understanding the dependence on ℓ of the $L^p \to L^q$ operator norms of $\mathcal{R}_d^\ell$. We then apply such results to obtain new, sharp restriction inequalities for a collection of "degenerate" hypersurfaces (i.e. hypersurfaces whose curvature vanishes on some nonempty set). We are motivated by the recent success of the first author [13] (cf. [4]) in directly deducing sharp estimates for model convolution operators by using a generalization of this approach.

The natural interpretation of ellipticity in this context leads to a slight generalization of the traditional notion of ellipticity formulated by Tao–Vargas–Vega [16]. We introduce some additional notation, letting

$$A^\ell(\xi_1, \dots, \xi_d) := (l_1\xi_1, \dots, l_d\xi_d), \qquad \ell \in (0,\infty)^d,$$

and

$$\mathbb{1} := (1, \dots, 1).$$

Definition 1.1. Let $\ell \in (0,\infty)^d$ and let g be a C^{N+2}_{loc} function on Q^ℓ for some $\ell \in (0,\infty]^d$, with $N \geq 0$ sufficiently large, possibly infinite. Assume that $D^2 g$ is positive definite throughout Q^ℓ, and let $0 < \varepsilon_0 \leq \frac{1}{2}$. We say that g is elliptic over Q^ℓ (with parameters N, ε_0) if $g(\xi) = |\xi|^2 + h(\xi)$, where the perturbation h satisfies

$h(0) = 0$, $\nabla h(0) = 0$, $D^2 h(0) = 0$, and

$$\|(D^2 h) \circ A^{\tilde{\ell}}\|_{C^N(Q^1)} < \varepsilon_0,$$

for every bounded $Q^{\tilde{\ell}}$ contained in Q^ℓ. If g is elliptic over Q^ℓ with parameters N, ε_0, we will also say that the surface

$$\Sigma_g := \{(g(\xi), \xi) : \xi \in Q^\ell\}$$

is elliptic over Q^ℓ (with parameters N, ε_0).

This definition of ellipticity is invariant under parabolic rescalings in the sense that g is elliptic over Q^ℓ if and only if $\lambda^2 g(\lambda^{-1}\cdot)$ is elliptic over $Q^{\lambda\ell}$. In the special cases that $d = 1$ or $\ell = \mathbb{1} = (1, \dots, 1)$, our definition of ellipticity coincides with that in [16], but ours is strictly more general in the sense that a surface elliptic over some Q^ℓ may not be coverable by a bounded (independent of ℓ) number of surfaces elliptic in the sense of [16]. We will see the utility of this generalization once we turn to applications.

Associated to a function g elliptic over some Q^ℓ are the familiar restriction and extension operators,

$$\mathcal{R}_g^\ell f(\xi) := \widehat{f}(g(\xi), \xi), \qquad \xi \in Q^\ell$$
$$\mathcal{E}_g^\ell f(t,x) := \int_{Q^\ell} e^{i(t,x)(g(\xi),\xi)} f(\xi)\, d\xi, \qquad (t,x) \in \mathbb{R}^{1+d}.$$

Since these operators are dual to one another, it suffices to state our results for the extension operator.

Ellipticity over some Q^ℓ is a more general concept than ellipticity in the sense of Tao–Vargas–Vega, and the following conjecture seems to be a reasonable generalization of the corresponding conjecture for elliptic hypersurfaces.

Conjecture 1.2. *For N sufficiently large, $0 < \varepsilon_0 < \frac{1}{2}$, and $1 \le p, q \le \infty$ in the range $q = \frac{d+2}{d} p' > p$, there exists a constant $C_{p,q,d} < \infty$ such that for any $\ell \in (0,\infty]^d$, $\|\mathcal{E}_g^\ell\|_{L^p \to L^q} \le C_{p,q,d}$, for any g elliptic over Q^ℓ with parameters N, ε_0.*

This conjecture is already verified in the case $d = 1$ by Fefferman–Stein [6] and Zygmund [19], and its deduction in the bilinear range in higher dimensions is relatively straightforward (Theorem 1.3). The authors have not investigated whether the results of [10, 11, 12, 18] extend to imply progress toward Conjecture 1.2, though the possibility of such an extension seems likely.

Theorem 1.3. *Conjecture 1.2 holds for all $d \ge 1$ and $q > \frac{2(d+3)}{d+1}$.*

In certain cases, this theorem is already known [3, 6, 14, 16, 19]; we will give the short deduction of the remaining cases in Section 4.

As promised, we turn now to the dependence of operator norms on the side-lengths.

Conjecture 1.4. *Let $\ell \in (0,\infty]^d$ satisfy $l_1 \le \cdots \le l_d$. For g elliptic over Q^ℓ with parameters N sufficiently large and $0 < \varepsilon_0 < \frac{1}{2}$, depending on d, p, q, we have the following operator norm bounds for $\mathcal{E}_g^l$, with implicit constants independent of g and ℓ. If $q > p$ satisfy $q = \frac{d-j-\theta+2}{d-j-\theta} p'$, for some $0 \le j < d$ and $0 \le \theta \le 1$, then*

$$\|\mathcal{E}_g^\ell\|_{L^p \to L^q} \lesssim (l_1 \cdots l_j l_{j+1}^\theta)^{\frac{1}{p'} - \frac{1}{q}}; \tag{1.1}$$

in particular, this quantity is finite whenever $l_{j+1} < \infty$. *If* $l_d < \infty$, *we have, in addition:*

$$\|\mathcal{E}_g^\ell\|_{L^p\to L^q} \lesssim_\varepsilon (l_1\cdots l_d)^{\frac{1}{q}-\frac{1}{p}}(\tfrac{l_d}{l_{j+1}})^\varepsilon (l_1\cdots l_j l_{j+1}^\theta)^{1-\frac{2}{q}}, \tag{1.2}$$

for $q = \frac{2(d-j-\theta+1)}{d-j-\theta} \le p$, $0 < \theta \le 1$, $\varepsilon > 0$, *and* $j = 0,\dots,d-2$; *and*

$$\|\mathcal{E}_g^\ell\|_{L^p\to L^q} \lesssim (l_1\cdots l_{d-1})^{\frac{1}{p'}-\frac{1}{q}} l_d^{1-\frac{3}{q}-\frac{1}{p}}, \tag{1.3}$$

for $q > 4$ *and* $p \ge (\frac{q}{3})'$.

Modulo the precise definition of ellipticity, the two-dimensional version of this conjecture was essentially formulated by Buschenhenke–Müller–Vargas in [2] (one must rescale).

We have the following positive result.

Theorem 1.5. *Conjecture 1.2 implies Conjecture 1.4. In particular, Conjecture 1.4 holds unconditionally for* $q > \frac{10}{3}$ *for all* $d \ge 2$, *and, when* $d \ge 3$ *and*

$$P_k := (1 - \tfrac{k+2}{k}\cdot\tfrac{k+1}{2(k+3)}, \tfrac{k+1}{2(k+3)}), \tag{1.4}$$

for $(\frac{1}{p},\frac{1}{q})$ *in the convex hull of* $[(1,0),P_{d-j})\cup[(1,0),P_{d-j+1})$, *for each* $1 \le j < d-1$.

More precise statements of the conditional part of Theorem 1.5 may be found in the lemmas leading to the proof of Theorem 1.5.

In addition, we prove that Conjecture 1.4, if true, is essentially optimal, excepting the precise asymptotics as $\frac{l_d}{l_{j+1}} \to \infty$ in the region $q \le p$, $q \le 4$.

Theorem 1.6. *Let* $\ell \in (0,\infty]^d$ *satisfy* $l_1 \le \cdots \le l_d$. *Let* g *be elliptic over* Q^ℓ *with parameters* $N \ge 2$ *and* $0 < \varepsilon_0 < \frac{1}{2}$. *Then* $\mathcal{E}_g^l$ *does not extend as a bounded linear operator for* (p,q) *lying outside of the region* $q \ge \frac{d+2}{d}p'$, $q > \frac{2(d+1)}{d}$. *If* $l_k = \infty$, *some* $1 \le k \le d$, *then* $\mathcal{E}_g^l$ *does not extend as a bounded operator from* L^p *to* L^q *for any* $p \ge q$ *nor* $q \le \frac{d-k+3}{d-k+1}p'$. *More precisely, if* $q > p$ *satisfy* $q = \frac{d-j-\theta+2}{d-j-\theta}p'$, *for some* $0 \le j < d$ *and* $0 \le \theta \le 1$, *then*

$$\|\mathcal{E}_g^\ell\|_{L^p\to L^q} \gtrsim (l_1\cdots l_j l_{j+1}^\theta)^{\frac{1}{p'}-\frac{1}{q}}. \tag{1.5}$$

If $q = \frac{2(d-j-\theta+1)}{d-j-\theta} \le p$, $0 < \theta \le 1$, *and* $j \in \{0,\dots,d-2\}$, *then*

$$\|\mathcal{E}_g^\ell\|_{L^p\to L^q} \gtrsim (l_1\cdots l_d)^{\frac{1}{q}-\frac{1}{p}}\alpha(\tfrac{l_d}{l_{j+1}})(l_1\cdots l_j l_{j+1}^\theta)^{1-\frac{2}{q}}. \tag{1.6}$$

Here α *depends on* d, p, *and* q; $\alpha \gtrsim 1$; *and* $\alpha(r)\to\infty$ *as* $r\to\infty$. *Finally, for* $q > 4$ *and* $p \ge (\frac{q}{3})'$,

$$\|\mathcal{E}_g^\ell\|_{L^p\to L^q} \gtrsim (l_1\cdots l_{d-1})^{\frac{1}{p'}-\frac{1}{q}} l_d^{1-\frac{3}{q}-\frac{1}{p}}. \tag{1.7}$$

Attribution for the statement of Conjecture 1.4 and prior progress toward Theorems 1.5 and 1.6 is somewhat ambiguous, particularly as some prior progress on these questions was not formalized into precisely stated theorems, the hypotheses and generality elsewhere differ, and the implications of earlier methods and results seem not to have been fully exploited. We give a recounting of the progress of which we are aware. For the fully conditional part of Theorem 1.5, we use an elementary deduction, which was used to obtain an alternate proof of the restriction inequality for the cone in [5]. In two dimensions, under a more restrictive hypothesis, lower bounds matching those from Theorem 1.6 in the region $2p' \le q \le 3p'$, $p < q$ were

obtained, as were the lower bounds in the region $3 < q < 4$, modulo the additional gain in $\frac{l_d}{l_{j+1}}$; it was also remarked that the methods in [9] (which, in turn, attributes the method to [5]) lead to the conditional result in this region. Nevertheless, the question seems not to have been formulated in this level of generality (particularly with regard to dimension), some of our lower bounds seem to be new in all dimensions, and some of our positive results (in the bilinear range) are obtained by means that also seem to be new in this context.

Two natural open questions are whether, for particular values of ℓ, there is a larger range of exponents for which unconditional progress toward Theorem 1.5 can be made, and whether unconditional results could be extended along horizontal lines in greater generality than just the bilinear range in two dimensions.

Our main application is to determine new inequalities and give a simpler proof of known inequalities for a class of degenerate hypersurfaces. Given $\beta \in (1,\infty)^d$, we define an extension operator

$$\mathcal{E}_\beta f(t,x) = \int_{Q^1} e^{i(t,x)(g_\beta(\xi),\xi)} f(\xi)\, d\xi, \qquad g_\beta(\xi) := \sum_{j=1}^d |\xi_j|^{\beta_j}.$$

In the case $d = 2$, this extension operator was considered in [9] in the Stein–Tomas range and in [2] in the bilinear range.

