



Minimal volume entropy of free-by-cyclic groups and 2-dimensional right-angled Artin groups

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Abstract

Let G be a free-by-cyclic group or a 2-dimensional right-angled Artin group. We provide an algebraic and a geometric characterization for when each aspherical simplicial complex with fundamental group isomorphic to G has minimal volume entropy equal to 0. In the nonvanishing case, we provide a positive lower bound to the minimal volume entropy of an aspherical simplicial complex of minimal dimension for these two classes of groups. Our results rely upon a criterion for the vanishing of the minimal volume entropy for 2-dimensional groups with uniform uniform exponential growth. This criterion is shown by analyzing the fiber π_1 -growth collapse and non-collapsing assumptions of Babenko–Sabourau (Minimal volume entropy and fiber growth, [arXiv:2102.04551](https://arxiv.org/abs/2102.04551), 2020).

1 Introduction

The *volume entropy* of a finite simplicial complex X equipped with a piecewise Riemannian metric g is defined as

$$\text{ent}(X, g) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \text{vol}(B_{x_0}(t), \tilde{g})$$

where $B_{x_0}(t)$ is the ball of radius t centered at some point x_0 in the universal cover $\tilde{X}$ and $\tilde{g}$ is the pull-back metric on $\tilde{X}$. This limit always exists and does not depend

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on the choice of x_0 . Initially defined as a Riemannian manifold invariant, the volume entropy measures the exponential growth rate of the volume of balls in the universal cover and is related to the growth of the fundamental group (Švarc [35] and Milnor [26]) and to the dynamics of the geodesic flow. Specifically, in this context, Dinaburg showed that the volume entropy gives a lower bound on the topological entropy of the geodesic flow [14]. Manning further showed that if the sectional curvatures for the metric are all nonpositive, then the volume entropy equals the topological entropy of the geodesic flow [24].

In order to obtain a topological invariant of X , it is natural to optimize the volume entropy over all piecewise Riemannian metrics. To get an invariant that is nondegenerate, we must take into account the effect of scaling the metric by a constant and counteract this by multiplying the volume entropy by an appropriate root of the volume. This leads to the notion of minimal volume entropy, introduced by Gromov originally in the context of Riemannian manifolds [18]. To this end, we set

$$\omega(X, g) = \text{ent}(X, g) \text{vol}(X, g)^{1/\dim(X)}.$$

The *minimal volume entropy* of a finite simplicial complex X is defined by

$$\omega(X) = \inf_g \omega(X, g)$$

where g runs over all piecewise Riemannian metrics on X .

When M is a closed, orientable n -manifold, Gromov showed that $\omega(M)^n \geq c_n \|M\|$ where $\|M\|$ is the simplicial volume of M and $c_n > 0$ is a constant that only depends on the dimension [18]. In dimensions at most 3, the invariants $\omega(M)^n$ and $\|M\|$ are proportional, as we explain below. It is unknown whether or not the reverse inequality holds up to a constant in higher dimensions. Nevertheless, it is in this sense that $\omega(X)^{\dim(X)}$ can be viewed as a substitute for simplicial volume for X when there is no natural choice of fundamental class.

Katok was the first to realize that minimal volume entropy could select an optimal metric, up to scale. He proved that if M is a closed surface with negative Euler characteristic then $\omega(M, g) \geq \omega(M, g_{\text{hyp}})$ where g_{hyp} is any hyperbolic metric, with equality if and only if g has constant curvature [21]. This was extended by Besson–Courtois–Gallot to closed, real hyperbolic manifolds of any dimension. [5].

For simplicial complexes that are not manifolds, there are few results. When X is a finite connected graph and every vertex has degree at least 3, Lim gave an explicit description of a metric g_0 so that $\omega(X) = \omega(X, g_0)$ [22]. Analogous to the results for closed real hyperbolic manifolds mentioned above, Lim additionally proves that this metric is unique up to scale. McMullen gave an alternate proof of this result [25]; I. Kapovich–Nagnibeda gave a proof of this result when every vertex in the graph has degree 3 [20].

Other general results regarding minimal volume entropy for simplicial complexes include the fiber π_1 -growth collapsing/non-collapsing assumptions recently provided by Babenko–Sabourau that are useful in showing whether or not $\omega(X)$ vanishes [1].

These will play a key role in this paper and are discussed in more detail later on in the Introduction and in Sect. 3.

As mentioned above, the volume entropy is related to the growth of the fundamental group in that it—or a slight variation—can be used to determine the growth type: polynomial or exponential. However, in general, the minimal volume entropy of a simplicial complex does depend on more than the fundamental group, as originally observed by Babenko [2]. (Although it will not play a role in what follows, in the context of manifolds there are circumstances where the minimal volume entropy is determined by the fundamental group; see the works of Babenko [2] and Brunnbauer [8].)

This leads into the central object of study in this paper. For a fixed group G we study the minimal volume entropy of a G -complex i.e., a finite aspherical simplicial complex X such that $\pi_1(X) \cong G$. By taking the infimum over G -complexes with minimal dimension, we obtain an invariant of a group G of finite type. We thus define the *minimal volume entropy of G* as

$$\omega(G) = \inf_X \omega(X)$$

where X runs over all G -complexes with $\dim(X)$ equal to the geometric dimension, $\text{gd}(G)$, i.e., the minimal dimension of a G -complex. For free groups, it was observed by I. Kapovich–Nagnibeda [20], Lim [22] and McMullen [25] that if X is a finite graph and $\pi_1(X)$ is isomorphic to a free group of rank n , then $\omega(X) \geq (3n - 3) \log 2$ with equality if and only if every vertex in X has degree 3. Cast in the above language, this gives $\omega(F_n) = (3n - 3) \log 2$.

In this paper we study the minimal volume entropy when G is either a free-by-cyclic group or a 2-dimensional right-angled Artin group (RAAG). Each such group G admits a 2-dimensional aspherical G -complex. In each case, we prove that either the minimal volume entropy vanishes for every G -complex or $\omega(G)$ is uniformly bounded from below. Moreover, as we will describe below, whether or not $\omega(G)$ vanishes is directly related to whether or not G is *tubular*, i.e., whether it admits a graph of groups decomposition with vertex groups equal to $\mathbb{Z}^2$ and edge groups equal to $\mathbb{Z}$.

We state our results in these two cases and then describe how we apply the fiber π_1 -growth collapsing/non-collapsing assumptions of Babenko–Sabourau.

Free-by-cyclic groups Every free-by-cyclic group is determined by a finite rank free group F_n and an element $\phi \in \text{Out}(F_n)$, the outer automorphism group of F_n . Denote by G_ϕ the free-by-cyclic group associated to ϕ . Specifically, the group G_ϕ is given by the presentation:

$$G_\phi = \langle F_n, t \mid txt^{-1} = \Phi(x) \rangle$$

where $\Phi \in \text{Aut}(F_n)$ represents ϕ .

We are able give an explicit description—up to passing to a power—for which outer automorphisms give rise to free-by-cyclic groups with vanishing minimal volume entropy. We call outer automorphisms with such a description *geometrically linear unipotent* (GLU) (see Definition 5.4). We prove:

Theorem 1.1 *Suppose that ϕ is an outer automorphism of a finitely generated free group. The following are equivalent:*

1. $\omega(X) = 0$ for every G_ϕ -complex X .
2. G_ϕ is virtually tubular.
3. Some power of ϕ is geometrically linear unipotent.

If none of these conditions hold, then $\omega(G_\phi) \geq \frac{\log 3}{12 \cdot 10^6}$.

There is an established connection between free-by-cyclic groups and mapping tori M_f of homeomorphisms of closed orientable surfaces $f: S \rightarrow S$. GLU automorphisms have linear growth, and via this connection are reminiscent of a multi-twist homeomorphism of a closed orientable surface. Pieroni showed that $\omega(M)^3$, for M a closed orientable 3-manifold, equals two times the sum of the volumes of the hyperbolic components in the JSJ decomposition [28]. In particular, by the celebrated result of Thurston, the minimal volume entropy of M_f vanishes if and only if some power of f is homotopic to a multi-twist [34]. In this way, Theorem 1.1 result can be regarded as a free group analogue. However, in contrast to the case of a mapping class on a closed surface, not all subexponentially growing outer automorphisms of free groups have linear growth, and not all G_ϕ with linearly growing ϕ have vanishing minimal volume entropy.

The L^2 -torsion $-\rho^{(2)}(\cdot)$ is an analytic invariant of certain groups that may also play the role of volume. Indeed, if M is a closed orientable 3-manifold, then Lück–Schick proved that $-\rho^{(2)}(\pi_1(M))$ equals $\frac{1}{6\pi}$ times the sum of the volumes of the hyperbolic components in the JSJ decomposition [23]. By combining the work of Gromov, Soma and Thurston, we have that $\|M\|$ also equals a constant times the sum of the volumes of the hyperbolic components in the JSJ decomposition in this case [18,32,33]. Thus we see that these three notions of volume—minimal volume entropy, L^2 -torsion, and simplicial volume—are all proportional for closed orientable 3-manifolds, in particular, for mapping tori of homeomorphisms of closed orientable surfaces.

As there is no well-defined fundamental class for a free-by-cyclic group, there is no natural way to define the simplicial volume. However, it is interesting to compare the minimal volume entropy and the L^2 -torsion for free-by-cyclic groups. The second author proved that $-\rho^{(2)}(G_\phi)$ vanishes when ϕ is polynomially growing [12]—conjecturally, the converse holds as well. As Theorem 1.1 shows that most free-by-cyclic groups with polynomially growing monodromy have nonvanishing minimal volume entropy, we see that these two invariants are not proportional in this setting. The second author provided an upper bound on $-\rho^{(2)}(G_\phi)$ using the dynamics of ϕ [12], it would be interesting to find an upper bound on the minimal volume entropy of G_ϕ as well.

Theorem 1.1 provides a characterization of free-by-cyclic groups that are virtually tubular. We note that Button has provided a characterization of tubular groups that are free-by-cyclic [9].

Right-angled Artin groups Let Γ be a finite simplicial graph. The right-angled Artin group A_Γ is the group whose generators are the vertices of Γ and whose relations are commutations between generators when the vertices are incident on an edge in Γ .

That is, A_Γ is given by the presentation:

$$A_\Gamma = \langle V\Gamma \mid vw = wv \text{ if } v \text{ and } w \text{ are incident on an edge in } \Gamma \rangle.$$

Right-angled Artin groups, though simple to define, form an essential class of groups in low-dimensional topology and geometric group theory. Partly, this is due to the surprising richness of their subgroups, their role as an interpolation between free groups and free abelian groups and also the frequency at which they arise as subgroups of geometrically defined groups.

The group A_Γ has geometric dimension equal to 2 if and only if Γ has at least one edge (i.e., A_Γ is not free) and Γ has no triangles (i.e., K_3 is not a subgraph of Γ) [10, Corollary 1.4.2]. In this case an A_Γ -complex, known as the Salvetti complex S_Γ , is built out of unions of circles S^1 and tori $S^1 \times S^1$ that are identified along certain cyclic subgroups. This structure seems reminiscent of a tubular group, however, not all 2-dimensional right-angled Artin groups are (free products of) tubular groups. In fact, whether or not a 2-dimensional right-angled Artin group is tubular is directly related to its minimal volume entropy.

Theorem 1.2 *Suppose that A_Γ is a right-angled Artin group with $\text{gd}(A_\Gamma) = 2$. The following are equivalent.*

1. $\omega(X) = 0$ for every A_Γ -complex X .
2. A_Γ is a free product of tubular groups and a free group.
3. Γ is a forest.

If none of these conditions hold, then $\omega(A_\Gamma) \geq \frac{\log 3}{2 \cdot 10^6}$.

We remark that according to Droms, A_Γ is a 3-manifold group exactly when Γ is a disjoint union of trees and triangles [15]. Since a triangle corresponds to $\mathbb{Z}^3$, Γ is a forest exactly when A_Γ is a 3-manifold group with geometric dimension at most 2.

It seems likely a characterization for the vanishing of minimal volume entropy of right-angled Artin groups of arbitrary dimension is possible, although the statement may not be so neat.

