2020 59th IEEE Conference on Decision and Control (CDC)
Jeju Island, Republic of Korea, December 14-18, 2020

Non-asymptotic Closed-Loop System Identification using Autoregressive Processes and Hankel Model Reduction

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Abstract— **One of the primary challenges of system identification is determining how much data is necessary to adequately fit a model. Non-asymptotic characterizations of the performance of system identification methods provide this knowledge. Such characterizations are available for several algorithms performing open-loop identification. Often times, however, data is collected in closed-loop. Application of open-loop identification methods to closed-loop data can result in biased estimates. One method to eliminate these biases involves first fitting a long-horizon autoregressive model and then performing model reduction. The asymptotic behavior of such algorithms is well characterized, but the non-asymptotic behavior is not. This work provides a non-asymptotic characterization of one particular variant of these algorithms. More specifically, we provide non-asymptotic upper bounds on the generalization error of the produced model, as well as high probability bounds on the difference between the produced model and the finite horizon Kalman Filter.**

I. Introduction

One of the first steps in the control design process is to obtain a model for the system of interest. In cases where knowledge of the system is nonexistent or incomplete, models must be identified from input/output data. This process can be viewed as a learning problem in which models are optimized in order to give the best fit for the data [1]. The quality of the model can be assessed via 1) generalization error, which measures how well the model fits unseen data, and 2) model error, which measures how far the identified model is from the "true" model. (In many cases, analysis of model error is an idealization, since the real system falls out side the class of models studied.)

While system identification can be viewed as a learning problem, correlations in the data lead to several challenges. Typical machine learning problems assume that the data are independent [2]. Using independence, learning theory provides *non-asymptotic* bounds on the generalization error obtained from finite amounts of data. In contrast, the data from system identification are correlated due to 1) internal system dynamics, 2) temporal correlations in the inputs, and 3) feedback from the outputs to the inputs. The result is that most traditional analyses of system identification methods focus on *asymptotic* bounds, which can only guarantee low generalization error in the limit of infinite data [1]. There have, however, been recent efforts to provide non-asymptotic bounds on the performance of system identification methods.

This work was supported in part by NSF CMMI-1727096

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Most work on non-asymptotic system identification focuses on open-loop problems. Early works give non-asymptotic analyses for the identification of transfer functions [3] and autoregressive models [4]. Recently, several works have provided non-asymptotic analyses of various open-loop system identification problems for linear time invariant systems with direct state measurement, [5], [6], and with partial state measurement [7], [8]. Each of these works assumes that the engineer may choose inputs to excite the system. In [9], bounds are derived for the identification of linear time invariant systems with no measured inputs.

The work in [10], [11] provides a non-asymptotic method for output error identification of linear models. Unlike the works mentioned above, the data could be collected in closed-loop. However, these works utilize the non-probabilistic framework of online optimization [12], [13], and are not directly comparable to the work on generalization bounds. Additionally, the models identified in these works are restricted to stable systems.

As discussed above, the recent works on non-asymptotic identification have focused on open-loop identification methods. However, for many systems, the plant is impossible to isolate from its controller or is unstable in open-loop. Furthermore, identification is most successful when performed in circumstances that match the desired application, which often includes a feedback controller [14]. This drives the study of methods that are effective with closed-loop data.

The task of developing identification methods that work on closed-loop data is nontrivial, as the correlation between past output noise and future inputs produces a bias in model estimates for many identification methods. This is particularly troublesome for subspace approaches [15]. In [16], it is demonstrated how subspace algorithms may be applied to closed-loop data by fitting high order vector autoregressive models with exogenous inputs (VARX models). The work of [17] proposed a subspace technique which used the VARX parameter estimates to recover the Kalman Filter. This helped to develop algorithms such as the well known predictor based subspace identification (PBSID) algorithm [18]. For summaries on the advancements of subspace approaches for closed-loop identification, see [19] and [20].

Our contribution is to analyze an algorithm for system identification in which a VARX model is fit, followed by balanced model reduction. This approach has been described in [20], [21], and it was shown that its asymptotic properties match those of a familiar subspace method. The primary difference of our analysis from prior non-asymptotic system identification characterizations is that we allow the presence

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of a feedback controller.

We note that the bound we obtain for the generalization error requires some prior knowledge. In particular, we require an upper bound on the of several system parameters such as the $\mathcal{H}_\infty$ norm of the closed loop system. In practice, we would need to estimate these from data. We also note that the generalization error bound obtained is loose when tested in simulation (see [22]). Thus, the result is more interesting from a theoretical perspective in that it quantifies how various parameters of the identification algorithm and the system impact the predictive capabilities of the model.

The paper is organized as follows. In Section II, we present the algorithm, precisely define the problem, and provide the main result: a non-asymptotic bound on the generalization error of the produced model. The proof of this result is available in Section III. Proofs for several supporting lemmas are omitted. They may be found in the full report [22]. Section IV presents a related result regarding the high probability bounds on the $\mathcal{H}_\infty$ norm of the error system from the identified model to the finite horizon Kalman Filter, and highlights several practical considerations of the bounds.