Varchenko's height [17] associated to these surfaces is the quantity h defined by

$$\tfrac{1}{h} := \tfrac{1}{\beta_1} + \cdots + \tfrac{1}{\beta_d}.$$

In determining bounds for $\mathcal{E}_\beta$, intermediate dimensional versions of the height become relevant. Thus, taking the convention that $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_d$, we also define

$$J_n := \tfrac{1}{\beta_1} + \cdots + \tfrac{1}{\beta_n}, \qquad 0 \leq n \leq d.$$

We obtain an essentially optimal conditional result for the operators $\mathcal{E}_\beta$. To facilitate its statement, we let T_d denote the set of all $(p,q) \in [1,\infty]^2$ for which the local elliptic extension operator is conjectured to be bounded, that is,

$$T_d := \{(p,q) \in [1,\infty]^2 : q > \tfrac{2(d+1)}{d},\ q \geq \tfrac{d+2}{d} p'\}. \tag{1.8}$$

Theorem 1.7. *Assume that Conjecture 1.4 holds for all $(\tilde p, \tilde q)$ in a relatively open subset $V \subseteq T_d$ containing (p,q). Then $\mathcal{E}_\beta$ extends as a bounded operator from L^p to L^q if at least one of the following conditions hold:*
(i) *$q > p$ and $\frac{q}{p'} \geq 1 + \frac{1}{J_n + \frac{d-n}{2}}$, for all $0 \leq n \leq d$;*
(ii) *$q \leq p$ and $\frac{1+J_n+d-n}{q} < \frac{J_n}{p'} + \frac{d-n}{2}$, for all $0 \leq n \leq d$; or*
(iii) *$q = p$, $\frac{1+J_d}{q} = \frac{J_d}{p'}$, and $\frac{1+J_n+d-n}{q} < \frac{J_n}{p'} + \frac{d-n}{2}$, for all $0 \leq n < d$.*
Furthermore, $\mathcal{E}_\beta$ is of restricted weak type (p,q) if
(iv) *$q \leq p \leq \infty$, $\frac{1+J_d}{q} = \frac{J_d}{p'}$, and $\frac{1+J_n+d-n}{q} < \frac{J_n}{p'} + \frac{d-n}{2}$, for all $0 \leq n < d$.*

Here we use the not-completely-standard definition that a linear operator T, initially defined on $L^1(\mathbb{R}^d)$, is of restricted weak type (p,q) if

$$|\langle T f_E, g_F\rangle| \lesssim |E|^{\frac{1}{p}} |F|^{\frac{1}{q'}},$$

for all measurable, finite measure E, F and measurable functions $|f_E| \leq \chi_E$ and $|g_F| \leq \chi_F$. (It will be convenient to note that we may equivalently replace '$\leq$' by '$\sim$' in the conditions on f_E, g_F.) For finite p, this is equivalent to the usual definition of restricted weak type boundedness.

Conditional on the restriction conjecture above rectangles, both the strong and restricted weak type estimates arising in Theorem 1.7 are sharp.

Proposition 1.8. *If $(p,q) \in [1,\infty]^2$ does not satisfy any of Conditions* (i-iii) *of Theorem 1.7, then $\mathcal{E}_\beta$ is not of strong type (p,q). If $(p,q) \in [1,\infty]^2$ does not satisfy any of Conditions* (i-iv) *of Theorem 1.7, then $\mathcal{E}_\beta$ is not of restricted weak type (p,q).*

When $d = 2$, Proposition 1.8 is due to [2], and Theorem 1.7 is due to [2] in the bilinear range (where it is unconditional) and to [9] in the Stein–Tomas range. For $d > 2$, Theorem 1.7 is due to [8] in the Stein-Tomas range. Our main new contribution is a direct deduction of the result from Conjecture 1.2, which leads to a simpler approach that avoids the complicated step of obtaining bilinear restriction estimates between rectangles at different scales. This simplification enables us to address the higher dimensional case, as well as the case when some exponents β_i are less than 2.

The region $(\frac{1}{p}, \frac{1}{q})$ described in Theorem 1.7 and Proposition 1.8 can be somewhat difficult to visualize, so we make a few simple observations. We see the familiar conditions $q \geq \frac{d+2}{d}p'$ and $q > \frac{2(d+1)}{d}$ in the $n = 0$ case of each of the constraints. The lower bound on $\frac{q}{p'}$ in (i) of Theorem 1.7 is strongest when $J_n + \frac{d-n}{2}$ is minimal, which occurs when $n = n_0$, the minimal index for which $\beta_i < 2$ for all $i > n_0$. Thus the constraint in (i) is strictly stronger than that in the elliptic restriction conjecture unless $n_0 = 0$, i.e. $\beta_i \leq 2$ for all i.

The condition (ii) may introduce some vertices in the Riesz diagram; these all lie on or above the line $\frac{1}{q} = \frac{1}{p}$. For each n, the lines $\frac{q}{p'} = 1 + \frac{1}{J_n + \frac{d-n}{2}}$ (seen in (i)) and $\frac{1+J_n+d-n}{q} = \frac{J_n}{p'} + \frac{d-n}{2}$ (seen in (ii)) intersect when $q = p = 2 + \frac{1}{J_n + \frac{d-n}{2}}$, and these two lines are equal when $n = d$. The slope of the line $\frac{1+J_n+d-n}{q} = \frac{J_n}{p'} + \frac{d-n}{2}$ is $-\frac{J_n}{1+\frac{1+d-n}{J_n}}$, which equals 0 when $n = 0$ and decreases as n increases. The intersection point of such a line with $q = p$ is $q = p = 2 + \frac{1}{J_n + \frac{d-n}{2}}$, which moves closer to $(0,0)$ as n increases until n reaches n_0, at which point it begins to increase. Thus only those lines with $n \leq n_0$ play a role in determining the boundary of the region.

Notation. Admissible constants may depend on the dimension d, the exponents p, q, and the d-tuple β in the definition of $\mathcal{E}_\beta$, as well as any operator norms on whose finiteness results may be conditioned. For nonnegative real numbers A, B, we will use the notation $A \lesssim B$, $B \gtrsim A$ to mean that $A \leq CB$ for an admissible constant C, which is allowed to change from line to line; $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. We will occasionally subscript constants or the $\lesssim$ notation to indicate dependence on an additional parameter. For $\lambda \in \mathbb{R}$, λ_+ denotes the positive part, $\lambda_+ := \max\{\lambda, 0\}$.

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2. Ellipticity over subsets of rectangles and other technical lemmas

In this section we will prove a handful of technical lemmas extending known results to surfaces elliptic over rectangles (and slices thereof). Most of these results use only basic calculus.

The statements of the results will be simpler if we generalize the notion of ellipticity.

Definition 2.1. Let K be a convex subset of $\mathbb{R}^d$ with nonempty interior, and let $g \in C^{N+2}_{\text{loc}}(K)$, with $D^2 g$ positive definite throughout K. We say that g is elliptic over K, with parameters N, ε_0, if there exist $\xi_0 \in K$, $U \in O(d)$, and $\ell \in (0,\infty]^d$ such that

$$K \subseteq \sqrt{2}[D^2 g(\xi_0)]^{-\frac{1}{2}} U Q^\ell + \xi_0,$$

and the functions

$$\tilde{g}(\xi) := g(\sqrt{2} D^2 g(\xi_0)^{-\frac{1}{2}} U\xi + \xi_0) - g(\xi_0) - \sqrt{2} D^2 g(\xi_0)^{-\frac{1}{2}} U\xi \cdot \nabla g(\xi_0) \tag{2.1}$$

and $\tilde{h}(\xi) := \tilde{g}(\xi) - |\xi|^2$ obey

$$\|\ell^\alpha \partial^\alpha D^2 \tilde{h}\|_{C^N(\tilde{K})} < \varepsilon_0, \tag{2.2}$$

where

$$\tilde{K} := \tfrac{1}{\sqrt{2}} D^2 g(\xi_0)^{\frac{1}{2}} U^T (K - \xi_0)). \tag{2.3}$$

2.1. **Dicing.** Here we will prove that the restriction of a function elliptic over a rectangle is elliptic over smaller rectangles, with improved parameters. This result will allow us to assume that ε_0 is sufficiently small in later arguments. More speculatively, such a result is potentially of use in induction on scales type arguments.

Lemma 2.2. *Let $\ell \in (0,\infty]^d$ and let g be elliptic over Q^ℓ with parameters (N, ε_0), some $N \geq 1$. Let $K \subseteq Q^\ell$ be a convex set with nonempty interior, and assume that $\varepsilon^{-1}(K - \xi_0) \subseteq Q^\ell$ for some $\xi_0 \in K$ and $0 < \varepsilon \leq 1$. Then g is elliptic over K with parameters $N, C_{N,d}\varepsilon\varepsilon_0$.*

Proof of dicing lemma. By taking limits, we may assume that K is compact. By the John ellipsoid theorem, there exists $\tilde{\ell} \in (0,\infty)^d$ and $U \in O(d)$ such that

$$c_d U Q^{\tilde{\ell}} \subseteq \tfrac{1}{\sqrt{2}} D^2 g(\xi_0)^{\frac{1}{2}} (K - \xi_0) \subseteq U Q^{\tilde{\ell}}.$$

Define $\tilde{g}$ as in (2.1) and $\tilde{K}$ as in (2.3). For $|\alpha| \geq 1$,

$$\begin{aligned} \|\tilde{\ell}^\alpha \partial^\alpha D^2 \tilde{g}\|_{C^0(\tilde{K})} &= \|2U^T D^2 g(\xi_0^{-\frac{1}{2}} (\sqrt{2} D^2 g(\xi_0)^{-\frac{1}{2}} U\tilde{\ell})^\alpha \partial^\alpha D^2 g D^2 g(\xi_0)^{-\frac{1}{2}} U\|_{C^0(K)} \\ &\leq 2\|D^2 g(\xi_0)^{-\frac{1}{2}}\|^2 \|\theta^\alpha \partial^\alpha D^2 g\|_{C^0(Q^\ell)}, \end{aligned}$$

where

$$\theta := \sqrt{2} D^2 g(\xi_0)^{-\frac{1}{2}} U\tilde{\ell} \in K - \xi_0 \subseteq c_d^{-1} \varepsilon Q^\ell.$$

As $\|D^2 g(\xi_0)^{-\frac{1}{2}}\| \leq C_d$, in the case $|\alpha| \geq 1$, (2.2) follows from the ellipticity hypothesis. In the case $\alpha = 0$, (2.2) follows from the $|\alpha| = 1$ case, $D^2\tilde{g}(0) = 2\mathbb{I}_d$, and the fundamental theorem of calculus. □

2.2. **Slicing.** Here we will show that the restriction of a function elliptic over a rectangle to some lower-dimensional slice of the rectangle is also elliptic, with comparable parameters. This result is essential in the (conditional) proof of Theorem 1.5.

Lemma 2.3. *Let $\ell \in (0,\infty]^d$ and let g be elliptic over Q^ℓ with parameters N, ε_0. Let $P \subseteq \mathbb{R}^d$ be an affine k-plane, and assume that $P \cap Q^\ell$ has nonempty interior in P. Let $\xi_0 \in P \cap Q^\ell$, and let $\{u_1, \cdots, u_k\}$ be an orthonormal basis for $P - \xi_0$. Then*

$$g^\flat(\eta_1, \ldots, \eta_k) := g(\xi_0 + \sum_{j=1}^k \eta_j u_j)$$

is elliptic over $K := \{(\eta_1, \ldots, \eta_k) : \xi_0 + \sum_{j=1}^k \eta_j u_j \in P \cap Q^\ell\}$.

Proof. By taking limits, we may assume that K is compact. By the John ellipsoid theorem, there exists $\tilde{\ell} \in (0,\infty)^k$ and $V \in O(k)$ such that

$$c_d V Q^{\tilde{\ell}} \subseteq \tfrac{1}{\sqrt{2}} D^2 g^\flat(0)^{\frac{1}{2}} K \subseteq V Q^{\tilde{\ell}}.$$

Set $\tilde{K} := \frac{1}{\sqrt{2}} V^T D^2 g^\flat(0)^{\frac{1}{2}} K$,

$$\tilde{g}^\flat(\eta) := g^\flat(\sqrt{2} D^2 g^\flat(0)^{-\frac{1}{2}} V\eta) - g^\flat(0) - \sqrt{2} D^2 g^\flat(0)^{-\frac{1}{2}} V\eta \cdot \nabla g^\flat(0),$$

and $\tilde{h}^\flat(\eta) := \tilde{g}^\flat(\eta) - |\eta|^2$, $\eta \in \mathbb{R}^k$.

Extend the given basis for $P - \xi_0$ to an orthonormal basis $\{u_1, \cdots, u_d\}$ of $\mathbb{R}^d$ and set $U := (u_1, \ldots, u_d) \in O(d)$. Then

$$c_d U[\sqrt{2} D^2 \tilde{g}(0)^{-\frac{1}{2}} V Q^{\tilde{\ell}} \times \{0\}] \subseteq (P \cap Q^\ell) - \xi_0 \subseteq U[\sqrt{2} D^2 \tilde{g}(0)^{-\frac{1}{2}} V Q^{\tilde{\ell}} \times \{0\}].$$

Let $|\alpha| \geq 1$ be a multiindex. By the chain rule,

$$\begin{aligned} \|\tilde{\ell}^\alpha \partial^\alpha D^2 \tilde{g}^\flat\|_{C^0(\tilde{K})} &\leq \|\sqrt{2} D^2 g^\flat(0)^{-\frac{1}{2}}\|^2 \|(\sqrt{2} D^2 g(0)^{-\frac{1}{2}} V\tilde{\ell})^\alpha \partial^\alpha D^2 g^\flat\|_{C^0(K)} \\ &\leq \|\sqrt{2} D^2 g^\flat(0)^{-\frac{1}{2}}\|^2 \|[U(\sqrt{2} D^2 \tilde{g}(0)^{-\frac{1}{2}} V\tilde{\ell}, 0)]^\alpha \partial^\alpha D^2 g\|_{C^0(Q^\ell)}. \end{aligned}$$

Finally, since

$$U(\sqrt{2} D^2 \tilde{g}(0)^{-\frac{1}{2}} V\tilde{\theta}, 0) \in U[\sqrt{2} D^2 \tilde{g}(0)^{-\frac{1}{2}} V Q^{\tilde{\ell}} \times \{0\}] \subseteq c_d^{-1}((P \cap Q^\ell) - \xi_0) \subseteq c_d^{-1} Q^\ell,$$

inequality (2.2) holds for $|\alpha| \geq 1$; the case $|\alpha| = 0$ follows analogously, by considering $\tilde{h}^\flat$. □

2.3. **Morse Lemma.** Next, we note that the following version of the Morse lemma follows readily by adapting standard undergraduate-level proofs of the Morse lemma to functions elliptic over rectangles. (We omit the details of this elementary adaptation.) This result allows us to invoke the classical arguments involving stationary phase, including the Stein–Tomas and Strichartz theorems and the wave packet decomposition of [14].