Fiber π_1 -growth assumptions Both theorems above are consequences of the fiber π_1 -growth collapsing/non-collapsing assumptions of Babenko–Sabourau [1]. These assumptions relate the vanishing of the minimal volume entropy of a simplicial complex X to the existence or non-existence of maps $f: X \rightarrow P$ to lower dimensional complexes, based on the π_1 -growth of fibers. Briefly, the two assumptions are:

- Fiber π_1 -growth collapsing assumption (FCA)—For some simplicial map $f: X \rightarrow P$, every induced subgroup $\pi_1(f^{-1}(x)) \subseteq \pi_1(X)$ is subexponentially growing with subexponential growth rate less than $1 - \frac{\dim P}{\dim X}$.
- Fiber π_1 -growth non-collapsing assumption (FNCA)—There is a constant $\delta > 0$ such that for every simplicial map $f: X \rightarrow P$, some induced subgroup $\pi_1(f^{-1}(x)) \subseteq \pi_1(X)$ has uniform exponential growth rate at least δ .

Although the two criteria are not *a priori* complementary, we show that they are in the case of free-by-cyclic groups and 2-dimensional right-angled Artin groups.

Moreover, the two assumptions are complementary when the fundamental group of X has uniform exponential growth and satisfies a technical condition on subexponentially growing subgroups (see Definition 2.1 and Proposition 3.10).

When X is a 2-dimensional simplicial complex, the fiber π_1 -growth assumptions consider maps $f: X \rightarrow P$ where P is a finite simplicial graph. Applying standard geometric group theoretic techniques, when X satisfies the FCA there is an induced graph of groups decomposition on $\pi_1(X)$ where the vertex and edge groups are all subexponentially growing (Proposition 4.1).

When a group G has uniform exponential growth (denoted by $\underline{\delta}(G) > 0$ in the following), we prove the following vanishing criteria:

Theorem 1.3 *Let G be a group with $\text{gd}(G) = 2$. Suppose $\underline{\delta}(G) > 0$ and that the subexponentially growing subgroups of G belong to the collection $\{\{1\}, \mathbb{Z}, \mathbb{Z}^2, BS(1, -1)\}$. Then $\omega(X) = 0$ for every G -complex X if and only if G is the fundamental group of a graph of groups where the edge groups belong to the collection $\{\{1\}, \mathbb{Z}\}$ and the vertex groups belong to the collection $\{\mathbb{Z}, \mathbb{Z}^2, BS(1, -1)\}$.*

Remark 1.4 The Baumslag–Solitar group $BS(1, -1) = \langle a, t \mid tat^{-1} = a^{-1} \rangle$ is the fundamental group of the Klein bottle. By a result of Degrijse, if G has cohomological dimension equal to 2, has subexponential growth and the group algebra $\mathbb{C}[G]$ does not have zero-divisors, then G is either $\mathbb{Z}^2$ or $BS(1, -1)$ [13, Theorem B]. Conjecturally, if G is torsion-free then $\mathbb{C}[G]$ does not contain any zero-divisors, which would render this hypothesis unnecessary. Therefore conjecturally, Theorem 1.3 applies to any group with $\text{gd}(G) = 2$ and $\underline{\delta}(G) > 0$.

Theorem 1.3 applies to the case of groups G with $\text{gd}(G) = 2$ that act freely and cocompactly on CAT(0) cube complexes with isolated flats by recent work of Gupta–Jankiewicz–Ng [19].

1.1 Outline of paper

In Sect. 2, we discuss notions of growth in groups and show that the vanishing of $\omega(X)$ is a homotopy invariant of G -complexes of minimal dimension. Section 3 recalls the fiber π_1 -growth collapsing/non-collapsing assumptions of Babenko–Sabourau [1] and proves that these are complementary when G has Property U . After briefly reviewing graphs of groups, in Sect. 4 we prove Theorem 1.3. In Sect. 5 we prove Theorem 1.1 regarding the minimal volume entropy of free-by-cyclic groups. Finally, in Sect. 6, we prove Theorem 1.2 regarding the minimal volume entropy of 2-dimensional right-angled Artin groups.

2 Entropy and volume in groups

In this section we discuss growth in groups and the relation between the minimal volume entropy of a group and the minimal volume entropy of a finite index subgroup.

2.1 Growth in groups

Let G be a finitely generated group and suppose that $S \subset G$ is a finite generating set. For an element $h \in G$, by $\|h\|_S$ we denote the word length of h with respect to S .

The *growth rate of G with respect to S* is the quantity

$$\delta(G, S) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \#\{h \in G \mid \|h\|_S \leq t\}.$$

We observe that if X is the Cayley graph of G with respect to the generating set S and g is the piecewise Riemannian metric on X for which each edge of X is isometric to the unit interval, then $\text{ent}(X, g) = \delta(G, S)$. If $\delta(G, S) > 0$ for some finite generating set $S \subset G$, then it is known that $\delta(G, S') > 0$ for all finite generating sets $S' \subset G$. In this case, the group G is said to have *exponential growth*. Else, the group is said to have *subexponential growth*. If further we have that $\#\{h \in G \mid \|h\|_S \leq t\}$ is bounded by a polynomial, the group G is said to have *polynomial growth*.

In the case of subexponential growth, we consider the *subexponential growth rate of G* which is defined by

$$\nu(G) = \lim_{t \rightarrow \infty} \frac{\log \log \#\{h \in G \mid \|h\|_S \leq t\}}{\log t}.$$

This quantity is independent of S and satisfies $0 \leq \nu(G) \leq 1$. We remark that if G has polynomial growth, then $\nu(G) = 0$.

In the case of exponential growth, to get a quantity that is independent of the generating set, we can take the infimum. This leads to the *uniform growth rate of G* which is defined by

$$\delta(G) = \inf_S \delta(G, S)$$

where S runs over all finite generating sets for S . The group G is said to have *uniform exponential growth* if $\delta(G) > 0$. There are examples of finitely generated groups with exponential growth, but not uniform exponential growth [36].

Taking this concept one step further, we can consider the infimum over all finitely generated exponentially growing subgroups of G as well. This leads to the *uniform growth rate of G* which is defined by

$$\underline{\delta}(G) = \inf_H \delta(H)$$

where H runs over all finitely generated subgroups of G with exponential growth. The group G is said to have *uniform uniform exponential growth* if $\underline{\delta}(G) > 0$.

Finally, the following property is relevant to the sequel.

Definition 2.1 A finitely generated group G has *Property U* if $\underline{\delta}(G) > 0$ and if H is a subgroup of G with subexponential growth, then $\nu(H) = 0$.

In particular, if $\underline{\delta}(G) > 0$ and every subexponentially growing subgroup of G has polynomial growth, then G has Property U.

2.2 Monotone maps and finite index subgroups

Let X and Y be simplicial complexes with $\dim(X) = \dim(Y) = m$. A simplicial map $f: Y \rightarrow X$ is said to be n -monotone for $n \geq 0$ if the preimage of any open m -simplex in X consists of at most n open m -simplices in Y . The following lemma gives a relation between the minimal volume entropy of X and Y using a n -monotone map $f: Y \rightarrow X$. This lemma appears in a paper by Brunnbauer [8, Lemma 4.1]. The proof is attributed to Babenko [2] and appears in a paper by Sabourau [29, Lemma 3.5].

Lemma 2.2 *Let $f: Y \rightarrow X$ be an n -monotone map between m -dimensional finite simplicial complexes. If $f_*: \pi_1(Y) \rightarrow \pi_1(X)$ is injective, then*

$$n^{1/m} \cdot \omega(X) \geq \omega(Y).$$

There are two useful consequences of this bound.

Proposition 2.3 *Let G be a group of finite type and let X and Y be G -complexes with $\dim(X) = \dim(Y)$. Then $\omega(X) = 0$ if and only if $\omega(Y) = 0$.*

Proof Let $f: X \rightarrow Y$ be a homotopy equivalence and let $f': Y \rightarrow X$ be the homotopy inverse to f . By the simplicial approximation theorem, we may assume that both of these maps are simplicial. Let m denote the common dimension of X and Y . By finiteness, each m -simplex of Y has at most n preimages for some $n > 0$ under f . Similarly, there exists $n' > 0$ such that each m -simplex of X has at most n' preimages under f' . The proposition now follows from Lemma 2.2, since n and n' are both positive. $\square$

We record the following corollary of Proposition 2.3.

Corollary 2.4 *Let G be a group of finite type and suppose that $\omega(X) = 0$ for some G -complex with $\dim(X) = \text{gd}(G)$. Then $\omega(X) = 0$ for every G -complex X with $\dim(X) = \text{gd}(G)$.*

The other useful consequence is with regards to finite index subgroups.

Proposition 2.5 *Suppose that G is a group of finite type. If H is a subgroup of G and $[G : H] = n$, then*

$$n^{1/\text{gd}(G)} \cdot \omega(G) \geq \omega(H).$$

Proof Let G and H be as in the statement. As G has finite type, so does H and moreover $\text{gd}(G) = \text{gd}(H)$.

Suppose that X is a G -complex with $\dim(X) = \text{gd}(G)$. Let $f: Y \rightarrow X$ be the cover corresponding to the subgroup H . Then f is n -monotone and $f_*: \pi_1(Y) \rightarrow \pi_1(X)$ is injective. Hence by Lemma 2.2 we have that

$$n^{1/\text{gd}(G)} \cdot \omega(X) \geq \omega(Y).$$

As X is an arbitrary G -complex with $\dim(X) = \text{gd}(G)$ and $\omega(Y) \geq \omega(H)$ for any H -complex Y with $\dim(Y) = \text{gd}(H)$, the result follows. $\square$

In particular, if $\omega(H) > 0$ for some finite index of a group G of finite type, then $\omega(G) > 0$ as well.

3 Fiber π_1 -growth assumptions

In this section we recall the fiber π_1 -growth collapsing and non-collapsing assumptions, introduced by Babenko–Sabourau [1]. The collapsing assumption provides a sufficient condition for the minimal volume entropy to vanish, while the non-collapsing assumption guarantees the minimal volume entropy is nonzero, and also provides a lower bound.

3.1 Fiber π_1 -growth collapsing assumption

First we discuss the collapsing assumption. Let X be a simplicial complex. A closed subset $F \subseteq X$ has *subexponential growth* if for every connected component $F_0 \subseteq F$, the inclusion induced image of $\pi_1(F_0)$ in $\pi_1(X)$ has subexponential growth.

Definition 3.1 (Babenko–Sabourau [1]) A simplicial complex X of dimension m satisfies the *fiber π_1 -growth collapsing assumption* (FCA) if there exists a simplicial map $f: X \rightarrow P$ to a finite simplicial complex P of dimension k such that for every $p \in P$, the fiber $f^{-1}(p)$ has subexponential growth with subexponential growth rate less than $\frac{m-k}{m}$.

Babenko–Sabourau prove that the FCA is sufficient to ensure that the minimal volume entropy vanishes. We will provide a proof of a weaker version that is sufficient for our needs. Namely, we will show that satisfying the FCA with the stronger assumption that the subexponential growth rate of the fibers is less than $1/\dim(X)$ implies that the minimal volume entropy vanishes. Our proof is based on their outline but the assumption about the subexponential growth rate of the fibers simplifies the argument.

To do this, we need some facts about subexponential functions. Let $\phi: [0, \infty) \rightarrow [0, \infty)$ be a non-decreasing, subexponential function. By definition, for every $0 < \lambda \leq 1$, there exists $T_\lambda \in [0, \infty)$ such that for all $t \geq T_\lambda$

$$\phi(t) \leq \exp(\lambda t).$$

We may assume T_λ is the largest t such that $\phi(t) = \exp(\lambda t)$. The next lemma describes the dependence of T_λ on λ .

Lemma 3.2 Suppose that $\phi: [0, \infty) \rightarrow [0, \infty)$ is a non-decreasing, subexponential function and that $\phi(t_0) > 1$ for some $t_0 \in (0, \infty)$. The following statements hold.

1. The function $\lambda \mapsto T_\lambda$ is strictly decreasing on $(0, \lambda_0]$ where $\lambda_0 = \min\{1, \frac{1}{t_0} \log \phi(t_0)\}$.
2. $\lim_{\lambda \rightarrow 0^+} T_\lambda = \infty$.

3. Suppose $(t_n) \subset [t_0, \infty)$ and $(\lambda_n) \subset (0, \lambda_0]$ are sequences such that $t_n \rightarrow \infty$ and $T_{\lambda_n} = t_n$. Then for any $\epsilon > 0$ we have

$$\limsup_{n \rightarrow \infty} \frac{\lambda_n}{t_n^{v+\epsilon-1}} \leq 1$$

for where v is the subexponential growth rate of $\phi(t)$.

