Fiscal stimulus as an optimal control problem

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Abstract

During the Great Recession, Democrats in the United States argued that government spending could be utilized to "grease the wheels" of the economy in order to create wealth and to increase employment; Republicans, on the other hand, contended that government spending is wasteful and discourages investment, thereby increasing unemployment. This past year we have found ourselves in the midst of another crisis where government spending and fiscal stimulus is again being considered as a solution. In the present paper, we address this question by formulating an optimal control problem generalizing the model of Radner and Shepp (1996). The model allows for the company to borrow continuously from the government. We prove that there exists an optimal strategy; rigorous verification proofs for its optimality are provided. We proceed to prove that government loans increase the expected value of a company. We also examine the consequences of different profit-taking behaviors among firms who receive fiscal stimulus.
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1. Introduction

The purpose of this paper is to mathematically model optimal fiscal policy with the hopes of contributing to the ongoing debate between United States' Democrats and Republicans on

what government measures should be taken to stimulate the economy. The model was originally developed by the second named and the third named authors in 2010, shortly after the Great Recession and the U.S. government's subsequent bailout of large banks. In the midst of another major recession from the COVID-19 pandemic, this work offers a timely addition to the third named author's seminal contributions to mathematical finance and optimal control.[2]

The key macroeconomic question considered in this paper is how much – if at all – should the government inject into the private sector to improve aggregate wealth of the economy? To this end, we assume that a company can be characterized by four parameters, (x, μ, σ, r): the present cash reserve, $x > 0$, the profit rate, $\mu > 0$, the riskiness, $\sigma > 0$, and the prevailing interest rate, $r \geq 0$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq0}, \mathsf{P})$ be a filtered probability space, where $\mathcal{F}_t$ represents the market information available at time t. Let $W(t)$ be a Wiener process w.r.t. $\{\mathcal{F}_t\}_{t\geq0}$, representing uncertainty, and $Z(t)$ be the total dividends subtracted from the fortune up until time t. The cash reserve of the company at time t, denoted by $X(t)$, evolves according to the dynamics

$$X(t) = x + \mu t + \sigma W(t) - Z(t), \quad t \geq 0, \tag{1}$$

where $W(0) = 0$ and $Z(0) \geq 0$.

To model fiscal policies that can "grease the wheels" of the private sector, we assume that the government may choose to provide a loan to the firm in order to increase the firm's expected profit rate from μ to μ^*, where it is assumed that $\mu^* \geq \mu > 0$. This loan is to be repaid at the interest rate r. It is assumed that the firm may borrow continuously at a limited rate from the government.[3] The above considerations lead us to write the process $Z(t)$ as $Z(t) = Z_+(t) - Z_-(t)$ where $Z_+(t)$ and $Z_-(t)$, respectively, represent the total dividends (which may be taxed) paid up to time t and the total loans taken up to time t. Hence, we require $Z_\pm(t) \geq 0$ for any t and rewrite Eq. (1) as

$$X(t) = x + \mu t + \sigma W(t) + Z_-(t) - Z_+(t). \tag{2}$$

Let $\mathcal{A}$ be the collection of admissible controls (Z_+, Z_-) which satisfy assumptions (A1)–(A3) below:

(A1) $Z_+(t)$ is a nondecreasing and càdlàg process adapted to $\{\mathcal{F}_t\}_{t\geq0}$.
(A2) $Z_+(0) \in [0, x]$, and $\Delta Z_+(t) = Z_+(t) - Z_+(t-) \leq X(t-)$ for every $t > 0$.
(A3) $Z_-(t)$ is a nondecreasing and differentiable process adapted to $\{\mathcal{F}_t\}_{t\geq0}$ with $Z_-(0) = 0$ and $dZ_-(t)/dt \leq \mu^* - \mu$ for some $\mu^* \geq \mu$.

Condition (A2) implies that the company cannot make a lump-sum dividend payment greater than its current fortune. Condition (A3) means that the company may be supplied any amount less than or equal to $(\mu^* - \mu)dt$ in each interval dt. It is worth noting that the restriction we put on $Z_-(t)$ is different from the usual setup of dividend problems where singular controls are used to represent instantaneous large-scale capital injections. We require that the government loan is paid back at interest rate r, and the objective of the firm is to maximize its expected net value (to be defined) by choosing the policy (Z_+, Z_-) optimally. Mathematically, this means

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[2] This paper has been written for the special issue in memory of the third author, Larry Shepp, who passed away on April 23rd, 2013. During his later years, one of Shepp's key research interests concerned stochastic modeling in finance and economics.

[3] It has been shown in the economics literature that government loans and subsidies to firms can be an effective means of fiscal stimulus [21].

we want to compute the value function defined as[4]

$$V(x) = V(x; \mu, \mu^*) = \sup_{(Z_+, Z_-) \in \mathcal{A}} \mathsf{E}_x \int_{0-}^{\tau_0} e^{-rt} [dZ_+(t) - dZ_-(t)], \tag{3}$$

where $\tau_0 = \inf\{t : X(t) \le 0\}$ is the time that the company goes bankrupt and the notation $\int_{0-}^{\tau} dZ(t)$ should be understood as $Z(0) + \int_0^{\tau} dZ(t)$. The definition implies that $V(0) = 0$. Further, $V(x) \ge x$ for any x since one strategy for taking profit is to take the fortune immediately; bankruptcy occurs at time $\tau_0 = 0$.

If we replace $\mathcal{A}$ with a smaller admissible class $\mathcal{A}' = \{(Z_+, Z_-) \in \mathcal{A} : Z_-(t) \equiv 0\}$ and define the corresponding value function by

$$\bar{V}(x) = \sup_{(Z_+, Z_-) \in \mathcal{A}'} \mathsf{E}_x \int_{0-}^{\tau_0} e^{-rt} dZ_+(t), \tag{4}$$

the problem then reduces to that considered in the seminal paper of Radner and Shepp [24]. But this is equivalent to considering the function $V(x; \mu, \mu)$, which corresponds to the case where the maximum loan rate is zero. Since $\mathcal{A}' \subset \mathcal{A}$, we have

$$V(x; \mu, \mu^*) \ge V(x; \mu, \mu) = \bar{V}(x). \tag{5}$$

The question is as follows: if $\mu^* > \mu$, whether $Z_-(t) = 0$ is strictly suboptimal, i.e., whether we have $V(x) > \bar{V}(x)$? It turns out that we do indeed have strict inequality, which implies that the expected additional dividend payouts from having the government funds are strictly greater than the loan cost (until bankruptcy), provided that the company takes profits in an optimal way to maximize its presumed objective. More generally, we have $V(x; \mu, \mu^*) > V(x; \mu, \mu')$ for any $\mu^* > \mu' \ge \mu$; that is, the more the fiscal stimulus offered by the government, the larger net value of the company (see Section 3.)

Now, what if the company, after borrowing from the government, chooses a "greedy" policy that maximizes its own dividend payouts without caring to repay the loans? In Section 5.2, we will show that such a strategy is socially undesirable in that the expected net value of the company could be smaller than that with no government loans and, moreover, the expected dividend payouts may not even cover the loan cost. This represents an interesting caveat to the results of our model: in order to ensure that the mathematically optimal and socially optimal solution is achieved, the government must play some role in enforcing how the firms who take government money operate.

The paper is organized as follows. In Section 2 we review the related literature and, in particular, the results of the seminal work of Radner and Shepp [24], which can be viewed as a baseline model where the government does not offer any loans to companies. In Section 3 we derive the corresponding free-boundary problem for the value function given in Eq. (3) and prove the existence of the solution. Further optimal control results for our problem are provided in Section 4, including how the optimal dividend payout policy changes with the model parameters. In Section 5, we discuss the economic implications of different dividend payout policies. Section 6 concludes the paper with the requisite technical proofs.

