DIFFUSIVE LIMIT OF TRANSPORT EQUATION IN 3D CONVEX DOMAINS

LEI WU

Abstract. Consider neutron transport equations in 3D convex domains with in-flow boundary. We mainly study the asymptotic limits as the Knudsen number $\epsilon\to 0^{+}$. Using Hilbert expansion, we rigorously justify that the solution of steady problem converges to that of Laplace's equation, and the solution of unsteady problem converges to that of heat equation. This is the most difficult case of a long-term project on asymptotic analysis of kinetic equations in bounded domains. The proof relies on a detailed analysis on the boundary layer effect with geometric correction. The upshot of this paper is a novel boundary layer decomposition argument in 3D and $L^2-L^{2m}-L^\infty$ bootstrapping method for time-dependent problem.

Keywords: neutron transport; boundary layer; Milne problem; geometric correction

Contents

1. Introduction

1.1. **Problem Presentation.** We consider the steady neutron transport equation in a three-dimensional bounded convex domain with in-flow boundary. In the spacial domain $\vec x=(x_1,x_2,x_3)\in\Omega$ where $\partial\Omega\in C^3$ and the velocity domain $\vec w=(w_1,w_2,w_3)\in\mathbb{S}^2$, the neutron density $u^\epsilon(\vec x,\vec w)$ satisfies

$$
\left\{
\begin{array}{l}
\epsilon\vec w\cdot\nabla_x u^\epsilon+u^\epsilon-\bar u^\epsilon=0\ \text{ in }\ \Omega\times\mathbb{S}^2,\\[2ex]
u^\epsilon(\vec x_0,\vec w)=g(\vec x_0,\vec w)\ \text{ for }\ \vec w\cdot\vec\nu<0\ \text{ and }\ \vec x_0\in\partial\Omega,
\end{array}
\right.
\tag{1.1}
$$

where

$$
\bar u^\epsilon(\vec x)=\frac{1}{4\pi}\int_{\mathbb{S}^2}u^\epsilon(\vec x,\vec w)\mathrm{d}\vec w,
\tag{1.2}
$$

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$\vec{\nu}$ is the outward unit normal vector, with the Knudsen number $0 < \epsilon << 1$.

Also, we consider the unsteady counterpart. Taking additional temporal domain $t \in \mathbb{R}^+$ into account, the neutron density $u^{\epsilon}(t, \vec{x}, \vec{w})$ satisfies

$$
\left\{
\begin{array}{l}
\epsilon^2 \partial_t u^{\epsilon} + \epsilon \vec{w} \cdot \nabla_x u^{\epsilon} + u^{\epsilon} - \bar{u}^{\epsilon} = 0 \ \text{ in } \ \mathbb{R}^+ \times \Omega \times \mathbb{S}^2, \\
\\
u^{\epsilon}(0, \vec{x}, \vec{w}) = h(\vec{x}, \vec{w}) \ \text{ in } \ \Omega \times \mathbb{S}^2, \\
\\
u^{\epsilon}(t, \vec{x}_0, \vec{w}) = g(t, \vec{x}_0, \vec{w}) \ \text{ for } \ t \in \mathbb{R}^+, \ \ \vec{w} \cdot \vec{\nu} < 0 \ \text{ and } \ \vec{x}_0 \in \partial\Omega,
\end{array}
\right.
\tag{1.3}
$$

where

$$
\bar{u}^{\epsilon}(t, \vec{x}) = \frac{1}{4\pi} \int_{\mathbb{S}^2} u^{\epsilon}(t, \vec{x}, \vec{w}) \mathrm{d}\vec{w}. \tag{1.4}
$$

The initial and boundary data satisfy the compatibility condition

$$
h(\vec{x}_0, \vec{w}) = g(0, \vec{x}_0, \vec{w}) \ \text{ for } \ \vec{x}_0 \in \partial\Omega \ \text{ and } \ \vec{w} \cdot \vec{\nu} < 0. \tag{1.5}
$$

In both cases, we intend to study the behavior of u^{ϵ} as $\epsilon \to 0$.

Based on the flow direction, we can divide the boundary $\Gamma = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega \ \text{ and } \ \vec{w} \in \mathbb{S}^2\}$ into the in-flow boundary Γ^-, the out-flow boundary Γ^+ and the grazing set Γ^0 as

$$
\Gamma^- = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega, \ \vec{w} \cdot \vec{\nu} < 0\} \tag{1.6}
$$

$$
\Gamma^+ = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega, \ \vec{w} \cdot \vec{\nu} > 0\} \tag{1.7}
$$

$$
\Gamma^0 = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega, \ \vec{w} \cdot \vec{\nu} = 0\} \tag{1.8}
$$

It is easy to see that $\Gamma = \Gamma^+ \cup \Gamma^- \cup \Gamma^0$. Hence, the boundary condition is only prescribed for Γ^-. This is usually called the in-flow or absorbing boundary condition.

1.2. **Background and Methods.** We have been involved in this long-term project to justify the diffusive limits for neutron transport equations(see [24], [21], [25], [6], [7], [22] and [23]) and the hydrodynamic limits for nonlinear Boltzmann equation (see [20] and [19]). In the past decade, we have pushed the results in many aspects: geometry of domain, dimension, time-dependence, etc. This paper concerns the most physically relevant and most mathematically challenging case, 3D problem with in-flow boundary.

1.2.1. *Previous Results.* Diffusive limits, or more general hydrodynamic limits, are central to connecting kinetic theory and fluid mechanics. The basic idea is to consider the asymptotic behaviors of the solutions to Boltzmann equation, transport equation, or Vlasov systems. Since the early 20^{th} century, this type of problems have been extensively studied in many different settings: steady or unsteady, linear or nonlinear, strong solution or weak solution, etc.

Among all these variations, one of the simplest but most important models — neutron transport equation in bounded domains, has attracted a lot of attention since the dawn of the atomic age. Besides its significance in nuclear sciences and medical imaging, the neutron transport equation is usually regarded as a linear prototype of the more complicated nonlinear Boltzmann equation, and thus, is an ideal starting point to develop new theories and techniques. The early development focuses on formal expansion, explicit solution and numerical methods. We refer to [8], [9], [10], [11], [12], [13], [14], [15], [16] for more details.

Our whole project starts from [24] and [25], where we introduced a fresh formulation of kinetic boundary layers to justify diffusive limit in 2D disk/annulus. As opposed to the classical theory (see [1]) which only works in the geometry of flat boundary, our new method can be implemented in both flat and curved domains.

Later, we developed several novel techniques to apply this idea to more general cases:

- Steady problem:
  - In [6], we extended the idea to 2D general convex domains (which do not have to be a disk/annulus) with diffusive boundary .
  - In [7], we extended the idea to 3D general convex domains with diffusive boundary.
  - In [23], we extend the idea to 2D general convex domains with in-flow boundary.

  In this paper, we will cover the last and most difficult piece, i.e. 3D general convex domains with in-flow boundary.

- Unsteady problem:
  - In [21], we considered the 2D disk/annulus with in-flow boundary. This is a unsteady counterpart of [24].The diffusive boundary counterpart is actually simpler.
  - In [22], we considered the 2D general convex domains with diffusive boundary. This is a unsteady counterpart of [6].

All in all, there are fewer studies on unsteady problems and there are many pieces missing. In this paper, we will skip the intermediate steps and directly attack the 3D general convex domains with in-flow boundary, which will cover all simpler cases.

1.2.2. *General Ideas.* We use 2D steady neutron transport equation (1.1) (i.e. $\vec{x}\in\Omega\subset\mathbb{R}^2$ and $\vec{w}\in\mathbb{S}^1$) to present the main idea. Generally speaking, the solution u^ϵ varies smoothly and slowly in the interior of Ω, and behaves like $u^\epsilon-\bar{u}^\epsilon=0$ which ignores $\vec{w}\cdot\nabla_x u^\epsilon$. However, its value changes severely when approaching the boundary $\partial\Omega$ and in this regime, $\vec{w}\cdot\nabla_x u^\epsilon$ plays a crucial role. The smaller ϵ is, the more violently u^ϵ changes.

This phenomenon indicates that u^ϵ can actually be described in two distinct regimes with different scalings, namely, the interior solution U and the boundary layer $\mathscr{U}$. The interior solution satisfies certain thermodynamic equations, and the boundary layer satisfies a half-space kinetic equation, which decays rapidly when it is away from the boundary.

The justification of this approximation, i.e. the so-called diffusive limit usually involves two steps:

(1) **Hilbert expansion**: expanding $U=\displaystyle\sum_{k=0}^{\infty}\epsilon^k U_k$ and $\mathcal{U}^B=\displaystyle\sum_{k=0}^{\infty}\epsilon^k\mathcal{U}_k^B$ as power series of ϵ and proving the coefficients U_k and $\mathcal{U}_k^B$ are well-defined.
Traditionally, the estimates of the interior solutions U_k are relatively straightforward. On the other hand, boundary layers $\mathcal{U}_k^B$ satisfy one-dimensional half-space problems which lose some key structures of the original equations. The well-posedness of boundary layer equations are sometimes extremely difficult and it is possible that they are actually ill-posed (e.g. certain type of Prandtl layers [5]).

For kinetic equations, the boundary layer is described by so-called Milne problem with geometric correction

$$\sin\phi\frac{\partial f}{\partial\eta}+\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial f}{\partial\phi}+f-\bar{f}=S, \tag{1.9}$$

where R_κ is the curvature of boundary. To close the proof, we need to bound the second-order derivative of f in L^∞. However, as pointed out in [4], even weighted $W^{2,\infty}$ estimates of general kinetic equations are not available. The resolution of this difficulty mainly involves two techniques:

- Weighted $W^{1,\infty}$ estimates: this is achieved by carefully designing weights and a detailed analysis along the characteristics. It is firstly used in [6].
- Boundary layer decomposition: we delicately decompose the boundary layer into two parts: the regular boundary can attain $W^{2,\infty}$ estimates and the singular boundary layer can only attain $W^{1,\infty}$ estimates with extremely small support. It is firstly used in [23].

(2) **Remainder estimates**: proving that $R=u^\epsilon-U_0-\mathcal{U}_0^B=o(1)$ as $\epsilon\to0$.
Ideally, this should be done just by expanding to the leading-order level U_0 and $\mathcal{U}_0^B$. However, in singular perturbation problems, the estimates of the remainder R usually involve negative powers of ϵ, which requires an expansion to higher-order terms U_N and $\mathcal{U}_N^B$ for $N\geq1$ such that we have a sufficient power of ϵ. In other words, we define $R=u^\epsilon-\displaystyle\sum_{k=0}^{N}\epsilon^kU_k-\sum_{k=0}^{N}\epsilon^k\mathcal{U}_k^B$ for $N\geq1$ instead of $R=u^\epsilon-U_0-\mathcal{U}_0^B$ to get better estimates of R.

In detail, for the remainder equation

$$\epsilon\vec{w}\cdot\nabla_xR+R-\bar{R}=S, \tag{1.10}$$

so-far the best estimates in 3D read

$$\|R\|_{L^\infty(\Omega\times\mathbb{S}^2)}\lesssim\frac{1}{\epsilon^{2+\frac{3}{2m}}}\|S\|_{L^{\frac{2m}{2m-1}}}+\text{higher-order terms}, \tag{1.11}$$

for $m\geq2$. This is actually gradually improved from [24] to [6] and [23].

1.2.3. *Key Difficulties.* These techniques have been matured in 2D. However, we encounter essential difficulties in deriving their 3D counterparts:

- **Boundary Layer Decomposition is Still Not Enough.**
In 2D steady problem, we may show L^{2m} estimates for arbitrary $m\in\mathbb{N}$. However, in 3D, due to the restriction of Sobolev embedding [23, (4.125)] and trace theorem [23, (4.130)], we can at most attain $m=3$ (i.e. L^6) estimates. Also, when bootstrapping from L^{2m} to L^∞, we lost even more power of ϵ. Hence, in (1.10), we can at most obtain (see [7])

$$\|R\|_{L^\infty(\Omega\times\mathbb{S}^2)}\lesssim\frac{1}{\epsilon^{\frac{5}{2}}}\|S\|_{L^{\frac{6}{5}}(\Omega\times\mathbb{S}^2)}+\text{higher-order terms.}\tag{1.12}$$

Hence, 3D problem requires $\epsilon^{\frac{5}{2}}\times\epsilon^{1-\frac{5}{6}}=\epsilon^{\frac{8}{3}}$, which is fatal since the boundary layer decomposition can at most provide $\epsilon^{\frac{5}{2}}$ extra power (see [23]). This is the major difficulty in 3D steady problem.
- **Remainder Estimates is Not Available.**
In 3D unsteady problem, an additional difficulty is that the traditional technique to show remainder estimates fails. Our bounds of the remainder is actually based on an improved averaging lemma. The proof highly relies on a series of delicately chosen test functions in the weak formulation. However, in 3D unsteady case, such test functions are not attainable. Also, the L^∞ estimates are obtained through a bootstrapping argument, which requires an application of double Duhamel's principle and a smart change of variables. In contrast to steady problem, such an argument breaks down in 3D due to lack of variables for substitution.

  Here, we face a dilemma: L^2 estimates for the unsteady problem are available, but the ϵ power is not sufficient to close the proof; L^6 estimates can do the job, but we must try a new approach to show the averaging-lemma-type estimates and recover the boostrapping method.

1.2.4. *Major Upshot.* In order to tackle the two difficulties mentioned above, we introduce the following fresh techniques:

- **Improved Estimates of Regular Boundary Layer.**
In 2D boundary layer decomposition (see [23] and above introduction), the regular and singular boundary layers have contradictory requirements on the cutoff $\sim\epsilon^\alpha$ of boundary data. Consider the boundary data decomposition $g=\mathscr{G}+\mathfrak{G}$ and the corresponding regular boundary layer $\mathscr{U}$ and singular boundary layer $\mathfrak{U}$:
  - to obtain $W^{2,\infty}$ estimate of $\mathscr{U}$ with data $\mathscr{G}$, we want α to be as small as possible; the smaller α is (better smoothness of $\mathscr{G}$), the better estimates we get;
  - to obtain $W^{1,\infty}$ estimate of $\mathfrak{U}$ with data $\mathfrak{G}$, we want α to be as large as possible; the larger α is (smaller support of $\mathfrak{G}$), the better estimates we get.

  The resulting optimal $\alpha=\dfrac{1}{2}$ is a balance of the above two arguments. In this paper, we remove the above restriction from the regular boundary layer. We improve the estimates from

$$\|\mathscr{U}\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\sim C\epsilon^{2+\alpha}\sim C\epsilon^{\frac{5}{2}},\tag{1.13}$$

  to

$$\|\mathscr{U}\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\sim C\epsilon^{2+\alpha}\sim \epsilon^{3}.\tag{1.14}$$

  Henceforth, α can be enlarged all the way to 1, which is the critical value in the framework. Then this is sufficient to cover the $\epsilon^{\frac{8}{3}}$ loss in remainder estimates and obtain $\epsilon^{\frac{1}{3}}$ asymptotic convergence.
- **Mixed $L^2-L^6-L^\infty$ Remainder Estimates.**
For the unsteady remainder equation

$$\epsilon^2\partial_tR+\epsilon\vec{w}\cdot\nabla_xR+R-\bar{R}=S,\tag{1.15}$$

  we propose three ideas to recover and further improve the remainder estimates:
  - Firstly, we introduce a combination of L^2 and L^6 estimates to bound the remainder R, i.e. to bound R quantities with L^6 and ∂_tR quantities with L^2. Here, we need to redefine the energy and dissipation and utilize an intricate interpolation estimates. In particular, $\|\partial_tR\|_{L^2(\Omega\times\mathbb{S}^2)}$

requires the bounds of

$$\left\|\epsilon^{-1}\vec{w}\cdot\nabla_x R\right\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}+\left\|\epsilon^{-2}(R-\bar{R})\right\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}+\left\|\epsilon^{-2}S\right\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}. \tag{1.16}$$

Here the presence of ϵ^{-2} involves a delicate balance of ϵ power, which relies on a special application of the orthogonality relation.

– Secondly, as [23] and [22] state, the traditional remainder estimate for unsteady neutron transport equation is based on an L^6 kernel estimates and L^2 energy estimates, which can at most provide

$$\|R(t)\|_{L^2(\Omega\times\mathbb{S}^2)}\leq\frac{C}{\epsilon^2}\|S\|_{L^{\frac{6}{5}}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}+\text{higher-order terms}. \tag{1.17}$$

In 3D, L^2 is far from enough to close the proof. Here, we introduce another L^6 energy estimates to improve the bounds to

$$\|R(t)\|_{L^6(\Omega\times\mathbb{S}^2)}\leq\frac{C}{\epsilon^{\frac{13}{6}}}\|S\|_{L^{\frac{6}{5}}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}+\text{higher-order terms}. \tag{1.18}$$

Here the improvement from L^2 to L^6 is far more crucial than the loss of $\epsilon^{\frac{1}{6}}$ in power.

– Thirdly, we introduce a new boostrapping argument that relies on triple Duhamel's principle. Roughly speaking, we "borrow" one velocity variable from the next-level mild formulation and fulfil the requirement of substitution. Definitely, this comes with a price. We now have to further decompose the integral domain and handle the singular Jacobian associated with the substitution. At the end of the day, we obtain

$$\|R\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq\frac{C}{\epsilon^{\frac{1}{2}}}\|R(t)\|_{L^6(\Omega\times\mathbb{S}^2)}+\text{higher-order terms}. \tag{1.19}$$

This is in sharp contrast with the L^2 version (see [22])

$$\|R\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq\frac{C}{\epsilon^{\frac{3}{2}}}\|R(t)\|_{L^2(\Omega\times\mathbb{S}^2)}+\text{higher-order terms}. \tag{1.20}$$

In total, we obtain the unsteady remainder estimate

$$\|R\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq\frac{C}{\epsilon^{\frac{8}{3}}}\|S\|_{L^{\frac{6}{5}}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}+\text{higher-order terms}. \tag{1.21}$$

Hence, we lose $\epsilon^{\frac{8}{3}}\times\epsilon^{1-\frac{5}{6}}=\epsilon^{\frac{17}{6}}$, but the above boundary layer decomposition provides ϵ^3. We eventually obtain $\epsilon^{\frac{1}{6}}$ asymptotic convergence.

1.3. Main Theorem.

1.3.1. *Steady Problem.*

Theorem 1.1 (Steady Diffusive Limit)**.** *Assume $g(\vec{x}_0,\vec{w})\in C^3(\Gamma^-)$. Then for the steady neutron transport equation* (1.1)*, there exists a unique solution $u^\epsilon(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathbb{S}^2)$. Moreover, for any $0<\delta<<1$, the solution obeys the estimate*

$$\left\|u^\epsilon-U-\mathcal{U}^B\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}\leq C(\delta)\epsilon^{\frac{1}{3}-\delta}, \tag{1.22}$$

where $U(\vec{x})$ satisfies the Laplace equation with Dirichlet boundary condition

$$\begin{cases}\Delta_x U(\vec{x})=0 & \text{in}\ \ \Omega,\\ U(\vec{x}_0)=D(\vec{x}_0) & \text{on}\ \ \partial\Omega,\end{cases} \tag{1.23}$$

and $\mathcal{U}^B(\eta,\iota_1,\iota_2,\phi,\psi)$ satisfies the ϵ-Milne problem with geometric correction

$$\begin{cases}\sin\phi\dfrac{\partial\mathcal{U}^B}{\partial\eta}-\left(\dfrac{\sin^2\psi}{R_1(\iota_1,\iota_2)-\epsilon\eta}+\dfrac{\cos^2\psi}{R_2(\iota_1,\iota_2)-\epsilon\eta}\right)\cos\phi\dfrac{\partial\mathcal{U}^B}{\partial\phi}+\mathcal{U}^B-\bar{\mathcal{U}}^B=0,\\ \mathcal{U}^B(0,\iota_1,\iota_2,\phi,\psi)=g(\iota_1,\iota_2,\phi,\psi)-D(\iota_1,\iota_2)\quad\text{for}\quad\sin\phi>0,\\ \mathcal{U}^B(L,\iota_1,\iota_2,\phi,\psi)=\mathcal{U}^B(L,\iota_1,\iota_2,\mathscr{R}[\phi],\psi),\end{cases} \tag{1.24}$$

for $L=\epsilon^{-\frac{1}{2}}$, $\mathscr{R}[\phi]=-\phi$, η *the rescaled normal variable,* (ι_1,ι_2) *the tangential variable, and* (ϕ,ψ) *the velocity variables (see Section 2.1.2).*

Remark 1.2. *Note that the effect of the boundary layer decays very fast when it is away from the boundary. Roughly speaking, this theorem states that for* $\vec{x}$ *not very close to the boundary,* $u^{\epsilon}(\vec{x},\vec{w})$ *can be approximated by the solution of a Laplace equation with Dirichlet boundary condition.*

Remark 1.3. *Here the Dirichlet boundary* $D(\vec{x}_0)$ *and* $D(\iota_1,\iota_2)$ *are identical quantities written in different coordinate systems. They are uniquely determined by the boundary data* g*.*

1.3.2. *Unsteady Problem.*

Theorem 1.4 (Unsteady Diffusive Limit)**.** *Assume* $h(\vec{x},\vec{w})\in C(\Omega\times\mathbb{S}^2)$ *and* $g(t,\vec{x}_0,\vec{w})\in C^3(\mathbb{R}^+\times\Gamma^-)$*. Then for the unsteady neutron transport equation* (1.3)*, there exists a unique solution* $u^{\epsilon}(t,\vec{x},\vec{w})\in L^{\infty}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)$*. Moreover,*

$$\lim_{\epsilon\to0}\left\|\mathrm{e}^{K_0t}\Big(u^{\epsilon}-U-\mathcal{U}^I-\mathcal{U}^B\Big)\right\|_{L^{\infty}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}=0, \tag{1.25}$$

where $U(\vec{x})$ *satisfies the heat equation with Dirichlet boundary condition*

$$\begin{cases}\partial_tU-\Delta_xU(\vec{x})=0 \quad in \quad \mathbb{R}^+\times\Omega,\\ U(0,\vec{x})=\dfrac{1}{4\pi}\displaystyle\int_{\mathbb{S}^2}h(\vec{x},\vec{w})\mathrm{d}\vec{w} \quad in \quad \Omega,\\ U(t,\vec{x}_0)=D(t,\vec{x}_0) \quad on \quad \partial\Omega,\end{cases} \tag{1.26}$$

$\mathcal{U}^I(\tau,\vec{x},\vec{w})$ *satisfies*

$$\mathcal{U}^I(\tau,\vec{x},\vec{w})=\mathrm{e}^{-\tau}\left(h(\vec{x},\vec{w})-\frac{1}{4\pi}\int_{\mathbb{S}^2}h(\vec{x},\vec{w})\mathrm{d}\vec{w}\right), \tag{1.27}$$

for τ *the rescaled time variable, and* $\mathcal{U}^B(t,\eta,\tau,\phi)$ *satisfies the* ϵ*-Milne problem with geometric correction*

$$\begin{cases}\sin\phi\dfrac{\partial\mathcal{U}^B}{\partial\eta}-\left(\dfrac{\sin^2\psi}{R_1(\iota_1,\iota_2)-\epsilon\eta}+\dfrac{\cos^2\psi}{R_2(\iota_1,\iota_2)-\epsilon\eta}\right)\cos\phi\dfrac{\partial\mathcal{U}^B}{\partial\phi}+\mathcal{U}^B-\bar{\mathcal{U}}^B=0,\\ \mathcal{U}^B(t,0,\iota_1,\iota_2,\phi,\psi)=g(t,\iota_1,\iota_2,\phi,\psi)-D(t,\iota_1,\iota_2) \quad for \quad \sin\phi>0,\\ \mathcal{U}^B(t,L,\iota_1,\iota_2,\phi,\psi)=\mathcal{U}^B(t,L,\iota_1,\iota_2,\mathscr{R}[\phi],\psi),\end{cases} \tag{1.28}$$

for $L=\epsilon^{-\frac{1}{2}}$, $\mathscr{R}[\phi]=-\phi$, η *the rescaled normal variable,* ι_1,ι_2 *the tangential variables, and* ϕ, ψ *the velocity variables (see Section 3.1.2 and 3.1.3).*

Remark 1.5. *Note that the effects of the boundary layer decays very fast when it is away from the boundary. Roughly speaking, this theorem states that for* $\vec{x}$ *not very close to the boundary,* $u^{\epsilon}(t,\vec{x},\vec{w})$ *can be approximated by the solution of a heat equation with Dirichlet boundary condition.*

1.4. **Notation and Convention.** Throughout this paper, $C>0$ denotes a constant that only depends on the domain Ω, but does not depend on the data or ϵ. It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity z. We write $a\lesssim b$ to denote $a\leq Cb$.

This paper is organized as follows: in Section 2, we study the steady problem, and in Section 3, we study the unsteady problem. Section 4 focuses on the analysis of boundary layer equation, i.e. the ϵ-Milne problem with geometric correction.

2. Steady Neutron Transport Equation

In this section, we prove the diffusive limit of the steady neutron transport equation (1.1).

2.1. **Asymptotic Expansions.**

2.1.1. *Interior Expansion.* We define the interior expansion as follows:

$$U(\vec{x},\vec{w}) \sim U_0(\vec{x},\vec{w}) + \epsilon U_1(\vec{x},\vec{w}) + \epsilon^2 U_2(\vec{x},\vec{w}), \tag{2.29}$$

where U_k can be determined by comparing the order of ϵ via plugging (2.29) into the equation (1.1). Thus we have

$$U_0 - \bar{U}_0 = 0, \tag{2.30}$$

$$U_1 - \bar{U}_1 = -\vec{w}\cdot\nabla_x U_0, \tag{2.31}$$

$$U_2 - \bar{U}_2 = -\vec{w}\cdot\nabla_x U_1. \tag{2.32}$$

Plugging (2.30) into (2.31), we obtain

$$U_1 = \bar{U}_1 - \vec{w}\cdot\nabla_x \bar{U}_0. \tag{2.33}$$

Plugging (2.33) into (2.32), we get

$$\begin{aligned} U_2 - \bar{U}_2 =& -\vec{w}\cdot\nabla_x(\bar{U}_1 - \vec{w}\cdot\nabla_x\bar{U}_0) \\ =& -\vec{w}\cdot\nabla_x\bar{U}_1 + \Big(w_1^2\partial_{x_1x_1}\bar{U}_0 + w_2^2\partial_{x_2x_2}\bar{U}_0 + w_3^2\partial_{x_3x_3}\bar{U}_0\Big) \\ &+ 2\Big(w_1w_2\partial_{x_1x_2}\bar{U}_0 + w_1w_3\partial_{x_1x_3}\bar{U}_0 + w_2w_3\partial_{x_2x_3}\bar{U}_0\Big). \end{aligned} \tag{2.34}$$

Integrating (2.34) over $\vec{w}\in\mathbb{S}^2$, we have the final form

$$\Delta_x\bar{U}_0 = 0, \tag{2.35}$$

where all cross terms vanish due to the symmetry of $\mathbb{S}^2$. Hence, $U_0(\vec{x},\vec{w})$ satisfies the equation

$$\begin{cases} U_0 = \bar{U}_0, \\ \Delta_x\bar{U}_0 = 0. \end{cases} \tag{2.36}$$

In a similar fashion, for $k=1,2$, we may define that U_k satisfies

$$\begin{cases} U_k = \bar{U}_k - \vec{w}\cdot\nabla_x U_{k-1}, \\ \Delta_x\bar{U}_k = -\displaystyle\int_{\mathbb{S}^2}\vec{w}\cdot\nabla_x U_{k-1}\mathrm{d}\vec{w}. \end{cases} \tag{2.37}$$

It is easy to see that $\bar{U}_k$ satisfies an elliptic equation. However, the boundary condition of $\bar{U}_k$ is unknown at this stage, since generally U_k does not necessarily satisfy the in-flow boundary condition of (1.1). Therefore, we have to resort to boundary layer analysis.

2.1.2. *Quasi-Spherical Coordinate System.* While the interior solution can be well-defined using the standard Cartesian coordinate system, the boundary layer requires a local description in a neighborhood of the physical boundary $\partial\Omega$. We call it a quasi-spherical coordinate system. The construction can be divided into the following substitutions:

Substitution 1: Spacial Substitution:
Consider the three-dimensional transport operator $\vec{w}\cdot\nabla_x$. By standard differential geometry, in a neighborhood of $\vec{x}_0\in\partial\Omega$, we can always define an orthogonal curvilinear coordinates system (ι_1,ι_2) such that at $\vec{x}_0$ the coordinate lines coincide with the principal directions (if $\vec{x}_0$ is not a umbilical point, then we can extend such properties to the whole neighborhood; note that the equation we derive is invariant under the change of coordinate system since it only depends on normal vector and tangential plane, so we only need to choose the simplest one). The boundary surface is $\vec{r}=\vec{r}(\iota_1,\iota_2)$. In addition, $\partial_1\vec{r}$ and $\partial_2\vec{r}$ denote two orthogonal tangential vectors. Then the outward unit normal vector is

$$\vec{\nu} = \frac{\partial_1\vec{r}\times\partial_2\vec{r}}{|\partial_1\vec{r}\times\partial_2\vec{r}|}. \tag{2.38}$$

Here $|\cdot|$ denotes the length and ∂_i denotes the derivative with respect to ι_i. Let

$$P = |\partial_1\vec{r}\times\partial_2\vec{r}| = |\partial_1\vec{r}|\,|\partial_2\vec{r}| = P_1P_2, \tag{2.39}$$

for $P_i=|\partial_i\vec r|$ with the unit tangential vectors

$$\vec\varsigma_1=\frac{\partial_1\vec r}{P_1},\quad \vec\varsigma_2=\frac{\partial_2\vec r}{P_2}. \tag{2.40}$$

Then consider the new coordinate system (μ,ι_1,ι_2), where μ denotes the normal distance to boundary surface $\partial\Omega$, i.e.

$$\vec x=\vec r-\mu\vec\nu. \tag{2.41}$$

The transport operator becomes

$$\begin{aligned}\vec w\cdot\nabla_x=&-\frac{\Big((\partial_1\vec r-\mu\partial_1\vec\nu)\times(\partial_2\vec r-\mu\partial_2\vec\nu)\Big)\cdot\vec w}{\Big((\partial_1\vec r-\mu\partial_1\vec\nu)\times(\partial_2\vec r-\mu\partial_2\vec\nu)\Big)\cdot\vec\nu}\frac{\partial}{\partial\mu}\\&+\frac{\Big((\partial_2\vec r-\mu\partial_2\vec\nu)\times\vec\nu\Big)\cdot\vec w}{\Big((\partial_1\vec r-\mu\partial_1\vec\nu)\times(\partial_2\vec r-\mu\partial_2\vec\nu)\Big)\cdot\vec\nu}\frac{\partial}{\partial\iota_1}-\frac{\Big((\partial_1\vec r-\mu\partial_1\vec\nu)\times\vec\nu\Big)\cdot\vec w}{\Big((\partial_1\vec r-\mu\partial_1\vec\nu)\times(\partial_2\vec r-\mu\partial_2\vec\nu)\Big)\cdot\vec\nu}\frac{\partial}{\partial\iota_2}.\end{aligned} \tag{2.42}$$

As usual, at $\vec x_0$, define the first fundamental form $(E,F,G)=\Big(\partial_1\vec r\cdot\partial_1\vec r,\partial_1\vec r\cdot\partial_2\vec r,\partial_2\vec r\cdot\partial_2\vec r\Big)$ and second fundamental form $(L,M,N)=\Big(\partial_{11}\vec r\cdot\vec\nu,\partial_{12}\vec r\cdot\vec\nu,\partial_{22}\vec r\cdot\vec\nu\Big)$. Then we have $F=M=0$ due to the orthogonality and the principal curvatures are

$$\kappa_1=\frac{L}{E},\quad \kappa_2=\frac{N}{G}. \tag{2.43}$$

Note that κ_1 and κ_2 depend on (ι_1,ι_2) and can change from point to point on $\partial\Omega$. Also,

$$\partial_1\vec\nu=\kappa_1\partial_1\vec r,\quad \partial_2\vec\nu=\kappa_2\partial_2\vec r. \tag{2.44}$$

Hence, direct computation using (2.38), (2.39) and (2.44) reveals that

$$\Big((\partial_1\vec r-\mu\partial_1\vec\nu)\times(\partial_2\vec r-\mu\partial_2\vec\nu)\Big)\cdot\vec\nu=(1-\kappa_1\mu)(1-\kappa_2\mu)(\partial_1\vec r\times\partial_2\vec r)\cdot\vec\nu=(1-\kappa_1\mu)(1-\kappa_2\mu)P, \tag{2.45}$$

$$\Big((\partial_1\vec r-\mu\partial_1\vec\nu)\times(\partial_2\vec r-\mu\partial_2\vec\nu)\Big)\cdot\vec w=(1-\kappa_1\mu)(1-\kappa_2\mu)(\partial_1\vec r\times\partial_2\vec r)\cdot\vec w=(1-\kappa_1\mu)(1-\kappa_2\mu)P(\vec w\cdot\vec\nu), \tag{2.46}$$

and

$$\Big((\partial_2\vec r-\mu\partial_2\vec\nu)\times\vec\nu\Big)\cdot\vec w=(1-\kappa_2\mu)(\partial_2\vec r\times\vec\nu)\cdot\vec w=(1-\kappa_2\mu)P_2(\vec w\cdot\vec\varsigma_1), \tag{2.47}$$

$$\Big((\partial_1\vec r-\mu\partial_1\vec\nu)\times\vec\nu\Big)\cdot\vec w=(1-\kappa_1\mu)(\partial_1\vec r\times\vec\nu)\cdot\vec w=-(1-\kappa_1\mu)P_1(\vec w\cdot\vec\varsigma_2). \tag{2.48}$$

Hence, plugging (2.45), (2.46), (2.47) and (2.48) into (2.42), we have the transport operator

$$\vec w\cdot\nabla_x=-(\vec w\cdot\vec\nu)\frac{\partial}{\partial\mu}-\frac{\vec w\cdot\vec\varsigma_1}{P_1(\kappa_1\mu-1)}\frac{\partial}{\partial\iota_1}-\frac{\vec w\cdot\vec\varsigma_2}{P_2(\kappa_2\mu-1)}\frac{\partial}{\partial\iota_2}. \tag{2.49}$$