Proof For $0 < \lambda < \mu \leq \lambda_0$, as $\exp(\lambda t) < \exp(\mu t)$ for $t > 0$, we find that $\exp(\lambda T_\mu) < \exp(\mu T_\mu) = \phi(T_\mu)$. Thus $T_\lambda > T_\mu$. This shows that T_λ is strictly decreasing on $(0, \lambda_0]$, equivalently, strictly increasing as $\lambda \rightarrow 0^+$. This completes the proof of (1).

Suppose $T_\lambda \rightarrow T_0 < \infty$ as $\lambda \rightarrow 0^+$. Since T_λ is strictly decreasing on $(0, \lambda_0]$, we have that $T_0 > T_{\lambda_0} \geq t_0$. Therefore for all $\lambda > 0$, we have $\phi(T_0) < \exp(\lambda T_0)$ and hence $\phi(T_0) \leq \lim_{\lambda \rightarrow 0^+} \exp(\lambda T_0) = 1$. However, $\phi(T_0)$ is strictly greater than 1, since $t_0 < T_0$ and ϕ is increasing, a contradiction. Therefore we conclude that $\lim_{\lambda \rightarrow 0^+} T_\lambda = \infty$. This shows (2).

Finally, suppose that $0 \leq v \leq 1$ is the subexponential growth rate of $\phi(t)$. Thus for any $\epsilon > 0$ we have that $\phi(t) \leq \exp(t^{v+\epsilon})$ for large t . Hence for large enough n , we have $\exp(\lambda_n t_n) = \phi(t_n) \leq \exp(t_n^{v+\epsilon})$. This gives $\lambda_n \leq t_n^{v+\epsilon-1}$ for large enough n and thus $\limsup_{n \rightarrow \infty} \frac{\lambda_n}{t_n^{v+\epsilon-1}} \leq 1$. This shows (3). $\square$

Given a simplicial map $f: X \rightarrow P$, we will call an edge e of X *long* if $f(e)$ is an edge of P , and *short* otherwise, in which case $f(e)$ is a vertex of P .

Theorem 3.3 (Babenko–Sabourau [1, Theorem 2.6]) *Let X be a finite, connected, simplicial complex. If X satisfies the FCA, then $\omega(X) = 0$.*

Proof when subexponential growth rate of fibers is less than $1/\dim(X)$ Let m denote the dimension of X . Suppose $f: X \rightarrow P$ is a simplicial map where $\dim P < m$ and for every $p \in P$, the fiber $f^{-1}(p)$ has subexponential growth with subexponential growth rate v where $v < 1/m$. Without loss of generality, we may assume that $f: X \rightarrow P$ is surjective and has connected fibers (see, for example, Proposition 2.1 of [1]). Fix piecewise Riemannian metrics g_X, g_P on X and P respectively, where the metrics on each simplex agree with that of a Euclidean simplex whose edges all have length 1. We can pull back the metric $f^*(g_P)$ to X , where it is everywhere degenerate because the dimension of P is strictly smaller than m . Consider a new metric g_s for $s > 0$ defined pointwise by

$$g_s = f^*(g_P) + s^2 g_X.$$

Since $f^*(g_P)$ is everywhere degenerate, we clearly have $\lim_{s \rightarrow 0^+} \text{vol}(X, g_s) = 0$. However, it is not the case in general that $\text{ent}(X, g_s)$ stays bounded as $s \rightarrow 0^+$ and so we need to analyze these quantities further. We will prove the theorem by finding a sequence of positive real numbers (s_n) such that $\omega(X, g_{s_n}) = \text{ent}(X, g_{s_n}) \text{vol}(X, g_{s_n})^{1/m}$ goes to 0 as $n \rightarrow \infty$.

We start with the following inequality.

Claim 3.4 *There is a constant V such that $\text{vol}(X, g_s) \leq sV$.*

Proof of Claim Let σ be a simplex of dimension m in X . We will show that the volume of σ with respect to the restriction of g_s is bounded by a constant times s . As there are only finitely many simplices in X , the claim follows.

At each point of σ , we can express the metric g_X as a matrix $[x_1 | \dots | x_m]$ where each $x_i \in \mathbb{R}^m$. Likewise, we can express $f^*(g_P)$ as a matrix $[y_1 | \dots | y_m]$ where each $y_i \in \mathbb{R}^m$. Let Σ_m denote the collection of subsets of $\{1, \dots, m\}$. We then compute

$$\begin{aligned} \det(g_s) &= \det([y_1 + s^2 x_1 | \dots | y_m + s^2 x_m]) \\ &= \sum_{S \in \Sigma_m} s^{2|S|} \det([z_{S,1} | \dots | z_{S,m}]) \end{aligned}$$

where $z_{S,i} = x_i$ if $i \in S$ and $z_{S,i} = y_i$ if $i \notin S$. As $f^*(g_P)$ is everywhere degenerate, we have $\det([y_1 | \dots | y_m]) = 0$. Hence every term in the above summation expressing $\det(g_s)$ contains s^{2n} for some $n \geq 1$. As σ is compact, each term $\det([z_{S,1} | \dots | z_{S,m}])$ is bounded on σ . From this it follows that $\det(g_s)$ is bounded by a constant times s^2 on σ and thus the volume of σ is bounded by a constant times s , as desired. $\square$

To calculate the volume entropy we will estimate the number of homotopy classes in X of g_s -length at most t , as t becomes large.

Let us first estimate the number of homotopy classes in fibers. Since g_s reduces to $s^2 g_X$ along fibers, we can choose s sufficiently small so that each fiber has diameter at most $\frac{1}{2}$. Suppose $F_v = f^{-1}(v)$ is a fiber over some vertex $v \in P$. Fix a basepoint $x \in F_v$, and for $t \geq 0$ let $\mathcal{N}_s(F_v, x; t)$ denote the number of homotopy classes of loops in F_v based at x whose g_s -length is at most t . Then since g_s scales the $g_X (= g_1)$ -length of edges in F_v by s , we have

$$\mathcal{N}_s(F_v, x; t) = \mathcal{N}_1\left(F_v, x; \frac{t}{s}\right).$$

As each fiber is subexponentially growing by assumption and since there are only finitely many vertices in P , for every $0 < \lambda \leq 1$, there exists T_λ such that for all $t \geq T_\lambda$ and vertex $v \in P$, we have

$$\mathcal{N}_1(F_v, x; t) \leq \exp(\lambda t).$$

Define $\mathcal{N}(t) = \max_{v \in P} \mathcal{N}_1(F_v, x; t)$. We define $C_\lambda = \mathcal{N}(T_\lambda)$, so that for all vertices $v \in P$ and for all $t \geq 0$,

$$\mathcal{N}_1(F_v, x; t) \leq C_\lambda \exp(\lambda t).$$

For fixed s , we can let λ depend on s . Suppose that $\mathcal{N}(t_0) > 1$ for some $t_0 > 0$ so that $\mathcal{N}(t)$ satisfies the hypotheses of Lemma 3.2. Let $\lambda_0 = \min\{1, \frac{1}{t_0} \log \mathcal{N}(t_0)\}$ as defined in Lemma 3.2. Define s_0 to be equal to $\frac{1}{T_{\lambda_0}}$. Since T_λ is strictly decreasing on $(0, \lambda_0]$ and $\lim_{\lambda \rightarrow 0^+} T_\lambda = \infty$, there is decreasing sequence of positive real numbers

(s_n) starting with s_0 such that $s_n \rightarrow 0$ as $n \rightarrow \infty$ and for which there exist positive real numbers (λ_n) such that $T_{\lambda_n} = \frac{1}{s_n}$. If $\mathcal{N}(t) \leq 1$ for all t , we take $s_n = 1/n$ and $\lambda_n = 0$.

An arbitrary loop γ in X can be represented as an edge path in the 1-skeleton. Decompose such a path γ as

$$\gamma = \varepsilon_1 \sigma_1 \dots \varepsilon_k \sigma_k,$$

where the ε_i are long edges, and the σ_i are edge paths consisting of edges in some fiber F_{v_i} .

Connect the endpoints of each σ_i to the basepoint $x_i \in F_{v_i}$ to form a loop $\bar{\sigma}_i$ at the expense of adjoining to paths of length at most $\text{diam}(F_{v_i}, g_{s_n}) < \frac{1}{2}$. Hence if σ_i has g_{s_n} -length t_i then the length of $\bar{\sigma}_i$ is at most $t_i + 2 \text{diam}(F_{v_i}, g_{s_n}) \leq t_i + 1$. Up to homotopy, the number of possible σ_i of length at most t_i is then bounded above by

$$\mathcal{N}_s(F_{v_i}, x_i; t_i + 2 \text{diam}(F_{v_i}, g_{s_n})) \leq \mathcal{N}_1\left(F_{v_i}, x_i; \frac{t_i + 1}{s_n}\right) \leq C_{\lambda_n} \exp\left(\lambda_n \frac{t_i + 1}{s_n}\right).$$

Let N_e be the total number of edges in X . On the one hand, each long edge ε_i has g_{s_n} -length at least 1, hence k is at most the length of γ . On the other hand, we also have $\sum_{i=1}^k t_i$ is less than the length of γ . Thus, we can bound the total number of possible paths of g_{s_n} -length at most some integer t by

$$\begin{aligned} N_e^t \prod_{i=1}^t C_{\lambda_n} \exp\left(\lambda_n \frac{t_i + 1}{s_n}\right) &= N_e^t C_{\lambda_n}^t \exp\left(\frac{\lambda_n}{s_n} \sum_{i=1}^t t_i\right) \exp\left(\frac{\lambda_n t}{s_n}\right) \\ &\leq N_e^t C_{\lambda_n}^t \exp\left(\frac{\lambda_n t}{s_n}\right) \exp\left(\frac{\lambda_n t}{s_n}\right). \end{aligned}$$

Taking the logarithm, dividing by t and letting $t \rightarrow \infty$ we obtain:

$$\text{ent}(X, g_{s_n}) \leq \log(N_e) + \log(C_{\lambda_n}) + \frac{2\lambda_n}{s_n}$$

Recall now that $C_{\lambda_n} = \mathcal{N}(T_{\lambda_n}) = \mathcal{N}\left(\frac{1}{s_n}\right)$. Let $\epsilon = \frac{1}{2}(1/m - v)$ so that $v + 2\epsilon = 1/m$. By Lemma 3.2(3) we have $\lambda_n \leq s_n^{1-(v+\epsilon)}$ for sufficiently large n . Thus, for such n , we have

$$\text{ent}(X, g_{s_n}) \leq \log(N_e) + \log \mathcal{N}\left(\frac{1}{s_n}\right) + \frac{2}{s_n^{v+\epsilon}}.$$

By Claim 3.4, there is a constant V such that $\text{vol}(X, g_{s_n}) \leq s_n V$. Therefore, the normalized volume of g_{s_n} is bounded by $s_n^{1/m} V^{1/m}$. Multiplying the above by this we get

$$\text{ent}(X, g_{s_n}) \text{vol}(X, g_{s_n})^{\frac{1}{m}} \leq s_n^{1/m} V^{1/m} \log(N_e) + s_n^{1/m} V^{1/m} \log \mathcal{N}\left(\frac{1}{s_n}\right) + 2s_n^\epsilon V^{1/m}.$$

Recall that $s_n \rightarrow 0$ as $n \rightarrow \infty$. Thus, the first and third terms on the right hand side clearly go to 0 as $n \rightarrow \infty$. Likewise, the middle term goes to 0 as $n \rightarrow \infty$ because $\mathcal{N}$ is a subexponential function with subexponential growth rate less than $1/m$. Thus, the left hand side must go to 0 as $n \rightarrow \infty$, so the sequence of metrics (X, g_{s_n}) shows that $\omega(X)$ equals 0. $\square$

Remark 3.5 The Gap Conjecture of Grigorchuk states that if a finitely generated group G satisfies $\nu(G) < 1/2$ then G has polynomial growth [17]. If this conjecture is true, then the proof of Theorem 3.3 only applies to the case when the fibers have polynomial growth.

As a consequence of Theorem 3.3, we get the following strengthening of Corollary 2.4.

Proposition 3.6 *Let G be a group of finite type. Then the following are equivalent:*

1. $\omega(X) = 0$ for some G -complex X with $\dim(X) = \text{gd}(G)$.
2. $\omega(X) = 0$ for every G -complex X .

