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ORIGINAL PAPER

Scattering resonances for a three-dimensional subwavelength hole

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Abstract
This work aims to investigate scattering resonances and the field amplification at resonant frequencies for a subwavelength hole of width ε embedded in a sound hard slab. We apply the integral equation approach and asymptotic analysis to derive the asymptotic expansions of scattering resonances and quantitatively analyze the corresponding field amplifications. It is shown that the complex-valued scattering resonances attain imaginary parts of order $O(\varepsilon^2)$. The field enhancement inside the hole and in the far field is of order $O(1/\varepsilon^2)$ at the resonant frequencies, which is much stronger the enhancement order in the two-dimensional subwavelengt hole of the same width.

Keywords Scattering resonances · Subwavelength holes · Acoustic wave · Helmholtz equation

Mathematics Subject Classification 35C20 · 35P30 · 35Q60

1 Introduction

Wave scattering by subwavelength apertures and holes has attracted a lot of attention in recent years due to its important applications in biological and chemical sensing [8,9,14,15,27,28]. The so-called extraordinary optical transmission (EOT) through the holes provides a label-free and highly sensitive manner to detect biomolecular events efficiently. The EOT transmission anomaly is related to a variety of resonances of the underlying subwavelength

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structures. Significant progress has been made in the last several years on the quantitative analysis of the resonances as well as the induced enhanced transmission for the two-dimensional structures. The readers are referred to [7,11,12,18–25,29] for the detailed investigation of different resonant phenomena for several typical subwavelength structures. Other related mathematical studies on the subwavelength resonant wave scattering and their applications can be found in [2–5] and references therein. In this work, we consider the three-dimensional problem and present quantitative analysis of scattering resonances for the acoustic wave scattering by a subwavelength hole embedded in a sound hard slab. We also study the wave field amplification when the frequency of the incident wave coincides with the real part of the complex-valued resonances. It is shown that the enhancement order $O(1/\varepsilon^2)$ is much stronger than the enhancement in the two-dimensional hole, which attains an order of $O(1/\varepsilon)$ [20].

The hole is bore through a sound hard material slab, and its geometry is shown in Fig. 1. The slab occupied the domain $\{(x_1.x_2, x_3) \mid 0 < x_3 < L\}$, and the hole is a cuboid given by $C_\varepsilon := \{(x_1.x_2, x_3) \mid 0 < x_1 < \varepsilon, 0 < x_2 < \varepsilon, 0 < x_3 < L\}$. We consider the case when the length and width ε of the hole is much smaller than the thickness of the slab and the wavelength of the incident wave λ, i.e., $\varepsilon \ll L \sim \lambda$. Without loss of generality, in what follows we scale the geometry of the problem by assuming that the slab thickness $L = 1$. Let us denote the upper and lower aperture of the hole by Γ^+ and Γ^- respectively, and semi-infinite domains Ω^+ and Ω^- above and below the slab respectively. The exterior domain is given by $\Omega_\varepsilon = \Omega^+ \cup \Omega^- \cup \bar{C}_\varepsilon$.

We consider the scattering when the plane wave u^i is incident upon the structure, where $u^i = e^{ik(d(x-x_0))}$ is the incident field. Here $(d_1, d_2, -d_3)$ is the incident direction with $d_3 > 0$, k is the wave number, and $x_0 = (0, 0, L)$. In the absence of hole, the total field in the domain Ω^+, consists of the incident field u^i and reflected field $u^r = e^{ik(d'(x-x_0))}$ where $d' = (d_1, d_2, d_3)$, while the field in the domain Ω^- is zero. In the presence of hole C_ε, the total field u_ε in the upper domain Ω^+ consists of u^i, u^r and the scattered field u_ε^s radiating from Γ^+. In the domain Ω_ε^-, u_ε only consists of the transmitted field through the lower aperture Γ^-. In addition, the Neumann boundary condition $\partial_\nu u_\varepsilon = 0$ is imposed on $\partial\Omega_\varepsilon$ for the sound hard material, where ν is the unit outward normal pointing to Ω_ε. Finally, the scattered field u_ε^s satisfies the Sommerfeld radiation condition at the semi-infinite domains [13]. In summary, the total field u_ε satisfies the following scattering problem:

$$\Delta u_\varepsilon + k^2 u_\varepsilon = 0, \quad \text{in } \Omega_\varepsilon, \tag{1.1}$$

$$\frac{\partial u_\varepsilon}{\partial \nu} = 0, \quad \text{on } \partial\Omega_\varepsilon, \tag{1.2}$$

$$u_\varepsilon = u_\varepsilon^s + u^i + u^r, \quad \text{in } \Omega^+, \tag{1.3}$$

$$u_\varepsilon = u_\varepsilon^s, \quad \text{in } \Omega^-, \tag{1.4}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u_\varepsilon^s}{\partial r} - iku_\varepsilon^s\right) = 0, \quad r = |x|. \tag{1.5}$$

For all complex wavenumbers k with $Imk \geq 0$, it can be shown that the above scattering problem has a unique solution. By analytic continuation, the resolvent $R(k) := (\Delta + k^2)^{-1}$ of the scattering problem (1.1)–(1.5) can be extended to the whole complex plane except at a countable number of poles. These poles are called the scattering resonances of the scattering problem. In this paper, we prove the existence of scattering resonances, derive the asymptotic expansions of those resonances, and present the quantitative analysis of the field amplification at the resonant frequencies. By reformulating the scattering problem (1.1)–(1.5) as the equivalent integral equation system, the resonances reduce to the characteristic values

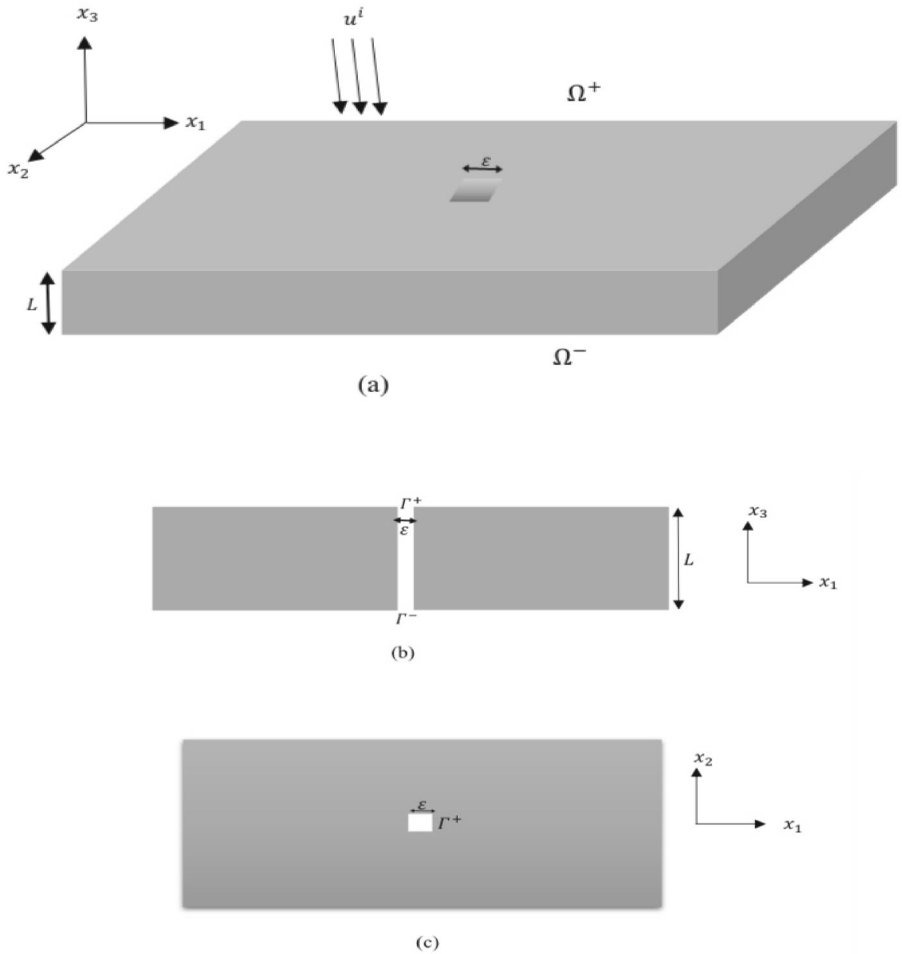

Fig. 1 **a** Geometry of the problem. The hole C_ε has a cuboid shape with height L and width ε. The domains above and below the hard sound slab are denoted as Ω^+ and Ω^- respectively and the exterior domain $\Omega_\varepsilon = \Omega^+ \cup \Omega^- \cup \bar{C}_\varepsilon$ is denoted by Ω_ε. The upper and lower aperture of the hole are denoted by Γ^+ and Γ^- respectively. **b**, **c** Vertical and horizontal cross section of the subwavelenth structure

of the certain integral operators. We apply the asymptotic analysis of the integral operators and the simplified Gohberg-Sigal theory to obtain scattering resonances. It is shown that the complex-valued scattering resonances attain imaginary parts of order $O(\varepsilon^2)$. We also analyze the field amplification at resonant frequencies and show that the enhancement is of order $O(1/\varepsilon^2)$.

The rest of the paper is organized as follows. In Sect. 2, we reformulate the scattering problem (1.1)–(1.5) by the boundary integral equation. Section 3 presents the asymptotic expansions of the boundary integral operators. Section 4 is devoted to the asymptotic expansion of the scattering resonances. The quantitative analysis of the field enhancement at the resonant frequencies is given in Sects. 5, and 6 proves the invertibility of the integral operator K used in the quantitative analysis of resonances.

2 Boundary integral equation formulation

The scattering problem (1.1)–(1.5) can be formulated equivalently as a system of boundary-integral equations. The development in this section is standard, see for instance [5,6,17]. Let $g^{\mathrm{e}}(k;x,y)$ and $g^{\mathrm{i}}_{\varepsilon}(k;x,y)$ be the Green's functions for the Helmholtz equations with the Neumann boundary condition in Ω^+, Ω^- and C_ε respectively. They satisfy the following equations:

$$\Delta g^{\mathrm{e}}(k;x,y) + k^2 g^{\mathrm{e}}(k;x,y) = \delta(x-y), \quad x,y \in \Omega^{\pm},$$
$$\Delta g^{\mathrm{i}}_{\varepsilon}(k;x,y) + k^2 g^{\mathrm{i}}_{\varepsilon}(k;x,y) = \delta(x-y), \quad x,y \in C_\varepsilon.$$

In addition $\frac{\partial g^{\mathrm{e}}(k;x,y)}{\partial \nu_y} = 0$ for $y_3 = 1$ and $y_3 = 0$, and $\frac{\partial g^{\mathrm{i}}(k;x,y)}{\partial \nu_y} = 0$ on ∂C_ε.

The Green's function in $\Omega^{\pm}$ is given by

$$g^{\mathrm{e}}(k;x,y) = -\frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|} - \frac{1}{4\pi}\frac{e^{ik|x'-y|}}{|x'-y|},$$

where

$$x' = \begin{cases} (x_1, x_2, 2-x_3) & \text{if } x,y \in \Omega^+, \\ (x_1, x_2, -x_3) & \text{if } x,y \in \Omega^-. \end{cases}$$

The interior Green function $g^{\mathrm{i}}_{\varepsilon}(x,y)$ in the hole C_ε with the Neumann boundary condition is

$$g^{i}_{\varepsilon}(k;x,y) = \sum_{m,n,l=0}^{\infty} c_{mnl}\varphi_{mnl}(x)\phi_{mnl}(y),$$

where $c_{mnl} = \frac{1}{k^2-(m\pi/\varepsilon)^2-(n\pi/\varepsilon)^2-(l\pi)^2}$, $\phi_{mnl} = \sqrt{\frac{\alpha_{mnl}}{\varepsilon}}\cos(\frac{m\pi x_1}{\varepsilon})\cos(\frac{n\pi x_2}{\varepsilon})\cos(l\pi x_3)$ and

$$a_{mnl} = \begin{cases} 1 \ mnl \in Z_1, \\ 2 \ mnl \in Z_2, \\ 4 \ mnl \in Z_3, \\ 8 \ mnl \in Z_4. \end{cases}$$

In the above $Z_1 = \{mnl \mid m = n = l = 0\}$, $Z_2 = \{mnl \mid m = n = 0, l \geq 1 \quad \text{or} \quad n = l = 0, m \geq 1 \quad \text{or} \quad m = l = 0, n \geq 1\}$, $Z_3 = \{mnl \mid m = 0, n \geq 1, l \geq 1 \quad \text{or} \quad n = 0, m \geq 1, l \geq 1 \quad \text{or} \quad l = 0, m \geq 1, n \geq 1\}$ and $Z_4 = \{mnl \mid m \geq 1, n \geq 1, l \geq 1\}$.