II. Problem and Results

We now describe the problem, and present the generalization error bound obtained. Subsection II-A summarizes the notation used throughout the paper. In Subsection II-B, we highlight the details and assumptions of the closed-loop system. The algorithm to be analyzed is presented in II-C, along with the main result: a non-asymptotic bound on the generalization error of the obtained model.

A. Notation and Terminology

Random variables are denoted using bold symbols. The expected value of a random variable, $\mathbf{x}$, is denoted by $\mathbb{E}[\mathbf{x}]$, while the probability of an event S is given by $\mathbb{P}(S)$.

The Euclidean norm of a vector, x, is denoted by $\|x\|$. The Frobenius norm of a matrix, G, is denoted by $\|G\|_F$, while its induced 2 norm is denoted by $\|G\|$. The minimal eigenvalue of a symmetric matrix, X, is denoted by $\lambda_{\min}(X)$. The trace of a square matrix G is given by $\mathrm{Tr}(G)$.

The *power* of a stationary process, $\mathbf{y}_t$, is defined by $\|\mathbf{y}\|_{\mathcal{P}}^2 = \mathbb{E}[\mathbf{y}_t^\top \mathbf{y}_t]$.

The forward shift operator is denoted by q, i.e. $q\mathbf{x}_t = \mathbf{x}_{t+1}$. If $G(q)$ is a time-domain operator defined in terms of shifts, we will identify it with its corresponding transfer matrix, $G(z)$. The $\mathcal{H}_\infty$ norm of a transfer matrix, $G(z)$, is denoted by $\|G\|_\infty$. The notation $\mathbf{x}_{i:j}$ represents the sequence starting from $\mathbf{x}_i$ and up to, but not including $\mathbf{x}_j$.

B. Problem Setup

Consider a linear time-invariant (LTI) system of the form

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + B\mathbf{u}_t + B_w\mathbf{w}_t \tag{1a}$$
$$\mathbf{y}_t = C\mathbf{x}_t + D_w\mathbf{w}_t, \tag{1b}$$

Where $\mathbf{x}_t$ is the n_x dimensional state, $\mathbf{u}_t$ is the n_u dimensional known input, $\mathbf{w}_t$ is Gaussian white noise, and $\mathbf{y}_t$ is the n_y dimensional measurement.

When the steady-state Kalman filter exists, the statistics of (1) are equivalent the statistics of the following system in *innovations form*:

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + B\mathbf{u}_t + K\mathbf{e}_t \tag{2a}$$
$$\mathbf{y}_t = C\mathbf{x}_t + \mathbf{e}_t. \tag{2b}$$

Here K is the Kalman gain and $\mathbf{e}_t$ is Gaussian white noise with dimension n_y. For details about the Kalman filter and innovation form, see section 3.3 of [23].

For compact notation, we set $\mathbf{z}_t = \begin{bmatrix}\mathbf{u}_t^\top & \mathbf{y}_t^\top\end{bmatrix}^\top \in \mathbb{R}^{n_z}$. For an integer p, we also define $\mathbf{d}_t = \mathbf{z}_{t-p:t}$. For later analysis, we have assumed that the system is strictly proper in the known inputs, $\mathbf{u}_t$.

We will assume that $\mathbf{u}_t$ can be represented as a linear feedback with excitatory noise:

$$\mathbf{s}_{t+1} = A^F\mathbf{s}_t + B_1^F\mathbf{y}_t + B_2^F\mathbf{v}_t \tag{3a}$$
$$\mathbf{u}_t = C^F\mathbf{s}_t + D_1^F\mathbf{y}_t + D_2^F\mathbf{v}_t. \tag{3b}$$

Here $\mathbf{v}_t$ is identity covariance Gaussian white noise with dimension n_u which is independent of $\mathbf{e}_t$, and $\mathbf{s}_t$ is the state of the controller with dimension n_s. A summary of the system is shown in Fig. 1.

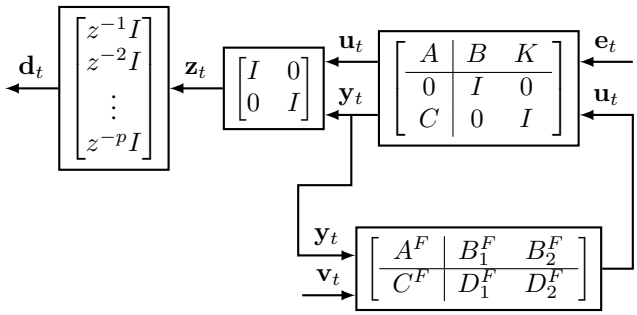

Fig. 1. The overall system.