Lemma 2.4. *Let g be elliptic over Q^ℓ, $\ell \in (0,\infty)^d$, with parameters $N \geq 1$ and $0 < \varepsilon \leq \varepsilon_d$. Then exist $U \subseteq \mathbb{R}^d$ and a C^N diffeomorphism F of U onto $Q^{\mathbb{1}}$ such that*

$$g(l_1 F_1, \ldots, l_d F_d) = \sum_j (l_j u_j)^2$$

and $\|F(\eta) - \eta\|_{C^N_\eta(U)} < C(\varepsilon)$.

3. Negative results: The proof of Theorem 1.6

For simplicity, we give a complete proof of Theorem 1.6, recalling that it is already known in some cases. We will actually prove a slightly stronger result. Let

$$\|\mathcal{E}_g^\ell\|_{L^p\to L^q}^{\mathrm{RWT}} := \sup_{E,F} \frac{|\langle \mathcal{E}_g^\ell f_E, g_F\rangle|}{|E|^{\frac{1}{p}}|F|^{\frac{1}{q'}}},$$

where the supremum is taken over measurable sets E, F with positive, finite measures and measurable functions $|f_E| \le \chi_E$, $|g_F| \le \chi_F$.

Proposition 3.1. *The conclusions of Theorem 1.6 hold with $\|\mathcal{E}_g^\ell\|_{L^p\to L^q}^{\mathrm{RWT}}$ in place of $\|\mathcal{E}_g^\ell\|_{L^p\to L^q}$.*

The rest of this section will be devoted to the proof of Proposition 3.1. We will use the convention that references to equations in the statement of Theorem 1.6 shall be superscripted with 'RWT.'

That $\|\mathcal{E}_g^\ell\|_{L^p\to L^q}^{\mathrm{RWT}}$ is infinite whenever $q < \frac{d+2}{d}p'$ follows from the classical Knapp example. The case when $q \le \frac{2(d+1)}{d}$ follows from a slight modification of the argument from [1], which will be given in Lemma 3.4. The assertion regarding unboundedness of $\mathcal{E}_g^\ell$ when some l_j are infinite follows from the lower bounds (1.5-1.7)$^{\mathrm{RWT}}$ and the elementary inequality

$$\|\mathcal{E}_g^\ell\|_{L^q\to L^p}^{\mathrm{RWT}} \ge \|\mathcal{E}_g^{\tilde{\ell}}\|_{L^q\to L^p}^{\mathrm{RWT}}, \text{ for } \tilde{l}_j \le l_j,\ j = 1,\dots,d. \tag{3.1}$$

It remains to prove the lower bounds in the case that each l_j is finite. This will be carried out in three lemmas, one for each numbered inequality.

Lemma 3.2. *The lower bound $(1.5)^{\mathrm{RWT}}$ is valid in the range $q > p$.*

Proof. The argument is an elementary generalization of the Knapp example. Let $j \in \{0,\dots,d-1\}$, $0 \le \theta \le 1$, and assume that $q > p$ satisfy $q = \frac{d-j-\theta+2}{d-j-\theta}p'$.

Let ϕ be a smooth, nonnegative function with $\operatorname{supp}\phi \subseteq Q^{\mathbb{1}}$ and $\int \phi = 1$. Set

$$\phi^{j,\ell}(\xi) = \phi(\tfrac{\xi_1}{l_1},\dots,\tfrac{\xi_j}{l_j},\tfrac{\xi_{j+1}}{l_{j+1}},\dots,\tfrac{\xi_d}{l_{j+1}}).$$

Then $|\mathcal{E}_g^\ell \phi^{j,\ell}| \gtrsim l_1\cdots l_j l_{j+1}^{d-j}$ on a rectangle with volume $(l_1\cdots l_j)^{-1}l_{j+1}^{-(d-j+2)}$. After a little arithmetic

$$\|\mathcal{E}_g^\ell\|_{L^p\to L^q}^{\mathrm{RWT}} \gtrsim \frac{l_1\cdots l_j l_{j+1}^{d-j}[(l_1\cdots l_j)^{-1}l_{j+1}^{-(d-j+2)}]^{\frac{1}{q}}}{(l_1\cdots l_j l_{j+1}^{d-j})^{\frac{1}{p}}} = (l_1\cdots l_j)^{\frac{1}{p'}-\frac{1}{q}} l_{j+1}^{(d-j)(\frac{1}{p'}-\frac{1}{q})-\frac{2}{q}}. \tag{3.2}$$

All that remains is the arithmetic verification of $(d-j)(\frac{1}{p'}-\frac{1}{q})-\frac{2}{q} = \theta(\frac{1}{p'}-\frac{1}{q})$ when $q = \frac{d-j-\theta+2}{d-j-\theta}p'$; we leave this to the reader. □

Lemma 3.3. *The lower bound $(1.7)^{\mathrm{RWT}}$ is valid in the range $q \le p$, $q > 4$.*

Proof. We assume $q > 4$ and $p \ge q$. We use the same ϕ^ℓ from the proof of Lemma 3.2, and inequality (3.2), which can be rearranged into (1.7), remains valid. □

Lemma 3.4. *The lower bound $(1.6)^{\mathrm{RWT}}$ is valid in the range $q \le p$, $q \le 4$. Moreover, $\|\mathcal{E}_g^\ell\|_{L^p\to L^q}^{\mathrm{RWT}} = \infty$ whenever $q \le \frac{2(d+1)}{d}$.*

Proof. We begin with the case $q = \frac{2(d-j-\theta+1)}{d-j-\theta}$ and $p \geq q$ for some $0 \leq j \leq d-2$ and $0 < \theta \leq 1$. We will argue by adapting the Kakeya-like argument of [1].

Let $N \gg 1$. Assume that $\frac{l_d}{l_{j+1}} > N^3$. By parabolic rescaling, we may assume that $l_{j+1} = \frac{1}{N}$. We cover (at least half of) Q^ℓ by pairwise disjoint rectangles $R \in \mathcal{R}$ congruent to $Q^{(l_1,\dots,l_j,1/N,\dots,1/N,1)}$ and further decompose each R into a disjoint union of rectangles $\kappa \in \mathcal{K}_R$, each congruent to $Q^{(l_1,\dots,l_j,1/N,\dots,1/N)}$. By the usual Knapp argument,

$$|\mathcal{E}_g^\ell \chi_\kappa(t,x)| \gtrsim |\kappa|$$

on a tube T_κ of volume $|T_\kappa| \sim |\kappa|^{-1} l_{j+1}^{-2}$. We will prove that for each R, there exist $\{(t_\kappa, x_\kappa)\}_{\kappa \in \mathcal{K}_R}$ such that

$$|\bigcup_{\kappa \in \mathcal{K}_R} T_\kappa + (t_\kappa, x_\kappa)| < o_N(1) \# \mathcal{K}_R |T_\kappa|, \tag{3.3}$$

where $o_N(1) \to 0$ as $N \to \infty$ and is otherwise independent of ℓ.

Define

$$F := \sum_{R \in \mathcal{R}} e^{-i(t_R,x_R)(g(\xi),\xi)} \sum_{\kappa \in \mathcal{K}_R} \omega_\kappa e^{-i(t_\kappa,x_\kappa)(g(\xi),\xi)} \chi_\kappa,$$

with $\{(t_R, x_R)\}_{R \in \mathcal{R}} \subseteq \mathbb{R}^{1+d}$ and $\{\omega_R\}_{\kappa \in \mathcal{K}_R} \subseteq \{\pm 1\}$ to be determined shortly. Of course, $|F| \lesssim \chi_{Q^\ell}$. For the (t_R, x_R) sufficiently widely spaced (depending on the (t_κ, x_κ)), the $L^{q,\infty}$ norms decouple:

$$\|\mathcal{E}_g^\ell F\|_{L^{q,\infty}} \gtrsim \big(\sum_{R \in \mathcal{R}} \|\mathcal{E}_g^\ell F_R\|_{L^{q,\infty}}^q \big)^{\frac{1}{q}}, \qquad F_R := \sum_{\kappa \in \mathcal{K}_R} \omega_\kappa e^{-i(t_\kappa,x_\kappa)(g(\xi),\xi)} \chi_\kappa.$$

By Khintchine's inequality, we may choose the ω_κ such that

$$\|\mathcal{E}_g^\ell F_R\|_{L^{q,\infty}} \gtrsim \| \big(\sum_{\kappa \in \mathcal{K}_R} |\mathcal{E}_g^\ell F_\kappa|^2 \big)^{\frac{1}{2}} \|_{L^{q,\infty}}, \qquad F_\kappa := e^{-i(t_\kappa,x_\kappa)(g(\xi),\xi)} \chi_\kappa.$$

Applying our pointwise lower bound on $\mathcal{E}_g^\ell \chi_\kappa$, Hölder's inequality, and (3.3),

$$\begin{aligned} \| \big(\sum_{\kappa \in \mathcal{K}_R} |\mathcal{E}_g^\ell F_\kappa|^2 \big)^{\frac{1}{2}} \|_{L^{q,\infty}} &\gtrsim |\kappa| \| \sum_{\kappa \in \mathcal{K}_R} \chi_{T_\kappa + (t_\kappa,x_\kappa)} \|_{L^{\frac{q}{2},\infty}}^{\frac{1}{2}} \\ &\gtrsim |\kappa| | \bigcup_{\kappa \in \mathcal{K}_R} T_\kappa + (t_\kappa,x_\kappa)|^{\frac{1}{q}-\frac{1}{2}} \| \sum_\kappa \chi_{T_\kappa+(t_\kappa,x_\kappa)} \|_{L^1}^{\frac{1}{2}} \gtrsim |\kappa| o_N(1)^{-1} (\# \mathcal{K}_R |T_\kappa|)^{\frac{1}{q}}. \end{aligned}$$

It takes a little arithmetic to put the pieces together:

$$\begin{aligned} \|\mathcal{E}_g^\ell\|_{L^p \to L^q}^{\mathrm{RWT}} \gtrsim \frac{\|\mathcal{E}_g^\ell F\|_{q,\infty}}{|Q^\ell|^{\frac{1}{p}}} &\gtrsim |Q^\ell|^{-\frac{1}{p}} |\kappa| o_N(1)^{-1} (|Q^\ell||\kappa|^{-2} \ell_{j+1}^{-2})^{\frac{1}{q}} \\ &\sim o_N(1)^{-1} |Q^\ell|^{\frac{1}{q}-\frac{1}{p}} (l_1 \cdots l_j l_{j+1}^\theta)^{1-\frac{2}{q}}. \end{aligned}$$

Thus the lemma is proved, modulo the Kakeya-like inequality (3.3).

For $\xi \in R$, we use Taylor's theorem to estimate

$$\begin{aligned} g(\xi) = g(\xi_R) + (\xi - \xi_R) \cdot \nabla g(\xi_R) + \tfrac{1}{2} (\xi - \xi_R)^T D^2 g(\xi_R) (\xi - \xi_R) \\ + O\big(\sum_{|\alpha|=3} |(\xi - \xi_R)^\alpha| \|\partial^\alpha g\|_{C^0(Q^\ell)} \big), \end{aligned}$$

where ξ_R denotes the center of R. We examine the error term. If $\alpha_d = 0$ and $\alpha_i \neq 0$,

$$|(\xi - \xi_R)^\alpha| \|\partial^\alpha g\|_{C^0(Q^\ell)} \leq \tfrac{1}{N^2} \ell_i \|\partial_i D^2 g\|_{C^0(Q^\ell)} < \tfrac{\varepsilon_0}{N^2},$$

since $|(\xi-\xi_R)_k| < \min\{\frac{1}{N}, l_k\}$, $1 \le k < d$. If $\alpha_d \neq 0$,

$$|(\xi-\xi_R)^\alpha| \|\partial^\alpha g\|_{C^0(Q^\ell)} \le \tfrac{1}{N^2} l_d \|\partial_d D^2 g\|_{C^0(Q^\ell)} < \tfrac{\varepsilon_0}{N^2},$$

since $|\xi-\xi_R| \le 1 < \frac{1}{N^2} l_d$. Take $\xi \in \kappa \subseteq R$ and let ξ_κ denote the center of κ. From the preceding and the definition of κ, for $\xi \in \kappa$,

$$g(\xi) = (\xi-\xi_\kappa)\cdot(\nabla g(\xi_R) + D^2 g(\xi_R)(\xi_\kappa - \xi_R)) + c(\kappa) + O(\tfrac{1}{N^2}),$$

where $c(\kappa)$ is independent of ξ.

