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Graviton Self-Energy from Gravitons in Cosmology

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ABSTRACT

Although matter contributions to the graviton self-energy $-i[{}^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ must be separately conserved on x^μ and x'^μ, graviton contributions obey the weaker constraint of the Ward identity, which involves a divergence on both coordinates. On a general homogeneous and isotropic background this leads to just four structure functions for matter contributions but nine structure functions for graviton contributions. We propose a convenient parameterization for these nine structure functions. We also apply the formalism to explicit one loop computations of $-i[{}^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ on de Sitter background, one of the contributions from a massless, minimally coupled scalar and the other for the contribution from gravitons in the simplest gauge. We also specialize the linearized, quantum-corrected Einstein equation to the graviton mode function and to the gravitational response to a point mass.

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This paper is dedicated to Stanley Deser on the occasion of his 90th birthday.

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1 Introduction

The graviton self-energy $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ is the 1PI (one-particle-irreducible) 2-graviton function. It can be used to quantum-correct the linearized Einstein equation,

$$\mathcal{L}^{\mu\nu\rho\sigma} h_{\rho\sigma}(x) - \int d^4x' \Big[^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') h_{\rho\sigma}(x') = \frac{1}{2}\kappa T^{\mu\nu}_{\rm lin}(x) \; , \tag{1}$$

where $h_{\mu\nu}(x)$ is the graviton field, $\mathcal{L}^{\mu\nu\rho\sigma}$ is the Lichnerowicz operator in the appropriate background geometry, $T^{\mu\nu}_{\rm lin}$ is the linearized stress tensor density and $\kappa^2 \equiv 16\pi G$ is the loop counting parameter of quantum gravity. Equation (1) can be used to study how quantum corrections change the propagation of gravitational radiation and also how they affect gravitational forces. Note that equation (1) does not quite represent a semi-classical approach to gravity because the graviton self-energy receives contributions from the 0-point fluctuations of gravity as well as matter.

Quantum corrections on flat space background make no change at all to the kinematics of gravitons, and corrections to gravitational potentials only become significant at the Planck length [1–17]. However, the situation can be very different in cosmology, especially during primordial inflation. Accelerated expansion rips light scalars and gravitons from the vacuum, causing secular enhancements of the graviton field strength [18], and changes in gravitational potentials that grow nonperturbatively strong at large distances and at late times [19].

The purpose of this paper is to develop a technique for representing the graviton self-energy on a general homogeneous, isotropic and spatially flat background, with scale factor $a(\eta)$ and Hubble parameter $H(\eta)$,

$$ds^2 = a^2(\eta)\Big[-d\eta^2 + d\vec{x}\cdot d\vec{x}\Big] \qquad \Longrightarrow \qquad H(\eta) \equiv \frac{\partial_0 a}{a^2} \; . \tag{2}$$

Our representation consists of a sum of tensor differential operators acting on four scalar structure functions. Similar representations have been already given for matter contributions to the graviton self-energy on de Sitter background [20, 21] but it is cumbersome to infer the structure functions from the primitive result, and then use them in the effective field equation (1). Another problem is that graviton contributions to the graviton self-energy require five new structure functions.

To understand the difference between matter contributions and those from gravity itself, first use general tensor analysis on the background (2) to construct 21 basis tensors $[{}^{\mu\nu}\mathcal{D}^{\rho\sigma}]$ from $\delta^{\mu}_{\ 0}$, the spatial part of the Minkowski metric $\overline{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} + \delta^{\mu}_{\ 0}\delta^{\nu}_{\ 0}$ and the spatial derivative operator $\overline{\partial}^{\mu} \equiv \partial^{\mu} + \delta^{\mu}_{\ 0}\partial_0$. The graviton self-energy can be expressed as a sum of these operators acting on scalar functions of η, η' and $\|\vec{x} - \vec{x}'\|$,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') = \sum_{i=1}^{21} \Big[{}^{\mu\nu}\mathcal{D}^{\rho\sigma}_i\Big] \times T^i(x;x') \; . \qquad (3)$$

The 21 basis tensors are listed in Table 1.

i	$[{}^{\mu\nu}\mathcal{D}^{\rho\sigma}_i]$	i	$[{}^{\mu\nu}\mathcal{D}^{\rho\sigma}_i]$	i	$[{}^{\mu\nu}\mathcal{D}^{\rho\sigma}_i]$
1	$\overline{\eta}^{\mu\nu}\overline{\eta}^{\rho\sigma}$	8	$\overline{\partial}^{\mu}\overline{\partial}^{\nu}\overline{\eta}^{\rho\sigma}$	15	$\delta^{(\mu}_{\ \ 0}\overline{\partial}^{\nu)}\delta^{\rho}_{\ 0}\delta^{\sigma}_{\ 0}$
2	$\overline{\eta}^{\mu(\rho}\overline{\eta}^{\sigma)\nu}$	9	$\delta^{(\mu}_{\ \ 0}\overline{\eta}^{\nu)(\rho}\delta^{\sigma)}_{\ \ 0}$	16	$\delta^{\mu}_{\ 0}\delta^{\nu}_{\ 0}\overline{\partial}^{\rho}\overline{\partial}^{\sigma}$
3	$\overline{\eta}^{\mu\nu}\delta^{\rho}_{\ 0}\delta^{\sigma}_{\ 0}$	10	$\delta^{(\mu}_{\ \ 0}\overline{\eta}^{\nu)(\rho}\overline{\partial}^{\sigma)}$	17	$\overline{\partial}^{\mu}\overline{\partial}^{\nu}\delta^{\rho}_{\ 0}\delta^{\sigma}_{\ 0}$
4	$\delta^{\mu}_{\ 0}\delta^{\nu}_{\ 0}\overline{\eta}^{\rho\sigma}$	11	$\overline{\partial}^{(\mu}\overline{\eta}^{\nu)(\rho}\delta^{\sigma)}_{\ \ 0}$	18	$\delta^{(\mu}_{\ \ 0}\overline{\partial}^{\nu)}\delta^{(\rho}_{\ \ 0}\overline{\partial}^{\sigma)}$
5	$\overline{\eta}^{\mu\nu}\delta^{(\rho}_{\ \ 0}\overline{\partial}^{\sigma)}$	12	$\overline{\partial}^{(\mu}\overline{\eta}^{\nu)(\rho}\overline{\partial}^{\sigma)}$	19	$\delta^{(\mu}_{\ \ 0}\overline{\partial}^{\nu)}\overline{\partial}^{\rho}\overline{\partial}^{\sigma}$
6	$\delta^{(\mu}_{\ \ 0}\overline{\partial}^{\nu)}\overline{\eta}^{\rho\sigma}$	13	$\delta^{\mu}_{\ 0}\delta^{\nu}_{\ 0}\delta^{\rho}_{\ 0}\delta^{\sigma}_{\ 0}$	20	$\overline{\partial}^{\mu}\overline{\partial}^{\nu}\delta^{(\rho}_{\ \ 0}\overline{\partial}^{\sigma)}$
7	$\overline{\eta}^{\mu\nu}\overline{\partial}^{\rho}\overline{\partial}^{\sigma}$	14	$\delta^{\mu}_{\ 0}\delta^{\nu}_{\ 0}\delta^{(\rho}_{\ \ 0}\overline{\partial}^{\sigma)}$	21	$\overline{\partial}^{\mu}\overline{\partial}^{\nu}\overline{\partial}^{\rho}\overline{\partial}^{\sigma}$

Table 1: The 21 basis tensors used in expression (3). Note that the pairs $(3,4)$, $(5,6)$, $(7,8)$, $(10,11)$, $(14,15)$, $(16,17)$ and $(19,20)$ are related by reflection.

Now note that 7 of the scalar coefficient functions are related by reflection invariance,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') = -i\Big[{}^{\rho\sigma}\Sigma^{\mu\nu}\Big](x';x) \; . \qquad (4)$$

The various relations are listed in Table 2.

i	Relation	i	Relation
$3,4$	$T^4(x;x') = +T^3(x';x)$	$14,15$	$T^{15}(x;x') = -T^{14}(x';x)$
$5,6$	$T^6(x;x') = -T^5(x';x)$	$16,17$	$T^{17}(x;x') = +T^{16}(x';x)$
$7,8$	$T^8(x;x') = +T^7(x';x)$	$19,20$	$T^{20}(x;x') = -T^{19}(x';x)$
$10,11$	$T^{11}(x;x') = -T^{10}(x';x)$		

Table 2: Scalar coefficient functions in expression (3) which are related by reflection.

The 14 algebraically independent scalar coefficient functions $T^i(x;x')$ are related by differential equations whose number depends upon whether the contributions to $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ come from matter or from gravity itself. To understand these relations it is useful to define the *Ward Operator*,

$$\mathcal{W}^{\mu}{}_{\alpha\beta}(x) \equiv \delta^{\mu}{}_{(\alpha}\partial_{\beta)} + Ha\delta^{\mu}{}_{0}\eta_{\alpha\beta} \; . \tag{5}$$

Because matter interacts with gravity through its conserved stress tensor, matter contributions to the graviton self-energy must be annihilated by the Ward operator acting on either point,

$$0 = \mathcal{W}^{\mu}{}_{\alpha\beta}(x) \times -i\Big[^{\alpha\beta}\Sigma^{\rho\sigma}\Big](x;x') = 0 = \sum_{i=1}^{10} \Big[^{\mu}\mathcal{D}^{\rho\sigma}\Big] \times S^i(x;x') \; . \tag{6}$$

The 10 independent tensor factors $[^{\mu}\mathcal{D}^{\rho\sigma}]$ are listed in Table 3.

i	$[^{\mu}\mathcal{D}^{\rho\sigma}_i]$	i	$[^{\mu}\mathcal{D}^{\rho\sigma}_i]$
1	$\delta^{\mu}{}_{0}\delta^{\rho}{}_{0}\delta^{\sigma}{}_{0}$	6	$2\overline{\eta}^{\mu(\rho}\delta^{\sigma)}{}_{0}$
2	$2\delta^{\mu}{}_{0}\delta^{(\rho}{}_{0}\overline{\partial}^{\sigma)}$	7	$2\overline{\partial}^{\mu}\overline{\partial}^{(\rho}\delta^{\sigma)}{}_{0}$
3	$\delta^{\mu}{}_{0}\overline{\eta}^{\rho\sigma}$	8	$2\overline{\eta}^{\mu(\rho}\overline{\partial}^{\sigma)}$
4	$\delta^{\mu}{}_{0}\overline{\partial}^{\rho}\overline{\partial}^{\sigma}$	9	$\overline{\partial}^{\mu}\overline{\eta}^{\rho\sigma}$
5	$\overline{\partial}^{\mu}\delta^{\rho}{}_{0}\delta^{\sigma}{}_{0}$	10	$\overline{\partial}^{\mu}\overline{\partial}^{\rho}\overline{\partial}^{\sigma}$

Table 3: Scalar coefficient functions in expression (3) which are related by reflection.

From (6) we see that matter contributions to the graviton self-energy are characterized by $14-10=4$ independent structure functions. Gravity does not interact with itself through a conserved vertex. Hence graviton contributions to the graviton self-energy obey the weaker condition that they are annihilated by acting the Ward operator on *both* points,

$$0 = \mathcal{W}^{\mu}{}_{\alpha\beta}(x) \times \mathcal{W}^{\rho}{}_{\gamma\delta}(x') \times -i\Big[{}^{\alpha\beta}\Sigma^{\gamma\delta}\Big](x;x') = \delta^{\mu}{}_{0}\delta^{\rho}{}_{0} \times R^1(x;x') + \overline{\eta}^{\mu\rho} \\ \times R^2(x;x') + \delta^{\mu}{}_{0}\overline{\partial}^{\rho} \times R^3(x;x') + \overline{\partial}^{\mu}\delta^{\rho}{}_{0} \times R^4(x;x') + \overline{\partial}^{\mu}\overline{\partial}^{\rho} \times R^5(x;x') \; . \quad (7)$$

Because expression (7) involves 5 independent tensors we see that graviton contributions to $-i[{}^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ require $14-5=9$ structure functions. Our purpose is to propose a convenient representation for these structure functions and to elucidate their role in the effective field equation (1).

Section 2 derives insights from the vacuum polarization, on flat space and in cosmology, and from the graviton self-energy on flat space. Our representation is given in section 3. We also work out the equations for quantum corrections to the graviton mode function, and for the two potentials that describe the response to a point mass. Section 4 derives explicit results on de Sitter background for a dimensionally regulated computation of the contribution from a massless, minimally coupled (MMC) scalar [20], and for a $D=4$ computation of the contribution from gravitons away from coincidence ($x^{\mu} \neq x'^{\mu}$) [22]. Section 5 discusses how to extend the $D=4$ computation to a fully renormalized result. Our conclusions comprise section 6.

2 Other Bi-Tensor 1PI 2-Point Functions

The purpose of this section is to motivate our representation for the graviton self-energy in cosmology by reviewing simpler bi-tensor 1PI 2-point functions and simpler backgrounds. The section begins with the vacuum polarization on flat space background. We then turn to the graviton self-energy on flat space background. The section concludes with the vacuum polarization on a general cosmological background (2).

2.1 Vacuum Polarization on Flat Space

The 1PI 2-photon function $i[{}^{\mu}\Pi^{\rho}](x;x')$ has the evocative name, "vacuum polarization". A cumbersome and foolish way of expressing it would be to give all $4^2 = 16$ of its tensor components as functions of the two points x^μ and x'^μ. A much better way is to consolidate the number of functions by using general tensor analysis and reflection invariance. On flat space background this results in the form,

$$i\Big[{}^\mu\Pi^\rho_{\rm flat}\Big](x;x') = \eta^{\mu\rho}\times A(\Delta x^2) + \partial^\mu\partial^\rho\times B(\Delta x^2) \; , \qquad (8)$$

where the invariant interval in a Feynman propagator is,

$$\Delta x^2 \equiv \Big\Vert \vec{x}-\vec{x}'\Big\Vert^2 - \Big(|\eta-\eta'|-i\epsilon\Big)^2 \; . \qquad (9)$$

Because photons couple to a conserved current the vacuum polarization is transverse on each index,

$$0 = \partial_\mu \times i\Big[{}^\mu\Pi^\rho_{\rm flat}\Big](x;x') = \partial^\rho\Big[A(\Delta x^2) + \partial^2 B(\Delta x^2)\Big] \; . \qquad (10)$$

Conservation implies $A = -\partial^2 B$, which allows us to express the flat space vacuum polarization in terms of a single structure function,

$$i\Big[{}^\mu\Pi^\rho_{\rm flat}\Big](x;x') = \Big[\partial^\mu\partial^\rho - \eta^{\mu\rho}\partial^2\Big]B(\Delta x^2) \equiv \Pi^{\mu\rho}B(\Delta x^2) \; . \qquad (11)$$

It is more usual in the literature of quantum field theory to refer to the structure function $B(\Delta x^2)$ by the symbol $i\Pi(\Delta x^2)$.

2.2 Graviton Self-Energy on Flat Space

The advantages of using structure functions are even greater for the graviton self-energy. It would be fatuous to express this by giving all $4^4 = 256$ components. Just as with the vacuum polarization, it is more efficient to exploit symmetries of the background, reflection invariance to express $-i[{}^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ in terms of five basis tensors,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm flat}\Big](x;x') = \eta^{\mu\nu}\eta^{\rho\sigma}\times A(\Delta x^2) + \eta^{\mu(\rho}\eta^{\sigma)\nu}\times B(\Delta x^2) + \Big[\eta^{\mu\nu}\partial^\rho\partial^\sigma \\ + \partial^\mu\partial^\nu\eta^{\rho\sigma}\Big]\times C(\Delta x^2) + \partial^{(\mu}\eta^{\nu)(\rho}\partial^{\sigma)}\times D(\Delta x^2) + \partial^\mu\partial^\nu\partial^\rho\partial^\sigma\times E(\Delta x^2) \; . \qquad (12)$$

In this expression and henceforth parenthesized indices are symmetrized, for example, $\eta^{\mu(\rho}\eta^{\sigma)\nu} \equiv \frac{1}{2}[\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}]$.