Above computation is valid in a neighborhood $\Sigma\subset\partial\Omega$ of the boundary surface. Let

$$R_{\min}=\min_{\iota_1,\iota_2}\{R_1(\iota_1,\iota_2),R_2(\iota_1,\iota_2)\}, \tag{2.50}$$

where $R_1(\iota_1,\iota_2)=\dfrac{1}{\kappa_1(\iota_1,\iota_2)}$ and $R_2(\iota_1,\iota_2)=\dfrac{1}{\kappa_2(\iota_1,\iota_2)}$. Therefore, under the substitution $(x_1,x_2,x_3)\to(\mu,\iota_1,\iota_2)$ for $0\leq\mu<R_{\min}$, the equation (1.1) is transformed into

$$
\left\{
\begin{aligned}
&\epsilon\bigg(-(\vec{w}\cdot\vec{\nu})\frac{\partial u^\epsilon}{\partial\mu}-\frac{\vec{w}\cdot\vec{\varsigma}_1}{P_1(\kappa_1\mu-1)}\frac{\partial u^\epsilon}{\partial\iota_1}-\frac{\vec{w}\cdot\vec{\varsigma}_2}{P_2(\kappa_2\mu-1)}\frac{\partial u^\epsilon}{\partial\iota_2}\bigg)+u^\epsilon-\bar{u}^\epsilon=0\ \text{ in }\ (0,R_{\min})\times\Sigma\times\mathbb{S}^2,\\
&u^\epsilon(0,\iota_1,\iota_2,\vec{w})=g(\iota_1,\iota_2,\vec{w})\ \text{ for }\ \vec{w}\cdot\vec{\nu}<0.
\end{aligned}
\right.
\tag{2.51}
$$

Substitution 2: Velocity Substitution:
Define the orthogonal velocity substitution

$$
\left\{
\begin{aligned}
-\vec{w}\cdot\vec{\nu}&=\sin\phi,\\
\vec{w}\cdot\vec{\varsigma}_1&=\cos\phi\sin\psi,\\
\vec{w}\cdot\vec{\varsigma}_2&=\cos\phi\cos\psi,
\end{aligned}
\right.
\tag{2.52}
$$

for $\phi\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ and $\psi\in[-\pi,\pi]$. Then using chain rule (See [18]), we may directly compute

$$
\frac{\partial}{\partial\iota_1}\to\frac{\partial}{\partial\iota_1}-\kappa_1P_1\sin\psi\frac{\partial}{\partial\phi}+\left(-\frac{\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)}{P_2}+\kappa_1P_1\tan\phi\cos\psi\right)\frac{\partial}{\partial\psi},
\tag{2.53}
$$

$$
\frac{\partial}{\partial\iota_2}\to\frac{\partial}{\partial\iota_2}-\kappa_2P_2\cos\psi\frac{\partial}{\partial\phi}+\left(\frac{\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)}{P_1}-\kappa_2P_2\tan\phi\sin\psi\right)\frac{\partial}{\partial\psi}.
\tag{2.54}
$$

Then inserting (2.53) and (2.54) into (2.51), we obtain the transport operator

$$
\begin{aligned}
\vec{w}\cdot\nabla_x=&\sin\phi\frac{\partial}{\partial\mu}-\left(\frac{\sin^2\psi}{R_1-\mu}+\frac{\cos^2\psi}{R_2-\mu}\right)\cos\phi\frac{\partial}{\partial\phi}\\
&+\left(\frac{\cos\phi\sin\psi}{P_1(1-\kappa_1\mu)}\frac{\partial}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\kappa_2\mu)}\frac{\partial}{\partial\iota_2}\right)\\
&+\Bigg(\frac{\sin\psi}{1-\kappa_1\mu}\bigg(\cos\phi\Big(\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\
&+\frac{\cos\psi}{1-\kappa_2\mu}\bigg(-\cos\phi\Big(\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial}{\partial\psi}.
\end{aligned}
\tag{2.55}
$$

Hence, under substitution $(w_1,w_2,w_3)\to(\phi,\psi)$ for $\phi\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ and $\psi\in[-\pi,\pi]$, the equation (1.1) is transformed into

$$
\left\{
\begin{aligned}
&\epsilon\sin\phi\frac{\partial u^\epsilon}{\partial\mu}-\epsilon\left(\frac{\sin^2\psi}{R_1-\mu}+\frac{\cos^2\psi}{R_2-\mu}\right)\cos\phi\frac{\partial u^\epsilon}{\partial\phi}\\
&\ +\epsilon\left(\frac{\cos\phi\sin\psi}{P_1(1-\kappa_1\mu)}\frac{\partial u^\epsilon}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\kappa_2\mu)}\frac{\partial u^\epsilon}{\partial\iota_2}\right)\\
&\ +\epsilon\Bigg(\frac{\sin\psi}{1-\kappa_1\mu}\bigg(\cos\phi\Big(\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\
&\ +\frac{\cos\psi}{1-\kappa_2\mu}\bigg(-\cos\phi\Big(\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial u^\epsilon}{\partial\psi}\\
&+u^\epsilon-\bar{u}^\epsilon=0\ \text{ in }\ (0,R_{\min})\times\Sigma\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[-\pi,\pi],\\
&u^\epsilon(0,\iota_1,\iota_2,\phi,\psi)=g(\iota_1,\iota_2,\phi,\psi)\ \text{ for }\ \sin\phi>0.
\end{aligned}
\right.
\tag{2.56}
$$

Substitution 3: Scaling Substitution:

Define the scaled variable $\eta=\dfrac{\mu}{\epsilon}$, which implies $\dfrac{\partial}{\partial\mu}=\dfrac{1}{\epsilon}\dfrac{\partial}{\partial\eta}$. Then, under the substitution $\mu\rightarrow\eta$, the equation (1.1) is transformed into

$$
\left\{
\begin{aligned}
&\sin\phi\frac{\partial u^{\epsilon}}{\partial\eta}-\epsilon\bigg(\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\bigg)\cos\phi\frac{\partial u^{\epsilon}}{\partial\phi}\\
&+\epsilon\bigg(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial u^{\epsilon}}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial u^{\epsilon}}{\partial\iota_2}\bigg)\\
&+\epsilon\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec\varsigma_1\cdot\Big(\vec\varsigma_2\times(\partial_{12}\vec r\times\vec\varsigma_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\
&+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec\varsigma_2\cdot\Big(\vec\varsigma_1\times(\partial_{12}\vec r\times\vec\varsigma_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial u^{\epsilon}}{\partial\psi}+u^{\epsilon}-\bar u^{\epsilon}=0\\
&+u^{\epsilon}-\bar u^{\epsilon}=0\ \ \text{in}\ \ \left(0,\frac{R_{\min}}{\epsilon}\right)\times\Sigma\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[-\pi,\pi],\\
&u^{\epsilon}(0,\iota_1,\iota_2,\phi,\psi)=g(\iota_1,\iota_2,\phi,\psi)\ \ \text{for}\ \ \sin\phi>0.
\end{aligned}
\right.
\tag{2.57}
$$

2.1.3. *Boundary Layer Expansion.* We define the boundary layer expansion as follows:

$$
\mathscr{U}^B(\eta,\iota_1,\iota_2,\phi,\psi)\sim\mathscr{U}^B_0(\eta,\iota_1,\iota_2,\phi,\psi)+\epsilon\mathscr{U}^B_1(\eta,\iota_1,\iota_2,\phi,\psi),
\tag{2.58}
$$

where $\mathscr{U}^B_k$ can be defined by comparing the order of ϵ via plugging (2.58) into the equation (2.57). Thus, in a neighborhood of the boundary, we have

$$
\sin\phi\frac{\partial\mathscr{U}^B_0}{\partial\eta}-\epsilon\bigg(\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\bigg)\cos\phi\frac{\partial\mathscr{U}^B_0}{\partial\phi}+\mathscr{U}^B_0-\bar{\mathscr{U}}^B_0=0,
\tag{2.59}
$$

$$
\sin\phi\frac{\partial\mathscr{U}^B_1}{\partial\eta}-\epsilon\bigg(\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\bigg)\cos\phi\frac{\partial\mathscr{U}^B_1}{\partial\phi}+\mathscr{U}^B_1-\bar{\mathscr{U}}^B_1=-G[\mathscr{U}^B_0],
\tag{2.60}
$$

where

$$
\begin{aligned}
G[\mathscr{U}^B_0]=&\bigg(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathscr{U}^B_0}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial\mathscr{U}^B_0}{\partial\iota_2}\bigg)\\
&+\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec\varsigma_1\cdot\Big(\vec\varsigma_2\times(\partial_{12}\vec r\times\vec\varsigma_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\
&+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec\varsigma_2\cdot\Big(\vec\varsigma_1\times(\partial_{12}\vec r\times\vec\varsigma_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial\mathscr{U}^B_0}{\partial\psi},
\end{aligned}
\tag{2.61}
$$

and

$$
\bar{\mathscr{U}}^B_k(\eta,\iota_1,\iota_2)=\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathscr{U}^B_k(\eta,\iota_1,\iota_2,\phi,\psi)\cos\phi\,\mathrm{d}\phi\mathrm{d}\psi.
\tag{2.62}
$$

We call this type of equations the ϵ-Milne problem with geometric correction.

2.1.4. *Decomposition and Modification.* In this section, we prove the important decomposition of boundary data, which can greatly improve the regularity. The idea is adapted from [17] for the flat Milne problem.

Consider the ϵ-Milne problem with geometric correction with $L=\epsilon^{-\frac{1}{2}}$ and $\mathscr{R}[\phi]=-\phi$,

$$
\left\{
\begin{aligned}
&\sin\phi\frac{\partial f}{\partial\eta}-\epsilon\bigg(\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\bigg)\cos\phi\frac{\partial f}{\partial\phi}+f-\bar f=0,\\
&f(0,\phi,\psi)=g(\phi,\psi)\ \ \text{for}\ \ \sin\phi>0,\\
&f(L,\phi,\psi)=f(L,\mathscr{R}[\phi],\psi).
\end{aligned}
\right.
\tag{2.63}
$$

We assume that $g(\phi,\psi)\in C^2$ is not a constant (if g is indeed a constant, then the leading-order boundary layer vanishes and it reduces to the case in [7].) and $0\leq g(\phi,\psi)\leq 1$. This is always achievable and we do not lose the generality since the equation is linear. For some $\alpha>0$ which will be determined later, define two C^∞ auxiliary functions for $\phi\in\left[0,\dfrac{\pi}{2}\right]$ and $\psi\in[-\pi,\pi]$,

$$g_1(\phi,\psi)=\begin{cases}0 & \text{for}\quad \phi\in(0,\epsilon^\alpha],\\ g(\phi,\psi) & \text{for}\quad \phi\in\left[2\epsilon^\alpha,\dfrac{\pi}{2}\right],\end{cases} \tag{2.64}$$

and

$$g_2(\phi,\psi)=\begin{cases}1 & \text{for}\quad \phi\in(0,\epsilon^\alpha],\\ g(\phi,\psi) & \text{for}\quad \phi\in\left[2\epsilon^\alpha,\dfrac{\pi}{2}\right].\end{cases} \tag{2.65}$$

A standard construction using mollifier justifies the existence of g_i for $i=1,2$. Also, we can easily obtain $\left\|\dfrac{\partial g_i}{\partial\phi}\right\|_{L^\infty_{\phi,\psi}}\leq C\epsilon^{-\alpha}$ and $\left\|\dfrac{\partial^2 g_i}{\partial\phi^2}\right\|_{L^\infty_{\phi,\psi}}\leq C\epsilon^{-2\alpha}$. In addition, it is easy to verify that $\left\|\dfrac{\partial g_i}{\partial\psi}\right\|_{L^\infty_{\phi,\psi}}\leq C$. Let $f_1(\eta,\phi,\psi)$ and $f_2(\eta,\phi,\psi)$ be the solutions to the equation (2.63) with in-flow data $g_1(\phi,\psi)$ and $g_2(\phi,\psi)$ respectively. Then by Theorem 4.1, we know f_1 and f_2 are well-defined in $L^\infty_{\eta,\phi,\psi}$. By Theorem 4.3, they satisfy the maximum principle, which means

$$f_1(0,0^+,\psi)-\bar f_1(0)=-\bar f_1(0)<0, \tag{2.66}$$

$$f_2(0,0^+,\psi)-\bar f_2(0)=1-\bar f_2(0)>0. \tag{2.67}$$

Therefore, there exists a constant $0<\lambda<1$ such that

$$\lambda\Big(f_1(0,0^+,\psi)-\bar f_1(0)\Big)+(1-\lambda)\Big(f_2(0,0^+,\psi)-\bar f_2(0)\Big)=0. \tag{2.68}$$

Let $g_\lambda(\phi)=\lambda g_1(\phi)+(1-\lambda)g_2(\phi)$ and the corresponding solution to the equation (2.63) is $f_\lambda(\eta,\phi)$. We have

$$f_\lambda(0,0^+,\psi)-\bar f_\lambda(0)=0. \tag{2.69}$$

Since for $\phi\in(0,\epsilon^\alpha]$, $g_\lambda=1-\lambda$ is a constant, we naturally have $\dfrac{\partial g_\lambda}{\partial\phi}=0$. We may solve from the equation (2.63) that at $\eta=0,\phi\in[0,\epsilon^\alpha],\psi\in[-\pi,\pi]$,

$$\frac{\partial f_\lambda}{\partial\eta}=\frac{1}{\sin\phi}\Bigg(\epsilon\bigg(\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\bigg)\cos\phi\frac{\partial g_\lambda}{\partial\phi}-\Big(f_\lambda-\bar f_\lambda\Big)\Bigg)=0. \tag{2.70}$$

Note that $g_\lambda(\phi,\psi)=g(\phi,\psi)$ for $\phi\in\left[2\epsilon^\alpha,\dfrac{\pi}{2}\right]$, so our modification is restricted to a small region near the grazing set and we can smoothen the normal derivative at the boundary.

This method can be easily generalized to treat other $g(\phi,\psi)$. In principle, for $g(\phi,\psi)\in C^1$, we can define a decomposition

$$g(\phi,\psi)=\mathscr{G}(\phi,\psi)+\mathfrak{G}(\phi,\psi), \tag{2.71}$$

such that $\mathfrak{G}(\phi,\psi)=0$ for $\sin\phi\geq 2\epsilon^\alpha$, and the solution to the equation (2.63) with in-flow data $\mathscr{G}(\phi)$ has L^∞ normal derivative at η=0. Such a decomposition comes with a price. Originally, we have $\left\|\dfrac{\partial g}{\partial\phi}\right\|_{L^\infty_{\phi,\psi}}\leq C$. However, now we only have $\left\|\dfrac{\partial\mathscr{G}}{\partial\phi}\right\|_{L^\infty_{\phi,\psi}}\leq C\epsilon^{-\alpha}$ and $\left\|\dfrac{\partial\mathfrak{G}}{\partial\phi}\right\|_{L^\infty_{\phi,\psi}}\leq C\epsilon^{-\alpha}$ due to the short-ranged cut-off function.

For either $\mathscr{G}$ and $\mathfrak{G}$, we may define the corresponding boundary layer $\mathscr{U}$ and $\mathfrak{U}$. We call $\mathscr{U}$ the regular boundary layer and expand it up to $O(\epsilon)$, i.e.

$$\mathscr{U}(\eta,\iota_1,\iota_2,\phi,\psi)\sim\mathscr{U}_0(\eta,\iota_1,\iota_2,\phi,\psi)+\epsilon\mathscr{U}_1(\eta,\iota_1,\iota_2,\phi,\psi). \tag{2.72}$$

Also, we call $\mathfrak{U}$ the singular boundary layer and only expand it to $O(1)$, i.e.

$$\mathfrak{U}(\eta,\iota_1,\iota_2,\phi,\psi)\sim\mathfrak{U}_0(\eta,\iota_1,\iota_2,\phi,\psi). \tag{2.73}$$

They should both satisfy the ϵ-Milne problem with geometric correction.

2.1.5. *Matching Procedure.* The bridge between the interior solution and boundary layer is the boundary condition of (1.1), so we consider the boundary expansion:

$$U_0(\vec{x}_0,\vec{w})+\mathscr{U}_0(\vec{x}_0,\vec{w})+\mathfrak{U}_0(\vec{x}_0,\vec{w})=g(\vec{x}_0,\vec{w}) \quad \text{for} \quad \vec{x}_0\in\partial\Omega, \tag{2.74}$$

$$U_1(\vec{x}_0,\vec{w})+\mathscr{U}_1(\vec{x}_0,\vec{w})=0 \quad \text{for} \quad \vec{x}_0\in\partial\Omega. \tag{2.75}$$

The construction and determination of asymptotic expansion are as follows:

Step 0: Preliminaries.
Define the force

$$F(\epsilon;\eta,\iota_1,\iota_2,\psi)=-\epsilon\bigg(\frac{\sin^2\psi}{R_1(\iota_1,\iota_2)-\epsilon\eta}+\frac{\cos^2\psi}{R_2(\iota_1,\iota_2)-\epsilon\eta}\bigg). \tag{2.76}$$

Define the length of boundary layer $L=\epsilon^{-n}$ for $0<n<\dfrac{1}{2}$. For $\phi\in\Big[-\dfrac{\pi}{2},\dfrac{\pi}{2}\Big]$, denote $\mathscr{R}[\phi]=-\phi$.

Step 1: Construction of $\mathscr{U}_0$, $\mathfrak{U}_0$ and U_0.
Define the zeroth-order regular boundary layer as

$$\left\{\begin{array}{l}\mathscr{U}_0(\eta,\iota_1,\iota_2,\phi,\psi)=\mathscr{F}_0(\eta,\iota_1,\iota_2,\phi,\psi)-\mathscr{F}_{0,L}(\iota_1,\iota_2),\\ \sin\phi\dfrac{\partial\mathscr{F}_0}{\partial\eta}+F(\epsilon;\eta,\iota_1,\iota_2,\psi)\cos\phi\dfrac{\partial\mathscr{F}_0}{\partial\phi}+\mathscr{F}_0-\bar{\mathscr{F}}_0=0,\\ \mathscr{F}_0(0,\iota_1,\iota_2,\phi,\psi)=\mathscr{G}(\iota_1,\iota_2,\phi,\psi) \quad \text{for} \quad \sin\phi>0,\\ \mathscr{F}_0(L,\iota_1,\iota_2,\phi,\psi)=\mathscr{F}_0(L,\iota_1,\iota_2,\mathscr{R}[\phi],\psi),\end{array}\right. \tag{2.77}$$

with $\mathscr{F}_{0,L}(\iota_1,\iota_2)$ is defined as in Theorem 4.1.

Define the zeroth-order singular boundary layer as

$$\left\{\begin{array}{l}\mathfrak{U}_0(\eta,\iota_1,\iota_2,\phi,\psi)=\mathfrak{F}_0(\eta,\iota_1,\iota_2,\phi,\psi)-\mathfrak{F}_{0,L}(\iota_1,\iota_2),\\ \sin\phi\dfrac{\partial\mathfrak{F}_0}{\partial\eta}+F(\epsilon;\eta,\iota_1,\iota_2,\psi)\cos\phi\dfrac{\partial\mathfrak{F}_0}{\partial\phi}+\mathfrak{F}_0-\bar{\mathfrak{F}}_0=0,\\ \mathfrak{F}_0(0,\iota_1,\iota_2,\phi,\psi)=\mathfrak{G}(\iota_1,\iota_2,\phi,\psi) \quad \text{for} \quad \sin\phi>0,\\ \mathfrak{F}_0(L,\iota_1,\iota_2,\phi,\psi)=\mathfrak{F}_0(L,\iota_1,\iota_2,\mathscr{R}[\phi],\psi),\end{array}\right. \tag{2.78}$$

with $\mathfrak{F}_{0,L}(\iota_1,\iota_2)$ is defined as in Theorem 4.1.

Also, define the zeroth-order interior solution $U_0(\vec{x},\vec{w})$ as

$$\left\{\begin{array}{l}U_0(\vec{x},\vec{w})=\bar{U}_0(\vec{x}),\\ \Delta_x\bar{U}_0(\vec{x})=0 \quad \text{in} \quad \Omega,\\ \bar{U}_0(\vec{x}_0)=\mathscr{F}_{0,L}(\iota_1,\iota_2)+\mathfrak{F}_{0,L}(\iota_1,\iota_2) \quad \text{on} \quad \partial\Omega.\end{array}\right. \tag{2.79}$$

Step 2: Construction of $\mathscr{U}_1$ and U_1.
Define the first-order regular boundary layer as

$$\left\{\begin{array}{l}\mathscr{U}_1(\eta,\iota_1,\iota_2,\phi,\psi)=\mathscr{F}_1(\eta,\iota_1,\iota_2,\phi,\psi)-\mathscr{F}_{1,L}(\iota_1,\iota_2),\\ \sin\phi\dfrac{\partial\mathscr{F}_1}{\partial\eta}+F(\epsilon;\eta,\iota_1,\iota_2,\psi)\cos\phi\dfrac{\partial\mathscr{F}_1}{\partial\phi}+\mathscr{F}_1-\bar{\mathscr{F}}_1=G[\mathscr{U}_0],\\ \mathscr{F}_1(0,\iota_1,\iota_2,\phi,\psi)=\vec{w}\cdot\nabla_xU_0(0,\iota_1,\iota_2) \quad \text{for} \quad \sin\phi>0,\\ \mathscr{F}_1(L,\iota_1,\iota_2,\phi,\psi)=\mathscr{F}_1(L,\iota_1,\iota_2,\mathscr{R}[\phi],\psi),\end{array}\right. \tag{2.80}$$

with $\mathscr{F}_{1,L}(\iota_1,\iota_2)$ is defined as in Theorem 4.1, and G_0 is defined in (2.61).

Then define the first-order interior solution $U_1(\vec{x},\vec{w})$ as

$$\left\{\begin{array}{l} U_1(\vec{x},\vec{w})=\bar{U}_1(\vec{x})-\vec{w}\cdot\nabla_xU_0(\vec{x},\vec{w}),\\ \Delta_x\bar{U}_1(\vec{x})=-\displaystyle\int_{\mathbb{S}^1}\Big(\vec{w}\cdot\nabla_xU_0(\vec{x},\vec{w})\Big)\mathrm{d}\vec{w}\ \text{ in }\ \Omega,\\ \bar{U}_1(\vec{x}_0)=\mathscr{F}_{1,L}(\iota_1,\iota_2)\ \text{ on }\ \partial\Omega. \end{array}\right. \tag{2.81}$$

Note that we do not define $\mathfrak{U}_1$ here.

Step 3: Construction of U_2.
Since we do not expand to $\mathscr{U}_2$ and $\mathfrak{U}_2$, simply define the second-order interior solution as

$$\left\{\begin{array}{l} U_2(\vec{x},\vec{w})=\bar{U}_2(\vec{x})-\vec{w}\cdot\nabla_xU_1(\vec{x},\vec{w}),\\ \Delta_x\bar{U}_2(\vec{x})=-\displaystyle\int_{\mathbb{S}^1}\Big(\vec{w}\cdot\nabla_xU_1(\vec{x},\vec{w})\Big)\mathrm{d}\vec{w}\ \text{ in }\ \Omega,\\ \bar{U}_2(\vec{x}_0)=0\ \text{ on }\ \partial\Omega. \end{array}\right. \tag{2.82}$$

Here, we might have $O(\epsilon^3)$ error in this step due to the trivial boundary data. Thanks to the remainder estimate, it will not affect the diffusive limit.

2.2. **Remainder Estimate.** In this section, we consider the remainder equation for $u(\vec{x},\vec{w})$ as

$$\left\{\begin{array}{l} \epsilon\vec{w}\cdot\nabla_xu+u-\bar{u}=S(\vec{x},\vec{w})\ \text{ in }\ \Omega\times\mathbb{S}^2,\\ u(\vec{x}_0,\vec{w})=g(\vec{x}_0,\vec{w})\ \text{ for }\ \vec{w}\cdot\vec{\nu}<0\ \text{ and }\ \vec{x}_0\in\partial\Omega. \end{array}\right. \tag{2.83}$$

Define the L^p norms with $1\leq p<\infty$ and L^∞ norm in $\Omega\times\mathbb{S}^2$ as usual:

$$\|f\|_{L^p(\Omega\times\mathbb{S}^2)}=\bigg(\int_\Omega\int_{\mathbb{S}^2}|f(\vec{x},\vec{w})|^p\,\mathrm{d}\vec{w}\mathrm{d}\vec{x}\bigg)^{\frac{1}{p}}, \tag{2.84}$$

$$\|f\|_{L^\infty(\Omega\times\mathbb{S}^2)}=\mathrm{esssup}_{(\vec{x},\vec{w})\in\Omega\times\mathbb{S}^2}\left|f(\vec{x},\vec{w})\right|. \tag{2.85}$$

Define the L^p norm with $1\leq p<\infty$ and L^∞ norm on the boundary $\Gamma=\partial\Omega\times\mathbb{S}^2$ as follows:

$$\|f\|_{L^p(\Gamma^\pm)}=\bigg(\iint_{\Gamma^\pm}|f(\vec{x},\vec{w})|^p\,|\vec{w}\cdot\vec{\nu}|\,\mathrm{d}\vec{w}\mathrm{d}\vec{x}\bigg)^{\frac{1}{p}}, \tag{2.86}$$

$$\|f\|_{L^\infty(\Gamma^\pm)}=\mathrm{esssup}_{(\vec{x},\vec{w})\in\Gamma^\pm}\left|f(\vec{x},\vec{w})\right|. \tag{2.87}$$

In particular, we denote $\mathrm{d}\gamma=(\vec{w}\cdot\vec{\nu})\mathrm{d}\vec{w}\mathrm{d}\vec{x}$ on the boundary.

The 3D steady remainder estimate follows from a similar argument as [23, Section 3] (also see [18]), so we omit the details here and only provide the main theorem.

Theorem 2.1. *The unique solution $u(\vec{x},\vec{w})$ to the equation* (2.83) *satisfies for integer $1\leq m\leq 3$,*

$$\begin{aligned}\|u\|_{L^\infty(\Omega\times\mathbb{S}^2)}\leq&C\bigg(\frac{1}{\epsilon^{1+\frac{3}{2m}}}\|S\|_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{2+\frac{3}{2m}}}\|S\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^2)}+\|S\|_{L^\infty(\Omega\times\mathbb{S}^2)}\\&+\frac{1}{\epsilon^{\frac{1}{2}+\frac{3}{2m}}}\|g\|_{L^2(\Gamma^-)}+\frac{1}{\epsilon^{\frac{3}{2m}}}\|g\|_{L^{\frac{4m}{3}}(\Gamma^-)}+\|g\|_{L^\infty(\Gamma^-)}\bigg).\end{aligned} \tag{2.88}$$

2.3. **Diffusive Limit.**

2.3.1. *Analysis of Regular Boundary Layer.* In this subsection, we will justify that the regular boundary layers are all well-defined for $0<\alpha\leq 1$. In the following, we will not show ϵ, ι_i and ψ dependence when there is no confusion.

Step 1: Well-Posedness of $\mathscr{U}_0$.

Based on (2.77), $\mathscr{U}_0$ satisfies the ϵ-Milne problem with geometric correction

$$\left\{\begin{array}{l}\sin\phi\dfrac{\partial\mathscr{U}_0}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathscr{U}_0}{\partial\phi}+\mathscr{U}_0-\bar{\mathscr{U}}_0=0,\\ \mathscr{U}_0(0,\phi)=\mathscr{G}(\phi)-\mathscr{F}_{0,L}\quad\text{for}\quad\sin\phi>0,\\ \mathscr{U}_0(L,\phi)=\mathscr{U}_0(L,\mathscr{R}[\phi]),\end{array}\right. \tag{2.89}$$

where

$$F(\eta)=-\epsilon\bigg(\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\bigg). \tag{2.90}$$

Therefore, since $\|\mathscr{G}\|_{L^\infty_{+,\phi,\psi}}\leq C$, applying Theorem 4.2 to the equation (2.89), we know

$$\left\|\mathrm{e}^{K_0\eta}\mathscr{U}_0\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C. \tag{2.91}$$

Step 2: Tangential Derivatives of $\mathscr{U}_0$.
For $i=1,2$, the ι_i derivative $W_i=\dfrac{\partial\mathscr{U}_0}{\partial\iota_i}$ satisfies

$$\left\{\begin{array}{l}\sin\phi\dfrac{\partial W_i}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial W_i}{\partial\phi}+W_i-\bar{W}_i=-\epsilon\left(\dfrac{\partial_{\iota_i}R_1\sin^2\psi}{(R_1-\epsilon\eta)^2}+\dfrac{\partial_{\iota_i}R_2\cos^2\psi}{(R_2-\epsilon\eta)^2}\right)\cos\phi\dfrac{\partial\mathscr{U}_0}{\partial\phi},\\ W_i(0,\phi)=\dfrac{\partial\mathscr{G}}{\partial\iota_i}(\phi)-\dfrac{\partial\mathscr{F}_{0,L}}{\partial\iota_i}\quad\text{for}\quad\sin\phi>0,\\ W_i(L,\phi)=W_i(L,\mathscr{R}[\phi]).\end{array}\right. \tag{2.92}$$

Since

$$\begin{aligned}\left|\frac{\partial_{\iota_i}R_1\sin^2\psi}{(R_1-\epsilon\eta)^2}+\frac{\partial_{\iota_i}R_2\cos^2\psi}{(R_2-\epsilon\eta)^2}\right|&\leq\max\left\{\left|\frac{\partial_{\iota_i}R_1}{R_1-\epsilon\eta}\right|,\left|\frac{\partial_{\iota_i}R_2}{R_2-\epsilon\eta}\right|\right\}\left|\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\right|\\&\leq C\left|\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\right|,\end{aligned} \tag{2.93}$$

we have

$$\left|-\epsilon\left(\frac{\partial_{\iota_i}R_1\sin^2\psi}{(R_1-\epsilon\eta)^2}+\frac{\partial_{\iota_i}R_2\cos^2\psi}{(R_2-\epsilon\eta)^2}\right)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right|\leq C\left|F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right|. \tag{2.94}$$

Applying Theorem 4.5 to the equation (2.89), we know

$$\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8\left(\|\mathscr{G}\|_{L^\infty_{+,\phi,\psi}}+\left\|\epsilon\frac{\partial\mathscr{G}}{\partial\phi}\right\|_{L^\infty_{+,\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\mathscr{U}_0\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right)\leq C\left|\ln(\epsilon)\right|^8. \tag{2.95}$$

Note that here although $\left\|\dfrac{\partial\mathscr{G}}{\partial\phi}\right\|_{L^\infty_{\phi,\psi}}\leq C\epsilon^{-\alpha}$, with the help of ϵ, we can get rid of this negative power. Therefore, applying Theorem 4.2 to the equation (2.92), we have

$$\left\|\mathrm{e}^{K_0\eta}W_i\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8. \tag{2.96}$$

Step 3: Velocity Derivatives of $\mathscr{U}_0$.

The ψ derivative $H=\dfrac{\partial\mathscr{U}_0}{\partial\psi}$ satisfies

$$\left\{\begin{array}{l}\sin\phi\dfrac{\partial H}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial H}{\partial\phi}+H=\epsilon\bigg(\dfrac{2\sin\psi\cos\psi}{R_1-\epsilon\eta}-\dfrac{2\sin\psi\cos\psi}{R_2-\epsilon\eta}\bigg)\cos\phi\dfrac{\partial\mathscr{U}_0}{\partial\phi},\\ H(0,\phi)=\dfrac{\partial\mathscr{G}}{\partial\psi}(\phi)\ \ \text{for}\ \ \sin\phi>0,\\ H(L,\phi)=H(L,\mathscr{R}[\phi]).\end{array}\right.\tag{2.97}$$

Since

$$\left|\frac{2\sin\psi\cos\psi}{R_1-\epsilon\eta}-\frac{2\sin\psi\cos\psi}{R_2-\epsilon\eta}\right|\leq C\left|\frac{1}{\max\{R_1,R_2\}-\epsilon\eta}\right|\leq C\left|\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\right|,\tag{2.98}$$

we have

$$\left|\epsilon\bigg(\frac{2\sin\psi\cos\psi}{R_1-\epsilon\eta}-\frac{2\sin\psi\cos\psi}{R_2-\epsilon\eta}\bigg)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right|\leq C\left|F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right|.\tag{2.99}$$

Therefore, using (2.95), applying Theorem 4.6 to the equation (2.97) (by the construction of $\mathscr{U}_0$, H must satisfy the requirement of Theorem 4.6), we have

$$\left\|\mathrm{e}^{K_0\eta}H\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8.\tag{2.100}$$

Step 4: Well-Posedness of $\mathscr{U}_1$.
$\mathscr{U}_1$ satisfies the ϵ-Milne problem with geometric correction

$$\left\{\begin{array}{l}\sin\phi\dfrac{\partial\mathscr{U}_1}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathscr{U}_1}{\partial\phi}+\mathscr{U}_1-\bar{\mathscr{U}}_1=G[\mathscr{U}_0],\\ \mathscr{U}_1(0,\phi)=\vec{w}\cdot\nabla_xU_0(\vec{x}_0,\vec{w})-\mathscr{F}_{1,L}\ \ \text{for}\ \ \sin\phi>0,\\ \mathscr{U}_1(L,\phi)=\mathscr{U}_1(L,\mathscr{R}[\phi]),\end{array}\right.\tag{2.101}$$

where

$$\begin{aligned}G[\mathscr{U}_0]=&\bigg(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}W_1+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}W_2\bigg)\\&+\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\&+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}H\\=&G_{W_1}W_1+G_{W_2}W_2+G_HH.\end{aligned}\tag{2.102}$$

Here we use G_{W_1}, G_{W_2} and G_H to denote the quantities in front of W_1, W_2 and H. We may directly bound

$$|G_{W_1}|+|G_{W_2}|+|G_H|\leq C.\tag{2.103}$$

Using (2.96) and (2.100), we have

$$\left\|\mathrm{e}^{K_0\eta}G[\mathscr{U}_0]\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8.\tag{2.104}$$

Therefore, applying Theorem 4.2 to the equation (2.101), we know

$$\left\|\mathrm{e}^{K_0\eta}\mathscr{U}_1\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8.\tag{2.105}$$

Step 5: Tangential and Velocity Derivatives of $\mathscr{U}_1$.