By construction, $\xi_\kappa - \xi_R = \frac{n_\kappa}{N} e_d$, some $n_\kappa \in \{-(N-1), \dots, 0, \dots, N-1\}$. Thus for $|t| < cN^2$, c sufficiently small,

$$(t,x)(g(\xi)-c(\kappa), \xi-\xi_\kappa) = c\,O(1) + (\xi-\xi_\kappa)(t\nabla g(\xi_R) + t\tfrac{n_\kappa}{N}\partial_d \nabla g(\xi_R) + x).$$

We therefore see that

$$|\mathcal{E}_g^\ell \chi_\kappa(t,x)| \gtrsim |\kappa|$$

on

$$T_\kappa := \{(t,x) : |t| < cN^2,\ |t\partial_i g(\xi_R) + t\tfrac{n_\kappa}{N}\partial_d\partial_i g(\xi_R) + x_i| < cL_i\}$$

where $L_i = \ell_i^{-1}$, if $i \le j$, and $L_i = N$ if $i > j$.

The linear transformation

$$A_R(t,s,y) := (t, -t\nabla g(\xi_R) - s\partial_d \nabla g(\xi_R) + (y,0)), \qquad (t,s,y) \in \mathbb{R}^{1+1+(d-1)}$$

has determinant $|\det A_R| = \partial_d^2 g(\xi_R) \sim 1$ and maps

$$T_\kappa^\flat := \{(t,s,y) : |t| < cN^2,\ |s - t\tfrac{n_\kappa}{N}| < cN, |y_i| < cL_i\}$$

onto T_κ, recalling that constants are allowed to change from line to line. Using (e.g.) Fefferman's construction [7], there exist $\{(t_\kappa, s_\kappa)\}_{\kappa\in\mathcal{K}_R}$ such that

$$|\bigcup_{\kappa\in\mathcal{K}_R} T_\kappa^\flat + (t_\kappa, s_\kappa, 0)| < o_N(1) \sum_{\kappa\in\mathcal{K}_R} |T_\kappa^\flat|.$$

Finally, (t_κ, x_κ) with

$$x_\kappa := -t_\kappa \nabla g(\xi_R) - s_\kappa \partial_d \nabla g(\xi_R)$$

gives (3.3).

The proof of the necessity of $q > \frac{2(d+1)}{d}$ is similar. By rescaling and using monotonicity in ℓ of the operator norms, we may assume that $\ell = \mathbb{1}$. We cover $Q^{\mathbb{1}}$ by rectangles congruent to $Q^{(1/N,\dots,1/N,1)}$ and cover these by smaller cubes κ congruent to $Q^{(1/N,\dots,1/N)}$. Following the argument above,

$$\|\mathcal{E}_g^{\mathbb{1}}\|_{L^p\to L^q}^{\mathrm{RWT}} \gtrsim o_N(1)^{-1} N^{\frac{2(d+1)}{q} - d},$$

which is unbounded as $N \to \infty$ for any $q \le \frac{2(d+1)}{d}$. □

4. Upper bounds above rectangles

In this section, we provide details for the deduction of Theorem 1.3 from known results and prove Theorem 1.5. By Fatou's lemma, it suffices to prove these results when $\ell \in (0,\infty)^d$. By Lemma 2.2, it suffices to prove Theorem 1.3 under the hypothesis that ε_0 is sufficiently small.

We recall the convention that $l_1 \le \cdots \le l_d$.

Lemma 4.1 (Bilinear extension over rectangles)**.** *Let g be elliptic over the rectangle Q^ℓ with parameters N sufficiently large, $\varepsilon_0 > 0$ sufficiently small, and $l_d \geq 1$, and let B_1, B_2 be two balls of radius 1, separated by a distance 1 and intersecting Q^ℓ. For functions $f_j \in L^2(Q^\ell)$, $\operatorname{supp} f_j \subseteq B_j$ and $q > \frac{2(d+3)}{d+1}$,*

$$\|\mathcal{E}_g^\ell f_1 \, \mathcal{E}_g^\ell f_2\|_{\frac{q}{2}} \lesssim \|f_1\|_2 \|f_2\|_2.$$

Proof. Since g is also elliptic over any rectangle contained in Q^ℓ, by shrinking the long sidelengths if needed, we may assume that $l_1 \leq \cdots \leq l_d = 1$.

If $l_1 \sim 1$, then g is elliptic in the sense of Tao–Vargas–Vega, so Lemma 4.1 is simply the main result of [14]. In the case $l_1 \sim \cdots \sim l_{d-1} \ll l_d$, the result follows from Theorem 1.4 of [3]. (The quantity d_0 in the statement of that theorem may be taken to equal l_1, as one can see from the discussion above the statement.)

In the case of general sidelengths, the proof is a relatively simple adaptation of that in [14], to which we now turn. We will argue by applying the methods and results of [14], in combination with an induction on $\mathcal{K}^\ell$, which we define to be the number of dyadic scales at which the sidelengths lie; that is,

$$\mathcal{K}^\ell := 1 + \#\{1 \leq j < d : l_j < \tfrac{1}{2} l_{j+1}\}.$$

That the lemma holds in the base case, $\mathcal{K}^\ell = 1$, was already established above. We suppose that $1 < \mathcal{K}^\ell \leq d$ and that our lemma has been proven up to $\mathcal{K}^\ell - 1$ dyadic scales of the sidelengths.

By Lemma 2.4, we obtain the decay estimates necessary to carry out the ε-removal arguments from [15]. More precisely, $|\mathcal{E}_g^\ell \psi_j(t,x)| \lesssim \langle (x,t) \rangle^{-\frac{d}{2}}$, whenever $\psi_j \in \mathcal{S}(\mathbb{R}^d)$ is a bump adapted to B_j, $j = 1, 2$. Thus it suffices to prove that for every $\varepsilon > 0$ and $R \geq 1$,

$$\|\mathcal{E}_g^l f_1 \, \mathcal{E}_g^\ell f_2\|_{L^{\frac{d+3}{d+1}}(Q_R)} \lesssim_\varepsilon R^\varepsilon \|f_1\|_2 \|f_2\|_2, \qquad f_j \in L^2(B_j),\ j = 1, 2. \tag{4.1}$$

Here Q_R denotes the cube of sidelength R centered at 0. By assumption (and Hölder's inequality), (4.1) holds for rectangles whose sidelengths have up to $\mathcal{K}^\ell - 1$ dyadic scales.

In proving (4.1), we consider two cases, $R \lesssim l_1^{-2}$ and $R \gg l_1^{-2}$. Suppose that $R \lesssim l_1^{-2}$. Let j_0 denote the least index j such that $l_j < \frac{1}{2} l_{j+1}$. We split our coordinates as $\mathbb{R}^d = \mathbb{R}^{j_0} \times \mathbb{R}^{d-j_0} ; \xi =: (\xi', \xi'')$, and define

$$\tilde{g}(\xi) = g(0, \xi'') + \xi' \cdot \nabla_{\xi'} g(0, \xi'') + |\xi'|^2.$$

Then $\tilde{g}$ is elliptic over $Q^{\tilde{\ell}}$, with $\tilde{\ell} := (l_{j_0+1}, \ldots, l_{j_0+1}, \ell'')$. As $\mathcal{K}^{\tilde{\ell}} = \mathcal{K}^\ell - 1$, (4.1) holds for $\tilde{g}$, by assumption. On the other hand,

$$\Sigma_g := \{(g(\xi), \xi) : \xi \in Q^\ell\}$$

lies within an $O(R^{-1})$-neighborhood of

$$\Sigma_{\tilde{g}} := \{(\tilde{g}(\xi), \xi) : \xi \in Q^{\tilde{\ell}}\},$$

so we can transfer (4.1) from $\tilde{g}$ to g. Indeed, let $\phi \in \mathcal{S}(\mathbb{R})$ with $|\phi| \gtrsim 1$ on $[-1, 1]$ and $\widehat{\phi}$ supported in $[-1, 1]$. Set $\psi_R(t,x) := \phi(\frac{t}{R}) \prod_{j=1}^d \phi(\frac{x_j}{R})$. Then

$$\begin{aligned}
\|\mathcal{E}_g^\ell f_1 \, \mathcal{E}_g^\ell f_2\|_{L^{\frac{d+3}{d+1}}(Q_R)} &\lesssim \|(\psi_R \mathcal{E}_g^\ell f_1)\,(\psi_R \mathcal{E}_g^\ell f_2)\|_{L^{\frac{d+3}{d+1}}} \\
&\lesssim \iint \|(\mathcal{E}_{\tilde{g}}^{\tilde{\ell}} f_1^\tau)(\mathcal{E}_{\tilde{g}}^{\tilde{\ell}} f_2^{\tau'}) \psi_R\|_{L^{\frac{d+3}{d+1}}} \, d\tau \, d\tau' \lesssim \iint \|f_1^\tau\|_2 \|f_2^{\tau'}\|_2 \, d\tau \, d\tau',
\end{aligned}$$

where we have used Hölder's inequality and the induction hypothesis for the final inequality; here

$$f_j^\tau(\xi) := \int_{B_j} f_j(\eta)\widehat{\psi_R}(\tau + \tilde{g}(\xi) - g(\eta), \xi - \eta)\, d\eta.$$

By Young's inequality and $|\tilde{g}(\xi) - g(\eta)| \lesssim R^{-1}$, $\|f_j^\tau\|_2 \lesssim R\chi_{[-\frac{C}{R}, \frac{C}{R}]} \|f_j\|_2$. Inequality (4.1) follows.

It remains to prove (4.1) in the case $R \gg l_1^{-2}$, to which we indicate the necessary (minor) change to the approach of Tao in [14, Section 9]. Namely, we take $R = Cl_1^{-2}$ as the base case in the induction on scales argument, having already established (4.1) in this case. For $R \gg l_1^{-2}$, our surface is elliptic over balls of radius $R^{-\frac{1}{2}}$, and so the wave packets in the decomposition associated to this scale obey the expected decay estimates. The remainder of the argument proceeds precisely, line-for-line according to the scheme from [14]. In particular, the condition that the normal vectors of one surface patch are transverse to the cones defined by the normal vectors to the second surface patch along the (codimension-two) intersection of the second surface patch with a translate of the first is a direct consequence of the smallness of $D^2 h$ (i.e. the second derivative of the perturbation term) in C^0.

This closes the induction on $\mathcal{K}^\ell$, completing the proof of the lemma for all values of ℓ. □

Theorem 1.3 follows from Lemma 4.1 by the bilinear-to-linear method of Tao–Vargas–Vega; no change is needed to their arguments in the case of surfaces elliptic over rectangles. (Later on, we will use a slightly different implementation of the method of Tao–Vargas–Vega which is adapted to rectangles in two dimensions.)

We now turn to the proof of Theorem 1.5, beginning with bounds corresponding directly to lower-dimensional restriction theorems.

Lemma 4.2 ([5])**.** *Given $d \geq 2$, $1 \leq k \leq d$, and exponents $q = \frac{k+2}{k} p' > q$, validity of Conjecture 1.2 in dimension k with exponents (p, q) implies that inequality* (1.1) *holds for this exponent pair in dimension d, for any ℓ.*

Combining Lemmas 4.1 and 4.2, Conjecture 1.4 holds unconditionally for $(\frac{1}{p}, \frac{1}{q})$ lying on any of the half-open line segments $[(1,0), P_{d-j})$, $0 \leq j < d$, with P_k defined as in (1.4). In particular, inequality (1.1) holds unconditionally on the segment $q = 3p' > p$.