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[4] Throughout this paper, we will frequently use notations such as $V(x; \mu, \mu^*)$ to indicate that the value function depends on the drift parameters. But parameters σ and r are always omitted since they are not of direct interest and can be treated as fixed.

2. Preliminaries

2.1. Solution to the Radner–Shepp model

Consider the problem with $\mu^* = \mu$ and the value function $\bar{V}(x)$ defined in Eq. (4), which we shall refer to as the Radner–Shepp model [24]. The solution was found by Dutta and Radner [11], Jeanblanc-Picqué and Shiryaev [15] and Asmussen and Taksar [4], and Dutta and Radner [11] further showed that the company that follows the optimal policy will go bankrupt in a finite time with probability 1. The optimal policy is to pay out dividends at a reflection barrier $\bar{a}$: if $X(t) \leq \bar{a}$, no payment is made; otherwise, an instantaneous payment is made so that $X(t)$ drops to $\bar{a}$. By solving the Hamilton–Jacobi–Bellman equation

$$\max\{\mathcal{L}v(x), 1 - v'(x)\} = 0, \qquad \mathcal{L} = -r + \mu\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2},$$

with the initial condition $v(0) = 0$, we obtain the solution

$$\bar{V}(x) = \begin{cases} \dfrac{e^{\gamma_+ x} - e^{\gamma_- x}}{\gamma_+ e^{\gamma_+ \bar{a}} - \gamma_- e^{\gamma_- \bar{a}}}, & 0 \leq x \leq \bar{a}, \\ \bar{V}(\bar{a}) + (x - \bar{a}), & \bar{a} < x < \infty, \end{cases} \tag{6}$$

where γ_+, γ_- are the roots of the indicial equation, $\gamma_\pm = (-\mu \pm \sqrt{\mu^2 + 2r\sigma^2})/\sigma^2$. The unknown optimal threshold $\bar{a}$ can be most easily determined by the smooth-fit heuristic, $\bar{V}''(\bar{a}) = 0$, which yields

$$\bar{a} = \frac{2}{\gamma_+ - \gamma_-} \log\left|\frac{\gamma_-}{\gamma_+}\right|, \tag{7}$$

given $\mu \geq 0$. If $\mu < 0$, then one can show $\bar{a} = 0$ and $\bar{V}(x) = x$; that is, it is always optimal to "take the money and run". Without loss of generality, henceforth we assume $\mu > 0$, which means the company is *profitable*. Interestingly, the threshold $\bar{a}$ goes to zero either as $\mu \downarrow 0$ or as $\mu \uparrow \infty$. We also point out for further reference that

$$\bar{V}(\bar{a}) = \frac{e^{\gamma_+ \bar{a}} - e^{\gamma_- \bar{a}}}{\gamma_+ e^{\gamma_+ \bar{a}} - \gamma_- e^{\gamma_- \bar{a}}} = \frac{\gamma_+ + \gamma_-}{\gamma_+ \gamma_-} = \frac{\mu}{r}. \tag{8}$$

To study the optimal payout policy, we introduce the notation $M_a(t; \mu)$, which denotes the unique solution to the Skorokhod reflection problem of the process $x + \mu t + \sigma W(t)$ reflected at the barrier $a > 0$ [31,32]. That is,

$$M_a(t; \mu) = \sup_{0 \leq s \leq t} (x + \mu s + \sigma W(s) - a)^+, \tag{9}$$

where $w^+ = w \vee 0$ denotes the positive part of w. The supremum in Eq. (4) is attained at $Z_+(t) = M_{\bar{a}}(t; \mu)$ [24].

For an arbitrary barrier payout policy with level a, we define the expected time-discounted total dividend payouts as[5]

$$D_a(x) = D_a(x; \mu) = \mathsf{E}_x\left[M_a(0; \mu) + \int_0^{\tau_0(a,\mu)} e^{-rt} dM_a(t; \mu)\right]. \tag{10}$$

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[5] We write $\tau_0 = \tau_0(a, \mu)$ to indicate that the bankruptcy time depends on μ and a.

As shown in [28], the formula in Eq. (6) is still applicable with $\bar{a}$ replaced by a, from which we obtain

$$
D_a(x; \mu) =
\begin{cases}
\dfrac{e^{\gamma_+ x} - e^{\gamma_- x}}{\gamma_+ e^{\gamma_+ a} - \gamma_- e^{\gamma_- a}}, & x \in [0, a], \\[3mm]
D_a(a; \mu) + (x - a), & x > a.
\end{cases}
\tag{11}
$$

In general, D_a does not satisfy the smooth-fit condition at the boundary a. For any $x \geq 0$, the mapping $a \mapsto D_a(x; \mu)$ is maximized at the optimal policy $a = \bar{a}$ and $D_{\bar{a}}(x; \mu) = \bar{V}(x)$.

2.2. Literature review

Both Dutta and Radner [11] and Radner and Shepp [24] allow a (μ, σ)-pair among $\{(\mu_i, \sigma_i) : i = 1, 2, \ldots, n\}$ to be part of the control, and Radner and Shepp [24] found which pair to use at any given value of $X(t)$. The solution gave rise to some surprising results, for example that if the company is nearly bankrupt then it should be very conservative and use the (μ_i, σ_i)-pair with the smallest σ_i which seems to be paradoxical to many economists; see the work of Sheth et al. [27]. For simplicity, in discussing the present question we will limit the company to only one corporate direction, i.e., $n = 1$.

Many variants of the Radner–Shepp model have been proposed in the literature [1,5,29]. Décamps and Villeneuve [10] extended the Radner–Shepp model to include a singular control process representing an investment, and recently, De Angelis and Ekström [9] found the optimal policy for the finite horizon case. For optimal dividend distribution problems with general diffusion models, see [22,23]. It should be noted that for insurance companies, this dividend distribution problem also involves finding an optimal reinsurance policy, which gives rise to another control component that can affect both the profit rate (drift) and the riskiness (volatility) of the underlying fortune process; see, for example, [3,7,12,14,30].

Another important generalization of the dividend problem is to incorporate capital injections, which may be in the form of equity issuance. There is a vast literature on this topic; see, among others, [6,13,17,19,20,25,26,33–35]. For a recent advance in the general theory, see [2]. The model we will propose in the present paper allows fiscal stimulus in the form of a loan from the government. However, in all the above references, the capital injection process is allowed to be singular and, in particular, be of barrier type; in the present paper, we restrict the maximum rate of government loans (and thus require that the capital injection process be continuous.)

3. Calculation of the value function $V(x; \mu, \mu^*)$

We now state the main result of this paper. For our value function defined in Eq. (3), under the optimal control, the dividend payout process is still of barrier type, though the threshold is different from that of Radner–Shepp model, and the company continuously borrows at the maximum possible rate, $c = \mu^* - \mu$.