For $j=1,2$, the ι_j derivative $V_j=\dfrac{\partial\mathscr{U}_1}{\partial\iota_j}$ satisfies

$$
\begin{cases}
\sin\phi\dfrac{\partial V_j}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial V_j}{\partial\phi}+V_j-\bar{V}_j=S_1+S_2+S_3,\\
V_j(0,\phi)=\dfrac{\partial}{\partial\iota_j}\Big(\vec{w}\cdot\nabla_xU_0(\vec{x}_0,\vec{w})-\mathscr{F}_{1,L}\Big)\ \ \text{for}\quad\sin\phi>0,\\
V_j(L,\phi)=V_j(L,\mathscr{R}[\phi]),
\end{cases}
\tag{2.106}
$$

where

$$
S_1=-\,\epsilon\left(\frac{\partial_{\iota_j}R_1\sin^2\psi}{(R_1-\epsilon\eta)^2}+\frac{\partial_{\iota_j}R_2\cos^2\psi}{(R_2-\epsilon\eta)^2}\right)\cos\phi\frac{\partial\mathscr{U}_1}{\partial\phi},
\tag{2.107}
$$

$$
S_2=\frac{\partial G_{W_1}}{\partial\iota_j}W_1+\frac{\partial G_{W_2}}{\partial\iota_j}W_2+\frac{\partial G_H}{\partial\iota_j}H,
\tag{2.108}
$$

$$
S_3=G_{W_1}\frac{\partial W_1}{\partial\iota_j}+G_{W_2}\frac{\partial W_2}{\partial\iota_j}+G_H\frac{\partial H}{\partial\iota_j}.
\tag{2.109}
$$

Using (2.93), we know

$$
\left\|\mathrm{e}^{K_0\eta}S_1\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_1}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}.
\tag{2.110}
$$

Using (2.96) and (2.100), applying Theorem 4.5 to the equation (2.101), we have

$$
\begin{aligned}
&\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_1}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\\
\leq& C\left(\left\|\mathrm{e}^{K_0\eta}\Big(G_{W_1}W_1+G_{W_2}W_2+G_HH\Big)\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial}{\partial\eta}\Big(G_{W_1}W_1+G_{W_2}W_2+G_HH\Big)\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right)\\
\leq& C\bigg(\left\|\mathrm{e}^{K_0\eta}W_1\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}W_2\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}H\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\\
&+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W_1}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W_2}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial H}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\bigg)\\
\leq& C\left(1+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W_1}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W_2}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial H}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right).
\end{aligned}
\tag{2.111}
$$

Here the weight function ζ is defined in (4.462). Combining (2.110) and (2.111), we know

$$
\left\|\mathrm{e}^{K_0\eta}S_1\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left(1+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W_1}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W_2}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial H}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right).
\tag{2.112}
$$

On the other hand, using (2.96) and (2.100), we directly estimate

$$
\left\|\mathrm{e}^{K_0\eta}S_2\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\Big(\left\|\mathrm{e}^{K_0\eta}W_1\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}W_2\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}H\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\Big)\leq C.
\tag{2.113}
$$

Also, we know

$$
\left\|\mathrm{e}^{K_0\eta}S_3\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left(\left\|\mathrm{e}^{K_0\eta}\frac{\partial W_1}{\partial\iota_j}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\frac{\partial W_2}{\partial\iota_j}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\frac{\partial H}{\partial\iota_j}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right).
\tag{2.114}
$$

Summarizing (2.112), (2.113) and (2.114) and applying Theorem 4.2, we know

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}V_j\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}} \leq&\left\|\mathrm{e}^{K_0\eta}S_1\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}S_2\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}S_3\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\\ \leq&C\bigg(1+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W_1}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W_2}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial H}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\\ &+\left\|\mathrm{e}^{K_0\eta}\frac{\partial W_1}{\partial\iota_j}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\frac{\partial W_2}{\partial\iota_j}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\frac{\partial H}{\partial\iota_j}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\bigg).\end{aligned}\tag{2.115}$$

Therefore, in order to bound V_j, we need the η and ι_i derivative estimates of W_i and H.

Using a similar argument, we obtain the estimate of ψ derivative $M=\dfrac{\partial\mathscr{U}_1}{\partial\psi}$:

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}M\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}} \leq&C\bigg(1+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W_1}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W_2}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial H}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\\ &+\left\|\mathrm{e}^{K_0\eta}\frac{\partial W_1}{\partial\psi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\frac{\partial W_2}{\partial\psi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\frac{\partial H}{\partial\psi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\bigg).\end{aligned}\tag{2.116}$$

Therefore, in order to bound M, we need the η and ψ derivative estimates of W_i and H.

Step 6: Tangential and Velocity Derivatives of W_i and H.
The ι_j derivative $\Theta_{ij}=\dfrac{\partial W_i}{\partial\iota_j}$ satisfies

$$\left\{\begin{aligned}&\sin\phi\frac{\partial\Theta_{ij}}{\partial\eta}+F(\eta)\cos\phi\frac{\partial\Theta_{ij}}{\partial\phi}+\Theta_{ij}-\bar\Theta_{ij}=T_1+T_2,\\ &\Theta_{ij}(0,\phi)=\frac{\partial^2\mathscr{G}}{\partial\iota_i\partial\iota_j}(\phi)-\frac{\partial^2\mathscr{F}_0}{\partial\iota_i\partial\iota_j}\ \ \text{for}\ \ \sin\phi>0,\\ &\Theta_{ij}(L,\phi)=\Theta_{ij}(L,\mathscr{R}[\phi]),\end{aligned}\right.\tag{2.117}$$

where

$$T_1=-\epsilon\left(\frac{\partial_{\iota_j}R_1\sin^2\psi}{(R_1-\epsilon\eta)^2}+\frac{\partial_{\iota_j}R_2\cos^2\psi}{(R_2-\epsilon\eta)^2}\right)\cos\phi\bigg(\frac{\partial W_i}{\partial\phi}+\frac{\partial W_j}{\partial\phi}\bigg),\tag{2.118}$$

$$T_2=-\epsilon\frac{\partial}{\partial\iota_j}\left(\frac{\partial_{\iota_i}R_1\sin^2\psi}{(R_1-\epsilon\eta)^2}+\frac{\partial_{\iota_i}R_2\cos^2\psi}{(R_2-\epsilon\eta)^2}\right)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}.\tag{2.119}$$

Using (2.93), we have

$$\left\|\mathrm{e}^{K_0\eta}T_1\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial W_i}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}.\tag{2.120}$$

By a similar argument as (2.93), using (2.95), we know

$$\left\|\mathrm{e}^{K_0\eta}T_2\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left\|F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8.\tag{2.121}$$

Applying Theorem 4.2 to the equation (2.117), using (2.120) and (2.121), we have

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}\Theta_{ij}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq&C\bigg(\left\|\mathrm{e}^{K_0\eta}T_1\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}T_2\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\bigg)\\ \leq&C\left|\ln(\epsilon)\right|^8+C\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial W_i}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}.\end{aligned}\tag{2.122}$$

Using a similar argument, we obtain the estimate of ψ derivative $\Lambda_i=\dfrac{\partial W_i}{\partial\psi}$:

$$\left\|\mathrm{e}^{K_0\eta}\Lambda_i\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8+C\left(\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial W_i}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial H}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right). \tag{2.123}$$

Using a similar argument with Theorem 4.6, we obtain the estimate of ι_j derivative $\Upsilon_j=\dfrac{\partial H}{\partial\iota_j}$:

$$\left\|\mathrm{e}^{K_0\eta}\Upsilon_j\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8+C\left(\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial W_j}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial H}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right). \tag{2.124}$$

Using a similar argument with Theorem 4.6, we obtain the estimate of ψ derivative $\Xi=\dfrac{\partial H}{\partial\psi}$:

$$\left\|\mathrm{e}^{K_0\eta}\Xi\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8+C\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial H}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}. \tag{2.125}$$

Inserting (2.122), (2.123), (2.124), (2.125) into (2.115) and (2.116), we know

$$\begin{aligned}&\left\|\mathrm{e}^{K_0\eta}V_j\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}M\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\\ \leq&C\Bigg(1+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W_1}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W_2}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial H}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\\ &+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial W_1}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial W_2}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial H}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\Bigg).\end{aligned} \tag{2.126}$$

Hence, we need the regularity estimate of W_i and H. However, this cannot be done directly. We will first study the normal derivative of $\mathscr{U}_0$.

Step 7: Regularity of Normal Derivative.
The normal derivative $A=\dfrac{\partial\mathscr{U}_0}{\partial\eta}$ satisfies

$$\begin{cases}\sin\phi\dfrac{\partial A}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial A}{\partial\phi}+A-\bar{A}=\dfrac{\partial F}{\partial\eta}\cos\phi\dfrac{\partial\mathscr{U}_0}{\partial\phi},\\ A(0,\phi)=\dfrac{1}{\sin\phi}\left(F(\eta)\cos\phi\dfrac{\partial\mathscr{G}}{\partial\phi}(\phi)-\mathscr{G}(0,\phi)+\bar{\mathscr{U}}_0(0,\phi)\right)\ \text{ for }\ \sin\phi>0,\\ A(L,\phi)=A(L,\mathscr{R}[\phi]).\end{cases} \tag{2.127}$$

This is where the cut-off in $\mathscr{G}$ plays a role. For $0<\phi<\epsilon^{\alpha}$, we know $A(0,\phi)=0$. Here note the fact that $\left|\dfrac{\partial F}{\partial\eta}\right|\leq C\epsilon\left|F\right|$.

Based on the construction of $\mathscr{G}$, we know $\left\|\dfrac{\partial\mathscr{G}}{\partial\phi}\right\|_{L^\infty_{+,\phi,\psi}}\leq C\epsilon^{-\alpha}$ and $\left|F(\eta)\right|\leq C\epsilon$. Hence, we have $\left\|A(0,\phi)\right\|_{L^\infty_{+,\phi,\psi}}\leq C\epsilon^{-\alpha}$ and $\left\|\epsilon\dfrac{\partial A}{\partial\phi}(0,\phi)\right\|_{L^\infty_{+,\phi,\psi}}\leq C\epsilon^{-\alpha}$. Therefore, applying Theorem 4.2 to the equation (2.127) and using (2.95), we have

$$\left\|\mathrm{e}^{K_0\eta}A\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left(\left\|A(0,\phi)\right\|_{L^\infty_{+,\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right)\leq C\epsilon^{-\alpha}. \tag{2.128}$$

Applying Theorem 4.5 to the equation (2.127) and using (2.95), we know

(2.129)
$$
\begin{aligned}
&\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial A}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial A}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\\
\leq& C\left|\ln(\epsilon)\right|^8\left(\left\|\epsilon\frac{\partial A}{\partial\phi}(0,\phi)\right\|_{L^\infty_{+,\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\frac{\partial F}{\partial\eta}\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial}{\partial\eta}\left(\frac{\partial F}{\partial\eta}\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right)\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right)\\
\leq& C\left|\ln(\epsilon)\right|^8\left(\epsilon^{-\alpha}+\epsilon\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\epsilon\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial A}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right)\\
\leq& C\left|\ln(\epsilon)\right|^8\left(\epsilon^{-\alpha}+\epsilon\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial A}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right).
\end{aligned}
$$

Then we may absorb $\epsilon\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial A}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}$ into the left-hand side to obtain

$$
\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial A}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial A}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\epsilon^{-\alpha}\left|\ln(\epsilon)\right|^8. \tag{2.130}
$$

Step 8: Regularity of Velocity Derivative.
The velocity derivative $B=\dfrac{\partial\mathscr{U}_0}{\partial\phi}$ satisfies

$$
\begin{cases}
\sin\phi\dfrac{\partial B}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial B}{\partial\phi}+B=-\cos\phi\dfrac{\partial\mathscr{U}_0}{\partial\eta}+F(\eta)\sin\phi\dfrac{\partial\mathscr{U}_0}{\partial\phi},\\
B(0,\phi)=\dfrac{\partial\mathscr{G}}{\partial\phi}(\phi)\quad\text{for}\quad\sin\phi>0,\\
B(L,\phi)=B(L,\mathscr{R}[\phi]).
\end{cases} \tag{2.131}
$$

Applying Theorem 4.6 to the equation (2.131), using (2.128), we obtain

(2.132)
$$
\begin{aligned}
\left\|\mathrm{e}^{K_0\eta}B\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq& C\bigg(\left\|\frac{\partial\mathscr{G}}{\partial\phi}\right\|_{L^\infty_{+,\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\cos\phi\frac{\partial\mathscr{U}_0}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\sin\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\bigg)\\
\leq& C\bigg(\epsilon^{-\alpha}+\left\|\mathrm{e}^{K_0\eta}A\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\epsilon\left\|\mathrm{e}^{K_0\eta}B\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\bigg)\leq C\bigg(\epsilon^{-\alpha}\left|\ln(\epsilon)\right|^8+\epsilon\left\|\mathrm{e}^{K_0\eta}B\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\bigg).
\end{aligned}
$$

Then we may absorb $\epsilon\left\|\mathrm{e}^{K_0\eta}B\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}$ into the left-hand side to obtain

$$
\left\|\mathrm{e}^{K_0\eta}B\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\epsilon^{-\alpha}\left|\ln(\epsilon)\right|^8. \tag{2.133}
$$

Then applying Theorem 4.7 to the equation (2.131), using (2.128), (2.130) and (2.133), we have

(2.134)
$$
\begin{aligned}
&\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial B}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial B}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\\
\leq&C\left|\ln(\epsilon)\right|^8\Bigg(\left\|\epsilon\frac{\partial^2\mathscr{G}}{\partial\phi^2}\right\|_{L^\infty_{+,\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}B\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\cos\phi\frac{\partial\mathscr{U}_0}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\sin\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\\
&+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial}{\partial\eta}\left(\cos\phi\frac{\partial\mathscr{U}_0}{\partial\eta}\right)\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial}{\partial\eta}\left(F(\eta)\sin\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right)\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\Bigg)\\
\leq&C\left|\ln(\epsilon)\right|^8\left(\epsilon^{-\alpha}+\epsilon^{-\alpha}+\left\|\mathrm{e}^{K_0\eta}A\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\epsilon\left\|\mathrm{e}^{K_0\eta}B\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial A}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\epsilon\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial B}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right)\\
\leq&C\left|\ln(\epsilon)\right|^8\left(\epsilon^{-\alpha}+\epsilon\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial B}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right).
\end{aligned}
$$

Then we may absorb $\epsilon\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial B}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}$ into the left-hand side to obtain

$$
\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial B}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial B}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\epsilon^{-\alpha}\left|\ln(\epsilon)\right|^8. \tag{2.135}
$$

Step 9: Regularity of Mixed Derivative.
The weighted mixed derivative $\mathcal{C}=\zeta\dfrac{\partial A}{\partial\phi}$ satisfies

$$
\begin{cases}
\sin\phi\dfrac{\partial\mathcal{C}}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathcal{C}}{\partial\phi}+\mathcal{C}=\zeta\dfrac{\partial F}{\partial\eta}\cos\phi\dfrac{\partial B}{\partial\phi}-\zeta\dfrac{\partial F}{\partial\eta}\sin\phi B-\zeta\cos\phi\dfrac{\partial A}{\partial\eta}+\zeta F(\eta)\sin\phi\dfrac{\partial A}{\partial\phi},\\
\mathcal{C}(0,\phi)=\zeta\dfrac{\partial}{\partial\phi}\left(\dfrac{1}{\sin\phi}\left(F(\eta)\cos\phi\dfrac{\partial\mathscr{G}}{\partial\phi}(\phi)-\mathscr{G}(0,\phi)+\bar{\mathscr{U}}_0(0,\phi)\right)\right)\ \text{ for }\ \sin\phi>0,\\
\mathcal{C}(L,\phi)=\mathcal{C}(L,\mathscr{R}[\phi]).
\end{cases} \tag{2.136}
$$

Applying Theorem 4.2 to the equation (2.136), using (2.130), (2.133) and (2.135), we have

$$
\begin{aligned}
\left\|\mathrm{e}^{K_0\eta}\mathcal{C}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq&C\Bigg(\epsilon^{-\alpha}+\epsilon\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial B}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\epsilon^2\left\|\mathrm{e}^{K_0\eta}B\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\\
&+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial A}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\epsilon\left\|\mathrm{e}^{K_0\eta}\mathcal{C}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\Bigg)\\
\leq&C\left(\epsilon^{-\alpha}\left|\ln(\epsilon)\right|^8+\epsilon\left\|\mathrm{e}^{K_0\eta}\mathcal{C}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right).
\end{aligned} \tag{2.137}
$$

Then we may absorb $\epsilon\left\|\mathrm{e}^{K_0\eta}\mathcal{C}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}$ into the left-hand side to obtain

$$
\left\|\mathrm{e}^{K_0\eta}\mathcal{C}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq\epsilon^{-\alpha}\left|\ln(\epsilon)\right|^8. \tag{2.138}
$$

Step 10: Regularity of Tangential and Velocity Derivative.
We turn to the regularity of W_i. Applying Theorem 4.5 to the equation (2.92), using (2.95) and (2.138),

noting $|F|\leq C\epsilon$, we have

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W_i}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial W_i}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}&\\ \leq C\left|\ln(\epsilon)\right|^8\left(\left\|\epsilon\frac{\partial^2\mathscr{G}}{\partial\iota_i\partial\phi}\right\|_{L^\infty_{+,\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\frac{\partial F}{\partial\iota_i}\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial}{\partial\eta}\left(\frac{\partial F}{\partial\iota_i}\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right)\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right)&\\ \leq C\left|\ln(\epsilon)\right|^8\left(1+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\mathcal{C}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\right)&\\ \leq C\left|\ln(\epsilon)\right|^{16}\left(1+\epsilon^{1-\alpha}\right)\leq C\left|\ln(\epsilon)\right|^{16}.&\end{aligned}\tag{2.139}$$

A similar argument holds for the regularity of H. Applying Theorem 4.7 to the equation (2.97), we have

$$\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial H}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial H}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^{16}.\tag{2.140}$$

Summary.
Inserting (2.139) and (2.140) into (2.126), we obtain

$$\left\|\mathrm{e}^{K_0\eta}V_j\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}M\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^{16}.\tag{2.141}$$

This is actually

$$\left\|\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_1}{\partial\iota_1}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_1}{\partial\iota_2}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_1}{\partial\psi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\leq C\left|\ln(\epsilon)\right|^{16}.\tag{2.142}$$

Theorem 2.2. *For $K_0>0$ sufficiently small, the regular boundary layer satisfies*

$$\begin{aligned}&\left\|\mathrm{e}^{K_0\eta}\mathscr{U}_0\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C, &&\left\|\mathrm{e}^{K_0\eta}\mathscr{U}_1\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8,\\ &\left\|\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_0}{\partial\iota_i}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8, &&\left\|\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_1}{\partial\iota_i}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^{16},\\ &\left\|\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_0}{\partial\psi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8, &&\left\|\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_1}{\partial\psi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^{16}.\end{aligned}\tag{2.143}$$

2.3.2. *Analysis of Singular Boundary Layer.* This subsection is similar to [23, Section 6.2] (also see [18]), so we only present the main theorem.

Theorem 2.3. *Let*

$$\begin{cases}\chi_1: & 0\leq\zeta<2\epsilon^\alpha,\\ \chi_2: & 2\epsilon^\alpha\leq\zeta\leq 1.\end{cases}\tag{2.144}$$

For $K_0>0$ sufficiently small, the singular boundary layer satisfies

$$\begin{aligned}&\left\|\mathrm{e}^{K_0\eta}(\chi_1\mathfrak{U}_0)\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C, &&\left\|\mathrm{e}^{K_0\eta}(\chi_2\mathfrak{U}_0)\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\epsilon^\alpha,\\ &\left\|\mathrm{e}^{K_0\eta}\frac{\partial(\chi_1\mathfrak{U}_0)}{\partial\iota_i}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8, &&\left\|\mathrm{e}^{K_0\eta}\frac{\partial(\chi_2\mathfrak{U}_0)}{\partial\iota_i}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\epsilon^\alpha\left|\ln(\epsilon)\right|^8,\\ &\left\|\mathrm{e}^{K_0\eta}\frac{\partial(\chi_1\mathfrak{U}_0)}{\partial\psi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8, &&\left\|\mathrm{e}^{K_0\eta}\frac{\partial(\chi_2\mathfrak{U}_0)}{\partial\psi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C\epsilon^\alpha\left|\ln(\epsilon)\right|^8.\end{aligned}\tag{2.145}$$

2.3.3. *Analysis of Interior Solution.* This subsection is similar to [23, Section 6.3] (also see [18]), so we only present the main theorem.

Theorem 2.4. *The interior solution satisfies*

$$\left\|U_0\right\|_{H^3(\Omega)}\leq C,\quad \left\|U_1\right\|_{H^3(\Omega)}\leq C\left|\ln(\epsilon)\right|^8,\quad \left\|U_2\right\|_{H^3(\Omega)}\leq C\left|\ln(\epsilon)\right|^8.\tag{2.146}$$

2.3.4. *Proof of Theorem 1.1.* Based on Theorem 2.1, we know there exists a unique $u^\epsilon(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathbb{S}^2)$, so we focus on the diffusive limit.

Step 1: Remainder definitions.
We define the remainder as

$$R = u^\epsilon - \sum_{k=0}^{2}\epsilon^k U_k - \sum_{k=0}^{1}\epsilon^k \mathscr{U}_k - \mathfrak{U}_0 = u^\epsilon - Q - \mathscr{Q} - \mathfrak{Q}, \tag{2.147}$$

where

$$Q = U_0 + \epsilon U_1 + \epsilon^2 U_2, \tag{2.148}$$

$$\mathscr{Q} = \mathscr{U}_0 + \epsilon \mathscr{U}_1, \tag{2.149}$$

$$\mathfrak{Q} = \mathfrak{U}_0. \tag{2.150}$$

Noting the equation (2.57) is equivalent to the equation (1.1), we write $\mathcal{L}$ to denote the neutron transport operator as follows:

$$\begin{aligned}\mathcal{L}[u] =& \epsilon\vec{w}\cdot\nabla_x u + u - \bar{u}\\ =& \sin\phi\frac{\partial u}{\partial\eta} - \epsilon\left(\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\right)\cos\phi\frac{\partial u}{\partial\phi}\\ &+\epsilon\left(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial u}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial u}{\partial\iota_2}\right)\\ &+\epsilon\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec{\varsigma}_1\cdot\big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\ &+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec{\varsigma}_2\cdot\big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial u}{\partial\psi}+u-\bar{u}.\end{aligned} \tag{2.151}$$

Step 2: Estimates of $\mathcal{L}[Q]$.
The interior contribution can be estimated as

$$\mathcal{L}[Q] = \epsilon\vec{w}\cdot\nabla_x Q + Q - \bar{Q} = \epsilon^3\vec{w}\cdot\nabla_x U_2. \tag{2.152}$$

Based on Theorem 2.4, we have

$$\left\|\mathcal{L}[Q]\right\|_{L^\infty(\Omega\times\mathbb{S}^2)} \leq \left\|\epsilon^3\vec{w}\cdot\nabla_x U_2\right\|_{L^\infty(\Omega\times\mathbb{S}^2)} \leq C\epsilon^3\left\|\nabla_x U_2\right\|_{L^\infty(\Omega\times\mathbb{S}^2)} \leq C\epsilon^3\left|\ln(\epsilon)\right|^8. \tag{2.153}$$

This implies

$$\left\|\mathcal{L}[Q]\right\|_{L^2(\Omega\times\mathbb{S}^2)} \leq C\epsilon^3\left|\ln(\epsilon)\right|^8, \tag{2.154}$$

$$\left\|\mathcal{L}[Q]\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^2)} \leq C\epsilon^3\left|\ln(\epsilon)\right|^8, \tag{2.155}$$

$$\left\|\mathcal{L}[Q]\right\|_{L^\infty(\Omega\times\mathbb{S}^2)} \leq C\epsilon^3\left|\ln(\epsilon)\right|^8. \tag{2.156}$$

Step 3: Estimates of $\mathcal{L}\mathscr{Q}$.

We need to estimate $\mathscr{U}_0+\epsilon\mathscr{U}_1$. The boundary layer contribution can be estimated as

$$
\begin{aligned}
\mathcal{L}[\mathscr{U}_0+\epsilon\mathscr{U}_1] =& \sin\phi\frac{\partial(\mathscr{U}_0+\epsilon\mathscr{U}_1)}{\partial\eta}-\epsilon\bigg(\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\bigg)\cos\phi\frac{\partial(\mathscr{U}_0+\epsilon\mathscr{U}_1)}{\partial\phi}\\
&+\epsilon\bigg(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial(\mathscr{U}_0+\epsilon\mathscr{U}_1)}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial(\mathscr{U}_0+\epsilon\mathscr{U}_1)}{\partial\iota_2}\bigg)\\
&+\epsilon\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec\varsigma_1\cdot\Big(\vec\varsigma_2\times(\partial_{12}\vec r\times\vec\varsigma_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\
&+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec\varsigma_2\cdot\Big(\vec\varsigma_1\times(\partial_{12}\vec r\times\vec\varsigma_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial(\mathscr{U}_0+\epsilon\mathscr{U}_1)}{\partial\psi}\\
&+(\mathscr{U}_0+\epsilon\mathscr{U}_1)-(\bar{\mathscr{U}_0}+\epsilon\bar{\mathscr{U}_1})\\
=&\epsilon^2\bigg(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathscr{U}_1}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial\mathscr{U}_1}{\partial\iota_2}\bigg)\\
&+\epsilon^2\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec\varsigma_1\cdot\Big(\vec\varsigma_2\times(\partial_{12}\vec r\times\vec\varsigma_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\
&+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec\varsigma_2\cdot\Big(\vec\varsigma_1\times(\partial_{12}\vec r\times\vec\varsigma_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial\mathscr{U}_1}{\partial\psi}.
\end{aligned}
$$

Based on Theorem 2.2, we have

$$
\begin{aligned}
&\left\|\epsilon^2\bigg(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathscr{U}_1}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial\mathscr{U}_1}{\partial\iota_2}\bigg)\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}\\
\leq& C\epsilon^2\left(\left\|\frac{\partial\mathscr{U}_1}{\partial\iota_1}\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}+\left\|\frac{\partial\mathscr{U}_1}{\partial\iota_2}\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}\right)\leq C\epsilon^2\left|\ln(\epsilon)\right|^8.
\end{aligned}
\tag{2.157}
$$

Similarly, we can show that

$$
\begin{aligned}
&\Bigg\|\epsilon^2\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec\varsigma_1\cdot\Big(\vec\varsigma_2\times(\partial_{12}\vec r\times\vec\varsigma_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\
&+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec\varsigma_2\cdot\Big(\vec\varsigma_1\times(\partial_{12}\vec r\times\vec\varsigma_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial\mathscr{U}_1}{\partial\psi}\Bigg\|_{L^\infty(\Omega\times\mathbb{S}^2)}\\
\leq& C\epsilon^2\left\|\frac{\partial\mathscr{U}_1}{\partial\psi}\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}\leq C\epsilon^2\left|\ln(\epsilon)\right|^8.
\end{aligned}
\tag{2.158}
$$

Also, the exponential decay of $\dfrac{\partial\mathscr{U}_1}{\partial\iota_i}$ in Theorem 2.2 and the rescaling $\eta=\dfrac{\mu}{\epsilon}$ implies

(2.159)

$$
\begin{aligned}
&\left\|\epsilon^2\left(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathscr{U}_1}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial\mathscr{U}_1}{\partial\iota_2}\right)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\leq C\epsilon^2\left(\left\|\frac{\partial\mathscr{U}_1}{\partial\iota_1}\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\left\|\frac{\partial\mathscr{U}_1}{\partial\iota_2}\right\|_{L^2(\Omega\times\mathbb{S}^2)}\right)\\
\leq&\epsilon^2\left(\int_0^{R_{\min}}(R_{\min}-\mu)\left(\left\|\frac{\partial\mathscr{U}_1}{\partial\iota_1}(\mu)\right\|_{L^\infty_{\phi,\psi}}^2+\left\|\frac{\partial\mathscr{U}_1}{\partial\iota_2}(\mu)\right\|_{L^\infty_{\phi,\psi}}^2\right)\mathrm{d}\mu\right)^{\frac{1}{2}}\\
\leq&\epsilon^{\frac{5}{2}}\left(\int_0^{\frac{R_{\min}}{\epsilon}}(R_{\min}-\epsilon\eta)\left(\left\|\frac{\partial\mathscr{U}_1}{\partial\iota_1}(\eta)\right\|_{L^\infty_{\phi,\psi}}^2+\left\|\frac{\partial\mathscr{U}_1}{\partial\iota_2}(\eta)\right\|_{L^\infty_{\phi,\psi}}^2\right)\mathrm{d}\eta\right)^{\frac{1}{2}}\\
\leq&C\epsilon^{\frac{5}{2}}\left|\ln(\epsilon)\right|^8\left(\int_0^\infty\mathrm{e}^{-2K_0\eta}\mathrm{d}\eta\right)^{\frac{1}{2}}\\
\leq&C\epsilon^{\frac{5}{2}}\left|\ln(\epsilon)\right|^8.
\end{aligned}
$$

Similarly, we have

$$
\left\|\epsilon^2\left(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathscr{U}_1}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial\mathscr{U}_1}{\partial\iota_2}\right)\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^2)}\leq C\epsilon^{3-\frac{1}{2m}}\left|\ln(\epsilon)\right|^8. \tag{2.160}
$$

Using similar arguments, we may justify

(2.161)

$$
\begin{aligned}
&\Bigg\|\epsilon^2\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec\varsigma_1\cdot\big(\vec\varsigma_2\times(\partial_{12}\vec r\times\vec\varsigma_2)\big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\
&+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec\varsigma_2\cdot\big(\vec\varsigma_1\times(\partial_{12}\vec r\times\vec\varsigma_1)\big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial\mathscr{U}_1}{\partial\psi}\Bigg\|_{L^2(\Omega\times\mathbb{S}^2)}\leq C\epsilon^{\frac{5}{2}}\left|\ln(\epsilon)\right|^8,
\end{aligned}
$$

and

(2.162)

$$
\begin{aligned}
&\Bigg\|\epsilon^2\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec\varsigma_1\cdot\big(\vec\varsigma_2\times(\partial_{12}\vec r\times\vec\varsigma_2)\big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\
&+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec\varsigma_2\cdot\big(\vec\varsigma_1\times(\partial_{12}\vec r\times\vec\varsigma_1)\big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial\mathscr{U}_1}{\partial\psi}\Bigg\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^2)}\leq C\epsilon^{3-\frac{1}{2m}}\left|\ln(\epsilon)\right|^8.
\end{aligned}
$$

In total, we have

$$
\left\|\mathcal{L}[\mathscr{Q}]\right\|_{L^2(\Omega\times\mathbb{S}^2)}\leq C\epsilon^{\frac{5}{2}}\left|\ln(\epsilon)\right|^8, \tag{2.163}
$$

$$
\left\|\mathcal{L}[\mathscr{Q}]\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^2)}\leq C\epsilon^{3-\frac{1}{2m}}\left|\ln(\epsilon)\right|^8, \tag{2.164}
$$

$$
\left\|\mathcal{L}[\mathscr{Q}]\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}\leq C\epsilon^{2}\left|\ln(\epsilon)\right|^8. \tag{2.165}
$$

Step 4: Estimates of $\mathcal{L}\mathfrak{Q}$.