The closed-loop system is assumed to be stable. This implies that the signal power, $\|\mathbf{z}\|_{\mathcal{P}}$, is finite. Additionally, we will assume that the joint covariance of the noise is positive definite:

$$\mathbb{E}\left[\begin{bmatrix}\mathbf{e}_t \\ D_2^F\mathbf{v}_t\end{bmatrix}\begin{bmatrix}\mathbf{e}_t \\ D_2^F\mathbf{v}_t\end{bmatrix}^\top\right] = \begin{bmatrix}\Psi & 0 \\ 0 & \Omega\end{bmatrix} = \Gamma \succ 0.$$

This ensures that identifiability conditions hold, as in traditional system identification [1]. Note that we do not assume that the open-loop system is stable.

The finite horizon Kalman Filter represents the output estimate for (2) provided the p previous time steps as $\mathbf{y}^\star_{t|t-p:t} = \mathbb{E}[\mathbf{y}_t|\mathbf{d}_t]$, where the notation follows that mentioned previously; the sequence $x_{i:j}$ does not include x_j. This indicates that the finite horizon Kalman Filter estimate depends only upon data collected at times k with $t-p \le k < t$. The estimate is a linear function of $\mathbf{d}_t$. We define G_{OPT} as the linear transformation relating the two:

$$\mathbf{y}^\star_{t|t-p:t} = G_{OPT}\mathbf{d}_t. \tag{4}$$

We also define the operator $H^{OPT}(q)$ such that $\mathbf{y}^\star_{t|t-p:t} = H^{OPT}(q)\mathbf{z}_t$. The steady state Kalman Filter operator $H^\star(q)$

is given as $\mathbf{y}^\star_{t|-\infty:t} = H^\star(q)\mathbf{z}_t$. Due to the special form of the innovations model, we can write the steady state Kalman Filter as

$$\begin{aligned}\mathbf{x}_{t+1} &= (A-KC)\mathbf{x}_t + B\mathbf{u}_t + K\mathbf{y}_t\\ \mathbf{y}^\star_{t|-\infty:t} &= C\mathbf{x}_t,\end{aligned}$$

and the associated expected squared error, $\mathbb{E}[\|\mathbf{y}_t - \mathbf{y}^\star_{t|-\infty:t}\|^2]$, is $\|\mathbf{e}\|^2_{\mathcal{P}}$.

C. The REDAR Algorithm and its Prediction Error

The method of this paper is termed the REDAR (pronounced "reader") algorithm. See Alg. 1. Here $\mathbf{H}^A$ is the system corresponding to the least-squares model, while $\mathbf{H}^R$ is the result of balanced model reduction subject to $\mathcal{H}_\infty$ error tolerance ϕ. See Chapter 7 of [24] for a description balanced reduction with limited error tolerance. Our final predictor is given by $\hat{\mathbf{y}}_t = \mathbf{H}^R(q)\mathbf{z}_t$. Additionally, given the state-space realization of $\mathbf{H}^R$, all of the parameters of the innovation form model, (2), can be estimated.

Algorithm 1 The REDuced AutoRegressive (REDAR) algorithm

1: Given signals $\mathbf{u}_{1-p:T+1}$, $\mathbf{y}_{1-p:T+1}$, VARX order $p > 0$, a regularization paramater $\alpha > 0$, and a reduction error $\phi > 0$
2: Let $\mathbf{z}_t = \begin{bmatrix}\mathbf{u}_t^\top & \mathbf{y}_t^\top\end{bmatrix}^\top$ and $\mathbf{d}_t = \mathbf{z}_{t-p:t}$
3: Solve the VARX identification problem $\mathbf{G}_T = \operatorname{argmin}_G \sum_{t=1}^T \|\mathbf{y}_t - G\mathbf{d}_t\|^2 + \alpha\|G\|_F^2$ using least squares
4: Construct a state-space operator, $\mathbf{H}^A$, such that $\mathbf{H}^A(q)\mathbf{z}_t = \mathbf{G}_T\mathbf{d}_t$
5: Apply balanced model reduction to find $\mathbf{H}^R$ such that $\|\mathbf{H}^A - \mathbf{H}^R\|_\infty \le \phi$
6: Compute estimates $(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}}, \hat{\mathbf{K}})$ by
$$\left[\begin{array}{c|cc}\hat{\mathbf{A}} - \hat{\mathbf{K}}\hat{\mathbf{C}} & \hat{\mathbf{B}} & \hat{\mathbf{K}}\\ \hline \hat{\mathbf{C}} & \hat{\mathbf{D}} & 0\end{array}\right] = \mathbf{H}^R$$

The general scheme of the REDAR algorithm has been proposed in closed-loop system identification literature [20], [21]. However, its finite-sample behavior has not been characterized. Our main result gives such a characterization:

Theorem 1. *Suppose there exists $L > 0$ and $\rho < 1$ such that for all $|z| \ge \rho$, $\|H^\star(z)\| \le L$. Then there are constants k and T_0 depending upon n_u, n_y, p, α, the $\mathcal{H}_\infty$ norm of the closed loop system, and $\lambda_{min}(\Gamma)$ such that for all $T \ge T_0$,*

$$\begin{aligned}&\mathbb{E}[\|\mathbf{y}_t - \hat{\mathbf{y}}_t\|^2]\\ &\quad\le \|\mathbf{e}\|^2_{\mathcal{P}} + \left(\frac{L\rho^{p+1}}{1-\rho}\right)^2 \|\mathbf{z}\|^2_{\mathcal{P}} + 2\phi\|\mathbf{z}\|^2_{\mathcal{P}} + \frac{2kp}{\sqrt{T}}\|\mathbf{z}\|^2_{\mathcal{P}}.\end{aligned}$$

where $\mathbf{y}_t$ is the output generated by running the process (2) *in closed loop, and $\hat{\mathbf{y}}_t$ is the one step ahead prediction generated by the model output of Algorithm 1.*

It should be noted that the identification algorithm is only run on a single data set consisting of $u_{1-p:T+1}$ and $y_{1-p:T+1}$ to construct a model which is capable of predicting the output y_t from the previous input and output data. Therefore, if the data set was fixed, and the engineer was trying to evaluate their model, the quantity of interest would be $\mathbb{E}[\|y_t - \hat{y}_t|u_{1-p:T+1}, y_{1-p:T+1}]$, or the mean squared prediction error of their model. This is not the quantity that we are interested in bounding. Instead, we are interested in bounding the expectation of this quantity over the possible data sets to determine the tradeoffs in the modeling parameters and data requirements prior to running the identification algorithm.

III. Proof of Theorem 1

The proof of Theorem 1 has several stages. In Subsection III-A, the expected squared prediction error is decomposed into terms due to 1) noise, 2) finite autoregressive order, 3) model reduction, and 4) a limited amount of data. The error due to finite autoregressive order is bounded in Subsection III-B. In order to bound the errors due to limited data, some non-asymptotic convergence results are derived in Subsection III-C. These results are used to bound the error due to limited data in Subsection III-D. Finally, the errors due to model reduction are bounded in Subsection III-E. Complete proofs may be found in [22].

A. Decomposition

The expected squared prediction error of Alg. 1 is now decomposed into the following components: the optimal prediction error given the true model, terms resulting from the limited model complexity determined by parameters p and ϕ, and a component from the limited amount of data.

Lemma 1. *Let $\mathbf{y}_t^A = \mathbf{G}_T\mathbf{d}_t$ be the output of the VARX model. Then the prediction error of the REDAR algorithm can be decomposed as*

$$\begin{aligned}\mathbb{E}[\|\mathbf{y}_t - \hat{\mathbf{y}}_t\|^2] \le \|\mathbf{e}\|^2_{\mathcal{P}} &+ \mathbb{E}[\|\mathbf{y}^\star_{t|-\infty:t} - \mathbf{y}^\star_{t|t-p:t}\|^2]\\ &+ 2\mathbb{E}[\|\mathbf{y}^\star_{t|t-p:t} - \mathbf{y}_t^A\|^2] + 2\mathbb{E}[\|\mathbf{y}_t^A - \hat{\mathbf{y}}_t\|^2].\end{aligned}$$

B. Finite Model Order Error

Here, we bound the term arising from Lemma 1 that results from the finite model order:

$$\mathbb{E}[\|\mathbf{y}^\star_{t|-\infty:t} - \mathbf{y}^\star_{t|t-p:t}\|^2]. \tag{5}$$

Recall that $H^\star(q)$ is the Kalman filter operator. Note that $H^\star(q)$ can be written as

$$H^\star(q) = C\sum_{i=1}^\infty \tilde{A}^{i-1}[B \quad K]q^{-i},$$

where $\tilde{A} = A - KC$. Let $H^{\mathbf{Head}}$ be the truncation of $H^\star(q)$ to p terms:

$$H^{\mathbf{Head}}(q) = C\sum_{i=1}^p \tilde{A}^{i-1}[B \quad K]q^{-i}.$$

Then the difference between these two systems is

$$H^{\mathbf{Tail}}(q) = (H^\star - H^{\mathbf{Head}})(q) = \sum_{i=p+1}^\infty C\tilde{A}^{i-1}[B \quad K]q^{-i}.$$

To simplify notation, let $H_i = C\tilde{A}^{i-1}[B \quad K]$.

Note that $\mathbb{E}[\|\mathbf{y}^\star_{t|-\infty:t} - \mathbf{y}^\star_{t|t-p:t}\|^2] \leq \mathbb{E}[\|\mathbf{y}^\star_{t|-\infty:t} - \mathbf{y}^{\mathbf{Head}}_t\|^2]$, where $\mathbf{y}^{\mathbf{Head}}_t = H^{\mathbf{Head}}(q)\mathbf{z}_t$. We therefore opt to bound the term on the right hand side. This may be written as $\mathbb{E}[\|\mathbf{y}^\star_{t|-\infty:t} - \mathbf{y}^{\mathbf{Head}}_t\|^2] = \mathbb{E}[\|H^{\mathbf{Tail}}(q)\mathbf{z}_t\|^2]$.