Theorem 1 (*Verification*). *Consider the free-boundary problem*

$$
\begin{cases}
\mathcal{L}^* v(x) = c, & x \in [0, a], \\
v(0) = 0, & \\
v'(x) = 1, & x \in [a, \infty), \\
v''(x) = 0, & x \in [a, \infty),
\end{cases}
$$

where both a and v are unknown, $c = \mu^* - \mu \geq 0$ and

$$\mathcal{L}^* = -r + \mu^* \frac{\partial}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2}.$$

Let $\hat{V} \in C^2$ and $\hat{a} \geq 0$ be the solution to this problem. Then $\hat{V}(x) = V(x; \mu, \mu^*)$, the value function defined in Eq. (3). The associated optimal policy $(\hat{Z}_+, \hat{Z}_-)$ is

$$\hat{Z}_+(t) = M_{\hat{a}}(t; \mu^*) = \sup_{0 \leq s \leq t} (x + \mu^* s + \sigma W(s) - \hat{a})^+, \quad \hat{Z}_-(t) = ct.$$

Proof. The proof is given in Section 6.1. $\square$

Next, we show that the solution to the free-boundary problem given in Theorem 1 always exists. We first notice that $\mathcal{L}^* v = c$ is just a second-order linear equation with constant coefficients. Hence, standard differential equation results yield that for $x \in [0, \hat{a}]$, $\hat{V}(x)$ is of the form

$$-\frac{c}{r} + A_+ e^{\gamma_+^* x} + A_- e^{\gamma_-^* x},$$

where constants γ_+^*, γ_-^* are obtained by solving $\sigma^2 \gamma^2 / 2 + \mu^* \gamma - r = 0$,

$$\gamma_\pm^* = \frac{-\mu^* \pm \sqrt{(\mu^*)^2 + 2r\sigma^2}}{\sigma^2}.$$

Using boundary conditions $\hat{V}'(\hat{a}) = 1$ and $\hat{V}''(\hat{a}) = 0$, we find that

$$A_+ = \frac{-\gamma_-^* e^{-\gamma_+^* \hat{a}}}{\gamma_+^* (\gamma_+^* - \gamma_-^*)}, \quad A_- = \frac{\gamma_+^* e^{-\gamma_-^* \hat{a}}}{\gamma_-^* (\gamma_+^* - \gamma_-^*)}.$$

Since $\hat{V}$ is linear on $[\hat{a}, \infty)$, the solution can be written as

$$\hat{V}(x) = A_+ e^{\gamma_+^* x} + A_- e^{\gamma_-^* x} - (c/r), \quad x \in [0, \hat{a}], \tag{12}$$

$$\hat{V}(x) = \hat{V}(\hat{a}) + (x - \hat{a}), \quad x \in [\hat{a}, \infty). \tag{13}$$

Finally, the boundary condition $\hat{V}(0) = 0$ implies that $\hat{a}$ must satisfy

$$f(\hat{a}) = 0, \quad \text{where } f(a; \mu, \mu^*) = \frac{(-\gamma_-^*) e^{-\gamma_+^* a}}{(\gamma_+^*)(\gamma_+^* - \gamma_-^*)} + \frac{(\gamma_+^*) e^{-\gamma_-^* a}}{\gamma_-^* (\gamma_+^* - \gamma_-^*)} - \frac{c}{r}. \tag{14}$$

We denote the solution to the equation $f(a; \mu, \mu^*) = 0$ by $\hat{a} = \hat{a}(\mu, \mu^*)$ to emphasize its dependence on μ and μ^*. In Proposition 1, we prove that such a solution always exists and is unique given that $\mu > 0$ and $\mu^* \geq \mu$. Hence, by solving $f(a) = 0$ we obtain the optimal reflection barrier $\hat{a}$, but, unlike in the Radner–Shepp model, it does not have a closed-form expression. The only exception is the special case $\mu^* = \mu$, where we have $\hat{a}(\mu, \mu) = \bar{a}$, the latter of which is as defined in Eq. (7).

Proposition 1. *Assume $\mu > 0$ and $\mu^* \geq \mu$. The free-boundary problem in Theorem 1 has a unique solution $(\hat{V}, \hat{a})$ such that $\hat{V} \in C^2$ and $\hat{a} > 0$.*

Proof. It suffices to show that $f(a; \mu, \mu^*) = 0$ has only one solution and it is positive, and then the rest follows from (12) and (13). Using (14), it is straightforward to verify that f is monotone decreasing, $f(0) = \mu/r > 0$, and $f(\infty) = -\infty$. So there is one and only one root on $(0, \infty)$. $\square$

From (5), we already know that by borrowing money from the government, the company has an expected net value at least as large as in the Radner–Shepp model without government loans. Now we prove another key result of this work: as long as the optimal policy is used, the net value of the company is strictly larger; furthermore, the more money a company can borrow from the government, the larger net value it has.

Theorem 2. *Consider $x > 0$ and two pairs of parameters (μ_1, μ_1^*) and (μ_2, μ_2^*).*

(i) *If $\mu_1 = \mu_2 = \mu$ and $\mu_1^* > \mu_2^* \geq \mu$, then $V(x; \mu, \mu_1^*) > V(x; \mu, \mu_2^*)$.*
(ii) *If $\mu_1^* = \mu_2^* = \mu^*$ and $\mu_2 < \mu_1 \leq \mu^*$, then $V(x; \mu_1, \mu^*) > V(x; \mu_2, \mu^*)$.*

Proof. The proof for part (i) relies on the verification techniques used for proving Theorem 1 and requires the result of Proposition 2. Hence, the proof for this part is relegated to Section 6.2.

To prove part (ii), let $\hat{a}_2 = \hat{a}(\mu_2, \mu^*)$ be the solution to $f(a; \mu_2, \mu^*) = 0$. Using the definition of the barrier payout process given in (9), the optimal control for a company characterized by the five-tuple $(x, \mu_2, \mu^*, \sigma, r)$ can be denoted by $(M_{\hat{a}_2}(t; \mu^*), (\mu^* - \mu_2)t)$. For another company characterized by $(x, \mu_1, \mu^*, \sigma, r)$, the policy $(M_{\hat{a}_2}(t; \mu^*), (\mu^* - \mu_1)t)$ is admissible, which means to make dividend payments at reflection barrier $\hat{a}_2$ but borrow money at rate $\mu^* - \mu_1 < \mu^* - \mu_2$. Hence,

$$V(x; \mu_1, \mu^*) \geq \mathsf{E}_x \left[\int_{0-}^{\tau_0} e^{-rt} dM_{\hat{a}_2}(t; \mu^*) - \int_0^{\tau_0} (\mu^* - \mu_1)e^{-rt} dt \right]$$

$$= V(x; \mu_2, \mu^*) + \mathsf{E}_x \left[\int_0^{\tau_0} (\mu_1 - \mu_2)e^{-rt} dt \right].$$

Given any $x > 0$, we have $\tau_0 > 0$ and thus part (ii) of the theorem follows from the assumption $\mu_1 > \mu_2$. $\square$

4. Choice of the dividend payout barrier

We compare $\hat{a} = \hat{a}(\mu, \mu^*)$ with two suboptimal choices:

$$\bar{a} = \hat{a}(\mu, \mu), \qquad a^* = \hat{a}(\mu^*, \mu^*).$$

The threshold $\bar{a}$ is the same as that defined in Eq. (7), which is the optimal threshold for the Radner–Shepp model where borrowing is not allowed. The threshold a^* represents the greedy strategy of a firm that has original profit rate μ and borrows at rate $c = \mu^* - \mu$; the firm does not care about repaying the loan and thus chooses to use a^* to maximize the expected total dividend payouts (of course, a^* is also the optimal threshold for the Radner–Shepp model where the profit rate of the firm is μ^*.)

We first prove that $\hat{a}$ is always less than the Radner–Shepp threshold $\bar{a}$. An immediate consequence is that since $\hat{a} \in (0, \bar{a})$, $\hat{a}$ can be computed numerically using a standard one-dimensional optimization algorithm.

