RESTRICTION INEQUALITIES FOR THE HYPERBOLIC HYPERBOLOID

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Abstract. In this article we establish new inequalities, both conditional and unconditional, for the restriction problem associated to the hyperbolic, or one-sheeted, hyperboloid in three dimensions, endowed with a Lorentz-invariant measure. These inequalities are unconditional (and optimal) in the bilinear range $q > \frac{10}{3}$.

Résumé. Nous obtenons des nouvelles inégalités pour le problème de restriction de la transformée de Fourier associé à l'hyperboloïde hyperbolique (ou à une nappe), équipé avec une mesure invariante par transformations de Lorentz. Ces inégalités sont optimales et inconditionnelles dans le régime bilinéaire $q > \frac{10}{3}$.

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1. Introduction

This article concerns the boundedness of the Fourier restriction operator associated to the hyperbolic, or one-sheeted, hyperboloid in $\mathbb{R}^{1+2}$,

$$\Gamma := \{(\tau, \xi) \in \mathbb{R}^{1+2} : 1 + \tau^2 = |\xi|^2\}.$$

This surface is invariant under the Lorentz transformations

$$L_\nu : (\tau, \xi) \mapsto (\langle \nu \rangle \tau - \nu \cdot \xi, \xi^\perp + \langle \nu \rangle \xi^\parallel - \nu \tau), \qquad \nu \in \mathbb{R}^2, \tag{1.1}$$

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where $\langle \nu \rangle := \sqrt{1+|\nu|^2}$ and $\xi^\perp, \xi^\parallel$ are the perpendicular and parallel components of ξ with respect to ν. We endow the surface with the unique (up to scalar multiples) Lorentz-invariant measure, which coincides with what is known as the affine surface measure,

$$\int_\Gamma f \, \mathrm{d}\sigma = \int_{\{|\xi|>1\}} (f(-\langle\!\langle \xi \rangle\!\rangle, \xi) + f(\langle\!\langle \xi \rangle\!\rangle, \xi)) \, \tfrac{\mathrm{d}\xi}{\langle\!\langle \xi \rangle\!\rangle}, \quad \text{where} \quad \langle\!\langle \xi \rangle\!\rangle := \sqrt{|\xi|^2 - 1}, \; |\xi| \geq 1.$$

Various geometric features of this surface make it potentially interesting from the perspective of Fourier restriction/extension. Though the Gaussian curvature is nonvanishing, the principal curvatures have different signs, which presents challenges at all scales because the restriction theory for hyperbolic surfaces is much less well-developed than that for elliptic surfaces. One of the main contributions of the present article is an adaptation of the techniques of [12, 16, 20] to establish unconditional, global restriction inequalities in the bilinear range. In particular, we establish the first extension inequalities on the parabolic scaling line $q = 2p'$ beyond the Stein–Tomas range (i.e. with $p > 2$) for any negatively curved surface that is not the hyperbolic paraboloid. The above-mentioned techniques are directly applicable in the low-frequency region $\{|\xi| \lesssim 1\}$, but at high frequencies, the surface is asymptotic to the cone, presenting some additional complications. In this region, we use conic decoupling and interpolation with bilinear inequalities to prove a conditional result that boosts local restriction inequalities on the low-frequency region to global ones in a range that is non-optimal but, nevertheless, offers the possibility of improvement over that obtainable directly from bilinear restriction. Our explorations of the conic region also suggest possible future applications of some (surprisingly, still open) questions about the restriction operator associated to the cone in $1+2$ dimensions.

We turn now to statements of our main results, given in terms of the Fourier extension operator $\mathcal{E}f := \widehat{f \mathrm{d}\sigma}$, and its local version $\mathcal{E}_0 f := \mathcal{E}(\mathbb{1}_{\{|\xi|\lesssim 1\}} f)$. We say that $\mathcal{R}^*(p \to q)$ holds if there exists a universal constant C such that $\|\mathcal{E}f\|_{L^q(\mathbb{R}^3)} \leq C\|f\|_{L^p(\Gamma;\mathrm{d}\sigma)}$, for all $f \in C^\infty_{\mathrm{cpct}}(\mathbb{R}^3)$; we say that $\mathcal{R}^*_0(p \to q)$ holds when the analogous statement holds with $\mathcal{E}_0$ in place of $\mathcal{E}$.

Theorem 1.1. *For $(p, q) \neq (4, 4)$ obeying $2p' \leq q \leq 3p'$, $q \geq p$, and $q > \frac{10}{3}$, $\mathcal{R}^*(p \to q)$ holds. Moreover, for $3 < q_0 < \frac{10}{3}$, $\mathcal{R}^*_0((\frac{q_0}{2})' \to q_0)$ implies $\mathcal{R}^*(p \to q)$ for all exponent pairs obeying $q_0 < q \leq \frac{10}{3}$, $(\frac{q}{2})' \leq p \leq q$, and*

$$\frac{1}{p} > \frac{2}{5} \cdot \frac{1/q - 3/10}{1/q_0 - 3/10} + \frac{1}{10}.$$

In particular, the first author proved in [4] (see also Remark 5.2) that $\mathcal{R}^*_0((\frac{q_0}{2})' \to q_0)$ holds for $q_0 > 3.25$, and so our conditional result implies that $\mathcal{R}^*(p \to q)$ holds for $q \leq \frac{10}{3}$, $(\frac{q}{2})' \leq p \leq q$, and

$$\frac{1}{p} > \frac{52}{q} - \frac{31}{2}.$$

(The upper line segment of this region has endpoints $(\frac{1}{p}, \frac{1}{q}) = (\frac{31}{102}, \frac{31}{102})$ and $(\frac{7}{18}, \frac{11}{36})$.) Because of the loss in the range of q, we expect that our conditionality in Theorem 1.1 is not optimal. This suggests a potential application of improvements (or, rather, the techniques used to obtain those improvements) to the range of $L^p \times L^p \to L^q$ bilinear

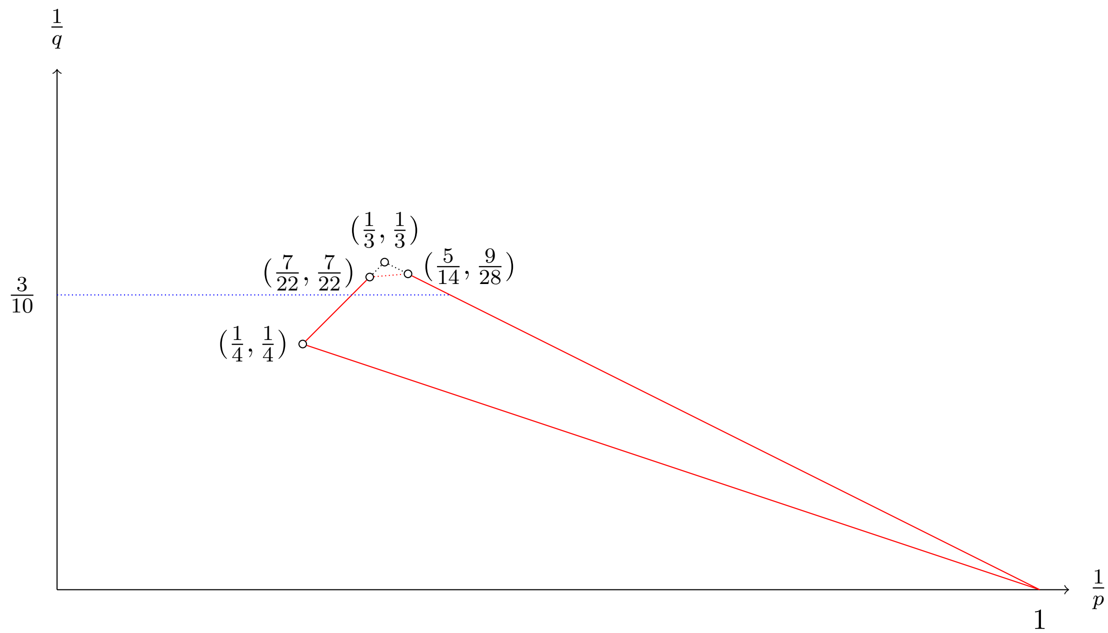

FIGURE 1. By Theorem 1.1, the full restriction conjecture for the low-frequency region would imply global restriction estimates for exponent pairs (p^{-1}, q^{-1}) within the red quadrilateral. Unconditional estimates hold in the bilinear range $q > \frac{10}{3}$.

extension inequalities for the cone in $\mathbb{R}^3$, should such inequalities become available in the future.

By contrast with Theorem 1.1, we note the following negative result.

Proposition 1.2. *For $(p, q) \in \{(3, 3), (4, 4)\}$ and for (p^{-1}, q^{-1}) lying outside of the triangle*

$$T := \{(p^{-1}, q^{-1}) : 2p' \leq q \leq 3p',\ q \geq p\},$$

$\mathcal{R}^*(p \to q)$ fails.

We note in particular that there are exponent pairs along the diagonal $q = p$ at which $\mathcal{R}^*(p \to q)$ holds; the authors had not expected this. The question of improved estimates at the endpoint $(4, 4)$ looks to be potentially interesting for further study. Indeed, the Kakeya-like example of [2] rules out even a restricted weak-type inequality at the endpoint $(3, 3)$, but we are not able to exclude the possibility that some weaker inequality (such as a restricted weak-type bound) might be valid at the endpoint $(4, 4)$, and, in fact, the analogous question for the extension operator associated to the cone also seems to be open.

Overview. We prove the negative result, Proposition 1.2, in Section 2 via familiar Knapp and Kakeya-like examples. In Section 3, we give a brief, self-contained proof of Theorem 1.1 in the classical range, $q > 4$. We also record a family of L^2-based mixed norm (Strichartz) inequalities which will be useful later on. In moving beyond the classical range, we begin with our unconditional result: that $\mathcal{R}_0^*(p \to q)$ holds in the bilinear range $q \geq 2p'$ and $q > \frac{10}{3}$. This argument will occupy Section 4, in which we establish the bilinear-to-linear deduction for this surface, and Section 5, in which we prove an $L^2 \times L^2 \to L^{q/2}$ bilinear extension theorem for appropriately separated "tiles". The geometry of the surface, namely the double ruling, plays a critical role, because it enables us to define a bi-parameter family of "tiles" that is quite close to that which naturally arises in the case of the hyperbolic paraboloid. In Section 6, we note that, via a Lorentz boost, $\mathcal{R}_0^*(p \to q)$ implies bounds on unit width "sectors" at high frequencies, and we use bilinear extension estimates (similar to those for the cone) to deduce from our unconditional result uniform bounds for the extension from dyadic frusta

$$\Gamma_N := \{(\tau, \xi) \in \Gamma : |\xi| \sim 2^N\}.$$

In Section 7, we use conic decoupling to extend the deduction in Section 6 and obtain a conditional result in a larger (but likely non-optimal) range. Finally, in Section 8, we prove that uniform estimates for the extension from dyadic frusta imply global bounds for $\mathcal{E}$.

Notation. We will use throughout the standard notation $A \lesssim B$ to mean that $A \leq CB$, for a constant C that is allowed to depend on the Lebesgue exponents in question and also, in the case of conditional results, on assumed finite bounds on the operator norms of the extension operator. The expression $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

2. The negative result: Proof of Proposition 1.2

Proof of necessity of $q \geq 2p'$. We apply the usual Knapp example. Indeed, if f_δ is a smooth bump function of radius $0 < \delta < 1$ on Γ, centered at $(0,0,1)$, then $\|f_\delta\|_p \sim \delta^{\frac{2}{p}}$, while $|\mathcal{E} f_\delta| \sim \delta^2$ on a tube of length $c\delta^{-2}$ and width $c\delta^{-1}$, yielding $\|\mathcal{E} f_\delta\|_q \gtrsim \delta^{2-\frac{4}{q}}$. In more detail, set $\phi(\zeta) := \sqrt{1 + \zeta_1^2 - \zeta_2^2}$, and note that $\Gamma \supseteq \{(\zeta_1, \zeta_2, \phi(\zeta)) : \zeta_2^2 < \frac{1}{4}(1+\zeta_1^2)\}$. Our Lorentz-invariant measure on Γ is expressed in these coordinates by $\mathrm{d}\sigma(\zeta) = \frac{\mathrm{d}\zeta}{\phi(\zeta)}$. Given sufficiently small $\delta > 0$, let

$$\mathcal{C}_\delta = \{(\zeta, \phi(\zeta)) \in \Gamma : |\zeta| \leq \delta\} \tag{2.1}$$

denote the cap on Γ of radius δ centered at $(0,0,1)$, and consider its indicator function $\mathbb{1}_\delta = \mathbb{1}_{\mathcal{C}_\delta}$. Then

$$\|\mathbb{1}_\delta\|_{L^p(\Gamma; \mathrm{d}\sigma)} = \sigma(\mathcal{C}_\delta)^{\frac{1}{p}} \sim \delta^{\frac{2}{p}}, \tag{2.2}$$

whereas

$$\mathcal{E}(\mathbb{1}_\delta)(t,x) = e^{ix_2} \int_{|\zeta| \leq \delta} e^{i(t,x_1,x_2)\cdot(\zeta_1, \zeta_2, \phi(\zeta) - 1)} \frac{\mathrm{d}\zeta}{\phi(\zeta)}.$$

Since $|\zeta_1|, |\zeta_2| \leq \delta$, it follows that $|\phi(\zeta) - 1| \leq C\delta^2$. Consequently, if $|t|, |x_1| \leq C_1^{-1}\delta^{-1}$ and $|x_2| \leq C_1^{-1}\delta^{-2}$, and C_1 is sufficiently large, then

$$\begin{aligned} |\mathcal{E}(\mathbb{1}_\delta)(t,x)| &= \left| \int_{|\zeta| \leq \delta} e^{i(t,x_1,x_2)\cdot(\zeta_1,\zeta_2,\phi(\zeta)-1)} \frac{\mathrm{d}\zeta}{\phi(\zeta)} \right| \\ &\geq \int_{|\zeta| \leq \delta} \cos(t\zeta_1 + x_1\zeta_2 + x_2(\phi(\zeta) - 1)) \frac{\mathrm{d}\zeta}{\phi(\zeta)} \geq \frac{\delta^2}{2}, \end{aligned}$$

and therefore

$$\|\mathcal{E}(\mathbb{1}_\delta)\|_{L^q(\mathbb{R}^3)}^q = \int_{\mathbb{R}^3} |\mathcal{E}(\mathbb{1}_\delta)(t,x)|^q \mathrm{d}t\mathrm{d}x \gtrsim \delta^{2q}(\delta^{-1}\delta^{-1}\delta^{-2}) = \delta^{2q-4}. \tag{2.3}$$

If $\mathcal{E} : L^p(\Gamma; \mathrm{d}\sigma) \to L^q(\mathbb{R}^3)$ defines a bounded operator, then from (2.2) and (2.3) it follows that

$$\delta^{2-\frac{4}{q}} \lesssim \|\mathcal{E}(\mathbb{1}_\delta)\|_{L^q(\mathbb{R}^3)} \lesssim \|\mathbb{1}_\delta\|_{L^p(\Gamma;\mathrm{d}\sigma)} \sim \delta^{\frac{2}{p}}.$$

Sending $\delta \searrow 0$ implies $\frac{2}{p} \le 2 - \frac{4}{q}$, as claimed. □

Proof of necessity of $q \le 3p'$. We apply a conic Knapp example. Details are analogous to the previous paragraph, so we shall be brief. For $r > 0$ sufficiently small and $\lambda > 0$ sufficiently large, consider the set

$$\Gamma_{r,\lambda} := \left\{(\tau, \xi) \in \Gamma : \tau \sim \lambda,\ |\tfrac{\xi}{|\xi|} - e_1| < r\right\}, \tag{2.4}$$

where $e_1 \in \mathbb{R}^2$ denotes the first coordinate vector. Let $f_{r,\lambda}$ be a smooth bump function adapted to $\Gamma_{r,\lambda}$. Then $\|f_{r,\lambda}\|_p \sim (\lambda r)^{\frac{1}{p}}$, and $|\mathcal{E}f_{r,\lambda}| \sim \lambda r$ on a slab of length $c(\lambda r^2)^{-1}$ (perpendicular to $\Gamma_{r,\lambda}$), width $c(\lambda r)^{-1}$ (tangent to $\Gamma_{r,\lambda}$ in the angular direction), and mini width $c\lambda^{-1}$ (tangent to $\Gamma_{r,\lambda}$ in the radial direction). Thus $\|\mathcal{E}f_{r,\lambda}\|_q \gtrsim (\lambda r)^{1-\frac{3}{q}}$. Holding r fixed and sending $\lambda \to \infty$ yields $\frac{1}{p} \ge 1 - \frac{3}{q}$, as claimed. □

Proof of necessity of $q \ge p$. We apply the standard example of summing many disjoint, highly modulated caps whose L^p and L^q norms are all comparable to one another. For $k \ge 1$, consider the functions $g_k(\tau, \xi) := e^{i(t_k,x_k)(\tau,\xi)} f_{2^{-k},2^k}(\tau, \xi)$, with the (t_k, x_k) to be determined. Here, $f_{2^{-k},2^k}$ is a smooth bump function adapted to $\Gamma_{2^{-k},2^k}$; recall (2.4). Then the previous paragraph implies that $\|g_k\|_p \sim 1$, while $\|\mathcal{E}g_k\|_q \gtrsim 1$. For (t_k, x_k) sufficiently widely separated, we then have that $\|\sum_{k=1}^N g_k\|_p \sim N^{\frac{1}{p}}$ and $\|\mathcal{E}(\sum_{k=1}^N g_k)\|_q \gtrsim N^{\frac{1}{q}}$, from which we see the necessity of $q \ge p$. □

Proof of necessity of $(p, q) \ne (3, 3), (4, 4)$. This follows by either using parabolic, resp. conic scaling, Fatou's lemma, and the fact that the corresponding inequalities do not hold for the hyperbolic paraboloid nor for the cone, or by directly using stationary phase. At the endpoint $(3, 3)$, the Kakeya-like example of [2] rules out the possibility of even a restricted weak-type inequality, but the authors have not been able to exclude the possibility that weaker inequalities might hold at the endpoint $(4, 4)$. □

3. Proof in the classical range $q > 4$

We will use the mixed-norm Strichartz inequality

$$\begin{gathered} \|\mathcal{E}f\|_{L^r_t L^s_x(\mathbb{R}^{1+2})} \lesssim \|\langle\langle\xi\rangle\rangle^{\frac{1}{r}-\frac{1}{s}} f\|_{L^2(\Gamma;\mathrm{d}\sigma)}, \\ 2 \le r, s; \quad s < \infty; \quad \frac{2}{r} + \frac{1+\theta}{s} = \frac{1+\theta}{2}, \text{ for some } \theta \in [0, 1]. \end{gathered} \tag{3.1}$$

This classical estimate follows from a straightforward modification of the methods in [11]. As (3.1) implies boundedness of $\mathcal{E}$ in the range $p = 2$, $4 \le q \le 6$, by interpolation it will suffice to restrict attention to the conic line $q = 3p'$.