2.2.1 From Matter

Because matter couples to gravitons through its conserved stress-energy tensor, matter contributions to the graviton self-energy must be transverse on each index,

$$0 = \partial_\nu \times -i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm flat}\Big](x;x') = \partial^\mu \eta^{\rho\sigma}\Big[A + \partial^2 C\Big] \\ + \eta^{\mu(\rho}\partial^{\sigma)}\Big[B + \frac{1}{2}\partial^2 D\Big] + \partial^\mu\partial^\rho\partial^\sigma\Big[C + \frac{1}{2}D + \partial^2 E\Big] \; . \qquad (13)$$

Conservation (13) allows us to express A, C and E in terms of C and D,

$$\Big(\text{Eqn. } 13\Big) \Longrightarrow A = -\partial^2 C \quad , \quad B = -\frac{1}{2}\partial^2 D \quad , \quad E = -\frac{1}{\partial^2}\Big(C + \frac{1}{2}D\Big) \; . \qquad (14)$$

Substituting (14) in (12) results in another familiar form,

$$\Big(\text{Eqn. } 13\Big) \Longrightarrow -i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm flat}\Big](x;x') = \Pi^{\mu\nu}\Pi^{\rho\sigma}\Big(-\frac{1}{\partial^2}C\Big) + \Pi^{\mu(\rho}\Pi^{\sigma)\nu}\Big(-\frac{1}{2\partial^2}D\Big) \; , \qquad (15)$$

where $\Pi^{\alpha\beta} \equiv \partial^\alpha\partial^\beta - \eta^{\alpha\beta}\partial^2$ was introduced in (11).

2.2.2 From Gravitons

Because the couplings of gravitons to themselves are not conserved, the divergence on a single index group does not vanish. Of course one can still use general tensor analysis to parameterize it in terms of three scalar functions,

$$\partial_\nu \times -i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm flat}\Big](x;x') = \partial^\mu\eta^{\rho\sigma}F(\Delta x^2) + \eta^{\mu(\rho}\partial^{\sigma)}G(\Delta x^2) + \partial^\mu\partial^\rho\partial^\sigma\mathcal{H}(\Delta x^2) \; . \qquad (16)$$

The Ward identity requires gravitational contributions to the graviton self energy to vanish when a divergence is taken on both index groups,

$$0 = \partial_\nu\partial_\sigma \times -i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm flat}\Big](x;x') = \eta^{\mu\rho}\Big[\frac{1}{2}\partial^2 G\Big] + \partial^\mu\partial^\rho\Big[F + \frac{1}{2}G + \partial^2\mathcal{H}\Big] \; . \qquad (17)$$

Expression (17) implies,

$$\Big(\text{Eqn. } 17\Big) \qquad \Longrightarrow \qquad F = -\partial^2\mathcal{H} \qquad , \qquad G = 0 \; . \qquad (18)$$

Of course the initial parameterization (12) of the graviton self-energy pertains to both gravitational and matter contributions. Hence expression

(13) is still valid for the result of a single divergence. Comparing (13) with (16) allows us to express the scalar coefficient functions A, B and E in terms of C, D and $\mathcal{H}$,

$$\Bigl(\text{Eqn. 17}\Bigr) \Longrightarrow A = -\partial^2 \Bigl(C + \mathcal{H}\Bigr) \; , \; B = -\frac{\partial^2 D}{2} \; , \; E = \frac{1}{\partial^2} \Bigl(-C - \frac{D}{2} + \mathcal{H}\Bigr) \; . \tag{19}$$

Substituting (19) into (12) gives,

$$\Bigl(\text{Eqn. 17}\Bigr) \qquad \Longrightarrow \qquad -i\Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm flat}\Bigr](x;x') = \Pi^{\mu\nu}\Pi^{\rho\sigma}\Bigl(-\frac{1}{\partial^2}C\Bigr)$$
$$+\Pi^{\mu(\rho}\Pi^{\sigma)\nu}\Bigl(-\frac{1}{2\partial^2}D\Bigr) + \Bigl[\partial^\mu\partial^\nu\partial^\rho\partial^\sigma - \eta^{\mu\nu}\eta^{\rho\sigma}\partial^4\Bigr]\Bigl(\frac{1}{\partial^2}\mathcal{H}\Bigr) \; . \tag{20}$$

2.2.3 An Explicit Example

In section 4 we will reconstruct the structure functions functions from an explicit computation of $-i[{}^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ on de Sitter background [22]. That result was derived using a de Sitter breaking gauge in which the graviton propagator consists of three constant tensor factors, constructed from $\eta^{\mu\nu}$ and δ^μ_0, which multiply scalar propagators whose expansions in $D=4$ spacetime dimensions have at most two terms [23, 24]. In 1979 Capper used the flat space limit of this same gauge, with dimensional regularization, to compute the one loop contribution to the graviton self-energy [25],

$$-i\Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm flat}\Bigr](x;x') = \frac{-\kappa^2}{4(D^2-1)}\Biggl\{T_1\partial^\mu\partial^\nu\partial^\rho\partial^\sigma + T_2\eta^{\mu\nu}\eta^{\rho\sigma}\partial^4 + 2T_3\eta^{\mu(\rho}\eta^{\sigma)\nu}\partial^4$$
$$T_4\Bigl[\eta^{\mu\nu}\partial^2\partial^\rho\partial^\sigma + \partial^\mu\partial^\nu\eta^{\rho\sigma}\partial^2\Bigr] + 4T_5\partial^{(\mu}\eta^{\nu)(\rho}\partial^{\sigma)}\partial^2\Biggr\}\Bigl[i\Delta(x;x')\Bigr]^2 . \tag{21}$$

Here $i\Delta(x;x')$ is the massless propagator in flat space,

$$i\Delta(x;x') = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}}\frac{1}{\Delta x^{D-2}} \; . \tag{22}$$

Capper's results for the coefficients $T_i(D)$ are [25],

$$T_1(D) \;\; = \;\; \frac{9}{16}D^4 - \frac{21}{16}D^3 - \frac{9}{8}D^2 \; , \tag{23}$$

$$T_2(D) \;\; = \;\; \frac{\frac{9}{16}D^5 - \frac{39}{16}D^4 - \frac{25}{8}D^3 + \frac{123}{8}D^2 + \frac{33}{4}D - 8}{D-2} \; , \tag{24}$$

$$T_3(D) \;=\; \frac{\frac14 D^4 + \frac{17}{16} D^3 - \frac{97}{16} D^2 - \frac{17}{8} D + 4}{D-2} = -T_5(D) \; , \qquad (25)$$

$$T_4(D) \;=\; \frac{-\frac{9}{16} D^5 + \frac{43}{16} D^4 + \frac{15}{8} D^3 - \frac{119}{8} D^2 - \frac{25}{4} D + 8}{D-2} \; , \qquad (26)$$

Comparing expressions (12) and (21) allows us to identify two of the structure functions,

$$-\frac{1}{\partial^2} C = \frac{\kappa^2 T_4}{4(D^2-1)} \Bigl[i\Delta^2(x;x') \Bigr]^2 \quad , \quad -\frac{1}{2\partial^2} D = \frac{\kappa^2 4 T_5}{4(D^2-1)} \Bigl[i\Delta^2(x;x') \Bigr]^2 \; . \qquad (27)$$

The final structure function derives from a comparison of (16) with the divergence of (21),

$$\frac{1}{\partial^2} H = -\frac{\kappa^2 (T_1 + T_4 + 2T_5)}{4(D^2-1)} \Bigl[i\Delta^2(x;x') \Bigr]^2 \; . \qquad (28)$$

Substituting (27) and (28) in (20) gives,

$$-i \Bigl[{}^{\mu\nu} \Sigma^{\rho\sigma}_{\rm flat} \Bigr](x;x') = \frac{-\kappa^2}{4(D^2-1)} \Bigl\{ -T_4 \Pi^{\mu\nu} \Pi^{\rho\sigma} - 2T_5 \Pi^{\mu(\rho} \Pi^{\sigma)\nu} + (T_1 + T_4 + 2T_5) \Bigl[\partial^\mu \partial^\nu \partial^\rho \partial^\sigma - \eta^{\mu\nu} \eta^{\rho\sigma} \partial^4 \Bigr] \Bigr\} \Bigl[i\Delta(x;x') \Bigr]^2 . \qquad (29)$$

Expression (29) is the dimensionally regulated, primitive contribution. To renormalize we first isolate ultraviolet divergences using the expansion [26, 27],

$$\Bigl[i\Delta(x;x') \Bigr]^2 = \frac{\Gamma(\frac{D}{2} - 1)}{4\pi^{\frac{D}{2}}} \frac{\mu^{D-4} i\delta^D(x-x')}{2(D-3)(D-4)} - \frac{\partial^2}{32\pi^4} \Bigl[\frac{\ln(\mu^2 \Delta x^2)}{\Delta x^2} \Bigr] + O(D-4) \; . \qquad (30)$$

At $D = 4$ the coefficients $T_4(4) = -\frac{23}{2}$ and $T_5(4) = -\frac{61}{4}$ are nonzero, but the final coefficient vanishes,

$$T_1(D) + T_4(D) + 2T_5(D) = -\frac14 (D-4) D (D+1) \; . \qquad (31)$$

This means that renormalization requires only the Ricci-squared and Weyl-squared counterterms. Note also that (29) must be recovered in the flat space limit of the graviton self-energy on de Sitter background [22]. The final, unregulated result is,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}\Big](x;x') = \frac{\kappa^2}{1920\pi^4}\Big[\frac{23}{2}\Pi^{\mu\nu}\Pi^{\rho\sigma} + \frac{61}{2}\Pi^{\mu(\rho}\Pi^{\sigma)\nu}\Big]\partial^2\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big] + \frac{\kappa^2}{96\pi^2}\Big[\partial^\mu\partial^\nu\partial^\rho\partial^\sigma - \eta^{\mu\nu}\eta^{\rho\sigma}\partial^4\Big] i\delta^4(x-x') \; . \qquad (32)$$

Except for their numerical coefficients, the two nonlocal terms on the first line of (32) could have come from matter contributions; the local term on the last line is only possible from gravitational contributions to the graviton self-energy because its divergence on a single index group is nonzero, $\partial_\nu[\partial^\mu\partial^\nu\partial^\rho\partial^\sigma - \eta^{\mu\nu}\eta^{\rho\sigma}\partial^4] = \partial^\mu\partial^2\Pi^{\rho\sigma}$. In cosmological backgrounds (2) we will see that distinctly gravitational contributions are much more varied, that they can harbor divergences, and that they can be nonlocal.

2.3 Vacuum Polarization in Cosmology

Now consider the photon self-energy (aka, the "vacuum polarization") on a general cosmological background (2). The symmetries of cosmology are homogeneity and isotropy. This means that the initial reduction involves two additional tensors from (8) and that the coefficient functions depend on η, η' and $\|\vec{x} - \vec{x}'\|$. When account is also taken of reflection invariance we can write,[1]

$$i\Big[{}^\mu\Pi^\rho_{\rm cos}\Big](x;x') = \overline{\eta}^{\mu\rho}A(x;x') + \delta^\mu_{~0}\delta^\rho_{~0}B(x;x') + \delta^\mu_{~0}\overline{\partial}^\rho C(x;x') - \overline{\partial}^\mu\delta^\rho_{~0}C(x';x) + \overline{\partial}^\mu\overline{\partial}^\rho D(x;x') \; . \qquad (33)$$

The scalar coefficient functions A, B and D are all reflection invariant,

$$A(x;x') = A(x';x) \quad , \quad B(x;x') = B(x';x) \quad , \quad D(x;x') = D(x';x) \; . \qquad (34)$$

[1] Rather than factors of the spatial gradient ∂^i, the primitive expression contains one or two factors of the spatial coordinate interval $\Delta x^i \equiv x^i - x'^i$ multiplied by functions of $\|\Delta\vec{x}\|^2$. These are then written in terms of spatial gradients using the identities,

$$\Delta x^i f(\|\vec{x}\|^2) = \frac{1}{2}\partial^i I[f] \; ,$$
$$\Delta x^i \Delta x^j f(\|\vec{x}\|^2) = \frac{1}{4}\partial^i\partial^j I^2[f] - \frac{1}{2}\delta^{ij} I[f] \; ,$$

where $I[f]$ represents the indefinite integral of $f(\|\vec{x}\|^2)$ with respect to $\|\vec{x}\|^2$.

We also remind the reader that the presence of a bar over a tensor indicates suppression of its temporal components,

$$\overline{\eta}^{\mu\rho} \equiv \eta^{\mu\rho} + \delta^{\mu}{}_{0}\delta^{\rho}{}_{0} \qquad , \qquad \overline{\partial}^{\mu} \equiv \partial^{\mu} + \delta^{\mu}{}_{0}\partial_{0} \; . \tag{35}$$

Because the vacuum polarization is a bi-vector density, its divergence on each index group must vanish, on any background geometry. On cosmological backgrounds (2) the divergence produces two independent tensors,

$$0 = \partial_{\mu} \times i\Bigl[{}^{\mu}\Pi^{\rho}_{\rm cos}\Bigr](x;x') = \overline{\partial}^{\rho}\Bigl[A(x;x') + \partial_0 C(x;x') \\ + \nabla^2 D(x;x')\Bigr] + \delta^{\rho}{}_{0}\Bigl[\partial_0 B(x;x') - \nabla^2 C(x';x)\Bigr] \; , \tag{36}$$

where $\nabla^2 \equiv \partial^i \partial^i$ is the flat space Laplacian. Expression (36) allows us to solve for two of the coefficient functions,

$$A(x;x') = -\partial_0 C(x;x') - \nabla^2 D(x;x') \qquad , \qquad B(x;x') = \frac{\nabla^2}{\partial_0} C(x';x) \; . \tag{37}$$

The reflection invariance (34) of A and B also implies an important relation for reflecting $C(x;x')$,

$$\partial_0 C(x;x') = \partial_0' C(x';x) \qquad \Longrightarrow \qquad C(x';x) = \frac{\partial_0}{\partial_0'} C(x;x') \; . \tag{38}$$

Note that $\frac{1}{\partial_0'} C(x;x')$ is reflection invariant.

Substituting (37) and (38) into (33) shows how the cosmological vacuum polarization can be expressed using two structure functions,

$$i\Bigl[{}^{\mu}\Pi^{\rho}_{\rm cos}\Bigr](x;x') = \Bigl[-\overline{\eta}^{\mu\rho}\partial_0\partial_0' + \delta^{\mu}{}_{0}\partial_0'\overline{\partial}^{\rho} - \overline{\partial}^{\mu}\delta^{\rho}{}_{0}\partial_0 + \delta^{\mu}{}_{0}\delta^{\rho}{}_{0}\nabla^2\Bigr]\frac{1}{\partial_0'} C(x;x') \\ + \Bigl[\overline{\partial}^{\mu}\overline{\partial}^{\rho} - \overline{\eta}^{\mu\rho}\nabla^2\Bigr] D(x;x') \; , \tag{39}$$

$$= \Bigl[\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\Bigr]\partial_{\nu}\partial_{\sigma}'\Bigl[\frac{1}{\partial_0'} C(x;x')\Bigr] \\ + \Bigl[\overline{\eta}^{\mu\nu}\overline{\eta}^{\rho\sigma} - \overline{\eta}^{\mu\sigma}\overline{\eta}^{\nu\rho}\Bigr]\partial_{\nu}\partial_{\sigma}'\Bigl[D(x;x') - \frac{1}{\partial_0'} C(x;x')\Bigr] \; . \tag{40}$$

The representation (40) was first employed in studying charged MMC scalar contributions to the vacuum polarization on de Sitter background [28, 29]. The procedure for transforming to other representations has been given in detail [30, 31].

3 Graviton Self-Energy in Cosmology

The purpose of this section is to present our formalism for representing the graviton self-energy in cosmology. We give a unified derivation which applies to the cases of matter contributions and to those from gravity itself. We then specialize the effective field equations to the cases of the graviton mode function and to the two potentials that represent the gravitational response to a point mass.

3.1 Our Representation

As discussed in the Introduction, the symmetries of cosmology permit us to represent the graviton self-energy as the sum (3) of the 21 tensor differential operators $[{}^{\mu\nu}\mathcal{D}_i^{\rho\sigma}]$ of Table 1 acting on scalar coefficient functions $T^i(x;x')$,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') = \sum_{i=1}^{21} \Big[{}^{\mu\nu}\mathcal{D}^{\rho\sigma}\Big] \times T^i(x;x') \; . \qquad (41)$$

Seven of the coefficient functions are related to others by reflection invariance, as described in Table 2. Acting the Ward operator (5) on a single index group produces a sum (6) of the 10 tensor differential operators of Table 3 acting on scalar coefficient functions $S^i(x;x')$,

$$\mathcal{W}^{\mu}{}_{\alpha\beta}(x) \times -i\Big[{}^{\alpha\beta}\Sigma^{\rho\sigma}\Big](x;x') = \sum_{i=1}^{10} \Big[{}^{\mu}\mathcal{D}^{\rho\sigma}\Big] \times S^i(x;x') \; . \qquad (42)$$

Relation (6) can be used to express each of the $S^i(x;x')$ in terms of the 14 algebraically distinct $T^j(x;x')$. The expansions are given in Table 4.