Proof of Lemma 4.2. Because the lemma was not stated as such in [5], we give the complete proof. When $q = \infty$, the result is a direct application of Hölder's inequality. Now we let $0 \leq j \leq d-1$, let $q = \frac{d-j+2}{d-j} p' > p > 1$, and assume that Conjecture 1.2 is valid in dimension $d-j$ for this exponent pair. Given $f \in C_0^\infty(Q^\ell)$, we take the Fourier transform of $\mathcal{E}_g^\ell$ in the x' variables to obtain,

$$\mathcal{F}_{x'} \mathcal{E}_g^\ell f(t, x'')(\xi') = \mathcal{E}_{g_{\xi'}}^{\ell''} f_{\xi'}(t, x''),$$

where we have split the coordinates as $x = (x', x'') \in \mathbb{R}^j \times \mathbb{R}^{d-j}$, and we are writing $h_{\xi'}(\xi'') = h(\xi)$, for h a function on $\mathbb{R}^d$. After making a linear transformation, which amounts to replacing $g_{\xi'}(\xi'')$ with

$$g_{\xi'}(\xi'') - g_{\xi'}(0'') - \xi'' \cdot \nabla'' g_{\xi'}(0''),$$

and applying Lemma 2.3 (which implies that the lower-dimensional extension operator is elliptic), we see that the hypothesized Conjecture 1.2 applies uniformly to $\mathcal{E}^{\ell''}_{g_{\xi'}}$, $\xi' \in Q^{\ell'}$.

Now applying Hausdorff–Young (using $q > 2$), then Minkowski's inequality (using $q' < q$), then the hypothesized validity of Conjecture 1.2, and finally Hölder's inequality (using $p > q'$),

$$\begin{aligned}\|\mathcal{E}^{\ell}_{g} f\|_q &\le \|\mathcal{E}^{\ell''}_{g_{\xi'}} f_{\xi'}\|_{L^q_{t,x}(L^{q'}_{\xi'})} \le \|\mathcal{E}^{\ell''}_{g_{\xi'}} f_{\xi'}\|_{L^{q'}_{\xi'}(L^q_{t,x''})} \lesssim \|f\|_{L^{q'}_{\xi'}(L^p_{\xi''})} \\ &\le |Q^{\ell'}|^{\frac{1}{q'}-\frac{1}{p}} \|f\|_p = (l_1 \cdots l_j)^{\frac{1}{p'}-\frac{1}{q}} \|f\|_p.\end{aligned}$$

This inequality is equivalent to (1.1) in the case $\theta = 0$ (and in the case $j-1, \theta = 1$). □

Lemma 4.3. *Let $d \ge 2$ and $1 \le j < d$. If Conjecture 1.2 is valid in dimensions $d-j-1$ and $d-j$, with exponents $q_1 = \frac{d-j+1}{d-j-1} p_1' > p_1$ and $q_0 = \frac{d-j+2}{d-j} p_0' > p_0$, respectively, then* (1.1) *holds for every exponent pair (p^{-1}, q^{-1}) on the line segment joining (p_0^{-1}, q_0^{-1}) and (p_1^{-1}, q_1^{-1}).*

Combining Lemma 4.3 with Lemma 4.2 (and the remarks after the latter), we obtain the conditional results in Theorem 1.5 for all exponents p, q satisfying $\frac{d+2}{d} p' \ge q \ge 3p'$ and $q > p$, and the unconditional results for $(\frac{1}{p}, \frac{1}{q})$ lying in any of the triangles with vertices $(1,0), P_{d-j}, P_{d-j+1}$, $1 \le j < d$.

Proof of Lemma 4.3. The argument is by the obvious interpolation. By our hypothesis and Lemma 4.2,

$$\|\mathcal{E}^{\ell}_{g}\|_{L^{p_i} \to L^{q_i}} \lesssim (l_1 \cdots l_j l_{j+1}^i)^{\frac{1}{p_i'}-\frac{1}{q_i}}, \qquad i = 0, 1.$$

Setting

$$\tfrac{1}{p_\theta} := \tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1}, \qquad \tfrac{1}{q_\theta} := \tfrac{1-\theta}{q_0} + \tfrac{\theta}{q_1},$$

as usual, interpolation gives

$$\|\mathcal{E}^{\ell}_{g}\|_{L^{p_\theta} \to L^{q_\theta}} \lesssim (l_1 \cdots l_j)^{\frac{1}{p_\theta'}-\frac{1}{q_\theta}} l_{j+1}^{\frac{\theta}{p_1'}-\frac{\theta}{q_1}}.$$

Inequality (1.1) in the claimed region thus follows once we prove that the equation

$$\nu(\tfrac{1}{p_\theta'} - \tfrac{1}{q_\theta}) = \theta(\tfrac{1}{p_1'} - \tfrac{1}{q_1}) \tag{4.2}$$

is valid for the quantity $\nu = \nu_\theta$ defined implicitly by

$$q_\theta =: \tfrac{d-j-\nu+2}{d-j-\nu} p_\theta'.$$

(In other words, ν is the 'θ' from (1.1).) Indeed, taking the convex combination of the scaling equations for (p_0, q_0) and (p_1, q_1) yields

$$\tfrac{d-j}{p_\theta'} - \tfrac{\theta}{p_1'} = \tfrac{d-j+2}{q_\theta} - \tfrac{\theta}{q_1}, \tag{4.3}$$

while the definition of ν can be rearranged as

$$\tfrac{d-j}{p_\theta'} - \tfrac{\nu}{p_\theta'} = \tfrac{d-j+2}{q_\theta} - \tfrac{\nu}{q_\theta}. \tag{4.4}$$

Subtracting (4.4) from (4.3) and rearranging yields (4.2). □

Lemma 4.4. *In the region $\frac{2(d+1)}{d} < q \le 4$, $q \le p$, validity of Conjecture 1.2 in dimensions $d-j-1$ and $d-j$ implies validity of Conjecture 1.4 on the region $\frac{2(d-j+1)}{d-j} < q \le \frac{2(d-j)}{d-j-1}$, for $0 \le j \le d-2$.*

This completes the proof of Theorem 1.5 in the range $q \leq \frac{10}{3}$.

Proof. By Hölder's inequality, for $q > p$,

$$\|\mathcal{E}_g^\ell\|_{L^p \to L^q} \leq (l_1 \cdots l_d)^{\frac{1}{q}-\frac{1}{p}} \|\mathcal{E}_g^\ell\|_{L^q \to L^q},$$

so it suffices to verify the theorem on the line $q = p$. By the Drury–Guo dimension reduction argument used in the proof of Lemma 4.2,

$$\|\mathcal{E}_g^\ell\|_{L^q \to L^q} \leq (l_1 \cdots l_j)^{\frac{1}{q'}-\frac{1}{q}} \sup_{\xi' \in Q^{\ell'}} \|\mathcal{E}_{g_{\xi'}}^{\ell''}\|_{L^q \to L^q}.$$

This reduces matters to the case $j = 0$. By parabolic rescaling (which we recall leaves Conjecture 1.4 invariant), it suffices to consider the case when $l_1 = 1$.

Write $q = \frac{2(d-\theta+1)}{d-\theta}$, with $0 < \theta \leq 1$. By hypothesis and Lemma 4.3, for all $0 < \nu < \theta$,

$$\|\mathcal{E}_g^\ell\|_{L^{p_\nu} \to L^q} \lesssim 1, \text{ where } p_\nu := (\tfrac{d-\nu}{d-\nu+2} q)'.$$

Thus by Hölder's inequality,

$$\|\mathcal{E}_g^\ell\|_{L^q \to L^q} \lesssim (l_1 \cdots l_d)^{\frac{1}{p_\nu}-\frac{1}{q}} \lesssim l_d^{(d-1)(\frac{1}{p_\nu}-\frac{1}{q})}.$$

Setting $\varepsilon := (d-1)(\frac{1}{p_\nu} - \frac{1}{q})$ and sending $\nu \nearrow \theta$ completes the proof. □

We now turn to the fully unconditional results.

Lemma 4.5. *Theorem 1.5 holds in the region $q > 4$.*

Proof. The proof is a direct deduction from Lemma 4.2 (applied in the case $q = 3p'$) via Hölder's inequality:

$$\|\mathcal{E}_g^\ell f\|_q \lesssim (l_1 \cdots l_{d-1})^{\frac{3}{q}-\frac{1}{q}} \|f\|_{(\frac{q}{3})'} \lesssim (l_1 \cdots l_{d-1})^{\frac{3}{q}-\frac{1}{q}} |Q^\ell|^{1-\frac{3}{q}-\frac{1}{p}} \|f\|_p,$$

and the right hand side equals that of (1.3). □

Lemma 4.6. *Theorem 1.5 holds in the region $q > \frac{10}{3}$.*

Proof. We begin with a series of reductions. Let $q > \frac{10}{3}$. By Lemma 4.5, we may assume that $q \leq 4$. By Lemma 4.4, we may assume that $q > p$. By the dimension-reduction argument from the proof of Lemma 4.2, we may assume that $d = 2$. By parabolic rescaling, we may assume that $1 = l_1 \leq l_2$. In summary, it remains to prove that when $d = 2$, $1 = l_1 \leq l_2$, $\frac{10}{3} < q \leq 4$, and $q > \max\{2p', p\}$, we have

$$\|\mathcal{E}_g^\ell\|_{L^p \to L^q} \lesssim 1.$$

The above conditions on p, q imply that $p > 2$. By interpolation with the (known) inequality $\|\mathcal{E}_g^\ell\|_{L^p \to L^q} \lesssim 1$ on the line $q = 3p'$, $q > p$, we may assume, in addition, that $\frac{8}{q} + \frac{2}{p} > 3$. Finally, by real interpolation, it suffices to bound $\|\mathcal{E}_g^\ell f_\Omega\|_q$ for $|f_\Omega| \sim \chi_\Omega$, Ω a finite measure set.

We adapt the argument of Tao–Vargas–Vega [16]. Making a partial Whitney decomposition,

$$Q^\ell \times Q^\ell = \bigcup_{N=0}^{\infty} \bigcup_{\tau \sim \tau' \in \mathcal{D}_N} \tau \times \tau',$$

where $\mathcal{D}_N$ denotes a finitely overlapping collection of width 1, height 2^N rectangles contained in Q^ℓ and $\tau \sim \tau'$ if $N = 0$ and $\operatorname{dist}(\tau, \tau') \lesssim 1$ or $N > 0$ and $\operatorname{dist}(\tau, \tau') \sim$

2^N. (Thus squares of sidelength 1 are allowed to equal one another.) Making a partition of unity, we can write

$$(\mathcal{E}f_\Omega)^2 = \sum_{N=0}^{\infty} \sum_{\tau\sim\tau'\in\mathcal{D}_N} \mathcal{E}f_{\Omega\cap\tau}\mathcal{E}f_{\Omega\cap\tau'},$$

where $|f_{\Omega\cap\tau}|$ is bounded by the characteristic function of $\Omega\cap\tau$.

Letting $\widetilde{\tau} := \{(g(\xi),\xi) : \xi\in\tau\}$, we see that the convex hulls of the $\widetilde{\tau}+\widetilde{\tau}'$, with $\tau\sim\tau'\in\mathcal{D}_N$, are finitely overlapping as τ,τ',N vary. Thus by Lemma 6.1 of [16],

$$\|\mathcal{E}f_\Omega\|_q^q \lesssim \sum_{N=0}^{\infty} \sum_{\tau\sim\tau'\in\mathcal{D}_N} \|\mathcal{E}f_{\Omega\cap\tau}\mathcal{E}f_{\Omega\cap\tau'}\|_{\frac{q}{2}}^{\frac{q}{2}}.$$

A volume preserving affine transformation maps the $\{(g(\xi),\xi) : \xi\in\tau\}$, $\tau\in\mathcal{D}_0$ to surfaces elliptic over $Q^{\mathbb{1}}$. Thus in the case $N=0$, we may apply Cauchy–Schwarz, the inequality $\|\mathcal{E}_g^{\mathbb{1}}\|_{L^p\to L^q}\lesssim 1$ (which follows immediately from Theorem 1.3 via Hölder's inequality), and Hölder's inequality to see that

$$\sum_{\tau\sim\tau'\in\mathcal{D}_0} \|\mathcal{E}f_{\Omega\cap\tau}\mathcal{E}f_{\Omega\cap\tau'}\|_{\frac{q}{2}}^{\frac{q}{2}} \lesssim \sum_{\tau\in\mathcal{D}_0} |\Omega\cap\tau|^{\frac{q}{p}} \lesssim |\Omega|^{\frac{q}{p}}.$$

It thus remains to bound the terms with $N>0$.