Proposition 3.1. *Theorem 1.1 holds on the line $q = 3p'$.*

This result can be proved by slicing (see [10, 14]). For the convenience of the reader, we include some details.

Proof of Proposition 3.1 via slicing. By interpolation, it suffices to restrict attention to $2 \leq p < 4$ and $q = 3p'$. Since $4 < q \leq 6$, it follows that $p < q$. In polar coordinates, we have that

$$\mathcal{E}f(t,x) = \int_{|\xi|>1} f(\xi)e^{i(t,x)\cdot(\langle\langle\xi\rangle\rangle,\xi)}\frac{\mathrm{d}\xi}{\langle\langle\xi\rangle\rangle} = \int_1^\infty \left(\int_{\mathbb{S}^1} f(r\omega)e^{irx\cdot\omega}\mathrm{d}\gamma(\omega)\right) e^{it\sqrt{r^2-1}}\frac{r}{\sqrt{r^2-1}}\mathrm{d}r,$$

where γ denotes the usual arc length measure on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. Changing variables $\sqrt{r^2-1} = s$, and applying the Lorentz space version of the Hausdorff–Young inequality together with Minkowski's integral inequality,[1] yields

$$\begin{aligned}\|\mathcal{E}f\|_{L^q(\mathbb{R}^3)} &\lesssim \left\| \int_{\mathbb{S}^1} e^{i\sqrt{1+s^2}x\cdot\omega} f(\sqrt{1+s^2}\omega)\mathrm{d}\gamma(\omega)\right\|_{L^q_x L^{q',q}_s} \\ &\lesssim \left\| \int_{\mathbb{S}^1} e^{i\sqrt{1+s^2}x\cdot\omega} f(\sqrt{1+s^2}\omega)\mathrm{d}\gamma(\omega)\right\|_{L^{q',q}_s L^q_x}. \end{aligned} \tag{3.2}$$

A further change of variables $y = \sqrt{1+s^2}x$ allows us to estimate the inner norm on the right-hand side of (3.2) as follows:

$$\begin{aligned}\left\| \int_{\mathbb{S}^1} e^{i\sqrt{1+s^2}x\cdot\omega} f(\sqrt{1+s^2}\omega)\mathrm{d}\gamma(\omega)\right\|_{L^q_x} &= (1+s^2)^{-1/q}\left\| \int_{\mathbb{S}^1} e^{iy\cdot\omega} f(\sqrt{1+s^2}\omega)\mathrm{d}\gamma(\omega)\right\|_{L^q_y} \\ &\lesssim (1+s^2)^{-1/q}\|f(\sqrt{1+s^2}\cdot)\|_{L^p(\mathbb{S}^1)}\end{aligned}$$

where the latter estimate follows from the $L^p(\mathbb{S}^1;\mathrm{d}\gamma) \to L^q(\mathbb{R}^2)$ adjoint restriction inequality on the unit circle $\mathbb{S}^1$; see [21]. Going back to (3.2), we then have that

$$\begin{aligned}\|\mathcal{E}f\|_{L^q(\mathbb{R}^3)} &\lesssim \left\|(1+s^2)^{-1/q}\|f(\sqrt{1+s^2}\cdot)\|_{L^p(\mathbb{S}^1)}\right\|_{L^{q',q}_s} \\ &\lesssim \left\|(1+s^2)^{-1/q}\|f(\sqrt{1+s^2}\cdot)\|_{L^p(\mathbb{S}^1)}\right\|_{L^{q',p}_s}\end{aligned}$$

where the latter estimate holds since $p < q$. Denote

$$F(s) := (1+s^2)^{-1/q} \text{ and } G(s) := \|f(\sqrt{1+s^2}\cdot)\|_{L^p(\mathbb{S}^1)},$$

and let $\alpha > 0$ be such that $\frac{1}{q'} = \frac{1}{\alpha} + \frac{1}{p}$. Then the Lorentz space version of Hölder's inequality implies

$$\|\mathcal{E}f\|_{L^q(\mathbb{R}^3)} \lesssim \|F\|_{L^{\alpha,\infty}_s}\|G\|_{L^{p,p}_s}.$$

To check that $F \in L^{\alpha,\infty}_s$, simply note that $2\alpha = q$ since $\frac{1}{q'} = \frac{1}{\alpha} + \frac{1}{p}$ and $q = 3p'$. Finally, reverting back to the original variable $r = \sqrt{1+s^2}$, we see that

$$\|G\|^p_{L^{p,p}_s} = \|G\|^p_{L^p_s} = \int_0^\infty \|f(\sqrt{1+s^2}\cdot)\|^p_{L^p(\mathbb{S}^1)}\mathrm{d}s$$

[1] These are valid moves since $\max\{q, q'\} = q > 2$.

$$= \int_1^\infty \int_{\mathbb{S}^1} |f(r\omega)|^p \mathrm{d}\gamma(\omega) \frac{r}{\sqrt{r^2-1}} \mathrm{d}r = \|f\|^p_{L^p(\Gamma;\mathrm{d}\sigma)}.$$

This shows the boundedness of the operator $\mathcal{E} : L^p(\Gamma; \mathrm{d}\sigma) \to L^q(\mathbb{R}^3)$ whenever $2 \le p < 4$ and $q = 3p'$, as desired. □

4. Unconditional bounds at low frequencies in the bilinear range

We turn now to the heart of the article, the proof of Theorem 1.1 beyond the classical range. We begin by bounding the low-frequency extension operator $\mathcal{E}_0$ in the bilinear range ($q > \frac{10}{3}$, $p \ge (\frac{q}{2})'$), which will occupy the next two sections. The companion article, [4], bounds $\mathcal{E}_0$ in the polynomial range ($q > 3.25$, $p \ge (\frac{q}{2})'$), except on the scaling line $p = (\frac{q}{2})'$. Utilizing the results of this and the next section, we can extend the strictly local ($p > (\frac{q}{2})'$) inequalities of [4] to the scaling line. We sketch this argument in Section 5; see Remark 5.2.

It will suffice to prove extension estimates for a small region contained in a rotated version of the hyperboloid. Let

$$\Sigma := \Big\{ \Big(\sqrt{1 + \xi_1^2 - \xi_2^2}, \xi \Big) \in \mathbb{R} \times \mathbb{R}^2 : |\xi| \le \frac{1}{2} \Big\}, \tag{4.1}$$

and let U be a small neighborhood of the origin that we will choose. We will consider the subset of Σ that lies above U. Abusing notation, we define the extension operator $\mathcal{E}_0$ by

$$\mathcal{E}_0 f(t,x) := \int_U e^{i(t,x)\cdot(\sqrt{1+\xi_1^2-\xi_2^2},\xi)} f(\xi) \mathrm{d}\xi.$$

This definition of $\mathcal{E}_0$ is not quite the same as the one given in Section 1; however, the two operators obey the same range of $L^p \to L^q$ estimates, as one can see by using the triangle inequality and symmetries of the operator. We aim to prove the following result.

Theorem 4.1. *If $q > 10/3$, then $\|\mathcal{E}_0 f\|_q \lesssim \|f\|_{(q/2)'}$ for all $f \in L^{(q/2)'}(U)$.*

Our starting point will be the L^2-based bilinear theory for Σ, which we will obtain by rescaling a result of Lee [12].

4.1. **Related tiles.** To state the bilinear estimate, we must first define "related tiles", i.e. pairs of subsets of U adapted to the transversality conditions that arise in the bilinear method. Here, the geometry of the hyperbolic hyperboloid will play a distinguished role, particularly the double ruling. Given $(\tau, \xi) \in \Sigma$, the lines in Σ that contain (τ, ξ) are parametrized by the formulae

$$\ell^\pm_{(\tau,\xi)}(t) := (\tau, \xi) + t(\xi_1\tau \mp \xi_2, 1 + \xi_1^2, \xi_1\xi_2 \pm \tau),$$

and their projections to the spatial coordinates are given by

$$\ell^\pm_\xi(t) := \xi + t\Big(1 + \xi_1^2, \xi_1\xi_2 \pm \sqrt{1 + \xi_1^2 - \xi_2^2}\Big). \tag{4.2}$$

Fix an integer $n \ge 10$, and set $I := [-2^{-n}, 2^{-n})$ and $Q := I \times I$. Let $D := D(0, 1/10) \subseteq \mathbb{R}^2$ denote the open disc of radius $\frac{1}{10}$ centered at the origin. Define maps $\Phi : Q \to \mathbb{R}^2$ and $\pi^\pm : D \to \mathbb{R}$ by

$$\Phi(\zeta) := \frac{(\zeta_1\sqrt{1+\zeta_2^2} + \zeta_2\sqrt{1+\zeta_1^2}, \zeta_2 - \zeta_1)}{\sqrt{1+\zeta_1^2} + \sqrt{1+\zeta_2^2}}$$

and

$$\pi^{\pm}(\xi) := \xi_1 - \frac{\xi_2(1+\xi_1^2)}{\xi_1\xi_2 \pm \sqrt{1+\xi_1^2-\xi_2^2}}. \tag{4.3}$$

Then (possibly after increasing n) Φ is a diffeomorphism satisfying $\frac{1}{3} \leq \det \nabla\Phi \leq 1$, $\|\Phi\|_{C^1} \leq 3$, and $\Phi(Q) \subseteq D(0, 2^{-n+5})$. Indeed, Φ can be viewed as a perturbation of the rotation $\zeta \mapsto \frac{1}{2}(\zeta_1+\zeta_2, \zeta_2-\zeta_1)$. Likewise, the maps $\pi^{\pm}$ are submersions satisfying $\|\pi^{\pm}\|_{C^1} \leq 3$. We now set $U := \Phi(Q)$.

Lemma 4.2. *The maps Φ and $\pi^{\pm}$ satisfy the following geometric properties:*

(1) $\{\Phi(\zeta)\} = \ell^+_{(\zeta_1,0)} \cap \ell^-_{(\zeta_2,0)}$ *and* $(\pi^{\pm}(\xi), 0) \in \ell^{\pm}_{\xi}$ *for every* $\zeta \in Q$ *and* $\xi \in D$.

(2) *The fibers of* $\pi^{\pm}$ *are precisely the line segments* $\ell^{\pm}_{\xi} \cap D$ *with* $\xi \in D$.

(3) $\Phi^{-1} = (\pi^+ \times \pi^-)|_U$, *where* $\pi^+ \times \pi^-(\xi) := (\pi^+(\xi), \pi^-(\xi))$.

Proof. Property (1) can be verified by a straightforward calculation. It is helpful to reparametrize (4.2) so that the second coordinates of $\ell^+_{(\zeta_1,0)}(t)$ and $\ell^-_{(\zeta_2,0)}(t)$ are t and $-t$, respectively.

Property (2) is a consequence of property (1) and the following claim: If $|\eta|, |\eta'| \leq 1/2$ and $\eta' \in \ell^{\pm}_{\eta}$, then $\ell^{\pm}_{\eta'} = \ell^{\pm}_{\eta}$. Indeed, assume the claim holds, and let $\xi \in D$ and $c \in \mathbb{R}$ satisfy $\pi^{\pm}(\xi) = c$. Then $\xi' \in \ell^{\pm}_{\xi} \cap D$ implies that $\ell^{\pm}_{\xi'}$ and $\ell^{\pm}_{\xi}$ are identical and thus have the same x-intercept. Consequently, $(\pi^{\pm})^{-1}(c) \supseteq \ell^{\pm}_{\xi} \cap D$ by property (1). If $\tilde{\xi}$ is another point such that $\pi^{\pm}(\tilde{\xi}) = c$, then applying the claim twice more shows that $\ell^{\pm}_{\tilde{\xi}} = \ell^{\pm}_{(c,0)} = \ell^{\pm}_{\xi}$. Thus, $(\pi^{\pm})^{-1}(c) = \ell^{\pm}_{\xi} \cap D$. It remains to prove the claim. Define $F : \Sigma \to D(0,1/2)$ by $F(\tau, \xi) := \xi$. Then F is an invertible map such that $F^{-1}(\ell^{\pm}_{\xi} \cap D(0,1/2))$ is a line in Σ for every $|\xi| \leq 1/2$. Suppose for contradiction that $\ell^+_{\eta'} \neq \ell^+_{\eta}$. Then the lines $\ell^+_{\eta'}, \ell^-_{\eta'}, \ell^+_{\eta}$ are distinct (as one can easily check) and intersect at η', implying that $F^{-1}(\eta')$ belongs to three lines in Σ. However, no three lines in the hyperbolic hyperboloid intersect at a common point. Thus, we must have $\ell^+_{\eta'} = \ell^+_{\eta}$ and, by a similar argument, $\ell^-_{\eta'} = \ell^-_{\eta}$.

Property (3) is a consequence of properties (1) and (2). □

We also record that

$$\angle(\ell^{\pm}_{\eta}, \mathbb{R}(1, \pm 1)) \leq 10^\circ \tag{4.4}$$

for all $\eta \in D$; in particular, we always have $\angle(\ell^+_{\eta}, \ell^-_{\eta'}) \geq 70^\circ$.

For each integer $j > n$, let $\mathcal{I}_j$ denote the set of dyadic intervals of length 2^{-j} contained in I; that is,

$$\mathcal{I}_j := \{[m2^{-j}, (m+1)2^{-j}) : m \in \mathbb{Z} \cap [-2^{j-n}, 2^{j-n})\}.$$

Given $I_j, I_j' \in \mathcal{I}_j$, we write $I_j \sim I_j'$ if I_j and I_j' are non-adjacent but have adjacent dyadic parents.

Definition 4.3. A *tile* is any set of the form $\Phi(I_j \times I_k)$ with $(I_j, I_k) \in \mathcal{I}_j \times \mathcal{I}_k$ and $j, k > n$. We denote by $\Theta_{j,k}$ the set of $2^{-j} \times 2^{-k}$ tiles. Given $\theta, \theta' \in \Theta_{j,k}$, we write $\theta \sim \theta'$, and say that θ and θ' are *related*, if $\pi^+(\theta) \sim \pi^+(\theta')$ and $\pi^-(\theta) \sim \pi^-(\theta')$. (Note that if $\theta = \Phi(I_j \times I_k)$, then $\pi^+(\theta) = I_j$ and $\pi^-(\theta) = I_k$.) Finally, given $C > 0$, we define $C\theta := \Phi(C\Phi^{-1}(\theta) \cap Q)$, where $C\Phi^{-1}(\theta)$ is the C-fold dilate of the rectangle $\Phi^{-1}(\theta)$ with respect to its center.

We can now state the bilinear restriction theorem for related tiles, which we will prove in the next section.

Theorem 4.4. *Let $\theta_1, \theta_2 \in \Theta_{j,k}$ be related tiles. If $q > 10/3$, then*

$$\|\mathcal{E}_0 f \mathcal{E}_0 g\|_{q/2} \lesssim 2^{(j+k)(\frac{4}{q}-1)} \|f\|_2 \|g\|_2$$

for all $f \in L^2(\theta_1)$ and $g \in L^2(\theta_2)$.

Next, we establish several properties of the tiles θ, most of which are easy consequences of analogous properties of their rectangular counterparts $I_j \times I_k$.

Definition 4.5. Given a measurable set $\Omega \subseteq U$, we call any set of the form $\ell_\xi^\pm \cap \Omega$ with $\xi \in \Omega$ a *$\pi^\pm$-fiber* of Ω. The *length* of $\ell_\xi^\pm \cap \Omega$ is $\mathcal{H}^1(\ell_\xi^\pm \cap \Omega)$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Given an integer $K \geq 0$, we define two sets

$$\Omega(K)^\pm := \{\xi \in \Omega : 2^{-K} \leq \mathcal{H}^1(\ell_\xi^\pm \cap \Omega) < 2^{-K+1}\},$$

and say that Ω has *constant $\pi^\pm$-fiber length* 2^{-K} if $\Omega = \Omega(K)^\pm$.