$S^i(x;x')$	Expansion in $T^j = T^j(x;x')$ and $T^{jR} = T^j(x';x)$
S^1	$(D-1)aHT^3 + (\partial_0 - aH)T^{13} - \frac{1}{2}\nabla^2 T^{14R} + aH\nabla^2 T^{16R}$
S^2	$(\frac{D-1}{2})aHT^5 + \frac{1}{4}T^9 - \frac{1}{2}aHT^{10R}$ $+\frac{1}{2}(\partial_0 - aH)T^{14} + \frac{1}{4}\nabla^2 T^{18} - \frac{1}{2}aH\nabla^2 T^{19R}$
S^3	$(D-1)aHT^1 + aHT^2 + (\partial_0 - aH)T^{3R} - \frac{1}{2}\nabla^2 T^{5R} + aH\nabla^2 T^{7R}$
S^4	$(D-1)aHT^7 + \frac{1}{2}T^{10} + aHT^{12}$ $+(\partial_0 - aH)T^{16} + \frac{1}{2}\nabla^2 T^{19} + aH\nabla^2 T^{21}$
S^5	$T^3 - \frac{1}{2}\partial_0 T^{14R} + \nabla^2 T^{16R}$
S^6	$\frac{1}{4}\partial_0 T^9 - \frac{1}{4}\nabla^2 T^{10R}$
S^7	$\frac{1}{2}T^5 - \frac{1}{4}T^{10R} + \frac{1}{4}\partial_0 T^{18} - \frac{1}{2}\nabla^2 T^{19R}$
S^8	$\frac{1}{2}T^2 + \frac{1}{4}\partial_0 T^{10} + \frac{1}{4}\nabla^2 T^{12}$
S^9	$T^1 - \frac{1}{2}\partial_0 T^{5R} + \nabla^2 T^{7R}$
S^{10}	$T^7 + \frac{1}{2}T^{12} + \frac{1}{2}\partial_0 T^{19} + \nabla^2 T^{21}$

Table 4: Expansion of the coefficients $S^i(x;x')$ of equation (6) in terms of the coefficients $T^j(x;x')$ of the initial representation (3).

Our strategy for representing the $T^i(x;x')$ is motivated by the flat space analog considered in section 2.2. It is the same for both matter and gravity: we use the ten relations of Table 4 to solve for the coefficient functions in terms of the functions S^i and a "minimal" set of T^i's consisting of T^{12}, T^{16}, T^{18} and T^{19}. Each of the ten S^i must vanish for matter contributions, whereas they can be nonzero for contributions from gravity itself. However, because we must get zero from acting the Ward operator on both index groups, and because this action results in five distinct tensor operators (7), the ten S^i are subject to five relations given in Table 5.

$R^i(x;x')$	Expansion in $S^j = S^j(x;x')$
R^1	$(\partial_0' - a'H')S^1 - \nabla^2 S^2 + (D-1)a'H'S^3 + a'H'\nabla^2 S^4$
R^2	$\partial_0' S^6 - \nabla^2 S^8$
R^3	$\partial_0' S^2 - S^3 - \nabla^2 S^4$
R^4	$(\partial_0' - a'H')S^5 - S^6 - \nabla^2 S^7 + 2a'H'S^8 + (D-1)a'H'S^9 + a'H'\nabla^2 S^{10}$
R^5	$\partial_0' S^7 - S^8 - S^9 - \nabla^2 S^{10}$

Table 5: Expansion of the coefficients $R^i(x;x') = 0$ of the Ward identity (7) in terms of the coefficients $S^j(x;x')$ of the action (6) of the Ward operator on a single index group.

We can eliminate S^3 using $R^3(x;x') = 0$,

$$S^3 = \partial_0' S^2 - \nabla^2 S^4 \; . \tag{43}$$

Combining this with $R^1(x;x') = 0$ gives S^1,[2]

$$S^1 = \frac{1}{\partial_0' - a'H'} \Bigl\{ - \Bigl[(D-1) a'H' \partial_0' - \nabla^2 \Bigr] S^2 + (D-2) a'H' \nabla^2 S^4 \Bigr\} \; . \tag{45}$$

The relations $R^2(x;x') = 0$ and $R^5(x;x') = 0$ imply,

$$S^6 = \frac{\nabla^2}{\partial_0'} S^8 \qquad , \qquad S^9 = \partial_0' S^7 - S^8 - \nabla^2 S^{10} \; . \tag{46}$$

and substituting (46) in $R^4(x;x') = 0$ gives,

$$S^5 = \frac{1}{\partial_0' - a'H'} \Biggl\{ - \Bigl[(D-1) a'H' \partial_0' - \nabla^2 \Bigr] S^7 + \Bigl[(D-3) a'H' + \frac{\nabla^2}{\partial_0'} \Bigr] S^8 + (D-2) a'H' \nabla^2 S^{10} \Biggr\} \; . \tag{47}$$

So our structure functions consist of T^{12}, T^{16}, T^{18} and T^{19}, plus (for graviton contributions) S^2, S^4, S^7, S^8 and S^{10}.

[2] The inverse of $\partial_0' - a'H'$ can be expressed as a simple integral with respect to η',

$$\frac{1}{\partial_0' - a'H'} = a' \times \frac{1}{\partial_0'} \times \frac{1}{a'} \; . \tag{44}$$

Because the ten relations in Table 4 are coupled they are best solved in four stages. First use the relations for S^5, S^8, S^9 and S^{10} to write,

$$T^3 = \frac{1}{2}\partial_0 T^{14R} - \nabla^2 T^{16R} + S^5 \; , \qquad (48)$$

$$T^2 = -\frac{1}{2}\partial_0 T^{10} - \frac{1}{2}\nabla^2 T^{12} + 2S^8 \; , \qquad (49)$$

$$T^1 = \frac{1}{2}\partial_0 T^{5R} - \nabla^2 T^{7R} + S^9 \; , \qquad (50)$$

$$\nabla^2 T^{21} = -T^7 - \frac{1}{2}T^{12} - \frac{1}{2}\partial_0 T^{19} + S^{10} \; . \qquad (51)$$

Because T^{21} and T^{12} are symmetric we can use relation 51) to solve for the anti-symmetric part of $T^7 \equiv T^{7S} + T^{7A}$,

$$T^{7A} \equiv \frac{1}{2}(T^7 - T^{7R}) = -\frac{1}{4}(\partial_0 T^{19} - \partial_0' T^{19R}) + S^{10A} \; . \qquad (52)$$

The next step is using the S^2 and S^7 relations to solve for T^5 and T^9,

$$T^5 = \frac{1}{2}T^{10R} - \frac{1}{2}\partial_0 T^{18} + \nabla^2 T^{19R} + 2S^7 \; , \qquad (53)$$

$$T^9 = -(D-3)aHT^{10R} - 2(\partial_0 - aH)T^{14} + \Big[(D-1)aH\partial_0 - \nabla^2\Big]T^{18} - 2(D-2)aH\nabla^2 T^{19R} + 4S^2 - 4(D-1)aHS^7 \; . \qquad (54)$$

Relation (53) could be used in (50) to reduce T^1 but we postpone this. In the 3rd stage the S^1 and S^4 relations give T^{13} and T^{10},

$$T^{13} = \frac{1}{\partial_0 - aH}\Bigg\{-\frac{1}{2}\Big[(D-1)aH\partial_0 - \nabla^2\Big]T^{14R} + (D-2)aH\nabla^2 T^{16R} + S^1 - (D-1)aHS^5\Bigg\} \; , \qquad (55)$$

$$T^{10} = -2(D-2)aHT^7 - aHT^{12} - 2(\partial_0 - aH)T^{16} + (aH\partial_0 - \nabla^2)T^{19} + 2S^4 - 2aHS^{10} . \qquad (56)$$

We now use relations (53) and (56) to update T^1, T^2, T^5 and T^9,

$$T^1 = -\Big[\Big(\frac{D-2}{2}\Big)\partial_0 aH + \nabla^2\Big]T^7 - \frac{1}{4}\partial_0 aHT^{12} - \frac{1}{2}\partial_0(\partial_0 - aH)T^{16} - \frac{1}{4}\partial_0\partial_0' T^{18} + \frac{1}{4}(\partial_0 aH - \nabla^2)\partial_0 T^{19} + \frac{1}{2}\nabla^2\partial_0' T^{19R}$$

$$+\frac{1}{2}\partial_0 S^4 + \partial_0' S^7 + \partial_0 S^{7R} - S^8 - \frac{1}{2}\partial_0 aHS^{10} - \nabla^2 S^{10R} \; , \quad (57)$$

$$T^2 = (D-2)\partial_0 aHT^7 + \frac{1}{2}(\partial_0 aH - \nabla^2)T^{12} + \partial_0(\partial_0 - aH)T^{16} - \frac{1}{2}(\partial_0 aH - \nabla^2)\partial_0 T^{19} - \partial_0 S^4 + 2S^8 + \partial_0 aHS^{10} \; , \quad (58)$$

$$T^5 = -(D-2)a'H'T^{7R} - \frac{1}{2}a'H'T^{12} - (\partial_0' - a'H')T^{16R} - \frac{1}{2}\partial_0 T^{18} + \frac{1}{2}(a'H'\partial_0' + \nabla^2)T^{19R} + S^{4R} + 2S^7 - a'H'S^{10R} \; , \quad (59)$$

$$T^9 = 2(D-3)(D-2)aHa'H'T^{7R} + (D-3)aHa'H'T^{12} - 2(\partial_0 - aH)T^{14} + 2(D-3)aH(\partial_0' - a'H')T^{16R} + \Big[(D-1)aH\partial_0 - \nabla^2\Big]T^{18} - \Big[(D-3)aHa'H'\partial_0' + (D-1)aH\nabla^2\Big]T^{19R} + 4S^2 - 2(D-3)aHS^{4R} - 4(D-1)aHS^7 + 2(D-3)aHa'H'S^{10R} \; . \quad (60)$$

The final stage begins by noting that the S^3 and S^5 relations can be expressed in terms of two functions $A(x;x')$ and $B(x;x')$,

$$S^3 \quad \Longrightarrow \quad 0 = -\Big[(D-3)aH\partial_0 + \nabla^2\Big]A + \partial_0' B \; , \quad (61)$$

$$S^6 \quad \Longrightarrow \quad 0 = +\Big[(D-3)\partial_0 aH + \nabla^2\Big]A^R - \partial_0 B \; . \quad (62)$$

The functions A and B are,

$$A \equiv \frac{1}{2}(D-2)aHT^7 + \frac{1}{4}aHT^{12} + \frac{1}{2}(\partial_0 - aH)T^{16} - \frac{1}{4}(aH\partial_0 - \nabla^2)T^{19} - \frac{1}{2}S^4 - \frac{1}{\partial_0}S^{8R} + \frac{1}{2}aHS^{10} \; , \quad (63)$$

$$B \equiv \frac{1}{2}(\partial_0 - aH)T^{14} - \frac{1}{4}\Big[(D-1)aH\partial_0 - \nabla^2\Big]T^{18} + \frac{(D-2)}{2}aH\nabla^2 T^{19R} - S^2 + (D-1)aHS^7 - (D-3)aH\frac{1}{\partial_0'}S^8 \; . \quad (64)$$

We first solve equation (61) for T^{14},

$$T^{14} = \frac{1}{\partial_0 - aH}\Bigg\{\frac{1}{2}\Big[(D-1)aH\partial_0 - \nabla^2\Big]T^{18} - (D-2)aH\nabla^2 T^{19R} + 2S^2 - 2(D-1)aHS^7 + 2(D-3)aH\frac{1}{\partial_0'}S^8 + \Big[(D-3)aH\partial_0 + \nabla^2\Big]\frac{1}{\partial_0'}A\Bigg\} . \quad (65)$$

The final relation derives from combining (61) with (62),

$$\partial_0\Bigl(\text{Eqn. 61}\Bigr)-\partial_0'\Bigl(\text{Eqn. 62}\Bigr) = \Bigl[(D\!-\!3)\partial_0 aH\!+\!\nabla^2\Bigr]\Bigl[-\partial_0 A+\partial_0' A^R\Bigr] = 0 \; . \quad (66)$$

It follows that $\partial_0 A = \partial_0' A^R$, which allows us to solve for the symmetric part of T^7. Combining this with the antisymmetric part (52) gives,

$$T^7 = -\frac{\frac12 T^{12}}{D\!-\!2}-\frac12\partial_0 T^{19}+S^{10}-\frac{2}{D\!-\!2}\frac{1}{(\partial_0 aH-\partial_0' a'H')}\Bigl[\partial_0\Delta A\!-\!\partial_0'\Delta A^R\Bigr] \; . \quad (67)$$

Here the residual part of A is,

$$\Delta A \equiv \frac12(\partial_0 - aH)T^{16} - \frac14\Bigl[(D\!-\!1)aH\partial_0-\nabla^2\Bigr]T^{19} \\ -\frac12 S^4 - \frac{1}{\partial_0}S^{8R} + \frac{(D\!-\!1)}{2}aHS^{10} \; . \quad (68)$$

We should comment on how to invert the differential operator $\mathcal{D} = \partial_0 aH - \partial_0' a'H'$. This is accomplished by first factoring out $aH \times a'H'$,

$$\mathcal{D} = \frac{1}{aH\,a'H'} \times \frac{1}{aH\partial_0 - a'H'\partial_0'} \times aH\,a'H' \; . \quad (69)$$

Now change the time variable from η to u such that,

$$du \equiv \frac{d\eta}{aH} \; . \quad (70)$$

By employing "lightcone" variables,

$$u_\pm \equiv \frac12(u \pm u') \qquad \Longrightarrow \qquad \frac{\partial}{\partial u} - \frac{\partial}{\partial u'} = \frac{\partial}{\partial u_-} \; , \quad (71)$$

we can express (69) as an integration with respect to u_-,

$$\frac{1}{\partial_0 aH - \partial_0' a'H'} = \frac{1}{aH\,a'H'} \times \int du_- \times aH\,a'H' \; . \quad (72)$$

For the important special case of de Sitter we have $u = -\frac12\eta^2$.