For any operator H, $\mathbb{E}[\|H(q)\mathbf{x}_t\|^2] \leq \|H\|^2_\infty \|\mathbf{x}\|^2_{\mathcal{P}}$. Thus

$$\mathbb{E}[\|H^{\mathbf{Tail}}(q)\mathbf{z}_t\|^2] \leq \|H^{\mathbf{Tail}}\|^2_\infty \|\mathbf{z}\|^2_{\mathcal{P}}. \tag{6}$$

Lemma 2. *(modification of [4], Lemma 1). Assume that there are constants $\rho < 1$ and $L > 0$ such that the Kalman filter satisfies $\|H^\star(z)\|_2 \leq L$ for all $|z| \geq \rho$. Then the coefficients of $H^\star$ satisfy*

$$\|H_i\|_2 \leq L\rho^i \text{ for } i = 1, 2, \ldots$$

and the tail is bounded as

$$\|H^{\mathbf{Tail}}\|_\infty \leq \frac{L\rho^{p+1}}{1-\rho}.$$

Combining the result of Lemma 2 with (6), we have a bound for (5).

C. Convergence of Empirical Means

The least squares problem in Alg. 1 converges asymptotically to a steady state value. This subsection takes the first step in bounding the distance from the asymptotic value with a finite amount of data. In particular, probability bounds are provided for the difference of individual components of the least squares solution from their asymptotic value.

Recall the definition of $\mathbf{d}_t$ and the corresponding least-squares estimator, $\mathbf{G}_T$, from Alg. 1. The least-squares solution can be expressed as

$$\mathbf{Q}_T = \frac{1}{T}\sum_{t=1}^{T} \mathbf{d}_t\mathbf{d}_t^\top,\ \mathbf{N}_T = \frac{1}{T}\sum_{t=1}^{T} \mathbf{y}_t\mathbf{d}_t^\top,$$
$$\mathbf{G}_T = \mathbf{N}_T\left(\mathbf{Q}_T + \frac{\alpha}{T}I\right)^{-1}.$$

Then optimal solution defined in (4) may be written

$$Q = \mathbb{E}[\mathbf{Q}_T],\ N = \mathbb{E}[\mathbf{N}_T],\ G_{OPT} = NQ^{-1}.$$

We will denote $\mathbf{Q}_T - Q$ as $\Delta\mathbf{Q}$ and $\mathbf{N}_T - N$ as $\Delta\mathbf{N}$. The focus of this subsection will be to derive a bound on the probability that any element of $\Delta\mathbf{Q}$ or $\Delta\mathbf{N}$ exceed a given magnitude. This will then be used in the following subsection to bound the finite data error.

Let $J(q)$ be the closed-loop operator that maps

$$\mathbf{z}_t = J(q)\begin{bmatrix}\Psi^{-1/2}\mathbf{e}_t \\ \mathbf{v}_t\end{bmatrix}.$$

Here, we have re-normalized the innovation error signal so that the input to J has identity covariance.

Define $\mathbf{Z} = \begin{bmatrix}\mathbf{z}_{1-p}^\top & \ldots & \mathbf{z}_{T-1}^\top & \mathbf{z}_T^\top\end{bmatrix}^\top$. Let $R = \mathbb{E}\left[\mathbf{Z}\mathbf{Z}^\top\right]$, and $r_t = \mathbb{E}[\mathbf{z}_t\mathbf{z}_0^\top]$ be the autocorrelation function. Then $R_{t,\tau} = r_{t-\tau}$. Let $\Phi_{\mathbf{z}}(e^{j\omega})$ be the Fourier transform of r_t, which is the power spectral density. Note that $\Phi_{\mathbf{z}}(e^{j\omega}) = J(e^{j\omega})J(e^{j\omega})^*$, and so $\|\Phi_{\mathbf{z}}(e^{j\omega})\| \leq \|J\|^2_\infty$.

Lemma 3. *The covariance, R, satisfies $\|R\| \leq \|J\|^2_\infty$.*

Lemma 4. *For all symmetric S and all $\delta > 0$, the following bound holds for all $T \geq p$.*

$$\mathbb{P}\left(\mathbf{Z}^\top S\mathbf{Z} > \mathrm{Tr}(RS) + \delta T\right) \leq \exp\left(-T\min\left\{\frac{\delta^2}{32\|S\|^2\|J\|^4_\infty}, \frac{\delta}{8\|S\|\|J\|^2_\infty}\right\}\right).$$

Proof. Note that SR and $R^{1/2}SR^{1/2}$ have the same eigenvalues, so all of the eigenvalues of SR are real. For all $\eta > 0$ such that $\eta R^{1/2}SR^{1/2} \prec I$, Markov's inequality implies that

$$\begin{aligned}\mathbb{P}&\left(\mathbf{Z}^\top S\mathbf{Z} > \mathrm{Tr}(SR) + T\delta\right) \\ &\leq e^{-\frac{\eta}{2}(\mathrm{Tr}(SR)+T\delta)}\mathbb{E}\left[e^{\frac{\eta}{2}\mathbf{Z}^\top S\mathbf{Z}}\right] \\ &= e^{\left(-\frac{1}{2}(\eta\mathrm{Tr}(SR)+\eta T\delta+\log\det(I-\eta SR))\right)}.\end{aligned} \tag{7}$$

The equality follows from direct calculation.