Lemma 4.6. *The tiles θ satisfy the following properties:*

(1) *There exists a set $N \subset U \times U$ of measure zero such that*

$$(U \times U) \setminus N = \bigcup_{j,k>n} \bigcup_{\substack{\theta,\theta' \in \Theta_{j,k}: \\ \theta \sim \theta'}} \theta \times \theta';$$

moreover, the union is disjoint.

(2) *For each pair of related tiles $\theta \sim \theta' \in \Theta_{j,k}$, there exists a rectangle $R_{\theta,\theta'}$ such that $\theta + \theta' \subseteq R_{\theta,\theta'}$ and the collection $\{2R_{\theta,\theta'}\}_{\theta \sim \theta' \in \Theta_{j,k}}$ has bounded overlap.*

(3) *For every constant $C > 0$, the collection of dilates $C\theta$, with $\theta \in \Theta_{j,k}$, has bounded overlap.*

(4) *For every $\theta \in \Theta_{j,k}$ and constant $C > 0$, we have $|C\theta| \sim 2^{-j-k}$.*

(5) *For every $\theta \in \Theta_{j,k}$ and constant $C > 0$, the set $C\theta$ has π^+-fibers and π^--fibers of length $O(2^{-k})$ and $O(2^{-j})$, respectively.*

Proof. Property (1) is obtained by applying the diffeomorphism $(\zeta, \zeta') \mapsto (\Phi(\zeta), \Phi(\zeta'))$ to the disjoint union

$$(Q \times Q) \setminus M = \bigcup_{j,k>n} \bigcup_{\substack{I_j, I'_j \in \mathcal{I}_j : I_j \sim I'_j \\ I_k, I'_k \in \mathcal{I}_k : I_k \sim I'_k}} I_j \times I_k \times I'_j \times I'_k$$

with $M := \{(\zeta, \zeta') \in Q \times Q : \zeta_1 = \zeta'_1 \text{ or } \zeta_2 = \zeta'_2\}$.

Next, we prove property (2). We may assume that $j \geq k$. Fix $\theta \sim \theta' \in \Theta_{j,k}$ and set $c_\theta := \Phi(c_\theta^+, c_\theta^-)$, where $c_\theta^\pm$ is the center of the dyadic interval $\pi^\pm(\theta)$. We claim that there exists an $O(2^{-j}) \times O(2^{-k})$ rectangle $\tilde{R}_{\theta,\theta'}$ with center c_θ and major axis $\ell_{c_\theta}^+$ such that $\theta \cup \theta' \subseteq \tilde{R}_{\theta,\theta'}$. If $\zeta \in \Phi^{-1}(\theta \cup \theta')$, then $|\zeta_1 - c_\theta^+| \lesssim 2^{-j}$ and $|\zeta_2 - c_\theta^-| \lesssim 2^{-k}$, whence

$$\operatorname{dist}(\Phi(\zeta), \ell_{c_\theta}^+) \leq |\Phi(\zeta) - \Phi(c_\theta^+, \zeta_2)| \lesssim \|\Phi\|_{C^1} |\zeta_1 - c_\theta^+| \lesssim 2^{-j}$$

and similarly $\mathrm{dist}(\Phi(\zeta), \ell^-_{c_\theta}) \lesssim 2^{-k}$. Thus, $\theta \cup \theta'$ lies in the intersection of $O(2^{-j})$- and $O(2^{-k})$-neighborhoods of $\ell^+_{c_\theta}$ and $\ell^-_{c_\theta}$, respectively. By (4.4), this intersection is nearly a rectangle; it lies in an $O(2^{-j}) \times O(2^{-k})$ rectangle with center c_θ and major axis $\ell^+_{c_\theta}$, which we may take as $\tilde{R}_{\theta,\theta'}$. We can define the rectangle $R_{\theta,\theta'}$ as $R_{\theta,\theta'} := \tilde{R}_{\theta,\theta'} + \tilde{R}_{\theta,\theta'} = 2\tilde{R}_{\theta,\theta'} + c_\theta$. We need to show that the collection $\{2R_{\theta,\theta'}\}_{\theta\sim\theta'\in\Theta_{j,k}}$ has bounded overlap. Suppose $\theta_i \sim \theta_i' \in \Theta_{j,k}$, $i = 1, 2$, are such that $2R_{\theta_1,\theta_1'} \cap 2R_{\theta_2,\theta_2'} \neq \emptyset$. Then there exist points $\xi^i \in 2R_{\theta_i,\theta_i'} \cap (\ell^+_{c_{\theta_i}} + c_{\theta_i})$ such that $|\xi^2 - \xi^1| \lesssim 2^{-j}$. Since $c_{\theta_i} \in \ell^+_{c_{\theta_i}}$, it follows that $\xi^i/2 \in \ell^+_{c_{\theta_i}}$. Moreover, $|\xi^i - 2c_{\theta_i}| \lesssim 2^{-k}$, so if n (and therefore k) is sufficiently large, then $\xi^i/2 \in D$. Since π^+ is constant on $\ell^+_{c_{\theta_i}} \cap D$, we see that

$$|c^+_{\theta_2} - c^+_{\theta_1}| = |\pi^+(\xi^2/2) - \pi^+(\xi^1/2)| \leq \|\pi^+\|_{C^1} |\xi^2/2 - \xi^1/2| \lesssim 2^{-j}.$$

The assumption that $2R_{\theta_1,\theta_1'} \cap 2R_{\theta_2,\theta_2'} \neq \emptyset$ also implies that $|2c_{\theta_2} - 2c_{\theta_1}| \lesssim 2^{-k}$, whence

$$|c^-_{\theta_2} - c^-_{\theta_1}| \leq \|\pi^-\|_{C^1} |c_{\theta_2} - c_{\theta_1}| \lesssim 2^{-k}.$$

Since $c^+_{\theta_2}$ and $c^-_{\theta_2}$ are the centers of dyadic intervals of length 2^{-j} and 2^{-k}, respectively, we have shown the following: If θ_1 is fixed and $2R_{\theta_1,\theta_1'} \cap 2R_{\theta_2,\theta_2'} \neq \emptyset$, then θ_2 must be one of $O(1)$ possible tiles. Since any tile has at most $O(1)$ relatives, it then follows that the collection $\{2R_{\theta,\theta'}\}_{\theta\sim\theta'\in\Theta_{j,k}}$ has bounded overlap.

Property (3) follows from the dilated dyadic rectangles $C(I_j \times I_k)$ having bounded overlap.

Property (4) follows from the change of variables theorem and the fact that $|\det \nabla\Phi| \sim 1$.

Using property (3) in Lemma 4.2, one sees that $\ell^+_\xi \cap U = \Phi(\{\zeta_1\} \times I)$, where $\xi \in U$ and $\zeta := \Phi^{-1}(\xi)$. Hence, if $C > 0$, $\theta \in \Theta_{j,k}$, and $\xi \in C\theta$, then $\ell^+_\xi \cap C\theta = \Phi((\{\zeta_1\} \times I) \cap C\Phi^{-1}(\theta))$. The line segment $(\{\zeta_1\} \times I) \cap C\Phi^{-1}(\theta)$ has length at most $C2^{-k}$, and thus the bounds on $\nabla\Phi$ imply that $\ell^+_\xi \cap C\theta$ has length $O(2^{-k})$. A similar argument applies to the fibers $\ell^-_\xi \cap C\theta$, proving property (5). □

4.2. **Proof of Theorem 4.1.** Having defined related tiles and shown that they behave like dyadic rectangles, we are ready to prove Theorem 4.1. We adapt the argument of the third author in [16], with the fibers of π^+ and π^- now playing the roles of vertical and horizontal fibers. For the remainder of this section, we will assume that $\frac{10}{3} < q < 4$.

The main step is to prove a restricted strong-type inequality. We state and prove the below lemmas for characteristic functions, but the proofs are unchanged if we replace $\mathbb{1}_\Omega$ with a measurable function f_Ω with $|f_\Omega| \sim \mathbb{1}_\Omega$.

Proposition 4.7. *Let $\Omega \subseteq U$ have constant π^+-fiber length 2^{-K} for some integer $K \geq 0$. Then $\|\mathcal{E}_0 \mathbb{1}_{\Omega'}\|_q \lesssim |\Omega|^{1-\frac{2}{q}}$ for every measurable set $\Omega' \subseteq \Omega$.*

Proof. We essentially follow Vargas's argument in [20], but replace dyadic rectangles $I_j \times I_k$ with tiles θ. Fix a measurable set $\Omega' \subseteq \Omega$. Using property (1) of Lemma 4.6, the triangle inequality, almost orthogonality (combining [19, Lemma 6.1] and property (2) of Lemma 4.6), and finally Theorem 4.4, we have

$$\|\mathcal{E}_0 \mathbb{1}_{\Omega'}\|_q^2 = \Bigg\| \sum_{j,k>n} \sum_{\substack{\theta,\theta'\in\Theta_{j,k}: \\ \theta\sim\theta'}} \mathcal{E}_0(\mathbb{1}_{\Omega'\cap\theta}) \mathcal{E}_0(\mathbb{1}_{\Omega'\cap\theta'}) \Bigg\|_{q/2}$$

$$\lesssim \sum_{j,k>n} \Bigg(\sum_{\substack{\theta,\theta'\in\Theta_{j,k}: \\ \theta\sim\theta'}} \|\mathcal{E}_0(\mathbb{1}_{\Omega'\cap\theta})\mathcal{E}_0(\mathbb{1}_{\Omega'\cap\theta'})\|_{q/2}^{q/2} \Bigg)^{\frac{2}{q}}$$

$$\lesssim \sum_{j,k>n} 2^{(j+k)(\frac{4}{q}-1)} \Bigg(\sum_{\substack{\theta,\theta'\in\Theta_{j,k}: \\ \theta\sim\theta'}} |\Omega'\cap\theta|^{\frac{q}{4}}|\Omega'\cap\theta'|^{\frac{q}{4}} \Bigg)^{\frac{q}{2}}.$$

Since $10\theta \supseteq \theta'$ whenever θ and θ' are related, each tile has a bounded number of relatives, and the dilates 10θ have bounded overlap, it follows that

$$\|\mathcal{E}_0\mathbb{1}_{\Omega'}\|_q^2 \lesssim \sum_{j,k>n} 2^{(j+k)(\frac{4}{q}-1)} \Bigg(\sum_{\theta\in\Theta_{j,k}} |\Omega\cap 10\theta|^{\frac{q}{2}} \Bigg)^{\frac{2}{q}} \lesssim \sum_{j,k>n} 2^{(j+k)(\frac{4}{q}-1)} |\Omega|^{\frac{2}{q}} \max_{\theta\in\Theta_{j,k}} |\Omega\cap 10\theta|^{1-\frac{2}{q}}. \tag{4.5}$$

Let J be an integer such that $|\pi^+(\Omega)| \sim 2^{-J}$. By the coarea formula, the hypothesis on Ω, and property (5) in Lemma 4.6, we have $|\Omega| \sim 2^{-J-K}$ and

$$|\Omega\cap 10\theta| \lesssim |\pi^+(\Omega\cap 10\theta)| \sup_{\xi\in\Omega\cap 10\theta} \mathcal{H}^1(\ell_\xi^+ \cap \Omega \cap 10\theta)$$
$$\lesssim \min\{2^{-J}, 2^{-j}\} \min\{2^{-K}, 2^{-k}\},$$

for every $\theta \in \Theta_{j,k}$. Inserting this bound into (4.5) and summing the resulting (four) geometric series produces the required estimate. ☐

Proposition 4.8. *Let $\Omega \subseteq U$ have constant π^+-fiber length 2^{-K} for some integer $K \geq 0$, let J be an integer such that $|\Omega| \sim 2^{-J-K}$, and let ε be the smallest dyadic number such that $\|\mathcal{E}_0\mathbb{1}_{\Omega'}\|_q \leq \varepsilon^2|\Omega|^{1-\frac{2}{q}}$ for all measurable sets $\Omega' \subseteq \Omega$. Up to a set of measure zero, there exists a decomposition*

$$\Omega = \bigcup_{0<\delta\lesssim\varepsilon^{1/4}} \Omega_\delta,$$

where the union is taken over dyadic numbers, such that the following properties hold:

1. *$\|\mathcal{E}_0\mathbb{1}_{\Omega'}\|_q \lesssim \delta|\Omega|^{1-2/q}$ for every measurable set $\Omega' \subseteq \Omega_\delta$, and*
2. *$\Omega_\delta \subseteq \bigcup_{\theta\in\Theta_\delta}\theta$, where $\Theta_\delta \subseteq \Theta_{J,K}$ with $\#\Theta_\delta \lesssim \delta^{-C_0}$ for some constant C_0.*

Proof. The construction of the sets Ω_δ proceeds in three steps.

Step 1. Let $S := \pi^+(\Omega)$. By the coarea formula, $|S| \sim 2^{-J}$. Let ξ_1 be a Lebesgue point of S and $0 < \eta \leq \varepsilon$ a dyadic number. Define $I_\eta(\xi_1)$ to be the maximal dyadic interval I such that $\xi_1 \in I$ and

$$\frac{|I\cap S|}{|I|} \geq \eta^C, \tag{4.6}$$

where C is a constant (to be chosen); such an interval exists by the Lebesgue differentiation theorem. Since we may exclude a set of measure zero in our decomposition, we assume without loss of generality that S is equal to its set of Lebesgue points. We note that $|I_\eta(\xi_1)| \lesssim \eta^{-C}2^{-J}$. Let

$$T_\eta := \{\xi_1 \in S : |I_\eta(\xi_1)| \geq \eta^C 2^{-J}\},$$

and let $S_\varepsilon := T_\varepsilon$ and $S_\eta := T_\eta \setminus T_{2\eta}$ for $\eta < \varepsilon$. Then every point of S is contained in a unique S_η. We set $\Omega^1_\eta := \Omega \cap (\pi^+)^{-1}(S_\eta)$.

Lemma 4.9. *For every $0 < \eta \leq \varepsilon$, the set Ω^1_η is contained in a union of $O(\eta^{-3C})$ tiles in $\Theta_{J,n}$, and for every measurable set $\Omega' \subseteq \Omega^1_\eta$, we have*

$$\|\mathcal{E}_0 \mathbb{1}_{\Omega'}\|_q \lesssim \eta^2 |\Omega|^{1-\frac{2}{q}}.$$

Proof. By its definition, S_η is covered by dyadic intervals I of length $|I| \gtrsim \eta^C |S|$, in each of which S has density obeying (4.6). The density of each such I in S is

$$\frac{|I \cap S|}{|S|} = \frac{|I \cap S|}{|I|} \cdot \frac{|I|}{|S|} \gtrsim \eta^{2C}.$$

Thus, a minimal-cardinality covering of S_η by these I (which are necessarily pairwise disjoint) has size $O(\eta^{-2C})$. Additionally, each I satisfies $|I| \lesssim \eta^{-C} 2^{-J}$, and thus S_η is covered by $O(\eta^{-3C})$ intervals in $\mathcal{I}_J$. Consequently, Ω^1_η is contained in a union of $O(\eta^{-3C})$ tiles in $\Theta_{J,n}$.

We turn to the extension estimate, fixing a measurable set $\Omega' \subseteq \Omega$. By the definition of ε, we may assume that $\eta < \varepsilon$. By the same argument that yields (4.5), we have

$$\|\mathcal{E}_0 \mathbb{1}_{\Omega'}\|_q^2 \lesssim \sum_{j,k>n} 2^{(j+k)(\frac{4}{q}-1)} |\Omega|^{\frac{2}{q}} \max_{\theta \in \Theta_{j,k}} |\Omega' \cap 10\theta|^{1-\frac{2}{q}}, \tag{4.7}$$

and the coarea formula implies that

$$|\Omega' \cap 10\theta| \lesssim \min\{2^{-J}, 2^{-j}\} \min\{2^{-K}, 2^{-k}\},$$

for every $\theta \in \Theta_{j,k}$. If $|j - J| < \frac{C}{4} \log_2 \eta^{-1}$, then the definition of Ω^1_η leads to the stronger estimate

$$|\Omega' \cap 10\theta| \lesssim \eta^{\frac{3C}{4}} \min\{2^{-J}, 2^{-j}\} \min\{2^{-K}, 2^{-k}\}. \tag{4.8}$$

Indeed, fix such a j. It suffices to prove (4.8) with some $\tilde{\theta} \in \Theta_{j-4,k-4}$ in place of θ, since each θ is contained in a union of four such tiles. Let $\tilde{\theta} =: \Phi(I_{j-4} \times I_{k-4})$, so that $\pi^+(\tilde{\theta}) = I_{j-4} \in \mathcal{I}_{j-4}$. We have

$$|I_{j-4}| \geq 16 \eta^{\frac{C}{4}} 2^{-J} \geq (2\eta)^C 2^{-J}$$

for η sufficiently small (which we may assume). Suppose that $I_{j-4} \cap S_\eta \neq \emptyset$. Then there exists $\xi_1 \in I_{j-4}$ such that $\xi_1 \notin T_{2\eta}$, whence

$$|I_{2\eta}(\xi_1)| < (2\eta)^C 2^{-J} \leq |I_{j-4}|.$$

Consequently, by the maximality of $I_{2\eta}(\xi_1)$ and the fact that $2^{-j} \leq \eta^{-\frac{C}{4}} 2^{-J}$, we have

$$|I_{j-4} \cap S_\eta| \leq |I_{j-4} \cap S| \leq (2\eta)^C |I_{j-4}| = 16 (2\eta)^C 2^{-j} \lesssim \eta^{\frac{3C}{4}} \min\{2^{-J}, 2^{-j}\}.$$

Thus, by the coarea formula,

$$|\Omega' \cap \tilde{\theta}| \lesssim |I_{j-4} \cap S_\eta| \min\{2^{-K}, 2^{-k}\} \lesssim \eta^{\frac{3C}{4}} \min\{2^{-J}, 2^{-j}\} \min\{2^{-K}, 2^{-k}\},$$

as claimed. Inserting this bound into (4.7) and summing the resulting (eight) geometric series leads to the estimate $\|\mathcal{E}_0 \mathbb{1}_{\Omega'}\|_q \lesssim \eta^{C'} |\Omega|^{1-2/q}$, where C' is a constant determined by C. We can choose C so that $C' = 2$. □

Step 2. For dyadic $0 < \eta \le \varepsilon$ and $0 < \rho \lesssim \eta^{1/4}$, define

$$\Omega^2_{\eta,\rho} := \{\xi \in \Omega^1_\eta : \rho^{4D}\eta^{-3C-D}2^{-J} \le \mathcal{H}^1(\ell^-_\xi \cap \Omega^1_\eta) < (2\rho)^{4D}\eta^{-3C-D}2^{-J}\},$$

where D is a constant to be chosen. Lemma 4.9 and the near-orthogonality of ℓ^+_ξ and $\ell^-_{\xi'}$ imply that $\mathcal{H}^1(\ell^-_\xi \cap \Omega^1_\eta) \lesssim \eta^{-3C}2^{-J}$ for every $\xi \in \Omega^1_\eta$. Thus, each $\xi \in \Omega^1_\eta$ belongs to a unique $\Omega^2_{\eta,\rho}$.