Using the second Green's identity in Ω^+ and noting that $\dfrac{\partial u^i}{\partial \nu} + \dfrac{\partial u^r}{\partial \nu} = 0$ on $x_3 = 1$, we obtain

$$u_\varepsilon(x) = \int_{\Gamma^+} g^{\mathrm{e}}(x,y)\frac{\partial u_\varepsilon}{\partial \nu}\mathrm{d}s_y + u^{\mathrm{i}}(x) + u^{\mathrm{r}}(x), \quad x \in \Omega^+. \tag{2.1}$$

By the continuity of single layer potential [17], we have

$$u_\varepsilon(x) = \int_{\Gamma^+} -\left(\frac{1}{2\pi}\right)\frac{e^{ik|x-y|}}{|x-y|}\frac{\partial u_\varepsilon}{\partial \nu}\mathrm{d}s_y + u^{\mathrm{i}}(x) + u^{\mathrm{r}}(x), \quad x \in \Gamma^+. \tag{2.2}$$

Similarly,

$$u_\varepsilon(x) = \int_{\Gamma^-} -\left(\frac{1}{2\pi}\right)\frac{e^{ik|x-y|}}{|x-y|}\frac{\partial u_\varepsilon}{\partial \nu}\mathrm{d}s_y, \quad x \in \Gamma^-. \tag{2.3}$$

The solution inside the hole can be expressed as

$$u_\varepsilon(x) = -\int_{\Gamma^+\cup\Gamma^-} g_\varepsilon^{\mathrm{i}}(x,y)\frac{\partial u_\varepsilon}{\partial\nu}\mathrm{d}s_y, \quad x \in C_\varepsilon.$$

Taking the limit when x approaches the hole apertures Γ^+ and Γ^-, there holds

$$u_\varepsilon(x) = -\int_{\Gamma^+\cup\Gamma^-} g_\varepsilon^{\mathrm{i}}(x,y)\frac{\partial u_\varepsilon}{\partial\nu}\mathrm{d}s_y, \quad x \in \Gamma^+\cup\Gamma^-. \tag{2.4}$$

By imposing continuity of the solution along the hole apertures, we obtain the boundary integral equations as follows

$$\begin{cases} \displaystyle\int_{\Gamma^+} -\left(\frac{1}{2\pi}\right)\frac{e^{ik|x-y|}}{|x-y|}\frac{\partial u_\varepsilon}{\partial\nu}\mathrm{d}s_y + \int_{\Gamma^+\cup\Gamma^-} g_\varepsilon^{i}(x,y)\frac{\partial u_\varepsilon}{\partial\nu}\mathrm{d}s_y + u^{\mathrm{i}}(x) + u^{\mathrm{r}}(x) = 0 \ \text{ on } \ \Gamma^+, \\ \displaystyle\int_{\Gamma^-} -\left(\frac{1}{2\pi}\right)\frac{e^{ik|x-y|}}{|x-y|}\frac{\partial u_\varepsilon}{\partial\nu}\mathrm{d}s_y + \int_{\Gamma^+\cup\Gamma^-} g_\varepsilon^{\mathrm{i}}(x,y)\frac{\partial u_\varepsilon}{\partial\nu}\mathrm{d}s_y = 0 \ \text{ on } \ \Gamma^-. \end{cases} \tag{2.5}$$

It is clear that $\frac{\partial u^\varepsilon}{\partial\nu}|_{\Gamma^+} = \frac{\partial u_\varepsilon}{\partial y_3}(y_1,y_2,1)$, $\frac{\partial u^\varepsilon}{\partial\nu}|_{\Gamma^-} = -\frac{\partial u_\varepsilon}{\partial y_3}(y_1,y_2,0)$, $(u^i+u^r)|_{\Gamma^+} = 2e^{ik(d_1x_1+d_2x_2)}$.

We rescale the functions by introducing $X_1 = \frac{x_1}{\varepsilon}$, $X_2 = \frac{x_2}{\varepsilon}$ and $Y_1 = \frac{y_1}{\varepsilon}$, $Y_2 = \frac{y_2}{\varepsilon}$, and define the following quantities:

$$\begin{aligned} \varphi_1(Y) &:= -\frac{\partial u_\varepsilon}{\partial y_3}(\varepsilon Y, 1); \quad \varphi_2(Y) := \frac{\partial u_\varepsilon}{\partial y_3}(\varepsilon Y, 0); \\ f(X) &:= (u^i + u^r)(\varepsilon X, 1) = 2e^{ik\varepsilon X(d_1+d_2)}; \\ G^{\mathrm{e}}(X,Y) &= -\frac{1}{2\pi}\frac{e^{ik\varepsilon|X-Y|}}{\varepsilon|X-Y|}; \\ G_\varepsilon^{\mathrm{i}}(X,Y) &:= g_\varepsilon^{\mathrm{i}}(k;\varepsilon X_1,\varepsilon X_2,1;\varepsilon Y_1,\varepsilon Y_2,1) = g_\varepsilon^{\mathrm{i}}(k;\varepsilon X_1,\varepsilon X_2,0;\varepsilon Y_1,\varepsilon Y_2,0) \\ &= \sum_{m,n,l=0}^{\infty}\frac{c_{mnl}\alpha_{mnl}}{\varepsilon^2}\cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2); \\ \tilde{G}^{\mathrm{i}}_{\varepsilon(X,Y)} &:= g_\varepsilon^{\mathrm{i}}(k;\varepsilon X_1,\varepsilon X_2,1;\varepsilon Y_1,\varepsilon Y_2,0) = g_\varepsilon^{\mathrm{i}}(k;\varepsilon X_1,\varepsilon X_2,0;\varepsilon Y_1,\varepsilon Y_2,1) \\ &= \sum_{m,n,l=0}^{\infty}(-1)^l\frac{c_{mnl}\alpha_{mnl}}{\varepsilon^2}\cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2). \end{aligned}$$

Let $R_1 := (0,1)\times(0,1)$, $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$. For $X \in R_1$, we define the integral operators:

$$(Q^{\mathrm{e}}\varphi)(X) = \varepsilon\int_{R_1} G_\varepsilon^{e}(X,Y)\varphi(Y)dY; \tag{2.6}$$

$$(Q^{\mathrm{i}}\varphi)(X) = \varepsilon\int_{R_1} G_\varepsilon^{i}(X,Y)\varphi(Y)dY; \tag{2.7}$$

$$(\tilde{Q}^{\mathrm{i}}\varphi)(X) = \varepsilon\int_{R_1} \tilde{G}_\varepsilon^{i}(X,Y)\varphi(Y)dY. \tag{2.8}$$

By the change of variables, the following proposition follows.

Proposition 2.1 *The system of integral equations (2.5) is equivalent to the system $\mathbb{Q}\boldsymbol{\varphi} = \mathbf{f}$, in which*

$$\mathbb{Q} = \begin{bmatrix} Q^{\mathrm{e}} + Q^{\mathrm{i}} & \tilde{Q}^{\mathrm{i}} \\ \tilde{Q}^{\mathrm{i}} & Q^{\mathrm{e}} + Q^{\mathrm{i}} \end{bmatrix}, \quad \boldsymbol{\varphi} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \frac{f}{\varepsilon} \\ 0 \end{bmatrix}. \tag{2.9}$$

3 Asymptotic expansion of the integral operators

First we introduce several notations below:

$$\begin{aligned} \beta(k,\varepsilon) &= -\frac{i\varepsilon k}{2\pi} + \frac{\cot k}{\varepsilon k}, \\ \tilde{\beta}(k,\varepsilon) &= \frac{1}{\varepsilon k \sin k}, \\ K_1(X,Y) &= -\frac{1}{2\pi\varepsilon|X-Y|}, \\ K_2(X,Y) &= -\frac{1}{\varepsilon}\sum_{m\geq 0, n\geq 0} \frac{2^j}{\pi\sqrt{m^2+n^2}} \cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2), \\ &\quad\text{where } \; j=1 \; \text{ for } \; m=0 \; \text{ or } \; n=0 \; \text{ and } \; j=2 \; \text{ for } \; m,n\geq 1. \\ \kappa(X,Y) &= \varepsilon(K_1(X,Y)+K_2(X,Y)). \end{aligned} \tag{3.1}$$

The asymptotic expansions of the kernels G^e_ε, G^i_ε, $\tilde{G}^i_\varepsilon$ are presented in the following lemma.

Lemma 3.1 *If $|k\varepsilon| \ll 1$ and $X, Y \in R_1$, then*

$$\begin{aligned} G^e_\varepsilon(X,Y) &= -\frac{1}{2\varepsilon\pi|X-Y|} - \frac{ik}{2\pi} + \kappa_{1,\varepsilon}(X,Y), \\ G^i_\varepsilon(X,Y) &= \frac{\cot k}{k\varepsilon^2} - K_2(X,Y) + \kappa_{2,\varepsilon}(X,Y) \\ \tilde{G}^i_\varepsilon(X,Y) &= \frac{1}{\varepsilon^2}\frac{1}{k\sin k} + \tilde{\kappa}_\infty(X,Y), \end{aligned}$$

where $\kappa_{1,\varepsilon}(X,Y) \sim O(k^2\varepsilon)$, $\kappa_{2,\varepsilon}(X,Y) \sim O(k^2\varepsilon)$, and $\tilde{\kappa}_\infty(X,Y) \sim O(\exp(-1/\varepsilon))$.

Proof The asymptotic expansion of $G^{\mathrm{e}}_\varepsilon(X,Y)$ is straightforward from the Taylor expansion:

$$\begin{aligned} G^e_\varepsilon(X,Y) &= -\frac{e^{ik\varepsilon|X-Y|}}{2\pi\varepsilon|X-Y|} \\ &= -\frac{1}{2\varepsilon\pi|X-Y|}\left[1 + ik\varepsilon|X-Y| + \frac{1}{2}(k\varepsilon)^2(X-Y)^2 + O(k\varepsilon)^3\right] \\ &= -\frac{1}{2\varepsilon\pi|X-Y|} - \frac{ik}{2\pi} + O(k^2\varepsilon). \end{aligned}$$

Recall that

$$G^i_\varepsilon(X,Y) = \frac{1}{\varepsilon^2}\sum_{m,n=0}^{\infty}\left(\sum_{l=0}^{\infty} c_{mnl}\alpha_{mnl}\right)\cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2) \tag{3.2}$$

Let $C_{mn} = \sum_{l=0}^{\infty} c_{mnl}\alpha_{mnl}$, then using the formulas in [16], we have

$$
\begin{aligned}
C_{00}(k) &= \sum_{l=1}^{\infty} \frac{2}{k^2-(l\pi)^2} + \frac{1}{k^2} = \frac{\cot k}{k}.\\
C_{m0}(k,\varepsilon) &= \sum_{l=1}^{\infty} \frac{4}{k^2-(m\pi/\varepsilon)^2-(l\pi)^2} + \frac{2}{k^2-(m\pi/\varepsilon)^2}\\
&= -\frac{2}{\sqrt{(m\pi/\varepsilon)^2-k^2}} \coth\left(\sqrt{(m\pi/\varepsilon)^2-k^2}\right)\\
&= -\frac{2\varepsilon}{m\pi} - \frac{k^2\varepsilon^3}{m^3\pi^3} + O\left(\frac{k^4\varepsilon^5}{m^5}\right), \quad m \ge 1.\\
C_{n0}(k,\varepsilon) &= \sum_{l=1}^{\infty} \frac{4}{k^2-(n\pi/\varepsilon)^2-(l\pi)^2} + \frac{2}{k^2-(n\pi/\varepsilon)^2}\\
&= -\frac{2}{\sqrt{(n\pi/\varepsilon)^2-k^2}} \coth\left(\sqrt{(n\pi/\varepsilon)^2-k^2}\right)\\
&= -\frac{2\varepsilon}{n\pi} - \frac{k^2\varepsilon^3}{n^3\pi^3} + O\left(\frac{k^4\varepsilon^5}{n^5}\right), \quad n \ge 1.
\end{aligned}
$$