Our final expressions for the coefficient functions can be simplified by using two symmetric auxiliary functions to absorb all the terms involving the inverse of $\mathcal{D}$. We first define the (not necessarily symmetric) function $\gamma(x;x')$,

$$\gamma \equiv \partial_0(\partial_0 - aH)T^{16} - \frac{1}{2}\Bigl[(D-1)\partial_0 aH - \nabla^2\Bigr]\partial_0 T^{19} \\ -\partial_0 S^4 - 2S^{8R} + (D-1)\partial_0 aH S^{10} \; . \tag{73}$$

The two symmetric functions are,

$$\alpha \equiv \frac{1}{\mathcal{D}}\Bigl[\gamma - \gamma^R\Bigr] \; , \tag{74}$$

$$\beta \equiv \frac{1}{2}\Bigl(\gamma + \gamma^R\Bigr) - \frac{1}{2}\Bigl(\partial_0 aH + \partial_0' a' H'\Bigr)\alpha \; . \tag{75}$$

Note that the function $\beta(x;x')$ can be written in two different ways,

$$\beta = \gamma - \partial_0 aH\alpha = \gamma^R - \partial_0' a' H' \alpha \; . \tag{76}$$

Also note that we can eliminate T^{16},

$$T^{16} = \frac{1}{\partial_0 - aH}\frac{1}{\partial_0}\Biggl\{\beta + \partial_0 aH\alpha + \frac{1}{2}\Bigl[(D-1)\partial_0 aH - \nabla^2\Bigr]\partial_0 T^{19} \\ +\partial_0 S^4 + 2S^{8R} - (D-1)\partial_0 aH S^{10}\Biggr\}. \tag{77}$$

The notation can be further simplified by introducing symbols to stand for three differential operators and an inverse operator that occur repeatedly,

$$D_0 \equiv \partial_0 aH \quad , \quad D_1 \equiv (D-1)\partial_0 aH - \nabla^2 \; , \tag{78}$$

$$\mathcal{I} \equiv \frac{1}{\partial_0 - aH}\frac{1}{\partial_0} \quad , \quad D_3 \equiv (D-3)\partial_0 aH + \nabla^2 \; . \tag{79}$$

Giving any of these operators a prime indicates that it is constructed from the same quantities at x'^μ instead of x^μ, for example, $D_0' = \partial_0' a' H'$. With these definitions our final expressions for the algebraically independent coefficient functions are,

$$T^1 = \frac{\nabla^2 T^{12}}{2(D-2)} - \frac{1}{4}\partial_0\partial_0' T^{18} + \frac{1}{2}\nabla^2\Bigl(\partial_0 T^{19} + \partial_0' T^{19R}\Bigr) + \frac{\nabla^2\alpha}{D-2} - \frac{1}{2}\beta \\ +\Bigl(\partial_0' S^7 + \partial_0 S^{7R}\Bigr) - \Bigl(S^8 + S^{8R}\Bigr) - \nabla^2\Bigl(S^{10} + S^{10R}\Bigr) \; , \tag{80}$$

$$T^2 = -\frac{1}{2}\nabla^2 T^{12} + \beta + 2\Big(S^8 + S^{8R}\Big) \; , \qquad (81)$$

$$T^3 = \mathcal{I}'\Bigg\{\frac{1}{4}D_1'\partial_0\partial_0' T^{18} - \frac{1}{2}(D-2)\nabla^2 D_0'\partial_0 T^{19} - \frac{1}{2}\nabla^2 D_1'\partial_0' T^{19R} - \nabla^2 D_0'\alpha$$

$$+\Big(\frac{1}{2}D_1' - D_0'\Big)\beta + \partial_0\partial_0' S^{2R} - \nabla^2\partial_0' S^{4R} - D_1'\partial_0' S^7 - (D-1)D_0'\partial_0 S^{7R}$$

$$+D_3' S^8 + (D-3)D_0' S^{8R} + (D-2)\nabla^2 D_0' S^{10} + (D-1)\nabla^2 D_0' S^{10R}\Bigg\}, \qquad (82)$$

$$T^5 = -\frac{1}{2}\partial_0 T^{18} + \nabla^2 T^{19R} - \frac{1}{\partial_0'}\beta + 2S^7 - \frac{2}{\partial_0'}S^8 \; , \qquad (83)$$

$$T^7 = -\frac{T^{12}}{2(D-2)} - \frac{1}{2}\partial_0 T^{19} - \frac{\alpha}{D-2} + S^{10} \; , \qquad (84)$$

$$T^9 = -\frac{2\nabla^2}{\partial_0\partial_0'}\beta \; , \qquad (85)$$

$$T^{10} = -\frac{2}{\partial_0}\beta - \frac{4}{\partial_0}S^{8R} \; , \qquad (86)$$

$$T^{13} = \mathcal{I}\mathcal{I}'\Bigg\{-\frac{1}{4}D_1 D_1'\partial_0\partial_0' T^{18} + \frac{1}{2}(D-2)\nabla^2\Big[D_1 D_0'\partial_0 T^{19} + D_0 D_1'\partial_0' T^{19R}\Big]$$

$$+(D-2)\nabla^2 D_0 D_0'\alpha - \frac{1}{2}\Big[(D-3)D_1 D_3' - D_3\nabla^2\Big]\beta - D_1'\partial_0\partial_0' S^2 - D_1\partial_0\partial_0' S^{2R}$$

$$+(D-2)\nabla^2\Big[D_0'\partial_0 S^4 + D_0\partial_0' S^{4R}\Big] + (D-1)\Big[D_0 D_1'\partial_0' S^7 + D_0' D_1\partial_0 S^{7R}\Big]$$

$$-(D-3)\Big[D_0 D_1' S^8 + D_0 D_1 S^{8R}\Big] - (D-2)(D-1)\nabla^2 D_0 D_0'(S^{10} + S^{10R})\Bigg\}, \qquad (87)$$

$$T^{14} = \mathcal{I}\Bigg\{\frac{1}{2}D_1\partial_0 T^{18} - (D-2)\nabla^2 D_0 T^{19R} + \frac{D_3}{\partial_0'}\beta$$

$$+2\partial_0 S^2 - 2(D-1)D_0 S^7 + 2(D-3)\frac{D_0}{\partial_0'}S^8\Bigg\}, \qquad (88)$$

$$T^{21} = -\frac{1}{2}\Big(\frac{D-3}{D-2}\Big)\frac{1}{\nabla^2}T^{12} + \frac{1}{D-2}\frac{1}{\nabla^2}\alpha \; . \qquad (89)$$

The cumbersome nature of these expressions prompts several comments on the issue of accuracy. First, the flat space limit agrees with the decomposition of section 2.2. Second, our results for $T^1(x;x')$, $T^2(x;x')$, $T^9(x;x')$ and $T^{13}(x;x')$ are reflection symmetric as they should be. Finally, the contributions proportional to T^{12}, T^{18}, T^{19}, T^{19R}, α, and β are each separately annihilated by the action of the Ward operator (5) on either index group, while the contributions proportional to S^2, S^4, S^7, S^8 and S^{10} are annihilated when the Ward operator is acted on both index groups.

3.2 Ricci and Weyl Operators

Our results (77) and (80-89) can be reorganized into a sum of products of tensor differential operators acting on the fundamental structure functions. Each of these tensor differential operators is separately annihilated by the action of either one or two Weyl operators. All the operators descend from $3+1$ decomposing the same products of the transverse projection operator that we encountered in section 2.2,

$$\Pi^{\mu\nu} \equiv \partial^\mu \partial^\nu - \partial^2 \eta^{\mu\nu} = \Pi^{\mu\nu}_A + \Pi^{\mu\nu}_B \; , \tag{90}$$

where the two projectors are,

$$\Pi^{\mu\nu}_A \;\; \equiv \;\; \overline{\partial}^\mu \overline{\partial}^\nu - \nabla^2 \overline{\eta}^{\mu\nu} \; , \tag{91}$$

$$\Pi^{\mu\nu}_B \;\; \equiv \;\; \nabla^2 \delta^\mu_{\ 0} \delta^\nu_{\ 0} - 2\partial_0 \delta^{(\mu}_{\ \ 0} \overline{\partial}^{\nu)} + \partial_0^2 \overline{\eta}^{\mu\nu} \; . \tag{92}$$

What we term the *Ricci operators* are simple extensions of (91) and (92),

$$\Pi^{\mu\nu}_A \longrightarrow \overline{\partial}^\mu \overline{\partial}^\nu - \nabla^2 \overline{\eta}^{\mu\nu} + \frac{(D-2)\nabla^2}{\partial_0 - aH} a H \delta^\mu_{\ 0} \delta^\nu_{\ 0} \equiv \mathcal{R}^{\mu\nu}_A(x) \; , \tag{93}$$

$$\Pi^{\mu\nu}_B \longrightarrow \frac{\delta^\mu_{\ 0}\delta^\nu_{\ 0}}{\partial_0 - aH} \Bigl[\nabla^2 - (D-1) a H \partial_0\Bigr] \partial_0 - 2\partial_0 \delta^{(\mu}_{\ \ 0} \overline{\partial}^{\nu)} + \partial_0^2 \overline{\eta}^{\mu\nu} \equiv \mathcal{R}^{\mu\nu}_B(x) \; . \tag{94}$$

It is straightforward to verify that acting the Ward operator annihilates each Ricci operator,

$$\mathcal{W}^\mu_{\ \alpha\beta}(x) \times \mathcal{R}^{\alpha\beta}_A(x) = 0 = \mathcal{W}^\mu_{\ \alpha\beta}(x) \times \mathcal{R}^{\alpha\beta}_B(x) \; . \tag{95}$$

The related *Weyl operators* come from extending products of two projection operators. The simplest is purely spatial,

$$\Pi^{\mu(\rho}_A \times \Pi^{\sigma)\nu}_A \longrightarrow \Pi^{\mu(\rho}_A \Pi^{\sigma)\nu}_A - \frac{\Pi^{\mu\nu}_A \Pi^{\rho\sigma}_A}{D-2} \equiv \mathcal{C}^{\mu\nu\rho\sigma}_{AA} \; . \tag{96}$$

Because $\mathcal{C}^{\mu\nu\rho\sigma}_{AA}$ is both transverse and traceless, it is annihilated when the Ward operator acts on either index group,

$$\mathcal{W}^\mu_{\ \alpha\beta}(x) \times \mathcal{C}^{\alpha\beta\rho\sigma}_{AA} = 0 = \mathcal{W}^\rho_{\ \gamma\delta}(x') \times \mathcal{C}^{\mu\nu\gamma\delta}_{AA} \; . \tag{97}$$

The second Weyl operator comes from extending the product of two B-type projectors,

$$\Pi_B^{\mu(\rho}\times\Pi_B^{\sigma)\nu}\longrightarrow\mathcal{C}_{BB}^{\mu\nu\rho\sigma}(x;x')\equiv\partial_0^2\partial_0'^2\overline{\eta}^{\mu(\rho}\overline{\eta}^{\sigma)\nu}-2\partial_0\partial_0'^2\delta^{(\mu}{}_0\overline{\eta}^{\nu)(\rho}\overline{\partial}^{\sigma)}$$
$$+2\partial_0^2\partial_0'\overline{\partial}^{(\mu}\overline{\eta}^{\nu)(\rho}\delta^{\sigma)}{}_0-2\partial_0\partial_0'\nabla^2\delta^{(\mu}{}_0\overline{\eta}^{\nu)(\rho}\delta^{\sigma)}{}_0-2\partial_0\partial_0'\delta^{(\mu}{}_0\overline{\partial}^{\nu)}\delta^{(\rho}{}_0\overline{\partial}^{\sigma)}$$
$$-\frac{\delta^\mu{}_0\delta^\nu{}_0}{\partial_0-aH}aH\partial_0^2\Bigl[\partial_0'^2\overline{\eta}^{\rho\sigma}+2\partial_0'\delta^{(\rho}{}_0\overline{\partial}^{\sigma)}\Bigr]-\Bigl[\partial_0^2\overline{\eta}^{\mu\nu}-2\partial_0\delta^{(\mu}{}_0\overline{\partial}^{\nu)}\Bigr]\frac{\delta^\rho{}_0\delta^\sigma{}_0}{\partial_0'-a'H'}a'H'\partial_0'^2$$
$$+\frac{\delta^\mu{}_0\delta^\nu{}_0}{\partial_0-aH}\partial_0\Bigl[\partial_0'\overline{\partial}^\rho\overline{\partial}^\sigma+2\nabla^2\delta^{(\rho}{}_0\overline{\partial}^{\sigma)}\Bigr]\partial_0'+\Bigl[\partial_0\overline{\partial}^\mu\overline{\partial}^\nu-2\nabla^2\delta^{(\mu}{}_0\overline{\partial}^{\nu)}\Bigr]\partial_0\frac{\delta^\rho{}_0\delta^\sigma{}_0}{\partial_0'-a'H'}\partial_0'$$
$$+\frac{\delta^\mu{}_0\delta^\nu{}_0\delta^\rho{}_0\delta^\sigma{}_0}{(\partial_0-aH)(\partial_0'-a'H')}\Bigl[(D-1)aH\partial_0a'H'\partial_0'-(aH\partial_0+a'H'\partial_0')\nabla^2+\nabla^4\Bigr]\partial_0\partial_0' . \qquad (98)$$

One can also show that $\mathcal{C}_{BB}^{\mu\nu\rho\sigma}$ is annihilated by the Ward operator acting on either index group,

$$\mathcal{W}^\mu{}_{\alpha\beta}(x)\times\mathcal{C}_{BB}^{\alpha\beta\rho\sigma}(x;x')=0=\mathcal{W}^\rho{}_{\gamma\delta}(x')\times\mathcal{C}_{BB}^{\mu\nu\gamma\delta}(x;x') . \qquad (99)$$

We can use the Ricci and Weyl operators to express matter contributions to the graviton self-energy,

$$-i\Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm mat}\Bigr](x;x')=-\frac{1}{2}\mathcal{C}_{AA}^{\mu\nu\rho\sigma}\times\frac{1}{\nabla^2}T^{12}-\frac{1}{4}\mathcal{R}_B^{\mu\nu}(x)\times\mathcal{R}_B^{\rho\sigma}(x')\times\frac{1}{\partial_0\partial_0'}T^{18}$$
$$+\frac{1}{2}\Bigl[\mathcal{R}_B^{\mu\nu}(x)\times\mathcal{R}_A^{\rho\sigma}(x')\times\frac{1}{\partial_0}T^{19}+\mathcal{R}_A^{\mu\nu}(x)\times\mathcal{R}_B^{\rho\sigma}(x')\times\frac{1}{\partial_0'}T^{19R}\Bigr]$$
$$+\frac{\mathcal{R}_A^{\mu\nu}(x)\times\mathcal{R}_A^{\rho\sigma}(x')}{D-2}\frac{1}{\nabla^2}\alpha+\Bigl[\mathcal{C}_{BB}^{\mu\nu\rho\sigma}(x;x')-\frac{1}{2}\mathcal{R}_B^{\mu\nu}(x)\times\mathcal{R}_B^{\rho\sigma}(x')\Bigr]\frac{1}{\partial_0^2\partial_0'^2}\beta . \qquad (100)$$

Expressing contributions from gravity itself requires the additional operators formed from suppressing the temporal components of $\mathcal{C}_{BB}^{\mu\nu\rho\sigma}(x;x')$ on either x^μ of x'^μ,

$$\mathcal{C}_{BB}^{\overline{\mu\nu}\rho\sigma}(x;x')\equiv\frac{\overline{\delta}^\mu{}_\alpha\overline{\delta}^\nu{}_\beta}{\partial_0^2}\mathcal{C}_{BB}^{\alpha\beta\rho\sigma}(x;x')\ \ ,\ \ \mathcal{C}_{BB}^{\mu\nu\overline{\rho\sigma}}(x;x')\equiv\frac{\overline{\delta}^\rho{}_\gamma\overline{\delta}^\sigma{}_\delta}{\partial_0'^2}\mathcal{C}_{BB}^{\mu\nu\gamma\delta}(x;x') , \qquad (101)$$

where $\overline{\delta}^\alpha{}_\beta\equiv\delta^\alpha{}_\beta-\delta^\alpha{}_0\delta^0{}_\beta$ is the spatial unit matrix. The contributions from gravity itself include all of the same terms in (100) with the addition of terms involving the structure functions S^2, S^4, S^7, S^8 and S^{10},

$$-i\Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm grav}\Bigr](x;x')=\Bigl(\text{Eqn. 100}\Bigr)$$

$$+\frac{\delta^\mu{}_0\delta^\nu{}_0}{\partial_0 - aH}\Big[\mathcal{R}^{\rho\sigma}_B(x')\frac{1}{\partial'_0}S^2 + \mathcal{R}^{\rho\sigma}_A(x')S^4\Big] + \frac{\delta^\rho{}_0\delta^\sigma{}_0}{\partial'_0 - a'H'}\Big[\mathcal{R}^{\mu\nu}_B(x)\frac{1}{\partial_0}S^{2R} + \mathcal{R}^{\mu\nu}_A(x)S^{4R}\Big]$$
$$+\Big[\overline{\eta}^{\mu\nu} - \frac{(D-1)\delta^\mu{}_0\delta^\nu{}_0}{\partial_0 - aH}aH\Big]\Big[\mathcal{R}^{\rho\sigma}_B(x')\frac{1}{\partial'_0}S^7 + \mathcal{R}^{\rho\sigma}_A(x')S^{10}\Big]$$
$$+\Big[\overline{\eta}^{\rho\sigma} - \frac{(D-1)\delta^\rho{}_0\delta^\sigma{}_0}{\partial'_0 - a'H'}a'H'\Big]\Big[\mathcal{R}^{\mu\nu}_B(x)\frac{1}{\partial_0}S^{7R} + \mathcal{R}^{\mu\nu}_A(x)S^{10R}\Big]$$
$$+\Big\{\mathcal{C}^{\overline{\mu\nu}\rho\sigma}_{BB}(x;x') - \Big[\overline{\eta}^{\mu\nu} - \frac{(D-3)\delta^\mu{}_0\delta^\nu{}_0}{\partial_0 - aH}aH\Big]\mathcal{R}^{\rho\sigma}_B(x')\Big\}\frac{1}{\partial_0'^2}S^8$$
$$+\Big\{\mathcal{C}^{\mu\nu\overline{\rho\sigma}}_{BB}(x;x') - \mathcal{R}^{\mu\nu}_B(x)\Big[\overline{\eta}^{\rho\sigma} - \frac{(D-3)\delta^\rho{}_0\delta^\sigma{}_0}{\partial'_0 - a'H'}a'H'\Big]\Big\}\frac{1}{\partial_0^2}S^{8R} . \quad (102)$$

3.3 Effective Field Equations

Of course the point of developing this representation for $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ is to facilitate solving the effective field equation (1). We here adapt it to two important special cases:

1. Plane wave gravitons; and
2. The response to a point mass.

In the first case the graviton field $h_{\mu\nu}(x)$ takes the form,

$$\text{Gravitons} \Longrightarrow h_{\mu\nu} = u(\eta,k)e^{i\vec{k}\cdot\vec{x}}\epsilon_{\mu\nu}(\vec{k}) \quad , \quad \epsilon_{0\mu} = 0 = k_i\epsilon_{i\mu} = \epsilon_{ii} \; , \quad (103)$$

and what we seek is an equation for the mode function $u(\eta,k)$. For the second the graviton field takes the form,

$$\text{Potentials} \Longrightarrow \kappa h_{00} = -2\Psi(\eta,r) \; , \; \kappa h_{0i} = 0 \; , \; \kappa h_{ij} = -2\Phi(\eta,r)\delta_{ij} \; , \quad (104)$$

and what we seek are equations for the two potentials $\Psi(\eta,r)$ and $\Phi(\eta,r)$.