Proposition 2. *Assume $\mu^* \geq \mu > 0$. Then $\hat{a} = \hat{a}(\mu, \mu^*) \leq \hat{a}(\mu, \mu) = \bar{a}$ where $\bar{a}$ is the optimal threshold of the Radner–Shepp model. Further, for any $\mu_1^* > \mu_2^* \geq \mu$, we have the strict inequality $\hat{a}(\mu, \mu_1^*) < \hat{a}(\mu, \mu_2^*)$.*

Proof. Recall that in (8) we showed that $V(\hat{a}(\mu, \mu); \mu, \mu) = \mu/r$. This is actually a special case of the identity

$$V(\hat{a}(\mu, \mu^*); \mu, \mu^*) = \frac{\mu}{r}, \qquad \text{for } \mu^* \geq \mu,$$

which can be straightforwardly verified using the boundary conditions in Theorem 1. By (5), $V(x; \mu, \mu^*) \geq V(x; \mu, \mu)$ for every x. Further, both $V(x; \mu, \mu)$ and $V(x; \mu, \mu^*)$ are monotone increasing in x. Hence, $V(\bar{a}; \mu, \mu) = V(\hat{a}; \mu, \mu^*) = \mu/r$ implies that $\hat{a} \leq \bar{a}$.

To prove the strict inequality, consider the function f defined in (14) and the mapping $\mu^* \mapsto f'(a; \mu, \mu^*)$ where f' denotes the derivative with respect to a. Routine but heavy calculation gives

$$\frac{\partial f'(a; \mu, \mu^*)}{\partial \mu^*} = -\frac{r\sigma^2 e^{-\gamma_+^* a}}{[(\mu^*)^2 + 2r\sigma^2]^{3/2}} \left\{ 1 + h(\mu^*) + e^{2h(\mu^*)}(h(\mu^*) - 1) \right\},$$

where $h(\mu^*) = a\sigma^{-2}\sqrt{(\mu^*)^2 + 2r\sigma^2}$. By computing the first two derivatives, one can verify that $h \mapsto 1 + h + e^{2h}(h - 1)$ is always positive on $(0, \infty)$. Hence, for any $a \geq 0$, $\mu^* \mapsto f'(a; \mu, \mu^*)$ is monotone decreasing. Combining this with the facts that $f(0; \mu, \mu^*) = \mu/r > 0$ and $f'(a; \mu, \mu^*) < 0$ for any $a \geq 0$, we conclude that $\mu^* \mapsto \hat{a}(\mu, \mu^*)$ is also strictly monotone decreasing. $\square$

We prove in Proposition 3 that $\hat{a}$ is also smaller than the greedy threshold a^* given $\mu^* > \mu$. Hence, in the absence of being held accountable, the firm taking the loan will pay out dividends later, which might be surprising.

Proposition 3. *Assume $\mu^* > \mu > 0$. Then $\hat{a} = \hat{a}(\mu, \mu^*) < \hat{a}(\mu^*, \mu^*) = a^*$ where a^* is the optimal threshold of the Radner–Shepp model with profit rate μ^*. More generally, for any $\mu^* \geq \mu_1 > \mu_2$, we have $\hat{a}(\mu_1, \mu^*) > \hat{a}(\mu_2, \mu^*)$.*

Proof. We only need to prove the general claim since $a^* = \hat{a}(\mu^*, \mu^*)$ is a special case. Recall that, for any $\mu \leq \mu^*$, $\hat{a}(\mu, \mu^*)$ is the solution to $f(a; \mu, \mu^*) = 0$ where f is as defined in (14), and $f(a; \mu, \mu^*)$ is monotone decreasing in a. Observe that in the expression for f, only the last term $c = \mu^* - \mu$ depends on μ. Since μ only changes the vertical shift, but not the shape, of the function $f(a; \mu, \mu^*)$, we conclude that $\hat{a}(\mu_1, \mu^*) > \hat{a}(\mu_2, \mu^*)$ as $\mu^* - \mu_1 < \mu^* - \mu_2$. $\square$

To gain further insights into the problem, here we give an alternative proof for the inequality $\hat{a} \leq a^*$. For our model, the net value of a company is the difference between the total dividend payouts and the total loan cost (both time-discounted). We use $(M_a(t; \mu^*), ct)$ to denote a policy that always borrows money at the maximum rate c and pays out whatever amount that exceeds a threshold $a > 0$. For such a policy, the expected net value can be written as

$$V_a(x; \mu, \mu^*) = D_a(x; \mu^*) - C_a(x; \mu, \mu^*) \tag{15}$$

where D_a (the total dividend payouts) is as defined in (10) and can be computed by (11), and the total loan cost is given by

$$C_a(x) = C_a(x; \mu, \mu^*) = \mathsf{E}_x \int_0^{\tau_0(a, \mu^*)} ce^{-rt} dt.$$

Note that since the company borrows at the maximum rate, both D_a and τ_0 (the bankruptcy time) do not depend on the original profit rate μ. Since $\hat{a}$ is the optimal barrier for the value function in Eq. (3), we have

$$D_{\hat{a}}(x; \mu^*) - C_{\hat{a}}(x; \mu, \mu^*) = V_{\hat{a}}(x) \geq V_{a^*}(x) = D_{a^*}(x; \mu^*) - C_{a^*}(x; \mu, \mu^*).$$

On the other hand, $a \mapsto D_a(x; \mu^*)$ is maximized at $a = a^*$ since a^* is optimal for the Radner–Shepp model with profit rate μ^*. Thus, $D_{\hat{a}}(x; \mu^*) \leq D_{a^*}(x; \mu^*)$, which further implies that
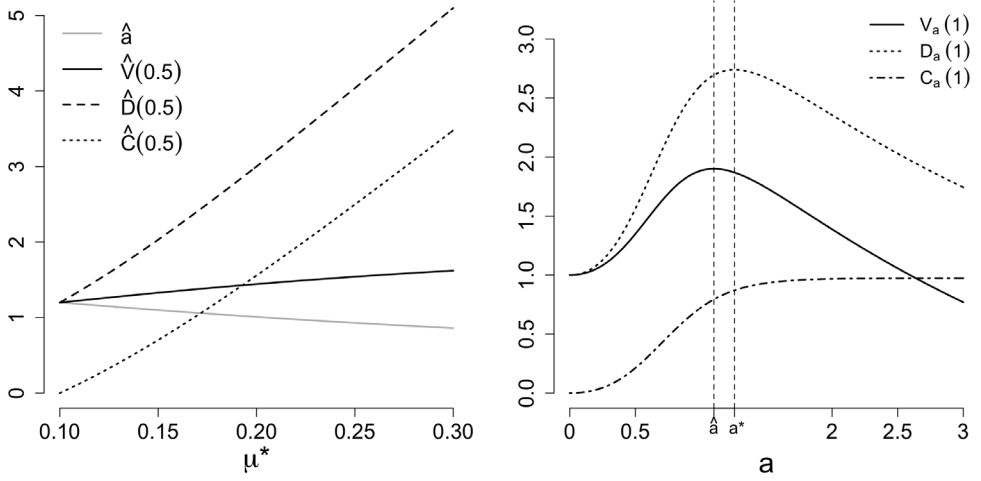
Fig. 1. Our model with $\mu = 0.1$, $r = 0.05$ and $\sigma = 0.3$. In the left panel, we fix $x = 0.5$, let μ^* range from 0.1 to 0.3, and plot $\hat{a} = \hat{a}(\mu, \mu^*)$, $\hat{V}(x) = V(x; \mu, \mu^*)$, $\hat{D}(x) = D_{\hat{a}}(x; \mu^*)$, $\hat{C}(x) = C_{\hat{a}}(x; \mu, \mu^*)$. Note that $\bar{a} = \hat{a}(\mu, \mu) = 1.200$. In the right, we fix $\mu^* = 0.15$, $x = 1$, let barrier a range from 0 to 3, and plot the expected net value $V_a(x) = V_a(x; \mu, \mu^*)$, the expected dividend payouts $D_a(x) = D_a(x; \mu^*)$, and the expected loan cost $C_a(x) = C_a(x; \mu, \mu^*)$. Note that V_a is maximized at $\hat{a} = 1.099$ and D_a is maximized at $a^* = 1.257$.