We need to estimate $\mathfrak{U}_0$. The boundary layer contribution can be estimated as

$$
\begin{aligned}
\mathcal{L}[\mathfrak{U}_0] =& \sin\phi\frac{\partial\mathfrak{U}_0}{\partial\eta} - \epsilon\bigg(\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\bigg)\cos\phi\frac{\partial\mathfrak{U}_0}{\partial\phi}\\
&+\epsilon\bigg(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathfrak{U}_0}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial\mathfrak{U}_0}{\partial\iota_2}\bigg)\\
&+\epsilon\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\
&+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial\mathfrak{U}_0}{\partial\psi}+\mathfrak{U}_0-\bar{\mathfrak{U}}_0\\
=&\epsilon\bigg(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathfrak{U}_0}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial\mathfrak{U}_0}{\partial\iota_2}\bigg)\\
&+\epsilon\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\
&+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial\mathfrak{U}_0}{\partial\psi}.
\end{aligned}
\tag{2.166}
$$

Based on Theorem 2.3, we have

$$
\begin{aligned}
&\left\|\epsilon\bigg(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathfrak{U}_0}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial\mathfrak{U}_0}{\partial\iota_2}\bigg)\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}\\
\leq& C\epsilon\bigg(\left\|\frac{\partial\mathfrak{U}_0}{\partial\iota_1}\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}+\left\|\frac{\partial\mathfrak{U}_0}{\partial\iota_2}\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}\bigg)\leq C\epsilon\left|\ln(\epsilon)\right|^8.
\end{aligned}
\tag{2.167}
$$

Similarly, we can show that

$$
\begin{aligned}
&\Bigg\|\epsilon\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\
&+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial\mathfrak{U}_0}{\partial\psi}\Bigg\|_{L^\infty(\Omega\times\mathbb{S}^2)}\\
\leq& C\epsilon\left\|\frac{\partial\mathfrak{U}_0}{\partial\psi}\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}\leq C\epsilon\left|\ln(\epsilon)\right|^8.
\end{aligned}
\tag{2.168}
$$

Also, the exponential decay of $\dfrac{\partial\mathfrak{U}_0}{\partial\iota_1}$ in Theorem 2.3 and the rescaling $\eta=\dfrac{\mu}{\epsilon}$ implies

$$
\begin{aligned}
&\left\|\epsilon\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathfrak{U}_0}{\partial\iota_1}\right\|_{L^2(\Omega\times\mathbb{S}^2)}\leq\epsilon\left\|\frac{\partial\mathfrak{U}_0}{\partial\iota_1}\right\|_{L^2(\Omega\times\mathbb{S}^2)} \qquad (2.169)\\
\leq&\epsilon\left(\int_0^{R_{\min}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\chi_1(R_{\min}-\mu)\left\|\frac{\partial\mathfrak{U}_0}{\partial\iota_1}(\mu)\right\|_{L^\infty_{\phi,\psi}}^2\mathrm{d}\phi\mathrm{d}\mu\right)^{\frac{1}{2}}\\
&+\epsilon\left(\int_0^{R_{\min}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\chi_2(R_{\min}-\mu)\left\|\frac{\partial\mathfrak{U}_0}{\partial\iota_1}(\mu)\right\|_{L^\infty_{\phi,\psi}}^2\mathrm{d}\phi\mathrm{d}\mu\right)^{\frac{1}{2}}\\
\leq&\epsilon^{\frac{3}{2}}\left(\int_0^{\frac{R_{\min}}{\epsilon}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\chi_1(R_{\min}-\epsilon\eta)\left\|\frac{\partial\mathfrak{U}_0}{\partial\iota_1}(\eta)\right\|_{L^\infty_{\phi,\psi}}^2\mathrm{d}\phi\mathrm{d}\eta\right)^{\frac{1}{2}}\\
&+\epsilon^{\frac{3}{2}}\left(\int_0^{\frac{R_{\min}}{\epsilon}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\chi_2(R_{\min}-\epsilon\eta)\left\|\frac{\partial\mathfrak{U}_0}{\partial\iota_1}(\eta)\right\|_{L^\infty_{\phi,\psi}}^2\mathrm{d}\phi\mathrm{d}\eta\right)^{\frac{1}{2}}\\
\leq&C\left(\epsilon^{1+\frac{3}{2}\alpha}+\epsilon^{\frac{3}{2}+\alpha}\right)\left|\ln(\epsilon)\right|^8\left(\int_{-\pi}^{\pi}\int_0^{\frac{R_{\min}}{\epsilon}}\mathrm{e}^{-2K_0\eta}\mathrm{d}\eta\right)^{\frac{1}{2}}\\
\leq&C\epsilon^{1+\frac{3}{2}\alpha}\left|\ln(\epsilon)\right|^8.
\end{aligned}
$$

Here the smallness of χ_1 quantity comes from the small domain $|\phi|\leq\epsilon^\alpha$ and $|\eta|\leq\epsilon^{2\alpha-1}$. The smallness of χ_2 quantity comes from the extra ϵ^α for $0<\alpha<1$. This can naturally be extended to treat ι_2 and ψ derivatives. Hence, we have

$$
\left\|\epsilon\left(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathfrak{U}_0}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial\mathfrak{U}_0}{\partial\iota_2}\right)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\leq C\epsilon^{1+\frac{3}{2}\alpha}\left|\ln(\epsilon)\right|^8, \tag{2.170}
$$

$$
\left\|\epsilon\left(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathfrak{U}_0}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial\mathfrak{U}_0}{\partial\iota_2}\right)\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^2)}\leq C\epsilon^{2-\frac{1}{2m}+\alpha}\left|\ln(\epsilon)\right|^8, \tag{2.171}
$$

Using similar arguments, we may justify

$$
\begin{aligned}
&\left\|\epsilon\left(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\left(\cos\phi\Big(\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\right)\right.\right.\\
&\left.\left.+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\left(-\cos\phi\Big(\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\right)\right)\frac{1}{P_1P_2}\frac{\partial\mathfrak{U}_0}{\partial\psi}\right\|_{L^2(\Omega\times\mathbb{S}^2)}\leq C\epsilon^{1+\frac{3}{2}\alpha}\left|\ln(\epsilon)\right|^8,
\end{aligned} \tag{2.172}
$$

and

$$
\begin{aligned}
&\left\|\epsilon\left(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\left(\cos\phi\Big(\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\right)\right.\right.\\
&\left.\left.+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\left(-\cos\phi\Big(\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\right)\right)\frac{1}{P_1P_2}\frac{\partial\mathfrak{U}_0}{\partial\psi}\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^2)}\leq C\epsilon^{2-\frac{1}{2m}+\alpha}\left|\ln(\epsilon)\right|^8.
\end{aligned} \tag{2.173}
$$

In total, we have

$$
\left\|\mathcal{L}[\mathscr{Q}]\right\|_{L^2(\Omega\times\mathbb{S}^2)}\leq C\epsilon^{1+\frac{3}{2}\alpha}\left|\ln(\epsilon)\right|^8, \tag{2.174}
$$

$$
\left\|\mathcal{L}[\mathscr{Q}]\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^2)}\leq C\epsilon^{2-\frac{1}{2m}+\alpha}\left|\ln(\epsilon)\right|^8, \tag{2.175}
$$

$$
\left\|\mathcal{L}[\mathscr{Q}]\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}\leq C\epsilon\left|\ln(\epsilon)\right|^8. \tag{2.176}
$$

Step 5: Source Term and Boundary Condition.
In summary, since $\mathcal{L}[u^\epsilon]=0$, collecting estimates in Step 2 to Step 4 with $\alpha=1-\delta$ for arbitrarily small $\delta>0$, we can prove

$$\|\mathcal{L}[R]\|_{L^2(\Omega\times\mathbb{S}^2)} \leq C\epsilon^{\frac{5}{2}-\delta}\left|\ln(\epsilon)\right|^8, \tag{2.177}$$

$$\|\mathcal{L}[R]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^2)} \leq C\epsilon^{3-\frac{1}{2m}-\delta}\left|\ln(\epsilon)\right|^8, \tag{2.178}$$

$$\|\mathcal{L}[R]\|_{L^\infty(\Omega\times\mathbb{S}^2)} \leq C\epsilon\left|\ln(\epsilon)\right|^8. \tag{2.179}$$

We can directly obtain that the boundary data is satisfied up to $O(\epsilon)$, so we know for the boundary data R^B of R,

$$\left\|R^B\right\|_{L^2(\Gamma^-)} \leq C\epsilon^2, \tag{2.180}$$

$$\left\|R^B\right\|_{L^m(\Gamma^-)} \leq C\epsilon^2, \tag{2.181}$$

$$\left\|R^B\right\|_{L^\infty(\Gamma^-)} \leq C\epsilon^2 \tag{2.182}$$

Step 6: Diffusive Limit.
Hence, the remainder R satisfies the equation

$$\left\{\begin{array}{l} \epsilon\vec{w}\cdot\nabla_x R + R - \bar{R} = \mathcal{L}[R] \ \text{ in } \ \Omega\times\mathbb{S}^2,\\ \\ R = R^B \ \text{ for } \ \vec{w}\cdot\vec{\nu}<0 \ \text{ and } \ \vec{x}_0\in\partial\Omega. \end{array}\right. \tag{2.183}$$

By Theorem 2.1, we have for $1\leq m\leq 3$,

$$\begin{aligned} \|R\|_{L^\infty(\Omega\times\mathbb{S}^2)} \leq& C\bigg(\frac{1}{\epsilon^{1+\frac{3}{2m}}}\|\mathcal{L}[R]\|_{L^2(\Omega\times\mathbb{S}^2)} + \frac{1}{\epsilon^{2+\frac{3}{2m}}}\|\mathcal{L}[R]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^2)} + \|\mathcal{L}[R]\|_{L^\infty(\Omega\times\mathbb{S}^2)}\\ &+\frac{1}{\epsilon^{\frac{1}{2}+\frac{3}{2m}}}\left\|R^B\right\|_{L^2(\Gamma^-)} + \frac{1}{\epsilon^{\frac{3}{2m}}}\left\|R^B\right\|_{L^{\frac{4m}{3}}(\Gamma^-)} + \left\|R^B\right\|_{L^\infty(\Gamma^-)}\bigg)\\ \leq& C\bigg(\frac{1}{\epsilon^{1+\frac{3}{2m}}}\epsilon^{\frac{5}{2}-\delta}\left|\ln(\epsilon)\right|^8 + \frac{1}{\epsilon^{2+\frac{3}{2m}}}\epsilon^{3-\frac{1}{2m}-\delta}\left|\ln(\epsilon)\right|^8 + (\epsilon)\left|\ln(\epsilon)\right|^8\\ &+\frac{1}{\epsilon^{\frac{1}{2}+\frac{3}{2m}}}(\epsilon^2) + \frac{1}{\epsilon^{\frac{3}{2m}}}(\epsilon^2) + (\epsilon^2)\bigg)\\ \leq& C\epsilon^{1-\frac{2}{m}-\delta}\left|\ln(\epsilon)\right|^8. \end{aligned} \tag{2.184}$$

Here taking $m=3$, we get the estimates

$$\|R\|_{L^\infty(\Omega\times\mathbb{S}^2)} \leq C\epsilon^{\frac{1}{3}-\delta}\left|\ln(\epsilon)\right|^8.$$

Since it is easy to see

$$\left\|\sum_{k=1}^{2}\epsilon^k U_k + \sum_{k=1}^{1}\epsilon^k\mathscr{U}_k\right\|_{L^\infty(\Omega\times\mathbb{S}^2)} \leq C\epsilon, \tag{2.185}$$

our result naturally follows. We simply take $U=U_0$ and $\mathcal{U}^B=\mathscr{U}_0+\mathfrak{U}_0$. It is obvious that $\mathcal{U}^B$ satisfies the ϵ-Milne problem with geometric correction with the full boundary data $g(\phi,\psi,\iota_1,\iota_2)-\mathscr{F}_{0,L}(\iota_1,\iota_2)-\mathfrak{F}_{0,L}(\iota_1,\iota_2)$. This completes the proof of main theorem.

3. Unsteady Neutron Transport Equation

In this section, we prove the diffusive limit of the unsteady neutron transport equation (1.3)

3.1. **Asymptotic Expansions.**

3.1.1. *Interior Expansion.* We define the interior expansion as follows:

$$U(t,\vec{x},\vec{w})\sim\sum_{k=0}^{2}\epsilon^kU_k(t,\vec{x},\vec{w}), \tag{3.186}$$

where U_k can be defined by comparing the order of ϵ via plugging (3.186) into the equation (1.3). Thus, we have

$$U_0-\bar{U}_0=0, \tag{3.187}$$

$$U_1-\bar{U}_1=-\vec{w}\cdot\nabla_xU_0, \tag{3.188}$$

$$U_2-\bar{U}_2=-\partial_tU_0-\vec{w}\cdot\nabla_xU_1. \tag{3.189}$$

Plugging (3.187) into (3.188), we obtain

$$U_1=\bar{U}_1-\vec{w}\cdot\nabla_x\bar{U}_0. \tag{3.190}$$

Plugging (3.190) into (3.189), we get

$$\begin{aligned}U_2-\bar{U}_2=&-\partial_tU_0-\vec{w}\cdot\nabla_x(\bar{U}_1-\vec{w}\cdot\nabla_x\bar{U}_0)\\=&-\partial_tU_0-\vec{w}\cdot\nabla_x\bar{U}_1+\Big(w_1^2\partial_{x_1x_1}\bar{U}_0+w_2^2\partial_{x_2x_2}\bar{U}_0+w_3^2\partial_{x_3x_3}\bar{U}_0\Big)\\&+2\Big(w_1w_2\partial_{x_1x_2}\bar{U}_0+w_1w_3\partial_{x_1x_3}\bar{U}_0+w_2w_3\partial_{x_2x_3}\bar{U}_0\Big).\end{aligned} \tag{3.191}$$

Integrating (3.191) over $\vec{w}\in\mathbb{S}^2$, we achieve the final form

$$\partial_t\bar{U}_0-\frac{1}{3}\Delta_x\bar{U}_0=0, \tag{3.192}$$

where all cross terms vanish due to the symmetry of $\mathbb{S}^2$. Hence, $U_0(t,\vec{x},\vec{w})$ satisfies the equation

$$\left\{\begin{array}{l}U_0=\bar{U}_0,\\ \partial_t\bar{U}_0-\dfrac{1}{3}\Delta_x\bar{U}_0=0.\end{array}\right. \tag{3.193}$$

Similarly, we can derive that $U_k(t,\vec{x},\vec{w})$ for $k=1,2$ satisfies

$$\left\{\begin{array}{l}U_k=\bar{U}_k-\vec{w}\cdot\nabla_xU_{k-1},\\ \partial_t\bar{U}_k-\dfrac{1}{3}\Delta_x\bar{U}_k=0,\end{array}\right. \tag{3.194}$$

It is easy to see that $\bar{U}_k$ satisfies a parabolic equation. However, the initial and boundary conditions of $\bar{U}_k$ is unknown at this stage, since generally U_k does not necessarily satisfy the initial and boundary condition of (1.3). Therefore, we have to resort to initial and boundary layer analysis.

3.1.2. *Initial Layer Expansion.* In order to determine the initial condition for U_k, we need to define the initial layer expansion. Hence, we need a substitution:

Temporal Substitution:
We define the rescaled variable τ by making the scaling transform for $\tau=\dfrac{t}{\epsilon^2}$, which implies $\dfrac{\partial u^\epsilon}{\partial t}=\dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial\tau}$. Then, under the substitution $t\rightarrow\tau$, the equation (1.3) is transformed into

$$\left\{\begin{array}{l}\partial_\tau u^\epsilon+\epsilon\vec{w}\cdot\nabla_xu^\epsilon+u^\epsilon-\bar{u}^\epsilon=0\quad\text{for}\quad(\tau,\vec{x},\vec{w})\in\mathbb{R}^+\times\Omega\times\mathbb{S}^2,\\ u^\epsilon(0,\vec{x},\vec{w})=h(\vec{x},\vec{w})\quad\text{for}\quad(\vec{x},\vec{w})\in\Omega\times\mathbb{S}^2,\\ u^\epsilon(\tau,\vec{x}_0,\vec{w})=g(\tau,\vec{x}_0,\vec{w})\quad\text{for}\quad\tau\in\mathbb{R}^+,\quad\vec{x}_0\in\partial\Omega,\quad\text{and}\quad\vec{w}\cdot\vec{\nu}<0.\end{array}\right. \tag{3.195}$$

We define the initial layer expansion as follows:

$$\mathcal{U}^I(\tau,\vec{x},\vec{w})\sim\mathcal{U}_0^I(\tau,\vec{x},\vec{w})+\epsilon\mathcal{U}_1^I(\tau,\vec{x},\vec{w}), \tag{3.196}$$

where $\mathcal{U}_k^I$ can be determined by comparing the order of ϵ via plugging (3.196) into the equation (3.195). Thus, we have

$$\partial_\tau\mathcal{U}_0^I+\mathcal{U}_0^I-\bar{\mathcal{U}}_0^I=0, \tag{3.197}$$

$$\partial_\tau\mathcal{U}_1^I+\mathcal{U}_1^I-\bar{\mathcal{U}}_1^I=-\vec{w}\cdot\nabla_x\mathcal{U}_0^I. \tag{3.198}$$

Integrate (3.197) over $\vec{w}\in\mathbb{S}^2$, we have

$$\partial_\tau\bar{\mathcal{U}}_0^I=0, \tag{3.199}$$

which further implies

$$\bar{\mathcal{U}}_0^I(\tau,\vec{x})=\bar{\mathcal{U}}_0^I(0,\vec{x})\quad\text{for}\quad\tau\in\mathbb{R}^+. \tag{3.200}$$

Therefore, from (3.197), we can deduce

$$\mathcal{U}_0^I(\tau,\vec{x},\vec{w})=\mathrm{e}^{-\tau}\mathcal{U}_0^I(0,\vec{x},\vec{w})+\int_0^\tau\bar{\mathcal{U}}_0^I(s,\vec{x})\mathrm{e}^{s-\tau}\mathrm{d}s=\mathrm{e}^{-\tau}\mathcal{U}_0^I(0,\vec{x},\vec{w})+(1-\mathrm{e}^{-\tau})\bar{\mathcal{U}}_0^I(0,\vec{x}). \tag{3.201}$$

This means that we have

$$\left\{\begin{array}{l}\partial_\tau\bar{\mathcal{U}}_0^I=0,\\[2mm] \mathcal{U}_0^I(\tau,\vec{x},\vec{w})=\mathrm{e}^{-\tau}\mathcal{U}_0^I(0,\vec{x},\vec{w})+(1-\mathrm{e}^{-\tau})\bar{\mathcal{U}}_0^I(0,\vec{x}).\end{array}\right. \tag{3.202}$$

Similarly, we can derive that $\mathscr{U}_1^I(\tau,\vec{x},\vec{w})$ satisfies

$$\left\{\begin{array}{l}\partial_\tau\bar{\mathcal{U}}_1^I=-\displaystyle\int_{\mathbb{S}^2}\Big(\vec{w}\cdot\nabla_x\mathcal{U}_0^I\Big)\mathrm{d}\vec{w},\\[2mm] \mathcal{U}_1^I(\tau,\vec{x},\vec{w})=\mathrm{e}^{-\tau}\mathcal{U}_1^I(0,\vec{x},\vec{w})+\displaystyle\int_0^\tau\Big(\bar{\mathcal{U}}_1^I-\vec{w}\cdot\nabla_x\mathcal{U}_0^I\Big)(s,\vec{x},\vec{w})\mathrm{e}^{s-\tau}\mathrm{d}s.\end{array}\right. \tag{3.203}$$

3.1.3. *Boundary Layers Expansion.* Here, we implement the same geometric substitution as in steady problems.

Substitution 1: Spacial Substitution:
In a neighborhood of $\vec{x}_0\in\partial\Omega$, define an orthogonal curvilinear coordinates system (ι_1,ι_2) such that at $\vec{x}_0$ the coordinate lines coincide with the principal directions. The boundary surface is $\vec{r}=\vec{r}(\iota_1,\iota_2)$. Let

$$P=|\partial_1\vec{r}\times\partial_2\vec{r}|=|\partial_1\vec{r}|\,|\partial_2\vec{r}|=P_1P_2, \tag{3.204}$$

for $P_i=|\partial_i\vec{r}|$ with the unit tangential vectors

$$\vec{\varsigma}_1=\frac{\partial_1\vec{r}}{P_1},\quad\vec{\varsigma}_2=\frac{\partial_2\vec{r}}{P_2}. \tag{3.205}$$

Then consider the new coordinate system (μ,ι_1,ι_2), where μ denotes the normal distance to boundary surface $\partial\Omega$, i.e.

$$\vec{x}=\vec{r}-\mu\vec{\nu}. \tag{3.206}$$

Let κ_1 and κ_2 be principal curvatures and

$$R_{\min}=\min_{\iota_1,\iota_2}\{R_1(\iota_1,\iota_2),R_2(\iota_1,\iota_2)\}, \tag{3.207}$$

where $R_1(\iota_1,\iota_2)=\dfrac{1}{\kappa_1(\iota_1,\iota_2)}$ and $R_2(\iota_1,\iota_2)=\dfrac{1}{\kappa_2(\iota_1,\iota_2)}$. Therefore, under the substitution $(x_1,x_2,x_3)\to(\mu,\iota_1,\iota_2)$ for $0\leq\mu<R_{\min}$, the equation (1.3) is transformed into

(3.208)

$$\left\{\begin{array}{l}\epsilon^2\partial_tu^\epsilon+\epsilon\bigg(-(\vec{w}\cdot\vec{\nu})\dfrac{\partial u^\epsilon}{\partial\mu}-\dfrac{\vec{w}\cdot\vec{\varsigma}_1}{P_1(\kappa_1\mu-1)}\dfrac{\partial u^\epsilon}{\partial\iota_1}-\dfrac{\vec{w}\cdot\vec{\varsigma}_2}{P_2(\kappa_2\mu-1)}\dfrac{\partial u^\epsilon}{\partial\iota_2}\bigg)+u^\epsilon-\bar{u}^\epsilon=0\ \text{ in }\ \mathbb{R}^+\times(0,R_{\min})\times\Sigma\times\mathbb{S}^2,\\[3mm] u^\epsilon(0,\mu,\iota_1,\iota_2,\vec{w})=h(\mu,\iota_1,\iota_2,\vec{w})\ \text{ in }\ (0,R_{\min})\times\Sigma\times\mathbb{S}^2,\\[3mm] u^\epsilon(t,0,\iota_1,\iota_2,\vec{w})=g(t,\iota_1,\iota_2,\vec{w})\ \text{ for }\ t\in\mathbb{R}^+\ \text{ and }\ \vec{w}\cdot\vec{\nu}<0.\end{array}\right.$$

Substitution 2: Velocity Substitution:
Define the orthogonal velocity substitution

$$\begin{cases} -\vec{w}\cdot\vec{\nu}=\sin\phi,\\ \vec{w}\cdot\vec{\varsigma}_1=\cos\phi\sin\psi,\\ \vec{w}\cdot\vec{\varsigma}_2=\cos\phi\cos\psi, \end{cases} \tag{3.209}$$

for $\phi\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ and $\psi\in[-\pi,\pi]$. Hence, under substitution $(w_1,w_2,w_3)\rightarrow(\phi,\psi)$ for $\phi\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ and $\psi\in[-\pi,\pi]$, the equation (1.3) is transformed into

$$\begin{cases} \epsilon^2\partial_t u^\epsilon+\epsilon\sin\phi\dfrac{\partial u^\epsilon}{\partial\mu}-\epsilon\bigg(\dfrac{\sin^2\psi}{R_1-\mu}+\dfrac{\cos^2\psi}{R_2-\mu}\bigg)\cos\phi\dfrac{\partial u^\epsilon}{\partial\phi}\\ +\epsilon\bigg(\dfrac{\cos\phi\sin\psi}{P_1(1-\kappa_1\mu)}\dfrac{\partial u^\epsilon}{\partial\iota_1}+\dfrac{\cos\phi\cos\psi}{P_2(1-\kappa_2\mu)}\dfrac{\partial u^\epsilon}{\partial\iota_2}\bigg)\\ +\epsilon\Bigg(\dfrac{\sin\psi}{1-\kappa_1\mu}\bigg(\cos\phi\Big(\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\ +\dfrac{\cos\psi}{1-\kappa_2\mu}\bigg(-\cos\phi\Big(\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\dfrac{1}{P_1P_2}\dfrac{\partial u^\epsilon}{\partial\psi}\\ +u^\epsilon-\bar{u}^\epsilon=0\ \text{ in }\ \mathbb{R}^+\times(0,R_{\min})\times\Sigma\times\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\times[-\pi,\pi],\\ u^\epsilon(0,\mu,\iota_1,\iota_2,\phi,\psi)=h(\mu,\iota_1,\iota_2,\phi,\psi)\ \text{ in }\ (0,R_{\min})\times\Sigma\times\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\times[-\pi,\pi],\\ u^\epsilon(t,0,\iota_1,\iota_2,\phi,\psi)=g(t,\iota_1,\iota_2,\phi,\psi)\ \text{ for }\ t\in\mathbb{R}^+\ \text{ and }\ \sin\phi>0. \end{cases} \tag{3.210}$$

Substitution 3: Scaling Substitution:
Define the scaled variable $\eta=\dfrac{\mu}{\epsilon}$, which implies $\dfrac{\partial}{\partial\mu}=\dfrac{1}{\epsilon}\dfrac{\partial}{\partial\eta}$. Then, under the substitution $\mu\rightarrow\eta$, the equation (1.3) is transformed into

$$\begin{cases} \epsilon^2\partial_t u^\epsilon+\sin\phi\dfrac{\partial u^\epsilon}{\partial\eta}-\epsilon\bigg(\dfrac{\sin^2\psi}{R_1-\epsilon\eta}+\dfrac{\cos^2\psi}{R_2-\epsilon\eta}\bigg)\cos\phi\dfrac{\partial u^\epsilon}{\partial\phi}\\ +\epsilon\bigg(\dfrac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\dfrac{\partial u^\epsilon}{\partial\iota_1}+\dfrac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\dfrac{\partial u^\epsilon}{\partial\iota_2}\bigg)\\ +\epsilon\Bigg(\dfrac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\ +\dfrac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\dfrac{1}{P_1P_2}\dfrac{\partial u^\epsilon}{\partial\psi}+u^\epsilon-\bar{u}^\epsilon=0\\ +u^\epsilon-\bar{u}^\epsilon=0\ \text{ in }\ \mathbb{R}^+\times\left(0,\dfrac{R_{\min}}{\epsilon}\right)\times\Sigma\times\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\times[-\pi,\pi],\\ u^\epsilon(0,\eta,\iota_1,\iota_2,\phi,\psi)=h(\eta,\iota_1,\iota_2,\phi,\psi)\ \text{ in }\ \left(0,\dfrac{R_{\min}}{\epsilon}\right)\times\Sigma\times\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\times[-\pi,\pi],\\ u^\epsilon(t,0,\iota_1,\iota_2,\phi,\psi)=g(t,\iota_1,\iota_2,\phi,\psi)\ \text{ for }\ t\in\mathbb{R}^+\ \text{ and }\ \sin\phi>0. \end{cases} \tag{3.211}$$

We define the boundary layer expansion as follows:

$$\mathcal{U}^B(t,\eta,\iota_1,\iota_2,\phi,\psi)\sim\mathcal{U}_0^B(t,\eta,\iota_1,\iota_2,\phi,\psi)+\epsilon\mathcal{U}_1^B(t,\eta,\iota_1,\iota_2,\phi,\psi), \tag{3.212}$$

where $\mathcal{U}_k^B$ can be defined by comparing the order of ϵ via plugging ((3.212)) into the equation ((3.211)). Thus, in a neighborhood of the boundary, we have

$$\sin\phi\frac{\partial\mathcal{U}_0^B}{\partial\eta}-\epsilon\bigg(\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\bigg)\cos\phi\frac{\partial\mathcal{U}_0^B}{\partial\phi}+\mathcal{U}_0^B-\bar{\mathcal{U}}_0^B=0, \tag{3.213}$$

$$\sin\phi\frac{\partial\mathcal{U}_1^B}{\partial\eta}-\epsilon\bigg(\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\bigg)\cos\phi\frac{\partial\mathcal{U}_1^B}{\partial\phi}+\mathcal{U}_1^B-\bar{\mathcal{U}}_1^B=-G[\mathcal{U}_0^B], \tag{3.214}$$

where

$$\begin{aligned}G[\mathcal{U}_0^B]=&\bigg(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathcal{U}_0^B}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial\mathcal{U}_0^B}{\partial\iota_2}\bigg)\\&+\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\&+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial\mathcal{U}_0^B}{\partial\psi},\end{aligned} \tag{3.215}$$

and

$$\bar{\mathcal{U}}_k^B(\eta,\iota_1,\iota_2)=\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathcal{U}_k^B(\eta,\iota_1,\iota_2,\phi,\psi)\cos\phi\mathrm{d}\phi\mathrm{d}\psi. \tag{3.216}$$

3.1.4. *Matching Procedure.* Here we still define the boundary data decomposition

$$g(\phi,\psi)=\mathscr{G}(\phi,\psi)+\mathfrak{G}(\phi,\psi). \tag{3.217}$$

For either $\mathscr{G}$ and $\mathfrak{G}$, we may define the corresponding boundary layer $\mathscr{U}$ and $\mathfrak{U}$. We call $\mathscr{U}$ the regular boundary layer and expand it up to $O(\epsilon)$, i.e.

$$\mathscr{U}(\eta,\iota_1,\iota_2,\phi,\psi)\sim\mathscr{U}_0(\eta,\iota_1,\iota_2,\phi,\psi)+\epsilon\mathscr{U}_1(\eta,\iota_1,\iota_2,\phi,\psi). \tag{3.218}$$

Also, we call $\mathfrak{U}$ the singular boundary layer and only expand it to $O(1)$, i.e.

$$\mathfrak{U}(\eta,\iota_1,\iota_2,\phi,\psi)\sim\mathfrak{U}_0(\eta,\iota_1,\iota_2,\phi,\psi). \tag{3.219}$$

They should both satisfy the ϵ-Milne problem with geometric correction.

The bridge between the interior solution, initial layer and boundary layer is the initial and boundary conditions of (1.3), so we consider the initial and boundary expansion:

$$U_0(0,\vec{x},\vec{w})+\mathcal{U}_0^I(0,\vec{x},\vec{w})=h(\vec{x},\vec{w}), \tag{3.220}$$

$$U_0(t,\vec{x}_0,\vec{w})+\mathscr{U}_0(t,\vec{x}_0,\vec{w})+\mathfrak{U}_0(t,\vec{x}_0,\vec{w})=g(t,\vec{x}_0,\vec{w})\quad\text{for}\quad\vec{x}_0\in\partial\Omega, \tag{3.221}$$

$$U_1(t,\vec{x}_0,\vec{w})+\mathscr{U}_1(t,\vec{x}_0,\vec{w})=0\quad\text{for}\quad\vec{x}_0\in\partial\Omega. \tag{3.222}$$

The construction and determination of asymptotic expansion are as follows:

Step 0: Preliminaries.
Define the force

$$F(\epsilon;\eta,\iota_1,\iota_2,\psi)=-\epsilon\bigg(\frac{\sin^2\psi}{R_1(\iota_1,\iota_2)-\epsilon\eta}+\frac{\cos^2\psi}{R_2(\iota_1,\iota_2)-\epsilon\eta}\bigg). \tag{3.223}$$

Define the length of boundary layer $L=\epsilon^{-n}$ for $0<n<\frac{1}{2}$. For $\phi\in\Big[-\frac{\pi}{2},\frac{\pi}{2}\Big]$, denote $\mathscr{R}[\phi]=-\phi$.

Step 1: Construction of $\mathscr{U}_0$, $\mathfrak{U}_0$, $\mathcal{U}_0^I$ and U_0.

Define the zeroth-order regular boundary layer as

$$
\left\{
\begin{aligned}
&\mathscr{U}_0(t,\eta,\iota_1,\iota_2,\phi,\psi)=\mathscr{F}_0(t,\eta,\iota_1,\iota_2,\phi,\psi)-\mathscr{F}_{0,L}(t,\iota_1,\iota_2),\\
&\sin\phi\frac{\partial\mathscr{F}_0}{\partial\eta}+F(\epsilon;\eta,\iota_1,\iota_2,\psi)\cos\phi\frac{\partial\mathscr{F}_0}{\partial\phi}+\mathscr{F}_0-\bar{\mathscr{F}}_0=0,\\
&\mathscr{F}_0(t,0,\iota_1,\iota_2,\phi,\psi)=\mathscr{G}(t,\iota_1,\iota_2,\phi,\psi)\quad\text{for}\quad\sin\phi>0,\\
&\mathscr{F}_0(t,L,\iota_1,\iota_2,\phi,\psi)=\mathscr{F}_0(t,L,\iota_1,\iota_2,\mathscr{R}[\phi],\psi),
\end{aligned}
\right.
\tag{3.224}
$$

with $\mathscr{F}_{0,L}(t,\iota_1,\iota_2)$ is defined as in Theorem 4.1.

Define the zeroth-order singular boundary layer as

$$
\left\{
\begin{aligned}
&\mathfrak{U}_0(t,\eta,\iota_1,\iota_2,\phi,\psi)=\mathfrak{F}_0(t,\eta,\iota_1,\iota_2,\phi,\psi)-\mathfrak{F}_{0,L}(\iota_1,\iota_2),\\
&\sin\phi\frac{\partial\mathfrak{F}_0}{\partial\eta}+F(\epsilon;\eta,\iota_1,\iota_2,\psi)\cos\phi\frac{\partial\mathfrak{F}_0}{\partial\phi}+\mathfrak{F}_0-\bar{\mathfrak{F}}_0=0,\\
&\mathfrak{F}_0(t,0,\iota_1,\iota_2,\phi,\psi)=\mathfrak{G}(t,\iota_1,\iota_2,\phi,\psi)\quad\text{for}\quad\sin\phi>0,\\
&\mathfrak{F}_0(t,L,\iota_1,\iota_2,\phi,\psi)=\mathfrak{F}_0(t,L,\iota_1,\iota_2,\mathscr{R}[\phi],\psi),
\end{aligned}
\right.
\tag{3.225}
$$

with $\mathfrak{F}_{0,L}(t,\iota_1,\iota_2)$ is defined as in Theorem 4.1.

Define the zeroth-order initial layer as

$$
\left\{
\begin{aligned}
&\mathcal{U}_0^I(\tau,\vec{x},\vec{w})=\mathfrak{f}_0(\tau,\vec{x},\vec{w})-\mathfrak{f}_0(\infty,\vec{x})\\
&\partial_\tau\bar{\mathfrak{f}}_0=0,\\
&\mathfrak{f}_0(\tau,\vec{x},\vec{w})=\mathrm{e}^{-\tau}\mathfrak{f}_0(0,\vec{x},\vec{w})+(1-\mathrm{e}^{-\tau})\bar{\mathfrak{f}}_0(0,\vec{x}),\\
&\mathfrak{f}_0(0,\vec{x},\vec{w})=h(\vec{x},\vec{w}),\\
&\lim_{\tau\to\infty}\mathfrak{f}_0(\tau,\vec{x},\vec{w})=\mathfrak{f}_0(\infty,\vec{x}).
\end{aligned}
\right.
\tag{3.226}
$$

Also, define the zeroth-order interior solution $U_0(t,\vec{x},\vec{w})$ as

$$
\left\{
\begin{aligned}
&U_0(t,\vec{x},\vec{w})=\bar{U}_0(t,\vec{x}),\\
&\partial_t\bar{U}_0-\frac{1}{3}\Delta_x\bar{U}_0=0\quad\text{in}\quad\Omega,\\
&\bar{U}_0(0,\vec{x})=\mathfrak{f}_0(\infty,\vec{x})\quad\text{in}\quad\Omega,\\
&\bar{U}_0(t,\vec{x}_0)=\mathscr{F}_{0,L}(t,\iota_1,\iota_2)+\mathfrak{F}_{0,L}(t,\iota_1,\iota_2)\quad\text{on}\quad\partial\Omega.
\end{aligned}
\right.
\tag{3.227}
$$

Step 2: Construction of $\mathscr{U}_1,\mathcal{U}_1^I$ and U_1.
Define the first-order regular boundary layer as

$$
\left\{
\begin{aligned}
&\mathscr{U}_1(t,\eta,\iota_1,\iota_2,\phi,\psi)=\mathscr{F}_1(t,\eta,\iota_1,\iota_2,\phi,\psi)-\mathscr{F}_{1,L}(t,\iota_1,\iota_2),\\
&\sin\phi\frac{\partial\mathscr{F}_1}{\partial\eta}+F(\epsilon;\eta,\iota_1,\iota_2,\psi)\cos\phi\frac{\partial\mathscr{F}_1}{\partial\phi}+\mathscr{F}_1-\bar{\mathscr{F}}_1=G[\mathscr{U}_0],\\
&\mathscr{F}_1(t,0,\iota_1,\iota_2,\phi,\psi)=\vec{w}\cdot\nabla_xU_0(t,\vec{x}_0)\quad\text{for}\quad\sin\phi>0,\\
&\mathscr{F}_1(t,L,\iota_1,\iota_2,\phi,\psi)=\mathscr{F}_1(t,L,\iota_1,\iota_2,\mathscr{R}[\phi],\psi),
\end{aligned}
\right.
\tag{3.228}
$$

with $\mathscr{F}_{1,L}(t,\iota_1,\iota_2)$ is defined as in Theorem 4.1, and G_0 is defined in (3.215).