3.3.1 The Graviton Mode Function

On a general cosmological background the Lichnerowicz operator receives contributions from whatever matter source supports the geometry. On de Sitter background its action on a general graviton field takes the form [19],

$$\mathcal{L}^{\mu\nu\rho\sigma}h_{\rho\sigma} = \partial_\alpha\Big[a^2\mathcal{L}^{\mu\nu\rho\sigma\alpha\beta}\partial_\beta h_{\rho\sigma}\Big] + \partial_\alpha\Big[Ha^3\eta^{\mu\nu}h^{\alpha 0}\Big] - Ha^3\eta^{0(\mu}\partial^{\nu)}h \; , \quad (105)$$

where the tensor factor is,

$$\mathcal{L}^{\mu\nu\rho\sigma\alpha\beta} = \frac{1}{2}\eta^{\alpha\beta}(\eta^{\mu(\rho}\eta^{\sigma)\nu} - \eta^{\mu\nu}\eta^{\rho\sigma}) + \frac{1}{2}\eta^{\mu\nu}\eta^{\rho(\alpha}\eta^{\beta)\sigma} + \frac{1}{2}\eta^{\rho\sigma}\eta^{\mu(\alpha}\eta^{\beta)\nu} - \eta^{\alpha(\rho}\eta^{\sigma)(\mu}\eta^{\nu)\beta} . \tag{106}$$

Acting the Lichnerowicz operator on a plane wave graviton (103) gives,

$$\mathcal{L}^{\mu\nu\rho\sigma} u(\eta,k) e^{i\vec{k}\cdot\vec{x}} \epsilon_{\rho\sigma} = -\frac{1}{2}a^2\Big[\partial_0^2 + 2aH\partial_0 + k^2\Big] u(\eta,k) \times e^{i\vec{k}\cdot\vec{x}}\epsilon^{\mu\nu} \; . \tag{107}$$

Gravitons have zero stress tensor. Because their polarization tensor $\epsilon_{\mu\nu}$ is purely spatial, transverse and also traceless, the only one of the coefficient functions $T^i(x;x')$ that contributes is $T^2(x;x')$,[3]

$$\int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') h_{\rho\sigma}(x') = \epsilon^{\mu\nu}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \int d^4x' \, iT^2(x;x') u(\eta',k) e^{-i\vec{k}\cdot\Delta\vec{x}} \; . \tag{108}$$

Hence the equation for corrections to the graviton mode function is,

$$-\frac{1}{2}a^2\Big[\partial_0^2 + 2aH\partial_0 + k^2\Big] u(\eta,k) = \int d^4x' \, iT^2(x;x') u(\eta',k) e^{-i\vec{k}\cdot\Delta\vec{x}} \; . \tag{109}$$

This is the same for matter and gravity contributions, however, what T^2 is in terms for the fundamental structure functions differs according to relation (81). Note that it will be necessary to extract a number of derivatives from the structure functions; primed derivatives being acted on the mode function and unprimed derivatives pulled outside the integration.

3.3.2 Response to A Point Mass

Acting the Lichnerowicz operator on the potentials (104) produces,

$$\begin{aligned}\mathcal{L}^{\mu\nu\rho\sigma}\Big[-2\delta^0_\rho\delta^0_\sigma\Psi - 2\overline{\eta}_{\rho\sigma}\Phi\Big] &= a^2\delta^\mu_0\delta^\nu_0\Big[6a^2H^2\Psi - 2(\nabla^2 - 3aH\partial_0)\Phi\Big] \\ &\quad + 2a^2\delta^{(\mu}_0\overline{\partial}^{\nu)}\Big[2aH\Psi + 2\partial_0\Phi\Big] + a^2\overline{\partial}^\mu\overline{\partial}^\nu\Big[\Psi - \Phi\Big] \\ &\quad + a^2\overline{\eta}^{\mu\nu}\Big[-(\nabla^2 + 2aH\partial_0 + 6a^2H^2)\Psi + (\nabla^2 - 4aH\partial_0 - 2\partial_0^2)\Phi\Big]\end{aligned} \tag{110}$$

[3] Of course $T^2(x;x')$ may vanish for some theories, however, it is generally nonzero because gravity couples to all fields and the 0-point fluctuations of these fields can make nonzero contributions to $-i[{}^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ even if the stress-energy of the background vanishes.

The potentials $\Psi(\eta, r)$ and $\Phi(\eta, r)$ are the response to a static point mass M whose linearized stress tensor is,

$$\frac{1}{2}\kappa T^{\mu\nu}_{\rm lin}(\eta, \vec{x}) = -\frac{1}{2}\kappa M a(\eta)\delta^3(\vec{x})\delta^\mu_0\delta^\nu_0 \; . \tag{111}$$

The zeroth order response is,

$$\Psi_0(\eta, r) = \Phi_0(\eta, r) = -\frac{GM}{a(\eta) r} \; . \tag{112}$$

Loop corrections are sourced by the integral of the self-energy against lower order response,

$$\mathcal{L}^{\mu\nu\rho\sigma} h_{\rho\sigma}(x) = \int d^4x' \Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}\Bigr](x; x') h_{\rho\sigma}(x') \equiv \mathcal{S}^{\mu\nu}(x) \; . \tag{113}$$

Although we have worked out all components of $\mathcal{S}^{\mu\nu}(x)$, the only necessary ones are $\mathcal{S}^{0i}$ — which is ∂_i of something — and the $\partial_i \partial_j$ part of $\mathcal{S}^{ij}$. Comparison with (110) implies that the two potentials obey,

$$4a^3 H \Psi + 4a^2 \partial_0 \Phi = 2\int d^4x' \Biggl\{ iT^{14}(x'; x)\Psi(x') \\ + \Bigl[3iT^5(x'; x) - iT^{10}(x; x') - iT^{19}(x; x')\nabla^2\Bigr]\Phi(x')\Biggr\}, \tag{114}$$

$$a^2\Psi - a^2\Phi = -2\int d^4x' \Biggl\{ iT^{16}(x'; x)\Psi(x') \\ + \Bigl[3iT^7(x'; x) + iT^{12}(x; x')\Bigr]\Phi(x')\Biggr\}. \tag{115}$$

The same comments apply to these results as for the mode equation (109): these equations are valid for any contribution to the graviton self-energy, although what those contributions are in terms of the fundamental structure functions varies from matter to gravity according to relations (83), (84) and (86). And one should also note that derivatives will be extracted from the structure functions, with primed ones partially integrated onto the potentials and unprimed ones taken outside the integration.

4 Explicit Examples on de Sitter

The previous section described our formalism for representing the graviton self-energy in cosmology. The purpose of this section is to put this formalism in context with two explicit one loop results obtained on de Sitter background. As an example of matter contributions (100) we consider the dimensionally regulated result from a loop of massless, minimally coupled scalars [20]. The more complex relation for gravity itself (102) is exemplified by an old $D = 4$ computation [22] that was made in the simplest gauge [23] before it was understood how to apply dimensional regularization.

4.1 Contributions from a MMC Scalar

Suppose that $S[\varphi, g]$ represents the sum of the scalar and gravitational actions, and that $\Delta S[g]$ stands for the counter-action. The one scalar loop contributions to the graviton self-energy can be expressed as the expectation value of the sum of three variational derivatives of these quantities,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') = \Big\langle \Omega \Big| T^* \Big[\Big[\frac{i\delta S[\varphi,g]}{\delta h_{\mu\nu}(x)}\Big]_{\varphi\varphi} \Big[\frac{i\delta S[\varphi,g]}{\delta h_{\rho\sigma}(x')}\Big]_{\varphi\varphi} + \Big[\frac{i\delta^2 S[\varphi,g]}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big]_{\varphi\varphi} + \Big[\frac{i\delta^2 \Delta S[g]}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big]_1 \Big] \Big| \Omega \Big\rangle , \qquad (116)$$

where the subscripts indicate how many of the weak fields are retained and the T^*-ordering symbol means any derivatives are taken after time ordering the operators. Figure 2 shows the associated Feynman diagrams.

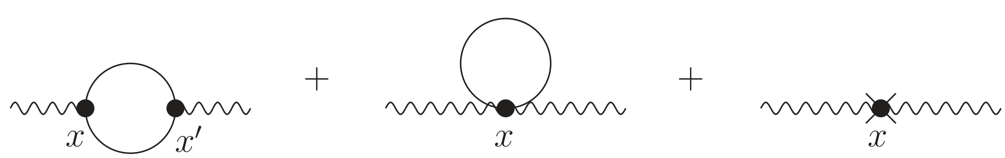

Figure 1: One scalar loop contributions to the graviton self-energy, shown in the same order, left to right, as the three contributions to (116). Scalar lines are straight and graviton lines are wavy.

The 4-point diagram (the central one of Figure 1) can be exactly canceled by the counterterm diagram (the right hand one of Figure 1). The 3-point diagram (the left hand one of Figure 1) takes the form [19, 20],

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm 3pt}\Big](x;x') = (aa')^{D+2}\Big\{ D^{\mu} D'^{(\rho} y \, D'^{\sigma)} D^{\nu} y \, \alpha(y) + D^{(\mu} y \, D^{\nu)} D'^{(\rho} y \, D'^{\sigma)} y \, \beta(y) + D^{\mu} y \, D^{\nu} y \, D'^{\rho} y \, D'^{\sigma} y \, \gamma(y) + H^4 g^{\mu\nu} g'^{\rho\sigma} \, \delta(y) + H^2 \Big[g^{\mu\nu} D'^{\rho} y \, D'^{\sigma} y + D^{\mu} y \, D^{\nu} y \, g'^{\rho\sigma} \Big] \epsilon(y) \Big\} \tag{117}$$

Here $y \equiv aa' H^2 \Delta x^2$ and its covariant derivatives are,

$$D^{\mu} y = -\frac{H}{a}\Big[y \delta^{\mu}_{\ 0} - 2a' H \Delta x^{\mu} \Big] \quad , \quad D'^{\rho} y = -\frac{H}{a'}\Big[y \delta^{\rho}_{\ 0} + 2a H \Delta x^{\rho} \Big] \; , \tag{118}$$

$$D^{\mu} D'^{\rho} y = \frac{H^2}{aa'} \Big[y \delta^{\mu}_{\ 0} \delta^{\rho}_{\ 0} + 2a \delta^{\mu}_{\ 0} H \Delta x^{\rho} - 2H \Delta x^{\mu} a' \delta^{\rho}_{\ 0} - 2\eta^{\mu\rho} \Big] \; . \tag{119}$$

The various coefficients in expression (117) are given in terms of a single function $A(y)$ whose first derivative is [26, 27],

$$A'(y) = -\frac{H^{D-2}}{4(4\pi)^{\frac{D}{2}}} \Big\{ \Gamma\Big(\frac{D}{2}\Big) \Big(\frac{4}{y}\Big)^{\frac{D}{2}} + \Gamma\Big(\frac{D}{2}+1\Big) \Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} + \sum_{n=0}^{\infty} \Big[\frac{\Gamma(n+\frac{D}{2}+2)}{\Gamma(n+3)} \Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2} - \frac{\Gamma(n+D)}{\Gamma(n+\frac{D}{2}+1)} \Big(\frac{y}{4}\Big)^{n} \Big] \Big\} . \tag{120}$$

The functions $\alpha(y)$, $\beta(y)$, $\gamma(y)$, $\delta(y)$ and $\epsilon(y)$ are,[4]

$$\alpha(y) = -\frac{\kappa^2}{2} A'^2 \;\; , \;\; \beta(y) = -\kappa^2 A' A'' \;\; , \;\; \gamma(y) = -\frac{\kappa^2}{2} A''^2 \; , \tag{121}$$

$$\delta(y) = -\frac{\kappa^2}{8} \Big[(4y{-}y^2)^2 A''^2 + 2(2{-}y)(4y{-}y^2) A' A'' + [4(D{-}4) - (4y{-}y^2)] A'^2 \Big] , \tag{122}$$

$$\epsilon(y) = \frac{\kappa^2}{4} \Big[(4y - y^2) A''^2 + 2(2-y) A' A'' - A'^2 \Big] \; . \tag{123}$$

We can express (117) in the basis of Table 1 by first 3 + 1 decomposing the tensors of expressions (118-119),

$$\eta^{\mu\nu} = \overline{\eta}^{\mu\nu} - \delta^{\mu}_{\ 0} \delta^{\nu}_{\ 0} \quad , \quad \Delta x^{\mu} = \overline{\Delta x}^{\mu} + \delta^{\mu}_{\ 0} \Delta \eta \; , \tag{124}$$

[4]Note that the scalar functions $\alpha(y)$, $\beta(y)$ and $\gamma(y)$ are unrelated to the bi-scalar densities $\alpha(x;x')$, $\beta(x;x')$ and $\gamma(x;x')$ defined in equations (73-75).

where $\Delta\eta \equiv x^0 - x'^0$. Factors of $\overline{\Delta x}^\mu$ are then expressed as derivatives using the rules,

$$\overline{\Delta x}^\alpha f(\Delta x^2) = \frac{\overline{\partial}^\mu}{2} I[f] \quad , \quad \overline{\Delta x}^\alpha \overline{\Delta x}^\beta f(\Delta x^2) = \frac{\overline{\partial}^\alpha \overline{\partial}^\beta}{4} I^2[f] - \frac{\overline{\eta}^{\alpha\beta}}{2} I[f] \; , \tag{125}$$

$$\overline{\Delta x}^\alpha \overline{\Delta x}^\beta \overline{\Delta x}^\gamma f(\Delta x^2) = \frac{1}{8} \overline{\partial}^\alpha \overline{\partial}^\beta \overline{\partial}^\gamma I^3[f] - \frac{3}{4} \overline{\eta}^{(\alpha\beta} \overline{\partial}^{\gamma)} I^2[f] \; , \tag{126}$$

$$\overline{\Delta x}^\alpha \overline{\Delta x}^\beta \overline{\Delta x}^\gamma \overline{\Delta x}^\delta f(\Delta x^2) = \frac{1}{16} \overline{\partial}^\alpha \overline{\partial}^\beta \overline{\partial}^\gamma \overline{\partial}^\delta I^4[f] - \frac{3}{4} \overline{\eta}^{(\alpha\beta} \overline{\partial}^\gamma \overline{\partial}^{\delta)} I^3[f] + \frac{3}{4} \overline{\eta}^{(\alpha\beta} \overline{\eta}^{\gamma\delta)} I^2[f] \; . \tag{127}$$

Here the operator $I[f]$ stands for the indefinite integral of $f(\Delta x^2)$ with respect to Δx^2. Table 6 gives the coefficient functions.