$C_{\hat{a}}(x) \leq C_{a^*}(x)$. For fixed μ, μ^*, the mapping $a \mapsto C_a(x)$ is monotone increasing since a larger value of the barrier would imply a longer expected "lifetime" of the company. Hence, we conclude that $\hat{a} \leq a^*$.

To compute the function C_a, note that

$$C_a(x; \mu, \mu^*) = \frac{c}{r}\left\{1 - g_a(x; \mu^*)\right\}, \quad \text{where } g_a(x; \mu^*) = \mathsf{E}_x\left[e^{-r\tau_0(a, \mu^*)}\right]. \quad (16)$$

As shown in [28], for $x \in [0, a]$, $g_a(x)$ is the solution to the differential equation $\mathcal{L}^* g = 0$ with boundary conditions $g'(a) = 0$ and $g(0) = 1$ (cf. [8,18]). Straightforward calculation then yields that

$$g_a(x; \mu^*) = \frac{\gamma_-^* e^{-\gamma_+^*(a-x)} - \gamma_+^* e^{-\gamma_-^*(a-x)}}{\gamma_-^* e^{-\gamma_+^* a} - \gamma_+^* e^{-\gamma_-^* a}}, \quad x \in [0, a].$$

For $x > a$, we have $g_a(x; \mu^*) = g_a(a; \mu^*)$ due to the initial dividend payment that forces $X(0) = a$. Hence, for each $a \geq 0$, we can explicitly compute the value of V_a defined in (15), and the mapping $a \mapsto V_a^*(x)$ must be maximized at $\hat{a} = \hat{a}(\mu, \mu^*)$.

5. Discussion

5.1. Analysis of the optimal payout policy

We present a numerical example in Fig. 1 illustrating the theoretical results proved in the previous section. By Theorem 2, when the "socially optimal" barrier $\hat{a} = \hat{a}(\mu, \mu^*)$ is used, the firm's expected net value increases monotonically as the size of the loan increases. The left panel of Fig. 1 shows the growth curve for a firm with original profit rate $\mu = 0.1$, as μ^* increases from 0.1 (no government loan) to 0.3. The dividend payouts increase at a slightly

faster rate than the cost of the loan, and thus the firm is able to add value by taking on the government loan, when cost is taken into account.

By Proposition 2, as the firm takes on an increasing amount of government loans, holding everything else constant, the barrier $\hat{a}$ actually decreases; see the gray line in the left panel of Fig. 1. In particular, we have $\bar{a} > \hat{a}$, where $\bar{a} = \hat{a}(\mu, \mu)$ is the optimal barrier when no money can be borrowed. From an economic perspective, this says that the firm will choose to pay dividends sooner if it receives more government funds. In the right panel of Fig. 1, μ^* is fixed at 0.15 and we examine the three functions given in (15), V_a, D_a and C_a, for different barrier levels. The expected net value V_a is maximized when the firm selects $\hat{a}$, which incorporates repayment of the loan. The function D_a is maximized at a^*, the optimal barrier of a firm with profit rate μ^* but without access to government loans. As shown in Proposition 3, we always have $\hat{a} < a^*$.

The observation that $\hat{a}$ is less than either $\bar{a}$ or a^* reveals what economists have known for years about perverse financial incentives but resulting from a purely mathematical perspective within the constructs of our optimal control problem. When a firm is given money without oversight, or knows it will be bailed out, it may act more selfishly and recklessly (economists refer to this "moral hazard"). This also justifies why, in reality, the government often requires, as a condition attached to stimulus packages or bailouts, that their loans must be paid back before companies can pay dividends to shareholders. The fiscal stimulus from the government in the form of a loan does boost the economy, but with the caveat that there are some greedy incentives at work and therefore requires oversight.

Nevertheless, we point out that $\hat{a} < \bar{a}$ does not mean that the firms may go bankruptcy faster when borrowing money. Actually, according to our numerical results (not shown here), we conjecture that for any given $\mu > r$ and $0 < x \leq \hat{a}(\mu, \mu)$, the mapping $\mu^* \mapsto g_{\hat{a}(\mu,\mu^*)}(x; \mu^*) \equiv \tilde{g}(\mu^*)$ is monotone decreasing on $[\mu, \infty)$, where g is as given in (16). Since $\tilde{g}(\mu^*) = \mathsf{E}_x\left[e^{-r\tau_0}\right]$ where τ_0 is the bankruptcy time of the firm using the barrier $\hat{a}$, this conjecture implies that the more money a firm can borrow, the longer lifetime it tends to have. Hence, government loans do improve the financial stability of firms in the sense that bankruptcy can be delayed.

5.2. Analysis of the greedy payout policy

If a company can borrow money at rate $\mu^* - \mu$ but does not care about repaying the loan, it would use the greedy threshold a^* since it maximizes the expected total dividend payouts. Numerical experiments were performed to investigate the consequences of such greedy policies, from which we have made two interesting observations.

First, the expected net value of a greedy company that does take the loan, $V_{a^*}(x; \mu, \mu^*)$, could be smaller than the value of a company that does not take the loan, $V(x; \mu, \mu)$. That is, for a greedy company, the additional profit gained by borrowing money may not be able to even cover the loan cost! In Fig. 2, we give a numerical example of two companies. We fix $x = 1$, $r = 0.05$, with an original profit rate $\mu = 0.08$ for Company 1 and $\mu = 0.06$ for Company 2. For Company 1 (solid line), $V_{a^*}(x; \mu, \mu^*)$ keeps increasing as μ^* increases, which means that the more money it borrows, an even larger increase in value can be obtained by using the greedy payout barrier a^*. However, for Company 2 (dashed line), when government loan is not allowed, it has $V(1; \mu, \mu) = 1.24$. But if $\mu^* = 0.16$, which means to borrow at rate 0.1, $V_{a^*}(1; \mu, \mu^*)$ drops to 1.17. Therefore, for Company 2, its greedy payout policy is *socially undesirable*.
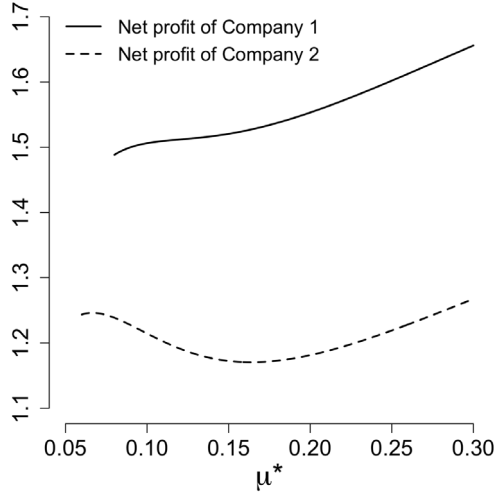
Fig. 2. Behavior of two "greedy" companies with $r = 0.05$ and $x = 1$. We plot $V_{a^*}(x; \mu, \mu^*)$ (the net value of a greedy policy) against the parameter μ^*. The first company has original profit rate $\mu = 0.08$ and the second has $\mu = 0.06$. Both have volatility $\sigma = 0.3$.