Lemma 4.10. *For every $0 < \eta \le \varepsilon$ and $0 < \rho \lesssim \eta^{1/4}$, we have $\|\mathcal{E}_0 \mathbb{1}_{\Omega'}\|_q \lesssim \rho^2 |\Omega|^{1-2/q}$ for every measurable set $\Omega' \subseteq \Omega^2_{\eta,\rho}$.*

Proof. If $\rho^{4D}\eta^{-3C-D} \ge \rho^{2D}$, then by Lemma 4.9, we have

$$\|\mathcal{E}_0 \mathbb{1}_{\Omega'}\|_q \lesssim \eta^2 |\Omega|^{1-\frac{2}{q}} \le \rho^{\frac{4D}{3C+D}} |\Omega|^{1-\frac{2}{q}} \lesssim \rho^2 |\Omega|^{1-\frac{2}{q}}$$

for D chosen sufficiently large. Thus, we may assume that $\rho^{4D}\eta^{-3C-D} \le \rho^{2D}$. Given $\theta \in \Theta_{j,k}$, the set $\Omega' \cap 10\theta$ has π^+- and π^--fibers of length at most $\min\{2^{-K}, 2^{-k}\}$ and $\min\{\rho^{2D}2^{-J}, 2^{-j}\}$, respectively, and the images of $\Omega' \cap 10\theta$ under π^+ and π^- have measure at most $\min\{2^{-J}, 2^{-j}\}$ and 2^{-k}, respectively. Thus, by the coarea formula,

$$|\Omega' \cap 10\theta| \lesssim \min\{2^{-J-K}, 2^{-j-K}, 2^{-j-k}, \rho^{2D}2^{-J-k}\}. \tag{4.9}$$

We define

$$R_1 := \{(j,k) : J - D\log_2 \rho^{-1} \ge j,\ K \ge k\} \cup \{(j,k) : J \ge j,\ K - D\log_2 \rho^{-1} \ge k\},$$
$$R_2 := \{(j,k) : j \ge J + D\log_2 \rho^{-1},\ K \ge k\} \cup \{(j,k) : j \ge J,\ K - D\log_2 \rho^{-1} \ge k\},$$
$$R_3 := \{(j,k) : j \ge J + D\log_2 \rho^{-1},\ k \ge K\} \cup \{(j,k) : j \ge J,\ k \ge K + D\log_2 \rho^{-1}\},$$
$$R_4 := \{(j,k) : J + D\log_2 \rho^{-1} \ge j,\ k + D\log_2 \rho^{-1} \ge K\}.$$

Each (j,k) belongs to some R_i, so by (4.7) and (4.9), we have

$$\|\mathcal{E}_0 \mathbb{1}_{\Omega'}\|_q^2 \lesssim \sum_{(j,k)\in R_1} 2^{(j+k)(\frac{4}{q}-1)} 2^{-(J+K)(1-\frac{2}{q})} |\Omega|^{\frac{2}{q}} + \sum_{(j,k)\in R_2} 2^{(j+k)(\frac{4}{q}-1)} 2^{-(j+K)(1-\frac{2}{q})} |\Omega|^{\frac{2}{q}}$$
$$+ \sum_{(j,k)\in R_3} 2^{(j+k)(\frac{4}{q}-1)} 2^{-(j+k)(1-\frac{2}{q})} |\Omega|^{\frac{2}{q}} + \sum_{(j,k)\in R_4} 2^{(j+k)(\frac{4}{q}-1)} \rho^{2D(1-\frac{2}{q})} 2^{-(J+k)(1-\frac{2}{q})} |\Omega|^{\frac{2}{q}}.$$

Summing these geometric series leads to the bound $\|\mathcal{E}_0 \mathbb{1}_{\Omega'}\|_q \lesssim \rho^{D'} |\Omega|^{1-2/q}$, where D' is a constant determined by D; increasing D if necessary, we can make $D' \ge 2$. □

Step 3. The final step of our decomposition is the same as the first, but with π^- in place of π^+. Indeed, each $\Omega^2_{\eta,\rho}$ has π^--fibers of (essentially) constant length $\rho^{4D}\eta^{-3C-D}2^{-J}$. For $0 < \eta \le \varepsilon$ and $0 < \rho \lesssim \eta^{1/4}$, let $S_{\eta,\rho} := \pi^-(\Omega^2_{\eta,\rho})$. Let $K_{\eta,\rho}$ be an integer such that $|S_{\eta,\rho}| \sim 2^{-K_{\eta,\rho}}$. Let ξ_1 be a Lebesgue point of $S_{\eta,\rho}$ and $0 < \delta \le \rho$ a dyadic number. Define $I_{\eta,\rho,\delta}(\xi_1)$ to be the maximal dyadic interval I such that $\xi_1 \in I$ and $|I \cap S_{\eta,\rho}| \ge \delta^C |I|$; as before, the Lebesgue differentiation theorem guarantees such an interval exists. Let

$T_{\eta,\rho,\delta} := \{\xi_1 \in S : |I_{\eta,\rho,\delta}(\xi_1)| \geq \delta^C 2^{-K_{\eta,\rho}}\}$, and set $S_{\eta,\rho,\rho} := T_{\eta,\rho,\rho}$ and $S_{\eta,\rho,\delta} := T_{\eta,\rho,\delta} \backslash T_{\eta,\rho,2\delta}$ for $\delta < \rho$. Finally, we let $\Omega^3_{\eta,\rho,\delta} := \Omega^2_{\eta,\rho} \cap (\pi^-)^{-1}(S_{\eta,\rho,\delta})$.

Lemma 4.11. *For every $0 < \eta \leq \varepsilon$ and $0 < \delta \leq \rho \lesssim \eta^{1/4}$, the set $\Omega^3_{\eta,\rho,\delta}$ is contained in a union of $O(\delta^{-15C-4D})$ tiles in $\Theta_{J,K}$, and for every measurable set $\Omega' \subseteq \Omega^3_{\eta,\rho,\delta}$, we have*

$$\|\mathcal{E}_0 \mathbb{1}_{\Omega'}\|_q \lesssim \delta^2 |\Omega|^{1-\frac{2}{q}}.$$

Proof. By an argument similar to the proof of Lemma 4.9, one can show that

$$\|\mathcal{E}_0 \mathbb{1}_{\Omega'}\|_q \lesssim \delta^2 |\Omega^2_{\eta,\rho}|^{1-2/q} \leq \delta^2 |\Omega|^{1-2/q}.$$

Likewise, one sees that $\Omega_{\eta,\rho,\delta}$ is contained in a union of $O(\delta^{-3C})$ tiles in $\Theta_{n,K_{\eta,\rho}}$. Since $\Omega^2_{\eta,\rho}$ has π^--fibers of length at least $\rho^{4D}\eta^{-3C-D}2^{-J}$ and volume at most 2^{-J-K}, we must have $2^{-K_{\eta,\rho}} \lesssim \rho^{-4D}2^{-K}$. Thus, $\Omega^3_{\eta,\rho,\delta}$ is contained in $O(\delta^{-3C-4D})$ tiles in $\Theta_{n,K}$. By Lemma 4.9 and the fact that $\rho \lesssim \eta^{1/4}$, we also know that $\Omega^3_{\eta,\rho,\delta}$ is contained in $O(\delta^{-12C})$ tiles in $\Theta_{J,n}$. The intersection of a tile in $\Theta_{J,n}$ and a tile in $\Theta_{n,K}$ is a tile in $\Theta_{J,K}$. □

We are now ready to complete the proof of Proposition 4.8. We set

$$\Omega_\delta := \bigcup_{\delta \leq \rho \lesssim \varepsilon^{1/4}} \bigcup_{\rho^4 \lesssim \eta \leq \varepsilon} \Omega^3_{\eta,\rho,\delta},$$

so that $\Omega = \bigcup_{0<\delta\lesssim\varepsilon^{1/4}} \Omega_\delta$. Since for fixed δ there are $O((\log\delta^{-1})^2)$ sets $\Omega^3_{\eta,\rho,\delta}$, properties (1) and (2) in the proposition follow from Lemma 4.11. □

Now, fix some $\Omega \subseteq U$, and for each K, let $J(K)$ be an integer such that $|\Omega(K)^+| \sim 2^{-J(K)-K}$. For each dyadic number ε, let $\mathcal{K}(\varepsilon)$ denote the collection of all integers $K \geq 0$ for which ε is the smallest dyadic number satisfying $\|\mathcal{E}_0\mathbb{1}_{\Omega'}\|_q \lesssim \varepsilon^2|\Omega(K)^+|^{1-2/q}$ for every measurable set $\Omega' \subseteq \Omega(K)^+$. For each $K \in \mathcal{K}(\varepsilon)$, Proposition 4.8 produces a decomposition $\Omega(K)^+ = \bigcup_{0<\delta\lesssim\varepsilon^{1/4}} \Omega(K)^+_\delta$ such that for each δ, we have $\Omega(K)^+_\delta \subseteq \bigcup_{\theta\in\Theta(K)_\delta}\theta$ for some $\Theta(K)_\delta \subseteq \Theta_{J(K),K}$ with $\#\Theta(K)_\delta \lesssim \delta^{-C_0}$.

Lemma 4.12. *For every $0 < \delta \lesssim \varepsilon^{1/4}$, we have*

$$\Big\| \sum_{K\in\mathcal{K}(\varepsilon)} \mathcal{E}_0 \mathbb{1}_{\Omega(K)^+_\delta} \Big\|_q^q \lesssim (\log\delta^{-1})^q \sum_{K\in\mathcal{K}(\varepsilon)} \|\mathcal{E}_0\mathbb{1}_{\Omega(K)^+_\delta}\|_q^q + \delta|\Omega|^{q-2}.$$

Proof. Let A be a constant to be chosen later, and divide $\mathcal{K}(\varepsilon)$ into $O(\log\delta^{-1})$ subsets $\mathcal{K}$ such that each is $A\log\delta^{-1}$-separated. It suffices to prove that

$$\Big\| \sum_{K\in\mathcal{K}} \mathcal{E}_0 \mathbb{1}_{\Omega(K)^+_\delta} \Big\|_q^q \lesssim \sum_{K\in\mathcal{K}} \|\mathcal{E}_0\mathbb{1}_{\Omega(K)^+_\delta}\|_q^q + \delta^2|\Omega|^{q-2}$$

for each $\mathcal{K}$. We recall that $q < 4$. Thus,

$$\Big\| \sum_{K\in\mathcal{K}} \mathcal{E}_0 \mathbb{1}_{\Omega(K)^+_\delta} \Big\|_q^q = \int \Big| \sum_{\boldsymbol{K}\in\mathcal{K}^4} \prod_{i=1}^4 \mathcal{E}_0 \mathbb{1}_{\Omega(K_i)^+_\delta} \Big|^{\frac{q}{4}}$$

$$\lesssim \sum_{K\in\mathcal{K}} \|\mathcal{E}_0 \mathbb{1}_{\Omega(K)_\delta^+}\|_q^q + \sum_{\boldsymbol{K}\in\mathcal{K}^4\setminus D(\mathcal{K}^4)} \Big\| \prod_{i=1}^4 \mathcal{E}_0 \mathbb{1}_{\Omega(K_i)_\delta^+} \Big\|_{\frac{q}{4}}^{\frac{q}{4}}, \tag{4.10}$$

where $D(\mathcal{K}^4) := \{\boldsymbol{K} \in \mathcal{K}^4 : K_1 = K_2 = K_3 = K_4\}$. To control the latter sum, we have the following lemma.

Lemma 4.13. *For all $K, K' \in \mathcal{K}$, we have*

$$\|\mathcal{E}_0(\mathbb{1}_{\Omega(K)_\delta^+})\mathcal{E}_0(\mathbb{1}_{\Omega(K')_\delta^+})\|_{q/2} \lesssim 2^{-c_0|K-K'|} \max\{|\Omega(K)^+|, |\Omega(K')^+|\}^{2-\frac{4}{q}}$$

for some constant $c_0 > 0$.

Proof. Set $\tilde\Omega := \Omega(K)_\delta^+$, $\tilde\Omega' := \Omega(K')_\delta^+$, $J := J(K)$, and $J' := J(K')$. By the Cauchy–Schwarz inequality and Proposition 4.7, we have

$$\|\mathcal{E}_0(\mathbb{1}_{\tilde\Omega})\mathcal{E}_0(\mathbb{1}_{\tilde\Omega'})\|_{q/2} \lesssim |\tilde\Omega|^{1-\frac{2}{q}}|\tilde\Omega'|^{1-\frac{2}{q}}.$$

If either (i) $K = K'$, (ii) $J = J'$, (iii) $J < J'$ and $K < K'$, or (iv) $J > J'$ and $K > K'$, then

$$|\tilde\Omega|^{1-\frac{2}{q}}|\tilde\Omega'|^{1-\frac{2}{q}} \lesssim 2^{-(1-\frac{2}{q})|K-K'|} \max\{|\tilde\Omega|, |\tilde\Omega'|\}^{2-\frac{4}{q}}.$$

Thus, by symmetry, we may assume that $K < K'$ and $J > J'$. By the bound $\#(\Theta(K)_\delta \times \Theta(K')_\delta) \lesssim \delta^{-2C_0}$ and the separation condition on $\mathcal{K}$ (with A sufficiently large), it suffices to prove that

$$\|\mathcal{E}_0(\mathbb{1}_{\tilde\Omega\cap\theta})\mathcal{E}_0(\mathbb{1}_{\tilde\Omega'\cap\sigma})\|_{q/2} \lesssim 2^{-c|K-K'|} \max\{|\tilde\Omega|, |\tilde\Omega'|\}^{2-\frac{4}{q}}$$

for all $\theta \in \Theta(K)_\delta$, $\sigma \in \Theta(K')_\delta$, and some constant $c > 0$.