For $m \ge 1, n \ge 1$,

$$
\begin{aligned}
C_{mn}(k,\varepsilon) &= \sum_{l=1}^{\infty} \frac{8}{k^2-(m\pi/\varepsilon)^2-(n\pi/\varepsilon)^2-(l\pi)^2} + \frac{4}{k^2-(m\pi/\varepsilon)^2-(n\pi/\varepsilon)^2}\\
&= -4\left(\sum_{l=1}^{\infty} \frac{2}{(m\pi/\varepsilon)^2+(n\pi/\varepsilon)^2+(l\pi)^2-k^2} + \frac{1}{(m\pi/\varepsilon)^2+(n\pi/\varepsilon)^2-k^2}\right)\\
&= \frac{-4}{\sqrt{(m^2+n^2)\pi^2/\varepsilon^2-k^2}} \coth\left(\sqrt{(m^2+n^2)\pi^2/\varepsilon^2-k^2}\right)\\
&= \frac{-4\varepsilon}{\pi\sqrt{m^2+n^2}}\left(1+\frac{-k^2\varepsilon^2}{\pi\sqrt{m^2+n^2}}\right)^{-\frac{1}{2}} = \frac{-4\varepsilon}{\pi\sqrt{m^2+n^2}} + O(k^2\varepsilon^3).
\end{aligned}
$$

Substituting these into (3.2),we obtain

$$
G_\varepsilon^i(X,Y) = \frac{1}{\varepsilon^2}\frac{\cot k}{k} + K_2(X,Y) + O(k^2\varepsilon).
$$

Similarly,

$$
\tilde{G}_\varepsilon^i(X,Y) = \frac{1}{\varepsilon^2}\sum_{m,n=0}^{\infty}\left(\sum_{l=0}^{\infty}(-1)^l c_{mnl}\alpha_{mnl}\right)
$$
$$
\cos(m\pi X_1)\cos(n\pi X_2)\cos(m\pi Y_1)\cos(n\pi Y_2). \tag{3.3}
$$

Let $\tilde{C}_{mn} = \sum_{l=0}^{\infty}(-1)^l c_{mnl}\alpha_{mnl}$, then

$$
\tilde{C}_{00} = \sum_{l=0}^{\infty} \frac{2(-1)^l}{k^2-(l\pi)^2} + \frac{1}{k^2} = \frac{1}{\sin k}.
$$

$$\begin{aligned}
\tilde{C}_{m0} &= \sum_{l=1}^{\infty} \frac{4(-1)^l}{k^2 - (m\pi/\varepsilon)^2 - (l\pi)^2} + \frac{2}{k^2 - (m\pi/\varepsilon)^2} \\
&= -\frac{2}{\sqrt{(m\pi/\varepsilon)^2 - k^2}\sinh\left(\sqrt{(m\pi/\varepsilon)^2 - k^2}\right)} \\
&= O\left(\frac{\varepsilon}{m\pi}\exp(-\frac{m\pi}{\varepsilon})\right), \quad m \geq 1. \\
\tilde{C}_{0n} &= \sum_{l=1}^{\infty} \frac{4(-1)^l}{k^2 - (n\pi/\varepsilon)^2 - (l\pi)^2} + \frac{2}{k^2 - (n\pi/\varepsilon)^2} \\
&= -\frac{2}{\sqrt{(n\pi/\varepsilon)^2 - k^2}\sinh\left(\sqrt{(n\pi/\varepsilon)^2 - k^2}\right)} \\
&= O\left(\frac{\varepsilon}{n\pi}\exp(-n\pi/\varepsilon)\right), \quad n \geq 1.
\end{aligned}$$

For $m \geq 1, n \geq 1$,

$$\begin{aligned}
\tilde{C}_{mn} &= \sum_{l=1}^{\infty} \frac{8(-1)^l}{k^2 - (m\pi/\varepsilon)^2 - (n\pi/\varepsilon)^2 - (l\pi)^2} + \frac{4}{k^2 - (m\pi/\varepsilon)^2 - (n\pi/\varepsilon)^2} \\
&= -\frac{4}{\sqrt{(m^2+n^2)\pi^2/\varepsilon^2 - k^2}\sinh\left(\sqrt{(m^2+n^2)\pi^2/\varepsilon^2 - k^2}\right)}.
\end{aligned}$$

Substituting into (3.3), we obtain

$$\tilde{G}_\varepsilon^i(X, Y) = \frac{1}{(k\sin k)\varepsilon^2} + O\left(\exp(-1/\varepsilon)\right).$$

□

Define the function spaces

$$V_1 = \tilde{H}^{-\frac{1}{2}}(R_1) := \{u = U|_{R_1} \;\big|\; U \in H^{-1/2}(\mathbf{R}) \text{ and } supp\, U \subset \bar{R}_1\} \quad \text{and} \quad V_2 = H^{\frac{1}{2}}(R_1),$$

where $H^{\frac{1}{2}}(R_1)$ and $H^{-1/2}(\mathbf{R})$ are the standard Sobolev spaces [1]. We define a projection operator $P : V_1 \to V_2$ such that

$$P\varphi(X) = \langle\varphi, 1\rangle 1,$$

where 1 is a function defined on R_1 and is equal to one therein. We denote by K, K_∞, $\tilde{K}_\infty$ the integral operators corresponding to the kernels $\kappa(X, Y)$, $\kappa_\infty(X, Y)$ and $\varepsilon\tilde{\kappa}_\infty(X, Y)$, respectively,where $\kappa(X, Y)$ is defined in (3.1), $\tilde{\kappa}_\infty(X, Y)$ is defined in Lemma 3.1, and $\kappa_\infty(X, Y) = \varepsilon(\kappa_{1,\varepsilon}(X, Y) + \kappa_{2,\varepsilon}(X, Y))$.

Lemma 3.2 *The operators $Q^{\mathrm{e}} + Q^{\mathrm{i}}$ and $\tilde{Q}^{\mathrm{i}}$ admit the decompositions*

$$Q^{\mathrm{e}} + Q^{\mathrm{i}} = \beta P + K + K_\infty, \quad \text{and} \quad \tilde{Q}^{\mathrm{i}} = \tilde{\beta} P + \tilde{K}_\infty.$$

Moreover,the operator $K : H^{-\frac{1}{2}}(R_1) \to H^{\frac{1}{2}}(R_1)$ is invertible, K_∞ and $\tilde{K}_\infty$ are bounded from V_1 to V_2 with the operator norms $\|K_\infty\| \lesssim \varepsilon^2$ and $\|\tilde{K}_\infty\| \lesssim \exp(-1/\varepsilon)$ uniformly for bounded k's respectively.

Proof By using the definition of operators in (2.6)–(2.8), and the decomposition in Lemma 3.1, we have

$$
\begin{aligned}
(Q^{\mathrm{e}} + Q^{\mathrm{i}})\varphi(X) = \varepsilon \int \Big[& -\frac{ik}{2\pi} + K_1(X,Y) + \kappa_{1,\varepsilon}(X,Y) \\
& + \frac{\cot k}{\varepsilon^2 k} + K_2(X,Y) + \kappa_{2,\varepsilon}(X,Y) \Big] \varphi(Y) dY. \\
= \beta P\varphi + K\varphi + K_\infty \varphi.
\end{aligned}
$$

The decomposition for $\tilde{Q}^i$ follows by similar calculations. The proof of the invertibility of K is postponed to Section 6. □

4 Asymptotic expansion of resonances

Note that the scattering problem (1.1)–(1.5) and the system (2.9) are equivalent. Thus the resonances of the scattering problem, which are the set of complex-valued frequencies for the homogeneous problem with zero incident field, are the characteristic frequencies k such that $\mathbb{Q}(k)\,\boldsymbol{\varphi} = 0$ attains non-trivial solutions in $(V_1)^2$.

Lemma 4.1 *Let $Q_+ = Q^{\mathrm{e}} + Q^{\mathrm{i}} + \tilde{Q}^{\mathrm{i}}$ and $Q_- = Q^{\mathrm{e}} + Q^{\mathrm{i}} - \tilde{Q}^{\mathrm{i}}$, then*

$$
\sigma(Q) = \sigma\big(Q_+\big) \cup \sigma\big(Q_-\big),
$$

where $\sigma(Q)$, $\sigma(Q_+)$ and $\sigma(Q_-)$ denote the sets of characteristic frequencies k of Q, Q_+ and Q_-, respectively.

Proof Decomposing function space $(V_1)^2$ as $(V_1)^2 = V_{\mathrm{even}} \oplus V_{\mathrm{odd}}$, where $V_{\mathrm{even}} = \{[\varphi_+, \varphi_+]^T \,;\, \varphi_+ \in V_1\}$ and $V_{\mathrm{odd}} = \{[\varphi_-, -\varphi_-]^T \,;\, \varphi_- \in V_1\}$ are invariant subspaces for Q. Thus $\sigma(Q) = \sigma(Q|_{V_{\mathrm{even}}}) \cup \sigma(Q|_{V_{\mathrm{odd}}})$. By observing that

$$
\begin{bmatrix} Q^{\mathrm{e}} + Q^{\mathrm{i}} & \tilde{Q}^{\mathrm{i}} \\ \tilde{Q}^{\mathrm{i}} & Q^{\mathrm{e}} + Q^{\mathrm{i}} \end{bmatrix} \begin{bmatrix} \varphi_+ \\ \varphi_+ \end{bmatrix} = \begin{bmatrix} Q_+\varphi_+ \\ Q_+\varphi_+ \end{bmatrix},
$$

it follows that $\sigma(Q|_{V_{\mathrm{even}}}) = \sigma\big(Q_+\big)$, and similarly $\sigma(Q|_{V_{\mathrm{odd}}}) = \sigma\big(Q_-\big)$. □

By the virtue of Lemma 3.2, we have

$$
\begin{aligned}
Q_\pm &= Q^{\mathrm{e}} + Q^{\mathrm{i}} \pm \tilde{Q}^{\mathrm{i}} \\
&= (\beta \pm \tilde{\beta})P + K + K_\infty \pm \tilde{K}_\infty =: P_\pm + L_\pm,
\end{aligned}
$$

where $P_\pm = (\beta \pm \tilde{\beta})P$ and $L_\pm = K + K_\infty \pm \tilde{K}_\infty$. Furthermore, the following lemma holds.

Lemma 4.2 *$L_\pm$ is invertible for sufficiently small ε, and there holds*

$$
\begin{aligned}
L_\pm &= K + K_\infty \pm \tilde{K}_\infty, \\
L_\pm^{-1} 1 &= K^{-1} 1 + O(\varepsilon^2), \\
\langle L_\pm^{-1} 1, 1 \rangle &= \gamma + O(\varepsilon^2), \qquad (4.1)
\end{aligned}
$$

where $\gamma := \langle K^{-1} 1, 1 \rangle_{L^2(R_1)}$.

We first solve

$$Q_+\varphi = (P_+ + L_+)\varphi = 0,$$

which is equivalent to

$$L_+^{-1} P_+\varphi + \varphi = 0. \tag{4.2}$$

Note that

$$\begin{aligned} P_+\varphi &= \left(\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right) P\varphi \\ &= \left(\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right) \langle \varphi, 1\rangle 1, \end{aligned}$$

it follows that

$$L_+^{-1} P_+\varphi = \left(\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right) L_+^{-1} 1 \langle \varphi, 1\rangle.$$

Substituting it into (4.2) and taking inner product with the constant function 1 yields

$$\left(\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right) \langle L_+^{-1} 1, 1\rangle \langle \varphi, 1\rangle + \langle \varphi, 1\rangle = 0.$$

We obtain the corresponding resonance condition

$$\theta_+(k, \varepsilon) := 1 + \left(\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right) \langle L_+^{-1} 1, 1\rangle = 0.$$

Similar calculations for the equation

$$Q_-\varphi(X) = (P_- + L_-)\varphi(X) = 0$$

yields the second resonance condition

$$\theta_-(k, \varepsilon) := 1 + \left(-\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right) \langle L_-^{-1} 1, 1\rangle = 0.$$

Lemma 4.3 *The resonances of the scattering problem are the roots of the functions* $\theta_\pm(k, \varepsilon) = 0$.