Now we will examine the exponent from (7). Let λ_i be the eigenvalues of SR for $i = 1, \ldots, T+p$. As discussed above, these are real and furthermore, $|\lambda_i| \leq \|SR\| \leq \|S\|\|R\| \leq \|S\|\|J\|^2_\infty$. Using the bounds on the eigenvalues, the exponent can be bounded as follows.

$$\begin{aligned}&\eta\mathrm{Tr}(SR) + \eta T\delta + \log\det(I - \eta SR) \\ &\quad= \sum_{i=1}^{T+p}(\eta\lambda_i + \log(1-\eta\lambda_i)) + \eta T\delta \\ &\quad= \eta T\delta - \sum_{i=1}^{T+p}\sum_{k=2}^{\infty}\frac{(\eta\lambda_i)^k}{k} \\ &\quad\geq \eta T\delta - 2T\sum_{k=2}^{\infty}\frac{(\eta\|S\|\|J\|^2_\infty)^k}{k} \\ &\quad\geq \eta T\delta - 2T\sum_{k=2}^{\infty}(\eta\|S\|\|J\|^2_\infty)^k \\ &\quad= \eta T\delta - 2T\frac{(\eta\|S\|\|J\|^2_\infty)^2}{1-\eta\|S\|\|J\|^2_\infty}.\end{aligned}$$

Now say that $\eta \leq 1/(2\|S\|_2\|J\|^2_\infty)$. Then the above expression can be bounded below by

$$T\left(\eta\delta - 4\eta^2(\|S\|_2\|J\|^2_\infty)^2\right). \tag{8}$$

Now we will see how to choose η to ensure that (8) is positive. For simple notation, let $a = 8(\|S\|_2\|J\|^2_\infty)^2$ and let $b = 1/(2\|S\|_2\|J\|^2_\infty)$. Then η can be chosen by maximizing $\delta\eta - \frac{a\eta^2}{2}$ over $0 \leq \eta \leq b$. The optimal solution is given by $\eta = \min\{\delta/a, b\}$. If $\eta = \delta/a$, then the optimal value is given by $\delta^2/(2a)$. If $\eta = b$, then we must have that $\delta \geq ab$ and so the optimal value satisfies $b\delta - ab^2/2 \geq b\delta - b\delta/2 = b\delta/2$. Thus, we get the final bound on (8) as

$$T\min\left\{\frac{\delta^2}{16\|S\|^2\|J\|^4_\infty}, \frac{\delta}{4\|S\|\|J\|^2_\infty}\right\}.$$

The lemma follows by plugging this into the exponential bound on the probability from (7). □

Note that every entry of $\mathbf{N}_T$ and $\mathbf{Q}_T$ is of the form

$$\frac{1}{T}\sum_{t=1}^{T}(\mathbf{z}_{t-k})_i(\mathbf{z}_{t-\ell})_j$$

for some $i,j \in \{1,\dots,n_z\}$ and $k,\ell \in \{0,\dots,p\}$. Recall that r_t is the autocorrelation function of $\mathbf{z}$. The next lemma shows that these empirical means converge to the corresponding autocorrelation values exponentially in probability. The proof follows from an application of Lemma 4.

Lemma 5. *For all $i,j \in \{1,\dots,n_z\}$, all $k,\ell \in \{0,\dots,p\}$, and all $T \geq p$, the following bound holds*

$$\mathbb{P}\left(\left|\frac{1}{T}\sum_{t=1}^{T}(\mathbf{z}_{t-k})_i(\mathbf{z}_{t-\ell})_i - (r_{\ell-k})_{ij}\right| > \delta\right) \leq 2\exp\left(-T\min\left\{\frac{\delta^2}{32\|J\|_\infty^4}, \frac{\delta}{8\|J\|_\infty^2}\right\}\right).$$

Now note that every element of $\Delta\mathbf{Q}$ and $\Delta\mathbf{N}$ may be expressed as

$$\frac{1}{T}\sum_{t=1}^{T}(\mathbf{z}_{t-k})_i(\mathbf{z}_{t-\ell})_j - (r_{l-k})_{ij}$$

for some $i,j \in \{1,\dots,n_z\}$ and $k,\ell \in \{0,\dots,p\}$. The following lemma uses this fact to bound the probability of elementwise deviations of $\Delta\mathbf{Q}$ and $\Delta\mathbf{N}$ from zero.