When $N>0$ and $\tau\sim\tau'$, rescaling Lemma 4.1 implies that

$$\|\mathcal{E}f_\tau \mathcal{E}f_{\tau'}\|_{\frac{q}{2}} \lesssim 2^{N(2-\frac{8}{q})}\|f_\tau\|_2\|f_{\tau'}\|_2.$$

Thus we obtain by following the argument of [16] that

$$\begin{aligned}\sum_{N=1}^{\infty} \sum_{\tau\sim\tau'\in\mathcal{D}_N} \|\mathcal{E}f_{\Omega\cap\tau}\mathcal{E}f_{\Omega\cap\tau'}\|_{\frac{q}{2}}^{\frac{q}{2}} &\lesssim \sum_{N=1}^{\infty} 2^{N(q-4)} \sum_{\tau\sim\tau'\in\mathcal{D}_N} |\Omega\cap\tau|^{\frac{q}{4}}|\Omega\cap\tau'|^{\frac{q}{4}} \\ &\lesssim \sum_{N=1}^{\infty} 2^{N(q-4)} \sum_{\tau\in\mathcal{D}_N} |\Omega\cap\tau|^{\frac{q}{2}} \lesssim \sum_{N=1}^{\infty} 2^{N(q-4)} \min\{|\Omega|,2^N\}^{\frac{q}{2}-1}|\Omega|;\end{aligned} \tag{4.5}$$

here we have used the fact that $|\Omega\cap\tau|\le \min\{|\Omega|,|\tau|\} = \min\{|\Omega|,2^N\}$, for each $\tau\in\mathcal{D}_N$.

Our proof now bifurcates into two cases, $|\Omega|\ge 1$ and $|\Omega|\le 1$. If $|\Omega|\le 1$, the right hand side of (4.5) is bounded by

$$|\Omega|^{\frac{q}{2}} \le |\Omega|^{\frac{q}{p}},$$

since $p\ge 2$. If $|\Omega|\ge 1$, the right hand side of (4.5) is bounded by

$$\sum_{N=1}^{\log|\Omega|} 2^{N(\frac{3q}{2}-5)}|\Omega| + \sum_{N=\log|\Omega|}^{\infty} 2^{N(q-4)}|\Omega|^{\frac{q}{2}} \lesssim |\Omega|^{\frac{3q}{2}-4} \le |\Omega|^{\frac{q}{p}},$$

where we have used $\frac{2}{p}+\frac{8}{q}\ge 3$ in the last inequality. □

5. The application: The proof of Theorem 1.7

We turn now to the proof of Theorem 1.7, to which we devote the entirety of this section. By the triangle inequality and the symmetry of changing the sign of any ξ_i, it suffices to bound the operator

$$\mathcal{E}_\beta f(t,x) := \int_{(0,1]^d} e^{i(t,x)(|\xi_1|^{\beta_1}+\cdots+|\xi_d|^{\beta_d},\xi)} f(\xi)\,d\xi.$$

We begin by making a dyadic decomposition,

$$(0,1]^d = \bigcup_{k\in\mathbb{N}^d} R^k, \qquad R^k := \{\xi : \xi_i \sim 2^{-2k_i},\ i = 1,\dots,d\},$$

which induces the decomposition

$$\mathcal{E}_\beta = \sum_{k\in\mathbb{N}^d} \mathcal{E}_\beta^k, \qquad \mathcal{E}_\beta^k f(t,x) := \int_{R^k} e^{i(t,x)(|\xi_1|^{\beta_1}+\cdots+|\xi_d|^{\beta_d},\xi)} f(\xi)\,d\xi.$$

For σ a permutation in S_d, we define

$$\mathcal{K}_\beta^\sigma := \{k \in \mathbb{N}^d : k_{\sigma(1)}\beta_{\sigma(1)} \geq \cdots \geq k_{\sigma(d)}\beta_{\sigma(d)}\}$$

and $\mathcal{E}_\beta^\sigma := \sum_{k\in\mathcal{K}_\beta^\sigma} \mathcal{E}_\beta^k$. By the triangle inequality, it suffices to bound each $\mathcal{E}_\beta^\sigma$. We will do this by first bounding the partial sums

$$\mathcal{E}_\beta^{\sigma,k_{\sigma(1)}} := \sum_{k'\in\mathcal{K}_\beta^\sigma(k_{\sigma(1)})} \mathcal{E}_\beta^k,$$

$$\mathcal{K}_\beta^\sigma(k_{\sigma(1)}) := \{(k'_{\sigma(j)})_{j=2}^d \in \mathbb{N}^{d-1} : k_{\sigma(1)}\beta_{\sigma(1)} \geq k'_{\sigma(2)}\beta_{\sigma(2)} \geq \cdots \geq k'_{\sigma(d)}\beta_{\sigma(d)}\}.$$

We restate Conjecture 1.4 and Proposition 1.8, as they apply to the $\mathcal{E}_\beta^k$.

Lemma 5.1. *Let $p,q \in [1,\infty]$, and assume that the conclusions of Conjecture 1.4 hold for this exponent pair. Let $k \in \mathcal{K}_\beta^\sigma$. If $q > p$ and $q = \frac{d-j-\theta+2}{d-j-\theta}p'$, for some $0 \leq j < d$ and $0 \leq \theta \leq 1$, then*

$$\|\mathcal{E}_\beta^k\|_{L^p\to L^q} \lesssim \tag{5.1}$$

$$\left[\Big(\prod_{m=1}^{j} 2^{-2k_{\sigma(m)}}\Big)\big(2^{k_{\sigma(j+1)}\beta_{\sigma(j+1)}[(1-\theta)-\frac{2}{\beta_{\sigma(j+1)}}]}\big)\Big(\prod_{m=j+2}^{d} 2^{k_{\sigma(m)}\beta_{\sigma(m)}(1-\frac{2}{\beta_{\sigma(m)}})}\Big)\right]^{\frac{1}{p'}-\frac{1}{q}}.$$

Additionally,

$$\|\mathcal{E}_\beta^k\|_{L^p\to L^q} \lesssim_\varepsilon \Big(\prod_{m=1}^{j} 2^{-2k_{\sigma(m)}(\frac{1}{p'}-\frac{1}{q})}\Big)\big(2^{k_{\sigma(j+1)}\beta_{\sigma(j+1)}[(1-\theta)(1-\frac{2}{q})-\frac{2}{\beta_{\sigma(j+1)}}(\frac{1}{p'}-\frac{1}{q})+\varepsilon]}\big)$$
$$\times \Big(\prod_{m=j+2}^{d} 2^{k_{\sigma(m)}\beta_{\sigma(m)}[(1-\frac{2}{q})-\frac{2}{\beta_{\sigma(m)}}(\frac{1}{p'}-\frac{1}{q})]}\Big)2^{-k_{\sigma(d)}\beta_{\sigma(d)}\varepsilon}. \tag{5.2}$$

for $q = \frac{2(d-j-\theta+1)}{d-j-\theta} \leq p$, $0 < \theta \leq 1$, $\varepsilon > 0$, and $j = 0,\dots,d-1$.

The result is true without the loss $2^{\epsilon(k_{\sigma(j+1)}\beta_{\sigma(j+1)}-k_{\sigma(d)}\beta_{\sigma(d)})}$ in the range $p \geq q > 4$, but, since this loss is harmless for our application, we have left it in to simplify the statement.

Proof. The lemma is proved by introducing coordinates, $\eta_i = 2^{(2-\beta_{\sigma(i)})k_{\sigma(i)}}\xi_{\sigma(i)}$, $\xi \in R^k$, producing the function

$$g_\beta^k(\eta) = \sum_{i=1}^{d} 2^{k_{\sigma(i)}(\beta_{\sigma(i)}-2)\beta_{\sigma(i)}}|\eta_i|^{\beta_{\sigma(i)}},$$

which is elliptic (to arbitrary order) over a rectangle $\{\eta_i \sim 2^{-k_{\sigma(i)}\beta_{\sigma(i)}}\}$. In the notation of Conjecture 1.4, this rectangle is congruent to Q^ℓ, where

$$\ell := \big(2^{-k_{\sigma(1)}\beta_{\sigma(1)}},\dots,2^{-k_{\sigma(d)}\beta_{\sigma(d)}}\big).$$

□

Proof of Theorem 1.7. We will give the details of the proof only in the case $\beta_i \neq 2$ for all i. In the case that $\beta_i = 2$ for some i, we may use a Galilean transformation in those coordinates ξ_i with $\beta_i = 2$ to see that

$$\|\mathcal{E}_\beta\|_{L^p\to L^q} \lesssim \|{\sum_{k\in\mathbb{N}^d}}' \mathcal{E}^k_\beta\|_{L^p\to L^q}, \tag{5.3}$$

where the prime indicates a sum taken over those $k \in \mathbb{N}^d$ with $k_i = 1$, for all i such that $\beta_i = 2$. Since the difficulty in the general case lies in summing over those k_i such that $\beta_i \neq 2$, we will give the complete details only in the case that $\beta_i \neq 2$ for all i. The change needed to handle the general case is just notational.

For most cases, we will use an interpolation lemma whose hypotheses will necessitate boundedness of $\mathcal{E}^k_\beta$ as an operator from $L^{\tilde p}$ to $L^{\tilde q}$ for $(\tilde p, \tilde q)$ lying in a neighborhood (in $\mathbb{R}^2$) of (p, q). Thus we begin by dispensing with those cases wherein (p, q) lies on the boundary of the region T_d defined in (1.8).

The case $q = \infty$ is elementary: $\mathcal{E}_\beta : L^p \to L^\infty$ for all $p \geq 1$, by Hölder's inequality. The case $p = \infty > q$ may arise under condition (ii) or (iv), but the claimed bounds for such pairs follow from the claimed bounds with finite p by Hölder's inequality, except possibly for the point $(p, q) = (\infty, 1 + \frac{1}{J_d})$, which can arise under condition (iv). The case $p < q = \frac{d+2}{d}p'$ is a little more involved. On the one hand, in the notation of Lemma 5.1, $j + \theta = 0$, so (5.1) reads

$$\|\mathcal{E}^k_\beta\|_{L^p\to L^{\frac{d+2}{d}p'}} \lesssim \prod_{j=1}^{d} 2^{-k_{\sigma(i)}(2-\beta_{\sigma(i)})(\frac{1}{p'}-\frac{1}{q})}. \tag{5.4}$$

On the other hand, from the hypothesis $\beta_i \neq 2$ and the ordering $\beta_1 \geq \cdots \geq \beta_d$, the difference $(J_n + \frac{d-n}{2}) - (J_{n-1} + \frac{d-n+1}{2}) = \frac{1}{\beta_n} - \frac{1}{2}$ is increasing in n and never zero. Therefore, the case $\frac{q}{p'} = \frac{d+2}{d} = 1 + \frac{1}{J_0+\frac{d-0}{2}}$ of Condition (i) of the theorem is only possible when $n \mapsto J_n + \frac{d-n}{2}$ has a strict minimum at 0, i.e. when $\beta_i < 2$ for all i. In this case, all of the exponents in (5.4) are negative, and it is elementary to sum.

We will specifically address the case $(p, q) = (\infty, 1 + \frac{1}{J_d})$ at the end of the proof, but for now, we may assume that the bounds in Lemma 5.1 hold for exponent pairs in a neighborhood (in $\mathbb{R}^2$) of (p, q). By the reductions above and real interpolation, it suffices to prove that the bounds expressed in the theorem hold for all pairs $q > \max\{\frac{d+2}{d}p', \frac{2(d+1)}{d}\}$ obeying, in addition, one of the conditions (i), (ii and $p < \infty$), or (iv).

We now complete the argument in the case of (i). Let $p < q = \frac{d-j-\theta+2}{d-j-\theta}p'$, for some $0 \leq j < d$ and $0 \leq \theta < 1$. We start by proving bounds for the $\mathcal{E}^{\sigma,k_{\sigma(1)}}_\beta$ for exponent pairs (p, q) lying in a neighborhood of some pair obeying (i); thus we do not yet assume that (i) holds.