It remains to comment on ultraviolet divergences and renormalization. Table 6 gives dimensionally regulated, primitive results. Comparing Table 6 with expressions (120) and (121-123) reveals that the fundamental structure functions have the following leading behaviors near coincidence:

$$T^{12} \sim \frac{1}{\Delta x^{2D-2}} \quad , \quad T^{16} \sim \frac{1}{\Delta x^{2D-2}} \quad , \quad T^{18} \sim \frac{1}{\Delta x^{2D-2}} \quad , \quad T^{19} \sim \frac{\Delta\eta}{\Delta x^{2D-2}} \; . \tag{128}$$

It must be recalled that the ultimate goal is to integrate $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ of x' in the quantum-corrected, linearized Einstein equation (1). Hence an expression such as $1/\Delta x^{2D-2}$ is quadratically divergent, while $1/\Delta x^{2D-4}$ is logarithmically divergent. We localize these divergences by extracting derivatives until the integrable power of $1/\Delta x^{2D-6}$ is reached, then adding zero in the form of the massless scalar propagator equation [26],

$$\frac{1}{\Delta x^{2D-2}} = \frac{\partial^2}{2(D-2)^2} \frac{1}{\Delta x^{2D-4}} = \frac{\partial^4}{4(D-2)^2(D-3)(D-4)} \frac{1}{\Delta x^{2D-6}} \; , \tag{129}$$

$$= \frac{\partial^4}{4(D-2)^2(D-3)(D-4)} \left[\frac{1}{\Delta x^{2D-6}} - \frac{\mu^{D-4}}{\Delta x^{D-2}} \right] + \frac{\mu^{D-4} \pi^{\frac{D}{2}} \partial^2 i\delta^D(x-x')}{(D-2)^2(D-3)(D-4)\Gamma(\frac{D}{2}-1)} , \tag{130}$$

$$= \frac{\mu^{D-4} \pi^{\frac{D}{2}} \partial^2 i\delta^D(x-x')}{(D-2)^2(D-3)(D-4)\Gamma(\frac{D}{2}-1)} - \frac{\partial^4}{32} \left[\frac{\ln(\mu^2 \Delta x^2)}{\Delta x^2} \right] + O(D-4) \; . \tag{131}$$

i	$T^i(x;x')$
1	$\delta - 2(a^2 + a'^2)H^2 I[\epsilon] + 4a^2 a'^2 H^4 I^2[\gamma]$
2	$4\alpha - 4aa'H^2 I[\beta] + 8a^2 a'^2 I^2[\gamma]$
3	$-\delta + (y + 2aH\Delta\eta)^2\epsilon$ $-2a'^2 H^2 I[\alpha + (2aH\Delta\eta + y)\beta + (4a^2H^2\Delta\eta^2 + 4aH\Delta\eta y + y^2)\gamma - \epsilon]$
5	$-2aa'^2 H^3 I^2[\beta + 2(2aH\Delta\eta + y)\gamma] + 2aHI[(2aH\Delta\eta + y)\epsilon]$
7	$-2a^2 a'^2 H^4 I^3[\gamma] + a^2 H^2 I^2[\epsilon]$
9	$-4[2 + y + 2aa'H^2\Delta\eta^2]\alpha - 2[y^2 - 2aa'H^2\Delta\eta^2(2-y)]\beta$ $+2aa'H^2 I[2\alpha + (2+3y)\beta + 4aa'H^2\Delta\eta^2\beta + 4y^2\gamma - 8aa'H^2\Delta\eta^2(2-y)\gamma]$
10	$2aHI[-2\alpha + (2a'H\Delta\eta - y)\beta] + 4a^2 a' H^3 I^2[\beta - 2(2a'H\Delta\eta - y)\gamma]$
12	$2aa'H^2 I^2[\beta] - 8a^2 a'^2 H^4 I^3[\gamma]$
13	$(2 + y + 2aa'H^2\Delta\eta^2)^2\alpha + (2 + y + 2aa'H^2\Delta\eta^2)$ $\times[y^2 - 2aa'H^2\Delta\eta^2(2-y)]\beta + (y^2 - 2aa'H^2\Delta\eta^2(2-y))]^2\gamma$ $+\delta - 2[y^2 + 2aa'H^2\Delta\eta^2(2+y) + 2a^2a'^2H^4\Delta\eta^4]\epsilon$
14	$2aHI[(2 + 2aa'H^2\Delta\eta^2)\alpha + y\alpha] - 4aa'H^2\Delta\eta(1 + aH\Delta\eta + aa'H^2\Delta\eta^2)I[\beta]$ $+2aHI[(1 - a'H\Delta\eta + 2aa'H^2\Delta\eta^2)y\beta + y^2\beta] + 8a^2a'^2H^4\Delta\eta^3 I[(2-y)\gamma]$ $-2aHI[2aa'H^2\Delta\eta^2(2-y)y\gamma + (2a'H\Delta\eta - y)y^2\gamma + (2aH\Delta\eta + y)\epsilon]$
16	$H^2 I^2[a^2\{\alpha - (2-y)\beta + (2-y)^2\gamma - \epsilon\} + aa'\{2\beta - 4(2-y)\gamma\} + 4a'^2\gamma]$
18	$aa'H^2 I^2[-2\alpha + 3(2-y)\beta - 32\gamma + 4(4y - y^2)\gamma]$ $+(a^2 + a'^2)H^2 I^2[-4\beta + 8(2-y)\gamma]$
19	$a^2 a' H^3 I^3[-\beta + 2(2-y)\gamma] - 4aa'^2 I^3[\gamma]$
21	$a^2 a'^2 H^4 I^4[\gamma]$

Table 6: Scalar contributions to the coefficient functions $T^i(x;x')$. The de Sitter length function is $y = aa'H^2\Delta x^2$, the various functions of it such as $\alpha(y)$ are defined in expressions (121-123), and the operator "I" indicates indefinite integration with respect to Δx^2.

We can similarly write,

$$\frac{\Delta\eta}{\Delta x^{2D-2}} = \frac{\mu^{D-4}\pi^{\frac{D}{2}}\partial_0 i\delta^D(x-x')}{(D-2)(D-3)(D-4)\Gamma(\frac{D}{2}-1)} - \frac{\partial_0\partial^2}{16}\Bigl[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Bigr] + O(D-4) \; . \tag{132}$$

Renormalization is accomplished by using local counterterms to cancel the divergent delta functions in expressions (131-132). (This sometimes leaves local residuals proportional to $\ln(a)$.) Because the derivatives on the contributions act on functions of $x^\mu - x'^\mu$, they can either be maintained as unprimed derivatives and pulled outside the x'^μ integration of the quantum-corrected Einstein equation (1), or they can be reflected into primed derivatives ($\partial_\mu \to -\partial'_\mu$) and then partially integrated onto the graviton field $h_{\rho\sigma}(x')$.

4.2 Contributions from Gravitons

Suppose $S[g]$ represents the classical action of gravity, $S_g[h,\overline{\theta},\theta]$ is the ghost and gauge fixing action, and $\Delta S[g]$ stands for the counter-action. The one loop graviton self-energy can be expressed as the expectation value of the sum of three variational derivatives of these quantities,

$$-i\Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}\Bigr](x;x') = \Bigl\langle\Omega\Bigl|T^*\Bigl[\Bigl[\frac{i\delta S[g]}{\delta h_{\mu\nu}(x)}\Bigr]_{hh}\Bigl[\frac{i\delta S[g]}{\delta h_{\rho\sigma}(x')}\Bigr]_{hh} + \Bigl[\frac{i\delta S[g]}{\delta h_{\mu\nu}(x)}\Bigr]_{\overline{\theta}\theta} \times\Bigl[\frac{i\delta S[g]}{\delta h_{\rho\sigma}(x')}\Bigr]_{\overline{\theta}\theta} + \Bigl[\frac{i\delta^2 S[g]}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Bigr]_{hh} + \Bigl[\frac{i\delta^2 \Delta S[g]}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Bigr]_{1}\Bigr]\Bigl|\Omega\Bigr\rangle , \tag{133}$$

where the subscripts indicate how many graviton fields are retained and the T^*-ordering symbol means any derivatives are taken after time ordering the operators. Figure 2 shows the associated Feynman diagrams.

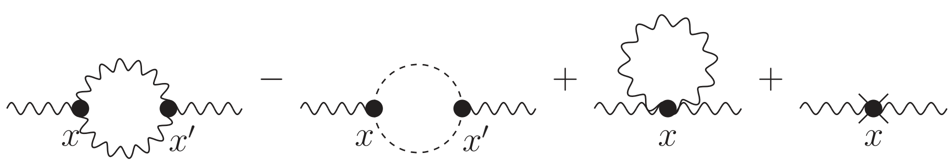

Figure 2: Diagrams contributing to the one loop graviton self-energy, shown in the same order, left to right, as the three contributions to (133). Graviton lines are wavy and ghost lines are dashed.

The actual computation [22] was made in $D=4$ dimensions before it was understood how to employ dimensional regularization, so it can only be used away from coincidence. At the end of this section we discuss how it might be extended to recover the full result. The computation was considerably more difficult than deriving the scalar contribution of section 4.1. It differs from the scalar result in three ways:

- It breaks de Sitter invariance, both inessentially through the use of a de Sitter breaking gauge [23, 24] and unavoidably through physical de Sitter breaking for gravitons [32–36] whose kinematics are the same as those of the massless, minimally coupled scalar [37, 38];
- The coefficient functions $T^i(x;x')$ involve not only the two scale factors and powers of Δx^2, but also up to a single factor of $\ln(H^2\Delta x^2)$; and
- The self-energy is not annihilated by the action of a single Ward operator so the coefficients $S^i(x;x')$ are nonzero.

The second point means that we can decompose the coefficient functions T^i and S^i into parts with and without a factor of $\ln(H^2\Delta x^2)$,

$$T^i(x;x') = T^i_N(x;x') + T^i_L(x;x') \times \ln(H^2\Delta x^2) , \qquad (134)$$

$$S^i(x;x') = S^i_N(x;x') + S^i_L(x;x') \times \ln(H^2\Delta x^2) . \qquad (135)$$

Tables 7 and 8 give our results for the algebraically independent $T^i_N(x;x')$ and $T^i_L(x;x')$, respectively. Tables 9 and 10 do the $S^i_N(x;x')$ and $S^i_L(x;x')$.

We also need the auxiliary functions defined in expressions (73), (74) and (75). The antisymmetric part of $\gamma(x;x')$ is,

$$\gamma - \gamma^R = (\eta+\eta')\Delta\eta \Biggl\{ -\frac{5376 aa'H^2\Delta\eta^2}{5\Delta x^{10}} - \frac{8aa'H^2[359+124aa'H^2\Delta\eta^2]}{5\Delta x^8} - \frac{4a^2a'^2H^4[181+60aa'H^2\Delta\eta^2]}{15\Delta x^6} + \frac{a^3a'^3H^6[24+11aa'H^2\Delta\eta^2]}{\Delta x^4} - \frac{2a^4a'^4H^8[5-2aa'H^2\Delta\eta^2]}{\Delta x^2} + \Biggl[\frac{96a^2a'^2H^4\Delta\eta^2}{\Delta x^8} + \frac{8a^2a'^2H^4[6+7aa'H^2\Delta\eta^2]}{\Delta x^6} + \frac{2a^3a'^3H^6[1+7aa'H^2\Delta\eta^2]}{\Delta x^4} - \frac{a^4a'^4H^8[31+8aa'H^2\Delta\eta^2]}{\Delta x^2}\Biggr] \ln(H^2\Delta x^2)\Biggr\} . \qquad (136)$$

This implies that $\alpha(x;x')$ and $\beta(x;x')$ are,

$$\alpha = \frac{\frac{672}{5}\Delta\eta^2}{\Delta x^8} + \frac{\frac{764}{15}+\frac{16}{3}aa'H^2\Delta\eta^2}{\Delta x^6} - \frac{aa'H^2[\frac{16}{3}+\frac{5}{3}aa'H^2\Delta\eta^2]}{\Delta x^4}$$

i	$T^i_N(x;x')$
1	$\frac{8(\frac{92}{5}-77aa'H^2\Delta\eta^2-12a^2a'^2H^4\Delta\eta^4)}{\Delta x^8}-\frac{\frac{4}{3}aa'(55+203aa'H^2\Delta\eta^2+48a^2a'^2H^4\Delta\eta^4)H^2}{\Delta x^6}-\frac{a^2a'^2(19+22aa'H^2\Delta\eta^2)H^4}{\Delta x^4}$
2	$\frac{\frac{1952}{5}-416aa'H^2\Delta\eta^2}{\Delta x^8}-\frac{\frac{16}{3}aa'(8+7aa'H^2\Delta\eta^2-6a^2a'^2H^4\Delta\eta^4)H^2}{\Delta x^6}+\frac{4a^2a'^2(14+3aa'H^2\Delta\eta^2)H^4}{\Delta x^4}$
3	$-\frac{\frac{64}{5}[23+(55a+90a')H\Delta\eta]\Delta\eta^2}{\Delta x^{10}}-\frac{8[23+7(16a+13a')H\Delta\eta-(a^2-36a'^2)H^2\Delta\eta^2]}{\Delta x^8}-\frac{4[52a^2+117aa'-\frac{163}{3}a'^2-16(a^3-a'^3)H\Delta\eta]H^2}{\Delta x^6}-\frac{aa'(18a^2+147aa'-20a'^2)H^4}{\Delta x^4}$
5	$\frac{\frac{16}{5}[23+(55a+90a')H\Delta\eta]\Delta\eta}{\Delta x^8}+\frac{\frac{8}{3}[86a+a'+(5a^2+12a'^2)H\Delta\eta]H}{\Delta x^6}-\frac{2a(6a^2-33aa'+4a'^2)H^3}{\Delta x^4}$
7	$-\frac{\frac{4}{15}[23+(55a+90a')H\Delta\eta]}{\Delta x^6}+\frac{(\frac{1}{3}a^2-6aa'+8a'^2)H^2}{\Delta x^4}+\frac{aa'(2a^2-6aa'+a'^2)H^4}{\Delta x^2}$
9	$-\frac{\frac{128}{5}(61-55aa'H^2\Delta\eta^2)\Delta\eta^2}{\Delta x^{10}}-\frac{16(61+10aa'H^2\Delta\eta^2-2a^2a'^2H^4\Delta\eta^4)}{\Delta x^8}-\frac{8aa'(70-aa'H^2\Delta\eta^2)H^2}{\Delta x^6}-\frac{12a^2a'^2H^4}{\Delta x^4}$
10	$\frac{\frac{16}{5}(61-40aa'H^2\Delta\eta^2)\Delta\eta}{\Delta x^8}+\frac{\frac{16}{3}[20a-16a'+(9a-2a')a'H\Delta\eta]H}{\Delta x^6}+\frac{2a'(a^2-19aa'-2a'^2)H^3}{\Delta x^4}$
12	$-\frac{\frac{8}{15}(61-20aa'H^2\Delta\eta^2)}{\Delta x^6}-\frac{\frac{2}{3}aa'(16+5aa'H^2\Delta\eta^2)H^2}{\Delta x^4}+\frac{2a^2a'^2(4-aa'H^2\Delta\eta^2)H^4}{\Delta x^2}$
13	$\frac{5376\Delta\eta^4}{\Delta x^{12}}+\frac{64(84+21aa'H^2\Delta\eta^2)\Delta\eta^2}{\Delta x^{10}}+\frac{8(126+434aa'H^2\Delta\eta^2+53a^2a'^2H^4\Delta\eta^4)}{\Delta x^8}+\frac{4aa'(409+36aa'H^2\Delta\eta^2-24a^2a'^2H^4\Delta\eta^4)H^2}{\Delta x^6}+\frac{a^2a'^2(557+24aa'H^2\Delta\eta^2)H^4}{\Delta x^4}$
14	$-\frac{\frac{5376}{5}\Delta\eta^3}{\Delta x^{10}}-\frac{16[42+(19a-15a')H\Delta\eta]\Delta\eta}{\Delta x^8}+\frac{8[-\frac{238}{3}a+56a'+\frac{1}{3}(13a^2+48a'^2)H\Delta\eta-6aa'(3a-4a')H^2\Delta\eta^2]H}{\Delta x^6}+\frac{2a(8a^2-139aa'-6a'^2)H^3}{\Delta x^4}$
16	$\frac{\frac{336}{5}\Delta\eta^2}{\Delta x^8}+\frac{4[\frac{13}{3}-(a+6a')H\Delta\eta)]}{\Delta x^6}+\frac{(\frac{67}{3}a^2+14aa'+6a'^2)H^2}{\Delta x^4}-\frac{aa'(2a^2-10aa'+3a'^2)H^4}{\Delta x^2}$
18	$\frac{\frac{1344}{5}\Delta\eta^2}{\Delta x^8}+\frac{8(\frac{29}{3}+6aa'H^2\Delta\eta^2)}{\Delta x^6}+\frac{2aa'(-46+5aa'H^2\Delta\eta^2)H^2}{\Delta x^4}$
19	$-\frac{\frac{112}{5}\Delta\eta}{\Delta x^6}+\frac{(\frac{26}{3}a+8a')H}{\Delta x^4}-\frac{a^2a'^2H^4\Delta\eta}{\Delta x^2}$
21	$\frac{\frac{14}{5}}{\Delta x^4}$