Fix two such tiles θ, σ, and set $\tau := \Phi^{-1}(\theta)$ and $\kappa := \Phi^{-1}(\sigma)$. Thus, τ and κ are dyadic rectangles of dimensions $2^{-J} \times 2^{-K}$ and $2^{-J'} \times 2^{-K'}$, respectively. We note that our assumptions on J, J', K, K' imply that τ is taller than κ and κ wider than τ. By translation, we may assume that the ζ_2- and ζ_1-axes intersect the centers of τ and κ, respectively. Define

$$\tau_k := \begin{cases} \tau \cap \{\zeta : |\zeta_2| \sim 2^{-k}\}, & k < K', \\ \tau \cap \{\zeta : |\zeta_2| \lesssim 2^{-K'}\}, & k = K' \end{cases} \quad \text{and} \quad \kappa_j := \begin{cases} \kappa \cap \{\zeta : |\zeta_1| \sim 2^{-j}\}, & j < J, \\ \kappa \cap \{\zeta : |\zeta_1| \lesssim 2^{-J}\}, & j = J \end{cases},$$

as well as $\theta_k := \Phi(\tau_k)$ and $\sigma_j := \Phi(\kappa_j)$. Thus,

$$\theta = \bigcup_{k=0}^{K'} \theta_k \quad \text{and} \quad \sigma = \bigcup_{j=0}^{J} \sigma_j,$$

so that by the triangle inequality,

$$\|\mathcal{E}_0(\mathbb{1}_{\tilde\Omega\cap\theta})\mathcal{E}_0(\mathbb{1}_{\tilde\Omega'\cap\sigma})\|_{q/2} \le \sum_{k=0}^{K'}\sum_{j=0}^{J} \|\mathcal{E}_0(\mathbb{1}_{\tilde\Omega\cap\theta_k})\mathcal{E}_0(\mathbb{1}_{\tilde\Omega'\cap\sigma_j})\|_{q/2}.$$

We first sum the terms with $k = K'$. By the Cauchy–Schwarz inequality, Proposition 4.7, and the fact that $|\det \nabla\Phi| \sim 1$, we have

$$\sum_{j=0}^{J} \|\mathcal{E}_0(\mathbb{1}_{\tilde\Omega\cap\theta_{K'}})\mathcal{E}_0(\mathbb{1}_{\tilde\Omega'\cap\sigma_j})\|_{q/2} \lesssim \sum_{j=0}^{J} |\theta_{K'}|^{1-\frac{2}{q}}|\sigma_j|^{1-\frac{2}{q}} \sim \sum_{j=0}^{J} |\tau_{K'}|^{1-\frac{2}{q}}|\kappa_j|^{1-\frac{2}{q}}.$$

Since κ has width $2^{-J'}$, there are at most two nonempty κ_j with $j \le J'$. This fact and the bound

$$|\kappa_j| \le \min\{2^{-(j-J')}, 1\}|\kappa| \tag{4.11}$$

imply that $\sum_{j=0}^{J} |\kappa_j|^{1-2/q} \lesssim |\kappa|^{1-2/q}$. Since $|\tau_{K'}| \lesssim 2^{-(K'-K)}|\tau|$, $|\tau| \sim |\tilde\Omega|$, and $|\kappa| \sim |\tilde\Omega'|$, we altogether have

$$\sum_{j=0}^{J} \|\mathcal{E}_0(\mathbb{1}_{\tilde\Omega\cap\theta_{K'}})\mathcal{E}_0(\mathbb{1}_{\tilde\Omega'\cap\sigma_j})\|_{q/2} \lesssim 2^{-(K'-K)(1-\frac{2}{q})}|\tilde\Omega|^{1-\frac{2}{q}}|\tilde\Omega'|^{1-\frac{2}{q}},$$

which is acceptable. A similar argument shows that

$$\begin{aligned}\sum_{k=0}^{K'} \|\mathcal{E}_0(\mathbb{1}_{\tilde\Omega\cap\theta_k})\mathcal{E}_0(\mathbb{1}_{\tilde\Omega'\cap\sigma_J})\|_{q/2} &\lesssim 2^{-(J-J')(1-\frac{2}{q})}|\tilde\Omega|^{1-\frac{2}{q}}|\tilde\Omega'|^{1-\frac{2}{q}}\\ &\sim 2^{-(K'-K)(1-\frac{2}{q})}|\tilde\Omega|^{2-\frac{4}{q}}.\end{aligned}$$

We now consider the terms with $k < K'$ and $j < J$. In this case, τ_k is contained in a union of four dyadic rectangles of dimensions $2^{-J} \times 2^{-\max\{K,k\}}$, and κ_j is contained in a union of four dyadic rectangles of dimensions $2^{-\max\{J',j\}} \times 2^{-K'}$. Moreover, these rectangles are separated by a distance of (at least) 2^{-k} and 2^{-j} in the vertical and horizontal directions, respectively. Thus, we can apply Theorem 4.4 to θ_k and σ_j to get

$$\|\mathcal{E}_0(\mathbb{1}_{\tilde\Omega\cap\theta_k})\mathcal{E}_0(\mathbb{1}_{\tilde\Omega'\cap\sigma_j})\|_{q/2} \lesssim 2^{(j+k)(\frac{4}{q}-1)}|\tilde\Omega\cap\theta_k|^{\frac12}|\tilde\Omega'\cap\sigma_j|^{\frac12}.$$

Using (4.11) and the analogous bound for $|\tau_k|$, we now get

$$\begin{aligned}\sum_{k=0}^{K'-1}\sum_{j=0}^{J-1} \|\mathcal{E}_0(\mathbb{1}_{\tilde\Omega\cap\theta_k})\mathcal{E}_0(\mathbb{1}_{\tilde\Omega'\cap\sigma_j})\|_{q/2} &\lesssim 2^{(J'+K)(\frac{4}{q}-1)}|\theta|^{\frac12}|\sigma|^{\frac12}\\ &\sim 2^{(J'-J+K-K')(\frac{2}{q}-\frac12)}|\tilde\Omega|^{1-\frac{2}{q}}|\tilde\Omega'|^{1-\frac{2}{q}}.\end{aligned}$$

By the relations $K < K'$ and $J > J'$ and the fact that $q < 4$, the lemma is proved. □

Returning to the proof of Lemma 4.12, we consider the second sum in (4.10). Given $\boldsymbol{K} \in \mathcal{K}^4 \setminus D(\mathcal{K}^4)$, let $p(\boldsymbol{K}) = (p_i(\boldsymbol{K}))_{i=1}^4$ be a permutation of $\boldsymbol{K}$ such that $|\Omega(p_1(\boldsymbol{K}))^+|$ is maximal among $|\Omega(K_i)^+|$, $1 \le i \le 4$, and such that $|K_i - K_j| \le 2|p_1(\boldsymbol{K}) - p_2(\boldsymbol{K})|$ for all $1 \le i, j \le 4$. Then by the Cauchy–Schwarz inequality, Lemma 4.12, the separation condition on $\mathcal{K}$, the fact that $q > 3$, and choosing A sufficiently large, we get

$$\begin{aligned}\sum_{\boldsymbol{K}\in\mathcal{K}^4\setminus D(\mathcal{K}^4)} \Big\|\prod_{i=1}^{4}\mathcal{E}_0\mathbb{1}_{\Omega(K_i)^+_\delta}\Big\|_{\frac{q}{4}}^{\frac{q}{4}} &\lesssim \sum_{\substack{\boldsymbol{K}\in\mathcal{K}^4\setminus D(\mathcal{K}^4)\\ \boldsymbol{K}=p(\boldsymbol{K})}} 2^{-c_0|p_1(\boldsymbol{K})-p_2(\boldsymbol{K})|}|\Omega(p_1(\boldsymbol{K}))^+|^{q-2}\\ &\lesssim \sum_{K_1\in\mathcal{K}}\sum_{K_2\in\mathcal{K}} |K_1-K_2|^2 2^{-c_0|K_1-K_2|}|\Omega(K_1)^+|^{q-2}\\ &\lesssim \delta^{\frac{c_0A}{2}}\sum_{K_1\in\mathcal{K}}|\Omega(K_1)^+|^{q-2} \lesssim \delta^2|\Omega|^{q-2}.\end{aligned}$$

This concludes the proof of Lemma 4.12. □

Proof of Theorem 4.1. By interpolation, it suffices to prove the analogous restricted strong-type estimate. Let $\Omega \subseteq U$ be a measurable set. We have the decomposition

$$\Omega = \bigcup_{0<\varepsilon\lesssim 1} \bigcup_{0<\delta\lesssim\varepsilon^{1/4}} \bigcup_{K\in\mathcal{K}(\varepsilon)} \Omega(K)^+_\delta.$$

Thus, by the triangle inequality, Lemma 4.12, Proposition 4.8, and the fact that $q > 3$, we obtain

$$\begin{aligned}
\|\mathcal{E}_0 \mathbb{1}_\Omega\|_q &\le \sum_{0<\varepsilon\lesssim 1} \sum_{0<\delta\lesssim\varepsilon^{1/4}} \Big\| \sum_{K\in\mathcal{K}(\varepsilon)} \mathcal{E}_0 \mathbb{1}_{\Omega(K)^+_\delta} \Big\|_q \\
&\lesssim \sum_{0<\varepsilon\lesssim 1} \sum_{0<\delta\lesssim\varepsilon^{1/4}} \Big((\log \delta^{-1})^q \sum_{K\in\mathcal{K}(\varepsilon)} \|\mathcal{E}_0 \mathbb{1}_{\Omega(K)^+_\delta}\|_q^q + \delta|\Omega|^{q-2} \Big)^{\frac{1}{q}} \\
&\lesssim \Bigg[\sum_{0<\varepsilon\lesssim 1} \sum_{0<\delta\lesssim\varepsilon^{1/4}} (\log\delta^{-1})\delta \Big(\sum_{K\in\mathcal{K}(\varepsilon)} |\Omega(K)^+|^{q-2} \Big)^{\frac{1}{q}} \Bigg] + |\Omega|^{1-\frac{2}{q}} \\
&\lesssim \Bigg[\sum_{0<\varepsilon\lesssim 1} \sum_{0<\delta\lesssim\varepsilon^{1/4}} (\log\delta^{-1})\delta |\Omega|^{1-\frac{2}{q}} \Bigg] + |\Omega|^{1-\frac{2}{q}} \lesssim |\Omega|^{1-\frac{2}{q}}.
\end{aligned}$$

□

5. Proof of Theorem 4.4

In this section, we prove Theorem 4.4, the bilinear restriction estimate for related tiles. As mentioned above, we proceed by rescaling a result of Lee [12].

We begin by defining some notation. The basic symmetries of the hyperbolic hyperboloid are the Lorentz transformations, which, given the parametrization (4.1), are the linear maps on $\mathbb{R} \times \mathbb{R}^2$ that preserve the quadratic form $(\tau, \xi) \mapsto \tau^2 - \xi_1^2 + \xi_2^2$. The Lorentz-invariant measure $\mathrm{d}\sigma$ on Σ takes the form

$$\int_\Sigma g \,\mathrm{d}\sigma := \int_U g(\phi(\xi), \xi) \frac{\mathrm{d}\xi}{\phi(\xi)},$$

where $\phi(\xi) = \sqrt{1 + \xi_1^2 - \xi_2^2}$ as before. If L is a Lorentz transformation and $\operatorname{supp} g \subseteq \Sigma$ and $L^{-1}(\operatorname{supp} g) \subseteq \Sigma$, then

$$\int_\Sigma (g \circ L) \mathrm{d}\sigma = \int_\Sigma g \,\mathrm{d}\sigma.$$

Let $\Omega := \{\xi \in \mathbb{R}^2 : 1 + \xi_1^2 - \xi_2^2 \ge 0\}$. Given a Lorentz transformation L and $\xi \in \Omega$, let

$$\overline{L}(\xi) := \pi(L(\phi(\xi), \xi)),$$

where $\pi(\tau', \xi') := \xi'$ is the projection to the spatial coordinates. If $\xi \in \Omega$ and $e_1 \cdot L(\phi(\xi), \xi) \ge 0$, where $e_1 = (1, 0, 0)$ denotes the first standard basis vector, then $\overline{ML}(\xi) = \overline{M}(\overline{L}(\xi))$ for any other Lorentz transformation M. In particular, if $E \subseteq \Omega$ and $e_1 \cdot L(\phi(\xi), \xi) \ge 0$ for all $\xi \in E$, then $\overline{L}$ is invertible on E with $\overline{L}^{-1}(\zeta) = \overline{L^{-1}}(\zeta)$ for $\zeta \in \overline{L}(E)$.

We now turn to the proof of Theorem 4.4. We may assume that $j \geq k$. Fix $\theta_1 \sim \theta_2 \in \Theta_{j,k}$ and $c \in \theta_1 \cup \theta_2$. Arguing as in the proof of Lemma 4.6, there exists an $O(2^{-j}) \times O(2^{-k})$ rectangle S centered at c with major axis ℓ_c^+ such that $\theta_1 \cup \theta_2 \subseteq S$. We define three Lorentz transformations,

$$R(\tau,\xi) := (-\omega_2\xi_2 + \omega_1\tau, \xi_1, \omega_1\xi_2 + \omega_2\tau), \qquad \omega := \left(\frac{\phi(c)}{\sqrt{1+c_1^2}}, -\frac{c_2}{\sqrt{1+c_1^2}}\right) \in \mathbb{S}^1,$$

$$B(\tau,\xi) := \Big(- c_1\xi_1 + \sqrt{1+c_1^2}\tau, \sqrt{1+c_1^2}\xi_1 - c_1\tau, \xi_2\Big),$$

$$D(\tau,\xi) := \frac{1}{2}\big(2\tau, (2^{\frac{k-j}{2}} + 2^{\frac{j-k}{2}})\xi_1 + (2^{\frac{k-j}{2}} - 2^{\frac{j-k}{2}})\xi_2, (2^{\frac{k-j}{2}} - 2^{\frac{j-k}{2}})\xi_1 + (2^{\frac{k-j}{2}} + 2^{\frac{j-k}{2}})\xi_2\big),$$

which correspond to a spatial rotation, a boost, and a dilation, respectively. The composition BR takes $(\phi(c), c)$ to $(1, 0, 0)$. We will show that $\overline{DBR}$ essentially takes θ_1, θ_2 to a pair of $O(2^{-\frac{j+k}{2}})$-squares near the origin, which we will then parabolically rescale to size 1. After checking that the separation of θ_1 and θ_2 is respected by these rescalings, we will apply Lee's result [12, Theorem 1.1] to finish the proof.

We turn to the details. Because Lorentz transformations preserve the hyperbolic hyperboloid and are linear, they must permute the lines in the surface. Thus, since $BR(\phi(c), c) = (1, 0, 0)$, we have either $\overline{BR}(\ell_c^+) = \ell_0^+$ or $\overline{BR}(\ell_c^+) = \ell_0^-$. It is easy to check that in fact $\overline{BR}(\ell_c^+) = \ell_0^+ = \mathbb{R}(1,1)$ and that $\|\nabla\overline{BR}\| \lesssim 1$ near the origin. Thus, by the definition of the rectangle S, it follows that $\overline{BR}(\theta_1 \cup \theta_2)$ is contained in an $O(2^{-j}) \times O(2^{-k})$ rectangle of slope 1 centered at the origin. Since $\overline{D}$ contracts by a factor comparable to $2^{\frac{k-j}{2}}$ in the direction of $(1, 1)$ and expands by a factor comparable to $2^{\frac{j-k}{2}}$ in the orthogonal direction, $\overline{D}(\overline{BR}(\theta_1 \cup \theta_2))$ lies in a disc V of radius $O(2^{-\frac{j+k}{2}})$ centered at 0. If j and k are sufficiently large (which we may assume), then $V \subseteq U$. It is easy to check that $e_1 \cdot BR(\phi(\xi), \xi) \geq 0$ for all $\xi \in U$. Thus, setting $L := (DBR)^{-1}$, we have

$$\overline{L^{-1}}(\theta_1 \cup \theta_2) = \overline{D}(\overline{BR}(\theta_1 \cup \theta_2)) \subseteq V. \tag{5.1}$$

Let $Q_i := \overline{L}^{-1}(\theta_i) := \{\xi \in \Omega : \overline{L}(\xi) \in \theta_i\}$. We claim that

$$Q_i = \overline{L^{-1}}(\theta_i). \tag{5.2}$$

Given a set $E \subseteq \Omega$, let $E^{\pm} := \{(\pm\phi(\xi), \xi) : \xi \in E\}$. Then we have

$$\begin{aligned} Q_i &= \{\xi \in \Omega : L(\phi(\xi), \xi) \in \theta_i^+ \cup \theta_i^-\} \\ &= \{\xi \in \Omega : (\phi(\xi), \xi) \in L^{-1}(\theta_i^+) \cup (-L^{-1}((-\theta_i)^+))\}. \end{aligned}$$

It is easy to check that $e_1 \cdot L^{-1}(\phi(\zeta), \zeta) > 0$ for all $\zeta \in U$. Thus, since $-\theta_i \subseteq U$ and $\phi \geq 0$, we have $(\phi(\xi), \xi) \notin -L^{-1}((-\theta_i)^+)$ for every ξ. Hence

$$Q_i = \{\xi \in \Omega : (\phi(\xi), \xi) \in L^{-1}(\theta_i^+)\} = \overline{L^{-1}}(\theta_i),$$

proving the claim.

Now, if f is supported in $\theta_1 \cup \theta_2$, then by the Lorentz invariance of the measure $\mathrm{d}\sigma$,

$$\mathcal{E}_0 f(t,x) = \mathcal{E}_0 f_L(L^*(t,x)),$$

where

$$f_L(\xi) := \frac{f(\overline{L}(\xi))\phi(\overline{L}(\xi))}{\phi(\xi)}.$$

We have $|f_L| \sim |f \circ \overline{L}|$ on $\operatorname{supp} f_L \subseteq Q_1 \cup Q_2 \subseteq V$. Additionally, $e_1 \cdot L(\phi(\xi), \xi) \geq 0$ for all $\xi \in V$, so we know that $\overline{L}$ is invertible on $Q_1 \cup Q_2$ with $\overline{L}^{-1}(\zeta) = \overline{L^{-1}}(\zeta)$ for $\zeta \in \overline{L}(Q_1 \cup Q_2)$. A straightforward calculation shows that $|\det \nabla \overline{L}^{-1}(\zeta)| \lesssim 1$ on $\overline{L}(Q_1 \cup Q_2)$. Combining these observations, we see that the estimate in Theorem 4.4 is equivalent to

$$\|\mathcal{E}_0 f \mathcal{E}_0 g\|_{q/2} \lesssim 2^{(j+k)(\frac{4}{q}-1)} \|f\|_2 \|g\|_2 \tag{5.3}$$

for all $f \in L^2(Q_1)$ and $g \in L^2(Q_2)$.