Define the first-order initial layer as

$$
\begin{cases}
\mathcal{U}_1^I(\tau,\vec{x},\vec{w})=\mathfrak{f}_1(\tau,\vec{x},\vec{w})-\mathfrak{f}_1(\infty,\vec{x})\\
\partial_\tau\bar{\mathfrak{f}}_1=-\int_{\mathbb{S}^2}\Big(\vec{w}\cdot\nabla_x\mathcal{U}_0^I\Big)\mathrm{d}\vec{w},\\
\mathfrak{f}_1(\tau,\vec{x},\vec{w})=\mathrm{e}^{-\tau}\mathfrak{f}_1(0,\vec{x},\vec{w})+\int_0^\tau\Big(\bar{\mathfrak{f}}_1-\vec{w}\cdot\nabla_x\mathcal{U}_0^I\Big)(s,\vec{x},\vec{w})\mathrm{e}^{s-\tau}\mathrm{d}s,\\
\mathfrak{f}_1(0,\vec{x},\vec{w})=\vec{w}\cdot\nabla_xU_0(0,\vec{x}),\\
\lim_{\tau\to\infty}\mathfrak{f}_1(\tau,\vec{x},\vec{w})=\mathfrak{f}_1(\infty,\vec{x}).
\end{cases}\tag{3.229}
$$

Then define the first-order interior solution $U_1(\vec{x},\vec{w})$ as

$$
\begin{cases}
U_1(\vec{x},\vec{w})=\bar{U}_1(\vec{x})-\vec{w}\cdot\nabla_xU_0(\vec{x},\vec{w}),\\
\partial_t\bar{U}_1-\frac{1}{3}\Delta_x\bar{U}_1=-\int_{\mathbb{S}^1}\Big(\vec{w}\cdot\nabla_xU_0(\vec{x},\vec{w})\Big)\mathrm{d}\vec{w}\ \text{ in }\ \Omega,\\
\bar{U}_1(0,\vec{x})=\mathfrak{f}_1(\infty,\vec{x})\ \text{ in }\ \Omega,\\
\bar{U}_1(t,\vec{x}_0)=\mathscr{F}_{1,L}(t,\iota_1,\iota_2)\ \text{ on }\ \partial\Omega.
\end{cases}\tag{3.230}
$$

Note that we do not define $\mathfrak{U}_1$ here.

Step 3: Construction of U_2.
Since we do not expand to $\mathscr{U}_2$ and $\mathfrak{U}_2$, simply define the second-order interior solution as

$$
\begin{cases}
U_2(\vec{x},\vec{w})=\bar{U}_2(\vec{x})-\vec{w}\cdot\nabla_xU_1(\vec{x},\vec{w}),\\
\partial_t\bar{U}_2-\frac{1}{3}\Delta_x\bar{U}_2=-\int_{\mathbb{S}^1}\Big(\vec{w}\cdot\nabla_xU_1(\vec{x},\vec{w})\Big)\mathrm{d}\vec{w}\ \text{ in }\ \Omega,\\
\bar{U}_2(0,\vec{x})=0\ \text{ in }\ \Omega,\\
\bar{U}_2(t,\vec{x}_0)=0\ \text{ on }\ \partial\Omega.
\end{cases}\tag{3.231}
$$

Here, we might have $O(\epsilon^3)$ error in this step due to the trivial boundary data. Thanks to the remainder estimate, it will not affect the diffusive limit.

3.2. **Remainder Estimate.** In this section, we consider the remainder equation for $u(t,\vec{x},\vec{w})$ as

$$
\begin{cases}
\epsilon^2\partial_tu+\epsilon\vec{w}\cdot\nabla_xu+u-\bar{u}=S\quad\text{for }\ (t,\vec{x},\vec{w})\in\mathbb{R}^+\times\Omega\times\mathbb{S}^2,\\
u(0,\vec{x},\vec{w})=h(\vec{x},\vec{w})\ \text{ for }\ (\vec{x},\vec{w})\in\Omega\times\mathbb{S}^2\\
u(t,\vec{x}_0,\vec{w})=g(t,\vec{x}_0,\vec{w})\ \text{ for }\ t\in\mathbb{R}^+,\ \ \vec{x}_0\in\partial\Omega\ \text{ and }\ \vec{w}\cdot\vec{\nu}<0.
\end{cases}\tag{3.232}
$$

The initial and boundary data satisfy the compatibility condition

$$
h(\vec{x}_0,\vec{w})=g(0,\vec{x}_0,\vec{w})\ \text{ for }\ \vec{x}_0\in\partial\Omega\ \text{ and }\ \vec{w}\cdot\vec{\nu}<0.\tag{3.233}
$$

Define the L^p norms with $1\leq p<\infty$ and L^∞ norm in $\mathbb{R}^+\times\Omega\times\mathbb{S}^2$ as usual:

$$
\|f\|_{L^p(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}=\bigg(\int_0^\infty\int_\Omega\int_{\mathbb{S}^2}|f(t,\vec{x},\vec{w})|^p\,\mathrm{d}\vec{w}\mathrm{d}\vec{x}\mathrm{d}t\bigg)^{\frac{1}{p}},\tag{3.234}
$$

$$
\|f\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}=\operatorname{esssup}_{(t,\vec{x},\vec{w})\in\mathbb{R}^+\times\Omega\times\mathbb{S}^2}|f(t,\vec{x},\vec{w})|\,.\tag{3.235}
$$

Define the L^p norm with $1\leq p<\infty$ and L^{∞} norm on the boundary $\Gamma=\partial\Omega\times\mathbb{S}^2$ as follows:

$$\|f\|_{L^p(\mathbb{R}^+\times\Gamma)}=\bigg(\int_0^{\infty}\iint_{\Gamma}\abs{f(t,\vec x,\vec w)}^p\abs{\vec w\cdot\vec\nu}\mathrm{d}\vec w\mathrm{d}\vec x\mathrm{d}t\bigg)^{\frac{1}{p}}, \tag{3.236}$$

$$\|f\|_{L^p(\mathbb{R}^+\times\Gamma^{\pm})}=\bigg(\int_0^{\infty}\iint_{\Gamma^{\pm}}\abs{f(t,\vec x,\vec w)}^p\abs{\vec w\cdot\vec\nu}\mathrm{d}\vec w\mathrm{d}\vec x\mathrm{d}t\bigg)^{\frac{1}{p}}, \tag{3.237}$$

$$\|f\|_{L^{\infty}(\mathbb{R}^+\times\Gamma)}=\mathrm{esssup}_{(t,\vec x,\vec w)\in\mathbb{R}^+\times\Gamma}\abs{f(t,\vec x,\vec w)}, \tag{3.238}$$

$$\|f\|_{L^{\infty}(\mathbb{R}^+\times\Gamma^{\pm})}=\mathrm{esssup}_{(t,\vec x,\vec w)\in\mathbb{R}^+\times\Gamma^{\pm}}\abs{f(t,\vec x,\vec w)}. \tag{3.239}$$

In particular, we denote $\mathrm{d}\gamma=(\vec w\cdot\vec\nu)\mathrm{d}\vec w\mathrm{d}\vec x$ on the boundary.

Similar notation also applies to the space $[0,t]\times\Omega\times\mathbb{S}^2$, $[0,t]\times\Gamma$, and $[0,t]\times\Gamma^{\pm}$.

3.2.1. *L^2 Estimate.*

Lemma 3.1 (Green's Identity)**.** *Assume $f(t,\vec x,\vec w),\ g(t,\vec x,\vec w)\in L^{\infty}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)$ and $\partial_tf+\vec w\cdot\nabla_xf,\ \partial_tg+\vec w\cdot\nabla_xg\in L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)$ with $f,\ g\in L^2(\mathbb{R}^+\times\Gamma)$. Then for almost all $s,t\in\mathbb{R}^+$,*

$$\begin{aligned}&\int_s^t\iint_{\Omega\times\mathbb{S}^2}\bigg((\partial_tf+\vec w\cdot\nabla_xf)g+(\partial_tg+\vec w\cdot\nabla_xg)f\bigg)\mathrm{d}\vec x\mathrm{d}\vec w\mathrm{d}r\\&=\int_s^t\int_{\Gamma}fg\mathrm{d}\gamma\mathrm{d}r+\iint_{\Omega\times\mathbb{S}^2}f(t)g(t)\mathrm{d}\vec x\mathrm{d}\vec w-\iint_{\Omega\times\mathbb{S}^2}f(s)g(s)\mathrm{d}\vec x\mathrm{d}\vec w.\end{aligned} \tag{3.240}$$

Proof. See [2, Chapter 9] and [3]. ☐

Theorem 3.2. *Assume $S(t,\vec x,\vec w)\in L^{\infty}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)$, $h(\vec x,\vec w)\in L^{\infty}(\Omega\times\mathbb{S}^2)$ and $g(t,x_0,\vec w)\in L^{\infty}(\mathbb{R}^+\times\Gamma^-)$. Then the neutron transport equation (3.232) has a unique solution $u(t,\vec x,\vec w)\in L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)$ satisfying*

$$\begin{aligned}&\|u(t)\|_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{2}}}\|u\|_{L^2([0,t]\times\Gamma^+)}+\|u\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\&\leq C\bigg(\frac{1}{\epsilon^2}\|S\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\|h\|_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{2}}}\|g\|_{L^2([0,t]\times\Gamma^-)}\bigg).\end{aligned} \tag{3.241}$$

Proof.
Step 1: Kernel Estimate.
Applying Lemma 3.1 to the equation (3.232). Then for any $\phi\in L^2([0,t]\times\Omega\times\mathbb{S}^2)$ satisfying $\epsilon\partial_t\phi+\vec w\cdot\nabla_x\phi\in L^2([0,t]\times\Omega\times\mathbb{S}^2)$ and $\phi\in L^2([0,t]\times\Gamma)$, we have

$$\begin{aligned}&-\epsilon^2\int_0^t\iint_{\Omega\times\mathbb{S}^2}\partial_t\phi u-\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\vec w\cdot\nabla_x\phi)u+\int_0^t\iint_{\Omega\times\mathbb{S}^2}(u-\bar u)\phi\\&=-\epsilon^2\iint_{\Omega\times\mathbb{S}^2}u(t)\phi(t)+\epsilon^2\iint_{\Omega\times\mathbb{S}^2}u(0)\phi(0)-\epsilon\int_0^t\int_{\Gamma}u\phi\mathrm{d}\gamma+\int_0^t\iint_{\Omega\times\mathbb{S}^2}S\phi.\end{aligned} \tag{3.242}$$

Our goal is to choose a particular test function ϕ. We first construct an auxiliary function $\zeta(t)$. Clearly, $u(t)\in L^2(\Omega\times\mathbb{S}^2)$ implies that $\bar u(t)\in L^2(\Omega)$. Define $\zeta(t,\vec x)$ on Ω satisfying

$$\left\{\begin{array}{l}\Delta_x\zeta(t)=\bar u(t)\ \text{ in }\ \Omega,\\ \zeta(t)=0\ \text{ on }\ \partial\Omega.\end{array}\right. \tag{3.243}$$

In the bounded domain Ω, based on the standard elliptic estimates, there exists a unique $\zeta(t)\in H^2(\Omega)$ such that

$$\|\zeta(t)\|_{H^2(\Omega)}\leq C\|\bar u(t)\|_{L^2(\Omega)}. \tag{3.244}$$

We plug the test function

$$\phi(t)=-\vec w\cdot\nabla_x\zeta(t) \tag{3.245}$$

into the weak formulation (3.242) and estimate each term there. By definition, we have

$$\|\phi(t)\|_{L^2(\Omega)}\leq C\|\zeta(t)\|_{H^1(\Omega)}\leq C\|\bar u(t)\|_{L^2(\Omega)}. \tag{3.246}$$

On the other hand, we decompose

$$-\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\vec{w}\cdot\nabla_x\phi)u=-\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\vec{w}\cdot\nabla_x\phi)\bar{u}-\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\vec{w}\cdot\nabla_x\phi)(u-\bar{u}). \tag{3.247}$$

For the first term on the right-hand side of (3.247), by (3.243) and (3.245), we have

$$\begin{aligned}&-\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\vec{w}\cdot\nabla_x\phi)\bar{u}\\ =&\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}\bar{u}\Big(w_1(w_1\partial_{11}\zeta+w_2\partial_{12}\zeta+w_3\partial_{13}\zeta)+w_2(w_1\partial_{21}\zeta+w_2\partial_{22}\zeta+w_3\partial_{23}\zeta)+w_3(w_1\partial_{31}\zeta+w_2\partial_{32}\zeta+w_3\partial_{33}\zeta)\Big)\\ =&\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}\bar{u}\Big(w_1^2\partial_{11}\zeta+w_2^2\partial_{22}\zeta+w_3^2\partial_{33}\zeta\Big)=\frac{4}{3}\epsilon\pi\int_0^t\int_\Omega\bar{u}(\partial_{11}\zeta+\partial_{22}\zeta+\partial_{33}\zeta)=\frac{4}{3}\epsilon\pi\left\|\bar{u}\right\|_{L^2([0,t]\times\Omega)}^2\\ =&\frac{1}{3}\epsilon\left\|\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}^2.\end{aligned} \tag{3.248}$$

Here ∂_i denotes the derivative with respect to x_i. In the second equality, above cross terms vanish due to the symmetry of the integral over $\mathbb{S}^2$.
For the second term on the right-hand side of (3.247), Hölder's inequality and (3.246) imply

$$\begin{aligned}\left|-\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\vec{w}\cdot\nabla_x\phi)(u-\bar{u})\right|&\leq C\epsilon\left\|\vec{w}\cdot\nabla_x\phi\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\ \leq C\epsilon\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\bigg(\int_0^t\left\|\zeta(s)\right\|_{H^2(\Omega)}^2\mathrm{d}s\bigg)^{\frac{1}{2}}&\leq C\epsilon\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\left\|\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}.\end{aligned} \tag{3.249}$$

Using the trace theorem, Hölder's inequality and (3.246), we have

$$\begin{aligned}\left|\epsilon\int_0^t\int_\Gamma u\phi\mathrm{d}\gamma\right|=&\epsilon\int_0^t\int_{\Gamma^-}g\phi\mathrm{d}\gamma+\epsilon\int_0^t\int_{\Gamma^+}u\phi\mathrm{d}\gamma\\ \leq&\epsilon\left\|\phi\right\|_{L^2([0,t]\times\Gamma)}\bigg(\left\|u\right\|_{L^2([0,t]\times\Gamma^+)}+\left\|g\right\|_{L^2([0,t]\times\Gamma^-)}\bigg)\\ \leq&\epsilon\left\|\phi\right\|_{H^1([0,t]\times\Omega\times\mathbb{S}^2)}\bigg(\left\|u\right\|_{L^2([0,t]\times\Gamma^+)}+\left\|g\right\|_{L^2([0,t]\times\Gamma^-)}\bigg)\\ \leq&\epsilon\left\|\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\bigg(\left\|u\right\|_{L^2([0,t]\times\Gamma^+)}+\left\|g\right\|_{L^2([0,t]\times\Gamma^-)}\bigg).\end{aligned} \tag{3.250}$$

Also, using Hölder's inequality and (3.246), we obtain

$$\left|\int_0^t\iint_{\Omega\times\mathbb{S}^2}(u-\bar{u})\phi\right|\leq C\left\|\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}, \tag{3.251}$$

and

$$\left|\int_0^t\iint_{\Omega\times\mathbb{S}^2}S\phi\right|\leq C\left\|\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}. \tag{3.252}$$

On the other hand, using Hölder's inequality and (3.246), we may directly estimate

$$\begin{aligned}\left|\epsilon^2\iint_{\Omega\times\mathbb{S}^2}u(t)\phi(t)\right|\leq&C\epsilon^2\left\|\phi(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\left\|u(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\ \leq&C\epsilon^2\left\|\bar{u}(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\left\|u(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\leq C\epsilon^2\left\|u(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}^2.\end{aligned} \tag{3.253}$$

Similarly, we know

$$\left|\epsilon^2\iint_{\Omega\times\mathbb{S}^2}u(0)\phi(0)\right|\leq C\epsilon^2\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}^2. \tag{3.254}$$

Then the only remaining term in (3.242) is

$$\left|-\epsilon^2\int_0^t\iint_{\Omega\times\mathbb{S}^2}\partial_t\phi u\right|=\left|\epsilon^2\int_0^t\iint_{\Omega\times\mathbb{S}^2}\partial_t\phi(u-\bar{u})\right| \tag{3.255}$$
$$\leq\epsilon^2\left\|\partial_t\phi\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\leq\epsilon^2\left\|\partial_t\nabla_x\zeta\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}.$$

Now we have to tackle $\left\|\partial_t\nabla_x\zeta\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}$. We will implement difference quotient.

For test function $\phi(\vec{x},\vec{w})$ which is independent of time t, in time interval $[t-\delta,t]$ the weak formulation in (3.242) can be simplified as

$$\epsilon^2\iint_{\Omega\times\mathbb{S}^2}u(t)\phi-\epsilon^2\iint_{\Omega\times\mathbb{S}^2}u(t-\delta)\phi-\epsilon\int_{t-\delta}^t\iint_{\Omega\times\mathbb{S}^2}(\vec{w}\cdot\nabla_x\phi)u+\int_{t-\delta}^t\iint_{\Omega\times\mathbb{S}^2}(u-\bar{u})\phi \tag{3.256}$$
$$=-\epsilon\int_{t-\delta}^t\int_\Gamma u\phi\mathrm{d}\gamma+\int_{t-\delta}^t\iint_{\Omega\times\mathbb{S}^2}S\phi.$$

Taking difference quotient as $\delta\to0$, we know

$$\frac{\epsilon^2\iint_{\Omega\times\mathbb{S}^2}u(t)\phi-\epsilon^2\iint_{\Omega\times\mathbb{S}^2}u(t-\delta)\phi}{\delta}\to\epsilon^2\iint_{\Omega\times\mathbb{S}^2}\partial_tu(t)\phi. \tag{3.257}$$

Then (3.256) can be simplified into

$$\epsilon^2\iint_{\Omega\times\mathbb{S}^2}\partial_tu(t)\phi=\epsilon\iint_{\Omega\times\mathbb{S}^2}(\vec{w}\cdot\nabla_x\phi)u(t)-\iint_{\Omega\times\mathbb{S}^2}\Big(u(t)-\bar{u}(t)\Big)\phi-\epsilon\int_\Gamma u(t)\phi\mathrm{d}\gamma+\iint_{\Omega\times\mathbb{S}^2}S(t)\phi. \tag{3.258}$$

For fixed t, taking $\phi=-\Phi(\vec{x})$ which satisfies

$$\begin{cases}\Delta_x\Phi=\partial_t\bar{u}(t) & \text{in }\ \Omega,\\ \Phi(t)=0 & \text{on }\ \partial\Omega,\end{cases} \tag{3.259}$$

which further implies $\Phi=\partial_t\zeta$. Then the left-hand side of (3.258) is actually

$$LHS=-\epsilon^2\iint_{\Omega\times\mathbb{S}^2}\Phi\partial_tu(t)=-\epsilon^2\iint_{\Omega\times\mathbb{S}^2}\Phi\partial_t\bar{u}=-\epsilon^2\iint_{\Omega\times\mathbb{S}^2}\Phi\Delta_x\Phi=\epsilon^2\iint_{\Omega\times\mathbb{S}^2}\left|\nabla_x\Phi\right|^2 \tag{3.260}$$
$$=\epsilon^2\left\|\partial_t\nabla_x\zeta(t)\right\|^2_{L^2(\Omega\times\mathbb{S}^2)}.$$

By a similar argument as above and Poincaré's inequality, the right-hand side of (3.258) can be bounded as

$$RHS\leq\left\|\partial_t\nabla_x\zeta(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\bigg(\left\|u(t)-\bar{u}(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\left\|S(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\bigg). \tag{3.261}$$

Note that the boundary terms vanish due to the construction of Φ. Therefore, combining (3.260) and (3.261), we have

$$\epsilon^2\left\|\partial_t\nabla_x\zeta(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\leq\left\|u(t)-\bar{u}(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\left\|S(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}. \tag{3.262}$$

(3.262) is true for all t. Then we can further integrate it over $[0,t]$ to obtain

$$\epsilon^2\left\|\partial_t\nabla_x\zeta\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\leq\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}. \tag{3.263}$$

Collecting terms in (3.248), (3.249), (3.250), (3.251), (3.252), (3.253), (3.254),(3.255) and (3.263), we have

(3.264)

$$\begin{aligned}
&\epsilon \left\|\bar{u}\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq& C\Bigg(\epsilon \left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} \left\|\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} + \epsilon \left\|\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} \bigg(\left\|u\right\|_{L^2([0,t]\times\Gamma^+)} + \left\|g\right\|_{L^2([0,t]\times\Gamma^-)}\bigg)\\
&+ \left\|\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} \left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} + \left\|\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} \left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\
&+ \epsilon^2 \left\|u(t)\right\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \epsilon^2 \left\|h\right\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} \bigg(\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} + \left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\bigg)\Bigg).
\end{aligned}$$

Applying Cauchy's inequality to each term on the right-hand side of (3.264), we obtain

$$\begin{aligned}
\epsilon \left\|\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} \leq& C\bigg(\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} + \epsilon^{\frac{3}{2}} \left\|u(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)} + \epsilon \left\|u\right\|_{L^2([0,t]\times\Gamma^+)}\\
&+ \left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} + \epsilon^{\frac{3}{2}} \left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)} + \epsilon \left\|g\right\|_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned} \tag{3.265}$$

Step 2: Energy Estimates.
In the weak formulation (3.242), we may take the test function $\phi = u$ to get the energy estimate

(3.266)

$$\frac{\epsilon^2}{2} \left\|u(t)\right\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \frac{\epsilon}{2} \left\|u\right\|^2_{L^2([0,t]\times\Gamma^+)} + \left\|u-\bar{u}\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} = \int_0^t \iint_{\Omega\times\mathbb{S}^2} Su + \frac{\epsilon^2}{2} \left\|h\right\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \frac{\epsilon}{2} \left\|g\right\|^2_{L^2([0,t]\times\Gamma^-)}.$$

On the other hand, we can square on both sides of (3.265) to obtain

$$\begin{aligned}
\epsilon^2 \left\|\bar{u}\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} \leq& C\bigg(\left\|u-\bar{u}\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} + \epsilon^3 \left\|u(t)\right\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \epsilon^2 \left\|u\right\|^2_{L^2([0,t]\times\Gamma^+)}\\
&+ \left\|S\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} + \epsilon^3 \left\|h\right\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \epsilon^2 \left\|g\right\|^2_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned} \tag{3.267}$$

Multiplying (3.267) by a sufficiently small constant and adding it to (3.266) to absorb $\left\|u-\bar{u}\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}$, $\epsilon^3 \left\|u(t)\right\|^2_{L^2(\Omega\times\mathbb{S}^2)}$, and $\epsilon^2 \left\|u\right\|^2_{L^2([0,t]\times\Gamma^+)}$, we deduce

$$\begin{aligned}
&\epsilon^2 \left\|u(t)\right\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \epsilon \left\|u\right\|^2_{L^2([0,t]\times\Gamma^+)} + \epsilon^2 \left\|\bar{u}\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} + \left\|u-\bar{u}\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq& C\bigg(\left\|S\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} + \int_0^t \iint_{\Omega\times\mathbb{S}^2} Su + \epsilon^2 \left\|h\right\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \epsilon \left\|g\right\|^2_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned} \tag{3.268}$$

Hence, we have

$$\begin{aligned}
&\epsilon^2 \left\|u(t)\right\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \epsilon \left\|u\right\|^2_{L^2([0,t]\times\Gamma^+)} + \epsilon^2 \left\|u\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq& C\bigg(\left\|S\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} + \int_0^t \iint_{\Omega\times\mathbb{S}^2} Su + \epsilon^2 \left\|h\right\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \epsilon \left\|g\right\|^2_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned} \tag{3.269}$$

A direct application of Cauchy's inequality leads to

$$\int_0^t \iint_{\Omega\times\mathbb{S}^2} Su \leq \frac{1}{4C_0\epsilon^2} \left\|S\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} + C_0\epsilon^2 \left\|u\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}. \tag{3.270}$$

Taking C_0 sufficiently small and inserting (3.270) into (3.269), we obtain

$$\begin{aligned}
&\epsilon^2 \left\|u(t)\right\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \epsilon \left\|u\right\|^2_{L^2([0,t]\times\Gamma^+)} + \epsilon^2 \left\|u\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq& C\bigg(\frac{1}{\epsilon^2} \left\|S\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)} + \epsilon^2 \left\|h\right\|^2_{L^2(\Omega\times\mathbb{S}^2)} + \epsilon \left\|g\right\|^2_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned} \tag{3.271}$$

Then we have

$$
\begin{aligned}
(3.272)\qquad &\|u(t)\|_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{2}}}\|u\|_{L^2([0,t]\times\Gamma^+)}+\|u\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq &C\bigg(\frac{1}{\epsilon^2}\|S\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\|h\|_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{2}}}\|g\|_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned}
$$

□

3.2.2. L^∞ *Estimate - First Round.*

Theorem 3.3. *Assume $S(t,\vec{x},\vec{w})\in L^\infty([0,t]\times\Omega\times\mathbb{S}^2)$, $h(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathbb{S}^2)$ and $g(t,x_0,\vec{w})\in L^\infty([0,t]\times\Gamma^-)$. Then the solution $u(t,\vec{x},\vec{w})$ to the neutron transport equation (3.232) satisfies*

$$
\begin{aligned}
(3.273)\qquad \|u\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\leq &C\bigg(\frac{1}{\epsilon^3}\|S\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\|S\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\\
&+\frac{1}{\epsilon^{\frac{3}{2}}}\|h\|_{L^2(\Omega\times\mathbb{S}^2)}+\|h\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\bigg)\\
&+\frac{1}{\epsilon^2}\|g\|_{L^2([0,t]\times\Gamma^-)}+\|g\|_{L^\infty([0,t]\times\Gamma^-)}.
\end{aligned}
$$

Proof.
Step 1: Mild formulation.
The characteristics $\Big(T(s),X(s),W(s)\Big)$ for $s\in\mathbb{R}$ of the equation (3.232) which goes through $(t,\vec{x},\vec{w})$ is defined by

$$
(3.274)\qquad \Big(T(0),X(0),W(0)\Big)=(t,\vec{x},\vec{w}),\quad \frac{\mathrm{d}T(s)}{\mathrm{d}s}=\epsilon^2,\quad \frac{\mathrm{d}X(s)}{\mathrm{d}s}=\epsilon W(s),\quad \frac{\mathrm{d}W(s)}{\mathrm{d}s}=0,
$$

which implies

$$
(3.275)\qquad T(s)=t+\epsilon^2 s,\quad X(s)=\vec{x}+(\epsilon\vec{w})s,\quad W(s)=\vec{w}.
$$

We rewrite the equation (3.232) along the characteristics as

$$
\begin{aligned}
(3.276)\qquad u(t,\vec{x},\vec{w})=&\mathbf{1}_{\{t>\epsilon^2t_b\}}g(t-\epsilon^2t_b,\vec{x}-\epsilon\vec{w}t_b,\vec{w})\mathrm{e}^{-t_b}+\mathbf{1}_{\{t=\epsilon^2t_b\}}h(\vec{x}-\epsilon\vec{w}t_b,\vec{w})\mathrm{e}^{-t_b}\\
&+\int_0^{t_b}S(t-\epsilon^2s,\vec{x}-\epsilon s\vec{w},\vec{w})\mathrm{e}^{-s}\mathrm{d}s+\int_0^{t_b}\bar{u}(t-\epsilon^2s,\vec{x}-\epsilon s\vec{w})\mathrm{e}^{-s}\mathrm{d}s\\
=&\mathbf{1}_{\{t>\epsilon^2t_b\}}g(t-\epsilon^2t_b,\vec{x}-\epsilon\vec{w}t_b,\vec{w})\mathrm{e}^{-t_b}+\mathbf{1}_{\{t=\epsilon^2t_b\}}h(\vec{x}-\epsilon\vec{w}t_b,\vec{w})\mathrm{e}^{-t_b}\\
&+\int_0^{t_b}S(t-\epsilon^2s,\vec{x}-\epsilon s\vec{w},\vec{w})\mathrm{e}^{-s}\mathrm{d}s+\frac{1}{4\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^2}u(t-\epsilon^2s,\vec{x}-\epsilon s\vec{w},\vec{w}_t)\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-s}\mathrm{d}s,
\end{aligned}
$$

where the backward exit time $t_b\in\big[0,t\epsilon^{-2}\big]$ is defined as

$$
(3.277)\qquad t_b(t,\vec{x},\vec{w})=\inf\big\{s\geq0:(t-\epsilon^2s,\vec{x}-\epsilon s\vec{w},\vec{w})\in([0,t]\times\Gamma^-)\cup(\{0\}\times\Omega\times\mathbb{S}^2)\big\},
$$

and $\vec{w}_t$ is a dummy variable for velocity.

For the last term in (3.276), we rewrite $u(t-\epsilon^2s,\vec{x}-\epsilon s\vec{w},\vec{w}_t)$ by tracking back along the characteristics again to obtain

(3.278)

$$
\begin{aligned}
u(t,\vec{x},\vec{w}) =&\mathbf{1}_{\{t>\epsilon^2 t_b\}} g(t-\epsilon^2 t_b,\vec{x}-\epsilon\vec{w}t_b,\vec{w})\mathrm{e}^{-t_b}+\mathbf{1}_{\{t=\epsilon^2 t_b\}} h(\vec{x}-\epsilon\vec{w}t_b,\vec{w})\mathrm{e}^{-t_b}+\int_0^{t_b} S(t',\vec{x}_t,\vec{w})\mathrm{e}^{-s}\mathrm{d}s\\
&+\frac{1}{4\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^2}\mathbf{1}_{\{t'>\epsilon^2 s_b\}} g(t'-\epsilon^2 s_b,\vec{x}_t-\epsilon\vec{w}s_b,\vec{w}_t)\mathrm{e}^{-s_b}\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-s}\mathrm{d}s,\\
&+\frac{1}{4\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^2}\mathbf{1}_{\{t'=\epsilon^2 s_b\}} h(\vec{x}_t-\epsilon\vec{w}s_b,\vec{w}_t)\mathrm{e}^{-s_b}\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-s}\mathrm{d}s,\\
&+\frac{1}{4\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^2}\bigg(\int_0^{s_b} S(t'-\epsilon^2 r,\vec{x}_t-\epsilon r\vec{w}_t,\vec{w}_t)\mathrm{e}^{-r}\mathrm{d}r\bigg)\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-s}\mathrm{d}s\\
&+\frac{1}{4\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^2}\bigg(\int_0^{s_b} \bar{u}(t'-\epsilon^2 r,\vec{x}_t-\epsilon r\vec{w}_t)\mathrm{e}^{-r}\mathrm{d}r\bigg)\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-s}\mathrm{d}s,
\end{aligned}
$$

where the exiting time from $(t',\vec{x}_t,\vec{w}_t)=(t-\epsilon^2 s,\vec{x}-\epsilon s\vec{w},\vec{w}_t)$ is defined as

$$
s_b(t,\vec{x},\vec{w};s,\vec{w}_t)=\inf\left\{r\geq 0:(t'-\epsilon^2 r,\vec{x}_t-\epsilon r\vec{w}_t,\vec{w}_t)\in([0,t]\times\Gamma^-)\cup(\{0\}\times\Omega\times\mathbb{S}^2)\right\}. \tag{3.279}
$$

We may reiterate (3.278) again along the characteristics to obtain

(3.280)

$$
\begin{aligned}
&u(t,\vec{x},\vec{w})\\
=&\mathbf{1}_{\{t>\epsilon^2 t_b\}} g(t-\epsilon^2 t_b,\vec{x}-\epsilon\vec{w}t_b,\vec{w})\mathrm{e}^{-t_b}+\mathbf{1}_{\{t=\epsilon^2 t_b\}} h(\vec{x}-\epsilon\vec{w}t_b,\vec{w})\mathrm{e}^{-t_b}+\int_0^{t_b} S(t',\vec{x}_t,\vec{w})\mathrm{e}^{-s}\mathrm{d}s\\
&+\frac{1}{4\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^2}\mathbf{1}_{\{t'>\epsilon^2 s_b\}} g(t'-\epsilon^2 s_b,\vec{x}_t-\epsilon\vec{w}s_b,\vec{w}_t)\mathrm{e}^{-s_b}\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-s}\mathrm{d}s,\\
&+\frac{1}{4\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^2}\mathbf{1}_{\{t'=\epsilon^2 s_b\}} h(\vec{x}_t-\epsilon\vec{w}s_b,\vec{w}_t)\mathrm{e}^{-s_b}\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-s}\mathrm{d}s,\\
&+\frac{1}{4\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^2}\bigg(\int_0^{s_b} S(t'',\vec{x}_s,\vec{w}_t)\mathrm{e}^{-r}\mathrm{d}r\bigg)\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-s}\mathrm{d}s\\
&+\left(\frac{1}{4\pi}\right)^2\int_0^{t_b}\bigg(\int_{\mathbb{S}^2}\bigg(\int_0^{s_b}\bigg(\int_{\mathbb{S}^2}\mathbf{1}_{\{t''>\epsilon^2 r_b\}} g(t''-\epsilon^2 r_b,\vec{x}_s-\epsilon\vec{w}_s r_b,\vec{w}_s)\mathrm{e}^{-r_b}\mathrm{d}\vec{w}_s\bigg)\mathrm{e}^{-r}\mathrm{d}r\bigg)\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-s}\mathrm{d}s\\
&+\left(\frac{1}{4\pi}\right)^2\int_0^{t_b}\bigg(\int_{\mathbb{S}^2}\bigg(\int_0^{s_b}\bigg(\int_{\mathbb{S}^2}\mathbf{1}_{\{t''=\epsilon^2 r_b\}} h(\vec{x}_s-\epsilon\vec{w}_s r_b,\vec{w}_s)\mathrm{e}^{-r_b}\mathrm{d}\vec{w}_s\bigg)\mathrm{e}^{-r}\mathrm{d}r\bigg)\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-s}\mathrm{d}s\\
&+\left(\frac{1}{4\pi}\right)^2\int_0^{t_b}\bigg(\int_{\mathbb{S}^2}\bigg(\int_0^{s_b}\bigg(\int_{\mathbb{S}^2}\int_0^{r_b} S(t''-\epsilon^2 q,\vec{x}_s-\epsilon q\vec{w}_s,\vec{w}_s)\mathrm{e}^{-q}\mathrm{d}q\mathrm{d}\vec{w}_s\bigg)\mathrm{e}^{-r}\mathrm{d}r\bigg)\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-s}\mathrm{d}s\\
&+\left(\frac{1}{4\pi}\right)^2\int_0^{t_b}\bigg(\int_{\mathbb{S}^2}\bigg(\int_0^{s_b}\bigg(\int_{\mathbb{S}^2}\int_0^{r_b} \bar{u}(t''-\epsilon^2 q,\vec{x}_s-\epsilon q\vec{w}_s)\mathrm{e}^{-q}\mathrm{d}q\mathrm{d}\vec{w}_s\bigg)\mathrm{e}^{-r}\mathrm{d}r\bigg)\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-s}\mathrm{d}s,
\end{aligned}
$$

where the dummy variable $\vec{w}_s$ and the exiting time from $(t'',\vec{x}_s,\vec{w}_s)=(t'-\epsilon^2 r,\vec{x}_t-\epsilon r\vec{w}_t,\vec{w}_s)$ is defined as

$$
r_b(t,\vec{x},\vec{w};s,\vec{w}_t;r,\vec{w}_s)=\inf\left\{q\geq 0:(t''-\epsilon^2 q,\vec{x}_s-\epsilon q\vec{w}_s,\vec{w}_s)\in([0,t]\times\Gamma^-)\cup(\{0\}\times\Omega\times\mathbb{S}^2)\right\}. \tag{3.281}
$$