We also find that $D_{a^*}(x; \mu^*)$ is not necessarily greater than or equal to $C_{a^*}(x; \mu, \mu^*)$; that is, it could be dangerous of the government to lend money to some firms since they may not even be able to repay the loan (in expectation), though this happens rarely according to our numerics (only when both μ and x are very small). For example, let $\mu = 0.005$, $\mu^* = 0.055$, $\sigma = 0.1$, $r = 0.05$ and $x = 0.05$. One can compute that $a^* = 0.416$ and $\hat{a} = 0.098$. Numerics show that as long as the payout barrier $a > 0.34$, the net value, $V_a(x; \mu, \mu^*)$, would be negative.

The above observations also lend support to our claim that government intervention may be needed in restricting the activities of the firms that receive stimulus until the loans are paid back. One way to incorporate the government intervention in our mathematical model is to require that the payout barrier has to be within some pre-specified range, say $[a_{\min}, a_{\max}]$. The parameters $a_{\min}$ and $a_{\max}$ should be calculated for each individual company separately in order to guarantee that the company is able to repay the loan. We note that a similar problem was considered by Paulsen [22] assuming no capital injections or fiscal stimulus.

5.3. Extensions of our model

We may interpret r in our model as an exogenous parameter that is charged by the government directly, as the government loans may not be funneled through the banking system and can come in the form of a direct subsidy. One may also consider a more general formulation of our problem with the value function (cf. [20])

$$V(x; \mu, \mu^*, \beta) = \sup_{(Z_+, Z_-) \in \mathcal{A}} \mathsf{E}_x \int_{0-}^{\tau_0} e^{-rt} [dZ_+(t) - \beta dZ_-(t)],$$

where $\beta \geq 1$ represents the proportional cost of borrowing money from the government. When $\beta = 1$, there is no additional cost and the value function reduces to the one defined in Eq. (3).

However, one can check that our argument cannot be straightforwardly extended to the case of $\beta > 1$, though it can be used for the case of $\beta < 1$, the case where we have some proportional cost of taking the dividends (see Section 6.3.) We leave the case of $\beta > 1$ to future study.

6. Proofs

6.1. Proof for Theorem 1

The verification consists of two steps. First, we need to check that $\hat{V}(x)$ is indeed the expected net value of the firm when we apply the candidate optimal control function $(\hat{Z}_+, \hat{Z}_-)$. Second, we need to prove that no other policy can do better. The latter requires the following lemma.

Lemma 1. *The solution $\hat{V}$ given in* (12) *and* (13) *satisfies $\hat{V}'(x) \geq 1$ and $\mathcal{L}^*\hat{V}(x) \leq c$ for any $x \geq 0$. Consequently, the following Hamilton–Jacobi–Bellman equation holds:*

$$\max\{\mathcal{L}^*\hat{V}(x) - c, \ 1 - \hat{V}'(x)\} = 0, \qquad \forall\, x \geq 0.$$

Proof. We first show that $\hat{V}''(x) \leq 0$ for any $x \geq 0$. By the expression of $\hat{V}$ given in (12) and (13), this is equivalent to proving that, for any $x \in [0, \hat{a}]$,

$$A_+(\gamma_+^*)^2 e^{(\gamma_+^* - \gamma_-^*)x} \leq -A_-(\gamma_-^*)^2. \tag{17}$$

Since by definition $\gamma_+^* > 0$ and $\gamma_-^* < 0$, we only need show (17) holds true for $x = \hat{a}$. But we already know that $V''(\hat{a}) = 0$ and thus $A_+(\gamma_+^*)^2 e^{(\gamma_+^* - \gamma_-^*)\hat{a}} = -A_-(\gamma_-^*)^2$. Hence we conclude $\hat{V}''(x) \leq 0$.

Since $\hat{V}'(x) = 1$ for any $x \geq \hat{a}$, the non-positivity of $\hat{V}''$ implies that $\hat{V}'(x) \geq 1$ for any $x \geq 0$. To prove $\mathcal{L}^*\hat{V}(x) \leq c$, notice that $\hat{V}(x) > \hat{V}(\hat{a})$, $\hat{V}'(x) = \hat{V}'(\hat{a})$, $\hat{V}''(x) = \hat{V}''(\hat{a})$ for any $x > \hat{a}$. The claim then follows from the condition that $\mathcal{L}^*\hat{V}(\hat{a}) = c$. $\square$

Remark 1. The smooth-fit condition, $\hat{V}''(\hat{a}) = 0$, is critical in the above proof. Assume all the other conditions in Theorem 1 are satisfied. Since $\hat{V}'(x) = 1$ for all $x \geq \hat{a}$, $\hat{V}''(\hat{a}+) = 0$. If $\hat{V}''(\hat{a}-) > 0$, then $\hat{V}'(\hat{a} - \epsilon) < 1$ for some $\epsilon > 0$ since $\hat{V}'(\hat{a}) = 1$. If $\hat{V}''(\hat{a}-) < 0$, one can show that $\mathcal{L}^*\hat{V}(\hat{a}+) > c$, since there is a jump increase in $\hat{V}''(x)$ at $x = \hat{a}$ and $\mathcal{L}^*\hat{V}(\hat{a}-) = c$. Therefore, the smooth-fit condition is necessary for the Hamilton–Jacobi–Bellman equation to hold true.

Now we present our verification proof.

Step 1. We use $\hat{X}$ to denote the cash reserve process when the candidate optimal control $(\hat{Z}_+, \hat{Z}_-)$ is applied, and let $\tau_0 = \tau_0^{\hat{X}}$ be the time of bankruptcy. First, consider the case $x \in [0, \hat{a}]$. The process $\hat{Z}_+(t) = M_{\hat{a}}(t; \mu^*)$ is just (a multiple of) the local time of the process $\hat{X}$ at level $\hat{a}$ ([16], Chap. 3.6). Define $\alpha_t = \tau_0^{\hat{X}} \wedge t$ and consider the process $e^{-r\alpha_t}\hat{V}(\hat{X}(\alpha_t))$. By Itô's formula,

$$e^{-r\alpha_t}\hat{V}(\hat{X}(\alpha_t))$$
$$= \hat{V}(x) + \int_0^{\alpha_t} e^{-rs}\mathcal{L}^*\hat{V}(\hat{X}(s))ds + \int_0^{\alpha_t} e^{-rs}\hat{V}'(\hat{X}(s))\Big[\sigma dW(s) - d\hat{Z}_+(s)\Big].$$

For any $s \in [0, \tau_0^{\hat{X}}]$, we always have $\hat{X}(s) \in [0, \hat{a}]$, and thus $\mathcal{L}^* \hat{V}(\hat{X}(s)) = c$ and $\hat{V}'(\hat{X}(s))$ stays bounded. Hence, the integral with respect to dW is a martingale. Taking expectations on both sides we obtain

$$\mathsf{E}_x \, e^{-r\alpha_t} \, \hat{V}(\hat{X}(\alpha_t)) = \hat{V}(x) + \mathsf{E}_x \int_0^{\alpha_t} e^{-rs} \left[c \, ds - d\hat{Z}_+(s) \right], \tag{18}$$

where we have also used the fact that $\hat{V}'(\hat{a}) = 1$, and $d\hat{Z}_+(t) = 0$ if $\hat{X}(t) \neq \hat{a}$. We now claim that

$$\lim_{t \to \infty} \mathsf{E}_x \, e^{-r\alpha_t} \, \hat{V}(\hat{X}(\alpha_t)) = 0. \tag{19}$$