Now, by parabolic rescaling, we have

$$\mathcal{E}_0 f(t,x) = 2^{-j-k} \mathcal{E}_0^{\psi}[f(2^{-\frac{j+k}{2}} \cdot)](2^{-j-k}t, 2^{-\frac{j+k}{2}}x),$$

for every f supported in $Q_1 \cup Q_2 \subseteq V$, where $\mathcal{E}_0^{\psi}$ is the extension operator associated to the phase $\psi(\xi) := 2^{j+k}\sqrt{1 + 2^{-j-k}(\xi_1^2 - \xi_2^2)}$. The estimate (5.3) now follows from [12, Theorem 1.1], provided the hypotheses of the latter are satisfied. Let $\tilde{Q}_i := 2^{\frac{j+k}{2}} Q_i$. We need to the check that

$$|\langle (\nabla^2 \psi(\xi''))^{-1} (\nabla\psi(\xi) - \nabla\psi(\zeta)), \nabla\psi(\xi') - \nabla\psi(\zeta') \rangle| \gtrsim 1,$$
$$|\langle (\nabla^2 \psi(\zeta''))^{-1} (\nabla\psi(\xi) - \nabla\psi(\zeta)), \nabla\psi(\xi') - \nabla\psi(\zeta') \rangle| \gtrsim 1,$$

for all $\xi, \xi', \xi'' \in \tilde{Q}_1$ and $\zeta, \zeta', \zeta'' \in \tilde{Q}_2$. Let $r(\xi) := (\xi_1, -\xi_2)$. Simple calculations and the mean value theorem show that

$$\nabla\psi(\xi) = r(\xi) + O(2^{-j-k}),$$
$$(\nabla^2\psi(\xi))^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + O(2^{-j-k}),$$

for all $|\xi| \leq 1$, and thus we only need to show that

$$|\langle \xi - \zeta, r(\xi' - \zeta') \rangle| \gtrsim 1 \tag{5.4}$$

for all $\xi, \xi' \in \tilde{Q}_1$ and $\zeta, \zeta' \in \tilde{Q}_2$.

Let $p(\xi)$ and $q(\xi)$ denote the orthogonal projections of ξ to the lines $\mathbb{R}(1,-1)$ and $\mathbb{R}(1,1)$, respectively. That is,

$$p(\xi) := \frac{1}{2}(\xi_1 - \xi_2, \xi_2 - \xi_1)$$
$$q(\xi) := \frac{1}{2}(\xi_1 + \xi_2, \xi_1 + \xi_2).$$

Assume for now that the following lemma holds:

Lemma 5.1. *For all $\xi \in Q_1$ and $\zeta \in Q_2$, we have*

$$|p(\xi) - p(\zeta)| \gtrsim 2^{-\frac{j+k}{2}},$$
$$|q(\xi) - q(\zeta)| \gtrsim 2^{-\frac{j+k}{2}}.$$

We immediately see that $|p(\xi-\zeta)| \gtrsim 1$ and $|q(\xi-\zeta)| \gtrsim 1$ for all $\xi \in \tilde{Q}_1$ and $\zeta \in \tilde{Q}_2$. We will use these bounds to prove (5.4). We have

$$\begin{aligned}\langle \xi-\zeta, r(\xi'-\zeta')\rangle &= \langle p(\xi-\zeta)+q(\xi-\zeta), p(r(\xi'-\zeta'))+q(r(\xi'-\zeta'))\rangle \\ &= \langle p(\xi-\zeta), p(r(\xi'-\zeta'))\rangle + \langle q(\xi-\zeta), q(r(\xi'-\zeta'))\rangle\end{aligned}$$

by orthogonality. Using the relations $p\circ r = r\circ q$ and $q\circ r = r\circ p$ and the fact that r is unitary, we thus have

$$\langle \xi-\zeta, r(\xi'-\zeta')\rangle = \langle r(p(\xi-\zeta)), q(\xi'-\zeta')\rangle + \langle q(\xi-\zeta), r(p(\xi'-\zeta'))\rangle. \tag{5.5}$$

Using the fact that the sets $q(\tilde{Q}_1)$ and $q(\tilde{Q}_2)$ (resp. $p(\tilde{Q}_1)$ and $p(\tilde{Q}_2)$) are disjoint and each of them is connected, one sees that $q(\xi-\zeta)$ and $q(\xi'-\zeta')$ are parallel, as are $p(\xi-\zeta)$ and $p(\xi'-\zeta')$. Since r is unitary, $r(p(\xi-\zeta))$ and $r(p(\xi'-\zeta'))$ are also parallel. Hence, both terms on the right-hand side of (5.5) have the same sign, and thus it suffices to bound one of them from below. We have

$$|\langle r(p(\xi-\zeta)), q(\xi'-\zeta')\rangle| = |p(\xi-\zeta)||q(\xi'-\zeta')| \gtrsim 1.$$

It remains to prove Lemma 5.1.

Proof of Lemma 5.1. We know that $Q_i = \overline{L^{-1}}(\theta_i) = \overline{D}(\overline{BR}(\theta_i))$, by (5.2) and (5.1). We claim, first, that

$$\begin{aligned}|\pi^+(\xi)-\pi^+(\zeta)| &\gtrsim 2^{-j}, \\ |\pi^-(\xi)-\pi^-(\zeta)| &\gtrsim 2^{-k}\end{aligned} \tag{5.6}$$

for all $\xi \in \overline{BR}(\theta_1)$ and $\zeta \in \overline{BR}(\theta_2)$, where $\pi^\pm$ are the projections used throughout the previous section (recall (4.3)). Using that $\|(BR)^{-1}\| \lesssim 1$, it is not difficult to show that $\overline{BR}^{-1}$ exists in a neighborhood of 0 (of constant size) and is given by $\overline{BR}^{-1}(\eta) = \overline{(BR)^{-1}}(\eta)$. We showed above that the sets $\overline{BR}(\theta_i)$ lie in a disc of radius $O(2^{-k})$ centered at 0. Fix $\xi \in \overline{BR}(\theta_1)$ and $\zeta \in \overline{BR}(\theta_2)$, and let $\xi' := \overline{BR}^{-1}(\xi)$, $\alpha := \overline{BR}^{-1}(\pi^+(\xi), 0)$ and $\zeta' := \overline{BR}^{-1}(\zeta)$, $\beta := \overline{BR}^{-1}(\pi^+(\zeta), 0)$. Since $(\pi^+(\xi), 0) \in \ell^+_\xi$ by Lemma 4.2, it follows that $\alpha \in \ell^+_{\xi'}$. Similarly, $\beta \in \ell^+_{\zeta'}$. Thus, since π^+ is constant along lines of the form ℓ^+_η, by Lemma 4.2, and $\theta_1 \sim \theta_2$, we have

$$\begin{aligned}2^{-j} &\lesssim |\pi^+(\xi')-\pi^+(\zeta')| \\ &= |\pi^+(\alpha)-\pi^+(\beta)| \\ &\lesssim |\alpha-\beta| \\ &= |\overline{BR}^{-1}(\pi^+(\xi),0) - \overline{BR}^{-1}(\pi^+(\zeta),0)| \\ &\lesssim |\pi^+(\xi)-\pi^+(\zeta)|.\end{aligned}$$

A similar argument gives the second estimate in (5.6).

We will now use (5.6) to prove the lemma. We only prove the first estimate; the second one follows by a similar argument. Fix $\xi \in Q_1$ and $\zeta \in Q_2$, and let $\alpha = \overline{D}^{-1}(p(\xi)) = 2^{\frac{k-j}{2}}p(\xi)$ and $\beta = \overline{D}^{-1}(p(\zeta)) = 2^{\frac{k-j}{2}}p(\zeta)$. It suffices to show that $|\alpha-\beta| \gtrsim 2^{-j}$. Let ξ' be the intersection of the lines ℓ^+_ξ and $\mathbb{R}(1,-1)$. The points $A := \xi$, $B := p(\xi)$, and $C := \xi'$

form a right triangle with hypotenuse AC. One easily checks that $\angle(\ell_\xi^+, \mathbb{R}(1,1)) \lesssim |\xi|$. Thus, $\angle CAB \lesssim 2^{-\frac{j+k}{2}}$ by the fact that $\xi \in Q_1 \subseteq V$. We also have $|AC| \lesssim 2^{-\frac{j+k}{2}}$, and thus $|p(\xi) - \xi'| = |BC| \lesssim 2^{-j-k}$. Let $\alpha' = \overline{D}^{-1}(\xi') = 2^{\frac{k-j}{2}}\xi'$. Then $|\alpha - \alpha'| = 2^{\frac{k-j}{2}}|p(\xi) - \xi'| \lesssim 2^{-\frac{3j+k}{2}}$. Because $\xi' \in \ell_\xi^+$, we have $\ell_{\xi'}^+ = \ell_\xi^+$ (see the proof of Lemma 4.2) and thus $\alpha' \in \overline{D}^{-1}(\ell_\xi^+) = \ell^+_{\overline{D}^{-1}(\xi)}$. Thus, since π^+ is constant along $\ell^+_{\overline{D}^{-1}(\xi)}$, we have

$$|\pi^+(\alpha) - \pi^+(\overline{D}^{-1}(\xi))| = |\pi^+(\alpha) - \pi^+(\alpha')| \lesssim |\alpha - \alpha'| \lesssim 2^{-\frac{3j+k}{2}},$$

and by a similar argument,

$$|\pi^+(\beta) - \pi^+(\overline{D}^{-1}(\zeta))| \lesssim 2^{-\frac{3j+k}{2}}.$$

Since $\overline{D}^{-1}(\xi) \in \overline{BR}(\theta_1)$ and $\overline{D}^{-1}(\zeta) \in \overline{BR}(\theta_2)$, the first estimate in (5.6) now gives the bound

$$2^{-j} \lesssim |\pi^+(\overline{D}^{-1}(\xi)) - \pi^+(\overline{D}^{-1}(\zeta))| = |\pi^+(\alpha) - \pi^+(\beta)| + O(2^{-\frac{3j+k}{2}}),$$

and consequently

$$|\alpha - \beta| \gtrsim |\pi^+(\alpha) - \pi^+(\beta)| \gtrsim 2^{-j},$$

which is what we needed to show. □

Remark 5.2. As an application, we pause to explain how the bilinear theory for $\mathcal{E}_0$ can be used to obtain further (conditional) linear estimates for $\mathcal{E}_0$ on the parabolic scaling line $p = (q/2)'$. Similar to the case of the hyperbolic paraboloid (see [16]), the proof of Theorem 4.1 can be adjusted to give the following conditional bilinear-to-linear result: Given $3 < q_0 < 4$, if there exists some $p_0 < (\frac{q_0}{2})'$ such that

$$\|\mathcal{E}_0 f \mathcal{E}_0 g\|_{q_0/2} \lesssim 2^{(j+k)(\frac{4}{q_0} + \frac{2}{p_0} - 2)} \|f\|_{p_0} \|g\|_{p_0},$$

for all functions f, g supported in related tiles in $\Theta_{j,k}$, then $\mathcal{E}_0$ is bounded from $L^{(q/2)'}$ to L^q for all $q > q_0$. In [4], the first author showed the following: If $q_0 > 3.25$, $p_0 > (\frac{q_0}{2})'$, and $0 < r \leq 1$, then $\|\mathcal{E}_0^r f\|_{q_0} \lesssim \|f\|_{p_0}$ uniformly in r, where

$$\mathcal{E}_0^r f(t,x) := \int_U e^{i(t,x)\cdot(r^{-2}\sqrt{1+r^2(\xi_1^2-\xi_2^2)},\xi)} f(\xi)\mathrm{d}\xi.$$

Using this result, the Cauchy–Schwarz inequality, a parabolic rescaling argument (utilizing the uniformity in r), and interpolation with Theorem 4.4, one can show that the hypothesis of the conditional version of Theorem 4.1 holds for each $q_0 > 3.25$. We conclude that $\mathcal{E}_0$ is bounded from $L^{(q/2)'}$ to L^q for every $q > 3.25$.

6. Bilinear adjoint restriction on annuli

In the next two sections we establish bounds for the extension operator associated to dyadic annuli in our hyperboloid. By invariance under cylindrical rotations and the triangle inequality, it suffices to consider subsets of these annuli with some angular restriction, and we abuse notation (relative to the introduction) by defining

$$\Gamma_N := \{(\tau, \xi) \in \Gamma : |\xi| \sim 2^N, \ |\tfrac{\xi}{|\xi|} - e_1| < 0.001\},$$

where e_1 denotes the usual first coordinate vector. We will use the notation f_N to denote a function supported on Γ_N.

The focus of this section will be on establishing bounds in the bilinear range, where our results are unconditional and our deduction is more straightforward. We will then turn to the conditional result in the next section, the proof of which will use some of the lemmas from this section.

Proposition 6.1. *Let* $(\frac{q}{2})' \leq p \leq q$ *and* $4 > q > \frac{10}{3}$. *Then*

$$\|\mathcal{E}f_N\|_q \lesssim \|f_N\|_p,$$

for all functions f_N *supported on* Γ_N.

The remainder of this section will be devoted to the proof of Proposition 6.1.

We will work on sectors of varying width contained in the Γ_N. Let $C \leq k \leq N$. By an (N,k)-*sector*, we mean a set of the form

$$\Gamma^{\omega}_{N,k} := \{(\tau,\xi) \in \Gamma_N : |\tfrac{\xi}{|\xi|} - \omega| < 2^{-k}\},$$

with $\omega \in \mathbb{S}^1$; we refer to 2^{-k} as the angular width of the sector.

We begin by establishing bounds on the thinnest sectors.

Lemma 6.2. *For any* p, q, *validity of* $\mathcal{R}^*_0(p \to q)$ *implies that*

$$\|\mathcal{E}f^{\omega}_{N,N}\|_q \lesssim \|f^{\omega}_{N,N}\|_p, \tag{6.1}$$

for every function $f^{\omega}_{N,N}$ *supported in an* (N,N)-*sector,* $N \geq 1$. *In particular,* (6.1) *holds for all* $q \geq 2p'$ *when* $q > \frac{10}{3}$.

Proof. We recall the definition (1.1) of the Lorentz boost L_ν and the Lorentz invariance of our measure. The deduction claimed in the lemma follows from the observation that if $\omega \in \mathbb{S}^1$ and $N \geq 1$, $L_{2^N\omega}$ maps $\Gamma^{\omega}_{N,N}$ into Γ_0. □

Now we turn to the deduction of bounds on the Γ_N from those on the $\Gamma^{\omega}_{N,N}$, for which we adapt the bilinear theory for the cone.

For $k < N$, we say that two (N,k)-sectors, $\Gamma^{\omega}_{N,k}$ and $\Gamma^{\omega'}_{N,k}$ are related, $\Gamma^{\omega}_{N,k} \sim \Gamma^{\omega'}_{N,k}$, when $2^{-k+4} \leq |\omega - \omega'| \leq 2^{-k+8}$. We say that two (N,N)-sectors, $\Gamma^{\omega}_{N,N}$ and $\Gamma^{\omega'}_{N,N}$ are related when $|\omega - \omega'| \leq 2^{-N+8}$.

We can deduce a near-optimal L^2-based bilinear adjoint restriction theorem for related (N,k)-sectors from results already in the literature. Namely, one may directly apply the bilinear restriction method from [18] (which was quickly observed to apply to conic surfaces) and conic rescaling, or else directly apply the results of [7] to obtain the following.

Theorem 6.3 ([7, 13, 18])**.** *Let* $C \leq k < N$, *let* $\Gamma^{\omega_1}_{N,k}$ *and* $\Gamma^{\omega_2}_{N,k}$ *be related* (N,k)-*sectors, and let* f_1, f_2 *be* L^2 *functions supported on* $\Gamma^{\omega_1}_{N,k}, \Gamma^{\omega_2}_{N,k}$, *respectively. Then*

$$\|\mathcal{E}f_1\mathcal{E}f_2\|_{L^{q/2}} \lesssim 2^{-(N-k)(\frac{6}{q}-1)}\|f_1\|_2\|f_2\|_2, \qquad q > \tfrac{10}{3}. \tag{6.2}$$

We state our bilinear-to-linear deduction in slightly more general terms than we need in this section in order to facilitate later arguments.

Lemma 6.4. *Let $3 < q < 4$, $(\frac{q}{2})' \le p \le q$, and $s \le p$. Assume that $\mathcal{R}_0^*(p \to q)$ holds and that for $C \le k < N$,*

$$\|\mathcal{E}f_1\mathcal{E}f_2\|_{q/2} \lesssim 2^{-(N-k)\alpha}\|f_1\|_s\|f_2\|_s, \tag{6.3}$$

whenever f_1 and f_2 are supported in related (N,k)-sectors. If $\alpha \ge \frac{2}{s} - \frac{2}{p}$, $\alpha > 0$, and either $\alpha \ne \frac{2}{s} - \frac{2}{q}$ or $p < q$, then

$$\|\mathcal{E}f_N\|_q \lesssim \|f_N\|_p,$$

for all measurable functions f_N satisfying $|f_N| \sim \mathbb{1}_{\Omega_N}$, for some $\Omega_N \subseteq \Gamma_N$.

Lemma 6.4 implies a restricted strong-type inequality for extension from the Γ_N, which can be interpolated to yield strong-type inequalities since we work with exponents obeying $q \ge p$.