Theorem 4.4 *The scattering resonances of (1.1) attain the following resonance expansions:*

$$k_n = n\pi + \frac{2n\pi}{\gamma}\varepsilon - in^2\pi\varepsilon^2 + O(\varepsilon^3), \quad n = 1, 2, 3, \ldots \quad \text{and} \quad n\varepsilon \ll 1, \tag{4.3}$$

where $\gamma = \langle K^{-1} 1, 1\rangle$ *is defined in Lemma 4.2.*

Proof We solve for the roots of

$$\theta_+(k, \varepsilon) = 1 + \left(\frac{1}{\varepsilon}\left(\frac{1}{k \sin k} + \frac{\cot k}{k}\right) - \frac{ik\varepsilon}{2\pi}\right) \langle L_+^{-1} 1, 1\rangle = 0,$$

or equivalently

$$p_+(k, \varepsilon) = \varepsilon\theta_+ = \varepsilon + \left[\left(\frac{1}{k \sin k} + \frac{\cot k}{k}\right) - \frac{ik\varepsilon^2}{2\pi}\right] (\gamma + r(k, \varepsilon)) = 0, \quad r(k, \varepsilon) \sim O(k^2\varepsilon^2).$$

Let

$$b(k) = \frac{1}{k \sin k} + \frac{\cot k}{k}.$$

The leading order term $b(k)$ of $\theta_+(k, \varepsilon)$ attains roots $k_0 = n\pi$ for odd integers n. To this end, we consider the domain for some fixed number $C \geq 0$

$$W_{\delta_o,\theta_o,C} = \{z \mid |z| \geq \delta\} \cup \{z : \; |z| \; \leq C, -(\pi - \theta_o) \leq argz \leq (\pi - \theta_o)\}, \quad \delta > 0.$$

To derive the leading order asymptotic term of k_n, define

$$p_{+,1}(k, \varepsilon) = \varepsilon + \left(b(k) - \frac{i\varepsilon^2 k}{2\pi}\right)\gamma. \tag{4.4}$$

Then

$$p_{+,1}(k, \varepsilon) = \varepsilon + \left[b^{'}(k_0)(k - k_0) + O(k - k_0)^2 - \frac{i\varepsilon^2}{2\pi}k_0\right]\gamma. \tag{4.5}$$

By a straightforward calculation, it is seen that $b^{'}(k_0) = -\frac{1}{2n\pi}$. We see that $p_{+,1}$ has simple roots in $W_{\delta_o,\theta_o,C}$ which are close to k_0's. By expanding the roots $k_{n,1}$ of $p_{+,1}$ in terms of ε, we obtain

$$k_{n,1} = n\pi + \frac{2n\pi}{\gamma}\varepsilon - in^2\pi\varepsilon^2 + O(\varepsilon^3).$$

To prove that $k_{n,1}$ is the leading order term of the asymptotic expansion of k_n, note that

$$p_+(k, \varepsilon) - p_{+,1}(k, \varepsilon) = (p_{+,1}(k, \varepsilon) - \varepsilon^2) \cdot O(\varepsilon).$$

One can find a constant $M > 0$ such that

$$|p_+(k, \varepsilon) - p_{+,1}(k, \varepsilon)| < |p_{+,1}(k, \varepsilon)|$$

for all k such that $|k - k_{n,1,1}| = M\varepsilon^3$. Hence we obtain the expansion (4.3) for odd integers n by the Rouches's theorem. By similar calculations for

$$\theta_-(k, \varepsilon) = 1 + \left(-\frac{1}{\varepsilon k \sin k} + \frac{\cot k}{\varepsilon k} - \frac{ik\varepsilon}{2\pi}\right)\langle L_-^{-1}1, 1\rangle = 0.$$

we obtain (4.3) for even n. □

5 Quantitative analysis of the field enhancement at the resonant frequencies

5.1 Solution of the system (2.9)

Decompose the system $\mathbb{Q}\boldsymbol{\varphi} = \mathbf{f}$ as

$$\mathbb{Q}\boldsymbol{\varphi}_{\text{even}} = \mathbf{f}_{\text{even}} \quad \text{and} \quad \mathbb{Q}\boldsymbol{\varphi}_{\text{odd}} = \mathbf{f}_{\text{odd}},$$

where $\boldsymbol{\varphi} = \boldsymbol{\varphi}_{\text{even}} + \boldsymbol{\varphi}_{\text{odd}}$ and $\mathbf{f} = \mathbf{f}_{\text{even}} + \mathbf{f}_{\text{odd}}$, with

$$\boldsymbol{\varphi}_{\text{even}} = \begin{bmatrix} \varphi_+ \\ \varphi_+ \end{bmatrix}, \quad \boldsymbol{\varphi}_{\text{odd}} = \begin{bmatrix} \varphi_- \\ -\varphi_- \end{bmatrix} \quad \text{and} \quad \mathbf{f}_{\text{even}} = \begin{bmatrix} \frac{f}{2\varepsilon} \\ \frac{f}{2\varepsilon} \end{bmatrix}, \quad \mathbf{f}_{\text{odd}} = \begin{bmatrix} \frac{f}{2\varepsilon} \\ -\frac{f}{2\varepsilon} \end{bmatrix}.$$

$\mathbb{Q}\boldsymbol{\varphi}_{\text{even}} = \mathbf{f}_{\text{even}}$ implies that

$$\begin{bmatrix} Q^{\mathrm{e}} + Q^{\mathrm{i}} & \tilde{Q}^{\mathrm{i}} \\ \tilde{Q}^{\mathrm{i}} & Q^{\mathrm{e}} + Q^{\mathrm{i}} \end{bmatrix} \begin{bmatrix} \varphi_+ \\ \varphi_+ \end{bmatrix} = \frac{1}{\varepsilon} \begin{bmatrix} \frac{f}{2} \\ \frac{f}{2} \end{bmatrix},$$

which is equivalent to solving

$$Q_+\varphi_+ = \varepsilon^{-1}\frac{f}{2}, \quad \text{where} \quad Q_+ = Q^{\mathrm{e}} + Q^{\mathrm{i}} + \tilde{Q}^{\mathrm{i}}.$$

$\mathbb{Q}\boldsymbol{\varphi}_{\text{odd}} = \mathbf{f}_{\text{odd}}$ implies that

$$\begin{bmatrix} Q^{\mathrm{e}} + Q^{\mathrm{i}} & \tilde{Q}^{\mathrm{i}} \\ \tilde{Q}^{\mathrm{i}} & Q^{\mathrm{e}} + Q^{\mathrm{i}} \end{bmatrix} \begin{bmatrix} \varphi_- \\ -\varphi_- \end{bmatrix} = \frac{1}{\varepsilon} \begin{bmatrix} \frac{f}{2} \\ -\frac{f}{2} \end{bmatrix},$$

which is equivalent to solving $Q_-\varphi_- = \varepsilon^{-1}\frac{f}{2}$, where $Q_- = Q^{\mathrm{e}} + Q^{\mathrm{i}} - \tilde{Q}^{\mathrm{i}}$. Recall that

$$p_\pm(k,\varepsilon) = \varepsilon\theta_+ = \varepsilon + \left[\left(\pm\frac{1}{k\sin k} + \frac{\cot k}{k}\right) - \frac{ik\varepsilon^2}{2\pi}\right](\gamma + r(k,\varepsilon)), \quad r(k,\varepsilon) \sim O(k^2\varepsilon^2).$$

Lemma 5.1 *The following asymptotic expansion holds for the solutions φ_+ and φ_- in V_1:*

$$\varphi_\pm = K^{-1}1\left(\frac{(d_1+d_2)}{2}O(k)\right) + \frac{1}{p_\pm}\left[K^{-1}1 + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right] + O(k^2\varepsilon),$$

where $d = (d_1, d_2, -d_3)$ is the incident direction. In addition,

$$\langle\varphi_\pm, 1\rangle = \frac{1}{p_\pm}\left(\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right). \tag{5.1}$$

Proof Let us consider $Q_+\varphi_+ = \varepsilon^{-1}\frac{f}{2}$. By the same calculations in Section 4, we have

$$L_+^{-1}P_+\varphi_+ + \varphi_+ = L_+^{-1}\frac{f}{2\varepsilon}, \tag{5.2}$$

or equivalently,

$$\tilde{b}(k)L_+^{-1}1\langle\varphi_+, 1\rangle + \varphi_+ = L_+^{-1}\frac{f}{2\varepsilon}, \tag{5.3}$$

where $\tilde{b}(k) = \frac{1}{\varepsilon}\left(\frac{1}{k\sin k} + \frac{\cot k}{k}\right) - \frac{ik\varepsilon}{2\pi}$. Thus by taking inner product with 1 on both sides of (5.3), we get

$$\langle\varphi_+, 1\rangle + \tilde{b}(k)\langle L_+^{-1}1, 1\rangle\langle\varphi_+, 1\rangle = \langle L_+^{-1}\frac{f}{2\varepsilon}, 1\rangle,$$

and it follows that

$$\langle\varphi_+, 1\rangle = \theta_+^{-1}\langle L_+^{-1}\frac{f}{2\varepsilon}, 1\rangle. \tag{5.4}$$

Note that

$$L_+^{-1}\frac{f}{2\varepsilon} = \frac{1}{\varepsilon}\left(1 + \frac{(d_1+d_2)}{2}O(k\varepsilon)\right)\left(K^{-1}1 + O(k\varepsilon)^2\right).$$

Substituting it into (5.4) yields

$$\langle \varphi_+, 1\rangle = \frac{1}{p_+}\left(\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right).$$

By substituting the above into (5.3), we have

$$\varphi_+ = \varepsilon^{-1}L_+^{-1}\frac{f}{2} + \frac{1-\theta_+}{\langle L_+^{-1}1, 1\rangle\theta_+}\langle L_+^{-1}\frac{f}{2\varepsilon}, 1\rangle \cdot L_+^{-1}1, \tag{5.5}$$

where

$$\begin{aligned} L_+^{-1} &= K^{-1}1 + O(k\varepsilon)^2, \\ \frac{f}{2\varepsilon} &= \frac{1}{\varepsilon}\cdot 1 + \frac{(d_1+d_2)}{2}O(k) \quad \text{in} \quad V_2 \times V_2. \end{aligned}$$

Therefore,

$$\begin{aligned} \varepsilon\varphi_+ = &\left(1 + \frac{(d_1+d_2)}{2}\cdot O(k\varepsilon)\right)K^{-1}1 + O(k^2\varepsilon^2) \\ &+\frac{1-\theta_+}{(\gamma + O(k\varepsilon)^2)\theta_+}\left[\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right][K^{-1}1 + O(k\varepsilon)^2] \\ = &\frac{(d_1+d_2)}{2}O(k\varepsilon)K^{-1}1 + \frac{1}{\theta_+}\left[K^{-1}1 + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right] + O(k\varepsilon)^2. \\ \varphi_+ = &K^{-1}1\left(\frac{(d_1+d_2)}{2}O(k)\right) + \frac{1}{p_+}\left[K^{-1}1 + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right] + O(k^2\varepsilon). \end{aligned}$$

Similarly,

$$\varphi_- = K^{-1}1\left(\frac{(d_1+d_2)}{2}O(k)\right) + \frac{1}{p_-}\left[K^{-1}1 + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right] + O(k^2\varepsilon).$$

□

Corollary 5.2 *Let* $\boldsymbol{\varphi} = [\varphi_1, \varphi_2]^T$ *be the solution of the system* $\mathbb{Q}\boldsymbol{\varphi} = \mathbf{f}$, *then* $\boldsymbol{\varphi} = [\varphi_+ + \varphi_-,\ \varphi_+ - \varphi_-]^T$, *where* $\varphi_\pm$ *are defined in Lemma 5.1. Furthermore,*

$$\begin{aligned} \langle \varphi_1, 1\rangle = \langle \varphi_+ + \varphi_-, 1\rangle &= \left(\frac{1}{p_+} + \frac{1}{p_-}\right)\left[\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right], \\ \langle \varphi_2, 1\rangle = \langle \varphi_+ - \varphi_-, 1\rangle &= \left(\frac{1}{p_+} - \frac{1}{p_-}\right)\left[\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right]. \end{aligned}$$