If bankruptcy happens, i.e., $\tau_0^{\hat{X}} < \infty$, then $\hat{V}(\hat{X}(t)) = 0$ for any $t \geq \tau_0^{\hat{X}}$; if $\tau_0^{\hat{X}} = \infty$, $\hat{V}(\hat{X}(t))$ always stays bounded and $e^{-r\alpha_t} \to 0$, from which (19) follows. For the integral on the right hand-side of (18), we have

$$\lim_{t \to \infty} \mathsf{E}_x \int_0^{\alpha_t} c e^{-rs} ds = \mathsf{E}_x \int_{0-}^{\tau_0^{\hat{X}}} c e^{-rs} ds \leq \frac{c}{r},$$

$$\lim_{t \to \infty} \mathsf{E}_x \int_0^{\alpha_t} e^{-rs} d\hat{Z}_+(s) = \mathsf{E}_x \int_{0-}^{\tau_0^{\hat{X}}} e^{-rs} d\hat{Z}_+(s),$$

by monotone convergence theorem. Hence, letting $t \to \infty$ in (18), we obtain

$$\hat{V}(x) = \mathsf{E}_x \int_{0-}^{\tau_0^{\hat{X}}} e^{-rt} \left[d\hat{Z}_+(t) - d\hat{Z}_-(t) \right]. \tag{20}$$

In particular, we have established the equality (20) for $x = \hat{a}$, which, together with the expression of $\hat{V}$ given in (13), can be used to show that (20) also holds for $x \in (\hat{a}, \infty)$.

Remark 2. The smooth-fit condition, $\hat{V}''(\hat{a}) = 0$, is not used in Step 1. One can verify that the function V_a given in (15) is the solution to the ordinary differential equation $\mathcal{L}^* v(x) = c$ for $x \in [0, a]$, with boundary conditions $v(0) = 0$, $v'(a) = 1$. Hence, the above argument also proves that V_a is indeed the expected net value of the policy $(M_a(t; \mu^*), (\mu^* - \mu)t)$.

Step 2. For any admissible policy $Z = (Z_+, Z_-)$, let

$$V_Z(x) = \mathsf{E}_x \int_{0-}^{\tau_0^X} e^{-rt} \left[dZ_+(t) - dZ_-(t) \right]$$

be the expected net value of the firm. Note that both X and τ_0^X now depend on (Z_+, Z_-), though this is not indicated explicitly in the notation. We need to show that $V_Z(x) \leq \hat{V}(x)$. By assumption, Z_- is a continuous process but Z_+ is not necessarily so. Therefore, we let Z_+^c denote the continuous part of Z_+. The dynamics of X is given by (2). Applying Itô's formula to $\hat{V}(X_t)$, we obtain

$$\hat{V}(x) = - \int_0^{t \wedge \tau_0^X} \sigma e^{-rs} \hat{V}'(X(s)) dW(s) + I_1 + I_2 - I_3 + I_4, \tag{21}$$

where

$$
I_1 = \int_0^{t \wedge \tau_0^X} e^{-rs} \left[\hat{V}'(X(s))(-dZ_-(s) + c\,ds) - \mathcal{L}^* \hat{V}(X(s))ds \right],
$$

$$
I_2 = \int_0^{t \wedge \tau_0^X} e^{-rs} \hat{V}'(X(s))dZ_+^c(s),
$$

$$
I_3 = \sum_{0 \leq s \leq (t \wedge \tau_0^X)} e^{-rs} \left\{ \hat{V}(X(s)) - \hat{V}(X(s-)) \right\},
$$

$$
I_4 = e^{-r(t \wedge \tau_0^X)} \hat{V}(X(t \wedge \tau_0^X)).
$$

(22)

From Lemma 1, we have $\hat{V}'(x) \geq 1$ and $\mathcal{L}^* \hat{V}(x) \leq c$. Moreover, since c is the maximum rate of loans, we have $-dZ_-(s) + c\,ds \geq 0$. We then obtain that

$$
I_1 \geq \int_0^{t \wedge \tau_0^X} e^{-rs} \left[-dZ_-(s) + c\,ds - c\,ds \right] = - \int_0^{t \wedge \tau_0^X} e^{-rs} dZ_-(s)
$$

and

$$
I_2 \geq \int_0^{t \wedge \tau_0^X} e^{-rs} dZ_+^c(s).
$$

Further, $\hat{V}'(x) \geq 1$ also implies that

$$
I_3 \leq - \sum_{0 \leq s \leq (t \wedge \tau_0^X)} e^{-rs} \Delta Z_+(s),
$$

where we have used the fact that $X(s) - X(s-) = -(Z_+(s) - Z_+(s-))$. Taking expectations on both sides of (21) and using the boundedness of $\hat{V}'$, we get

$$
\hat{V}(x) \geq \mathsf{E}_x \int_{0-}^{t \wedge \tau_0^X} e^{-rs} \left[dZ_+(s) - dZ_-(s) \right],
$$

since clearly $I_4 \geq 0$. Finally, by letting $t \to \infty$, applying monotone convergence theorem and noting that $\int_0^\infty e^{-rs} dZ_-(s) < \infty$, we obtain

$$
\hat{V}(x) \geq \lim_{t \to \infty} \mathsf{E}_x \int_{0-}^{t \wedge \tau_0^X} e^{-rs} \left[dZ_+(s) - dZ_-(s) \right]
$$

$$
= \mathsf{E}_x \int_{0-}^{\tau_0^X} e^{-rs} \left[dZ_+(s) - dZ_-(s) \right] = V_Z(x),
$$

which completes the proof.

6.2. Proof for Theorem 2(i)

Let $b = \hat{a}(\mu, \mu_2^*)$. The value function $V(x; \mu, \mu_2^*)$ is attained by the policy, $\hat{Z}_2 = (M_b(t; \mu_2^*), (\mu_2^* - \mu)t)$, where money is borrowed at rate $\mu_2^* - \mu$ and dividend payments are made at the barrier b. Clearly, this policy is also admissible to a company characterized by $(x, \mu, \mu_1^*, \sigma, r)$ since $\mu_1^* > \mu_2^*$, and we want to show that for any $x > 0$,

$$
V(x; \mu, \mu_1^*) > V(x; \mu, \mu_2^*) = V_{\hat{Z}_2}(x; \mu, \mu_1^*).
$$

Note that in the above verification proof, we have already shown $V(x; \mu, \mu_1^*) \geq V_{\hat{Z}_2}(x; \mu, \mu_1^*)$. Let $\hat{X}_2(t) = x + \mu t + \sigma W(t) + dM_b(t; \mu_2^*) - (\mu_2^* - \mu)t$ denote the cash reserve process when the sub-optimal control $\hat{Z}_2$ is applied. Define

$$\mathcal{L}_1^* = -r + \mu_1^* \frac{\partial}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2}.$$

By an argument wholly analogous to the Step 2 in Section 6.1, when $\hat{Z}_2$ is applied, we have

$$V(x; \mu, \mu_1^*) \geq \mathsf{E}_x \left(I_1 + I_2 - I_3 \right),$$

where

$$I_1 = \int_0^{t \wedge \tau_0^{\hat{X}_2}} e^{-rs} \left[V'(\hat{X}_2(s); \mu, \mu_1^*)(\mu_1^* - \mu_2^*)ds - \mathcal{L}_1^* V(\hat{X}_2(s); \mu, \mu_1^*)ds \right],$$

$$I_2 - I_3 \geq \int_{0-}^{t \wedge \tau_0^{\hat{X}_2}} e^{-rs} dM_b(s; \mu_2^*).$$