Proof of Lemma 6.4. We may choose $O(2^{-k})$-separated collections $\mathcal{D}_{N,k} \subseteq \mathbb{S}^1$, $C \le k \le N$, such that whenever $(\tau,\xi), (\tau',\xi') \in \Gamma_N$, there exists a pair of related (N,k)-sectors $\Gamma^{\omega}_{N,k} \ni (\tau,\xi)$ and $\Gamma^{\omega'}_{N,k} \ni (\tau',\xi')$, with $\omega, \omega' \in \mathcal{D}_{N,k}$. Here $k = N$ if $|\frac{\xi}{|\xi|} - \frac{\xi'}{|\xi'|}| \lesssim 2^{-N}$, and $2^{-k} \sim |\frac{\xi}{|\xi|} - \frac{\xi'}{|\xi'|}|$, otherwise. We will abuse notation by saying that for $\omega, \omega' \in \mathcal{D}_{N,k}$, $\omega \sim \omega'$ if $\Gamma^{\omega}_{N,k} \sim \Gamma^{\omega'}_{N,k}$. Thus we may decompose

$$\Gamma_N \times \Gamma_N := \bigcup_{k=C}^{N} \bigcup_{\omega\sim\omega'\in\mathcal{D}_{N,k}} \Gamma^{\omega}_{N,k} \times \Gamma^{\omega'}_{N,k}. \tag{6.4}$$

We will later use the geometric property that each $\Gamma^{\omega}_{N,k}$ is contained in a parallelepiped $P^{\omega}_{N,k}$, such that the sumsets $P^{\omega}_{N,k} + P^{\omega'}_{N,k}$ are finitely overlapping as the pair $\omega \sim \omega' \in \mathcal{D}_{N,k}$ varies.

Let f_N be a measurable function with $|f_N| \sim \mathbb{1}_{\Omega_N}$, for some subset $\Omega_N \subseteq \Gamma_N$. Using the decomposition (6.4) to make a partition of unity, we have

$$\begin{aligned}\|\mathcal{E}\mathbb{1}_{\Omega_N}\|_q^2 &= \|(\mathcal{E}\mathbb{1}_{\Omega_N})^2\|_{q/2} \\ &\le \Big\| \sum_{\omega\sim\omega'\in\mathcal{D}_{N,N}} \mathcal{E}f^{\omega}_{N,N}\mathcal{E}f^{\omega'}_{N,N}\Big\|_{q/2} + \Big\|\sum_{k=C}^{N-1}\sum_{\omega\sim\omega'\in\mathcal{D}_{N,k}} \mathcal{E}f^{\omega}_{N,k}\mathcal{E}f^{\omega'}_{N,k}\Big\|_{q/2} =: I_1 + I_2,\end{aligned}$$

where the $f^{\omega}_{N,k}$ are measurable functions supported on the $\Gamma^{\omega}_{N,k}$ with $|f^{\omega}_{N,k}| \lesssim |f_N|$.

We begin with the first term. By the Tao–Vargas–Vega orthogonality lemma [19, Lemma 6.1] and the finite overlap of sumsets, the Cauchy–Schwarz inequality, the hypothesis that $\mathcal{R}_0^*(p \to q)$ holds, and the fact that $q \ge p$, we have

$$\begin{aligned} I_1 &\lesssim \Big(\sum_{\omega\sim\omega'\in\mathcal{D}_{N,N}} \|\mathcal{E}f^{\omega}_{N,N}\mathcal{E}f^{\omega'}_{N,N}\|_{q/2}^{q/2}\Big)^{2/q} \le \Big(\sum_{\omega\sim\omega'\in\mathcal{D}_{N,N}} \|\mathcal{E}f^{\omega}_{N,N}\|_q^{q/2}\|\mathcal{E}f^{\omega'}_{N,N}\|_q^{q/2}\Big)^{2/q} \\ &\lesssim \Big(\sum_{\omega\sim\omega'\in\mathcal{D}_{N,N}} \|f^{\omega}_{N,N}\|_p^{q/2}\|f^{\omega'}_{N,N}\|_p^{q/2}\Big)^{2/q} \lesssim \Big(\sum_{\omega\in\mathcal{D}_{N,N}} \|f_N\mathbb{1}_{\Gamma^{\omega}_{N,N}}\|_p^q\Big)^{2/q} \lesssim \|f_N\|_p^2.\end{aligned}$$

Now we turn to the second term. Let $\Omega^{\omega}_{N,k} := \Omega_N \cap \Gamma^{\omega}_{N,k}$. By the triangle inequality, almost orthogonality, and the aforementioned finite overlap property of sumsets, and then (6.3) and some standard reindexing,

$$I_2 \lesssim \sum_{k=C}^{N-C} \left(\sum_{\omega\sim\omega'\in\mathcal{D}_{N,k}} \|\mathcal{E}f^{\omega}_{N,k}\mathcal{E}f^{\omega'}_{N,k}\|_{\frac{q}{2}}^{\frac{q}{2}} \right)^{\frac{2}{q}} \lesssim \sum_{k=C}^{N-C} 2^{-(N-k)\alpha} \Big(\sum_{\omega\in\mathcal{D}_{N,k}} \sigma(\Omega^{\omega}_{N,k})^{\frac{q}{s}} \Big)^{\frac{2}{q}}.$$

Thus by Hölder's inequality and the estimates

$$\sigma(\Omega^{\omega}_{N,k}) \le \min\{\sigma(\Omega_N), \sigma(\Gamma^{\omega}_{N,k})\} \quad \text{and} \quad \sigma(\Gamma^{\omega}_{N,k}) \sim 2^{N-k},$$

we see that

$$\begin{aligned} I_2 &\lesssim \sum_{j=C}^{N-C} 2^{-j\alpha} \min\{2^{j(\frac{2}{s}-\frac{2}{q})}, \sigma(\Omega_N)^{\frac{2}{s}-\frac{2}{q}}\} |\Omega_N|^{\frac{2}{q}} \\ &\le \sum_{j=C}^{\log_2(\sigma(\Omega_N))} 2^{j(\frac{2}{s}-\frac{2}{q}-\alpha)} \sigma(\Omega_N)^{\frac{2}{q}} + \sum_{j=\max\{C,\log_2(\sigma(\Omega_N))\}}^{N-C} 2^{-j\alpha}\sigma(\Omega_N)^{\frac{2}{s}} =: I_2' + I_2''. \end{aligned}$$

When $\sigma(\Omega_N) \le 1$, $I_2' = 0$ and $I_2'' \sim \sigma(\Omega_N)^{\frac{2}{s}} \le \sigma(\Omega_N)^{\frac{2}{p}}$, since $s \le p$. When $\sigma(\Omega_N) \ge 1$, $I_2'' \sim \sigma(\Omega_N)^{\frac{2}{s}-\alpha} \le \sigma(\Omega_N)^{\frac{2}{p}}$. If, in addition, $\frac{2}{s} - \frac{2}{q} - \alpha < 0$, $I_2' \sim \sigma(\Omega_N)^{\frac{2}{q}} \le \sigma(\Omega_N)^{\frac{2}{p}}$. Meanwhile, if $\frac{2}{s} - \frac{2}{q} - \alpha > 0$, $I_2' \sim \sigma(\Omega_N)^{\frac{2}{s}-\alpha} \le \sigma(\Omega_N)^{\frac{2}{p}}$. Finally, if $\alpha = \frac{2}{s} - \frac{2}{q}$ and $p < q$, then $I_2' \sim \log(\sigma(\Omega_N))\sigma(\Omega_N)^{\frac{2}{q}} \lesssim \sigma(\Omega_N)^{\frac{2}{p}}$.

In any case, combining our estimates for I_1 and I_2 gives $\|\mathcal{E}f_N\|_q^2 \lesssim \sigma(\Omega_N)^{\frac{2}{p}}$, completing the proof of the lemma. □

Theorem 6.3, Theorem 4.1, Lemma 6.2, Lemma 6.4 (with $q \ge p > 2$, $q \ge 2p'$, and $s = 2$), and real interpolation together imply that $\|\mathcal{E}f\|_q \lesssim \|f\|_p$ for all $f \in C^{\infty}_c(\Gamma_N)$ when $q > 10/3$ and $(\frac{q}{2})' \le p \le q$. Thus the proof of Proposition 6.1 is complete.

7. Reduction to bounds on Γ_0 via decoupling

In the previous section we showed how to deduce bounds for extension from the dyadic annuli Γ_N from those for extension from Γ_0 by using the bilinear adjoint restriction inequality (6.2). This approach is limited, since we do not currently know any such result with $q \le \frac{10}{3}$. In this section, we will use the conic decoupling theorem of Bourgain–Demeter to obtain new bounds for extension from Γ_N, conditional on further improvements to $\mathcal{R}_0^*(p \to q)$. The entirety of this section will be devoted to a proof of the following result.

Proposition 7.1. *Suppose that $\mathcal{R}_0^*((\frac{q_0}{2})' \to q_0)$ holds for some $q_0 < \frac{10}{3}$. Then for (p, q) obeying $(\frac{q}{2})' \le p \le q$ and*

$$\frac{1}{p} > \frac{2}{5} \cdot \frac{1/q - 3/10}{1/q_0 - 3/10} + \frac{1}{10}, \tag{7.1}$$

we have

$$\|\mathcal{E}f_N\|_q \lesssim \|f_N\|_p, \tag{7.2}$$

for all $f_N \in L^p(\Gamma_N)$, with bounds uniform in N.

Our main tool in the proof of Proposition 7.1 is the following consequence of Bourgain–Demeter's decoupling theorem for the cone.

Proposition 7.2. *Suppose that $\mathcal{R}_0^*(p \to q)$ holds for some $p \geq (\frac{q}{2})'$, $q \leq 4$. Then*

$$\|\mathcal{E}f\|_q \lesssim_\varepsilon 2^{(N-k)(\frac{1}{2}-\frac{1}{p}+\varepsilon)}\|f\|_p$$

for all functions f supported in an (N,k)-sector and all $\varepsilon > 0$.

Proof of Propostion 7.2. Let κ be an (N,k)-sector, let f_κ be supported in κ, and let $\mathcal{P}$ be a partition of κ into (N,N)-sectors. The estimate $\mathcal{R}_0^*(p \to q)$ and Lemma 6.2 imply that

$$\|\mathcal{E}f_\theta\|_q \lesssim \|f_\theta\|_p \tag{7.3}$$

for all f_θ supported in $\theta \in \mathcal{P}$. In particular, if $N - k \lesssim 1$, then $\#\mathcal{P} \lesssim 1$ and the required estimate is a consequence of the triangle inequality and (7.3). We may assume, therefore, that $N - k \geq C$ for some sufficiently large constant C.

We proceed by rescaling extension estimates on κ to those on a nearly conic set of angular width 1 in the region $|\xi| \sim 1$, where Bourgain–Demeter's conic decoupling theorem can be directly applied. We may assume by rotational symmetry that

$$\kappa = \{(\langle\langle\xi\rangle\rangle, \xi) : 2^N \leq |\xi| \leq 2^{N+1},\ \angle(\xi,(1,0)) \leq 2^{-k-1}\}. \tag{7.4}$$

Thus, κ lies in an $O(2^{-N})$-neighborhood of the conic sector

$$\kappa_c := \{(|\xi|, \xi) : 2^N \leq |\xi| \leq 2^{N+1},\ \angle(\xi,(1,0)) \leq 2^{-k-1}\}.$$

Let D be the conic dilation $D(\tau,\xi) := 2^{-N}(\tau,\xi)$. Then $D(\kappa_c)$ is a conic sector of angular width 2^{-k} in $C_0 := \{(|\xi|,\xi) : 1 \leq |\xi| \leq 2\}$ that contains the point $(1,1,0)$. Let L be the linear map satisfying

$$\begin{aligned} L(0,0,1) &= 2^k(0,0,1),\\ L(1,1,0) &= (1,1,0),\\ L(-1,1,0) &= 2^{2k}(-1,1,0). \end{aligned}$$

Geometrically, the vectors $(0,0,1)$, $(1,1,0)$, and $(-1,1,0)$ are respectively "angularly tangent," "radially tangent," and normal to C_0 at the point $(1,1,0)$. The map L preserves the cone and expands $D(\kappa_c)$ to angular width 1. Now, set $M = LD$ and $\delta = C'2^{2(k-N)}$, where C' is a constant. If C' is sufficiently large, then $M(\kappa)$ lies in the δ-neighborhood of a conic frustum $\tilde{C}_0$, a slight enlargement of C_0. Let $\mathrm{d}M_*\sigma$ be the pushforward measure on $M(\Gamma)$, given by

$$\int_{M(\Gamma)} g\,\mathrm{d}M_*\sigma := \int_\Gamma g \circ M\,\mathrm{d}\sigma,$$

and let $\mathcal{E}^M g := (g\mathrm{d}M_*\sigma)^\vee$. Let $\tilde{\mathcal{P}}$ be a partition of the δ-neighborhood of $\tilde{C}_0$ into sectors Δ of angular width $\delta^{1/2}$ and thickness δ. By conic decoupling, see [3, Theorem 1.2], the inequality

$$\|\mathcal{E}^M g\|_q \lesssim_\varepsilon \delta^{-\varepsilon}\Big(\sum_{\Delta\in\tilde{\mathcal{P}}} \|\mathcal{E}^M(g\mathbb{1}_{\Delta'})\|_q^2\Big)^{\frac{1}{2}} \tag{7.5}$$

holds for all g supported in $M(\kappa)$, where $\Delta' := \Delta \cap M(\kappa)$. We claim that every $\Delta \in \tilde{\mathcal{P}}$ obeys the bound

$$\#\{\theta \in \mathcal{P} : \theta \cap M^{-1}(\Delta') \neq \emptyset\} \lesssim 1. \tag{7.6}$$

Then, taking $g = f \circ M^{-1}$ in (7.5), rescaling, and applying (7.3) and Hölder's inequality, we get

$$\begin{aligned} \|\mathcal{E}f\|_q &\lesssim_\varepsilon \delta^{-\varepsilon} \Big(\sum_{\Delta \in \tilde{\mathcal{P}}} \|\mathcal{E}(f \mathbb{1}_{M^{-1}(\Delta')})\|_q^2 \Big)^{\frac{1}{2}} \\ &\lesssim \delta^{-\varepsilon} \Big(\sum_{\Delta \in \tilde{\mathcal{P}}} \sum_{\substack{\theta \in \mathcal{P}: \\ \theta \cap M^{-1}(\Delta') \neq \emptyset}} \|f\|^2_{L^p(\theta \cap M^{-1}(\Delta'))} \Big)^{\frac{1}{2}} \\ &\lesssim 2^{(N-k)(\frac{1}{2} - \frac{1}{p} + 2\varepsilon)} \|f\|_p. \end{aligned}$$

Since ε is arbitrary, the proof is complete modulo the claim (7.6).

To begin the proof of (7.6), we record the following notation: The angular separation of $\zeta, \zeta' \in \mathbb{R}^3$ is defined as

$$\operatorname{dist}_{\text{ang}}(\zeta, \zeta') := \left| \frac{(\zeta_2, \zeta_3)}{|(\zeta_2, \zeta_3)|} - \frac{(\zeta_2', \zeta_3')}{|(\zeta_2', \zeta_3')|} \right|.$$

Now, fix $\Delta \in \tilde{\mathcal{P}}$ and let $n := \#\{\theta \in \mathcal{P} : \theta \cap M^{-1}(\Delta') \neq \emptyset\}$. We need to show that $n \lesssim 1$, so we may assume that $n \geq 3$. Then there exist $\zeta, \zeta' \in \kappa \cap M^{-1}(\Delta')$ such that $\operatorname{dist}_{\text{ang}}(\zeta, \zeta') \gtrsim n 2^{-N}$. Since Δ' has angular width $O(2^{k-N})$, it suffices to show that $\operatorname{dist}_{\text{ang}}(M(\zeta), M(\zeta')) \gtrsim 2^k \operatorname{dist}_{\text{ang}}(\zeta, \zeta')$. Toward that end, it will be convenient to understand how M transforms the polar coordinates $(\langle\langle \xi \rangle\rangle, \xi) =: (\langle\langle r \rangle\rangle, r\cos\nu, r\sin\nu)$, where $\langle\langle r \rangle\rangle := \sqrt{r^2 - 1}$. We compute that $M(\langle\langle r \rangle\rangle, r\cos\nu, r\sin\nu) = 2^{-N-1}(m_i(r,\nu))_{i=1}^3$, where

$$\begin{aligned} m_1(r,\nu) &:= (1 + 2^{2k})\langle\langle r \rangle\rangle + (1 - 2^{2k}) r\cos\nu, \\ m_2(r,\nu) &:= (1 - 2^{2k})\langle\langle r \rangle\rangle + (1 + 2^{2k}) r\cos\nu, \\ m_3(r,\nu) &:= 2^{k+1} r \sin\nu. \end{aligned}$$

The polar angle associated to $M(\langle\langle r \rangle\rangle, r\cos\nu, r\sin\nu)$ is

$$A(r,\nu) := \arctan\left(\frac{m_3(r,\nu)}{m_2(r,\nu)} \right).$$

Thus, letting $\zeta =: (\langle\langle r \rangle\rangle, r\cos\nu, r\sin\nu)$ and $\zeta' =: (\langle\langle r' \rangle\rangle, r'\cos\nu', r'\sin\nu')$, we have

$$\operatorname{dist}_{\text{ang}}(M(\zeta), M(\zeta')) \sim |A(r,\nu) - A(r',\nu')|. \tag{7.7}$$