Proof Suppose that $\omega(X) = 0$ where $\dim(X) = \text{gd}(G)$ and let Y be a G -complex. If $\dim(Y) = \dim(X)$, then $\omega(Y) = 0$ by Corollary 2.4. Else we have that $\dim(Y) > \dim(X)$. Let $f: Y \rightarrow X$ be a homotopy equivalence. As in Proposition 2.3, we may assume that f is simplicial. As f is a homotopy equivalence, every fiber $f^{-1}(p)$ has subexponential growth with subexponential growth rate 0. Indeed, the inclusion induced image of $\pi_1(f^{-1}(p))$ in $\pi_1(Y)$ is trivial. Therefore Y satisfies the FCA and $\omega(Y) = 0$ by Theorem 3.3. This shows that (1) implies (2).

The other implication is obvious. $\square$

3.2 Fiber π_1 -growth non-collapsing assumption

Next, we discuss the non-collapsing assumption.

Definition 3.7 (Babenko–Sabourau [1]) A simplicial complex X of dimension m satisfies the *fiber π_1 -growth non-collapsing assumption* (FNCA) if there exists a constant $\delta = \delta(X) > 0$ such that for every simplicial map $f: X \rightarrow P$ to a finite simplicial complex P of dimension at most $m - 1$, there exists $p \in P$ and a connected component $F_0 \subseteq f^{-1}(p)$ such that the inclusion induced image of $\pi_1(F_0)$ in $\pi_1(X)$ has uniform exponential growth at least δ .

Babenko–Sabourau prove that the FNCA is sufficient to ensure non-vanishing of the minimal volume entropy and moreover provide a positive lower bound in this case.

Theorem 3.8 (Babenko–Sabourau [1, Theorem 3.6]) *If X is a connected, finite simplicial complex with dimension m satisfying the FNCA, then $\omega(X) > 0$. More precisely,*

we have

$$\omega(X) \geq \frac{\delta}{2 \cdot C_m}$$

where $\delta = \delta(X)$ and $C_m > 0$ is a constant depending only on the dimension m .

Remark 3.9 As we are primarily concerned with 2-dimensional simplicial complexes, we note that according to Papasoglu one may take $C_2 = 10^6$ [27].

3.3 FCA and FNCA are complementary for groups with Property U

As pointed out by Babenko–Sabourau, the definitions of FCA and FNCA are not complementary. The subtlety lies in the subexponential growth rate in the definition of the FCA and the uniformity of the constant δ in the definition of the FNCA. If we assume that the fundamental group of the complex has Property U, then this issue disappears.

Lemma 3.10 *Let G be a group of finite type and suppose that G has Property U. Then any G -complex either satisfies the FCA or the FCNA.*

Proof Let G be as in the statement and let X be a G -complex.

Suppose that X does not satisfy the FCA. Hence, given any simplicial map $f: X \rightarrow P$ where P is a simplicial complex with $\dim(P) < \dim(X)$, there is some point $p \in P$ and a component $F_0 \subseteq f^{-1}(p)$ such that the inclusion induced image of $\pi_1(F_0)$ in $\pi_1(X)$ has exponential growth. As $G \cong \pi_1(X)$, we must have that the uniform growth rate of the inclusion induced image of $\pi_1(F_0)$ is at least $\underline{\delta}(G)$. Thus we see that X satisfies the FNCA for $\delta(X) = \underline{\delta}(G)$. $\square$

Combining Lemma 3.10 with Theorem 3.3, Proposition 3.6 and Theorem 3.8 we obtain the following dichotomy for any group of finite type with Property U.

Proposition 3.11 *Let G be a group of finite type with $m = \text{gd}(G)$ and suppose that G has Property U. Then either*

1. $\omega(X) = 0$ for every G -complex, or
2. $\omega(G) \geq \frac{\underline{\delta}(G)}{2 \cdot C_m}$.

Proof If some G -complex X with $\dim(X) = \text{gd}(G)$ satisfies the FCA, then $\omega(X) = 0$ by Theorem 3.3. Thus by Proposition 3.6, we get that $\omega(X) = 0$ for every G -complex X and thus (1) holds.

Else, by Lemma 3.10, every G -complex X with $\dim(X) = \text{gd}(G)$ satisfies the FNCA with $\delta(X) = \underline{\delta}(G)$. Thus $\omega(X) \geq \frac{\underline{\delta}(G)}{2 \cdot C_m}$ by Theorem 3.8. As X is an arbitrary G -complex with $\dim(X) = \text{gd}(G)$, we see that (2) holds. $\square$

4 Vanishing criterion when $\text{gd}(G) = 2$

In this section we prove the first main result of this paper. Theorem 1.3 provides a characterization of when $\omega(X) = 0$ for every G -complex X , provided that $\text{gd}(G) = 2$,

$\delta(G) > 0$, and the subexponentially growing subgroups of G belong to the collection $\{\{1\}, \mathbb{Z}, \mathbb{Z}^2, BS(1, -1)\}$. (In particular, such a G has Property U .) Specifically, the minimal volume entropy vanishes for each G -complex precisely when G is the fundamental group of a graph of groups where the edge groups belong to the collection $\{\{1\}, \mathbb{Z}\}$ and the vertex groups belong to the collection $\{\mathbb{Z}, \mathbb{Z}^2, BS(1, -1)\}$. Before we prove this theorem in Sect. 4.3, we recall the definition of a graph of groups and set some notation in Sect. 4.1. Then in Sect. 4.2, we show that if a G -complex of minimal dimension satisfies the FCA and $\text{gd}(G) = 2$, then G is the fundamental group of a graph of groups where the vertex groups and edge groups are subexponentially growing. We complete the proof of Theorem 1.3 in Sect. 4.3.

4.1 Graphs of groups

General references for the material in this section are the works of Bass [3], Scott–Wall [30], and Serre [31].

A *graph of groups* consists of the following data.

1. A finite connected graph Y with vertex set VY and edge set EY . By $o(e)$ and $\tau(e)$ we denote the originating and terminal vertices of an edge e respectively, and $\bar{e}$ denotes the edge with opposite orientation. We have $\bar{\bar{e}} = e$ and $o(\bar{e}) = \tau(e)$.
2. For each vertex $v \in VY$, there is an associated group G_v .
3. For each edge $e \in EY$, there is an associated group G_e . We have $G_{\bar{e}} = G_e$.
4. For each edge $e \in EY$, there is an injective homomorphism $h_e: G_e \rightarrow G_{o(e)}$.

We will denote a graph of groups by $\mathcal{G} = (Y, \{G_v\}, \{G_e\}, \{h_e\})$.

Associated to a graph of groups $\mathcal{G} = (Y, \{G_v\}, \{G_e\}, \{h_e\})$ is the *fundamental group of the graph of groups*, denoted $\pi_1(\mathcal{G})$. Briefly, it is constructed by repeatedly taking amalgamated free products and HNN-extensions using the data in $\mathcal{G}$. In more detail, as a generating set of $\pi_1(\mathcal{G})$ we take the set

$$\left(\bigcup_{v \in VY} G_v \right) \cup \{x_e \mid e \in EY\}$$

where each x_e is an abstract letter.

All of the relations in the vertex groups hold plus some more that use the data in $\mathcal{G}$. To write down these additional relations for $\pi_1(\mathcal{G})$, we need to fix a maximal tree $T \subseteq Y$. Using the tree T , the additional relations for $\pi_1(\mathcal{G})$ are as follows:

$$\begin{aligned} x_e h_e(a) &= h_{\bar{e}}(a) x_{\bar{e}}, \text{ for each edge } e \in EY \text{ and element } a \in G_e = G_{\bar{e}} \\ x_e x_{\bar{e}} &= 1, \text{ for each edge } e \in EY \\ x_e &= 1, \text{ for each edge } e \in ET \end{aligned}$$

The isomorphism type of $\pi_1(\mathcal{G})$ does not depend on the choice of maximal tree $T \subseteq Y$.

Consider two graphs of groups $\mathcal{G} = (Y, \{G_v\}, \{G_e\}, \{h_e\})$ and $\mathcal{G}' = (Y', \{G'_v\}, \{G'_e\}, \{h'_e\})$ where

1. Y' is a subgraph of Y ,
2. $G'_v = G_v$ for each vertex $v \in VY'$,
3. $G'_e = G_e$ for each edge $e \in EY'$, and
4. $h'_e = h_e$ for each edge $e \in EY'$.

Then $\pi_1(\mathcal{G}')$ is isomorphic to a subgroup of $\pi_1(\mathcal{G})$.

Given a graph of groups $\mathcal{G} = (Y, \{G_v\}, \{G_e\}, \{h_e\})$, there is an associated *graph of spaces* $\mathcal{X} = (Y, \{X_v\}, \{X_e\}, \{f_e\})$, which is well-defined up to homotopy. For each vertex $v \in VY$, we set $X_v = K(G_v, 1)$. Likewise, for each edge we set $X_e = K(G_e, 1)$ and further we fix a map $f_e: X_e \rightarrow X_v$ so that $(f_e)_* = h_e$. There is an associated space $|\mathcal{X}|$, called the *realization of the graph of spaces*, obtained by gluing the spaces together using the graph Y . Specifically, we define

$$|\mathcal{X}| = \left(\bigcup_{v \in VY} X_v \cup \bigcup_{e \in EY} X_e \times [0, 1] \right) / \begin{array}{l} (x, 0) \in X_e \times [0, 1] \sim f_e(x) \in X_{o(e)} \text{ and} \\ (x, 1) \in X_e \times [0, 1] \sim (x, 1) \in X_{\bar{e}} \times [0, 1] \end{array}$$

We have that $\pi_1(|\mathcal{X}|) \cong \pi_1(\mathcal{G})$.

Let $\mathcal{G} = (Y, \{G_v\}, \{G_e\}, \{h_e\})$ be a graph of groups. Suppose that e_0 is an edge in Y so that $o(e_0) \neq \tau(e_0)$, i.e., e_0 is not a loop. If the inclusion map $h_{e_0}: G_{e_0} \rightarrow G_{o(e_0)}$ is an isomorphism, then we say that the edge e_0 is *collapsible*. In this case, we may collapse e_0 and obtain a new graph of groups $\mathcal{G}' = (Y', \{G'_v\}, \{G'_e\}, \{h'_e\})$. The underlying graph Y' is obtained by removing the edge e_0 from Y and identifying the vertices $o(e_0)$ and $\tau(e_0)$; we denote this image of these vertices by v' and define $G'_{v'} = G_{\tau(e_0)}$. All other vertices and edges of Y' correspond to a vertex or edge of Y and we define the vertex group G'_v or edge group G'_e accordingly. As the map $h_{e_0}: G_{e_0} \rightarrow G_{o(e_0)}$ is an isomorphism, we can consider $G_{o(e_0)}$ as a subgroup of $G_{\tau(e_0)}$ via $h_{\bar{e}_0} h_{e_0}^{-1}$. Thus for an edge e in Y where $o(e) = o(e_0)$, the injective homomorphism $h_e: G_e \rightarrow G_{o(e)}$ naturally defines an injective homomorphism $h'_e: G'_e \rightarrow G'_{o(e)}$. For all other edges, we have that $h'_e = h_e$. We say that $\mathcal{G}'$ is obtained from $\mathcal{G}$ by *collapsing the edge* e . This does not change the fundamental group, i.e., $\pi_1(\mathcal{G}) \cong \pi_1(\mathcal{G}')$. This follows because of the isomorphism $A *_C C \cong A$.

If no edge of Y is collapsible, we say that $\mathcal{G}$ is *reduced*. If $\mathcal{G}$ is not reduced, we may repeatedly collapse edges to obtain a reduced graph of groups decomposition whose fundamental group is $\pi_1(\mathcal{G})$.

There is a correspondence between decompositions of G as a graphs of groups, i.e., isomorphisms $G \cong \pi_1(\mathcal{G})$, and actions of G on simplicial trees $\tilde{Y}$. In this correspondence, the vertex groups and edge groups of $\mathcal{G}$ correspond to the conjugacy classes of the vertex stabilizers and edge stabilizers respectively for the action of G . The underlying graph of $\mathcal{G}$ is $G \backslash \tilde{Y}$.

4.2 FCA induces a graph of groups decomposition

We will now show how the FCA induces a graph of groups decomposition when $\text{gd}(G) = 2$.

Proposition 4.1 *Suppose that G is a group with $\text{gd}(G) = 2$. If X is a G -complex with $\dim(X) = 2$ and X satisfies the FCA, then G is isomorphic to the fundamental group of a graph of groups where the vertex groups and the edge groups are subexponentially growing.*

Proof Let G be as in the statement and let X be a G -complex that satisfies the FCA. Hence, there is a graph Γ and a simplicial map $f: X \rightarrow \Gamma$ with connected fibers such that for each $x \in \Gamma$, the image of $\pi_1(F_x)$ in $\pi_1(X)$ has subexponential growth where $F_x = f^{-1}(x)$.