To check that the above expression for I_1 agrees with (22), note that for $Z = \hat{Z}_2$ and $c = \mu_1^* - \mu$, we have

$$-dZ_-(s) + cds = -(\mu_2^* - \mu)ds + (\mu_1^* - \mu)ds = (\mu_1^* - \mu_2^*)ds.$$

To show that $\hat{Z}_2$ is strictly sub-optimal, we need a slightly finer argument than that used to prove Theorem 1. Write $V_1(x) = V(x; \mu, \mu_1^*)$ to simplify the notation. Recall that $V_1(x)$ is the solution to the free-boundary problem described in Theorem 1 and $\hat{a} = \hat{a}(\mu, \mu_1^*)$ is strictly smaller than $b = \hat{a}(\mu, \mu_2^*)$ by Proposition 2. Using Lemma 1, it is then straightforward to verify that

$$\begin{aligned} \mathcal{L}_1^* V_1(x) &< \mu_1^* - \mu, &\forall x \in (\hat{a}, b], \\ V_1'(x) &> 1, &\forall x \in (0, \hat{a}), \\ V_1'(x) &= 1, &\forall x \in [\hat{a}, b]. \end{aligned}$$

Further, the assumption $\mu_1^* - \mu_2^* > 0$ implies that for any $x \in (0, \hat{a}) \cup (\hat{a}, b]$,

$$\psi(x) = V_1'(x)(\mu_1^* - \mu_2^*) - \mathcal{L}_1^* V_1(x) + (\mu_2^* - \mu) > 0, \tag{23}$$

and $\psi(x) = 0$ if $x = \hat{a}$. For any set $A \subseteq [0, \infty)$, define

$$I_1(A) = \int_A e^{-rs} \left[V_1'(\hat{X}_2(s))(\mu_1^* - \mu_2^*)ds - \mathcal{L}_1^* V_1(\hat{X}_2(s))ds \right].$$

Choose an arbitrary $\delta > 0$, and let $A_0^\delta = [0, \delta \wedge \tau_0^{\hat{X}_2}]$ and $A_\delta^t = (\delta \wedge \tau_0^{\hat{X}_2}, t \wedge \tau_0^{\hat{X}_2}]$. For any $t \geq \delta$, we have $I_1 = I_1([0, t \wedge \tau_0^{\hat{X}_2}]) = I_1(A_0^\delta) + I_1(A_\delta^t)$. Consider

$$I_1(A_0^\delta) + \int_0^{\delta \wedge \tau_0^{\hat{X}_2}} e^{-rs}(\mu_2^* - \mu)ds = \int_0^{\delta \wedge \tau_0^{\hat{X}_2}} e^{-rs} \psi(\hat{X}_2(s))ds.$$

Since $\hat{X}_2(t)$ is a reflected Brownian motion, given any $\hat{X}_2(0) = x > 0$, we have, almost surely, the bankruptcy time $\tau_0^{\hat{X}_2} > 0$ and the Lebesgue measure of the set $\{0 \leq t \leq \tau_0 : \hat{X}_2(t) = \hat{a}(\mu, \mu_1^*)\}$ is zero. Recall that for a non-negative measurable function, its Lebesgue integral is

zero if and only if the function is zero almost everywhere. It then follows from (23) that

$$\mathsf{E}_x \int_0^{\delta \wedge \tau_0^{\hat{X}_2}} e^{-rs} \psi(\hat{X}_2(s)) ds = c_\delta(x) > 0, \quad \forall x > 0,$$

where $c_\delta(x)$ only depends on x. Using (23) again, we find that for any $t \geq \delta$,

$$\mathsf{E}_x(I_1) \geq c_\delta(x) - \mathsf{E}_x \int_0^{t \wedge \tau_0^{\hat{X}_2}} e^{-rs} (\mu_2^* - \mu) ds,$$

which further implies that

$$V(x) \geq \mathsf{E}_x(I_1 + I_2 - I_3) \geq c_\delta(x) + \mathsf{E}_x \int_{0-}^{t \wedge \tau_0} e^{-rs} \left[dM_b(s; \mu_2^*) - (\mu_2^* - \mu) ds \right].$$

Letting $t \to \infty$, we conclude that

$$V(x; \mu, \mu_1^*) \geq c_\delta(x) + V(x; \mu, \mu_2^*) > V(x; \mu, \mu_2^*), \quad \forall x > 0.$$

6.3. Extensions with proportional costs

For $\beta \in (0, 1]$, define the value function V_β by

$$V_\beta(x; \mu, \mu^*) = \sup_{(Z_+, Z_-) \in \mathcal{A}} \mathsf{E}_x \int_{0-}^{\tau_0} e^{-rt} [dZ_+(t) - \beta dZ_-(t)]. \tag{24}$$

Observe that

$$\beta^{-1} V_\beta(x; \mu, \mu^*) = \sup_{(Z_+, Z_-) \in \mathcal{A}} \mathsf{E}_x \int_{0-}^{\tau_0} e^{-rt} [\beta^{-1} dZ_+(t) - dZ_-(t)].$$

Hence, the problem (24) with $\beta < 1$ can be interpreted as an extension of the main problem defined in (3) where there is a proportional cost of taking the dividends. For this problem, the optimal policy is to borrow the money at maximum rate $c = \mu^* - \mu$ and make dividend payouts at some barrier $\hat{a}_\beta$. Further, $(V_\beta, \hat{a}_\beta)$ is the solution to the following free-boundary problem

$$\begin{cases} \mathcal{L}^* v(x) = \beta c, & x \in [0, a], \\ v(0) = 0, \\ v'(x) = 1, & x \in [a, \infty), \\ v''(x) = 0, & x \in [a, \infty). \end{cases}$$

By the argument following Theorem 1, the value function V_β can still be written in the form of (12) and (13). To check the existence of $\hat{a}_\beta$, one just need to verify the following function has a unique positive solution,

$$f_\beta(a) = \frac{(-\gamma_-^*) e^{-\gamma_+^* a}}{(\gamma_+^*)(\gamma_+^* - \gamma_-^*)} + \frac{(\gamma_+^*) e^{-\gamma_-^* a}}{\gamma_-^*(\gamma_+^* - \gamma_-^*)} - \frac{\beta c}{r}.$$

By Proposition 1, f_β is monotone decreasing to $-\infty$ and observe that $f_\beta(0) = [(1 - \beta)\mu^* + \beta\mu]/r > 0$. Hence, $\hat{a}_\beta$ exists uniquely. Similarly to Lemma 1, one can show that $\max\{\mathcal{L}^* V_\beta(x) - \beta c, 1 - V_\beta'(x)\} = 0$ for all $x \geq 0$.

The verification proof is almost the same as in the case of $\beta = 1$. The only step that does not follow immediately from the proof in Section 6.1 is how to bound the term I_1 defined

in (22). Note that since $V_\beta' \geq 1 \geq \beta$, $\mathcal{L}^* V_\beta \leq \beta c$ and $-dZ_-(s)/ds + c \geq 0$, we have

$$
\begin{aligned}
I_1 &= \int_0^{t \wedge \tau_0^X} e^{-rs} \left[V_\beta'(X(s))(-dZ_-(s) + c\,ds) - \mathcal{L}^* V_\beta(X(s))ds \right], \\
&\geq \int_0^{t \wedge \tau_0^X} e^{-rs} \left[\beta(-dZ_-(s) + c\,ds) - \beta c\,ds \right] = -\int_0^{t \wedge \tau_0^X} e^{-rs} \beta\, dZ_-(s).
\end{aligned}
$$

The rest then follows.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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