Table 7: Contributions to $T^i(x;x')$ that do not contain factors of $\ln(H^2\Delta x^2)$. Each of the tabulated terms must be multiplied by $-\frac{\kappa^2}{64\pi^4}$.

i	$T^i_L(x;x')$
1	$\frac{16aa'(a^2+a'^2)H^4\Delta\eta^2}{\Delta x^6} - \frac{4aa'(a^2-4aa'+a'^2)H^4}{\Delta x^4} + \frac{12a^3a'^3H^6}{\Delta x^2}$
2	$-\frac{32aa'(a^2+a'^2)H^4\Delta\eta^2}{\Delta x^6} - \frac{16a^2a'^2H^4}{\Delta x^4} - \frac{4a^3a'^3H^6}{\Delta x^2}$
3	$-\frac{16a^3a'^2H^5\Delta\eta^3}{\Delta x^6} + \frac{4a^2a'^2(3a-2a')H^5\Delta\eta}{\Delta x^4} - \frac{16a^3a'^3H^6}{\Delta x^2}$
5	$\frac{8a^2a'(a+a')H^4\Delta\eta}{\Delta x^4} + \frac{12a^3a'^2H^5}{\Delta x^2}$
7	$-\frac{2aa'(2a^2+aa'+a'^2)H^4}{\Delta x^2}$
9	$-\frac{1536aa'H^2\Delta\eta^4}{\Delta x^{10}} - \frac{96(a^2+10aa'+a'^2)H^2\Delta\eta^2}{\Delta x^8} - \frac{32(a^2+aa'+a'^2)H^2}{\Delta x^6} + \frac{4a^2a'^2H^4}{\Delta x^4}$
10	$\frac{192aa'H^2\Delta\eta^3}{\Delta x^8} + \frac{16(a^2+2aa'+3a'^2)H^2\Delta\eta}{\Delta x^6} - \frac{4a'(a^2-aa'+2a'^2)H^3}{\Delta x^4} + \frac{4a^2a'^3H^5}{\Delta x^2}$
12	$-\frac{32aa'H^2\Delta\eta^2}{\Delta x^6} - \frac{4(3a^2-4aa'+3a'^2)H^2}{\Delta x^4} + \frac{2aa'(2a^2+5aa'+2a'^2)H^4}{\Delta x^2}$
13	$-\frac{1536aa'H^2\Delta\eta^4}{\Delta x^{10}} + \frac{96(a^2-14aa'+a'^2)H^2\Delta\eta^2}{\Delta x^8} + \frac{32aa'(-3+7aa'H^2\Delta\eta^2+a^2a'^2H^4\Delta\eta^4)H^2}{\Delta x^6}$ $-\frac{4aa'(4a^2-19aa'+4a'^2)H^4}{\Delta x^4} + \frac{24a^3a'^3H^6}{\Delta x^2}$
14	$\frac{384aa'H^2\Delta\eta^3}{\Delta x^8} - \frac{32(a^2-6aa'-a'^2)H^2\Delta\eta}{\Delta x^6} - \frac{8a(2a^2+aa'-a'^2)H^3}{\Delta x^4} - \frac{16a^3a'^2H^5}{\Delta x^2}$
16	$-\frac{32aa'H^2\Delta\eta^2}{\Delta x^6} + \frac{4(a^2-3a'^2)H^2}{\Delta x^4} + \frac{2aa'(2a^2-a'^2)H^4}{\Delta x^2}$
18	$-\frac{96aa'H^2\Delta\eta^2}{\Delta x^6} - \frac{12(a^2+a'^2)H^2}{\Delta x^4} - \frac{10a^2a'^2H^4}{\Delta x^2}$
19	$\frac{8aa'H^2\Delta\eta}{\Delta x^4} + \frac{2aa'(a-3a')H^3}{\Delta x^2}$
21	0

Table 8: Contributions to $T^i(x;x')$ that contain factors of $\ln(H^2\Delta x^2)$. Each tabulated term must be multiplied by $-\frac{\kappa^2}{64\pi^4}$.

i	$S^i_N(x;x')$
1	$\frac{2560a'H\Delta\eta^4}{\Delta x^{12}} + \frac{128[42a-6a'+(9a^2+8a'^2)H\Delta\eta]H\Delta\eta^2}{\Delta x^{10}} + \frac{16[303a-193a'+(73a^2-33a'^2)H\Delta\eta+(12a^3+18a'^3)H^2\Delta\eta^2]H}{\Delta x^8} + \frac{8(20a^3+41a^2a'+50aa'^2-14a'^3)H^3}{\Delta x^6} + \frac{8a^2a'^2(5a+4a')H^5}{\Delta x^4}$
2	$-\frac{256a'H\Delta\eta^3}{\Delta x^{10}} - \frac{16[(42a-16a')H\Delta\eta+(9a^2+6a'^2)H^2\Delta\eta^2]}{\Delta x^8} - \frac{8H^2(17a^2+\frac{53}{3}aa'-5a'^2+4a^3H\Delta\eta)}{\Delta x^6} - \frac{2a^2a'H^4(5a'+12a)}{\Delta x^4}$
3	$\frac{-384(a^2-3aa'-a'^2)H^2\Delta\eta^3}{\Delta x^{10}} - \frac{16[57a+7a'-(5a^2-a'^2)H\Delta\eta+12a'^3H^2\Delta\eta^2]H}{\Delta x^8} + \frac{8(2a^3-\frac{65}{3}aa'^2+9a'^3)H^3}{\Delta x^6} - \frac{2a^2a'^2(4a+11a')H^5}{\Delta x^4}$
4	$\frac{32a'H\Delta\eta^2}{\Delta x^8} + \frac{8H[11a-5a'+(4a^2+\frac{1}{3}a'^2)H\Delta\eta]}{\Delta x^6} + \frac{2H^3a[4a^2+2aa'-\frac{11}{3}a'^2]}{\Delta x^4} + \frac{H^5a^2a'^2(4a+a')}{\Delta x^2}$
5	$-\frac{256a'H\Delta\eta^3}{\Delta x^{10}} + \frac{16[6a-16a'+(3a^2-8a'^2)H\Delta\eta]H\Delta\eta}{\Delta x^8} + \frac{8[7a^2-aa'-\frac{11}{3}a'^2-6a'^3H\Delta\eta]H^2}{\Delta x^6} - \frac{4aa'^2(a-2a')H^4}{\Delta x^4}$
6	$-\frac{256(3a-2a')H\Delta\eta^2}{\Delta x^{10}} - \frac{4[90a-50a'+(27a^2+3a'^2)H\Delta\eta]H}{\Delta x^8} - \frac{6aa'(9a-a')H^3}{\Delta x^6}$
7	$\frac{32a'H\Delta\eta^2}{\Delta x^8} - \frac{8H(a^3-a^2a'-\frac{10}{3}aa'^2+2a'^3)}{aa'\Delta x^6} + \frac{H^3aa'(a-5a')}{2\Delta x^4}$
8	$\frac{32(3a-2a')H\Delta\eta}{\Delta x^8} + \frac{(18a^2-20aa'+\frac{38}{3}a'^2)H^2}{\Delta x^6} + \frac{aa'^2(a'-\frac{9}{2}a)H^4}{\Delta x^4}$
9	$\frac{48(a^2-3aa'-a'^2)H^2\Delta\eta^2}{\Delta x^8} + \frac{8(-2a^2+2aa'-3a'^2+4a'^3H\Delta\eta)H^2}{\Delta x^6} + \frac{2aa'^2(2a-3a')H^4}{\Delta x^4}$
10	$-\frac{16a'H\Delta\eta}{3\Delta x^6} + \frac{2H^2a'(a-\frac{1}{3}a')}{\Delta x^4} + \frac{H^4a^2a'^2}{2\Delta x^2}$

Table 9: Contributions to $S^i(x;x')$ that do not contain a factor of $\ln(H^2\Delta x^2)$. Each of the tabulated terms must be multiplied by $-\frac{\kappa^2}{64\pi^4}$.

i	$S^i_L(x;x')$
1	$\frac{384a^2a'^2H^4\Delta\eta^3}{\Delta x^8}+\frac{64aa'(a^2+5aa'-3a'^2)H^4\Delta\eta}{\Delta x^6}+\frac{64a^3a'^2H^5}{\Delta x^4}$
2	$-\frac{64a^2a'^2H^4\Delta\eta^2}{\Delta x^6}-\frac{16a^3a'H^4}{\Delta x^4}$
3	$\frac{192a^2a'^2H^4\Delta\eta^3}{\Delta x^8}+\frac{32aa'(a^2-2aa'+4a'^2)H^4\Delta\eta}{\Delta x^6}-\frac{24a^3a'^2H^5}{\Delta x^4}$
4	$\frac{8a^2a'^2H^4\Delta\eta}{\Delta x^4}+\frac{4a^3a'^2H^5}{\Delta x^2}$

Table 10: Nonzero parts of $S^i(x;x')$ which are proportional to $\ln(H^2\Delta x^2)$. (The cases of $i=5,6,7,8,9,10$ vanish.) Each of the tabulated terms must be multiplied by $-\frac{\kappa^2}{64\pi^4}$.

$$+\frac{a^2a'^2H^4[4-aa'H^2\Delta\eta^2]}{\Delta x^2}+\Bigg[-\frac{16aa'H^2\Delta\eta^2}{\Delta x^6}-\frac{2aa'H^2[2+3aa'H^2\Delta\eta^2]}{\Delta x^4} +\frac{a^2a'^2H^4[9+2aa'H^2\Delta\eta^2]}{\Delta x^2}\Bigg]\ln(H^2\Delta x^2), \quad (137)$$

$$\beta=-\frac{[\frac{3904}{5}-704aa'H^2\Delta\eta^2]\Delta\eta^2}{\Delta x^{10}}-\frac{[\frac{488}{5}+272aa'H^2\Delta\eta^2-80a^2a'^2H^4\Delta\eta^4]}{\Delta x^8} -\frac{aa'H^2[\frac{280}{3}+\frac{332}{3}aa'H^2\Delta\eta^2]}{\Delta x^6}+\frac{4a^2a'^2H^4(6-aa'H^2\Delta\eta^2)}{\Delta x^4}+\Bigg[-\frac{768aa'H^2\Delta\eta^4}{\Delta x^{10}} -\frac{144aa'H^2\Delta\eta^2[4+aa'H^2\Delta\eta^2]}{\Delta x^8}-\frac{8aa'H^2[6+8aa'H^2\Delta\eta^2+2a^2a'^2H^4\Delta\eta^4]}{\Delta x^6} +\frac{2a^2a'^2[1+2aa'H^2\Delta\eta^2]}{\Delta x^4}-\frac{4a^3a'^3H^6}{\Delta x^2}\Bigg]\ln(H^2\Delta x^2). \quad (138)$$

Note that none of the structure functions differ in form from the primitive contributions. That is not some miracle of the de Sitter background; one can see that it must be true generally from equations (81) — which could be used to infer $\beta(x;x')$ — and equation (84) — which could be used to infer $\alpha(x;x')$. The absence of new functional forms is quite unlike what happened in previous representations of the scalar result [19, 20], neither of which could even be applied to contributions from gravity or to general cosmological backgrounds.

4.2.1 Recovering the Local Terms

The previous results determine the nine structure functions for all $x'^{\mu} \neq x^{\mu}$. However, there are still potentially important local contributions proportional to $i\delta^4(x-x')$. These terms dominate the fermion wave function [39,40] and the photon field strength [41, 42], and they make an important contribution to electromagnetic forces [43], so it is worth explaining how they can be recovered. Of course we could simply re-do the computation using dimensional regularization from the beginning using the D-dependent propagators [24] and vertices [23], but we have in mind a simpler approach based on understanding the three sources of local contributions:[5]

- From renormalization;
- From the 4-point diagram of Figure 2; and
- From differentiated propagators in the two 3-point diagrams of Figure 2.

Renormalization is the simplest case to understand. From Tables 7 and 8 we see that the most singular parts of the fundamental coefficient functions near coincidence are,

$$T^{12} \sim \frac{1}{\Delta x^6} \quad , \quad T^{16} \sim \frac{1}{\Delta x^6} \quad , \quad T^{18} \sim \frac{1}{\Delta x^6} \quad , \quad T^{19} \sim \frac{\Delta\eta}{\Delta x^6} \; . \tag{139}$$

In general D the ghost and graviton propagators involve functions of the same form as (120) in the scalar propagator [24],

$$i\Delta(x;x') \sim \Bigl(\frac{1}{aa'\Delta x^2}\Bigr)^{\frac{D}{2}-1} + \Bigl(\frac{1}{aa'\Delta x^2}\Bigr)^{\frac{D}{2}-2} + \dots \tag{140}$$

The generic vertex involves a factor of a^{D-2} with two derivatives [23]. So comparison with the leading scalar divergences (128) means that $D=4$ results like $1/\Delta x^6$ correspond to $1/\Delta x^{2D-2}$ in the dimensionally regulated

[5]The reader is free to dismiss the comments of this subsection as conjectural. However, they are based on the authors' great familiarity with the $D=4$ computation of the graviton loop contribution [22], and the close simularity of that contribution to the general D scalar loop contribution [20] reported in section 4.1. In particular, the graviton 3-point vertex takes the same $\kappa a^{D-2} h\partial h\partial h$ form as the scalar-graviton vertex $\kappa a^{D-2} h\partial\varphi\partial\varphi$. The most important part of the D-dimensional graviton propagator [23,24] is also just some constant tensors times the scalar propagator.

theory, and all D-dependent powers of the scale factor cancel between the two propagators and the two vertices. Hence we can extend the $D=4$ results from Tables 7, 8, 9 and 10 by the replacements such as,

$$\frac{1}{\Delta x^6} \longrightarrow \frac{\mu^{D-4}\pi^{\frac{D}{2}}\partial^2 i\delta^D(x-x')}{(D-2)^2(D-3)(D-4)\Gamma(\frac{D}{2}-1)} - \frac{\partial^4}{32}\Bigg[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Bigg] , \tag{141}$$

$$\frac{\Delta\eta}{\Delta x^6} \longrightarrow \frac{\mu^{D-4}\pi^{\frac{D}{2}}\partial_0 i\delta^D(x-x')}{(D-2)(D-3)(D-4)\Gamma(\frac{D}{2}-1)} - \frac{\partial_0\partial^2}{16}\Bigg[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Bigg] , \tag{142}$$

times non-negative integer powers of a and a'. Because counterterms are proportional to $a^{D-4}i\delta^D(x-x')$ times non-negative powers of a and a', it is possible to predict the finite factors of $\ln(a)\delta^4(x-x')$ that remain after renormalization. Note that we can also predict how to extract derivatives from the finite, nonlocal parts of the structure functions.

All the 4-point diagrams are local, and they are simple enough to compute directly. In dimensional regularization any D-dependent power of Δx^2 vanishes at coincidence. The coincident propagator comes from the integer sums in (120) and related propagator functions. These will never contribute D-dependent powers of a, hence the factor of a^{D-2} from the 4-point agrees with the D-dependent factor of a from the counterterm, so no finite factors of $\ln(a)$ can arise from this source. However, the coincident propagator can produce an easily-predictable and ultraviolet finite factor of $\ln(a)$.

Without regard to the tensor structure we can see that the generic 3-point contribution takes the form,

$$\Big(i\kappa a^{D-2}\partial^2\Big) \times i\Delta(x;x')i\Delta(x;x')\times\Big(i\kappa a'^{D-2}\partial'^2\Big) . \tag{143}$$

Acting two times derivatives on a propagator produces a delta function [26],

$$\partial_\mu\partial'_\nu i\Delta(x;x') = \frac{\delta^0{}_\mu\delta^0{}_\nu i\delta^D(x-x')}{a^{D-2}} + \text{Nonlocal Terms} . \tag{144}$$

The other propagator is taken to coincidence by the delta function, so the same considerations apply to it as for the 4-point contributions considered above. It turns out that acting the derivatives to produce the nonlocal terms is the rate-limiting step of the computation, so it is considerably simpler to access the local $\ln(a)$ term than to derive the dimensionally regulated nonlocal contributions.