Lemma 5.3 *If* $n\varepsilon \ll 1$, *then*

$$p_+(k,\varepsilon) = -\frac{in\gamma}{2}\varepsilon^2 + O(\varepsilon^3), \quad p_-(k,\varepsilon) = \left(\frac{\cos k - 1}{k\sin k}\right)\gamma + O(\varepsilon)$$

and

$$p_+(k,\varepsilon) = \left(\frac{\cos k + 1}{k\sin k}\right)\gamma + O(\varepsilon), \quad p_-(k,\varepsilon) = -\frac{in\gamma}{2}\varepsilon^2 + O(\varepsilon^3),$$

at the odd and even resonant frequencies $k = Rek_n$, *respectively.*

Proof Let us consider $p_+(k,\varepsilon)$. First assume that $|k - \text{Re}k_n| \le \varepsilon$ with odd integers n. From the definition of $p_{+,1}$ and its expansion, it follows that

$$\begin{aligned} p_+(k,\varepsilon) &= p_{+,1}(k,\varepsilon) + O(\varepsilon^3) \\ &= p_1'(k_n)(k-k_n) + O(k-k_n)^2 + O(\varepsilon^3) \\ &= \gamma b'(k_0)\cdot(k-k_n) + O(\varepsilon^3) \\ &= \frac{\gamma}{2n\pi}(k - \text{Re}k_n - i\text{Im}k_n) + O(\varepsilon^3). \end{aligned}$$

Since

$$\text{Im}k_n = \text{Im}k_{n,1} + O(\varepsilon^3) = -n^2 i\pi\varepsilon^2 + O(\varepsilon^3),$$

we obtain

$$p_+(k,\varepsilon) = -\frac{in\varepsilon^2\gamma}{2} + O(\varepsilon^3).$$

To derive the expression for $p_-(k,\varepsilon)$ at $k = Rek_n$ for odd n, recall that

$$p_-(k,\varepsilon) = p_{-,1}(k,\varepsilon) + O(\varepsilon^2) = \varepsilon + \left(c(k) - \frac{ik\varepsilon^2}{2\pi}\right)\gamma + O(\varepsilon^2).$$

$c(k) = \frac{\cot k}{k} - \frac{1}{k\sin k}$ is well defined for $k = \text{Re}k_n$, hence

$$p_-(k,\varepsilon) = \left(\frac{\cos k - 1}{k\sin k}\right)\gamma + O(\varepsilon).$$

The calculations for $p_\pm(k,\varepsilon)$ at the even resonant frequencies follow similarly. □

Proposition 5.4 *There hold $\varphi_1, \varphi_2 \sim O(1/\varepsilon^2)$ in V_1, and $\langle\varphi_1, 1\rangle, \langle\varphi_2, 1\rangle \sim O(1/\varepsilon^2)$ at the even and odd resonant frequencies $k = Rek_n$.*

5.2 Field enhancement in the hole

To investigate the field inside the hole, note that u_ε satisfies the following boundary value problem

$$\begin{cases} \Delta u_\varepsilon + k^2 u_\varepsilon = 0 \quad \text{in} \quad C_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial x_1} = 0, \quad \text{on} \quad x_1 = 0, x_1 = \varepsilon, \\ \frac{\partial u_\varepsilon}{\partial x_2} = 0, \quad \text{on} \quad x_2 = 0, x_2 = \varepsilon. \end{cases}$$

Then $u_\varepsilon(x)$ can be expanded as

$$\begin{aligned} u_\varepsilon(x) = a_{00}\cos(kx_3) + b_{00}\cos k(1-x_3) + \sum_{m,n\ge 0}\Big(a_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}\exp(-k_{mn}x_3) \\ + b_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}\exp(-k_{mn}(1-x_3))\Big), \end{aligned} \tag{5.6}$$

where $k_{mn} = \sqrt{(\frac{m\pi}{\varepsilon})^2 - (\frac{n\pi}{\varepsilon})^2 - k^2}$.

Lemma 5.5 *The following hold for the expansion coefficients in (5.6):*

$$a_{00} = \frac{1}{k\sin k}\left[\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right]\left(\frac{1}{p_+} + \frac{1}{p_-}\right),$$

$$
\begin{aligned}
b_{00} &= \frac{1}{k\sin k}\left[\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right]\left(\frac{1}{p_+} - \frac{1}{p_-}\right),\\
&\varepsilon\sqrt{m}|a_{m0}| \le C, \quad \varepsilon\sqrt{m}|b_{m0}| \le C \quad for \quad m \ge 1,\\
&\varepsilon\sqrt{n}|a_{0n}| \le C, \quad \varepsilon\sqrt{n}|b_{0n}| \le C \quad for \quad n \ge 1\\
&\sqrt{m+n}|a_{mn}| \le C, \quad \sqrt{m+n}|b_{mn}| \le C \quad for \quad m, n \ge 1,
\end{aligned}
$$

where C *is a positive constant independent of* ε, k, m *and* n.

Proof Taking the derivative of the expansion (5.6) with respect to x_3 and evaluating at $\Gamma_\pm$ gives

$$
\frac{\partial u_\varepsilon}{\partial x_3}(x_1, x_2, 1) = -a_{00}k\sin(k) + \sum_{m,n\ge 0}\left(-a_{mn}\exp(-k_{mn}) + b_{mn}\right)k_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}, \tag{5.7}
$$

$$
\frac{\partial u_\varepsilon}{\partial x_3}(x_1, x_2, 0) = b_{00}k\sin(k) + \sum_{m,n\ge 0}\left(-a_{mn} + b_{mn}\exp(-k_{mn})\right)k_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}. \tag{5.8}
$$

Integrating over the hole apertures and using Corollary 5.2 leads to

$$
\begin{aligned}
-a_{00}k\sin k &= \frac{1}{\varepsilon^2}\int_{\Gamma^+}\frac{\partial u_\varepsilon}{\partial x_3}(x_1, x_2, 1)\mathrm{d}x_1\mathrm{d}x_2 = -\int_{R_1}\varphi_1(X)\mathrm{d}x\\
&= -\left(\frac{1}{p_+} + \frac{1}{p_-}\right)\left(\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right),\\
b_{00}k\sin k &= \frac{1}{\varepsilon^2}\int_{\Gamma^-}\frac{\partial u_\varepsilon}{\partial x_3}(x_1, x_2, 0)\mathrm{d}x_1\mathrm{d}x_2 = \int_{R_1}\varphi_2(X)\mathrm{d}x\\
&= \left(\frac{1}{p_+} - \frac{1}{p_-}\right)\left(\gamma + \frac{(d_1+d_2)}{2}O(k\varepsilon) + O(k\varepsilon)^2\right).
\end{aligned}
$$

We obtain the desired formulas for a_{00} and b_{00}. For coefficients a_{m0}, b_{m0}, taking inner product of eqs (5.7) and (5.8) with $\cos\frac{m\pi x_1}{\varepsilon}$ and integrating over aperture yields

$$
\begin{aligned}
a_{m0}k_{m0} &= \frac{-2}{1-e^{-2k_{m0}}}\left(e^{-k_{m0}}\int_{R_1}\varphi_1(X)\cos(m\pi X)\mathrm{d}x + \int_{R_1}\varphi_2(X)\cos(m\pi X)\mathrm{d}x\right),\\
b_{m0}k_{m0} &= \frac{-2}{1-e^{-2k_{m0}}}\left(\int_{R_1}\varphi_1(X)\cos(m\pi X)\mathrm{d}x + e^{-k_{m0}}\int_{R_1}\varphi_2(X)\cos(m\pi X)\mathrm{d}x\right).
\end{aligned}
$$

Note that $k_{m0} = O(\frac{m}{\varepsilon})$ for $m \ge 1$, and

$$
\|\varphi_1\|_{V_1} \le \frac{1}{\varepsilon^2}, \quad \|\varphi_2\|_{V_1} \le \frac{1}{\varepsilon^2}, \quad \|\cos(m\pi X)\|_{V_2} \le \sqrt{m}.
$$

The estimate for a_{m0} and b_{m0} follows. A parallel calculation can also be applied for a_{0n} and b_{0n}.

For coefficients a_{mn}, b_{mn} taking inner product of (5.7) and (5.8) with $\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}$ and integrating over the aperture yields

$$
\begin{aligned}
a_{mn}k_{mn} = \frac{-4}{1-e^{-2k_{mn}}}\Big(&e^{-k_{mn}}\int_{R_1}\varphi_1(X)\cos(m\pi X_1)\cos(n\pi X_2)\mathrm{d}x_1\mathrm{d}x_2\\
&+ \int_{R_1}\varphi_2(X)\cos(m\pi X_1)\cos(n\pi X_2)\mathrm{d}x_1\mathrm{d}x_2\Big),
\end{aligned}
$$

$$
\begin{aligned}
b_{mn}k_{mn} = \frac{-4}{1-e^{-2k_{mn}}}\Big(\int_{R_1} \varphi_1(X)\cos(m\pi X_1)\cos(n\pi X_2)\mathrm{d}x_1 \\
+e^{-k_{mn}}\int_{R_1} \varphi_2(X)\cos(m\pi X_1)\cos(n\pi X_2)\mathrm{d}x_1\mathrm{d}x_2\Big).
\end{aligned}
$$

Since $k_{mn} = O(\sqrt{\frac{m^2+n^2}{\varepsilon^2}})$ and $\|\varphi_1\|_{V_1} \le \frac{1}{\varepsilon^2}$, $\|\varphi_2\|_{V_1} \le \frac{1}{\varepsilon^2}$, the desired formulas for a_{mn} and b_{mn} follow. □

Theorem 5.6 *The wave field in the hole $C_\varepsilon^{int} := \{x \in C_\varepsilon \mid x_3 \gg \varepsilon, 1 - x_3 \gg \varepsilon\}$ is given by*

$$
\begin{aligned}
u_\varepsilon(x) &= \left(\frac{1}{\varepsilon^2} + \frac{(d_1+d_2)}{2}O\left(\frac{1}{\varepsilon}\right) + O(1)\right)\frac{2i\cos(k(x_3-1/2))}{nk\sin(k/2)} \\
&\quad + \frac{\sin(k(x_3-1/2))}{\sin(k/2)} + O(\exp(-1/\varepsilon^2)), \\
u_\varepsilon(x) &= -\left(\frac{1}{\varepsilon^2} + \frac{(d_1+d_2)}{2}O\left(\frac{1}{\varepsilon}\right) + O(1)\right)\frac{2i\sin(k(x_3-1/2))}{nk\cos(k/2)} \\
&\quad + \frac{\cos(k(x_3-1/2))}{k\sin(k/2)} + O(\exp(-1/\varepsilon^2))
\end{aligned}
$$

at the odd and even resonant frequencies $k = Rek_n$ respectively.