The function $f: X \rightarrow \Gamma$ induces a graph of groups decomposition of G as in the statement of the proposition as we now recall. For further details, we refer the reader to the work of Dunwoody [16]. For each edge e of Γ , fix a point x_e in the interior of the edge. Let $\tilde{X}$ be the universal cover of X and let $p: \tilde{X} \rightarrow X$ be the covering map. We consider the lift of the fibers F_{x_e} to $\tilde{X}$ and define the two subsets of $\tilde{X}$

$$\mathcal{E} = \cup\{p^{-1}(F_{x_e}) \mid e \text{ is an edge of } \Gamma\} \text{ and } \mathcal{V} = \tilde{X} - \mathcal{E}.$$

For each component $\varepsilon \in \pi_0(\mathcal{E})$, there are exactly two components of $\mathcal{V}$ whose closures in $\tilde{X}$ contain ε . Therefore, in the obvious way, we get a graph T with vertices $\pi_0(\mathcal{V})$ and edges $\pi_0(\mathcal{E})$. The graph T is clearly connected and as each $\varepsilon \in \pi_0(\mathcal{E})$ separates $\tilde{X}$, we see that T is a tree.

The stabilizer of a component of $\mathcal{E}$ is a conjugate of the image of $\pi_1(F_{x_e})$ in $\pi_1(X)$ for some edge e of Γ . Given a component $c \in \pi_0(\mathcal{V})$, the subset $p(c)$ deformation retracts onto F_v for some vertex v of Γ . Hence the stabilizer of a component of $\mathcal{V}$ is a conjugate of the image of $\pi_1(F_v)$ in $\pi_1(X)$ for some vertex v of Γ .

Therefore G acts on a tree where the stabilizer of any point is subexponentially growing. As stated in Sect. 4.1, by Bass–Serre theory this implies that G is isomorphic to the fundamental group of a graph of groups where the vertex groups and edge groups are subexponentially growing. $\square$

4.3 Proof of Theorem 1.3

Before can prove Theorem 1.3, we need a lemma that shows that certain subgroups are prohibited in groups with $\text{gd}(G) = 2$.

Lemma 4.2 *Suppose that H_1 , H_2 and K belong to the collection $\{\mathbb{Z}^2, BS(1, -1)\}$. The geometric dimension of an amalgamated free product $H_1 *_K H_2$ is equal to 3 if both inclusions are proper. The geometric dimension of an HNN-extension $H_1 *_K$ is equal to 3.*

Proof Let H_1 , H_2 and K be as in the statement. For each of $H_1 *_K H_2$ and $H_1 *_K$, the respective graphs of spaces using $S^1 \times S^1$ for each $\mathbb{Z}^2$ and the Klein bottle for each $BS(1, -1)$ have dimension equal to 3. Since $S^1 \times S^1$ and the Klein bottle are aspherical, the respective graphs of spaces are also aspherical [30, Proposition 3.6(ii)]. Thus $\text{gd}(H_1 *_K H_2) \leq 3$ and $\text{gd}(H_1 *_K) \leq 3$.

The cohomological dimension of $H_1 *_K H_2$ is equal to 3 if both inclusions are proper. Likewise, the cohomological dimension $H_1 *_K$ is equal to 3. See the work of

Bieri [6, Corollaries 6.5 and 6.7] for complete details. As the geometric dimension is bounded from below by the cohomological dimension, the result follows. $\square$

Proof of Theorem 1.3 Suppose that G is a group with $\text{gd}(G) = 2$, $\underline{\delta}(G) > 0$, and that every subexponentially growing subgroup belongs to the collection $\{\{1\}, \mathbb{Z}, \mathbb{Z}^2, BS(1, -1)\}$.

First, we assume that $\omega(X) = 0$ for every G -complex X . As $\underline{\delta}(G) > 0$, by Theorem 3.8 and Lemma 3.10 there is some G -complex X that satisfies the FCA. Therefore by Proposition 4.1 we have that G is isomorphic to the fundamental group of a graph of groups $\mathcal{G} = (Y, \{G_v\}, \{G_e\}, \{h_e\})$ with subexponentially growing vertex groups and edge groups. By collapsing any collapsible edges, we may assume that $\mathcal{G}$ is reduced.

If G_v is trivial for some vertex $v \in VY$, then every edge incident on v is a loop as $\mathcal{G}$ is reduced. This implies that $G \cong \pi_1(\mathcal{G})$ is a free group. This is contrary to the assumption that $\text{gd}(G) = 2$. Hence the vertex groups of $\mathcal{G}$ belong to the collection $\{\mathbb{Z}, \mathbb{Z}^2, BS(1, -1)\}$.

By Lemma 4.2, the groups $\mathbb{Z}^2$ and $BS(1, -1)$ cannot appear as edge groups since $\mathcal{G}$ is reduced and $\text{gd}(G) = 2$. Thus, the edge groups of $\mathcal{G}$ belong to the collection $\{\{1\}, \mathbb{Z}\}$.

Next, we assume that G is isomorphic to the fundamental group of a graph of groups $\mathcal{G} = (Y, \{G_v\}, \{G_e\}, \{h_e\})$ where vertex groups belong to the collection $\{\mathbb{Z}, \mathbb{Z}^2, BS(1, -1)\}$ and the edge groups belong to the collection $\{\{1\}, \mathbb{Z}\}$. Let $\mathcal{X} = (Y, \{X_v\}, \{X_e\}, \{f_e\})$ be the corresponding graph of spaces built using a point, S^1 , $S^1 \times S^1$ and the Klein bottle respectively for each $\{1\}$, $\mathbb{Z}$, $\mathbb{Z}^2$ and $BS(1, -1)$ respectively. Then $|\mathcal{X}|$ is a G -complex with $\dim(|\mathcal{X}|) = 2 = \text{gd}(G)$ and there is a map $p: |\mathcal{X}| \rightarrow Y$ where each of the fibers either a point, S^1 , $S^1 \times S^1$ or $BS(1, -1)$. This shows that $|\mathcal{X}|$ satisfies the FCA and hence $\omega(|\mathcal{X}|) = 0$ by Theorem 3.3. By Proposition 3.6, we conclude that $\omega(X) = 0$ for every G -complex X . $\square$

5 Free-by-cyclic groups

In this section we examine the minimal volume entropy of free-by-cyclic groups and prove Theorem 1.1. To prove this theorem we must show that the following three statements are equivalent for a free-by-cyclic group G_ϕ .

1. $\omega(X) = 0$ for every G_ϕ -complex X .
2. G_ϕ is virtually tubular.
3. Some power of ϕ is geometrically linear unipotent power.

First we prove that (1) implies (3) in Proposition 5.7. This takes place in Sect. 5.2 after we formally define a geometrically linear unipotent outer automorphism. Following this, in Sect. 5.3 we complete the proof of Theorem 1.1 by showing that (3) implies (2) and observing that (2) implies (1) by Theorem 1.3 and Proposition 2.5.

Before we begin the proof of Theorem 1.1, in Sect. 5.1 we classify subexponentially growing subgroups of free-by-cyclic groups and show that free-by-cyclic groups have uniform uniform exponential growth (with a uniform constant).

5.1 Growth of subgroups of free-by-cyclic groups

Let ϕ be an outer automorphism of a finitely generated free group F_n and let G_ϕ be the corresponding free-by-cyclic group.

Lemma 5.1 *Any nontrivial finitely generated subgroup of G_ϕ with subexponential growth is isomorphic to $\mathbb{Z}$, $\mathbb{Z}^2$ or $BS(1, -1)$.*

Proof Write $G_\phi = F_n \rtimes_\Phi \mathbb{Z}$ where $\Phi \in \text{Aut}(F_n)$ represents the outer automorphism ϕ and let π be the projection onto the cyclic factor. Let H be a nontrivial finitely generated subgroup of G_ϕ that has subexponential growth. As H is nontrivial and has subexponential growth, $H \cap F_n$ is either trivial or isomorphic to $\mathbb{Z}$.

If $H \cap F_n$ is trivial, then π maps H injectively to $\mathbb{Z}$ hence $H \cong \mathbb{Z}$.

Otherwise, we have that $H \cap F_n = \langle a \rangle$ for some nontrivial $a \in F_n$. If $\pi(H)$ is trivial then $H = \langle a \rangle \cong \mathbb{Z}$. Else, let h be an element of H that generates $\pi(H) \cong \mathbb{Z}$. Then $hah^{-1}, h^{-1}ah \in H \cap F_n = \langle a \rangle$ and so $hah^{-1} = a^k$ and $h^{-1}ah = a^\ell$ for some nonzero integers k, ℓ . Since $a = h^{-1}(hah^{-1})h = a^{k\ell}$ we have that k equals either 1 or -1 and hence $hah^{-1} = a$ or $hah^{-1} = a^{-1}$. As h generates $\pi(H)$, this implies that $H \cong \mathbb{Z}^2$ in the case $hah^{-1} = a$ and that $H \cong BS(1, -1)$ in the case $hah^{-1} = a^{-1}$. $\square$

Lemma 5.2 *Suppose H is a finitely generated subgroup of G_ϕ that is exponentially growing. Then $\delta(H) \geq \frac{1}{6} \log 3$. In particular, $\underline{\delta}(G_\phi) \geq \frac{1}{6} \log 3$.*

Proof Write $G_\phi = F_n \rtimes_\Phi \mathbb{Z}$ where $\Phi \in \text{Aut}(F_n)$ represents the outer automorphism ϕ .

Suppose H is a finitely generated subgroup of G_ϕ with exponential growth and let S be a finite generating set for H . The commutator subgroup $[H, H]$ is contained in F_n and is normally generated by $C = \{[s_i, s_j] \mid s_i, s_j \in S\}$. As H has exponential growth we must have that $[H, H]$ is nontrivial.

Suppose that $[H, H] = \langle c \rangle$ is infinite cyclic, generated by some element $c \in F_n$. If so, there is a short exact sequence

$$1 \rightarrow \langle c \rangle \rightarrow H \rightarrow A \rightarrow 1$$

where A is a finitely generated abelian group. In particular, an index two subgroup of H is nilpotent. This implies that H has subexponential growth, contrary to our hypothesis.

Therefore, $[H, H]$ is not cyclic and hence free and nonabelian. If elements $[s_i, s_j], [s_k, s_\ell] \in C$ do not commute, then they generate a group isomorphic to F_2 . Else, C is a subset of an infinite cyclic subgroup $\langle c \rangle \subset F_n$. We must have that $s_k[s_i, s_j]s_k^{-1} \notin \langle c \rangle$ for some $[s_i, s_j] \in C$ and $s_k \in S$. Indeed, otherwise every conjugate of an element in C by an element in H lies in $\langle c \rangle$ and since C normally generates $[H, H]$ this would imply that $[H, H]$ is abelian, which is a contradiction. Thus $[s_i, s_j]$ and $s_k[s_i, s_j]s_k^{-1}$ generate a group isomorphic to F_2 . To summarize, there are two elements of $[H, H]$ of length at most 6 that generate a group isomorphic to F_2 . Therefore $\delta(H, S) \geq \frac{1}{6} \log 3$.

As S was an arbitrary generating set for H , we have $\delta(H) \geq \frac{1}{6} \log 3$. $\square$

Lemmas 5.1 and 5.2 show that free-by-cyclic groups have Property U .

5.2 Geometrically linear unipotent outer automorphisms

In this section we will define a geometrically linear unipotent outer automorphism of a free group and prove that if $\omega(X) = 0$ for every G_ϕ -complex X , then some power of ϕ is geometrically linear unipotent.

Definition 5.3 A *primitive free splitting* of F_n is a graph of groups decomposition of F_n where each vertex group is $\mathbb{Z}$ and each edge group is trivial.

Given a primitive free splitting of F_n , $\mathcal{F} = (Y', \{F_v\}, \{F_e\}, \{i_e\})$, a model for the corresponding graph of spaces is obtained by attaching a loop edge α_v to each vertex $v \in VY'$. We will denote this graph by $K(\mathcal{F})$. By choosing a maximal tree $T \subseteq Y'$ (equivalently a maximal tree in $K(\mathcal{F})$, and fixing generators $a_v \in F_v$, a subset $E_+(Y' - T) \subset EY' - ET$ that contains one edge from each pair $\{e, \bar{e}\} \subseteq EY' - ET$ and a vertex $v_0 \in VY'$, the set

$$\{a_v \mid v \in VY'\} \cup \{x_e \mid e \in E_+(Y' - T)\}$$

is a basis for F_n via the isomorphism $F_n \cong \pi_1(K(\mathcal{F}), v_0)$.