Since $\zeta \in \kappa$, we know that $2^N \leq r \leq 2^{N+1}$ and $|\nu| \leq 2^{-k-1}$ by (7.4). Consequently, one easily checks that $m_2(r,\nu) \sim 2^N$ and $|m_3(r,\nu)| \lesssim 2^N$; the same bounds hold for $m_2(r',\nu')$ and $m_3(r',\nu')$. Arguments using the mean value theorem and the preceding estimates show that

$$|A(r,\nu) - A(r,\nu')| \geq |\nu - \nu'| \inf_{|\varphi| \leq 2^{-k-1}} |\partial_2 A(r,\varphi)| \gtrsim 2^k \operatorname{dist}_{\text{ang}}(\zeta, \zeta')$$

and

$$|A(r,\nu') - A(r',\nu')| \le |r - r'| \sup_{2^N \le s \le 2^{N+1}} |\partial_1 A(s,\nu')| \lesssim 2^N 2^{2k-3N} \lesssim 2^{-C} 2^k \operatorname{dist}_{\text{ang}}(\zeta,\zeta').$$

Thus, if C is sufficiently large, then $|A(r,\nu) - A(r',\nu')| \gtrsim 2^k \operatorname{dist}_{\text{ang}}(\zeta,\zeta')$ by the triangle inequality. Plugging this estimate into (7.7) completes the proof. □

We are now ready to prove Proposition 7.1. By the hypothesis $\mathcal{R}_0^*((\frac{q_0}{2})' \to q_0)$, Proposition 7.2, and the Cauchy–Schwarz inequality, we have

$$\|\mathcal{E}f_1\mathcal{E}f_2\|_{q_0/2} \lesssim_\varepsilon 2^{-(N-k)(1-\frac{4}{q_0}-2\varepsilon)} \|f_1\|_{p_0}\|f_2\|_{p_0}$$

for all functions f_1, f_2 supported in (N,k)-sectors. Given $q_1 > \frac{10}{3}$, we also have

$$\|\mathcal{E}f_1\mathcal{E}f_2\|_{q_1/2} \lesssim 2^{-(N-k)(\frac{6}{q_1}-1)} \|f_1\|_2\|f_2\|_2$$

by Theorem 6.3, provided f_1 and f_2 are supported in related (N,k)-sectors. Interpolating these estimates, we see that

$$\|\mathcal{E}f_1\mathcal{E}f_2\|_{q_t/2} \lesssim_\varepsilon 2^{-(N-k)\alpha_t} \|f_1\|_{s_t}\|f_2\|_{s_t}, \tag{7.8}$$

for $0 \le t \le 1$, where

$$\left(\frac{1}{s_t}, \frac{1}{q_t}\right) := (1-t)\left(1 - \frac{2}{q_0}, \frac{1}{q_0}\right) + t\left(\frac{1}{2}, \frac{1}{q_1}\right),$$
$$\alpha_t := (1-t)\left(1 - \frac{4}{q_0} - 2\varepsilon\right) + t\left(\frac{6}{q_1} - 1\right).$$

We may apply Lemma 6.4 to obtain uniform restricted weak-type $L^p \to L^{q_t}$ bounds on dyadic annuli as long as $(\frac{q_t}{2})' \le p \le q_t$ and

$$\frac{1}{p} > \frac{1}{s_t} - \frac{\alpha_t}{2},$$

or, equivalently, after a bit of arithmetic, if

$$\frac{1}{p} > \left(\frac{3}{q_1} - \frac{1}{2} + \varepsilon\right)(1-t) + 1 - \frac{3}{q_1}. \tag{7.9}$$

Sending $q_1 \searrow \frac{10}{3}$ and $\varepsilon \searrow 0$, and substituting $1 - t = \frac{1/q_t - 1/q_1}{1/q_0 - 1/q_1}$ in (7.9) yields (7.1). Having proved restricted weak-type bounds in the claimed region, real interpolation completes the proof of Proposition 7.1.

8. Summing the bounds on annuli

The purpose of this section is to complete the proof of Theorem 1.1 by proving that uniform bounds for the extension from dyadic annuli imply global bounds on $\mathcal{E}$. Let $\mathcal{R}_{ann}^*(p \to q)$ denote the statement that for all $N \ge 1$ and measurable f_N supported on Γ_N,

$$\|\mathcal{E}f_N\|_q \lesssim \|f_N\|_p.$$

We will spend the majority of this section proving the following.

Lemma 8.1. *If $\mathcal{R}^*_{ann}(p_0 \to q_0)$ holds for some $(\frac{q_0}{2})' \leq p_0 \leq q_0$, then $\mathcal{R}^*(p \to q)$ holds for all $q > q_0$ and $p' = \frac{p_0'}{q_0} q$.*

Lemma 8.2. *If $\mathcal{R}^*_{ann}(q_0 \to q_0)$ holds for some $3 < q_0 < 4$, then $\mathcal{R}^*(q \to q)$ holds for all $q_0 < q < 4$.*

Before proving the lemmas in detail, we note that applying them in conjunction with Propositions 6.1 and 7.1 completes the proof of Theorem 1.1.

We will prove Lemmas 8.1 and 8.2 by proving that the hypotheses imply a bilinear extension estimate between annuli:

$$\|\mathcal{E}f_{N_1}\mathcal{E}f_{N_2}\|_{\frac{q}{2}} \lesssim 2^{-c_0|N_1-N_2|}\|f_{N_1}\|_p\|f_{N_2}\|_p, \tag{8.1}$$

for some $c_0 > 0$, and measurable functions $|f_{N_j}| \sim \mathbb{1}_{\Omega_{N_j}}$, $\Omega_{N_j} \subseteq \Gamma_{N_j}$, $j = 1, 2$. Indeed, assuming validity of such an estimate, for any $|f| \sim \mathbb{1}_\Omega$, by the triangle inequality and $q \leq 4$,

$$\begin{aligned}\|\mathcal{E}f\|_q^q &\lesssim \|f\|_p^q + \|\sum_{N\geq C}\mathcal{E}f_N\|_q^q \lesssim \|f\|_p^q + \sum_{N_1\geq N_2\geq N_3\geq N_4\geq C}\|\prod_{i=1}^4 \mathcal{E}f_{N_i}\|_{\frac{q}{4}}^{\frac{q}{4}} \\ &\lesssim \|f\|_p^q + \sum_{N_1\geq N_2\geq N_3\geq N_4\geq 1} 2^{-\frac{qc_0}{4}|N_1-N_4|}\prod_{i=1}^4 \|f\|_{L^p(\Gamma_{N_i})}^{\frac{q}{4}} \lesssim \sum_{N\geq 0}\|f\|_{L^p(\Gamma_N)}^q \lesssim \|f\|_{L^p}^q.\end{aligned}$$

Real interpolation leads to strong-type bounds.

Proof of Lemma 8.1. The Strichartz inequality (3.1) implies that

$$\begin{gathered}\|\mathcal{E}f\|_{L^r_tL^s_x} \lesssim \|\langle\langle\xi\rangle\rangle^{\frac{1}{r}-\frac{1}{s}}f\|_{L^2(\Gamma;\mathrm{d}\sigma)}, \\ 2 \leq r, s; \quad s < \infty; \quad \frac{2}{r} + \frac{2p_0'}{(q_0-p_0')s} = \frac{p_0'}{q_0-p_0'}.\end{gathered} \tag{8.2}$$

As (8.2) implies boundedness of $\mathcal{E}$ in the range $p = 2$, $4 \leq q \leq 6$, we may assume henceforth that $p_0 > 2$.

Let $q_2 = 2\frac{q_0}{p_0'}$, and choose some r_0, s_0, r_1, s_1 obeying (8.2), $r_0 < q_2 < s_0$, and

$$\frac{1}{q_2} = \frac{1}{2}(\frac{1}{r_0} + \frac{1}{r_1}) = \frac{1}{2}(\frac{1}{s_0} + \frac{1}{s_1}).$$

By the Cauchy–Schwarz inequality, for any $1 \leq N_1 \leq N_2$, we have the bilinear estimate

$$\begin{aligned}\|\mathcal{E}f_{N_1}\mathcal{E}f_{N_2}\|_{L^{\frac{q_2}{2}}} &\lesssim \|\mathcal{E}f_{N_1}\|_{L^{r_0}_tL^{s_0}_x}\|\mathcal{E}f_{N_2}\|_{L^{r_1}_tL^{s_1}_x} \lesssim 2^{N_1(\frac{1}{r_0}-\frac{1}{s_0})}2^{N_2(\frac{1}{r_1}-\frac{1}{s_1})}\|f_{N_1}\|_2\|f_{N_2}\|_2 \\ &= 2^{-(\frac{1}{r_0}-\frac{1}{s_0})|N_1-N_2|}\|f_{N_1}\|_2\|f_{N_2}\|_2.\end{aligned}$$

Inequality (8.1) follows by interpolation with the consequence

$$\|\mathcal{E}f_{N_1}\mathcal{E}f_{N_2}\|_{\frac{q_0}{2}} \lesssim \|f_{N_1}\|_{p_0}\|f_{N_2}\|_{p_0}$$

of our hypothesis. ∎

Proof of Lemma 8.2. We will prove a bilinear estimate between annuli as in (8.1), with $p = q$ and $C \leq N_1 \leq N_2 - C$ fixed. To do so, we will use three different bilinear extension estimates between sectors at different scales.

It is convenient to modify our Whitney decomposition slightly from earlier, though we will continue to use the convention that $|\frac{\xi}{|\xi|} - e_1| < c$ for all $\xi \in \Gamma_N$, for some sufficiently small c. For $C \leq k \leq N_1$, let $\mathcal{D}_k$ denote a 2^{-k}-separated subset of $\mathbb{S}^1$. For $\omega, \omega' \in \mathcal{D}_k$ and $k < N_1 - C$, we say that $\omega \sim \omega'$ if $2^{-k+C} \leq |\omega - \omega'| \leq 2^{-k+2C}$. Meanwhile, for $N_1 - C \leq k \leq N_1$, we say that $\omega \sim \omega'$ if $|\omega - \omega'| \leq 2^{-N_1+2C}$. Thus for $\xi_1 \in \Gamma_{N_1}$ and $\xi_2 \in \Gamma_{N_2}$, there is at least one and at most a bounded number of triples (k, ω, ω') with $\omega \sim \omega' \in \mathcal{D}_k$ and $\xi_1 \in \Gamma^{\omega}_{N_1,k}$ and $\xi_2 \in \Gamma^{\omega'}_{N_2,k}$.

By the hypothesis that $\mathcal{R}^*_{ann}(q_0 \to q_0)$ holds, interpolation, and the Cauchy–Schwarz inequality, for $k \geq C$ and $\omega, \omega' \in \mathcal{D}_k$, we have

$$\|\mathcal{E} f^{\omega}_{N_1,N_1}\, \mathcal{E} f^{\omega'}_{N_2,N_2}\|_{\frac{q}{2}} \lesssim \|f^{\omega}_{N_1,N_1}\|_q \|f^{\omega'}_{N_2,N_2}\|_q, \qquad q_0 \leq q < 4. \tag{8.3}$$

By Theorem 1.4 of [7] (see also [7, Theorem 1.10]), if $0 \leq k < N_1 - C$, $\omega \sim \omega' \in \mathcal{D}_k$, and $\frac{10}{3} < q_1 < 4$, then

$$\|\mathcal{E} f^{\omega}_{N_1,k}\, \mathcal{E} f^{\omega'}_{N_2,k}\|_{\frac{q_1}{2}} \lesssim 2^{-(N_1+N_2-2k)(\frac{3}{q_1}-\frac{1}{2})} 2^{-(\frac{3}{2}-\frac{5}{q_1})|N_1-N_2|} \|f^{\omega}_{N_1,k}\|_2 \|f^{\omega'}_{N_2,k}\|_2. \tag{8.4}$$

Finally, if $k \geq N_1 - C$ and $\omega \sim \omega' \in \mathcal{D}_k$, we claim that

$$\|\mathcal{E} f^{\omega}_{N_1,k}\, \mathcal{E} f^{\omega'}_{N_2,k}\|_2 \lesssim 2^{-\frac{1}{4}|N_1-N_2|} \|f^{\omega}_{N_1,k}\|_4 \|f^{\omega'}_{N_2,k}\|_4. \tag{8.5}$$

We now turn to the details of (8.5), which follow a well-established route. Let $(\langle\langle\xi\rangle\rangle, \xi) \in \Gamma^{\omega}_{N_1,k}$ and $(\langle\langle\eta\rangle\rangle, \eta) \in \Gamma^{\omega'}_{N_2,k}$. The coordinate change $\zeta = (\langle\langle\xi\rangle\rangle + \langle\langle\eta\rangle\rangle, \xi + \eta)$, $\beta = \xi^{\perp}$ (perpendicular direction taken with respect to ω) is finite-to-one, and has Jacobian determinant $|\frac{\partial(\zeta,\beta)}{\partial(\xi,\eta)}| \sim 2^{-2N_1}$. By Plancherel's identity, the change of variables formula (recall that we integrate with respect to $\mathrm{d}\sigma$), and Hölder's inequality (β varies over an interval of length at most 1), the right-hand side of (8.5) is bounded by

$$\|\widehat{\mathcal{E} f^{\omega}_{N_1,k}}\, \widehat{\mathcal{E} f^{\omega'}_{N_2,k}}\|_2 \lesssim \Big(\iint |f^{\omega}_{N_1,k}(\langle\langle\xi\rangle\rangle, \xi)\, f^{\omega'}_{N_2,k}(\langle\langle\eta\rangle\rangle, \eta) \tfrac{1}{\langle\langle\xi\rangle\rangle\langle\langle\eta\rangle\rangle} |\tfrac{\partial(\xi,\eta)}{\partial(\zeta,\beta)}||^2 \,\mathrm{d}\beta\, \mathrm{d}\zeta\Big)^{\frac{1}{2}}.$$

Changing variables back, estimating the various roughly constant terms that have arisen, and using Hölder's inequality again, the right-hand side of the preceding inequality is bounded by

$$2^{N_1} 2^{-\frac{1}{2}(N_1+N_2)} \sigma(\Gamma^{\omega}_{N_1,k})^{\frac{1}{4}} \sigma(\Gamma^{\omega'}_{N_2,k})^{\frac{1}{4}} \|f^{\omega}_{N_1,k}\|_4 \|f^{\omega'}_{N_2,k}\|_4. \tag{8.6}$$

Since $\sigma(\Gamma^{\omega}_{N_1,k}) \lesssim 1$, while $\sigma(\Gamma^{\omega'}_{N_2,k}) \lesssim 2^{N_2-N_1}$, inequality (8.6) implies (8.5).

Interpolating (8.3) and (8.4), yields, for all $\omega \sim \omega' \in \mathcal{D}_k$, $0 \leq k < N_1 - C$,

$$\|\mathcal{E} f^{\omega}_{N_1,k} \mathcal{E} f^{\omega'}_{N_2,k}\|_{\frac{q}{2}} \lesssim 2^{-(N_1+N_2-2k)\frac{\alpha}{2}} 2^{-\delta|N_1-N_2|} \|f^{\omega}_{N_1,k}\|_s \|f^{\omega'}_{N_2,k}\|_s, \tag{8.7}$$

for all $q_0 < q < 4$, some $s < q$, some $\alpha > \frac{2}{s} - \frac{2}{q}$, and some $\delta > 0$. Interpolating (8.5) and (8.3) (taking $q = q_0$ in the latter) yields, for $k = N_1 + C$ and $\omega \sim \omega' \in \mathcal{D}_{N_1,k}$,

$$\|\mathcal{E} f^{\omega}_{N_1,k} \mathcal{E} f^{\omega'}_{N_2,N_1}\|_{\frac{q}{2}} \lesssim 2^{-\delta|N_1-N_2|} \|f^{\omega}_{N_1,N_1}\|_q \|f^{\omega'}_{N_2,N_1}\|_q, \qquad q_0 < q < 4. \tag{8.8}$$

We adapt the bilinear to linear argument of Tao–Vargas–Vega [19]. Namely, if $|f_{N_j}| \sim \mathbb{1}_{\Omega_{N_j}}$, $j = 1, 2$, then using a partition of unity and almost orthogonality; using (8.7), (8.8), the Cauchy–Schwarz inequality, and reindexing; and finally summing as in the proof of Lemma 6.4,

$$\begin{aligned}
\|\mathcal{E} f_{N_1} \mathcal{E} f_{N_2}\|_{\frac{q}{2}} &\lesssim \sum_{k=C}^{N_1} \Big(\sum_{\omega \sim \omega' \in \mathcal{D}_k} \|\mathcal{E} f^{\omega}_{N_1,k} \mathcal{E} f^{\omega'}_{N_2,k}\|^{\frac{q}{2}}_{\frac{q}{2}} \Big)^{\frac{2}{q}} \\
&\lesssim 2^{-\delta|N_1 - N_2|} \left(\prod_{j=1}^{2} \Big(\sum_{\omega \in \mathcal{D}_{N_1}} \|f^{\omega}_{N_j,k}\|^q_q \Big)^{\frac{1}{q}} + \sum_{k=C}^{N_1} \prod_{j=1}^{2} 2^{-(N_j - k)\frac{\alpha}{2}} \Big(\sum_{\omega \in \mathcal{D}_k} \|f^{\omega}_{N_j,k}\|^q_s \Big)^{\frac{1}{q}} \right) \\
&\lesssim 2^{-\delta|N_1 - N_2|} \|f_{N_1}\|_q \|f_{N_2}\|_q.
\end{aligned}$$

□

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