Step 2: Estimates of all but the last term in (3.280).
Note the fact that $0\leq s\leq t_b$ and $0\leq r\leq s_b$. We can directly estimate

$$
\left|\mathbf{1}_{\{t>\epsilon^2 t_b\}} g(t-\epsilon^2 t_b,\vec{x}-\epsilon\vec{w}t_b,\vec{w})\mathrm{e}^{-t_b}\right|\leq\|g\|_{L^\infty([0,t]\times\Gamma^-)}, \tag{3.282}
$$

$$
\left|\mathbf{1}_{\{t=\epsilon^2 t_b\}} h(\vec{x}-\epsilon\vec{w}t_b,\vec{w})\mathrm{e}^{-t_b}\right|\leq\|h\|_{L^\infty(\Omega\times\mathbb{S}^2)}, \tag{3.283}
$$

$$\left|\int_0^{t_b} S(t-\epsilon^2 s,\vec x-\epsilon s\vec w,\vec w)\mathrm{e}^{-s}\mathrm{d}s\right|\leq \|S\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\,, \tag{3.284}$$

$$\left|\frac{1}{4\pi}\int_0^{t_b}\left(\int_{\mathbb{S}^2}\mathbf{1}_{\{t'>\epsilon^2 s_b\}}g(t'-\epsilon^2 s_b,\vec x_t-\epsilon\vec w s_b,\vec w_t)\mathrm{e}^{-s_b}\mathrm{d}\vec w_t\right)\mathrm{e}^{-s}\mathrm{d}s\right|\leq \|g\|_{L^\infty([0,t]\times\Gamma^-)}\,, \tag{3.285}$$

$$\left|\frac{1}{4\pi}\int_0^{t_b}\left(\int_{\mathbb{S}^2}\mathbf{1}_{\{t'=\epsilon^2 s_b\}}h(\vec x_t-\epsilon\vec w s_b,\vec w_t)\mathrm{e}^{-s_b}\mathrm{d}\vec w_t\right)\mathrm{e}^{-s}\mathrm{d}s\right|\leq \|h\|_{L^\infty(\Omega\times\mathbb{S}^2)}\,, \tag{3.286}$$

$$\left|\frac{1}{4\pi}\int_0^{t_b}\left(\int_{\mathbb{S}^2}\left(\int_0^{s_b} S(t'',\vec x_s,\vec w_t)\mathrm{e}^{-r}\mathrm{d}r\right)\mathrm{d}\vec w_t\right)\mathrm{e}^{-s}\mathrm{d}s\right|\leq \|S\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\,. \tag{3.287}$$

$$\left|\left(\frac{1}{4\pi}\right)^2\int_0^{t_b}\left(\int_{\mathbb{S}^2}\left(\int_0^{s_b}\left(\int_{\mathbb{S}^2}\mathbf{1}_{\{t''>\epsilon^2 r_b\}}g(t''-\epsilon^2 r_b,\vec x_s-\epsilon\vec w_s r_b,\vec w_s)\mathrm{e}^{-r_b}\mathrm{d}\vec w_s\right)\mathrm{e}^{-r}\mathrm{d}r\right)\mathrm{d}\vec w_t\right)\mathrm{e}^{-s}\mathrm{d}s\right|\leq \|g\|_{L^\infty([0,t]\times\Gamma^-)}\,, \tag{3.288}$$

$$\left|\left(\frac{1}{4\pi}\right)^2\int_0^{t_b}\left(\int_{\mathbb{S}^2}\left(\int_0^{s_b}\left(\int_{\mathbb{S}^2}\mathbf{1}_{\{t''=\epsilon^2 r_b\}}h(\vec x_s-\epsilon\vec w_s r_b,\vec w_s)\mathrm{e}^{-r_b}\mathrm{d}\vec w_s\right)\mathrm{e}^{-r}\mathrm{d}r\right)\mathrm{d}\vec w_t\right)\mathrm{e}^{-s}\mathrm{d}s\right|\leq \|h\|_{L^\infty(\Omega\times\mathbb{S}^2)}\,, \tag{3.289}$$

$$\left|\left(\frac{1}{4\pi}\right)^2\int_0^{t_b}\left(\int_{\mathbb{S}^2}\left(\int_0^{s_b}\left(\int_{\mathbb{S}^2}\int_0^{r_b} S(t''-\epsilon^2 q,\vec x_s-\epsilon q\vec w_s,\vec w_s)\mathrm{e}^{-q}\mathrm{d}q\mathrm{d}\vec w_s\right)\mathrm{e}^{-r}\mathrm{d}r\right)\mathrm{d}\vec w_t\right)\mathrm{e}^{-s}\mathrm{d}s\right|\leq \|S\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\,. \tag{3.290}$$

Step 3: Estimates of the last term in (3.280).
Now we decompose the last term in (3.278) as

$$\int_0^{t_b}\int_{\mathbb{S}^2}\int_0^{s_b}\int_{\mathbb{S}^2}\int_0^{r_b}=\int_0^{t_b}\int_{\mathbb{S}^2}\int_{0\leq r\leq\delta}\int_{\mathbb{S}^2}\int_0^{r_b}+\int_0^{t_b}\int_{\mathbb{S}^2}\int_{r\geq\delta}\int_{\mathbb{S}^2}\int_0^{r_b}=I_1+I_1^*, \tag{3.291}$$

for some $0<\delta<<1$ to be determined later. Since I_1 contains an integral in a very small region, we may directly estimate

$$\left|I_1\right|\leq \|u\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\left(\int_{r\leq\delta}\mathrm{e}^{-r}\mathrm{d}r\right)\leq C\delta\|u\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\,. \tag{3.292}$$

We may further decompose

$$I_1^*=\int_0^{t_b}\int_{\mathbb{S}^2}\int_{r\geq\delta}\int_{\mathbb{S}^2}\int_{0\leq q\leq\delta}+\int_0^{t_b}\int_{\mathbb{S}^2}\int_{r\geq\delta}\int_{\mathbb{S}^2}\int_{q\geq\delta}=I_2+I_2^*. \tag{3.293}$$

Similarly, we may bound

$$\left|I_2\right|\leq \|u\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\left(\int_{q\leq\delta}\mathrm{e}^{-q}\mathrm{d}r\right)\leq C\delta\|u\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\,. \tag{3.294}$$

Then we need to handle the most complicated term I_2^*,

$$\left|I_2^*\right|\leq C\int_0^{t_b}\int_{\mathbb{S}^2}\int_{r\geq\delta}\int_{\mathbb{S}^2}\int_{q\geq\delta}\left|\bar u(t''-\epsilon^2 q,\vec x_t-\epsilon r\vec w_t-\epsilon q\vec w_s)\right|\mathrm{e}^{-q}\mathrm{e}^{-r}\mathrm{e}^{-s}\mathrm{d}q\mathrm{d}\vec w_s\mathrm{d}r\mathrm{d}\vec w_t\Big)\mathrm{d}s. \tag{3.295}$$

By the definition of t_b and s_b, we always have

$$\vec x_t-\epsilon r\vec w_t-\epsilon q\vec w_s\in\Omega. \tag{3.296}$$

Hence, we may introduce the indicator function $\mathbf{1}_\Omega$ and apply Hölder's inequality to obtain
(3.297)

$$\begin{aligned}
\left|I_2^*\right| \leq& C\int_0^{t_b}\left(\int_{\mathbb{S}^2}\int_{r\geq\delta}\int_{\mathbb{S}^2}\int_{q\geq\delta}\mathbf{1}_\Omega(\vec{x}_t-\epsilon r\vec{w}_t-\epsilon q\vec{w}_s)\left|\bar{u}(t''-\epsilon^2 q,\vec{x}_t-\epsilon r\vec{w}_t-\epsilon q\vec{w}_s)\right|\mathrm{e}^{-q}\mathrm{e}^{-r}\mathrm{d}q\mathrm{d}\vec{w}_s\mathrm{d}r\mathrm{d}\vec{w}_t\right)\mathrm{e}^{-s}\mathrm{d}s\\
\leq& C\int_0^{t_b}\left(\left(\int_{\mathbb{S}^2}\int_{r\geq\delta}\int_{\mathbb{S}^2}\int_{q\geq\delta}\mathbf{1}_\Omega(\vec{x}_t-\epsilon r\vec{w}_t-\epsilon q\vec{w}_s)\left|\bar{u}(t''-\epsilon^2 q,\vec{x}_t-\epsilon r\vec{w}_t-\epsilon q\vec{w}_s)\right|^2\mathrm{e}^{-q}\mathrm{e}^{-r}\mathrm{d}q\mathrm{d}\vec{w}_s\mathrm{d}r\mathrm{d}\vec{w}_t\right)^{\frac{1}{2}}\right.\\
&\times\left.\left(\int_{\mathbb{S}^2}\int_{r\geq\delta}\int_{\mathbb{S}^2}\int_{q\geq\delta}\mathbf{1}_\Omega(\vec{x}_t-\epsilon r\vec{w}_t-\epsilon q\vec{w}_s)\mathrm{e}^{-q}\mathrm{e}^{-r}\mathrm{d}q\mathrm{d}\vec{w}_s\mathrm{d}r\mathrm{d}\vec{w}_t\right)^{\frac{1}{2}}\right)\mathrm{e}^{-s}\mathrm{d}s\\
\leq& C\int_0^{t_b}\left(\left(\int_{\mathbb{S}^2}\int_{r\geq\delta}\int_{\mathbb{S}^2}\int_{q\geq\delta}\mathbf{1}_\Omega(\vec{x}_t-\epsilon r\vec{w}_t-\epsilon q\vec{w}_s)\left|\bar{u}(t''-\epsilon^2 q,\vec{x}_t-\epsilon r\vec{w}_t-\epsilon q\vec{w}_s)\right|^2\mathrm{e}^{-q}\mathrm{e}^{-r}\mathrm{d}q\mathrm{d}\vec{w}_s\mathrm{d}r\mathrm{d}\vec{w}_t\right)^{\frac{1}{2}}\right)\mathrm{e}^{-s}\mathrm{d}s.
\end{aligned}$$

Note $\vec{w}_t,\vec{w}_s\in\mathbb{S}^2$, which can be parameterized as

$$\vec{w}_t=(\sin\phi\cos\psi,\sin\phi\sin\psi,\cos\phi), \tag{3.298}$$

$$\vec{w}_s=(\sin\phi'\cos\psi',\sin\phi'\sin\psi',\cos\phi'), \tag{3.299}$$

for $\phi,\phi'\in[0,\pi]$ and $\psi,\psi'\in[0,2\pi]$. Hence, we may write the integral

$$\int_{\mathbb{S}^2}\cdots\mathrm{d}\vec{w}_t=\int_0^{2\pi}\int_0^{\pi}\cdots\sin\phi\mathrm{d}\phi\mathrm{d}\psi,\quad \int_{\mathbb{S}^2}\cdots\mathrm{d}\vec{w}_s=\int_0^{2\pi}\int_0^{\pi}\cdots\sin\phi'\mathrm{d}\phi'\mathrm{d}\psi' \tag{3.300}$$

where $\sin\phi$ and $\sin\phi'$ are the Jacobian of spherical coordinates. Then we define the change of variable $[0,\pi]\times[0,2\pi]\times[0,2\pi]\to\Omega:(\phi,\psi,\psi')\to(y_1,y_2,y_3)=\vec{y}=\vec{x}_t-\epsilon r\vec{w}_t-\epsilon q\vec{w}_s$, i.e.

$$\left\{\begin{array}{rcl}
y_1&=&(x_t)_1-\epsilon r\sin\phi\cos\psi-\epsilon q\sin\phi'\cos\psi',\\
y_2&=&(x_t)_2-\epsilon r\sin\phi\sin\psi-\epsilon q\sin\phi'\sin\psi',\\
y_3&=&(x_t)_3-\epsilon r\cos\phi-\epsilon q\cos\phi'.
\end{array}\right. \tag{3.301}$$

The Jacobian is
(3.302)

$$\left|\frac{\partial(y_1,y_2,y_3)}{\partial(\phi,\psi,\psi')}\right|=\left\|\begin{array}{ccc}
-\epsilon r\cos\phi\cos\psi & \epsilon r\sin\phi\sin\psi & -\epsilon q\sin\phi'\sin\psi'\\
-\epsilon r\cos\phi\sin\psi & -\epsilon r\sin\phi\cos\psi & \epsilon q\sin\phi'\cos\psi'\\
\epsilon r\sin\phi & 0 & 0
\end{array}\right\|=\epsilon^3 r^2 q\sin^2\phi\sin\phi'\sin(\psi-\psi').$$

Now we consider the restriction on r and q. Naturally, $r,q\geq\delta$ implies $r^2q\geq\delta^3$. Therefore, we may bound the Jacobian

$$\left|\frac{\partial(y_1,y_2,y_3)}{\partial(\phi,\psi,\psi')}\right|\geq\epsilon^3\delta^3\sin^2\phi\left|\sin\phi'\sin(\psi-\psi')\right|. \tag{3.303}$$

$\sin\phi\sin\phi'$ can be cancelled out by (3.300). Now, we need to further restrict $\sin\phi\sin(\psi-\psi')$. Decompose

$$\begin{aligned}
I_2^*=&\int_0^{t_b}\int_{0\leq\sin\phi\leq\delta}\int_{r\geq\delta}\int_{\mathbb{S}^2}\int_{q\geq\delta}+\int_0^{t_b}\int_{\sin\phi\geq\delta}\int_{r\geq\delta}\int_{|\sin(\psi-\psi')|\leq\delta}\int_{q\geq\delta}\\
&+\int_0^{t_b}\int_{\sin\phi\geq\delta}\int_{r\geq\delta}\int_{|\sin(\psi-\psi')|\geq\delta}\int_{q\geq\delta}=I_3+I_4+I_4^*.
\end{aligned} \tag{3.304}$$

Similar to the estimate of I_1 and I_2 in (3.292) and (3.294), we may bound

$$|I_3|\leq\|u\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\left(\int_{0\leq\sin\phi\leq\delta}\sin\phi\mathrm{d}\phi\right)\leq C\delta\|u\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}, \tag{3.305}$$

and

$$|I_4|\leq\|u\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\left(\int_{|\sin(\psi-\psi')|\leq\delta}\sin\phi'\mathrm{d}\phi'\right)\leq C\delta\|u\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}. \tag{3.306}$$

Then in the estimate of I_4^*, using (3.303), we know

$$\left|\frac{\partial(y_1,y_2,y_3)}{\partial(\phi,\psi,\psi')}\right|\geq\epsilon^3\delta^5\sin\phi\sin\phi'. \tag{3.307}$$

Hence, we may bound

(3.308)

$$\left|I_4^*\right|\leq C\int_0^{t_b}\left(\int_{r\geq\delta}\int_{q\geq\delta}\left(\int_\Omega\frac{1}{\epsilon^3\delta^5}\left|\bar u(t''-\epsilon^2q,\vec y)\right|^2\mathrm{d}\vec y\right)\mathrm{e}^{-q}\mathrm{e}^{-r}\mathrm{d}q\mathrm{d}r\right)^{\frac{1}{2}}\mathrm{e}^{-s}\mathrm{d}s\leq\frac{C}{\epsilon^{\frac{3}{2}}\delta^{\frac{5}{2}}}\left\|\bar u(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}.$$

Here the time integral is finite due to the exponential terms.

Step 4: Synthesis.
Summarizing (3.282), (3.283), (3.284), (3.285), (3.286), (3.287), (3.288), (3.289), (3.290), (3.292), (3.294), (3.305), (3.306), (3.308), we have shown

(3.309)

$$|u|\leq C\bigg(\delta\left\|u\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\delta^{\frac{5}{2}}\epsilon^{\frac{3}{2}}}\left\|\bar u(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\left\|S\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}+\left\|h\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}+\left\|g\right\|_{L^\infty([0,t]\times\Gamma^-)}\bigg).$$

Taking supremum over all $(t,\vec x,\vec w)\in[0,t]\times\Omega\times\mathbb{S}^2$, we obtain

$$\begin{aligned}\left\|u\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\leq& C\bigg(\delta\left\|u\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\delta^{\frac{5}{2}}\epsilon^{\frac{3}{2}}}\left\|\bar u(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\&+\left\|S\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}+\left\|h\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}+\left\|g\right\|_{L^\infty([0,t]\times\Gamma^-)}\bigg).\end{aligned} \tag{3.310}$$

Then taking δ sufficiently small to absorb $C\delta\left\|u\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}$ into the left-hand side, we get

(3.311)

$$\left\|u\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\leq C\bigg(\frac{1}{\delta^{\frac{5}{2}}\epsilon^{\frac{3}{2}}}\left\|\bar u(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\left\|S\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}+\left\|h\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}+\left\|g\right\|_{L^\infty([0,t]\times\Gamma^-)}\bigg).$$

Using Theorem 3.2, we get

$$\begin{aligned}\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}\leq& C\bigg(\frac{1}{\epsilon^{\frac{7}{2}}}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\left\|S\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}\\&+\frac{1}{\epsilon^{\frac{3}{2}}}\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\left\|h\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^2}\left\|g\right\|_{L^2([0,t]\times\Gamma^-)}+\left\|g\right\|_{L^\infty(\Gamma^-)}\bigg).\end{aligned} \tag{3.312}$$

□

3.2.3. L^{2m} *Estimate.* In this subsection, we try to improve previous estimates. In the following, let $o(1)$ denote a sufficiently small constant.

Theorem 3.4. *Assume $S(t,\vec{x},\vec{w})\in L^{\infty}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)$, $h(\vec{x},\vec{w})\in L^{\infty}(\Omega\times\mathbb{S}^2)$ and $g(t,x_0,\vec{w})\in L^{\infty}(\mathbb{R}^+\times\Gamma^-)$. Then the solution $u(t,\vec{x},\vec{w})$ to the neutron transport equation (3.232) satisfies*

(3.313)
$$\begin{aligned}
&\|u(t)\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{2m}}}\|u\|_{L^2([0,t]\times\Gamma^+)}+\frac{1}{\epsilon^{\frac{1}{2m}}}\|u\|_{L^{2m}([0,t]\times\Gamma^+)}\\
&+\frac{1}{\epsilon^{\frac{1}{2m}}}\|\bar{u}\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{1}{2m}}}\|u-\bar{u}\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{m}}}\|u-\bar{u}\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq& C\bigg(\frac{1}{\epsilon^{2+\frac{1}{2m}}}\|S\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{1}{2m}}}\|S\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{1}{2m}}}\|S\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\
&+\frac{1}{\epsilon^{\frac{1}{2m}}}\|\partial_tS\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{2m}}}\|S(0)\|_{L^2(\Omega\times\mathbb{S}^2)}\\
&+\frac{1}{\epsilon^{\frac{1}{2m}}}\|h\|_{L^2(\Omega\times\mathbb{S}^2)}+\|h\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon^{1-\frac{1}{2m}}\|\vec{w}\cdot\nabla_xh\|_{L^2(\Omega\times\mathbb{S}^2)}\\
&+\frac{1}{\epsilon^{\frac{1}{2m}}}\|g\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{2m}}}\|g\|_{L^{2m}([0,t]\times\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{2m}}}\|g\|_{L^2([0,t]\times\Gamma^-)}+\epsilon^{\frac{3}{2}-\frac{1}{2m}}\|\partial_tg\|_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned}$$

Proof.
Step 1: Kernel Estimate.
Applying Lemma 3.1 to the equation (3.232). Then for any $\phi\in L^2([0,t]\times\Omega\times\mathbb{S}^2)$ satisfying $\epsilon\partial_t\phi+\vec{w}\cdot\nabla_x\phi\in L^2([0,t]\times\Omega\times\mathbb{S}^2)$ and $\phi\in L^2([0,t]\times\Gamma)$, we have

(3.314)
$$\epsilon^2\int_0^t\iint_{\Omega\times\mathbb{S}^2}\phi\partial_tu-\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\vec{w}\cdot\nabla_x\phi)u+\int_0^t\iint_{\Omega\times\mathbb{S}^2}(u-\bar{u})\phi=-\epsilon\int_0^t\int_{\Gamma}u\phi\mathrm{d}\gamma+\int_0^t\iint_{\Omega\times\mathbb{S}^2}S\phi.$$

Our goal is to choose a particular test function ϕ. We first construct an auxiliary function $\zeta(t)$. Since $u(t)\in L^{\infty}(\Omega\times\mathbb{S}^2)$, it implies $\bar{u}(t)\in L^{\infty}(\Omega)$. Define $\zeta(t,\vec{x})$ on Ω satisfying

$$\left\{\begin{array}{l}\Delta_x\zeta(t)=\bar{u}^{2m-1}(t)\ \text{ in }\ \Omega,\\ \zeta(t)=0\ \text{ on }\ \partial\Omega.\end{array}\right.\tag{3.315}$$

In the bounded domain Ω, based on the standard elliptic estimates, there exists a unique $\xi(t)\in W^{2,\frac{2m}{2m-1}}(\Omega)$ satisfying

$$\|\zeta(t)\|_{W^{2,\frac{2m}{2m-1}}(\Omega)}\leq C\left\|\bar{u}^{2m-1}(t)\right\|_{L^{\frac{2m}{2m-1}}(\Omega)}=C\|\bar{u}(t)\|_{L^{2m}(\Omega)}^{2m-1}.\tag{3.316}$$

We plug the test function

$$\phi=-\vec{w}\cdot\nabla_x\zeta\tag{3.317}$$

into the weak formulation (3.314) and estimate each term there. By Sobolev embedding theorem, we have

$$\|\phi(t)\|_{L^2(\Omega)}\leq C\|\zeta(t)\|_{H^1(\Omega)}\leq C\|\zeta(t)\|_{W^{2,\frac{2m}{2m-1}}(\Omega)}\leq C\|\bar{u}(t)\|_{L^{2m}(\Omega)}^{2m-1},\tag{3.318}$$
$$\|\phi(t)\|_{L^{\frac{2m}{2m-1}}(\Omega)}\leq C\|\zeta(t)\|_{W^{1,\frac{2m}{2m-1}}(\Omega)}\leq C\|\bar{u}(t)\|_{L^{2m}(\Omega)}^{2m-1}.\tag{3.319}$$

Note that the embedding $W^{2,\frac{2m}{2m-1}}(\Omega)\hookrightarrow H^1(\Omega)$ requires $m<3$.
On the other hand, we decompose

$$-\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\vec{w}\cdot\nabla_x\phi)u=-\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\vec{w}\cdot\nabla_x\phi)\bar{u}-\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\vec{w}\cdot\nabla_x\phi)(u-\bar{u}).\tag{3.320}$$

For the first term on the right-hand side of (3.320), by (3.315) and (3.317), we have

(3.321)

$$
\begin{aligned}
&-\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\vec{w}\cdot\nabla_x\phi)\bar{u}\\
=&\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}\bar{u}\Big(w_1(w_1\partial_{11}\zeta+w_2\partial_{12}\zeta+w_3\partial_{13}\zeta)+w_2(w_1\partial_{21}\zeta+w_2\partial_{22}\zeta+w_3\partial_{23}\zeta)+w_3(w_1\partial_{31}\zeta+w_2\partial_{32}\zeta+w_3\partial_{33}\zeta)\Big)\\
=&\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}\bar{u}\Big(w_1^2\partial_{11}\zeta+w_2^2\partial_{22}\zeta+w_3^2\partial_{33}\zeta\Big)=\frac{4}{3}\epsilon\pi\int_0^t\int_\Omega\bar{u}(\partial_{11}\zeta+\partial_{22}\zeta+\partial_{33}\zeta)\\
=&\frac{4}{3}\epsilon\pi\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega)}^{2m}=\frac{1}{3}\epsilon\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}^{2m}.
\end{aligned}
$$

In the second equality, above cross terms vanish due to the symmetry of the integral over $\mathbb{S}^2$.
For the second term on the right-hand side of (3.320), Hölder's inequality and (3.319) imply

(3.322)

$$
\begin{aligned}
&\left|-\epsilon\int_0^t\iint_{\Omega\times\mathbb{S}^2}(\vec{w}\cdot\nabla_x\phi)(u-\bar{u})\right|\leq C\epsilon\left\|u-\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\left(\int_0^t\left\|\nabla_x\phi\right\|_{L^{\frac{2m}{2m-1}}(\Omega)}^{\frac{2m}{2m-1}}\right)^{\frac{2m-1}{2m}}\\
\leq&C\epsilon\left\|u-\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\left(\int_0^t\left\|\zeta\right\|_{W^{2,\frac{2m}{2m-1}}(\Omega)}^{\frac{2m}{2m-1}}\right)^{\frac{2m-1}{2m}}\leq C\epsilon\left\|u-\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}^{2m-1}.
\end{aligned}
$$

Based on Sobolev embedding theorem, trace theorem and (3.318), we have

(3.323)

$$
\left\|\nabla_x\xi\right\|_{L^{\frac{4m}{4m-3}}(\Gamma)}\leq C\left\|\nabla_x\xi\right\|_{W^{\frac{1}{2m},\frac{2m}{2m-1}}(\Gamma)}\leq C\left\|\nabla_x\xi\right\|_{W^{1,\frac{2m}{2m-1}}(\Omega)}\leq C\left\|\xi\right\|_{W^{2,\frac{2m}{2m-1}}(\Omega)}\leq C\left\|\bar{u}\right\|_{L^{2m}(\Omega)}^{2m-1}.
$$

Using Hölder's inequality and (3.323), we obtain

$$
\begin{aligned}
\left|\epsilon\int_0^t\int_\Gamma u\phi\mathrm{d}\gamma\right|\leq&\left|\epsilon\int_0^t\int_{\Gamma^+}u\phi\mathrm{d}\gamma\right|+\left|\epsilon\int_0^t\int_{\Gamma^-}g\phi\mathrm{d}\gamma\right|\\
\leq&\epsilon\left\|\phi\right\|_{L^{\frac{4m}{4m-3}}(\Gamma)}\bigg(\left\|u\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^+)}+\left\|g\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^-)}\bigg)\\
\leq&\epsilon\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}^{2m-1}\bigg(\left\|u\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^+)}+\left\|g\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^-)}\bigg).
\end{aligned}\tag{3.324}
$$

Also, using Hölder's inequality and (3.318), we have

(3.325)

$$
\int_0^t\iint_{\Omega\times\mathbb{S}^2}(u-\bar{u})\phi\leq\left\|\phi\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\leq C\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}^{2m-1}\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)},
$$

and

$$
\int_0^t\iint_{\Omega\times\mathbb{S}^2}S\phi\leq C\leq\left\|\phi\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\leq C\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}^{2m-1}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}.\tag{3.326}
$$

Then the only remaining term is

$$
\begin{aligned}
\left|\epsilon^2\int_0^t\iint_{\Omega\times\mathbb{S}^2}\phi\partial_tu\right|=&\left|\epsilon^2\int_0^t\iint_{\Omega\times\mathbb{S}^2}\phi\partial_t(u-\bar{u})\right|\leq\epsilon^2\left\|\phi\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\left\|\partial_t(u-\bar{u})\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq&\epsilon^2\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}^{2m-1}\left\|\partial_t(u-\bar{u})\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}.
\end{aligned}\tag{3.327}
$$

Now we have to tackle $\left\|\partial_tu\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}$.

Taking ∂_t on both sides of the equation (3.232), we obtain

$$\left\{\begin{array}{l}\epsilon^2\partial_t(\partial_t u)+\epsilon\vec w\cdot\nabla_x(\partial_t u)+\partial_t u-\partial_t\bar u=\partial_t S\quad\text{for}\quad (t,\vec x,\vec w)\in\mathbb{R}^+\times\Omega\times\mathbb{S}^2,\\ \partial_t u(0,\vec x,\vec w)=\tilde h=\dfrac{1}{\epsilon^2}(-\epsilon\vec w\cdot\nabla_x h-h+\bar h+S)(0,\vec x,\vec w)\quad\text{for}\quad (\vec x,\vec w)\in\Omega\times\mathbb{S}^2\\ \partial_t u(t,\vec x_0,\vec w)=\partial_t g(t,\vec x_0,\vec w)\quad\text{for}\quad t\in\mathbb{R}^+,\quad \vec x_0\in\partial\Omega\quad\text{and}\quad \vec w\cdot\vec\nu<0,\end{array}\right.\tag{3.328}$$

Based on the proof of Theorem 3.2, we have

(3.329)

$$\begin{aligned}&\epsilon\left\|\partial_t u(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon^{\frac12}\left\|\partial_t u\right\|_{L^2([0,t]\times\Gamma^+)}+\left\|\partial_t(u-\bar u)\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\ \leq&C\bigg(\frac{1}{\epsilon}\left\|\partial_t S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon\left\|\tilde h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon^{\frac12}\left\|\partial_t g\right\|_{L^2([0,t]\times\Gamma^-)}\bigg)\\ \leq&C\bigg(\frac{1}{\epsilon}\left\|\partial_t S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon}\left\|S(0)\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\left\|\vec w\cdot\nabla_x h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon}\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon^{\frac12}\left\|\partial_t g\right\|_{L^2([0,t]\times\Gamma^-)}\bigg).\end{aligned}$$

Inserting (3.329) into (3.327), we obtain

$$\begin{aligned}\left|\epsilon^2\int_0^t\iint_{\Omega\times\mathbb{S}^2}\phi\partial_t u\right|\leq&C\left\|\bar u\right\|^{2m-1}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\bigg(\epsilon\left\|\partial_t S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon\left\|S(0)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\ &+\epsilon^2\left\|\vec w\cdot\nabla_x h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon^{\frac52}\left\|\partial_t g\right\|_{L^2([0,t]\times\Gamma^-)}\bigg).\end{aligned}\tag{3.330}$$

Collecting all terms in (3.321), (3.322), (3.324), (3.325), (3.326) and (3.330), we obtain

(3.331)

$$\begin{aligned}&\epsilon\left\|\bar u\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\ \leq&C\bigg(\epsilon\left\|u-\bar u\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\left\|u-\bar u\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon\left\|u\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^+)}+\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon\left\|\partial_t S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\ &+\epsilon\left\|S(0)\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon^2\left\|\vec w\cdot\nabla_x h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon\left\|g\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^-)}+\epsilon^{\frac52}\left\|\partial_t g\right\|_{L^2([0,t]\times\Gamma^-)}\bigg).\end{aligned}$$

Step 2: L^{2m} Energy Estimate.
Similar to the L^2 estimates, in the weak formulation (3.242), we may take the test function $\phi=u^{2m-1}$ to get the energy estimate

$$\begin{aligned}&\frac{\epsilon^2}{2m}\left\|u(t)\right\|^{2m}_{L^{2m}(\Omega\times\mathbb{S}^2)}+\frac{\epsilon}{2m}\left\|u\right\|^{2m}_{L^{2m}([0,t]\times\Gamma^+)}+\int_0^t\iint_{\Omega\times\mathbb{S}^2}(u-\bar u)u^{2m-1}\\ =&\int_0^t\iint_{\Omega\times\mathbb{S}^2}Su^{2m-1}+\frac{\epsilon^2}{2m}\left\|h\right\|^{2m}_{L^{2m}(\Omega\times\mathbb{S}^2)}+\frac{\epsilon}{2m}\left\|g\right\|^{2m}_{L^{2m}([0,t]\times\Gamma^-)}.\end{aligned}\tag{3.332}$$

Here, direct computation reveals that for $1\leq m\leq 3$,

$$\int_0^t\iint_{\Omega\times\mathbb{S}^2}(u-\bar u)u^{2m-1}\geq C\left\|u-\bar u\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}.\tag{3.333}$$

Hence, inserting (3.333) into (3.332), we obtain

$$\begin{aligned}&\epsilon^2\left\|u(t)\right\|^{2m}_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon\left\|u\right\|^{2m}_{L^{2m}([0,t]\times\Gamma^+)}+\left\|u-\bar u\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\ \leq&C\bigg(\int_0^t\iint_{\Omega\times\mathbb{S}^2}Su^{2m-1}+\epsilon^2\left\|h\right\|^{2m}_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon\left\|g\right\|^{2m}_{L^{2m}([0,t]\times\Gamma^-)}\bigg).\end{aligned}\tag{3.334}$$

Since

$$\left|u\right|^{2m-1}\leq C\bigg(\left|u-\bar u\right|^{2m-1}+\left|\bar u\right|^{2m-1}\bigg),\tag{3.335}$$

we have

$$\left|\int_0^t\iint_{\Omega\times\mathbb{S}^2}Su^{2m-1}\right|\leq C\bigg(\int_0^t\iint_{\Omega\times\mathbb{S}^2}\left|S\right|\left|u-\bar u\right|^{2m-1}+\int_0^t\iint_{\Omega\times\mathbb{S}^2}\left|S\right|\left|\bar u\right|^{2m-1}\bigg). \tag{3.336}$$

Using Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}\int_0^t\iint_{\Omega\times\mathbb{S}^2}\left|S\right|\left|u-\bar u\right|^{2m-1}\leq&\left\|\left|u-\bar u\right|^{2m-1}\right\|^{\frac{2m-1}{2m}}_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}\left\|S\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\ =&\left\|u-\bar u\right\|^{2m-1}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\left\|S\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\ \leq& C\bigg(\Big(o(1)\left\|u-\bar u\right\|^{2m-1}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\Big)^{\frac{2m}{2m-1}}+\Big(\left\|S\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\Big)^{2m}\bigg)\\ \leq& o(1)\left\|u-\bar u\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+C\left\|S\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)},\end{aligned} \tag{3.337}$$

and

$$\begin{aligned}\int_0^t\iint_{\Omega\times\mathbb{S}^2}\left|S\right|\left|\bar u\right|^{2m-1}\leq&\left\|\left|\bar u\right|^{2m-1}\right\|^{\frac{2m-1}{2m}}_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}\left\|S\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\ =&\left\|\bar u\right\|^{2m-1}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\left\|S\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\ =&\Big(\epsilon^{\frac{2m-1}{2m}}\left\|\bar u\right\|^{2m-1}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\Big)\Big(\epsilon^{-\frac{2m-1}{2m}}\left\|S\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\Big)\\ \leq& C\bigg(\Big(o(1)\epsilon^{\frac{2m-1}{2m}}\left\|\bar u\right\|^{2m-1}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\Big)^{\frac{2m}{2m-1}}+\Big(\epsilon^{-\frac{2m-1}{2m}}\left\|S\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\Big)^{2m}\bigg)\\ \leq& o(1)\epsilon\left\|\bar u\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{C}{\epsilon^{2m-1}}\left\|S\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}.\end{aligned} \tag{3.338}$$

Inserting (3.337) and (3.338) into (3.336), we obtain

$$\left|\int_0^t\iint_{\Omega\times\mathbb{S}^2}Su^{2m-1}\right|\leq o(1)\left\|u-\bar u\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+o(1)\epsilon\left\|\bar u\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{C}{\epsilon^{2m-1}}\left\|S\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}. \tag{3.339}$$