Definition 5.4 An outer automorphism $\phi \in \text{Out}(F_n)$ is *geometrically linear unipotent* (GLU) if there is a representative $\Phi \in \text{Aut}(F_n)$, a primitive free splitting of F_n , $\mathcal{F} = (Y', \{F_v\}, \{F_e\}, \{i_e\})$ of F_n , a maximal tree $T \subseteq Y'$ and vertex $v_0 \in VY'$ such that the following holds.

1. For every $e \in ET$ where $o(e)$ lies between v_0 and $\tau(e)$, there is an integer p_e .
2. For each $v \in VY'$, $\Phi(a_v) = w_v a_v w_v^{-1}$ where $(e_1, e_2, \dots, e_m)$ is the minimal length edge path in T from v_0 to v and

$$w_v = a_{o(e_1)}^{p_{e_1}} a_{o(e_2)}^{p_{e_2}} \dots a_{o(e_m)}^{p_{e_m}}.$$

3. For each $e \in E_+(Y' - T)$, $\Phi(x_e) = w_{o(e)} a_{o(e)}^{q_e} x_e a_{\tau(e)}^{r_e} w_{\tau(e)}^{-1}$ for some $q_e, r_e \in \mathbb{Z}$.

Example 5.5 Consider the primitive free splitting of $F_3 = \langle a, b, c \rangle$, where the underlying graph has two vertices v_1, v_2 and two (geometric) edges e_1, e_2 where $o(e_1) = \tau(e_2) = v_1$ and $\tau(e_1) = o(e_2) = v_2$, and where the vertex groups are $F_{v_1} = \langle a \rangle$ and $F_{v_2} = \langle b \rangle$. Let T be the single edge e_1 . See Fig. 1 below.

Fix integers p, q , and r . The automorphism

$$\Phi(a) = a; \quad \Phi(b) = a^p b a^{-p}; \quad \Phi(c) = a^p b^q c a^r$$

represents a GLU outer automorphism.

Remark 5.6 From the definition, we see that GLU outer automorphisms are linearly growing. Moreover, there exists a topological representative $f: K(\mathcal{F}) \rightarrow K(\mathcal{F})$ for ϕ defined by the following.

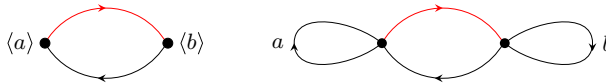


Fig. 1 The primitive free splitting of Example 5.5 is shown on the left, and its associated geometric realization on the right. The tree T is colored in red

1. For every $v \in VY'$, $f(\alpha_v) = \alpha_v$.
2. For every $e \in ET$ where $o(e)$ lies between v_0 and $\tau(e)$, $f(e) = \alpha_{o(e)}^{p_e} e$.
3. For every $e \in E_+(Y' - T)$, $f(e) = \alpha_{o(e)}^{q_e} e \alpha_{\tau(e)}^{r_e}$.

Conversely, any outer automorphism that has such a geometric representative is GLU.

Thus, each GLU automorphism has a geometric representative of a highly restrictive form. In the action of a GLU automorphism on the abelianization of F_n , it will be represented by a linearly growing unipotent matrix. Of course, not every automorphism with this property will be GLU. The terminology *geometrically linear unipotent* is meant to indicate that even on the level of homotopy the automorphism resembles a linear unipotent automorphism.

Proposition 5.7 *If $\omega(X) = 0$ for every G_ϕ -complex X , then some power of ϕ is geometrically linear unipotent.*

Proof Write $G_\phi = F_n \rtimes_\phi \mathbb{Z} = \langle F_n, t \mid tat^{-1} = \Phi(a), \forall a \in F_n \rangle$ where $\Phi \in \text{Aut}(F_n)$ represents the outer automorphism ϕ . Suppose that $\omega(X) = 0$ for every G_ϕ -complex X . We will show that the automorphism obtained by replacing Φ with a power and composing the result by an inner automorphism satisfies the conditions in Definition 5.4. On the level of the presentation, this is accomplished by replacing t with at^k for some $k \in \mathbb{Z}$ and $a \in F_n$.

As $\text{gd}(G_\phi) = 2$, Lemmas 5.1 and 5.2 show that we may apply Theorem 1.3 to G_ϕ . Since G_ϕ is 1-ended, Theorem 1.3 yields a graph of groups decomposition $\mathcal{G} = (Y, \{G_v\}, \{G_e\}, \{h_e\})$ of G_ϕ where all vertex groups belong to the collection $\{\mathbb{Z}, \mathbb{Z}^2, BS(1, -1)\}$ and every edge group is equal to $\mathbb{Z}$. We may assume that $\mathcal{G}$ is reduced. $\square$

Let $\tilde{Y}$ be the tree corresponding to the decomposition $G_\phi \cong \pi_1(\mathcal{G})$, so that $G_\phi \backslash \tilde{Y} = Y$. The normal subgroup $F_n \subset G_\phi$ acts on $\tilde{Y}$ and the quotient $Y' = F_n \backslash \tilde{Y}$ yields a graph of groups decomposition $\mathcal{F} = (Y', \{F_v\}, \{F_e\}, \{i_e\})$ of F_n . Let $\pi: Y' \rightarrow Y$ be the quotient map induced the action of $\langle t \rangle$. Following an observation of Brinkmann [7], we claim:

Claim 5.8 *The graph Y' is finite, each vertex group F_v belongs to the collection $\{\{1\}, \mathbb{Z}\}$ and each edge group F_e is trivial.*

Indeed, the quotient map π induces an injection on edge and vertex groups. In particular, as $\mathcal{F}$ is a decomposition of F_n and all vertex groups in $\mathcal{G}$ are either $\mathbb{Z}$, $\mathbb{Z}^2$ or $BS(1, -1)$, we must have that each vertex group of $\mathcal{F}$ is either $\mathbb{Z}$ or trivial.

For any edge e of Y , the map $\rho_e: Y \rightarrow Y_e$ that collapses the components of the complement of e induces a graph of groups decomposition $\mathcal{G}_e$ for G_ϕ with a single

edge where the edge group is $\mathbb{Z}$, i.e., a splitting over $\mathbb{Z}$. That is, there is a tree $\tilde{Y}_e$ and G_ϕ -equivariant map $\tilde{\rho}_e: \tilde{Y} \rightarrow \tilde{Y}_e$ with connected fibers inducing the map ρ_e on quotients where $G_\phi \backslash \tilde{Y}_e$ consists of a single edge. In particular, the edge group of e is the same in either graph of groups. Brinkmann showed that for any splitting of a free-by-cyclic group over $\mathbb{Z}$, the induced graph of groups decomposition on F_n , $\mathcal{F}_e = (Y'_e, \{F'_v\}, \{F'_e\}, \{i'_e\})$ has Y'_e a finite graph and $F'_e = \{1\}$ for each edge [7, Section 1]. Therefore for each e we have a pullback square

$$\begin{array}{ccc} Y' & \xrightarrow{\pi} & Y \\ \rho'_e \downarrow & & \downarrow \rho_e \\ Y'_e & \xrightarrow{\pi_e} & Y_e \end{array}$$

The map $\rho'_e: Y' \rightarrow Y'_e$ collapses the components of the complement of $\pi^{-1}(e)$. As above, the edge groups for $\mathcal{F}_e$ are the same as the corresponding edge groups in $\mathcal{F}$. Hence the edge group for an edge in $\pi^{-1}(e)$ is trivial. As e was arbitrary, this shows that the edges groups in $\mathcal{F}$ are all trivial. Further, as Y'_e is a finite graph, we see that $\pi^{-1}(e)$ consists of finitely many edge for any edge e of Y . This shows that Y' is a finite graph. This proves the claim. $\square$

Since Y' is finite, the stable letter t acts on Y' by a finite order automorphism. Thus some power of t , t^k , acts as the identity on Y' and also on each vertex group F_v since each such group has at most two automorphisms. We replace G_ϕ with the finite index subgroup G_{ϕ^k} . For the action of G_ϕ on $\tilde{Y}$ we now have that $G_\phi \backslash \tilde{Y} = Y' = F_n \backslash \tilde{Y}$. We continue to denote the graph of group decomposition of G_ϕ by $\mathcal{G}$.

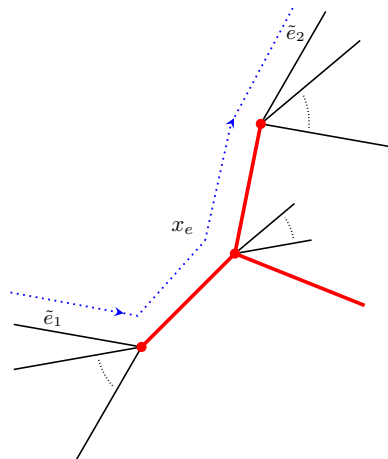
The graph of groups decomposition $\mathcal{F}$ may not be reduced. As t acts trivially on Y' , if an edge e is collapsible for $\mathcal{F}$, it is also collapsible for $\mathcal{G}$. This follows as the G_ϕ -stabilizer of an edge in $\tilde{Y}$ is generated by at for some $a \in F_n$ since t acts as the identity on Y' and such an element does not have a proper root. Therefore, we may collapse edges in Y' so that $\mathcal{F}$ is reduced. We will continue to denote by Y' the underlying graph.

If Y' has a single vertex v and F_v is trivial, then we collapse one of the incident loops and change the vertex group to $\mathbb{Z}$. Hence $F_v = \mathbb{Z}$ for all $v \in Y'$ and $F_e = \{1\}$ for all $e \in EY'$. In other words, $\mathcal{F}$ is a primitive free splitting.

Let $T \subseteq Y'$ be a maximal tree. Choose a basepoint $v_0 \in T$ and fix a subset $E_+(Y' - T) \subset EY' - ET$ that contains one edge from each pair $\{e, \bar{e}\}$. A lift $\tilde{T} \subseteq \tilde{Y}$ of T , determines for each $v \in \tilde{T}$ an element a_v that generates F_v . For each $e \in E_+(Y' - T)$ we obtain a hyperbolic element $x_e \in F_n$ that identifies an edge $\tilde{e}_1$ —which is a lift of e —at the lift of $o(e)$ in $\tilde{T}$, with an edge $\tilde{e}_2$ —which is also a lift of e —at the lift of $\tau(e)$ in $\tilde{T}$. See Fig. 2.

Since t acts trivially on Y' , for every point $x \in \tilde{Y}$, there exists $g \in F_n$ such that $(gt).x = x$. In particular, for each vertex $\tilde{v} \in \tilde{T}$, we can find $g_v \in F_n$ such that $(g_v t).\tilde{v} = \tilde{v}$. By the choice of t , this implies that $(g_v t)a_v(g_v t)^{-1} = a_v$ for all v , or in other words, that all vertex stabilizers are $\mathbb{Z}^2$. Moreover, replacing t with $g_{v_0}t$, we may assume the stabilizer of $\tilde{v}_0$ is $\langle a_0, t \rangle$.