Proof From Lemma 5.5,

$$
\begin{aligned}
u_\varepsilon(x) &= a_{00}\cos(kx_3) + b_{00}\cos k(1-x_3) + \sum_{m,n\ge 0}\Big(a_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}\exp(-k_{mn}x_3) \\
&\quad + b_{mn}\cos\frac{m\pi x_1}{\varepsilon}\cos\frac{n\pi x_2}{\varepsilon}\exp(-k_{mn}(1-x_3))\Big).
\end{aligned}
$$

For $\varepsilon \ll 1$,

$$
\begin{aligned}
u_\varepsilon(x) &= \left[\gamma + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2\right]\left[\left(\frac{1}{p_+} + \frac{1}{p_-}\right)\frac{\cos kx_3}{k\sin k} + \left(\frac{1}{p_+} - \frac{1}{p_-}\right)\frac{\cos k(1-x_3)}{k\sin k}\right] \\
&\quad + O(\exp(-1/\varepsilon^2)) \\
&= 2\left[\gamma + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2\right]\left[\frac{1}{p_+}\frac{\cos(k/2)\cos(k(x_3-1/2))}{k\sin k}\right. \\
&\quad \left. - \frac{1}{p_-}\frac{\sin(k/2)\sin(k(x_3-1/2))}{k\sin k}\right] \\
&\quad + O(\exp(-1/\varepsilon^2)).
\end{aligned}
$$

At the odd resonant frequencies $k = Rek_n$, it follows from Lemma 5.3 that $\frac{1}{p_+} = \frac{2i}{\gamma n\varepsilon^2}(1 + O(\varepsilon))$ and $\frac{1}{p_-} = \frac{k\sin k}{(\cos k - 1)\gamma}(1 + O(\varepsilon))$. Therefore,

$$
\begin{aligned}
u_\varepsilon(x) &= [1 + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2]\Big[\Big(\frac{1}{\varepsilon^2}\frac{2i\cos(k(x_3-1/2))}{nk\sin(k/2)}(1+O(\varepsilon)) \\
&\quad + \frac{\sin(k(x_3-1/2))}{k\sin(k/2)}(1+O(\varepsilon))\Big)\Big] + \exp(-1/\varepsilon^2) \\
&= \left(\frac{1}{\varepsilon^2} + \frac{(d_1+d_2)}{2}O(\frac{1}{\varepsilon}) + O(1)\right)\frac{2i\cos(k(x_3-1/2))}{nk\sin(k/2)} \\
&\quad + \frac{\sin(k(x_3-1/2))}{\sin(k/2)} + O(\exp(-1/\varepsilon^2)).
\end{aligned}
$$

Similarly, at the even resonant frequencies, $\frac{1}{p_+} = \frac{k\sin k}{(\cos k+1)\gamma}(1+O(\varepsilon))$ and $\frac{1}{p_-} = \frac{2i}{\gamma n\varepsilon^2}(1+O(\varepsilon))$. We obtain

$$u_\varepsilon(x) = -\left(\frac{1}{\varepsilon^2} + \frac{(d_1+d_2)}{2}O(\frac{1}{\varepsilon}) + O(1)\right)\frac{2i\sin(k(x_3-1/2))}{nk\cos(k/2)} + \frac{\cos(k(x_3-1/2))}{k\sin(k/2)} + O(\exp(-1/\varepsilon^2)).$$

□

5.3 Scattering enhancement in the far field

Consider the domain $\Omega^+ \setminus H_1^+$ above the hole, where $H_1^+ := \{x | x-(0,0,1) \leq 1\}$. Recall that

$$u_s^\varepsilon(x) = \int_{\Gamma^+} g^e(x,y)\frac{\partial u_\varepsilon}{\partial \nu}\mathrm{d}s_y, \quad x \in \Omega^+,$$

and

$$\frac{\partial u_\varepsilon}{\partial \nu}(x_1,x_2,1) = -\varphi_1(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}).$$

Therefore,

$$\begin{aligned} u_s^\varepsilon(x) &= -\int_{\Gamma^+} g^e(x,(y_1,y_2,1))\varphi_1(\frac{y_1}{\varepsilon},\frac{y_2}{\varepsilon})dy_1dy_2 \\ &= -\varepsilon^2\int_0^1\int_0^1 g^e(x,\varepsilon Y_1,\varepsilon Y_2,1)\varphi_1(Y_1,Y_2)dY_1dY_2. \end{aligned}$$

Note that

$$g^e(x,\varepsilon Y_1,\varepsilon Y_2,1) = g^e(x,(0,0,1))(1+O(\varepsilon)), \quad x \in \Omega^+ \setminus H_1^+.$$

and

$$\langle \varphi_1, 1\rangle_{L^2(R_1)} = \left(\frac{1}{p_+}+\frac{1}{p_-}\right)\left(\gamma + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2\right).$$

It follows that

$$u_s^\varepsilon(x) = -\varepsilon^2 g^e(x,(0,0,1))(1+O(\varepsilon))\left(\frac{1}{p_+}+\frac{1}{p_-}\right)\left(\gamma + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2\right).$$

From Lemma 5.3,

$$\frac{1}{p_+} = \frac{2i}{\gamma n\varepsilon^2}(1+O(\varepsilon)) \quad \text{and} \quad \frac{1}{p_-} = \frac{2i}{\gamma n\varepsilon^2}(1+O(\varepsilon))$$

when n is even and odd respectively. The corresponding scattered field is

$$u_s^\varepsilon(x) = -\frac{2i}{n}\cdot g^e(x,(0,0,1) + O(\varepsilon), \quad u_s^\varepsilon(x) = -\frac{2i}{n}\cdot g^e(x,(0,0,1) + O(\varepsilon).$$

Similarly, the scattered field in the domain $H_1^- := \{x | x-(0,0,0) \leq 1\}$ is

$$u_s^\varepsilon(x) = -\varepsilon^2 g^e(x,(0,0,0))(1+O(\varepsilon))\left(\frac{1}{p_+}-\frac{1}{p_-}\right)\left(\gamma + \frac{(d_1+d_2)}{2}O(\varepsilon) + O(\varepsilon)^2\right).$$

It follows that

$$u_s^\varepsilon(x) = -\frac{2i}{n} \cdot g^e(x, (0,0,0) + O(\varepsilon), \quad u_s^\varepsilon(x) = -\frac{2i}{n} \cdot g^e(x, (0,0,0) + O(\varepsilon),$$

at the odd and even resonant frequencies, respectively.

6 The invertibility of the operator *K*

Recall the integral operator

$$K\varphi(X) = \int_{R_1} \kappa(X,Y)\varphi(Y)dY, \quad \text{for} \quad X \in R_1 \quad \varphi \in H^{-\frac{1}{2}}(R_1). \tag{6.1}$$

Where $\kappa(X,Y)$ is given by (3.1).

We consider the integral equation $K\varphi = f$, where $f \in H^{\frac{1}{2}}(R_1)$. We extend the argument in [11] to show that K is invertible from $H^{-\frac{1}{2}}(R_1)$ to $H^{\frac{1}{2}}(R_1)$.

Let us consider the domain $\tilde{\Omega} = \tilde{\Omega}_e \cup \tilde{\Omega}_i$, where $\tilde{\Omega}_e = \mathbb{R}^3_+$, and $\tilde{\Omega}_i = (0,1)^2 \times \mathbb{R}_-$. Let $\tilde{\Omega}_\varepsilon = (0,1)^2 \times (0,\varepsilon)$ be the bounded domain inside the hole with the upper and lower boundary given by $\tilde{\Gamma} = (0,1)^2 \times \{0\}$ and $\tilde{\Gamma}_\varepsilon = (0,1)^2 \times \{-\varepsilon\}$, respectively. Let $u^\pm(X) = \lim_{t \to \pm 0} u(X + (t,0,0))$ for $x \in \tilde{\Gamma}$, and $u^\mp(X) = \lim_{t \to \pm 0} u(X + (t,0,0))$ on $X \in \tilde{\Gamma}_\varepsilon$. $[u]_{\tilde{\Gamma}}$ represents the jump $u^+(X) - u^-(X)$ for $X \in \tilde{\Gamma}$. The solution of the integral equation $K\varphi = f$ is related to the following transmission problem:

$$(A) \begin{cases} \Delta u(X) \ = \ 0, \quad \text{in } \tilde{\Omega}, \\ \frac{\partial u(X)}{\partial \nu} \ = \ 0, \quad \text{on } \partial\tilde{\Omega}, \quad \int_{\tilde{\Gamma}} u^-(X) \mathrm{d}s_X \ = \ 0, \quad [u]_{\tilde{\Gamma}} \ = \ f(X), \quad \left[\frac{\partial u(X)}{\partial X_3}\right]_{\tilde{\Gamma}} = 0, \\ u(X) - X_3 \int_{\tilde{\Gamma}} \frac{\partial u(X)}{\partial X_3} \mathrm{d}s_X \ = \ o(1), \quad X_3 \to -\infty \quad in \quad \tilde{\Omega}_i, \\ |\Delta(u(X) - X_3 \int_{\tilde{\Gamma}} \frac{\partial u(X)}{\partial X_3} \mathrm{d}s_X)| \ = \ o(1), \quad X_3 \to -\infty \quad in \quad \tilde{\Omega}_i, \\ u(X) - \frac{1}{\pi|X|} \int_{\tilde{\Gamma}} \frac{\partial u(X)}{\partial X_3} \mathrm{d}s_X \ = \ O\Big(\frac{1}{|X|^2}\Big), \quad |X| \to \infty \quad in \quad \tilde{\Omega}_e, \\ |\Delta u(X) \cdot \frac{X}{|X|} + \frac{1}{\pi|X|^2} \int_{\tilde{\Gamma}} \frac{\partial u(X)}{\partial X_3} \mathrm{d}s_X \ = \ O\Big(\frac{1}{|X|^3}\Big), \quad |X| \to \infty \quad in \quad \tilde{\Omega}_e. \end{cases}$$

(A) can be reformulated in the bounded domain $\tilde{\Omega}_\varepsilon$. To this end, we introduce the Green's function for the exterior domain $\tilde{\Omega}_e$:

$$\begin{cases} \Delta \tilde{G}_{\mathrm{e}}(X,Y) = \delta(X - Y), \quad \text{in} \quad \tilde{\Omega}_{\mathrm{e}}, \\ \frac{\partial \tilde{G}_e(X,Y)}{\partial \nu} \ = \ 0, \quad \text{on } \partial\tilde{\Omega}_e, \\ \tilde{G}_e(X,Y) + \frac{1}{\pi|X|} \ = \ O(\frac{1}{|X|^2}), \quad |X| \to \infty, \\ |\Delta_X \tilde{G}_{\mathrm{e}}(X,Y) \cdot \frac{X}{|X|} + \frac{1}{\pi|X|^2} \ = \ O\left(\frac{1}{|X|^3}\right), \quad |X| \to \infty. \end{cases}$$

The method of images shows that $\tilde{G}_e(X,Y) = -\frac{1}{4\pi}\frac{1}{|X-Y|} - \frac{1}{4\pi}\frac{1}{|X-Y'|}$, where $Y' = (Y_1, Y_2, -Y_3)$. The Green's function in $\tilde{\Omega}_i$ satisfies

$$\begin{cases} \Delta \tilde{G}_i(X,Y) = \delta(X-Y) \quad \text{in } \tilde{\Omega}_i, \\ \frac{\partial \tilde{G}_i(X,Y)}{\partial \nu} = 0, \quad \text{on } \partial\tilde{\Omega}_i, \\ \int_{\tilde{\Gamma}} \tilde{G}_i(X,Y)\mathrm{d}s_X = 0, \\ \tilde{G}_i(X,Y) = o(1), \quad |\Delta_X \tilde{G}_i(X,Y)| = o(1) \quad \text{as } X_3 \to \infty. \end{cases}$$

It can be shown that

$$\begin{aligned} \tilde{G}_{\mathrm{i}}(X,Y) = &- \sum_{m\geq 0, n\geq 0} \frac{2^j}{\pi\sqrt{m^2+n^2}}\left(e^{-\frac{\pi}{2}\sqrt{m^2+n^2}|X_3+Y_3|} + e^{-\frac{\pi}{2}\sqrt{m^2+n^2}|X_3-Y_3|}\right) \\ &. \cos(m\pi X_1)\cos(m\pi Y_1)\cos(n\pi X_2)\cos(n\pi Y_2), \\ &\text{where} \quad j=0 \quad \text{for} \quad m=0 \quad \text{or} \quad n=0 \quad \text{and} \quad j=1 \quad \text{for} \quad m,n\geq 1. \end{aligned}$$

We define two integral operators $\Theta : H^{-\frac{1}{2}}(\tilde{\Gamma}) \to H^{\frac{1}{2}}(\tilde{\Gamma})$ and $\Theta_\varepsilon : H^{-\frac{1}{2}}(\tilde{\Gamma}_\varepsilon) \to H^{\frac{1}{2}}(\tilde{\Gamma}_\varepsilon)$:

$$\begin{aligned} \Theta\varphi(X) &= \int_{\tilde{\Gamma}} \tilde{G}_e(X,Y)\varphi(Y)\mathrm{d}s_Y, \\ \Theta_\varepsilon\varphi(X) &= \int_{\tilde{\Gamma}_\varepsilon} \tilde{G}_i(X+(0,\varepsilon,0), Y+(0,\varepsilon,0))\varphi(Y)\mathrm{d}s_Y. \end{aligned}$$

Here Θ_ε does not depend on ε and $\Theta_\varepsilon 1(X) = \int_{\tilde{\Gamma}} \tilde{G}_i(X,Y)\mathrm{d}s_Y$. The bounded value problem in $\tilde{\Omega}_\varepsilon$ is formulated as follows:

$$(B) \begin{cases} \Delta u(X) = 0, \quad \text{in } \tilde{\Omega}_\varepsilon, \\ \int_{\tilde{\Gamma}} u(X)\mathrm{d}s_X = 0, \\ \frac{\partial u(X)}{\partial \nu} = 0, \quad \text{on} \quad X_1 = \{0,1\}, X_2 = \{0,1\}, \\ \Theta_\varepsilon\left(\frac{\partial u(X)}{\partial X_3}\right) + \varepsilon \int_{\tilde{\Gamma}} \frac{\partial u(X)}{\partial X_3}\mathrm{d}s_X = u(X), \quad \text{on} \quad \tilde{\Gamma}_\varepsilon, \\ -\Theta\left(\frac{\partial u(X)}{\partial X_3}\right) = u(X) + f(X), \quad \text{on } \tilde{\Gamma}. \end{cases}$$

As shown below, (A) and (B) are equivalent.