Since $1 - \frac{2}{\beta_i} \neq 0$ for all i, applying (5.1) and summing a geometric series,

$$\begin{aligned}\|\mathcal{E}^{\sigma,k_{\sigma(1)}}_\beta\|_{L^p\to L^q} &\leq \sum_{k'\in\mathcal{K}^\sigma_\beta(k_{\sigma(1)})} \|\mathcal{E}^k_\beta\|_{L^p\to L^q} \\ &\lesssim {\sum_{k_{\sigma(2)}}}' \cdots {\sum_{k_{\sigma(j+1)}}}' \left[\Big(\prod_{m=1}^{j} 2^{-2k_{\sigma(m)}}\Big) 2^{k_{\sigma(j+1)}\beta_{\sigma(j+1)}[-\theta+\sum_{i=j+1}^{d}(1-\frac{2}{\beta_{\sigma(i)}})_+]} \right]^{\frac{1}{p'}-\frac{1}{q}},\end{aligned}$$

where the $'$'s indicate sums over $1 \le k_{\sigma(m+1)} \le \frac{k_{\sigma(m)}\beta_{\sigma(m)}}{\beta_{\sigma(m+1)}}$. For $0 \le l < j$, if $-\theta - l + \sum_{i=j+1-l}^{d}(1 - \frac{2}{\beta_{\sigma(i)}})_+ > 0$,

$$\sum_{k_{\sigma(j+1-l)}}{}' \left[\Big(\prod_{m=1}^{j-l} 2^{-2k_{\sigma(m)}}\Big) 2^{k_{\sigma(j+1-l)}\beta_{\sigma(j+1-l)}[-\theta-l+\sum_{i=j+1-l}^{d}(1-\frac{2}{\beta_{\sigma(i)}})_+]} \right]^{\frac{1}{p'}-\frac{1}{q}}$$
$$\lesssim \left[\Big(\prod_{m=1}^{j-l-1} 2^{-2k_{\sigma(m)}}\Big) 2^{k_{\sigma(j-l)}\beta_{\sigma(j-l)}[-\theta-l-1+\sum_{i=j-l}^{d}(1-\frac{2}{\beta_{\sigma(i)}})_+]} \right]^{\frac{1}{p'}-\frac{1}{q}}.$$

By the preceding and a simple induction argument,

$$\|\mathcal{E}_\beta^{\sigma,k_{\sigma(1)}}\|_{L^p\to L^q} \lesssim 2^{k_{\sigma(1)}\beta_{\sigma(1)}[-\theta-j+\sum_{i=1}^{d}(1-\frac{2}{\beta_i})_+](\frac{1}{p'}-\frac{1}{q})}, \tag{5.5}$$

if $-\theta-l+\sum_{i=j+1-l}^{d}(1-\frac{2}{\beta_{\sigma(i)}})_+ > 0$, for all $0 \le l < j$, while if $-\theta-l+\sum_{i=j+1-l}^{d}(1-\frac{2}{\beta_{\sigma(i)}})_+ \le 0$ for some $0 \le l < j$, which we assume to be the least such l,

$$\|\mathcal{E}_\beta^{\sigma,k_{\sigma(1)}}\|_{L^p\to L^q} \lesssim \sum_{k_{\sigma(2)}}{}' \cdots \sum_{k_{\sigma(j-l)}}{}' k_{\sigma(j-l)} \prod_{m=1}^{j-l} 2^{-2k_{\sigma(m)}(\frac{1}{p'}-\frac{1}{q})}. \tag{5.6}$$

Suppose now that condition (i) holds. We may sum the right side of (5.5) in $k_{\sigma(1)}$ whenever $\theta + j > \sum_{i=1}^{d}(1-\frac{2}{\beta_i})_+$, and the right side of (5.6) may be summed in $k_{\sigma(1)}$ unconditionally. Condition (i) for our p, q is, after a bit of arithmetic, equivalent to $\theta + j \ge \sum_{i=1}^{d}(1-\frac{2}{\beta_i})_+$. Thus it remains to consider the case

$$\theta + l < \sum_{i=j+1-l}^{d}(1-\tfrac{2}{\beta_{\sigma(i)}})_+,\ 0 \le l < j, \text{ and } \theta + j = \sum_{i=1}^{d}(1-\tfrac{2}{\beta_i})_+. \tag{5.7}$$

Let $j_0 \le j = j_1$ and $0 < \theta_0, \theta_1 < 1$ with $j_0 + \theta_0 < j + \theta < j_1 + \theta_1$ and $|j + \theta - (j_i + \theta_i)|$ sufficiently small, $i = 0, 1$. Then with $p_i' := \frac{d-j_i-\theta_i}{d-j_i-\theta_i+2}q$, $q > p_i$, and inequality (5.1) holds at (p_i, q). Furthermore, since $j_1 = j$, (5.7) implies

$$\theta_1 + l < \sum_{i=j_1+1-l}^{d}(1-\tfrac{2}{\beta_{\sigma(i)}})_+, \qquad 0 \le l < j_1,$$

provided $|\theta-\theta_1|$ is sufficiently small, and we may argue similarly for θ_0 when $j_0 = j$. If $j_0 < j$, we may assume that $j_0 = j - 1$, so

$$\theta_0+l < \theta+l+1 < \sum_{i=j+1-(l+1)}^{d}(1-\tfrac{2}{\beta_{\sigma(i)}})_+ = \sum_{i=j_0+1-l}^{d}(1-\tfrac{2}{\beta_{\sigma(i)}})_+, \qquad 0 \le l+1 < j = j_0+1.$$

Therefore by (5.5),

$$\|\mathcal{E}_\beta^{\sigma,k_{\sigma(1)}}\|_{L^{p_i}\to L^q} \lesssim 2^{\alpha_i k_{\sigma(1)}},$$

where

$$\alpha_i := \beta_{\sigma(1)}(-\theta_i - j_i + \sum_{i=1}^{d}(1-\tfrac{2}{\beta_i})_+)(\tfrac{1}{p_i'}-\tfrac{1}{q}), \qquad i = 0, 1.$$

We observe that $\alpha_0 > 0 > \alpha_1$. Thus for f_Ω comparable to the characteristic function of a measurable set Ω,

$$\begin{aligned}\| \sum_{k_{\sigma(1)}} \mathcal{E}_\beta^{\sigma,k_{\sigma(1)}} f_\Omega \|_{L^q} &\lesssim \sum_{k_{\sigma(1)}} \min\{2^{\alpha_0 k_{\sigma(1)}} |\Omega|^{\frac{1}{p_0}}, 2^{\alpha_1 k_{\sigma(1)}} |\Omega|^{\frac{1}{p_1}}\} \\ &\lesssim |\Omega|^{\frac{\frac{\alpha_0}{p_1} - \frac{\alpha_1}{p_0}}{\alpha_0 - \alpha_1}} = |\Omega|^{\frac{1}{p}}\end{aligned}$$

(some arithmetic is needed for the last equality). This implies the restricted weak type inequality, and so completes the proof of the restricted-weak type inequality in the case $q > p$. Since the Riesz diagram lacks any vertex in the region $q > p$, by real interpolation, the proof is complete for $q > p$.

We now turn to the case $q = \frac{2(d-j-\theta+1)}{d-j-\theta} \leq p$. For any integer N,

$$\max_{0\leq n\leq N} n(1-\tfrac{2}{q}) - 2J_n(\tfrac{1}{p'} - \tfrac{1}{q}) = \sum_{i=1}^{N} [(1-\tfrac{2}{q}) - \tfrac{2}{\beta_i}(\tfrac{1}{p'} - \tfrac{1}{q})]_+.$$

Thus (iv) can be rewritten as

$$q \leq p \leq \infty, \text{ and } \sum_{i=1}^{d-1} [(1-\tfrac{2}{q}) - \tfrac{2}{\beta_i}(\tfrac{1}{p'} - \tfrac{1}{q})]_+ < d(1-\tfrac{2}{q}) - \tfrac{2}{q},$$

$$\text{and } \sum_{i=1}^{d} [(1-\tfrac{2}{q}) - \tfrac{2}{\beta_i}(\tfrac{1}{p'} - \tfrac{1}{q})] = d(1-\tfrac{2}{q}) - \tfrac{2}{q}.$$

This implies that $(1-\frac{2}{q}) - \frac{2}{\beta_d}(\frac{1}{p'} - \frac{1}{q}) > 0$, which, by our assumption that $-\frac{1}{\beta_1} \geq \cdots \geq -\frac{1}{\beta_d}$, further implies (since $p' < q$) that $(1-\frac{2}{q}) - \frac{2}{\beta_i}(\frac{1}{p'} - \frac{1}{q}) > 0$, for all i. Collecting these observations, and making similar (but simpler) manipulations in the case of (ii), we may rewrite conditions (ii and $p < \infty$) and (iv) as

$$\begin{cases} \text{(ii')} \ q \leq p < \infty, \text{ and } \sum_{i=1}^{d} [(1-\frac{2}{q}) - \frac{2}{\beta_i}(\frac{1}{p'} - \frac{1}{q})]_+ < d(1-\frac{2}{q}) - \frac{2}{q}, \\ \text{(iv')} \ q \leq p \leq \infty, \text{ and } (1-\frac{2}{q}) - \frac{2}{\beta_i}(\frac{1}{p'} - \frac{1}{q}) > 0, \text{ for all } i, \text{ and } \frac{1+J_d}{q} = \frac{J_d}{p'}. \end{cases}$$

Let us now assume that we are in case (ii'). Then by (5.2),

$$\begin{aligned}\|\mathcal{E}_\beta^{\sigma,k_{\sigma(1)}}\|_{L^p\to L^q} \lesssim_\varepsilon 2^{\varepsilon k_{\sigma(1)}\beta_{\sigma(1)}} {\sum_{k_{\sigma(2)}}}' \cdots {\sum_{k_{\sigma(j+1)}}}' \big[\big(\prod_{m=1}^{j} 2^{-2k_{\sigma(m)}(\frac{1}{p'}-\frac{1}{q})} \big) \\ \times 2^{k_{\sigma(j+1)}\beta_{\sigma(j+1)}(-\theta(1-\frac{2}{q}) + \sum_{m=j+1}^{d}[(1-\frac{2}{q}) - \frac{2}{\beta_{\sigma(m)}}(\frac{1}{p'}-\frac{1}{q})]_+)} \big],\end{aligned} \tag{5.8}$$

where the extra factor in front accounts for the loss in $2^{k_{\sigma(j+1)}\beta_{\sigma(j+1)}}$ coming from (5.2). Mimicking the inductive argument from before, if

$$-(\theta+l)(1-\tfrac{2}{q}) + \sum_{m=j+1-l}^{d} [(1-\tfrac{2}{q}) - \tfrac{2}{\beta_{\sigma(m)}}(\tfrac{1}{p'} - \tfrac{1}{q})]_+ \leq 0,$$

for any $0 \leq l < j$, the right hand side of (5.8) is summable in $k_{\sigma(1)}$. Also as before, if

$$-(\theta+l)(1-\tfrac{2}{q}) + \sum_{m=j+1-l}^{d} [(1-\tfrac{2}{q}) - \tfrac{2}{\beta_{\sigma(m)}}(\tfrac{1}{p'} - \tfrac{1}{q})]_+ > 0, \qquad 0 \leq l < j,$$

then we have

$$\|\mathcal{E}_\beta^{\sigma,k_{\sigma(1)}}\|_{L^p\to L^q} \lesssim_\varepsilon 2^{k_{\sigma(1)}\beta_{\sigma(1)}(-(\theta+j)(1-\frac{2}{q})+\sum_{m=1}^d[(1-\frac{2}{q})-\frac{2}{\beta_m}(\frac{1}{p'}-\frac{1}{q})]_+ +\varepsilon)}. \tag{5.9}$$

By inserting the definition of q into (ii') and taking ε sufficiently small, the exponent on the right hand side of (5.9) is negative, and so we may again sum in $k_{\sigma(1)}$.

We will turn in a moment to (iv'), but for now we assume only that $q = \frac{2(d-j-\theta+1)}{d-j-\theta} \le p$ and

$$(1-\tfrac{2}{q}) - \tfrac{2}{\beta_i}(\tfrac{1}{p'}-\tfrac{1}{q}) > 0, \text{ for all } i.$$

Taking $\varepsilon < (1-\frac{2}{q}) - \frac{2}{\beta_i}(\frac{1}{p'}-\frac{1}{q})$, $i=1,\dots,d$, (5.2) implies

$$\begin{aligned}\|\mathcal{E}_\beta^{\sigma,k_{\sigma(1)}}\|_{L^p\to L^q} \lesssim & \sum_{k_{\sigma(2)}}{}' \cdots \sum_{k_{\sigma(j+1)}}{}' \Big(\prod_{m=1}^j 2^{-2k_{\sigma(m)}(\frac{1}{p'}-\frac{1}{q})}\Big) \\ & \times 2^{k_{\sigma(j+1)}\beta_{\sigma(j+1)}(-\theta(1-\frac{2}{q})+\sum_{m=j+1}^d[(1-\frac{2}{q})-\frac{2}{\beta_{\sigma(m)}}(\frac{1}{p'}-\frac{1}{q})])}.\end{aligned} \tag{5.10}$$

Again arguing as before, if

$$-(\theta+l)(1-\tfrac{2}{q}) + \sum_{m=j+1-l}^d [(1-\tfrac{2}{q}) - \tfrac{2}{\beta_{\sigma(m)}}(\tfrac{1}{p'}-\tfrac{1}{q})] > 0, \qquad 0 \le l < j, \tag{5.11}$$

then (5.10) implies

$$\|\mathcal{E}_\beta^{\sigma,k_{\sigma(1)}}\|_{L^p\to L^q} \lesssim 2^{k_{\sigma(1)}\beta_{\sigma(1)}(-(\theta+j)(1-\frac{2}{q})+\sum_{m=1}^d[(1-\frac{2}{q})-\frac{2}{\beta_m}(\frac{1}{p'}-\frac{1}{q})])}, \tag{5.12}$$

while if (5.11) fails for some $0 \le l < j$, the right hand side of (5.10) may be summed in $k_{\sigma(1)}$.