4.2.2 The Gauge Issue

Graviton propagators require gauge fixing. The calculation [22] reported in section 4.2 was performed by adding a gauge fixing functional whose D-dimensional extension is [23, 24],

$$\mathcal{L}_{GF} = -\frac{a^{D-2}}{2} \eta^{\mu\nu} F_\mu F_\nu \quad , \quad F_\mu = \eta^{\rho\sigma} \Bigl(h_{\mu\rho,\sigma} - \frac{1}{2} h_{\rho\sigma,\mu} + (D-2) a H h_{\mu\rho} \delta^0{}_\sigma \Bigr) . \tag{145}$$

The special feature of this gauge is that it makes the propagator take the form of a sum of three constant tensor factors times scalar propagators,

$$i\Bigl[{}_{\mu\nu}\Delta_{\rho\sigma}\Bigr](x;x') = \sum_{I=A,B,C} \Bigl[{}_{\mu\nu}T^I_{\rho\sigma}\Bigr] \times i\Delta_I(x;x') . \tag{146}$$

Here the constant tensor factors are,

$$\Bigl[{}_{\mu\nu}T^A_{\rho\sigma}\Bigr] = 2\overline{\eta}_{\mu(\rho}\overline{\eta}_{\sigma)\nu} - \frac{2}{D-3}\overline{\eta}_{\mu\nu}\overline{\eta}_{\rho\sigma} \quad , \quad \Bigl[{}_{\mu\nu}T^B_{\rho\sigma}\Bigr] = -4\delta^0{}_{(\mu}\overline{\eta}_{\nu)(\rho}\delta^0{}_{\sigma)} , \tag{147}$$

$$\Bigl[{}_{\mu\nu}T^C_{\rho\sigma}\Bigr] = \frac{2E_{\mu\nu}E_{\rho\sigma}}{(D-2)(D-3)} \quad , \quad E_{\mu\nu} \equiv (D-3)\delta^0{}_\mu\delta^0{}_\nu + \overline{\eta}_{\mu\nu} . \tag{148}$$

And the three scalar propagators are all related to the function $A(y)$ of expression (120),

$$i\Delta_A(x;x') = A(y) + k\ln(aa') \qquad k \equiv \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} , \tag{149}$$

$$i\Delta_B(x;x') = B(y) \equiv -\frac{[(4y-y^2)A'(y) + (2-y)k]}{2(D-2)} , \tag{150}$$

$$i\Delta_C(x;x') = C(y) \equiv \frac{1}{2}(2-y)B(y) + \frac{k}{D-3} . \tag{151}$$

The flat space limit is obtained by taking the scale factor to unity and the Hubble parameter to zero. In this limit our de Sitter gauge reduces to,

$$\mathcal{L}_{GF} \longrightarrow -\frac{1}{2}\eta^{\mu\nu}F_\mu F_\nu \qquad , \qquad F_\mu = \eta^{\rho\sigma}\Bigl(h_{\mu\rho,\sigma} - \frac{1}{2}h_{\rho\sigma,\mu}\Bigr) , \tag{152}$$

and the corresponding propagator becomes,

$$i\Bigl[{}_{\mu\nu}\Delta_{\rho\sigma}\Bigr](x;x') \longrightarrow \Bigl(2\eta_{\mu(\rho}\eta_{\sigma)\nu} - \frac{2}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}\Bigr) \times i\Delta(x;x') , \tag{153}$$

where the massless scalar propagator of flat space was defined in expression (22). This is precisely the gauge Capper employed to derive the results reported in equations (21-26) [25]. It is straightforward to check that the flat space limits of the de Sitter results reported in Tables 7 and 8 agree with the specialization to $D = 4$ of Capper's results (21-26).

The advantages of our de Sitter gauge (145) are so great that it has been used for nine [22, 39, 41, 44–49] of the ten graviton loops which have so far been computed de Sitter background. The exception was a year-long *tour de force* made to check for gauge dependence in the vacuum polarization [50] using a cumbersome, 1-parameter family of de Sitter invariant gauges [51]. It would be quite challenging re-computing the graviton self-energy in this family of gauges. We have instead devised a 2-parameter deformation of the de Sitter breaking gauge (145) [52],

$$\mathcal{L}_{GF}^{\alpha\beta} = -\frac{a^{D-2}}{2\alpha}\eta^{\mu\nu}F_\mu F_\nu \quad , \quad F_\mu = \eta^{\rho\sigma}\Bigl(h_{\mu\rho,\sigma} - \frac{\beta}{2}h_{\rho\sigma,\mu} + (D-2)aHh_{\mu\rho}\delta^0{}_\sigma\Bigr) \; . \tag{154}$$

Although we have not yet computed the graviton self-energy in this gauge, Capper derived a result for its flat space limit [25]. The final, renormalized result takes the same form as (32) but with the numerical coefficients changed. In the general gauge the coefficient of $\Pi^{\mu\nu}\Pi^{\rho\sigma}$ becomes [25],

$$\frac{23}{2} \longrightarrow \frac{45}{2}\alpha^2 + \frac{113}{2} + \frac{15}{2}\frac{(\alpha-3)^2}{(\beta-2)^4} - \frac{135}{2}\frac{(\alpha-3)}{(\beta-2)^3} - \frac{25}{2}\frac{(2\alpha-19)}{(\beta-2)^2} + \frac{5}{2}\frac{(11\alpha+59)}{(\beta-2)} \; . \tag{155}$$

The coefficient of $\Pi^{\mu(\rho}\Pi^{\sigma)\nu}$ becomes [25],

$$\frac{61}{2} \longrightarrow \frac{15}{2}\alpha^2 + \frac{45}{4}\alpha - \frac{43}{4} + \frac{5}{2}\frac{(\alpha-3)^2}{(\beta-2)^4} - \frac{105}{4}\frac{(\alpha-3)}{(\beta-2)^3} - \frac{5}{4}\frac{(\alpha-51)}{(\beta-2)^2} + \frac{5}{4}\frac{(9\alpha-11)}{(\beta-2)} \; . \tag{156}$$

Expressions (155) and (156) can be made arbitrarily positive by taking β near 2. They do seem to be bounded below, but they can definitely change sign, and there are two real solutions which cause them both two vanish,

$$\alpha \simeq 0.551886 \qquad , \qquad \beta \simeq 1.42999 \; , \tag{157}$$

$$\alpha \simeq 2.24351 \qquad , \qquad \beta \simeq 1.78159 \; . \tag{158}$$

The gauge dependence we have exhibited in the flat space results (155-156) must of course be present in the de Sitter result. However, it still is not clear what happens to the parts of $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ which represent the effects of inflationary gravitons. To understand this better, consider the contributions to $T^{19}(x;x')$ from Tables 7 and 8,

$$T^{19}(x;x') = -\frac{\kappa^2}{64\pi^4}\Biggl\{-\frac{\frac{112}{5}\Delta\eta}{\Delta x^6} + \frac{(\frac{26}{3}a+8a')H}{\Delta x^4} - \frac{a^2a'^2H^4\Delta\eta}{\Delta x^2} + \Biggl[\frac{8aa'H^2\Delta\eta}{\Delta x^4} + \frac{2aa'(a-3a')H^3}{\Delta x^2}\Biggr]\ln(H^2\Delta x^2)\Biggr\} . \qquad (159)$$

The flat space result consists of just the first term. It is this term whose coefficient can be driven to infinity, or made to vanish by the gauge dependence of (155-156). This term has no effect on the graviton mode function, and induces fractional corrections to the potentials of the form κ^2/r^2. How the parameters α and β of the general de Sitter gauge (154) affect the other terms is not known. These other terms can potentially change the graviton mode function, and they typically induce fractional changes in the potentials of the form GH^2 times large temporal and/or spatial logarithms.

In flat space Donoghue and collaborators have shown how to extract unique, gauge independent results for the fractional κ^2/r^2 correction to the potentials [14,15]. Their technique [6,8] is to first compute the one loop scattering amplitude between two massive particles, then use inverse scattering theory to infer the exchange potential. It was recently discovered that this process can be short-circuited in order to directly purge the vacuum polarization of gauge dependence [53]. The procedure is to assemble the same diagrams whose sum would produce the scattering amplitude, however, one works in position space and employs a series of identities which permit the higher point diagrams to be viewed as corrections to the 1PI 2-point function. For example, one of the many diagrams which contribute to the scattering of two massive scalars consists of two graviton lines emerging from the vertex at x'^{μ} and attaching to the other massive scalar at points x^{μ} and y^{μ}. This diagram does not have the 2-point topology to be viewed as a contribution to the graviton self-energy, however, Donoghue and collaborators have derived a series of reductions that capture the nonanlytic parts of the full amplitude which are responsible for infrared phenomena [6, 8, 54, 55]. If $i\Delta_m(x;x')$ denotes the massive scalar propagator then the relevant Donoghue Identity for the 3-point contribution just described is [53],

$$i\Delta_m(x;y)i\Delta(x;x')i\Delta(y,x') \longrightarrow \frac{i\delta^D(x-y)}{2m^2}\Bigl[i\Delta(x;x')\Bigr]^2 . \qquad (160)$$

Applying (160) reduces the 3-point contribution to a 2-point form which can be viewed as a correction to the graviton self-energy. When all such corrections are combined, dependence upon α and β drops out and one is left with a unique and gauge independent result [53].

5 Epilogue

Quantum corrections from inflationary gravitons [39, 41, 45–50] modify how other particles propagate [40,42,56–59], and the force laws they mediate [43]. At one loop order these results involve a single graviton propagator, and it is principally the "tail" part of this propagator that engenders the most interesting effects [60, 61]. Quantum gravity corrections to gravity itself are even more interesting because they involve *two* graviton propagators at one loop order. The potential for gravity to mediate more interesting effects than matter can be seen from the factor of $\ln(H^2 \Delta x^2)$ which multiplies all the contributions of Table 8, and is absent from the analogous scalar contributions of Table 6.

The 1PI 2-graviton function $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ quantifies corrections to linearized gravity from matter and from gravity itself. The point of this paper has been to develop a representation for the tensor structure of this object in terms of differential operators acting on structure functions. Matter contributions must be annihilated when the Ward operator (5) acts on *either* coordinate, but gravity contributions are only annihilated when the Ward operator acts on *both* coordinates. On flat space background one requires two structure functions for matter contributions and three for contributions from gravity, as in expression (29). The absence of time translation invariance and Lorentz invariance in cosmology means that four structure functions are required for matter contributions whereas nine are needed for contributions from gravity. Our representations are given in expressions (100) for matter, and (102) for gravity.

Quantum field theory computations typically express $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ as a linear combination of basis tensors which do not individually obey the relevant Ward identity. For example, the flat space result (21) was originally reported [25] using a basis of five tensors, which can then be organized into three combinations (29) that obey the Ward identity. This procedure for passing from raw results to structure functions is known as *reconstruction.* Our reconstruction procedure for cosmology is based on first recasting the primitive result as a sum (3) of the 21 tensor differential operators $[^{\mu\nu}\mathcal{D}^{\rho\sigma}]$ listed in Table 1, each acting on a scalar coefficient function $T^i(x;x')$. The $[^{\mu\nu}\mathcal{D}^{\rho\sigma}]$ are constructed from $\delta^\mu_{\ 0}$ and the spatial parts of the Minkowski metric $\overline{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} + \delta^\mu_{\ 0}\delta^\nu_{\ 0}$ and the derivative operator $\overline{\partial}^\mu \equiv \partial^\mu + \delta^\mu_{\ 0}\partial_0$. A typical example is furnished by the scalar contribution (117) on de Sitter background [19, 20]. We first 3 + 1 decompose derivatives (118-119) of the de Sitter length function, then express factors of the spatial coordinate interval $\vec{x} - \vec{x}'$ as gradients using relations (125-127). Our fundamental structure functions for matter are $T^{12}(x;x')$, $T^{16}(x;x')$, $T^{18}(x;x')$ and $T^{19}(x;x')$. For gravity we express one action of the Ward operator in the form (6), involving the ten basis tensors of Table 3 acting on coefficient functions $S^i(x;x)$. The expansion of each S^i in terms of the T^i is given in Table 4. Because acting the Ward operator a second time must produce zero, the ten S^i obeys the five relations given in Table 5. We take the five new structure functions for gravity to be $S^2(x;x')$, $S^4(x;x')$, $S^7(x;x')$, $S^8(x;x')$ and $S^{10}(x;x')$. We have found it convenient to group some of the fundamental structure functions (T^{16}, T^{19}, S^4, S^8 and S^{10}) into two symmetric auxiliary functions, $\alpha(x;x')$ and $\beta(x;x')$, which are defined in expressions (73-75). Our final representations for the self-energy in terms of the fundamental structure functions are expressions (100) and (102).

The formalism we have derived for representing the graviton self-energy improves on previous results [20, 21] in three ways:

- It applies for contributions from gravitons in addition to contributions from matter;
- It is valid for any cosmological background (2), not just for de Sitter; and
- Its structure functions involve the same functional forms as the primitive result.

One can appreciate the final point from the explicit results for a loop of massless, minimally coupled scalars [19, 20]. Primitive contributions to the $T^i(x;x')$ consist of sums of products of non-negative powers of the two scale factors and the temporal separation $\Delta\eta$, times inverse powers of the Poincaré interval $\Delta x^2 \equiv (x-x')^2$. Because the fundamental structure functions of this new representation are just T^{12}, T^{16}, T^{18} and T^{19}, they of course have the same form. One might worry about the auxiliary functions α and β, but expressions (81) and (84) guarantee that they involve no new functional forms.

Compare that with what happens in the *simplest* of the previous representations. The renormalized spin zero structure function roughly equivalent to T^{18} and a combination of T^{16} and T^{19} is [21],

$$F_{0R} = \frac{\kappa^2}{9(4\pi)^4}\Biggl\{\frac{\partial^2}{2}\Biggl[\frac{\ln(H^2\Delta x^2)}{\Delta x^2}\Biggr] + a^2 a'^2 H^4\Biggl[-\frac{6}{y}+6 \\ \Bigl(-\frac{2}{y}+6-\frac{4}{4-y}\Bigr)\ln\Bigl(\frac{y}{4}\Bigr) + \frac{3}{2}(2-y)\Psi(y)\Biggr\}, \quad (161)$$

where $y \equiv aa'H^2\Delta x^2$ is the de Sitter length function and we define

$$\Psi(y) \equiv \frac{1}{2}\ln^2\Bigl(\frac{y}{4}\Bigr) - \ln\Bigl(1-\frac{y}{4}\Bigr)\ln\Bigl(\frac{y}{4}\Bigr) - \mathrm{Li}_2\Bigl(\frac{y}{4}\Bigr) , \quad (162)$$

where $\mathrm{Li}_2(x) \equiv -\int_0^x dt \ln(1-t)/t$ is the dilogarithm function. The same exotic functional forms appear in the two tensor structure functions, $F_{2R}(x;x')$ and $G_{2R}(x;x')$, which are roughly equivalent to $T^{12}(x;x')$ and a different combination of T^{16} and $T^{19}(x;x')$. Deriving these structure functions from the primitive result for $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ is a major undertaking because it entails solving partial differential equations. Those equations were barely tractable for the scalar contributions owing to the absence of de Sitter breaking, but they become hopelessly complicated when de Sitter invariance is lost with graviton contributions. Finally, it is of course difficult *using* the exotic structure functions to solve the effective field equations. The new formalism obviates all of these problems.

The point of devising this representation is to solve the effective field equations. Section 3.3 specializes the effective field equations for a graviton contribution to the cases of a spatial plane wave graviton (109) and the two scalar potentials (114-115) which represent the gravitational response to point mass. Both of these things have already been computed (using the old formalism) for the contribution of a massless, minimally coupled scalar [19, 20]. Although there are no changes in the graviton mode function [20], the response to a point mass acquires corrections which grow at late times and large distances [19],

$$\Psi(\eta, r) = -\frac{GM}{ar}\Biggl\{1 + \frac{G}{20\pi a^2 r^2} \\ +\frac{GH^2}{\pi}\Bigl[-\frac{1}{30}\ln(a) - \frac{3}{10}\ln(aHr)\Bigr] + O(G^2)\Biggr\}, \quad (163)$$

$$\Phi(\eta, r) = -\frac{GM}{ar}\Bigg\{1 - \frac{G}{60\pi a^2 r^2} + \frac{GH^2}{\pi}\Big[-\frac{1}{30}\ln(a) - \frac{3}{10}\ln(aHr) + \frac{2}{3}aHr\Big] + O(G^2)\Bigg\}. \qquad (164)$$

We are now in a position to study what gravity does to itself. Applying the Hartree approximation indicates that inflationary gravitons enhance the "electric" components of the Weyl field strength [18],

$$C^{1\ \text{loop}}_{0i0j}(\eta, k) \longrightarrow -\frac{8}{\pi} GH^2 \ln(a) \times C^{\text{tree}}_{0i0j}(\eta, k)\ . \qquad (165)$$

It would be very interesting to extend the $D = 4$ results of Tables 7, 8, 9 and 10 to recover fully renormalized results, and then employ them to solve equations (109) and (114-115).

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