Inserting (3.339) into (3.334) to absorb $o(1)\left\|u-\bar u\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}$ into the left-hand side, we have

$$\begin{aligned}&\epsilon^2\left\|u(t)\right\|^{2m}_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon\left\|u\right\|^{2m}_{L^{2m}([0,t]\times\Gamma^+)}+\left\|u-\bar u\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\ \leq& C\bigg(o(1)\epsilon\left\|\bar u\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{2m-1}}\left\|S\right\|^{2m}_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon^2\left\|h\right\|^{2m}_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon\left\|g\right\|^{2m}_{L^{2m}([0,t]\times\Gamma^-)}\bigg).\end{aligned} \tag{3.340}$$

Taking $(2m)^{th}$ root, we actually have

$$\begin{aligned}&\epsilon^{\frac{1}{m}}\left\|u(t)\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{2m}}\left\|u\right\|_{L^{2m}([0,t]\times\Gamma^+)}+\left\|u-\bar u\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\ \leq& C\bigg(o(1)\epsilon^{\frac{1}{2m}}\left\|\bar u\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|S\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{m}}\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{2m}}\left\|g\right\|_{L^{2m}([0,t]\times\Gamma^-)}\bigg).\end{aligned} \tag{3.341}$$

Multiplying (3.331) by $\epsilon^{-\frac{2m-1}{2m}}$, we have

$$\begin{aligned}&\epsilon^{\frac{1}{2m}}\left\|\bar u\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\ \leq& C\bigg(\epsilon^{\frac{1}{2m}}\left\|u-\bar u\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|u-\bar u\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{2m}}\left\|u\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^+)}\\ &+\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{2m}}\left\|\partial_tS\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{2m}}\left\|S(0)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\ &+\epsilon^{\frac{1}{2m}}\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon^{1+\frac{1}{2m}}\left\|\vec w\cdot\nabla_xh\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{2m}}\left\|g\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^-)}+\epsilon^{\frac{3}{2}+\frac{1}{2m}}\left\|\partial_tg\right\|_{L^2([0,t]\times\Gamma^-)}\bigg).\end{aligned} \tag{3.342}$$

Adding (3.342) to (3.341) to absorb $o(1)\epsilon^{\frac{1}{2m}}\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}$ and $\epsilon^{\frac{1}{2m}}\left\|u-\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}$ into the left-hand side, we have

(3.343)

$$\begin{aligned}
&\epsilon^{\frac{1}{m}}\left\|u(t)\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{2m}}\left\|u\right\|_{L^{2m}([0,t]\times\Gamma^+)}+\epsilon^{\frac{1}{2m}}\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\left\|u-\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq&C\bigg(\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{2m}}\left\|u\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^+)}\\
&+\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|S\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{2m}}\left\|\partial_tS\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{2m}}\left\|S(0)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\
&+\epsilon^{\frac{1}{2m}}\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{m}}\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon^{1+\frac{1}{2m}}\left\|\vec{w}\cdot\nabla_xh\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\
&+\epsilon^{\frac{1}{2m}}\left\|g\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^-)}+\epsilon^{\frac{1}{2m}}\left\|g\right\|_{L^{2m}([0,t]\times\Gamma^-)}+\epsilon^{\frac{3}{2}+\frac{1}{2m}}\left\|\partial_tg\right\|_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned}$$

Step 3: L^2 Energy Estimate.
Recall the standard L^2 estimates (3.266), where, in the weak formulation (3.242), we take the test function $\phi=u$ to get

$$\begin{aligned}
&\frac{\epsilon^2}{2}\left\|u(t)\right\|^2_{L^2(\Omega\times\mathbb{S}^2)}+\frac{\epsilon}{2}\left\|u\right\|^2_{L^2([0,t]\times\Gamma^+)}+\left\|u-\bar{u}\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\
=&\int_0^t\iint_{\Omega\times\mathbb{S}^2}Su+\frac{\epsilon^2}{2}\left\|h\right\|^2_{L^2(\Omega\times\mathbb{S}^2)}+\frac{\epsilon}{2}\left\|g\right\|^2_{L^2([0,t]\times\Gamma^-)}.
\end{aligned}\tag{3.344}$$

Naturally, this implies

$$\begin{aligned}
&\epsilon^2\left\|u(t)\right\|^2_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon\left\|u\right\|^2_{L^2([0,t]\times\Gamma^+)}+\left\|u-\bar{u}\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq&\int_0^t\iint_{\Omega\times\mathbb{S}^2}Su+\epsilon^2\left\|h\right\|^2_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon\left\|g\right\|^2_{L^2([0,t]\times\Gamma^-)}.
\end{aligned}\tag{3.345}$$

Multiplying (3.345) by $\epsilon^{-2+\frac{1}{m}}$, we have

$$\begin{aligned}
&\epsilon^{\frac{1}{m}}\left\|u(t)\right\|^2_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1-\frac{1}{m}}}\left\|u\right\|^2_{L^2([0,t]\times\Gamma^+)}+\frac{1}{\epsilon^{2-\frac{1}{m}}}\left\|u-\bar{u}\right\|^2_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq&\frac{1}{\epsilon^{2-\frac{1}{m}}}\int_0^t\iint_{\Omega\times\mathbb{S}^2}Su+\epsilon^{\frac{1}{m}}\left\|h\right\|^2_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1-\frac{1}{m}}}\left\|g\right\|^2_{L^2([0,t]\times\Gamma^-)}.
\end{aligned}\tag{3.346}$$

Note that

$$\left|\int_0^t\iint_{\Omega\times\mathbb{S}^2}Su\right|\leq\left|\int_0^t\iint_{\Omega\times\mathbb{S}^2}S(u-\bar{u})\right|+\left|\int_0^t\iint_{\Omega\times\mathbb{S}^2}S\bar{u}\right|.\tag{3.347}$$

Using Hölder's inequality and Cauchy's inequality, we have

$$\begin{aligned}
\left|\int_0^t\iint_{\Omega\times\mathbb{S}^2}S(u-\bar{u})\right|\leq&\left\|u-\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\left\|S\right\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq&o(1)\epsilon^{2-\frac{1}{m}}\left\|u-\bar{u}\right\|^2_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{C}{\epsilon^{2-\frac{1}{m}}}\left\|S\right\|^2_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)},
\end{aligned}\tag{3.348}$$

and

$$\begin{aligned}
\left|\int_0^t\iint_{\Omega\times\mathbb{S}^2}S\bar{u}\right|\leq&\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\left\|S\right\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq&o(1)\epsilon^2\left\|\bar{u}\right\|^2_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{C}{\epsilon^2}\left\|S\right\|^2_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}.
\end{aligned}\tag{3.349}$$

Inserting (3.348) and (3.349) into (3.347), we obtain

(3.350)

$$\left|\int_0^t\iint_{\Omega\times\mathbb{S}^2}Su\right|\leq o(1)\epsilon^{2-\frac{1}{m}}\left\|u-\bar{u}\right\|^2_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+o(1)\epsilon^2\left\|\bar{u}\right\|^2_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{C}{\epsilon^2}\left\|S\right\|^2_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}.$$

Inserting (3.350) into (3.346), we have

$$
\begin{aligned}
&\epsilon^{\frac{1}{m}}\left\|u(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}^2+\frac{1}{\epsilon^{1-\frac{1}{m}}}\left\|u\right\|_{L^2([0,t]\times\Gamma^+)}^2+\frac{1}{\epsilon^{2-\frac{1}{m}}}\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}^2\\
\leq&o(1)\left\|u-\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}^2+o(1)\epsilon^{\frac{1}{m}}\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}^2+\frac{C}{\epsilon^{4-\frac{1}{m}}}\left\|S\right\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}^2\\
&+\epsilon^{\frac{1}{m}}\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}^2+\frac{1}{\epsilon^{1-\frac{1}{m}}}\left\|g\right\|_{L^2([0,t]\times\Gamma^-)}^2.
\end{aligned}\tag{3.351}
$$

Taking square root in (3.351), we obtain

$$
\begin{aligned}
&\epsilon^{\frac{1}{2m}}\left\|u(t)\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{2}-\frac{1}{2m}}}\left\|u\right\|_{L^2([0,t]\times\Gamma^+)}+\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq&o(1)\left\|u-\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+o(1)\epsilon^{\frac{1}{2m}}\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{C}{\epsilon^{2-\frac{1}{2m}}}\left\|S\right\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}\\
&+\epsilon^{\frac{1}{2m}}\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{2}-\frac{1}{2m}}}\left\|g\right\|_{L^2([0,t]\times\Gamma^-)}.
\end{aligned}\tag{3.352}
$$

Multiplying (3.343) by a small constant and adding it to (3.352) to absorb $\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}$, $o(1)\left\|u-\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}$ and $o(1)\epsilon^{\frac{1}{2m}}\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}$ into the left-hand side, we obtain

$$
\begin{aligned}
&\epsilon^{\frac{1}{m}}\left\|u(t)\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{2}-\frac{1}{2m}}}\left\|u\right\|_{L^2([0,t]\times\Gamma^+)}+\epsilon^{\frac{1}{2m}}\left\|u\right\|_{L^{2m}([0,t]\times\Gamma^+)}\\
&+\epsilon^{\frac{1}{2m}}\left\|\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|u-\bar{u}\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\left\|u-\bar{u}\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq&C\bigg(\epsilon^{\frac{1}{2m}}\left\|u\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^+)}+\frac{1}{\epsilon^{2-\frac{1}{2m}}}\left\|S\right\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|S\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\
&+\epsilon^{\frac{1}{2m}}\left\|\partial_tS\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{2m}}\left\|S(0)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\
&+\epsilon^{\frac{1}{2m}}\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{m}}\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon^{1+\frac{1}{2m}}\left\|\vec{w}\cdot\nabla_xh\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\
&+\epsilon^{\frac{1}{2m}}\left\|g\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^-)}+\epsilon^{\frac{1}{2m}}\left\|g\right\|_{L^{2m}([0,t]\times\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{2}-\frac{1}{2m}}}\left\|g\right\|_{L^2([0,t]\times\Gamma^-)}+\epsilon^{\frac{3}{2}+\frac{1}{2m}}\left\|\partial_tg\right\|_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned}\tag{3.353}
$$

Using the interpolation estimate and Young's inequality, we know

$$
\begin{aligned}
\left\|u\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^+)}\leq&\left\|u\right\|_{L^{2m}([0,t]\times\Gamma^+)}^{\frac{2m-3}{2m-2}}\left\|u\right\|_{L^2([0,t]\times\Gamma^+)}^{\frac{1}{2m-2}}\\
\leq&o(1)\bigg(\left\|u\right\|_{L^{2m}([0,t]\times\Gamma^+)}^{\frac{2m-3}{2m-2}}\bigg)^{\frac{2m-2}{2m-3}}+C\bigg(\left\|u\right\|_{L^2([0,t]\times\Gamma^+)}^{\frac{1}{2m-2}}\bigg)^{2m-2}\\
\leq&o(1)\left\|u\right\|_{L^{2m}([0,t]\times\Gamma^+)}+C\left\|u\right\|_{L^2([0,t]\times\Gamma^+)}^{\frac{1}{2m-2}}.
\end{aligned}\tag{3.354}
$$

Hence, we know

$$
\begin{aligned}
\epsilon^{\frac{1}{2m}}\left\|u\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^+)}\leq&o(1)\epsilon^{\frac{1}{2m}}\left\|u\right\|_{L^{2m}([0,t]\times\Gamma^+)}+C\epsilon^{\frac{1}{2m}}\left\|u\right\|_{L^2([0,t]\times\Gamma^+)}\\
\leq&o(1)\epsilon^{\frac{1}{2m}}\left\|u\right\|_{L^{2m}([0,t]\times\Gamma^+)}+o(1)\frac{1}{\epsilon^{\frac{1}{2}-\frac{1}{2m}}}\left\|u\right\|_{L^2([0,t]\times\Gamma^+)}.
\end{aligned}\tag{3.355}
$$

Inserting (3.355) into (3.353) to absorb $\epsilon^{\frac{1}{2m}}\left\|u\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^+)}$ into the left-hand side, we have

(3.356)

$$\begin{aligned}
&\epsilon^{\frac{1}{m}}\left\|u(t)\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{2}-\frac{1}{2m}}}\left\|u\right\|_{L^2([0,t]\times\Gamma^+)}+\epsilon^{\frac{1}{2m}}\left\|u\right\|_{L^{2m}([0,t]\times\Gamma^+)}\\
&+\epsilon^{\frac{1}{2m}}\left\|\bar u\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|u-\bar u\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\left\|u-\bar u\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq&C\bigg(\frac{1}{\epsilon^{2-\frac{1}{2m}}}\left\|S\right\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1-\frac{1}{2m}}}\left\|S\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\
&+\epsilon^{\frac{1}{2m}}\left\|\partial_tS\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{2m}}\left\|S(0)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\
&+\epsilon^{\frac{1}{2m}}\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\epsilon^{\frac{1}{m}}\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon^{1+\frac{1}{2m}}\left\|\vec w\cdot\nabla_xh\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\
&+\epsilon^{\frac{1}{2m}}\left\|g\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^-)}+\epsilon^{\frac{1}{2m}}\left\|g\right\|_{L^{2m}([0,t]\times\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{2}-\frac{1}{2m}}}\left\|g\right\|_{L^2([0,t]\times\Gamma^-)}+\epsilon^{\frac{3}{2}+\frac{1}{2m}}\left\|\partial_tg\right\|_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned}$$

Hence, we get the desired estimate

(3.357)

$$\begin{aligned}
&\left\|u(t)\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{2m}}}\left\|u\right\|_{L^2([0,t]\times\Gamma^+)}+\frac{1}{\epsilon^{\frac{1}{2m}}}\left\|u\right\|_{L^{2m}([0,t]\times\Gamma^+)}\\
&+\frac{1}{\epsilon^{\frac{1}{2m}}}\left\|\bar u\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{1}{2m}}}\left\|u-\bar u\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{m}}}\left\|u-\bar u\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq&C\bigg(\frac{1}{\epsilon^{2+\frac{1}{2m}}}\left\|S\right\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{1}{2m}}}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{1}{2m}}}\left\|S\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\
&+\frac{1}{\epsilon^{\frac{1}{2m}}}\left\|\partial_tS\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{1}{2m}}}\left\|S(0)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\
&+\frac{1}{\epsilon^{\frac{1}{2m}}}\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon^{1-\frac{1}{2m}}\left\|\vec w\cdot\nabla_xh\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\
&+\frac{1}{\epsilon^{\frac{1}{2m}}}\left\|g\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{2m}}}\left\|g\right\|_{L^{2m}([0,t]\times\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{2m}}}\left\|g\right\|_{L^2([0,t]\times\Gamma^-)}+\epsilon^{\frac{3}{2}-\frac{1}{2m}}\left\|\partial_tg\right\|_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned}$$

$\square$

3.2.4. *L^∞ Estimate - Second Round.*

Theorem 3.5. *Assume $S(t,\vec x,\vec w)\in L^\infty(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)$, $h(\vec x,\vec w)\in L^\infty(\Omega\times\mathbb{S}^2)$ and $g(t,x_0,\vec w)\in L^\infty(\mathbb{R}^+\times\Gamma^-)$. Then the solution $u(t,\vec x,\vec w)$ to the neutron transport equation (3.232) satisfies*

(3.358)

$$\begin{aligned}
&\left\|u\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq&C\bigg(\frac{1}{\epsilon^{2+\frac{2}{m}}}\left\|S\right\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{2}{m}}}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{2}{m}}}\left\|S\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\left\|S\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\\
&+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|\partial_tS\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|S(0)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\
&+\epsilon^{1-\frac{2}{m}}\left\|\vec w\cdot\nabla_xh\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{3}{2m}}}\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\left\|h\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}\\
&+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|g\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^-)}+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|g\right\|_{L^{2m}([0,t]\times\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{2}+\frac{2}{m}}}\left\|g\right\|_{L^2([0,t]\times\Gamma^-)}+\left\|g\right\|_{L^\infty([0,t]\times\Gamma^-)}+\epsilon^{\frac{3}{2}-\frac{2}{m}}\left\|\partial_tg\right\|_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned}$$

Proof. Based on the analysis in proving Theorem (3.3), the key step is the estimate of I_4^*. We utilize the same substitution in (3.301) and apply Hölder's inequality with a different exponent to obtain

$$
\begin{aligned}
|I_4^*| \leq & C\int_0^{t_b}\left(\int_{r\geq\delta}\int_{q\geq\delta}\left(\int_\Omega\frac{1}{\epsilon^3\delta^5}\left|\bar{u}(t''-\epsilon^2q,\vec{y})\right|^{2m}\mathrm{d}\vec{y}\right)\mathrm{e}^{-q}\mathrm{e}^{-r}\mathrm{d}q\mathrm{d}r\right)^{\frac{1}{2m}}\mathrm{e}^{-s}\mathrm{d}s \\
\leq & \frac{C}{\epsilon^{\frac{3}{2m}}\delta^{\frac{5}{2m}}}\left\|\bar{u}(t)\right\|_{L^{2m}([0,t]\times\Omega)}.
\end{aligned} \tag{3.359}
$$

Summarizing (3.282), (3.283), (3.284), (3.285), (3.286), (3.287), (3.288), (3.289), (3.290), (3.292), (3.294), (3.305), (3.359), we have shown that for any $(t,\vec{x},\vec{w})\in[0,t]\times\bar{\Omega}\times\mathbb{S}^2$,

$$
\begin{aligned}
|u(t,\vec{x},\vec{w})| \leq & C\bigg(\delta\left\|u\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{3}{2m}}\delta^{\frac{5}{2m}}}\left\|\bar{u}(t)\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)} \\
& +\left\|S\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}+\left\|h\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}+\left\|g\right\|_{L^\infty([0,t]\times\Gamma^-)}\bigg).
\end{aligned} \tag{3.360}
$$

Let δ be sufficiently small such that $C\delta\leq\dfrac{1}{2}$. Taking supremum over $(t,\vec{x},\vec{w})\in[0,t]\times\Omega\times\mathbb{S}^2$ in (3.360) and using Theorem 3.4, we have

(3.361)

$$
\begin{aligned}
&\left\|u\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)} \\
\leq & C\bigg(\frac{1}{\epsilon^{2+\frac{2}{m}}}\left\|S\right\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{2}{m}}}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{2}{m}}}\left\|S\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}+\left\|S\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)} \\
& +\frac{1}{\epsilon^{\frac{2}{m}}}\left\|\partial_tS\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|S(0)\right\|_{L^2(\Omega\times\mathbb{S}^2)} \\
& +\epsilon^{1-\frac{2}{m}}\left\|\vec{w}\cdot\nabla_xh\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|h\right\|_{L^2(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{3}{2m}}}\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\left\|h\right\|_{L^\infty(\Omega\times\mathbb{S}^2)} \\
& +\frac{1}{\epsilon^{\frac{2}{m}}}\left\|g\right\|_{L^{\frac{4m}{3}}([0,t]\times\Gamma^-)}+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|g\right\|_{L^{2m}([0,t]\times\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{2}+\frac{2}{m}}}\left\|g\right\|_{L^2([0,t]\times\Gamma^-)}+\left\|g\right\|_{L^\infty([0,t]\times\Gamma^-)}+\epsilon^{\frac{3}{2}-\frac{2}{m}}\left\|\partial_tg\right\|_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned}
$$

□

3.2.5. *Superposition Argument.* In Theorem 3.5, the estimates depend on the derivative bounds of initial and boundary data. Here we can use superposition property to get rid of such dependence and get a cleaner form.

Theorem 3.6. *Assume $S(t,\vec{x},\vec{w})\in L^\infty(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)$, $h(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathbb{S}^2)$ and $g(t,x_0,\vec{w})\in L^\infty(\mathbb{R}^+\times\Gamma^-)$. Then the solution $u(t,\vec{x},\vec{w})$ to the neutron transport equation* (3.232) *satisfies*

$$
\begin{aligned}
&\left\|u\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)} \\
\leq & C\bigg(\frac{1}{\epsilon^{2+\frac{2}{m}}}\left\|S\right\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{2}{m}}}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{2}{m}}}\left\|S\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)} \\
& +\left\|S\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|\partial_tS\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|S(0)\right\|_{L^2(\Omega\times\mathbb{S}^2)} \\
& +\frac{1}{\epsilon^{\frac{3}{2m}}}\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\left\|h\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|g\right\|_{L^{2m}([0,t]\times\Gamma^-)}+\left\|g\right\|_{L^\infty([0,t]\times\Gamma^-)}\bigg).
\end{aligned} \tag{3.362}
$$

Proof. Since equation (3.232) is linear, we may decompose the solution $u=u_1+u_2$ satisfying

$$
\left\{
\begin{array}{l}
\epsilon^2\partial_tu_1+\epsilon\vec{w}\cdot\nabla_xu_1+u_1-\bar{u}_1=0\quad\text{for}\ \ (t,\vec{x},\vec{w})\in\mathbb{R}^+\times\Omega\times\mathbb{S}^2,\\
u_1(0,\vec{x},\vec{w})=h(\vec{x},\vec{w})\ \ \text{for}\ \ (\vec{x},\vec{w})\in\Omega\times\mathbb{S}^2\\
u_1(t,\vec{x}_0,\vec{w})=g(t,\vec{x}_0,\vec{w})\ \ \text{for}\ \ t\in\mathbb{R}^+,\ \ \vec{x}_0\in\partial\Omega\ \ \text{and}\ \ \vec{w}\cdot\vec{\nu}<0,
\end{array}
\right. \tag{3.363}
$$

and

$$\left\{\begin{array}{l}\epsilon^2\partial_t u_2+\epsilon\vec w\cdot\nabla_x u_2+u_2-\bar u_2=S\quad\text{for}\quad(t,\vec x,\vec w)\in\mathbb{R}^+\times\Omega\times\mathbb{S}^2,\\[1ex] u_2(0,\vec x,\vec w)=0\ \ \text{for}\ \ (\vec x,\vec w)\in\Omega\times\mathbb{S}^2\\[1ex] u_2(t,\vec x_0,\vec w)=0\ \ \text{for}\ \ t\in\mathbb{R}^+,\ \ \vec x_0\in\partial\Omega\ \ \text{and}\ \ \vec w\cdot\vec\nu<0.\end{array}\right.\tag{3.364}$$

Step 1: Estimate of u_1.
In Step 2 of proof of Theorem 3.4, we have shown

$$\begin{aligned}&\epsilon^2\left\|u_1(t)\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}^{2m}+\epsilon\left\|u_1\right\|_{L^{2m}([0,t]\times\Gamma^+)}^{2m}+\left\|u_1-\bar u_1\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}^{2m}\\ \leq&C\bigg(\epsilon^2\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}^{2m}+\epsilon\left\|g\right\|_{L^{2m}([0,t]\times\Gamma^-)}^{2m}\bigg).\end{aligned}\tag{3.365}$$

Take $(2m)^{th}$ root on both sides, we have

$$\begin{aligned}&\epsilon^{\frac1m}\left\|u_1(t)\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon^{\frac1{2m}}\left\|u_1\right\|_{L^{2m}([0,t]\times\Gamma^+)}+\left\|u_1-\bar u_1\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\ \leq&C\bigg(\epsilon^{\frac1m}\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\epsilon^{\frac1{2m}}\left\|g\right\|_{L^{2m}([0,t]\times\Gamma^-)}\bigg).\end{aligned}\tag{3.366}$$

Dividing $\epsilon^{\frac1m}$ on both sides, we arrive at

$$\begin{aligned}&\left\|u_1(t)\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac1{2m}}}\left\|u_1\right\|_{L^{2m}([0,t]\times\Gamma^+)}+\frac{1}{\epsilon^{\frac1m}}\left\|u_1-\bar u_1\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\ \leq&C\bigg(\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac1{2m}}}\left\|g\right\|_{L^{2m}([0,t]\times\Gamma^-)}\bigg).\end{aligned}\tag{3.367}$$

Then using the argument in the proof of Theorem 3.5, we have

$$\begin{aligned}\left\|u_1\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\leq&C\bigg(\frac{1}{\epsilon^{\frac3{2m}}}\left\|\bar u_1(t)\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\left\|h\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}+\left\|g\right\|_{L^\infty([0,t]\times\Gamma^-)}\bigg)\\ \leq&C\bigg(\frac{1}{\epsilon^{\frac3{2m}}}\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\left\|h\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac2m}}\left\|g\right\|_{L^{2m}([0,t]\times\Gamma^-)}+\left\|g\right\|_{L^\infty([0,t]\times\Gamma^-)}\bigg).\end{aligned}\tag{3.368}$$

Step 2: Estimate of u_2.
Here we refer to Theorem 3.5. Since initial and boundary data are all zero, the estimate is much simpler

$$\begin{aligned}&\left\|u_2\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\\ \leq&C\bigg(\frac{1}{\epsilon^{2+\frac2m}}\left\|S\right\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac2m}}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac2m}}\left\|S\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\ &+\left\|S\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac2m}}\left\|\partial_tS\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac2m}}\left\|S(0)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\bigg).\end{aligned}\tag{3.369}$$

Step 3: Synthesis.
Combining (3.368) and (3.369), we obtain

$$\begin{aligned}&\left\|u\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\leq\left\|u_1\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}+\left\|u_2\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}\\ \leq&C\bigg(\frac{1}{\epsilon^{2+\frac2m}}\left\|S\right\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac2m}}\left\|S\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac2m}}\left\|S\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\ &+\left\|S\right\|_{L^\infty([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac2m}}\left\|\partial_tS\right\|_{L^2([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac2m}}\left\|S(0)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\\ &+\frac{1}{\epsilon^{\frac3{2m}}}\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\left\|h\right\|_{L^\infty(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac2m}}\left\|g\right\|_{L^{2m}([0,t]\times\Gamma^-)}+\left\|g\right\|_{L^\infty([0,t]\times\Gamma^-)}\bigg).\end{aligned}\tag{3.370}$$

$\square$

Theorem 3.7. *Assume* $\mathrm{e}^{Kt}S(t,\vec{x},\vec{w})\in L^{\infty}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)$*,* $h(\vec{x},\vec{w})\in L^{\infty}(\Omega\times\mathbb{S}^2)$ *and* $\mathrm{e}^{Kt}g(t,x_0,\vec{w})\in L^{\infty}(\mathbb{R}^+\times\Gamma^-)$ *for some* $K>0$*. Then there exists* $0<K_0\leq K$ *such that the solution* $u(t,\vec{x},\vec{w})$ *to the neutron transport equation* (3.232) *satisfies*

$$
\begin{aligned}
&\left\|\mathrm{e}^{K_0t}u\right\|_{L^{\infty}([0,t]\times\Omega\times\mathbb{S}^2)}\\
\leq& C\bigg(\frac{1}{\epsilon^{2+\frac{2}{m}}}\left\|\mathrm{e}^{K_0t}S\right\|_{L^{\frac{2m}{2m-1}}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{2}{m}}}\left\|\mathrm{e}^{K_0t}S\right\|_{L^{2}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{2}{m}}}\left\|\mathrm{e}^{K_0t}S\right\|_{L^{2m}([0,t]\times\Omega\times\mathbb{S}^2)}\\
&+\left\|\mathrm{e}^{K_0t}S\right\|_{L^{\infty}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|\mathrm{e}^{K_0t}\partial_tS\right\|_{L^{2}([0,t]\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|S(0)\right\|_{L^{2}(\Omega\times\mathbb{S}^2)}\\
&+\frac{1}{\epsilon^{\frac{3}{2m}}}\left\|h\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\left\|h\right\|_{L^{\infty}(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|\mathrm{e}^{K_0t}g\right\|_{L^{2m}([0,t]\times\Gamma^-)}+\left\|\mathrm{e}^{K_0t}g\right\|_{L^{\infty}([0,t]\times\Gamma^-)}\bigg).
\end{aligned}
\tag{3.371}
$$

Proof. Let $v=\mathrm{e}^{K_0t}u$. Then v satisfies the equation

$$
\left\{
\begin{array}{l}
\epsilon^2\partial_tv+\epsilon\vec{w}\cdot\nabla_xv+v-\bar{v}=\mathrm{e}^{K_0t}S+K_0\epsilon^2v\ \text{ for }\ (t,\vec{x},\vec{w})\in\mathbb{R}^+\times\Omega\times\mathbb{S}^2,\\[1ex]
v(0,\vec{x},\vec{w})=h(\vec{x},\vec{w})\ \text{ for }\ (\vec{x},\vec{w})\in\Omega\times\mathbb{S}^2,\\[1ex]
v(t,\vec{x}_0,\vec{w})=\mathrm{e}^{K_0t}g(t,\vec{x}_0,\vec{w})\ \text{ for }\ t\in\mathbb{R}^+\ \ \vec{x}_0\in\partial\Omega\ \text{ and }\ \vec{w}\cdot\vec{\nu}<0.
\end{array}
\right.
\tag{3.372}
$$

Note that we have an extra term $K_0\epsilon^2v$. However, ϵ^2 helps to recover all the estimates in previous theorems and we can obtain exactly the same results. $\square$

3.3. **Asymptotic Analysis.**

3.3.1. *Analysis of Regular Boundary Layer.* Based on a similar analysis as in steady problems, we have the following:

Theorem 3.8. *For* $K_0>0$ *sufficiently small, the regular boundary layer satisfies*

$$
\begin{aligned}
&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\mathscr{U}_0(t)\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C, &&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\mathscr{U}_1(t)\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8,\\
&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_0}{\partial t}(t)\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8, &&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_1}{\partial t}(t)\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^{16},\\
&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_0}{\partial \iota_i}(t)\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8, &&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_1}{\partial \iota_i}(t)\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^{16},\\
&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_0}{\partial \psi}(t)\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8, &&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_1}{\partial \psi}(t)\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^{16}.
\end{aligned}
\tag{3.373}
$$

3.3.2. *Analysis of Singular Boundary Layer.* Based on a similar analysis as in steady problems, we have the following:

Theorem 3.9. *Let*

$$
\left\{
\begin{array}{ll}
\chi_1: & 0\leq\zeta<2\epsilon^{\alpha},\\
\chi_2: & 2\epsilon^{\alpha}\leq\zeta\leq 1.
\end{array}
\right.
\tag{3.374}
$$

For $K_0>0$ *sufficiently small, the singular boundary layer satisfies*

$$
\begin{aligned}
&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}(\chi_1\mathfrak{U}_0)(t)\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C, &&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}(\chi_2\mathfrak{U}_0)(t)\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\epsilon^{\alpha},\\
&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\frac{\partial(\chi_1\mathfrak{U}_0)(t)}{\partial t}\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8, &&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\frac{\partial(\chi_2\mathfrak{U}_0)(t)}{\partial t}\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\epsilon^{\alpha}\left|\ln(\epsilon)\right|^8,\\
&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\frac{\partial(\chi_1\mathfrak{U}_0)(t)}{\partial \iota_i}\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8, &&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\frac{\partial(\chi_2\mathfrak{U}_0)(t)}{\partial \iota_i}\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\epsilon^{\alpha}\left|\ln(\epsilon)\right|^8,\\
&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\frac{\partial(\chi_1\mathfrak{U}_0)(t)}{\partial \psi}\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\left|\ln(\epsilon)\right|^8, &&\left\|\mathrm{e}^{K_0t}\mathrm{e}^{K_0\eta}\frac{\partial(\chi_2\mathfrak{U}_0)}{\partial \psi}(t)\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C\epsilon^{\alpha}\left|\ln(\epsilon)\right|^8.
\end{aligned}
\tag{3.375}
$$

3.3.3. *Analysis of Initial Layer.* We divide the analysis into several steps:

Step 1: Analysis of $\mathscr{U}_0^I$.
Using (3.226), $\mathscr{U}_0^I$ satisfies

$$\mathscr{U}_0^I(\tau,\vec{x},\vec{w})=\mathrm{e}^{-\tau}(\mathscr{U}_0^I-\bar{\mathscr{U}}_0^I)=\mathrm{e}^{-\tau}(h-\bar{h}). \tag{3.376}$$

Naturally, it decays exponentially in τ.

Step 1: Analysis of $\mathscr{U}_1^I$.
Using (3.229), $\mathscr{U}_1^I$ satisfies

$$\partial_\tau\bar{\mathscr{U}}_1^I=-\int_{\mathbb{S}^2}\Big(\vec{w}\cdot\nabla_x\mathscr{U}_0^I\Big)\mathrm{d}\vec{w}=-\mathrm{e}^{-\tau}\int_{\mathbb{S}^2}\Big(\vec{w}\cdot\nabla_x(h-\bar{h})\Big)\mathrm{d}\vec{w}, \tag{3.377}$$

which, combined with $\bar{\mathscr{U}}_1^I(\infty,\vec{x})=0$, implies that

$$\bar{\mathscr{U}}_1^I(\tau,\vec{x})=\mathrm{e}^{-\tau}\int_{\mathbb{S}^2}\Big(\vec{w}\cdot\nabla_x(h-\bar{h})\Big)\mathrm{d}\vec{w}. \tag{3.378}$$

Then using (3.229), we have

$$\mathscr{U}_1^I(\tau,\vec{x},\vec{w})=\mathrm{e}^{-\tau}\bigg(\vec{w}\cdot\nabla_xU_0(0)+\int_{\mathbb{S}^2}\Big(\vec{w}\cdot\nabla_x(h-\bar{h})\Big)\mathrm{d}\vec{w}-\vec{w}\cdot\nabla_x(h-\bar{h})\bigg). \tag{3.379}$$

Naturally, it decays exponentially in τ.