Now we assume that (iv') holds. The equation $\frac{1+J_d}{q} = \frac{J_d}{p'}$ can be rewritten, after a little algebra, as

$$-(\theta+j)(1-\tfrac{2}{q}) + \sum_{m=1}^d [(1-\tfrac{2}{q}) - \tfrac{2}{\beta_m}(\tfrac{1}{p'}-\tfrac{1}{q})] = 0.$$

Our analysis now breaks into three cases. If $q < p < \infty$, we choose $q < p_0 < p < p_1$ with $|p-p_i|$ sufficiently small that (5.11) holds with (p_i, q) in place of (p,q). For f_Ω comparable to a characteristic function,

$$\|\mathcal{E}_\beta^\sigma f_\Omega\|_{L^q} \lesssim \sum_{k=1}^\infty \min_{i=0,1}\big\{2^{k\beta_{\sigma(1)}(-(\theta+j)(1-\frac{2}{q})+\sum_{m=1}^d[(1-\frac{2}{q})-\frac{2}{\beta_m}(\frac{1}{p_i'}-\frac{1}{q})])}|\Omega|^{\frac{1}{p_i}}\big\} \lesssim |\Omega|^{\frac{1}{p}},$$

which implies the restricted weak type inequality that we want.

If $q = p$, Condition (iii) holds, so we must prove a strong type inequality. On the line $\frac{1+J_d}{q} = \frac{J_d}{p'}$, equation (5.11) continues to hold for some $q < p$, and condition (i) holds for $q > p$. The strong type inequality follows by real interpolation and estimates already proved.

Finally, if $(p,q) = (\infty, \frac{1}{J_d}+1)$, we select $j_0+\theta_0 < j+\theta < j_1+\theta_1$ and set $q_i := \frac{2(d-j_i-\theta_i+1)}{d-j_i-\theta_i}$. We may choose j_i, θ_i so that $|j_i+\theta_i-(j+\theta)|$ is sufficiently small for $i=0,1$; thus inequality (5.11) holds with $\theta_i, q_i, j_i, \infty$ in place of θ, q, j, p, for each $i=0,1$. Therefore (5.12) holds with these same substitutions, and we may rewrite this inequality as

$$\|\mathcal{E}_\beta^{\sigma,k_{\sigma(1)}}\|_{L^\infty\to L^{q_i}} \lesssim 2^{k_{\sigma(1)}\beta_{\sigma(1)}[(d-\theta_i-j_i-2J_d)(1-\frac{2}{q_i})-\frac{2J_d}{q_i}]}.$$

As the exponent on the right is positive for $i=0$ and negative for $i=1$, we see (after some arithmetic) that for $f_{\Omega_1}, f_{\Omega_2}$ comparable to characteristic functions,

$$\langle \mathcal{E}^\sigma_\beta f_{\Omega_1}, f_{\Omega_2}\rangle \lesssim \sum_k \min_{i=0,1}\{2^{k\beta_{\sigma(1)}[(d-\theta_i-j_i-2J_d)(1-\frac{2}{q_i})-\frac{2J_d}{q_i}]}|\Omega_2|^{\frac{1}{q_i'}}\} \lesssim |\Omega_2|^{\frac{1}{q'}},$$

which implies the claimed restricted weak type inequality in the remaining case. □

6. The negative result: Proof of Proposition 1.8

We use the notation established at the beginning of the previous section. Rescaling the lower bounds in Theorem 1.6 (analogously to the proof of Lemma 5.1) yields the following lower bounds on the $\mathcal{E}^k_\beta$.

Lemma 6.1. *Assume that $k_1\beta_1 \geq k_2\beta_2 \geq \cdots \geq k_d\beta_d$. If $q = \frac{d-j-\theta+2}{d-j-\theta}p' > p$, for some $0 \leq j < d$ and $0 \leq \theta \leq 1$, then*

$$\|\mathcal{E}^k_\beta\|^{\mathrm{RWT}}_{L^p\to L^q} \gtrsim \left[\Big(\prod_{i=1}^{j} 2^{-2k_i}\Big)\Big(2^{k_{j+1}\beta_{j+1}[(1-\theta)-\frac{2}{\beta_{j+1}}]}\Big)\Big(\prod_{i=j+2}^{d} 2^{k_i\beta_i(1-\frac{2}{\beta_i})}\Big)\right]^{\frac{1}{p'}-\frac{1}{q}}. \quad (6.1)$$

Additionally, if $q = \frac{2(d-j-\theta+1)}{d-j-\theta} \leq p$, for some $0 < \theta \leq 1$ and $j = 0, \dots, d-1$,

$$\begin{aligned}\|\mathcal{E}^k_\beta\|^{\mathrm{RWT}}_{L^p\to L^q} \gtrsim &\Big(\prod_{i=1}^{j} 2^{-2k_i(\frac{1}{p'}-\frac{1}{q})}\Big)\Big(2^{k_{j+1}\beta_{j+1}[(1-\theta)(1-\frac{2}{q})-\frac{2}{\beta_{j+1}}(\frac{1}{p'}-\frac{1}{q})]}\Big)\\ &\times\Big(\prod_{i=j+2}^{d} 2^{k_i\beta_i[(1-\frac{2}{q})-\frac{2}{\beta_i}(\frac{1}{p'}-\frac{1}{q})]}\Big)\tilde{\alpha}\big(k_{j+1}\beta_{j+1}-k_d\beta_d\big), \quad (6.2)\end{aligned}$$

for some increasing $\tilde{\alpha}$ depending on p, q, d, satisfying $\tilde{\alpha}(0) = 1$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Proof of Proposition 1.8. Let $(p,q) \in [1,\infty]^2$, and assume that none of the conditions (i-iv) hold. We may assume that $(p,q) \in T_d$ and $p \neq 1$, $q \neq \infty$. We may define j, θ, depending on (p,q), such that q can be written in one of the forms given in Lemma 6.1.

Failure of conditions (i-iv) for $(p,q) \in T_d$ leads to a choice of an integer $n \geq 1$. Namely, if $q > p$, we choose $1 \leq n \leq d$ such that $\frac{q}{p'} < 1 + \frac{1}{J_n+\frac{d-n}{2}}$. If $q \leq p$, we choose $n = d$ if $\frac{1+J_d}{q} > \frac{J_d}{p'}$, and otherwise choose $n < d$ such that $\frac{1+J_n+d-n}{q} \geq \frac{J_n}{p'} + \frac{d-n}{2}$. A bit of arithmetic shows that in any of these cases, $n \geq j+\theta$.

Let $N > \beta_1$ sufficiently large and define $\tilde{k} = (\lfloor\frac{N}{\beta_1}\rfloor, \dots, \lfloor\frac{N}{\beta_n}\rfloor, 1 \dots, 1)$.

We consider first the case $q > p$. By (6.1)

$$\begin{aligned}\|\mathcal{E}^{\tilde{k}}_\beta\|^{\mathrm{RWT}}_{L^p\to L^q} &\gtrsim \left[\Big(\prod_{i=1}^{j} 2^{-k_i\beta_i[\frac{2}{\beta_i}]}\Big)2^{k_{j+1}\beta_{j+1}[(1-\theta)-\frac{2}{\beta_{j+1}}]}\Big(\prod_{m=j+2}^{d} 2^{k_i\beta_i[1-\frac{2}{\beta_i}]}\Big)\right]^{\frac{1}{p'}-\frac{1}{q}}\\ &\approx 2^{-N[\frac{2}{\beta_1}+\cdots+\frac{2}{\beta_n}-(n-j-\theta)](\frac{1}{p'}-\frac{1}{q})}.\end{aligned}$$

Thus, by choosing N large, we can make this term arbitrarily large if

$$2J_n - (n-j-\theta) < 0,$$

which, after a little algebra, is equivalent to

$$\frac{q}{p'} < 1 + \frac{1}{J_n + \frac{d-n}{2}}.$$

Next, we suppose $q \leq p$. By (6.2)

$$\begin{aligned} \|\mathcal{E}_\beta^{\tilde{k}}\|_{L^p \to L^q}^{\mathrm{RWT}} \gtrsim & \left(\prod_{i=1}^{j} 2^{-2k_i(\frac{1}{p'}-\frac{1}{q})} \right) 2^{-2k_{j+1}(\frac{1}{p'}-\frac{1}{q}) + k_{j+1}\beta_{j+1}(1+\theta)(1-\frac{2}{q})} \\ & \times \left(\prod_{i=j+2}^{d} 2^{k_i\beta_i[(1-\frac{2}{q}) - \frac{2}{\beta_i}(\frac{1}{p'}-\frac{1}{q})]} \right) \tilde{\alpha}(\tfrac{k_{j+1}\beta_{j+1}}{k_d\beta_d}). \end{aligned} \tag{6.3}$$

Thus, for all n such that $j + \theta \leq n < d$,

$$\|\mathcal{E}_\beta^{\tilde{k}}\|_{L^p \to L^q}^{\mathrm{RWT}} \gtrsim 2^{-N[2(\frac{1}{\beta_1}+\cdots+\frac{1}{\beta_n})(\frac{1}{p'}-\frac{1}{q}) - (n-j-\theta)(1-\frac{2}{q})]} \tilde{\alpha}(N),$$

which we can make arbitrarily large, for large N, if

$$2J_n(\tfrac{1}{p'}-\tfrac{1}{q}) - (n-j-\theta)(1-\tfrac{2}{q}) \leq 0,$$

which, after a little algebra, is equivalent to

$$\tfrac{1+J_n+d-n}{q} \geq \tfrac{J_n}{p'} + \tfrac{d-n}{2}.$$

In the case where $n = d$, (6.3) becomes

$$\|\mathcal{E}_\beta^{\tilde{k}}\|_{L^p \to L^q}^{\mathrm{RWT}} \gtrsim 2^{-N[2(\frac{1}{\beta_1}+\cdots+\frac{1}{\beta_d})(\frac{1}{p'}-\frac{1}{q}) - (d-j-\theta)(1-\frac{2}{q})]},$$

which we can make arbitrarily large, for large N, if

$$\tfrac{1+J_d}{q} > \tfrac{J_d}{p'}.$$

Lastly, we consider the case where conditions (i-iii) fail, but condition (iv) holds, implying that $q < p$, $\frac{1+J_d}{q} = \frac{J_d}{p'}$, and $\beta_i > 2$ for all i.

Let $\vec{k}_m = (\frac{Mm}{\beta_1}, ..., \frac{Mm}{\beta_d})$, where $M > 100 \max(\beta_1, ..., \beta_d)$, and let $\varphi_{R^{\vec{k}_m}}$ be a Schwartz function supported on $R^{\vec{k}_m}$ and satisfying $0 \leq \varphi_{R^{\vec{k}_m}} \leq 1$ and $\int \varphi_{R^{\vec{k}_m}} \approx |R^{\vec{k}_m}| \approx 2^{-2MmJ_d}$. Then $|\mathcal{E}_\beta \varphi_{R^{\vec{k}_m}}| \gtrsim |R^{\vec{k}_m}|$ on some dual rectangle $R^*_{\vec{k}_m}$, of dimensions $2^{\frac{2Mm}{\beta_1}} \times \cdots \times 2^{\frac{2Mm}{\beta_1}} \times 2^{2Mm}$, and decays rapidly away from $R^*_{\vec{k}_m}$.

Define $f(\xi) = \sum_{m=1}^{N} e^{i\vec{x}_m \cdot \xi} 2^{-\frac{2MmJ_d}{p}} \varphi_{R^{\vec{k}_m}}$, with $\vec{x}_m$ chosen so that the dual rectangles $R^*_{\vec{k}_m}$ are widely separated. Then

$$||f||_{L^p} \approx \left(\sum_{m=1}^{N} 2^{-2MmJ_d} |R^{\vec{k}_m}| \right)^{\frac{1}{p}} = N^{\frac{1}{p}},$$

and

$$\begin{aligned} ||\mathcal{E}_\beta f||_{L^q} &\gtrsim \left(\sum_{m=1}^{N} 2^{\frac{2MmJ_dq}{p}} |R^{\vec{k}_m}|^q |R^*_{\vec{k}_m}| \right)^{\frac{1}{q}} = \left(\sum_{m=1}^{N} 2^{-2MmJ_dq(1-\frac{1}{p})} 2^{2Mm(J_d+1)} \right)^{\frac{1}{q}} \\ &= \left(\sum_{m=1}^{N} 2^{-\frac{2MmJ_dq}{p'}} 2^{\frac{2MmJ_dq}{p'}} \right)^{\frac{1}{q}} = N^{\frac{1}{q}}, \end{aligned}$$

where in the last line, we used $\frac{1+J_d}{q} = \frac{J_d}{p'}$. Therefore, $||\mathcal{E}_\beta||_{L^p \to L^q} \gtrsim N^{\frac{1}{q}-\frac{1}{p}}$, which goes to infinity as $N \to \infty$, since $q < p$. Thus, $\mathcal{E}_\beta$ fails to have a strong type bound.

Conflict of Interest Statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

□

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