6.1 Equivalence of the well-posedness of (*A*) and the invertibility of *K*

Lemma 6.1 *The following two statements are equivalent:*

(1) K is invertible from $H^{-\frac{1}{2}}(\tilde{\Gamma}) \to H^{\frac{1}{2}}(\tilde{\Gamma})$,
(2) For any function $f \in H^{\frac{1}{2}}(\tilde{\Gamma})$, there exists a unique solution to (A).

Proof If (1) holds, given $f \in H^{\frac{1}{2}}(\tilde{\Gamma})$, let $\varphi_f(X) \in H^{-\frac{1}{2}}(\tilde{\Gamma})$ be a unique solution to $K\varphi_f(X) = f(x)$. Define u_f in $\tilde{\Omega}$ by

$$u_f(X) = \begin{cases} -\int_{\tilde{\Gamma}} \tilde{G}_e(X,Y)\varphi_f(Y)\mathrm{d}s_Y, & X \in \tilde{\Omega}_e, \\ X_3 \int_{\tilde{\Gamma}} \varphi_f(X)\mathrm{d}s_X + \int_{\tilde{\Gamma}} \tilde{G}_i(X,Y)\varphi_f(Y)\mathrm{d}s_Y, & X \in \tilde{\Omega}_i. \end{cases} \tag{6.2}$$

The function $u_f(X)$ is the solution to (A). To prove the uniqueness of the solution, let w_f be a solution to (A), with $[w_f] = f$ on $\tilde{\Gamma}$. Applying the Green's formula in $\tilde{\Omega}$, we have

$$w_f(X) = -\int_{\tilde{\Gamma}} \tilde{G}_e(X,Y)\frac{\partial w_f(Y)}{\partial Y_3}\mathrm{d}s_Y, \quad X \in \tilde{\Omega}_e. \tag{6.3}$$

$$w_f(X) = \int_{\tilde{\Gamma}} \tilde{G}_i(X,Y)\frac{\partial w_f(Y)}{\partial Y_3}\mathrm{d}s_Y + X_3\int_{\tilde{\Gamma}} \frac{\partial w_f(X)}{\partial X_3}\mathrm{d}s_X \quad X \in \tilde{\Omega}_i. \tag{6.4}$$

Taking trace of $w_f(X)$ on both sides of the boundary $\tilde{\Gamma}$ for $X = (X_1, X_2, 0) \in \tilde{\Gamma}$, we obtain

$$\begin{aligned} f(X) = &\int_0^1\int_0^1 \left(\tilde{G}_i(X_1,X_2,0;Y_1,Y_2,0) + \tilde{G}_e(X_1,X_2,0;Y_1,Y_2,0)\right)\frac{\partial w_f(Y_1,Y_2,0)}{\partial Y_3}dY_1dY_2 \\ =&K\left[\frac{\partial w_f}{\partial X_3}\right]. \end{aligned}$$

We infer from (1) that $\frac{\partial w_f}{\partial X_3} = \varphi_f(Y)$. By (6.3) and (6.4), it follows that $u_f = w_f$ in $\tilde{\Omega}$, which proves the uniqueness of the solution to (A).

Assume that (2) holds. Then from the above, we see that the solution w_f to (A) satisfies (6.3) and (6.4), and consequently

$$K\varphi = f \tag{6.5}$$

has at least one solution $\varphi(Y) = \frac{\partial w_f}{\partial X_3} \in H^{-\frac{1}{2}}(\tilde{\Gamma})$. For $f = 0$, let φ_0 be the corresponding solution of (6.5) and construct a solution u_0 to (A) by (6.2). Hence by (2), $u_0 \equiv 0$ implies $\varphi_0 = \frac{\partial u_0}{\partial \nu} = 0$. Thus the solution to (6.5) is unique.

6.2 The equivalence of (*A*) and (*B*)

Lemma 6.2 (A) *attains unique solution iff* (B) *has a unique solution.*

Proof Let u_f be the solution to (A). Applying the Green's formula in $\hat{\Omega} = (0,1)^2 \times (-\varepsilon, -\infty)$, we obtain

$$u_f(X) = \int_{\tilde{\Gamma}_\varepsilon} \tilde{G}_i(X + (0,\varepsilon,0), Y + (0,\varepsilon,0))\frac{\partial u_f(Y)}{\partial Y_3}\mathrm{d}s_Y + X_3\int_{\tilde{\Gamma}} \frac{\partial u_f(Y)}{\partial X_3}\mathrm{d}s_Y, \quad X \in \hat{\Omega}. \tag{6.6}$$

Taking the trace of (6.3) and (6.4) on $\tilde{\Gamma}$ and $\tilde{\Gamma}_\varepsilon$ respectively yields

$$f(X) = -\Theta(\frac{\partial u_f}{\partial X_3})(X) - u_f^-(X)$$

$$u_f^+(X) = \Theta_\varepsilon\frac{\partial u_f}{\partial X_3}(X) + \varepsilon\int_{\tilde{\Gamma}} \frac{\partial u_f(Y)}{\partial X_3}\mathrm{d}s_X.$$

Thus u_f is also a solution to (B).

Let $u(X)$ be the solution to (B), using Green's formula, (6.2) and (6.6), u can be extended continuously to $\tilde{\Omega}$. We claim that such extension is unique. Assume that there are two solutions u_1 and u_2 of (A) that coincides in $\tilde{\Omega}_\varepsilon$. Let $w = u_1 - u_2$ be the solution of the following system

$$\begin{cases} \Delta w(X) = 0, \quad \text{in} \quad \tilde{\Omega}\backslash\bar{\tilde{\Omega}}_\varepsilon, \\ \frac{\partial w(X)}{\partial \nu} = 0, \quad \partial\tilde{\Omega}_e \cup \{X_1 = \{0,1\}, X_2 = \{0,1\}\} \times (-\varepsilon, -\infty), \\ w(X) = 0, \quad \text{on} \quad \tilde{\Gamma} \cup \tilde{\Gamma}_\varepsilon, \\ w(X) = o(1) \quad |\nabla w(X)| = o(1), \qquad X_3 \to -\infty, \\ w(X) = O(\frac{1}{X^2}), \quad |\nabla w(X)| \cdot \frac{X}{|X|} = O(\frac{1}{|X|^3}), \qquad |X| \to \infty. \end{cases}$$

Let C_R^+ be the upper half sphere of radius R and center $(0, 0, 0)$ in $\tilde{\Omega}_e$. let $S_R^+ = \partial S_R^+ \cap \tilde{\Omega}_e$. We multiply Δw by $\bar{w}(X)$ and integrate by parts over C_R^+ to obtain

$$\int_{C_R^+} |\nabla w|^2 \mathrm{d}x = \int_{S_R^+} \nabla w(X) \cdot \frac{X}{|X|} \mathrm{d}s_X = O\Big(\frac{1}{R^3}\Big) \quad \text{as} \quad R \to +\infty.$$

Hence $w(X)$ is constant in C_R^+,therefore $w(X)$ is constant on $\tilde{\Omega}_e$. Since $w(X) = 0$ on $\tilde{\Gamma}$. we conclude that $w(X) = 0$ on $\tilde{\Omega}_e$. Let $P > \varepsilon$ be a positive constant. We multiply $\Delta w(X)$ by $\bar{w}(X)$ and integrating by parts over $(0, 1)^2 \times (-\varepsilon, -P) = \tilde{\Omega}_P$ to acquire

$$\int_{\tilde{\Omega}_P} |\nabla w|^2 \mathrm{d}x = \int_{\tilde{\Gamma}_P} \partial_{X_3} w(X_1, X_2, -P)\bar{w}(X_1, X_2, -P)\mathrm{d}x_1\mathrm{d}x_2 = o(1), \quad \text{as} \quad P \to \infty.$$

Thus $w(X)$ is constant in $\hat{\Omega}$. Since $w(X) = 0$ on $\tilde{\Gamma}_\varepsilon$, we deduce that $w(X) \equiv 0$ on $\hat{\Omega}$ which proves the uniqueness. □

6.3 Well-possedness of (*B*)

Define the function spaces:

$$H_0^{-\frac{1}{2}}(\tilde{\Gamma}) := \{\varphi(X) \in H^{-\frac{1}{2}}(\tilde{\Gamma}) : \int_{\tilde{\Gamma}} \varphi(X)\mathrm{d}s_X = 0\},$$
$$H_0^{\frac{1}{2}}(\tilde{\Gamma}) := \{\phi(X) \in H^{\frac{1}{2}}(\tilde{\Gamma}) : \int_{\tilde{\Gamma}} \phi(X)\mathrm{d}s_X = 0\}.$$

Lemma 6.3 *The operator Θ has a bounded inverse from $H^{\frac{1}{2}}(\tilde{\Gamma})$ to $H^{-\frac{1}{2}}(\tilde{\Gamma})$. In addition, the following inequality holds*

$$Re(\langle\Theta\varphi, \varphi\rangle_{\frac{1}{2},-\frac{1}{2}}) \geq 0, \quad \forall\varphi \in H^{-\frac{1}{2}}(\tilde{\Gamma}).$$

Proof First we show that $Re(\langle\Theta\varphi, \varphi\rangle_{\frac{1}{2},-\frac{1}{2}}) \geq 0$ for any $\varphi \in H^{-\frac{1}{2}}(\tilde{\Gamma})$. Assume $\varphi \in H^{-\frac{1}{2}}(\tilde{\Gamma})$ such that $\langle\Theta\varphi, \varphi\rangle_{\frac{1}{2},-\frac{1}{2}} \neq 0$. The function $w_\varphi(X) := \int_{\tilde{\Gamma}} \tilde{G}_e(X, Y)\varphi(Y)\mathrm{d}s_Y$ is a solution to

$$\begin{cases} \Delta w_\varphi(X) = 0, \quad in \quad \tilde{\Omega}_e, \\ \frac{\partial w_\varphi(X)}{\partial \nu} = 0, \quad \text{on} \quad \partial\tilde{\Omega}_e\backslash\tilde{\Gamma}, \\ \frac{\partial w_\varphi(X)}{\partial \nu} = \varphi(X), \quad \text{on} \quad \tilde{\Gamma}, \\ w_\varphi(X) = O(\frac{1}{|X|^2}), \quad |\nabla w_\varphi(X)| \cdot \frac{X}{|X|} = O(\frac{1}{|X|^3}), \qquad |X| \to \infty. \end{cases}$$

Let C_R^+ be the upper half sphere of radius R and centered at $(0,0,0)$ in $\tilde{\Omega}_e$, and $S_R^+ = \partial C_R^+ \cap \tilde{\Omega}_e$. Multiplying $\Delta \bar{w}_\varphi$ with w_φ and integrating by parts over C_R^+ yields

$$\begin{aligned}\int_{C_R^+} |\nabla w_\varphi|^2 \mathrm{d}x &= \int_{S_R^+} \nabla \bar{w}_\varphi(X) \cdot \frac{X}{|X|} w_\varphi(X) \mathrm{d}s_X + \langle \Theta\varphi, \varphi \rangle_{\frac{1}{2},-\frac{1}{2}} \\ &= \langle \Theta\varphi, \varphi \rangle_{\frac{1}{2},-\frac{1}{2}} + O\left(\frac{1}{R^3}\right), \quad R \to +\infty. \qquad (6.7)\end{aligned}$$