Fig. 2 The lifts $\tilde{e}_1$ and $\tilde{e}_2$ of $e \in E_+(Y' - T)$ are identified by the hyperbolic translation x_e . The axis for the action of x_e is the dotted line in blue, while a portion of $\tilde{T}$ is the thickened tripod in red



Let $\gamma_v = (e_1, \dots, e_m)$ be the minimal length edge path in T from v_0 to v . We claim that there exists integers p_e such that for the word $w_v = a_{o(e_1)}^{p_{e_1}} a_{o(e_2)}^{p_{e_2}} \dots a_{o(e_m)}^{p_{e_m}}$ we have $t \cdot \tilde{v} = w_v \cdot \tilde{v}$. Note, the integers p_e only depend on the edges and not the vertex v . We prove this by induction on $m + 1 \geq 0$, where w_{v_0} is understood to be trivial. In the base case, we have already chosen t so that $t \cdot \tilde{v}_0 = \tilde{v}_0$. Suppose now that the claim holds for some $m \geq 0$, and that γ_v has length $m + 1$. Then removing the last edge of γ_v is the path from v to v_m , so there exist integers p_e such that for the word $w_{v_m} = a_{o(e_1)}^{p_{e_1}} a_{o(e_2)}^{p_{e_2}} \dots a_{o(e_{m-1})}^{p_{e_{m-1}}}$ we have $t \cdot \tilde{v}_m = w_{v_m} \cdot \tilde{v}_m$. Let $\tilde{e}$ be the lift of e_m , which is the final edge of γ_v , to $\tilde{T}$. Since $w_{v_m}^{-1} t$ fixes $\tilde{v}_m$, there is some power $a_{v_m}^{-p_e}$ such that $a_{v_m}^{-p_e} (w_{v_m}^{-1} t) \cdot \tilde{e} = \tilde{e}$. Thus, $t \cdot \tilde{e} = (w_{v_m} a_{v_m}^{p_e}) \cdot \tilde{e}$ and hence

$$t \cdot \tilde{v} = \left(w_{v_m} a_{o(e_m)}^{p_{e_m}} \right) \cdot \tilde{v} = \left(a_{o(e_1)}^{p_{e_1}} \dots a_{o(e_{m-1})}^{p_{e_{m-1}}} a_{o(e_m)}^{p_{e_m}} \right) \cdot \tilde{v},$$

as desired. It follows that for each v , $(w_v^{-1} t) a_v (t^{-1} w_v) = a_v$. Hence, $t a_v t^{-1} = w_v a_v w_v^{-1}$. This proves that the t -action on the basis elements a_v has the form indicated in Definition 5.4.

Consider now an edge $e \in E_+(Y' - T)$ and let $\tilde{v}_1$ and $\tilde{v}_2$ be the lifts to $\tilde{T}$ of the vertices $v_1 = o(e)$ and $v_2 = \tau(e)$ respectively. There are lifts $\tilde{e}_1$ and $\tilde{e}_2$ of e where $o(\tilde{e}_1) = \tilde{v}_1$ and $\tau(\tilde{e}_2) = \tilde{v}_2$ and such that $x_e \cdot \tilde{e}_1 = \tilde{e}_2$. Since $w_{v_1}^{-1} t$ fixes $\tilde{v}_1$, there exists a power $a_{v_1}^{-q_e}$ such that $(a_{v_1}^{-q_e} w_{v_1}^{-1} t) \cdot \tilde{e}_1 = \tilde{e}_1$. Similarly, there exists a power $a_{v_2}^{r_e}$ such that $(a_{v_2}^{r_e} w_{v_2}^{-1} t) \cdot \tilde{e}_2 = \tilde{e}_2$. Then we obtain

$$\left((t^{-1} w_{v_1} a_{v_1}^{q_e}) x_e (a_{v_2}^{r_e} w_{v_2}^{-1} t) \right) \cdot \tilde{e}_1 = x_e \cdot \tilde{e}_2$$

Since edge stabilizers for the F_n -action are trivial we have that x_e is the unique element of F_n taking $\tilde{e}_1$ to $\tilde{e}_2$. Hence, recalling that $v_1 = o(e)$ and $v_2 = \tau(e)$, we conclude that

$$tx_e t^{-1} = w_{o(e)} a_{o(e)}^{q_e} x_e a_{\tau(e)}^{r_e} w_{\tau(e)}^{-1}.$$

As explained in the beginning of the proof, this shows that some power of ϕ is geometrically linear unipotent. $\square$

5.3 Proof of Theorem 1.1

There are two items left to show for Theorem 1.1. We must show that if some power of an outer automorphism is geometrically linear unipotent (3), then G_ϕ is virtually tubular (2) and that if G_ϕ is virtually tubular (2), then $\omega(X) = 0$ for every G_ϕ -complex X (1). Additionally, we must show that if none of these conditions holds that $\omega(G_\phi)$ is bounded away from zero. This last statement follows immediately from Proposition 3.11 and Lemma 5.2.

Proof of Theorem 1.1 First we show that (3) implies (2). Replacing ϕ with a power replaces G_ϕ with a finite-index subgroup. We will therefore assume ϕ is itself GLU, and prove that in this case G_ϕ is tubular. Let $\mathcal{F}$, $T \subseteq Y'$, $v_0 \in T$ and Φ be as in Definition 5.4. These data give us a basis $\{a_v, x_e\}$, where $v \in VY'$ runs over the vertex set of Y' , and $e \in E_+(Y' - T)$ runs over the edges in the complement of the maximal tree T .

First we show that the subgroup $G_0 = \langle t, a_v \mid v \in VY' \rangle$ is tubular. For each a_v we have $ta_v t^{-1} = \Phi(a_v) = w_v a_v w_v^{-1}$. Rearranging we obtain

$$(w_v^{-1} t) a_v (t^{-1} w_v) a_v^{-1} = [w_v^{-1} t, a_v] = 1,$$

or in other words $\langle a_v, w_v^{-1} t \rangle \cong \mathbb{Z}^2$. Let $\gamma_v = e_1, \dots, e_m$ be the minimal length edge path from v_0 to v in T . The condition that ϕ is GLU states that

$$w_v = a_{o(e_1)}^{p_{e_1}} \dots a_{o(e_m)}^{p_{e_m}} = w_{v'} a_{v'}^{p_{e_m}},$$

for some collection of integers p_e depending only on the edge e , and where $v' = o(e_m)$. In particular, $w_v^{-1} t \in \langle a_{v'}, w_{v'}^{-1} t \rangle$. We then build a graph of groups decomposition with graph T , vertex groups $\langle a_v, w_v^{-1} t \rangle$ and edge groups $\mathbb{Z}$: the edge between v and v' identifies $\langle w_v^{-1} t \rangle$ in each. Thus G_0 is tubular.

To finish the proof, we show that adding the elements x_e for each $e \in E_+(Y' - T)$ exhibits G_ϕ as an iterated HNN-extension of G_0 over $\mathbb{Z}$ edge groups. Each extension is independent of the others, so that the result is tubular. For each element x_e , we have $tx_e t^{-1} = \Phi(x_e) = w_{o(e)} a_{o(e)}^{q_e} x_e a_{\tau(e)}^{r_e} w_{\tau(e)}^{-1}$, or written differently:

$$x_e \left(a_{\tau(e)}^{r_e} w_{\tau(e)}^{-1} t \right) x_e^{-1} = w_{o(e)}^{-1} a_{o(e)}^{-q_e} t$$

Clearly, $a_{\tau(e)}^{r_e} w_{\tau(e)}^{-1} t \in \langle a_{\tau(e)}, w_{\tau(e)}^{-1} t \rangle$ while $w_{o(e)}^{-1} a_{o(e)}^{-q_e} t \in \langle a_{o(e)}, w_{o(e)}^{-1} t \rangle$. We see that the addition of each edge $e \in E_+(Y' - T)$ adds a generator conjugating an infinite cyclic subgroups of the $\tau(e)$ vertex group to one of the $o(e)$ vertex group. Hence G_ϕ is tubular.

Next, we observe that (2) implies (1). If G_ϕ is virtually tubular, then by Lemma 2.2, after passing to a tubular finite index subgroup, we have that $\omega(X) = 0$ for every G_ϕ -complex by Theorem 1.3 as $\delta(G_\phi) > 0$ (Lemma 5.2).

Finally, by Proposition 3.11 and Lemma 5.2, if $\omega(X) \neq 0$ for some G_ϕ -complex X then

$$\omega(G_\phi) \geq \frac{\delta(G_\phi)}{2 \cdot 10^6} = \frac{\log 3}{12 \cdot 10^6}.$$

□

6 2-Dimensional right-angled Artin groups

In this section we consider 2-dimensional right-angled Artin groups and prove Theorem 1.2. This theorem asserts the equivalence of the following three conditions for a 2-dimensional right-angled Artin group A_Γ .

1. $\omega(X) = 0$ for every A_Γ -complex X .
2. A_Γ is a free product of tubular groups and a free group.
3. Γ is a forest.

The equivalence of (1) and (2) will follow from Theorem 1.3. Indeed, that (1) is a consequence of (2) follows immediately from Theorem 1.3 once we show that $\delta(H) \geq \log 3$ for a nonabelian subgroup of a right-angled Artin group (Lemma 6.2). For the converse, we will show how to modify the graph of groups decomposition ensured by Theorem 1.3 when $\omega(X) = 0$ for every A_Γ -complex X in Lemma 6.3 to get the requirements on the vertex groups. In Lemmas 6.4 and 6.5 we establish the equivalence of (2) and (3). Before we prove these lemmas, we recall a theorem of Baudisch regarding subgroups of a right-angled Artin group generated by two elements which is important for our analysis.

6.1 Baudisch's theorem and consequences

Baudisch proved that the subgroup generated by two elements in a right-angled Artin group is either free or abelian [4]. We record two consequences of this fact.

Lemma 6.1 *Groups in the following classes are not isomorphic to a subgroup of a right-angled Artin group:*

1. Baumslag–Solitar groups $BS(p, q) = \langle a, t \mid ta^pt^{-1} = a^q \rangle$ if $p, q \neq 1$,
2. torus knot groups $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} = \langle x, y \mid x^p = y^q \rangle$ if $p \neq 1$ or $q \neq 1$, and
3. amalgamated free products $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}^2 = \langle x, y, z \mid x^p = w(y, z), yz = zy \rangle$ where $w(y, z)$ is a nontrivial element of the subgroup $\langle y, z \rangle \cong \mathbb{Z}^2$ if $p \neq 1$.

Proof The first two classes are immediate since they are generated by two elements and are neither free nor abelian. For the third class of groups, notice that either $w(y, z) \neq y$ or $w(y, z) \neq z$. Hence one of the subgroups $\langle x, y \rangle$ or $\langle x, z \rangle$ is neither free nor abelian if $p \neq 1$. □

Another consequence of Baudisch's theorem is that right-angled Artin groups have uniform exponential growth (with a uniform constant).

Lemma 6.2 *Let A_Γ be a right-angled Artin group. Suppose H is a finitely generated nonabelian subgroup of A_Γ . Then $\delta(H) \geq \log 3$. In particular, if A_Γ is nonabelian, then $\delta(A_\Gamma) \geq \log 3$.*

Proof Suppose A_Γ is a right-angled Artin group and that H is a finitely generated nonabelian subgroup of A_Γ . Let S be a finite generating set for H . As H is nonabelian, there are two elements $x, y \in S$ that do not commute. Hence as x and y do not commute, they must generate a nonabelian free group by Baudisch's theorem and therefore $\delta(H, S) \geq \log 3$. $\square$

In particular, any subexponentially growing subgroup of a right-angled Artin group is abelian and hence right-angled Artin groups have Property U .

6.2 Proof of Theorem 1.2

We can now prove the lemmas showing the equivalences of the three items in Theorem 1.2.

Lemma 6.3 *Suppose that A_Γ is a right-angled Artin group with $\text{gd}(A_\Gamma) = 2$. If $\omega(X) = 0$ for every A_Γ -complex X , then A_Γ is a free product of tubular groups and a free group.*

Proof Let A_Γ be a right-angled Artin group where $\text{gd}(A_\Gamma) = 2$ and let X be an A_Γ -complex where $\omega(X) = 0$. By Theorem 1.3 we have that A_Γ is the fundamental group of a graph of groups $\mathcal{G} = (Y, \{G_v\}, \{G_e\}, \{h_e\})$ where the edge groups belong to the collection $\{\{1\}, \mathbb{Z}\}$ and the vertex groups belong to the collection $\{\mathbb{Z}, \mathbb{Z}^2, BS(1, -1)\}$. As $BS(1, -1)$ is not isomorphic to a subgroup of A_Γ (Lemma 6.1), we can in fact conclude that the vertex groups belong to the collection $\{\mathbb{Z}, \mathbb{Z}^2\}$.

As in the proof of Theorem 1.3, we may assume that $\mathcal{G}$ is reduced. We will show that by altering $\mathcal{G}$ that we can arrange that all of the vertex groups are $\mathbb{Z}^2$. This will complete the proof of the lemma.

To this end, suppose that there is a vertex v in Y with $G_v = \mathbb{Z}$. Further suppose that there is an e with $o(e) = \tau(e) = v$ and $G_e = \mathbb{Z}$. Then vertex v and edge e correspond to an HNN-extension $\mathbb{Z} *_{\mathbb{Z}} \cong BS(p, q)$. By Lemma 6.1, we must have $p = q = 1$. Hence, we can remove the edge e and replace the vertex group G_v with $\mathbb{Z}^2$.