Lemma 6. *For all $i,j,k \in \{1,...pn_z\}$, all $l \in \{1,...n_y\}$, and all $T \geq p$, the following expression is satisfied.*

$$\mathbb{P}(\max_{i,j}\{|\Delta\mathbf{Q}_{ij}|\} > \delta \text{ or } \max_{k,l}\{|\Delta\mathbf{N}_{kl}|\} > \delta) \leq 2b\exp\left(-T\min\left\{\frac{\delta^2}{32\|J\|_\infty^4}, \frac{\delta}{8\|J\|_\infty^2}\right\}\right)$$

where

$$b = pn_yn_z + \frac{pn_z(pn_z+1)}{2}.$$

D. Finite Data Error

We now use the results from the previous subsection to bound $\mathbb{E}[\|\mathbf{y}^\star_{t|t-p:t} - \mathbf{y}^A_t\|^2] \leq \mathbb{E}[\|G_{OPT} - \mathbf{G}_T\|^2]\|\mathbf{z}\|_{\mathcal{P}}^2 p$. As $\|G_{OPT} - \mathbf{G}_T\|^2 \geq 0$, the expected value may be written

$$\mathbb{E}[\|\mathbf{G}_T - G_{OPT}\|^2] = \int_0^\infty \mathbb{P}[\|\mathbf{G}_T - G_{OPT}\|^2 > \epsilon]d\epsilon \quad (9)$$

An upper bound on this integral may be computed if, for any $\epsilon \geq 0$, we can bound $\mathbb{P}[\|\mathbf{G}_T - G_{OPT}\|^2 > \epsilon]$. To do so, define $\delta \geq 0$ such that

$$|\Delta\mathbf{N}_{ij}| \leq \delta \; i = 1,\dots,n_y \; j = 1,\dots,pn_z$$
$$|\Delta\mathbf{Q}_{ij}| \leq \delta \; i,j = 1,\dots,pn_z.$$

We will proceed by bounding $\|\mathbf{G}_T - G_{OPT}\|$ in terms of δ. It will then be possible to determine a value $\delta \geq 0$ corresponding to all sufficiently large ϵ such that

$$|\Delta\mathbf{Q}_{ij}| \leq \delta \text{ and } |\Delta\mathbf{N}_{ij}| \leq \delta \Rightarrow \|\mathbf{G}_T - G_{OPT}\|^2 \leq \epsilon.$$

Then Lemma 6 may be applied to bound the probability that the elementwise bounds hold.

The elementwise bounds above provide the following bounds on $\|\Delta\mathbf{N}\|$ and $\|\Delta\mathbf{Q}\|$.

$$\|\Delta\mathbf{N}\| \leq c_1\delta, \quad \|\Delta\mathbf{Q}\| \leq c_2\delta \quad (10)$$

where $c_1 = \sqrt{pn_yn_z}$ and $c_2 = pn_z$.

To simplify notation, we define $\xi = \lambda_{min}(\Gamma) \leq \|Q^{-1}\|^{-1}$. Then the above results may be applied to compute the following bound.

Lemma 7.

$$\|\mathbf{G}_T - G_{OPT}\| \leq \left(c_3\delta + \frac{c_4}{T}\right)\left\|(Q + \Delta\mathbf{Q} + \frac{\alpha}{T}I)^{-1}\right\|$$

where $c_3 = c_1 + \frac{\|J\|_\infty^2 c_2}{\xi}$ and $c_4 = \frac{\|J\|_\infty^2\alpha}{\xi}$.

We know that the following always holds

$$\|(Q + \Delta\mathbf{Q} + \frac{\alpha}{T}I)^{-1}\| \leq \frac{T}{\alpha}, \quad (11)$$

as $Q + \Delta\mathbf{Q} = \sum_{k=0}^{T}\mathbf{d}_t\mathbf{d}_t^T \succeq 0$. A tighter bound is available when δ is small.

Lemma 8. *For $\delta < \frac{(\xi - \frac{\alpha}{T})}{c_2}$,*

$$\|(Q + \Delta\mathbf{Q} + \frac{\alpha}{T}I)^{-1}\| \leq \frac{1}{\xi - c_2\delta - \frac{\alpha}{T}}$$

Lemma 9. *Assume $T \geq \max\left\{\frac{2\alpha}{\xi}, 1\right\}$. Let*

$$\epsilon_0 = \left(\frac{2\|J\|_\infty^2\alpha}{\xi^2T^{1/4}}\right)^2 \text{ and } \epsilon_1 = \left(\frac{2\|J\|_\infty^2 T}{\alpha}\right)^2.$$

For any $\epsilon \geq \epsilon_0$, we can find $\delta \geq 0$ such that

$$|\Delta\mathbf{Q}_{ij}| \leq \delta \text{ and } |\Delta\mathbf{N}_{ij}| \leq \delta \Rightarrow \|\mathbf{G}_T - G_{OPT}\|^2 \leq \epsilon,$$

by selecting

$$\delta = \begin{cases} \dfrac{(\xi T - \alpha)\sqrt{\epsilon} - c_4}{c_2T\sqrt{\epsilon} + c_3T} & \epsilon_0 \leq \epsilon \leq \epsilon_1 \quad (12a) \\ \dfrac{\alpha\sqrt{\epsilon} - c_4}{c_3T} & \epsilon \geq \epsilon_1 \quad (12b) \end{cases}$$

The reason for the two different expressions for δ in the above lemma is that (11) provides a tighter bound than Lemma 8 when δ becomes greater than $\frac{\xi - \frac{2\alpha}{T}}{c_2}$.