Hence $Re(\langle \Theta\varphi, \varphi \rangle_{\frac{1}{2},-\frac{1}{2}}) \geq 0$. Since $\Theta : H^{-\frac{1}{2}}(\tilde{\Gamma}) \to H^{\frac{1}{2}}(\tilde{\Gamma})$ is a compact operator, to show invertibility it is sufficient to prove the injectivity of Θ. Let $\varphi \in H_0^{-\frac{1}{2}}(\tilde{\Gamma})$ such that $\langle \Theta\varphi, \varphi \rangle_{\frac{1}{2},-\frac{1}{2}} = 0$ and by substituting it in (6.7), we see that $w_\varphi(X)$ is constant in $\tilde{\Omega}_e$. Since $w_\varphi(X) = O\left(\frac{1}{|X|}\right)$ large $|X|$, so $w_\varphi(X) \equiv 0$ on $\tilde{\Omega}_e$. By taking its normal derivative on $\tilde{\Gamma}$, we conclude that $\varphi \equiv 0$, which proves the claim and hence Θ is injective in $H_0^{-\frac{1}{2}}(\tilde{\Gamma})$. □

Lemma 6.4 *The operator Θ_ε is invertible from $H_0^{-\frac{1}{2}}(\tilde{\Gamma}_\varepsilon)$ to $H_0^{\frac{1}{2}}(\tilde{\Gamma}_\varepsilon)$. In addition, the following inequality holds :*

$$Re(\langle \Theta_\varepsilon\varphi, \varphi \rangle_{\frac{1}{2},-\frac{1}{2}}) \geq 0, \quad \forall \varphi \in H_0^{-\frac{1}{2}}(\tilde{\Gamma}_\varepsilon).$$

Proof Since Θ_ε is a compact operator, to prove the invertibility of the operator Θ_ε : $H_0^{-\frac{1}{2}}(\tilde{\Gamma}_\varepsilon) \to H_0^{\frac{1}{2}}(\tilde{\Gamma}_\varepsilon)$ amounts to proving its injectivity. Recall that

$$\Theta_\varepsilon\varphi(X) = \int_{\tilde{\Gamma}_\varepsilon} \tilde{G}_i(Y + (0,\varepsilon,0), X + (0,\varepsilon,0))\varphi(Y)\mathrm{d}s_Y = \int_{\tilde{\Gamma}} \tilde{G}_i(Y, X)\varphi(Y)\mathrm{d}s_Y.$$

Define the single layer potential

$$u(X) = \int_{\tilde{\Gamma}_\varepsilon} \tilde{G}_i(Y, X)\varphi(Y)\mathrm{d}s_Y \quad X \in \tilde{\Omega}_i \setminus \tilde{\Gamma}_\varepsilon.$$

Let us define $\tilde{\Omega}_i^+ = (0,1)^2 \times \mathbb{R}_+$ to be an extension of $\tilde{\Omega}_i$ to $+\infty$. We then multiply $\Delta \bar{u}$ by u and integrate over $\tilde{\Omega}_i^+$ and $\tilde{\Omega}_i$ respectively to obtain

$$\int_{\tilde{\Omega}_i^+} |\nabla u|^2(X)\mathrm{d}x = \int_{\partial\tilde{\Omega}_i^+} \partial_\nu \bar{u}_+(X)u(X)\mathrm{d}s_X = -\int_{\tilde{\Gamma}} \partial_{X_3}\bar{u}_+(X)u(X)\mathrm{d}s_X.$$

$$\int_{\tilde{\Omega}_i} |\nabla u|^2\mathrm{d}x = \int_{\partial\tilde{\Omega}_i} \partial_\nu \bar{u}_+(X)u(X)\mathrm{d}s_X = \int_{\tilde{\Gamma}} \partial_{X_3}\bar{u}_-(X)u(X)\mathrm{d}s_X.$$

Combining these results, we get

$$\int_{\tilde{\Omega}_i^+ \cup \tilde{\Omega}_i} |\nabla u|^2\mathrm{d}x = \int_{\tilde{\Gamma}} \left(\partial_{X_3}\bar{u}_-(X) - \partial_{X_3}\bar{u}_+(X)\right) u(X)\mathrm{d}s_X.$$

Since

$$\partial_{X_3}\bar{u}_-(X) - \partial_{X_3}\bar{u}_+(X) = \bar{\varphi}(X),$$

there holds

$$\int_{\tilde{\Omega}_i} |\nabla u|^2\mathrm{d}x = \int_{\tilde{\Gamma}} \Theta_\varepsilon\varphi(X)\bar{\varphi}(X)\mathrm{d}s_X \geq 0.$$

If $\langle \Theta_\varepsilon \varphi, \bar{\varphi} \rangle = 0$, it follows that $\nabla u = 0$ in $\tilde{\Omega}_i$ and $\mathbb{R}^3 \setminus \bar{\tilde{\Omega}}_i$ respectively. Hence $\varphi = \partial_\nu u_-(X) - \partial_\nu u_+(X) = 0$. □

To derive the variational formulation, we introduce the function space $V = \{w \in H^1(\tilde{\Omega}_\varepsilon) : \int_{\tilde{\Gamma}} w(X) \mathrm{d}s_X = 0\}$ for any test function $w \in V$, we multiply it by (B) and integrate by parts over $\tilde{\Omega}_\varepsilon$ to obtain

$$\int_{\tilde{\Omega}_\varepsilon} \nabla u(X) \nabla \bar{w}(X) \mathrm{d}x = \int_{\tilde{\Gamma}} \frac{\partial u(X)}{\partial X_3} \bar{w}(X) \mathrm{d}s_X - \int_{\tilde{\Gamma}_\varepsilon} \frac{\partial u(X)}{\partial X_3} \bar{w}(X) \mathrm{d}s_X. \tag{6.8}$$

The integral $\int_{\tilde{\Gamma}} \frac{\partial u(X)}{\partial X_3} \bar{w}(X) \mathrm{d}s_X$ in (6.8) can be understood as the dual product $\overline{\langle w, \frac{\partial u(X)}{\partial X_3} \rangle}_{\frac{1}{2}, -\frac{1}{2}}$. $\forall w|_{\tilde{\Gamma}} \in H^{\frac{1}{2}}(\tilde{\Gamma})$, $(\Theta^{-1} w)(X)$ is well defined in $H^{-\frac{1}{2}}(\tilde{\Gamma})$. Since the Green's function $\tilde{G}_e$ is symmetric, we can write $\overline{\langle w, \frac{\partial u(X)}{\partial X_3} \rangle}_{\frac{1}{2}, -\frac{1}{2}} = \langle \Theta(\frac{\partial u}{\partial X_3}), \Theta^{-1} w \rangle_{\frac{1}{2}, -\frac{1}{2}}$. Let u be the solution to (B), then

$$\int_{\tilde{\Gamma}} \frac{\partial u(X)}{\partial X_3} \bar{w}(X) \mathrm{d}s_X = -\int_{\tilde{\Gamma}} u(X)(\Theta^{-1} \bar{w})(X) \mathrm{d}s_X + \int_{\tilde{\Gamma}} u(X)(\Theta^{-1} f)(X) \mathrm{d}s_X. \tag{6.9}$$

For the integral $\int_{\tilde{\Gamma}_\varepsilon} \frac{\partial u(X)}{\partial X_3} \bar{w}(X) \mathrm{d}s_X$ in (6.8), we observe that the solution to (B) satisfies $\Theta_\varepsilon(\frac{\partial u}{\partial X_3})(X) + \varepsilon \langle 1, \frac{\partial u}{\partial X_3} \rangle = u(X)$ on $\tilde{\Gamma}_\varepsilon$. Integrating over $\tilde{\Gamma}_\varepsilon$ yields

$$4\varepsilon \langle 1, \frac{\partial u}{\partial X_3} \rangle = \int_{\tilde{\Gamma}_\varepsilon} u(X) \mathrm{d}x. \tag{6.10}$$

Let $t(w) = \frac{1}{4} \int_{\tilde{\Gamma}_\varepsilon} w \mathrm{d}s_X$, then

$$\int_{\tilde{\Gamma}_\varepsilon} \frac{\partial u(X)}{\partial X_3} \bar{w}(X) \mathrm{d}s_X = \int_{\tilde{\Gamma}_\varepsilon} \frac{\partial u(X)}{\partial X_3} (\bar{w} - t(\bar{w})) \mathrm{d}s_X + t(\bar{w}) \langle 1, \frac{\partial u}{\partial X_3} \rangle.$$

We deduce by the invertibility of $\Theta_\varepsilon : H^{-\frac{1}{2}}(\tilde{\Gamma}_\varepsilon) \to H^{-\frac{1}{2}}(\tilde{\Gamma}_\varepsilon)$, and the symmetry that

$$\begin{aligned} \int_{\tilde{\Gamma}_\varepsilon} \frac{\partial u(X)}{\partial X_3} \bar{w}(X)) \mathrm{d}s_X &= \int_{\tilde{\Gamma}_\varepsilon} \Theta_\varepsilon \Big(\frac{\partial u}{\partial X_3} \Big) \Theta_\varepsilon^{-1} (\bar{w} - t(\bar{w}) \mathrm{d}s_X + t(\bar{w}) \langle 1, \frac{\partial u}{\partial X_3} \rangle. \\ &= \int_{\tilde{\Gamma}_\varepsilon} (u - t(u)) \Theta_\varepsilon^{-1} (\bar{w} - t(\bar{w})) \mathrm{d}s_X + \frac{1}{\varepsilon} t(u) t(w). \end{aligned} \tag{6.11}$$

Therefore, by virtue of (6.8)-(6.11), we define the bilinear form $a(u, w)$ and the functional $F(w)$ as follows:

$$\begin{aligned} a(u, w) = &\int_{\tilde{\Omega}_\varepsilon} \nabla u \nabla \bar{w} \mathrm{d}x - \int_{\tilde{\Gamma}} u(\Theta^{-1} \bar{w}) \mathrm{d}s_X \\ &+ \int_{\tilde{\Gamma}_\varepsilon} (u - t(u)) \Theta_\varepsilon^{-1} (\bar{w} - t(\bar{w})) \mathrm{d}s_X + \varepsilon^{-1} t(u) t(\bar{w}), \end{aligned} \tag{6.12}$$

and

$$F(w) = \int_{\tilde{\Gamma}} (\Theta^{-1} f) \bar{w} \mathrm{d}s_X, \tag{6.13}$$

so that (6.8) reduces to

$$a(u, w) = F(w).$$

Theorem 6.5 *(1) $F(w)$ is a bounded linear functional from V to $\mathbb{C}$. The bilinear form $a(u, w)$ is bounded and coercive on $V \times V$: There exists constants $C_1 > 0$ and $C_2 > 0$, such that*

$$|a(u, w) \leq C_1 \|u\|_1 \|w\|_1, \quad \forall u, w \in V,$$

$$Re(a(u, u)) \geq C_2 \|u\|_1^2, \quad \forall u \in V.$$

(2) For any $f \in H^{\frac{1}{2}}(\tilde{\Gamma})$,there exists a unique solution to (B).

Proof First to show that $F(w)$ is bounded from V to $\mathbb{C}$. It follows from Lemma 6.3 that $\Theta^{-1} : H^{\frac{1}{2}}(\tilde{\Gamma}) \to H^{-\frac{1}{2}}(\tilde{\Gamma})$ is bounded operator. Since the operator on $\tilde{\Gamma}$ is continuous operator from V to $H^{\frac{1}{2}}(\tilde{\Gamma})$, $F(w)$ is continuous linear form from V to $\mathbb{C}$. The bilinear form $a(u, w)$ can easily seen to be bounded by using the trace theorem and the boundedness of Θ_ε^{-1} and Θ. Finally, for a fixed $u \in V$ as

$$a(u, u) = \int_{\Omega_\varepsilon} |\nabla u^2| \mathrm{d}x + \int_{\hat{\Gamma}} u(\Theta^{-1}\bar{u}) \mathrm{d}s_X + \int_{\hat{\Gamma}_\varepsilon} (u - t(u)) \Theta_\varepsilon^{-1}(\bar{u} - t(\bar{u})) \mathrm{d}s_X + \varepsilon^{-1}|t(u)|^2.$$

The coercivity of the bilinear form is a direct result of Poincare-Friedrichs inequality. Finally, from the Lax-Milgram theorem, (B) attains a unique solution $u \in H^{\frac{1}{2}}(\tilde{\Omega}_\varepsilon)$. □

Remark During the submission of the paper, the work [26] was brought to our attention, which uses a Fourier mode matching method to obtain the resonances of a sound hard slab with subwavelenth holes. The studies does not restrict the cross sectional shape of the hole.

Data availability Not applicable. No supporting data is used for the results reported in the article.

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