Next suppose that for every edge e with $o(e) = v$ we have $G_e = \{1\}$. In this case, we can add a new edge e' that is a loop based at v with group $G_{e'} = \{1\}$ and replace the vertex group G_v with $\{1\}$. The resulting graph of groups is not reduced. Indeed, if it were then every edge incident on v is loop which implies that A_Γ is free. Thus, there is some edge incident to v that is collapsible. When we collapse this edge we obtain a reduced graph of groups decomposition for A_Γ which has fewer vertices with vertex group $\mathbb{Z}$ and has not created a vertex with vertex group $\{1\}$.

Finally, we deal with the case that there are no loops at v with edge group equal to $\mathbb{Z}$ and that there is an edge e with $o(e) = v$ such that $G_e = \mathbb{Z}$. Then the subgroup

$G_{o(e)} *_{G_e} G_{\tau(e)}$ is either a torus knot group $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ or an amalgamated free product $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}^2$. By Lemma 6.1, we must have that $h_e: G_e \rightarrow G_{o(e)}$ or $h_{\bar{e}}: G_e \rightarrow G_{\tau(e)}$ is surjective, contradicting the assumption that the graph of groups decomposition is reduced. $\square$

Lemma 6.4 *Suppose that A_Γ is a right-angled Artin group with $\text{gd}(A_\Gamma) = 2$. If A_Γ is a free product of tubular groups and a free group, then Γ is a forest.*

Proof Let A_Γ be a 2-dimensional right-angled Artin group and suppose that $A_\Gamma \cong G_1 * \dots * G_k * F_n$ where $G_1, \dots, G_k$ are tubular groups and F_n is a free group. As tubular groups are one-ended, this decomposition must be the Grushko decomposition and thus we must have that $G_i \cong A_{\Gamma_i}$ for $i = 1, \dots, k$ where $\Gamma_1, \dots, \Gamma_k$ are the connected components of Γ that have at least two vertices. Therefore, it suffices to show that if Γ is connected and A_Γ is tubular, then Γ is a tree.

To this end, suppose that we have a graph of groups decomposition of A_Γ where each edge group is $\mathbb{Z}$ and each vertex group is $\mathbb{Z}^2$. We will use the calculation of the JSJ-decomposition of a one-ended right-angled Artin group over cyclic subgroups by the second author [11, Section 3]. The vertex groups of the JSJ-decomposition of A_Γ correspond to the biconnected components of Γ . Specifically, vertex groups are of the form A_{Γ_0} where $\Gamma_0 \subseteq \Gamma$ is a biconnected component. The most important property of this decomposition is that the vertex groups are subgroups of conjugates of the vertex groups for any graph of groups decomposition of A_Γ in which the edge groups are each equal to $\mathbb{Z}$ [11, Lemma 3.3]. Hence $A_{\Gamma_0} = \mathbb{Z}^2$ for each biconnected component $\Gamma_0 \subseteq \Gamma$ and thus every biconnected component is a single edge. Therefore every edge of Γ is separating and hence Γ is a tree. $\square$

Lemma 6.5 *Suppose that A_Γ is a right-angled Artin group with $\text{gd}(A_\Gamma) = 2$. If Γ is a forest, then A_Γ is the free product of tubular groups and a free group.*

Proof As in the proof of Lemma 6.4, it suffices to show that if Γ is a nontrivial tree, then A_Γ is tubular. We do so by showing that A_Γ is the fundamental group of a graph of groups $\mathcal{G} = (Y, \{G_v\}, \{G_e\}, \{h_e\})$ where all of edges groups are equal to $\mathbb{Z}$ and all of the vertex groups are equal to $\mathbb{Z}^2$.

Let $\hat{Y}$ be the graph obtained by subdivision of Γ . We will first construct a graph of groups $\hat{\mathcal{G}}$ with underlying graph $\hat{Y}$ so that $\pi_1(\hat{\mathcal{G}}) = A_\Gamma$. As $\hat{Y}$ is the subdivision of Γ , we can consider $V\Gamma$ as a subset of $V\hat{Y}$. For each vertex $v \in V\Gamma$ we set $G_v = \mathbb{Z}$ and for each vertex $v \in V\hat{Y} - V\Gamma$ we set $G_v = \mathbb{Z}^2$. In the latter case, there are exactly two edges $e, e' \in E\hat{Y}$ with $o(e) = o(e') = v$. Decompose G_v into the direct sum of two copies of $\mathbb{Z}$ denoted $\mathbb{Z}_e$ and $\mathbb{Z}_{e'}$ respectively so that $G_v = \mathbb{Z}_e \oplus \mathbb{Z}_{e'}$. For each edge $e \in E\hat{Y}$ we set $G_e = \mathbb{Z}$. The inclusion maps $h_e: G_e \rightarrow G_{o(e)}$ are defined as follows. If $o(e)$ lies in $V\Gamma$, then $G_{o(e)} = \mathbb{Z}$ and we define $h_e: G_e \rightarrow G_{o(e)}$ to be an isomorphism. Else we have that $G_{o(e)} = \mathbb{Z}^2 = \mathbb{Z}_e \oplus \mathbb{Z}_{e'}$ and we define $h_E: G_e \rightarrow G_{o(e)}$ to have $\mathbb{Z}_e$ as image. We set $\hat{\mathcal{G}}$ to be the graph of groups $(\hat{Y}, \{G_v\}, \{G_e\}, \{h_e\})$.

As Γ is a tree, the presentation for $\pi_1(\hat{\mathcal{G}})$ shows that $\pi_1(\hat{\mathcal{G}}) \cong A_\Gamma$.

Notice that $\hat{\mathcal{G}}$ is not reduced. Indeed, for every vertex in $V\Gamma$, there is an incident edge that is not a loop and for which the inclusion map is an isomorphism. For each vertex in $V\Gamma$, we fix one such edge and perform the collapse. Let $\mathcal{G} = (Y, \{G_v\}, \{G_e\}, \{h_e\})$

be the resulting graph of groups. We then observe that the vertex set VY corresponds to the set $V\widehat{Y} - V\Gamma$ and hence $G_v = \mathbb{Z}^2$ for each vertex $v \in VY$. The edge groups do not change and hence $G_e = \mathbb{Z}$ for each edge $e \in EY$. As $\pi_1(\mathcal{G}) = \pi_1(\widehat{\mathcal{G}}) \cong A_\Gamma$, we have shown that A_Γ is tubular. $\square$

Proof of Theorem 1.2 Let A_Γ be a right-angled Artin group where $\text{gd}(A_\Gamma) = 2$.

The equivalences of items (1) and (2) follow from Theorem 1.3 and Lemmas 6.2 and 6.3. The equivalences of items (2) and (3) follow immediately from Lemmas 6.4 and 6.5.

Finally, by Proposition 3.11 and Lemma 6.2, if $\omega(X) \neq 0$ for some A_Γ -complex X then

$$\omega(A_\Gamma) \geq \frac{\delta(A_\Gamma)}{2 \cdot 10^6} = \frac{\log 3}{2 \cdot 10^6}.$$

$\square$

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References

1. Babenko, I., Sabourau, S.: Minimal volume entropy and fiber growth (2020). [arXiv:2102.04551](https://arxiv.org/abs/2102.04551)
2. Babenko, I.K.: Asymptotic invariants of smooth manifolds. *Izv. Ross. Akad. Nauk Ser. Mat.* **56**(4), 707–751 (1992)
3. Bass, H.: Covering theory for graphs of groups. *J. Pure Appl. Algebra* **89**(1–2), 3–47 (1993)
4. Baudisch, A.: Subgroups of semifree groups. *Acta Math. Acad. Sci. Hung.* **38**(1–4), 19–28 (1981)
5. Besson, G., Courtois, G., Gallot, S.: Volume et entropie minimale des espaces localement symétriques. *Invent. Math.* **103**(2), 417–445 (1991)
6. Bieri, R.: Homological Dimension of Discrete Groups. Mathematics Department, Queen Mary College, Queen Mary College Mathematics Notes, London (1976)
7. Brinkmann, P.: Hyperbolic automorphisms of free groups. *Geom. Funct. Anal.* **10**(5), 1071–1089 (2000)
8. Brunnbauer, M.: Homological invariance for asymptotic invariants and systolic inequalities. *Geom. Funct. Anal.* **18**(4), 1087–1117 (2008)
9. Button, J.: Tubular free by cyclic groups act freely on CAT(0) cube complexes. *Can. Math. Bull.* **60**(1), 54–62 (2017)
10. Charney, R., Davis, M.W.: Finite $K(\pi, 1)$ s for Artin groups. In: *Prospects in Topology*, vol. 138 of *Ann. of Math. Stud.*, vol. 1995, pp. 110–124. Princeton University Press, Princeton (1994)
11. Clay, M.: When does a right-angled Artin group split over $\mathbb{Z}$? *Int. J. Algebra Comput.* **24**(6), 815–825 (2014)
12. Clay, M.: ℓ^2 -torsion of free-by-cyclic groups. *Q. J. Math.* **68**(2), 617–634 (2017)
13. Degrijse, D.: Amenable groups of finite cohomological dimension and the zero divisor conjecture (2020). [arXiv:1609.07635](https://arxiv.org/abs/1609.07635)
14. Dinaburg, E.I.: A connection between various entropy characterizations of dynamical systems. *Izv. Akad. Nauk SSSR Ser. Mat.* **35**, 324–366 (1971)
15. Droms, C.: Graph groups, coherence, and three-manifolds. *J. Algebra* **106**(2), 484–489 (1987)
16. Dunwoody, M.J.: The accessibility of finitely presented groups. *Invent. Math.* **81**(3), 449–457 (1985)

17. Grigorchuk, R.: Milnor's problem on the growth of groups and its consequences. In: *Frontiers in Complex Dynamics*, vol. 51 of Princeton Math. Ser., pp. 705–773. Princeton University Press, Princeton (2014)
18. Gromov, M.: Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.* **56**(1982), 5–99 (1983)
19. Gupta, R., Jankiewicz, K., Ng, T.: Groups acting on CAT(0) cube complexes with uniform exponential growth (2020). [arXiv:2006.03547](https://arxiv.org/abs/2006.03547)
20. Kapovich, I., Nagnibeda, T.: The Patterson-Sullivan embedding and minimal volume entropy for outer space. *Geom. Funct. Anal.* **17**(4), 1201–1236 (2007)
21. Katok, A.: Entropy and closed geodesics. *Ergodic Theory Dyn. Syst.* **2**, 3–4 (1982) [**339–365 (1983)**]
22. Lim, S.: Minimal volume entropy for graphs. *Trans. Am. Math. Soc.* **360**(10), 5089–5100 (2008)
23. Lück, W., Schick, T.: L^2 -torsion of hyperbolic manifolds of finite volume. *Geom. Funct. Anal.* **9**(3), 518–567 (1999)
24. Manning, A.: Topological entropy for geodesic flows. *Ann. Math. (2)* **110**(3), 567–573 (1979)
25. McMullen, C.T.: Entropy and the clique polynomial. *J. Topol.* **8**(1), 184–212 (2015)
26. Milnor, J.: A note on curvature and fundamental group. *J. Differ. Geom.* **2**, 1–7 (1968)
27. Papasoglu, P.: Urysohn width and volume. *Geom. Funct. Anal.* **30**(2), 574–587 (2020). <https://doi.org/10.1007/s00039-020-00533-5>
28. Pieroni, E.: Minimal volume entropy of 3-manifolds (2020). [arXiv:1902.09190](https://arxiv.org/abs/1902.09190)
29. Sabourau, S.: Systolic volume and minimal entropy of aspherical manifolds. *J. Differ. Geom.* **74**(1), 155–176 (2006)
30. Scott, P., Wall, T.T.: Topological methods in group theory. In: *Homological Group Theory (Proc. Sympos., Durham, 1977)*: vol. 36 of London Math. Soc. Lecture Note Ser., pp. 137–203. Cambridge University Press, Cambridge (1979)
31. Serre, J.-P.: *Trees*. Springer Monographs in Mathematics. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation. Springer, Berlin (2003)
32. Soma, T.: The Gromov invariant of links. *Invent. Math.* **64**(3), 445–454 (1981)
33. Thurston, W.P.: *The Geometry and Topology of 3-manifolds*. Lecture Notes. Princeton (1978)
34. Thurston, W. P.: Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Am. Math. Soc. (N. S.)* **6**(3), 357–381 (1982)
35. Švarc, A.S.: A volume invariant of coverings. *Dokl. Akad. Nauk SSSR (N. S.)* **105**, 32–34 (1955)
36. Wilson, J.S.: On exponential growth and uniformly exponential growth for groups. *Invent. Math.* **155**(2), 287–303 (2004)