Lemma 10. *For some k and T_0 depending on n_u, n_y, p, α, $\|J\|_\infty$, and $\lambda_{min}(\Gamma)$,*

$$\mathbb{E}[\|\mathbf{G}_T - G_{OPT}\|^2] \leq \frac{k}{\sqrt{T}} \quad \textit{for all } T \geq T_0.$$

Proof. Let $\delta_1(\epsilon)$ be given by (12a) and $\delta_2(\epsilon)$ be given by (12b). We obtain a bound on the right side of (9) by application of Lemma 9 along with Lemma 6.

$$\int_0^\infty (\mathbb{P}[\|\mathbf{G}_T - G_{OPT}\|^2 > \epsilon])d\epsilon \leq \underbrace{\int_0^{\epsilon_0} 1 d\epsilon}_{d_1}$$
$$+ \underbrace{\int_{\epsilon_0}^{\epsilon_1} 2b\exp\left(-\frac{T}{2}\min\left\{\left(\frac{\delta_1(\epsilon)}{4\|J\|_\infty^2}\right)^2, \frac{\delta_1(\epsilon)}{4\|J\|_\infty^2}\right\}\right)d\epsilon}_{d_2}$$
$$+ \underbrace{\int_{\epsilon_1}^{\infty} 2b\exp\left(-\frac{T}{2}\min\left\{\left(\frac{\delta_2(\epsilon)}{4\|J\|_\infty^2}\right)^2, \frac{\delta_2(\epsilon)}{4\|J\|_\infty^2}\right\}\right)d\epsilon}_{d_3},$$

where the integrand of d_1 results from the fact that the probability is at most 1. The bounds above were valid for $\mathrm{T} \geq \max\left\{\frac{2\alpha}{\xi}, p\right\}$. We now bound each term separately. We may solve d_1 explicitly. The others may be split up further using crude bounding techniques. We may then show d_2 and d_3 to be bounded by an expression of the form $aT^m \exp(-bT^n)$, which decay faster than $\frac{1}{\sqrt{T}}$. Thus the entire expression decays at a rate $\frac{1}{\sqrt{T}}$. □

E. Model Reduction Error

The only term that remains to be bounded is that from the model reduction step. The bound on this term is $\mathbb{E}[\|\mathbf{y}_t^A - \hat{\mathbf{y}}\|^2] \leq \phi \|\mathbf{z}\|_{\mathcal{P}}^2$, and results from step 5 of Alg. 1.

Theorem 1 now follows by the combining the bounds on the components of the decomposition from Lemma 1.

IV. Discussion

Another result following from the same analysis is now provided, along with a note about the bounds obtained.

Theorem 2. *For $0 < \theta \leq 1$, let*

$$\delta = 4\|J\|_\infty^2 \max\left\{\frac{2}{T}\log\frac{2b}{\theta}, \sqrt{\frac{2}{T}\log\frac{2b}{\theta}}\right\}.$$

Assume $T \geq p$. With probability at least $1-\theta$,

$$\|H^{OPT} - \mathbf{H}^R\|_\infty \leq \begin{cases} \dfrac{\left(c_3\delta + \frac{c_4}{T}\right)}{\xi - c_2\delta - \frac{\alpha}{T}}p + \phi & \delta \leq \dfrac{\xi - \frac{2\alpha}{T}}{c_2} \\ \dfrac{T\left(c_3\delta + \frac{c_4}{T}\right)}{\alpha}p + \phi & \delta > \dfrac{\xi - \frac{2\alpha}{T}}{c_2} \end{cases}$$

Remark 1. *There are free parameters in the bound from Theorem 1 which can greatly impact the quality of the bound. In particular, ρ may be chosen as any value between the spectral radius of the Kalman Filter and one. Smaller values of ρ will increase L, but decrease $\frac{\rho^{p+1}}{1-\rho}$. As such, we can optimize over ρ to obtain the tightest bound. Another free parameter is T_0, which may take values greater than that supplied in Lemma 10. Higher values of T_0 will decrease the value of k, but make the bound invalid for small T.*

V. Conclusion

The finite sample behavior of an algorithm known as REDAR was characterized for data generated in closed-loop. The algorithm first fits the data to a VARX model, and then finds the system model via a balanced reduction step. The simple nature of the algorithm allowed for the derivation of a non-asymptotic upper bound on the generalization error. The bound provides the engineer with a notion of the effectiveness of the model generated from a finite amount of data. Additionally, high probability bounds on the $\mathcal{H}_\infty$ norm of the error system from the estimated model to the finite horizon Kalman Filter are obtained. It may be possible to extend the results to bounds which may be applied in robust adaptive control.

VI. Acknowledgements

The authors thank Jianjun Yuan for helpful discussions.

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