Theorem 3.10. *For $K_0>0$ sufficiently small, the initial layer satisfies*

$$\begin{aligned}
&\left\|\mathrm{e}^{K_0\tau}\mathscr{U}_0^I(\vec{x})\right\|_{L^\infty(\mathbb{R}^+\times\mathbb{S}^2)}\leq C, &&\left\|\mathrm{e}^{K_0\tau}\mathscr{U}_1^I(\vec{x})\right\|_{L^\infty(\mathbb{R}^+\times\mathbb{S}^2)}\leq C,\\
&\left\|\mathrm{e}^{K_0\tau}\nabla_x\mathscr{U}_0^I(\vec{x})\right\|_{L^\infty(\mathbb{R}^+\times\mathbb{S}^2)}\leq C, &&\left\|\mathrm{e}^{K_0\tau}\nabla_x\mathscr{U}_1^I(\vec{x})\right\|_{L^\infty(\mathbb{R}^+\times\mathbb{S}^2)}\leq C.
\end{aligned} \tag{3.380}$$

3.3.4. *Analysis of Interior Solution.* In this subsection, we will justify that the interior solutions are all well-defined. We divide it into several steps:

Step 1: Well-Posedness of U_0.
U_0 satisfies a parabolic equation

$$\begin{cases}
U_0(t,\vec{x},\vec{w})=\bar{U}_0(t,\vec{x}),\\
\partial_t\bar{U}_0-\dfrac{1}{3}\Delta_x\bar{U}_0=0\ \text{ in }\ \Omega,\\
\bar{U}_0(0,\vec{x})=\mathfrak{f}_0(\infty,\vec{x})\ \text{ in }\ \Omega,\\
\bar{U}_0(t,\vec{x}_0)=\mathscr{F}_{0,L}(t,\iota_1,\iota_2)+\mathfrak{F}_{0,L}(t,\iota_1,\iota_2)\ \text{ on }\ \partial\Omega.
\end{cases} \tag{3.381}$$

Based on standard parabolic theory, we have

$$\left\|\mathrm{e}^{K_0t}U_0\right\|_{L^\infty_tH^3_xL^\infty_w([0,\infty)\times\Omega\times\mathbb{S}^2)}\leq C. \tag{3.382}$$

Step 2: Well-Posedness of U_1.
U_1 satisfies a parabolic equation

$$\begin{cases}
U_1(\vec{x},\vec{w})=\bar{U}_1(\vec{x})-\vec{w}\cdot\nabla_xU_0(\vec{x},\vec{w}),\\
\partial_t\bar{U}_1-\dfrac{1}{3}\Delta_x\bar{U}_1=-\displaystyle\int_{\mathbb{S}^1}\Big(\vec{w}\cdot\nabla_xU_0(\vec{x},\vec{w})\Big)\mathrm{d}\vec{w}\ \text{ in }\ \Omega,\\
\bar{U}_1(0,\vec{x})=\mathfrak{f}_1(\infty,\vec{x})\ \text{ in }\ \Omega,\\
\bar{U}_1(t,\vec{x}_0)=\mathcal{F}_{1,L}(t,\iota_1,\iota_2)\ \text{ on }\ \partial\Omega.
\end{cases} \tag{3.383}$$

Based on standard parabolic theory, we have

$$\left\|\mathrm{e}^{K_0t}U_1\right\|_{L^\infty_tH^3_xL^\infty_w([0,\infty)\times\Omega\times\mathbb{S}^2)}\leq C\left|\ln(\epsilon)\right|^8. \tag{3.384}$$

Step 3: Well-Posedness of U_2.
U_2 satisfies a parabolic equation

$$\left\{\begin{array}{l} U_2(\vec{x},\vec{w})=\bar{U}_2(\vec{x})-\vec{w}\cdot\nabla_xU_1(\vec{x},\vec{w}),\\ \partial_t\bar{U}_2-\dfrac{1}{3}\Delta_x\bar{U}_2=-\displaystyle\int_{\mathbb{S}^1}\Big(\vec{w}\cdot\nabla_xU_1(\vec{x},\vec{w})\Big)\mathrm{d}\vec{w}\ \ \text{in}\ \ \Omega,\\ \bar{U}_2(0,\vec{x})=0\ \ \text{in}\ \ \Omega,\\ \bar{U}_2(t,\vec{x}_0)=0\ \ \text{on}\ \ \partial\Omega. \end{array}\right. \tag{3.385}$$

Based on standard parabolic theory, we have

$$\left\|\mathrm{e}^{K_0t}U_2\right\|_{L^\infty_tH^3_xL^\infty_w([0,\infty)\times\Omega\times\mathbb{S}^2)}\leq C\left|\ln(\epsilon)\right|^8. \tag{3.386}$$

Theorem 3.11. *The interior solution satisfies*

$$\left\|\mathrm{e}^{K_0t}U_0\right\|_{L^\infty_tH^3_xL^\infty_w([0,\infty)\times\Omega\times\mathbb{S}^2)}+\left\|\mathrm{e}^{K_0t}\partial_tU_0\right\|_{L^\infty_tH^3_xL^\infty_w([0,\infty)\times\Omega\times\mathbb{S}^2)}+\left\|\mathrm{e}^{K_0t}\nabla_xU_0\right\|_{L^\infty_tH^3_xL^\infty_w([0,\infty)\times\Omega\times\mathbb{S}^2)}\leq C, \tag{3.387}$$

$$\left\|\mathrm{e}^{K_0t}U_1\right\|_{L^\infty_tH^3_xL^\infty_w([0,\infty)\times\Omega\times\mathbb{S}^2)}+\left\|\mathrm{e}^{K_0t}\partial_tU_1\right\|_{L^\infty_tH^3_xL^\infty_w([0,\infty)\times\Omega\times\mathbb{S}^2)}+\left\|\mathrm{e}^{K_0t}\nabla_xU_1\right\|_{L^\infty_tH^3_xL^\infty_w([0,\infty)\times\Omega\times\mathbb{S}^2)}\leq C\left|\ln(\epsilon)\right|^8, \tag{3.388}$$

$$\left\|\mathrm{e}^{K_0t}U_2\right\|_{L^\infty_tH^3_xL^\infty_w([0,\infty)\times\Omega\times\mathbb{S}^2)}+\left\|\mathrm{e}^{K_0t}\partial_tU_2\right\|_{L^\infty_tH^3_xL^\infty_w([0,\infty)\times\Omega\times\mathbb{S}^2)}+\left\|\mathrm{e}^{K_0t}\nabla_xU_2\right\|_{L^\infty_tH^3_xL^\infty_w([0,\infty)\times\Omega\times\mathbb{S}^2)}\leq C\left|\ln(\epsilon)\right|^8. \tag{3.389}$$

3.3.5. *Analysis of Initial-Boundary Layer.* The initial and boundary data satisfy the compatibility condition

$$h(\vec{x}_0,\vec{w})=g(0,\vec{x}_0,\vec{w})\ \ \text{for}\ \ \vec{x}_0\in\partial\Omega\ \ \text{and}\ \ \vec{w}\cdot\vec{\nu}<0. \tag{3.390}$$

Then in the half-space $\vec{w}\cdot\vec{\nu}<0$ at $(0,\vec{x}_0)$, the equation

$$\epsilon^2\partial_tu^\epsilon+\epsilon\vec{w}\cdot\nabla_xu^\epsilon+u^\epsilon-\bar{u}^\epsilon=0, \tag{3.391}$$

is valid, which implies that for arbitrary ϵ,

$$\epsilon^2\partial_tg(0,\vec{x}_0,\vec{w})+\epsilon\vec{w}\cdot\nabla_xh(\vec{x}_0,\vec{w})+h(\vec{x}_0,\vec{w})-\bar{h}(\vec{x}_0)=0. \tag{3.392}$$

Since g and h are both independent of ϵ, we must have that for $\vec{w}\cdot\vec{\nu}<0$,

$$\partial_tg(0,\vec{x}_0,\vec{w})=\vec{w}\cdot\nabla_xh(\vec{x}_0,\vec{w})=h(\vec{x}_0,\vec{w})-\bar{h}(\vec{x}_0)=0. \tag{3.393}$$

Then we obtain the improved compatibility condition

$$\left\{\begin{array}{l} h(\vec{x}_0,\vec{w})=g(0,\vec{x}_0,\vec{w})=C_0,\\ \partial_tg(0,\vec{x}_0,\vec{w})=\vec{w}\cdot\nabla_xh(\vec{x}_0,\vec{w})=h(\vec{x}_0,\vec{w})-\bar{h}(\vec{x}_0)=0, \end{array}\right.\quad\text{for}\ \ \vec{x}_0\in\partial\Omega\ \ \text{and}\ \ \vec{w}\cdot\vec{\nu}<0, \tag{3.394}$$

for some constant C_0.

Theorem 3.12. *The initial and boundary layers at* $\{0\}\times\partial\Omega$ *satisfy*

$$\mathscr{U}_0(0,\eta,\iota_1,\iota_2,\phi,\psi)=\mathfrak{U}_0(0,\eta,\iota_1,\iota_2,\phi,\psi)=0, \tag{3.395}$$

$$\mathcal{U}_0^I(t,\vec{x}_0,\vec{w})=0\ \ \textit{for}\ \ \vec{w}\cdot\vec{\nu}<0. \tag{3.396}$$

3.3.6. *Proof of Theorem 1.4.* Based on Theorem 3.7, we know there exists a unique $u^{\epsilon}(t,\vec{x},\vec{w}) \in L^{\infty}(\mathbb{R}^+ \times \Omega \times \mathbb{S}^2)$, so we focus on the diffusive limit.

Step 1: Remainder definitions.
We may rewrite the asymptotic expansion as follows:

$$u^{\epsilon} \sim \sum_{k=0}^{2} \epsilon^k U_k + \sum_{k=0}^{1} \epsilon^k \mathcal{U}_k^I + \sum_{k=0}^{1} \epsilon^k \mathscr{U}_k + \mathfrak{U}_0. \tag{3.397}$$

The remainder can be defined as

$$R = u^{\epsilon} - \sum_{k=0}^{2} \epsilon^k U_k - \sum_{k=0}^{1} \epsilon^k \mathcal{U}_k^I - \sum_{k=0}^{1} \epsilon^k \mathscr{U}_k - \mathfrak{U}_0 = u^{\epsilon} - Q - \mathcal{Q} - \mathscr{Q} - \mathfrak{Q}, \tag{3.398}$$

where

$$Q = \sum_{k=0}^{2} \epsilon^k U_k, \quad \mathcal{Q} = \sum_{k=0}^{1} \epsilon^k \mathcal{U}_k^I, \quad \mathscr{Q} = \sum_{k=0}^{1} \epsilon^k \mathscr{U}_k, \quad \mathfrak{Q} = \mathfrak{U}_0. \tag{3.399}$$

Noting the equation (1.3) is equivalent to the equations (3.195) and (3.215), we write $\mathcal{L}$ to denote the neutron transport operator as follows:

$$\begin{aligned}\mathcal{L}u =& \epsilon^2 \partial_t u + \epsilon \vec{w} \cdot \nabla_x u + u - \bar{u} = \partial_\tau u + \epsilon \vec{w} \cdot \nabla_x u + u - \bar{u} \\ =& \epsilon^2 \frac{\partial u}{\partial t} + \sin\phi \frac{\partial u}{\partial \eta} - \epsilon \bigg(\frac{\sin^2\psi}{R_1 - \epsilon\eta} + \frac{\cos^2\psi}{R_2 - \epsilon\eta} \bigg) \cos\phi \frac{\partial u}{\partial \phi} \\ &+ \epsilon \bigg(\frac{\cos\phi \sin\psi}{P_1(1-\epsilon\kappa_1\eta)} \frac{\partial u}{\partial \iota_1} + \frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)} \frac{\partial u}{\partial \iota_2} \bigg) \\ &+ \epsilon \Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta} \bigg(\cos\phi \Big(\vec{\varsigma}_1 \cdot \Big(\vec{\varsigma}_2 \times (\partial_{12}\vec{r} \times \vec{\varsigma}_2) \Big) \Big) - \kappa_1 P_1 P_2 \sin\phi\cos\psi \bigg) \\ &+ \frac{\cos\psi}{1-\epsilon\kappa_2\eta} \bigg(-\cos\phi \Big(\vec{\varsigma}_2 \cdot \Big(\vec{\varsigma}_1 \times (\partial_{12}\vec{r} \times \vec{\varsigma}_1) \Big) \Big) + \kappa_2 P_1 P_2 \sin\phi \sin\psi \bigg) \Bigg) \frac{1}{P_1 P_2} \frac{\partial u}{\partial \psi} + u - \bar{u}.\end{aligned} \tag{3.400}$$

Step 2: Estimates of $\mathcal{L}[Q]$.
The interior contribution can be estimated as

$$\mathcal{L}[Q] = \epsilon^3 \partial_t U_1 + \epsilon^4 \partial_t U_2 + \epsilon^3 \vec{w} \cdot \nabla_x U_2. \tag{3.401}$$

Using Theorem 3.11, we have

$$\|\mathcal{L}[Q]\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^3, \tag{3.402}$$

$$\|\mathcal{L}[Q]\|_{L^{\frac{2m}{2m-1}}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^3, \tag{3.403}$$

$$\|\mathcal{L}[Q]\|_{L^{2m}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^3, \tag{3.404}$$

$$\|\mathcal{L}[Q]\|_{L^{\infty}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^3, \tag{3.405}$$

$$\|\partial_t \mathcal{L}[Q]\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^3, \tag{3.406}$$

$$\|\mathcal{L}[Q](0)\|_{L^2(\Omega\times\mathbb{S}^2)} \leq C\epsilon^3. \tag{3.407}$$

Step 3: Estimates of $\mathcal{L}[\mathcal{Q}]$.
The initial layer contribution can be estimated as

$$\mathcal{L}[\mathcal{Q}] = \epsilon^2 \vec{w} \cdot \nabla_x \mathcal{U}_1^I. \tag{3.408}$$

Based on Theorem 3.10, we know

$$\left\| \epsilon^2 \vec{w} \cdot \nabla_x \mathcal{U}_1^I \right\|_{L^{\infty}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^2. \tag{3.409}$$

Note that $\mathscr{U}_1^I$ decays exponentially in τ and the scaling $\tau=\dfrac{t}{\epsilon^2}$, we have

$$\begin{aligned}\left\|\epsilon^2\vec{w}\cdot\nabla_x\mathcal{U}_1^I\right\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq&\epsilon^2\bigg(\int_0^\infty\int_{\Omega\times\mathbb{S}^2}\left|\nabla_x\mathcal{U}_1^I\right|^2(\tau,\vec{x},\vec{w})\mathrm{d}\vec{x}\mathrm{d}\vec{w}\mathrm{d}t\bigg)^{\frac{1}{2}}\\ \leq&C\epsilon^2\bigg(\int_0^\infty\mathrm{e}^{-\tau}\mathrm{d}t\bigg)^{\frac{1}{2}}=C\epsilon^3\bigg(\int_0^\infty\mathrm{e}^{-\tau}\mathrm{d}\tau\bigg)^{\frac{1}{2}}\leq C\epsilon^3.\end{aligned}\tag{3.410}$$

Similarly, we can derive that

$$\left\|\mathcal{L}[\mathcal{Q}]\right\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq C\epsilon^3,\tag{3.411}$$

$$\left\|\mathcal{L}[\mathcal{Q}]\right\|_{L^{\frac{2m}{2m-1}}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq C\epsilon^{4-\frac{1}{m}},\tag{3.412}$$

$$\left\|\mathcal{L}[\mathcal{Q}]\right\|_{L^{2m}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq C\epsilon^{2+\frac{1}{m}},\tag{3.413}$$

$$\left\|\mathcal{L}[\mathcal{Q}]\right\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq C\epsilon^2,\tag{3.414}$$

$$\left\|\partial_t\mathcal{L}[\mathcal{Q}]\right\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq C\epsilon^3,\tag{3.415}$$

$$\left\|\mathcal{L}[\mathcal{Q}](0)\right\|_{L^2(\Omega\times\mathbb{S}^2)}\leq C\epsilon^2.\tag{3.416}$$

Step 4: Estimates of $\mathcal{L}[\mathscr{Q}]$.
The regular boundary layer contribution can be estimated as

$$\begin{aligned}\mathcal{L}[\mathscr{Q}]=&\epsilon^2\partial_t\mathscr{U}_0+\epsilon^3\partial_t\mathscr{U}_1+\epsilon^2\bigg(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathscr{U}_1}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial\mathscr{U}_1}{\partial\iota_2}\bigg)\\ &+\epsilon^2\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\ &+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial\mathscr{U}_1}{\partial\psi}.\end{aligned}\tag{3.417}$$

Similarly to steady problems, based on Theorem 3.8 and Theorem 3.12, we have

$$\left\|\mathcal{L}[\mathscr{Q}]\right\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq C\epsilon^{\frac{5}{2}}\left|\ln(\epsilon)\right|^8,\tag{3.418}$$

$$\left\|\mathcal{L}[\mathscr{Q}]\right\|_{L^{\frac{2m}{2m-1}}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq C\epsilon^{3-\frac{1}{2m}}\left|\ln\epsilon\right|^8,\tag{3.419}$$

$$\left\|\mathcal{L}[\mathscr{Q}]\right\|_{L^{2m}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq C\epsilon^{2+\frac{1}{2m}}\left|\ln\epsilon\right|^8,\tag{3.420}$$

$$\left\|\mathcal{L}[\mathscr{Q}]\right\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq C\epsilon^2\left|\ln\epsilon\right|^8,\tag{3.421}$$

$$\left\|\partial_t\mathcal{L}[\mathscr{Q}]\right\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq C\epsilon^{\frac{5}{2}}\left|\ln(\epsilon)\right|^8,\tag{3.422}$$

$$\left\|\mathcal{L}[\mathscr{Q}](0)\right\|_{L^2(\Omega\times\mathbb{S}^2)}=0.\tag{3.423}$$

Step 5: Estimates of $\mathcal{L}[\mathfrak{Q}]$.
The singular boundary layer contribution can be estimated as

$$\begin{aligned}\mathcal{L}[\mathfrak{Q}]=&\epsilon^2\partial_t\mathfrak{U}_0+\epsilon\bigg(\frac{\cos\phi\sin\psi}{P_1(1-\epsilon\kappa_1\eta)}\frac{\partial\mathfrak{U}_0}{\partial\iota_1}+\frac{\cos\phi\cos\psi}{P_2(1-\epsilon\kappa_2\eta)}\frac{\partial\mathfrak{U}_0}{\partial\iota_2}\bigg)\\ &+\epsilon\Bigg(\frac{\sin\psi}{1-\epsilon\kappa_1\eta}\bigg(\cos\phi\Big(\vec{\varsigma}_1\cdot\Big(\vec{\varsigma}_2\times(\partial_{12}\vec{r}\times\vec{\varsigma}_2)\Big)\Big)-\kappa_1P_1P_2\sin\phi\cos\psi\bigg)\\ &+\frac{\cos\psi}{1-\epsilon\kappa_2\eta}\bigg(-\cos\phi\Big(\vec{\varsigma}_2\cdot\Big(\vec{\varsigma}_1\times(\partial_{12}\vec{r}\times\vec{\varsigma}_1)\Big)\Big)+\kappa_2P_1P_2\sin\phi\sin\psi\bigg)\Bigg)\frac{1}{P_1P_2}\frac{\partial\mathfrak{U}_0}{\partial\psi}.\end{aligned}$$

Similarly to steady problems, based on Theorem 3.9 and Theorem 3.12, we have

$$\left\|\mathcal{L}[\mathfrak{Q}]\right\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^{1+\frac{3}{2}\alpha}\left|\ln(\epsilon)\right|^8, \tag{3.424}$$

$$\left\|\mathcal{L}[\mathfrak{Q}]\right\|_{L^{\frac{2m}{2m-1}}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^{2-\frac{1}{2m}+\alpha}\left|\ln(\epsilon)\right|^8, \tag{3.425}$$

$$\left\|\mathcal{L}[\mathfrak{Q}]\right\|_{L^{2m}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^{1+\frac{1}{2m}+\alpha}\left|\ln(\epsilon)\right|^8, \tag{3.426}$$

$$\left\|\mathcal{L}[\mathfrak{Q}]\right\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon\left|\ln(\epsilon)\right|^8, \tag{3.427}$$

$$\left\|\partial_t\mathcal{L}[\mathfrak{Q}]\right\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^{1+\frac{3}{2}\alpha}\left|\ln(\epsilon)\right|^8, \tag{3.428}$$

$$\left\|\mathcal{L}[\mathfrak{Q}](0)\right\|_{L^2(\Omega\times\mathbb{S}^2)} = 0. \tag{3.429}$$

Step 6: Estimate of Source Term.
Summarizing all above, for $\alpha=1$, we have

$$\left\|\mathcal{L}[R]\right\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^{\frac{5}{2}}\left|\ln(\epsilon)\right|^8, \tag{3.430}$$

$$\left\|\mathcal{L}[R]\right\|_{L^{\frac{2m}{2m-1}}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^{3-\frac{1}{2m}}\left|\ln(\epsilon)\right|^8, \tag{3.431}$$

$$\left\|\mathcal{L}[R]\right\|_{L^{2m}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^{2+\frac{1}{2m}}\left|\ln(\epsilon)\right|^8, \tag{3.432}$$

$$\left\|\mathcal{L}[R]\right\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon\left|\ln(\epsilon)\right|^8, \tag{3.433}$$

$$\left\|\partial_t\mathcal{L}[R]\right\|_{L^2(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)} \leq C\epsilon^{\frac{5}{2}}\left|\ln(\epsilon)\right|^8, \tag{3.434}$$

$$\left\|\mathcal{L}[R](0)\right\|_{L^2(\Omega\times\mathbb{S}^2)} \leq C\epsilon^2. \tag{3.435}$$

Step 7: Estimate of Initial Data.
At $t=0$, the initial data of R is

$$R^I = \epsilon\mathscr{U}_1(0,\eta,\iota_1,\iota_2,\phi,\psi) + \epsilon^2\vec{w}\cdot\nabla_x U_1(0,\vec{x},\vec{w}). \tag{3.436}$$

Using Theorem 3.8 and Theorem 3.11, considering the spacial rescaling, we may derive

$$\left\|R^I\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)} \leq C\epsilon^{1+\frac{1}{2m}}, \tag{3.437}$$

$$\left\|R^I\right\|_{L^\infty(\Omega\times\mathbb{S}^2)} \leq C\epsilon. \tag{3.438}$$

Step 8: Estimate of Boundary Data.
At Γ^-, the boundary data or R is

$$R^B = \epsilon\mathcal{U}_1^I(\tau,\vec{x}_0,\vec{w}) + \epsilon^2\nabla_x U_1(t,\vec{x}_0,\vec{w}). \tag{3.439}$$

Using Theorem 3.10 and Theorem 3.11, considering the temporal rescaling, we may derive

$$\left\|R^B\right\|_{L^{2m}(\mathbb{R}^+\times\Gamma^-)} \leq C\epsilon^{1+\frac{1}{m}}, \tag{3.440}$$

$$\left\|R^B\right\|_{L^\infty((\mathbb{R}^+\times\Gamma^-))} \leq C\epsilon. \tag{3.441}$$

Step 9: Diffusive Limit.
Therefore, the remainder R satisfies the equation

$$\left\{
\begin{array}{l}
\epsilon^2\partial_t R + \epsilon\vec{w}\cdot\nabla_x R + R - \bar{R} = \mathcal{L}[R] \quad \text{for } \ (t,\vec{x},\vec{w})\in\mathbb{R}^+\times\Omega\times\mathbb{S}^2,\\
\\
R(0,\vec{x},\vec{w}) = R^I(\vec{x},\vec{w}) \ \text{ for } \ (\vec{x},\vec{w})\in\Omega\times\mathbb{S}^2\\
\\
R(t,\vec{x}_0,\vec{w}) = R^B(t,\vec{x}_0,\vec{w}) \ \text{ for } \ t\in\mathbb{R}^+, \ \ \vec{x}_0\in\partial\Omega, \ \ \text{and } \ \vec{w}\cdot\vec{\nu}<0.
\end{array}
\right. \tag{3.442}$$

By Theorem 3.5, for integer $1\leq m\leq 3$,

(3.443)

$$\begin{aligned}
&\|R\|_{L^{\infty}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\\
\leq&C\bigg(\frac{1}{\epsilon^{2+\frac{2}{m}}}\|\mathcal{L}[R]\|_{L^{\frac{2m}{2m-1}}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{2}{m}}}\|\mathcal{L}[R]\|_{L^{2}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{1+\frac{2}{m}}}\|\mathcal{L}[R]\|_{L^{2m}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}+\|\mathcal{L}[R]\|_{L^{\infty}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\\
&+\frac{1}{\epsilon^{\frac{2}{m}}}\|\partial_t\mathcal{L}[R]\|_{L^{2}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{2}{m}}}\|\mathcal{L}[R](0)\|_{L^{2}(\Omega\times\mathbb{S}^2)}\\
&+\frac{1}{\epsilon^{\frac{3}{2m}}}\left\|R^I\right\|_{L^{2m}(\Omega\times\mathbb{S}^2)}+\left\|R^I\right\|_{L^{\infty}(\Omega\times\mathbb{S}^2)}+\frac{1}{\epsilon^{\frac{2}{m}}}\left\|R^B\right\|_{L^{2m}(\mathbb{R}^+\times\Gamma^-)}+\left\|R^B\right\|_{L^{\infty}(\mathbb{R}^+\times\Gamma^-)}\bigg)\\
\leq&C\epsilon^{1-\frac{5}{2m}}\left|\ln(\epsilon)\right|^8\leq\epsilon^{\frac{1}{6}}\left|\ln(\epsilon)\right|^8.
\end{aligned}$$

Since it is obvious that

$$\left\|\epsilon U_1+\epsilon^2U_2+\epsilon\mathcal{U}_1^I+\epsilon\mathscr{U}_1\right\|_{L^{\infty}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq C\epsilon, \tag{3.444}$$

we naturally have for any $0<\delta<<1$,

$$\left\|u^{\epsilon}-U_0-\mathcal{U}_0^I-\mathscr{U}_0-\mathfrak{U}_0\right\|_{L^{\infty}(\mathbb{R}^+\times\Omega\times\mathbb{S}^2)}\leq C(\delta)\epsilon^{\frac{1}{6}-\delta}. \tag{3.445}$$

Step 10: Exponential Decay.
The exponential decay can be easily derived following a similar argument using Theorem 3.7. In particular, all the estimates remain the same with extra exponential terms. Then we simply take $U=U_0$, $\mathcal{U}^I=\mathcal{U}_0^I$ and $\mathcal{U}^B=\mathscr{U}_0+\mathfrak{U}_0$.

4. ϵ-Milne Problem with Geometric Correction

This section is very similar to [23, Section 4] and [23, Section 5] (also see [18]), so we omit the details here and only present the main theorems.

4.1. **Well-Posedness and Decay.** In this section, we temporarily ignore the superscript ϵ and hide the dependence on (ι_1,ι_2), i.e. we consider the ϵ-Milne problem with geometric correction for $f(\eta,\phi,\psi)$ in the domain $(\eta,\phi,\psi)\in[0,L]\times\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\times[-\pi,\pi]$:

$$\begin{cases}
\sin\phi\dfrac{\partial f}{\partial\eta}+F(\eta,\psi)\cos\phi\dfrac{\partial f}{\partial\phi}+f-\bar{f}=S(\eta,\phi,\psi),\\
f(0,\phi,\psi)=h(\phi,\psi)\ \ \text{for}\ \ \sin\phi>0,\\
f(L,\phi,\psi)=f(L,\mathscr{R}[\phi],\psi),
\end{cases} \tag{4.446}$$

where $L=\epsilon^{-n}$ for some $0<n<\dfrac{1}{2}$, $\mathscr{R}[\phi]=-\phi$,

$$F(\eta,\psi)=-\epsilon\bigg(\frac{\sin^2\psi}{R_1-\epsilon\eta}+\frac{\cos^2\psi}{R_2-\epsilon\eta}\bigg), \tag{4.447}$$

for R_1 and R_2 radium of two principal curvatures, and

$$\bar{f}(\eta)=\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}f(\eta,\phi,\psi)\cos\phi\mathrm{d}\phi\mathrm{d}\psi, \tag{4.448}$$

in which $\cos\phi$ shows up as the Jacobian of spherical coordinates in integration. Note that for $\phi\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$, we always have $\cos\phi\geq0$, which means this will not destroy the positivity of the integral. Here, all estimates we get will be uniform in ϵ, ι_1 and ι_2.

In this section, we need to introduce special notation to highlight the different contribution of space and velocity. Define the norms in the space $(\eta,\phi,\psi)\in[0,L]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[-\pi,\pi]$ as follows:

$$\|f\|_{L^2_\eta L^2_{\phi,\psi}}=\left(\int_0^L\int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|f(\eta,\phi,\psi)|^2\cos\phi\mathrm{d}\phi\mathrm{d}\psi\mathrm{d}\eta\right)^{\frac{1}{2}}, \tag{4.449}$$

$$\|f\|_{L^\infty_\eta L^\infty_{\phi,\psi}}=\mathrm{esssup}_{(\eta,\phi,\psi)\in[0,L]\times[-\frac{\pi}{2},\frac{\pi}{2}]\times[-\pi,\pi]}|f(\eta,\phi,\psi)|. \tag{4.450}$$

Define the norms at the in-flow boundary as

$$\|h\|_{L^2_{+,\phi,\psi}}=\left(\iint_{\sin\phi>0,\psi\in[-\pi,\pi]}|h(\phi,\psi)|^2\sin\phi\cos\phi\mathrm{d}\phi\mathrm{d}\psi\right)^{\frac{1}{2}}, \tag{4.451}$$

$$\|h\|_{L^\infty_{+,\phi,\psi}}=\mathrm{esssup}_{\sin\phi>0,\psi\in[-\pi,\pi]}|h(\phi,\psi)|. \tag{4.452}$$

Here we purposely use $+$ to indicate in-flow boundary to simplify the analysis, which is different from the convention in remainder estimates.

Assume the source term S and boundary data h satisfy

$$\|h\|_{L^\infty_{+,\phi,\psi}}\leq C, \tag{4.453}$$

and

$$\left\|\mathrm{e}^{K\eta}S\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C, \tag{4.454}$$

for some constants $C>0$ and $K>0$ uniform in ϵ and ι_i for $i=1,2$.

Theorem 4.1. *Assume S and h satisfy* (4.453) *and* (4.454)*. Then there exists a solution $f(\eta,\phi,\psi)$ to the equation* (4.446)*, satisfying that*

$$|f_L|\leq C,\qquad \|f-f_L\|_{L^2_\eta L^2_{\phi,\psi}}+\|f-f_L\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C, \tag{4.455}$$

where

$$f_L=\frac{\left\langle\sin^2\phi,f\right\rangle_{\phi,\psi}(L)}{\|\sin\phi\|^2_{L^2_{\phi,\psi}}}. \tag{4.456}$$

The solution is unique among functions such that above estimates hold.

Theorem 4.2. *Assume S and h satisfy* (4.453) *and* (4.454)*. Then there exists $K_0>0$ such that the solution $f(\eta,\phi,\psi)$ to the equation* (4.446) *satisfies*

$$\left\|\mathrm{e}^{K_0\eta}(f-f_L)\right\|_{L^2_\eta L^2_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}(f-f_L)\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C. \tag{4.457}$$

Theorem 4.3. *The solution $f(\eta,\phi,\psi)$ to the equation* (4.446) *with $S=0$ satisfies the maximum principle, i.e.*

$$\min_{\sin\phi>0}h(\phi,\psi)\leq f(\eta,\phi,\psi)\leq\max_{\sin\phi>0}h(\phi,\psi). \tag{4.458}$$

4.2. **Regularity.** For regularity estimates, we further assume the source term S and boundary data h satisfy

$$\|h\|_{L^\infty_{+,\phi,\psi}}+\left\|\frac{\partial h}{\partial\phi}\right\|_{L^\infty_{+,\phi,\psi}}+\left\|\frac{\partial h}{\partial\psi}\right\|_{L^\infty_{+,\phi,\psi}}\leq C, \tag{4.459}$$

and

$$\left\|\mathrm{e}^{K\eta}S\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K\eta}\frac{\partial S}{\partial\eta}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K\eta}\frac{\partial S}{\partial\phi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}+\left\|\mathrm{e}^{K\eta}\frac{\partial S}{\partial\psi}\right\|_{L^\infty_\eta L^\infty_{\phi,\psi}}\leq C, \tag{4.460}$$

for some constants $C>0$ and $K>0$ uniform in ϵ and ι_i for $i=1,2$.

$\mathscr{V}=f-f_L$ satisfies the ϵ-Milne problem with geometric correction

$$
\left\{
\begin{array}{l}
\sin\phi\dfrac{\partial\mathscr{V}}{\partial\eta}+F(\eta,\psi)\cos\phi\dfrac{\partial\mathscr{V}}{\partial\phi}+\mathscr{V}-\bar{\mathscr{V}}=S(\eta,\phi,\psi),\\[2ex]
\mathscr{V}(0,\phi,\psi)=p(\phi,\psi)=h(\phi,\psi)-f_L\quad\text{for}\quad\sin\phi>0,\\[2ex]
\mathscr{V}(L,\phi,\psi)=\mathscr{V}(L,\mathscr{R}[\phi],\psi).
\end{array}
\right.\tag{4.461}
$$

Define a weight function

$$
\zeta(\eta,\phi,\psi)=\left(1-\cos^2\phi\bigg(\frac{R_1-\epsilon\eta}{R_1}\bigg)^{2\sin^2\psi}\bigg(\frac{R_2-\epsilon\eta}{R_2}\bigg)^{2\cos^2\psi}\right)^{\frac{1}{2}}.\tag{4.462}
$$

Theorem 4.4. *The solution $\mathscr{V}$ to the difference equation* (4.461) *satisfies*

$$
\begin{aligned}
&\left\Vert\zeta\frac{\partial\mathscr{V}}{\partial\eta}\right\Vert_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}+\left\Vert F(\eta,\psi)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\Vert_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\\
\leq&C\left\vert\ln(\epsilon)\right\vert^8\left(\Vert p\Vert_{L^{\infty}_{+,\phi,\psi}}+\left\Vert\epsilon\frac{\partial p}{\partial\phi}\right\Vert_{L^{\infty}_{+,\phi,\psi}}+\Vert S\Vert_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}+\left\Vert\zeta\frac{\partial S}{\partial\eta}\right\Vert_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}+\Vert\mathscr{V}\Vert_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\right).
\end{aligned}\tag{4.463}
$$

Theorem 4.5. *For $K_0>0$ sufficiently small, the solution $\mathscr{V}$ to the difference equation* (4.461) *satisfies*

$$
\begin{aligned}
&\left\Vert\mathrm{e}^{K_0\eta}\zeta\frac{\partial\mathscr{V}}{\partial\eta}\right\Vert_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}+\left\Vert\mathrm{e}^{K_0\eta}F(\eta,\psi)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\Vert_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\\
\leq&C\left\vert\ln(\epsilon)\right\vert^8\left(\Vert p\Vert_{L^{\infty}_{+,\phi,\psi}}+\left\Vert\epsilon\frac{\partial p}{\partial\phi}\right\Vert_{L^{\infty}_{+,\phi,\psi}}+\left\Vert\mathrm{e}^{K_0\eta}S\right\Vert_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}+\left\Vert\mathrm{e}^{K_0\eta}\zeta\frac{\partial S}{\partial\eta}\right\Vert_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}+\left\Vert\mathrm{e}^{K_0\eta}\mathscr{V}\right\Vert_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\right).
\end{aligned}\tag{4.464}
$$

In the analysis of regular boundary layer, we actually need to estimate the solution to the following modified ϵ-Milne problem with geometric correction:

$$
\left\{
\begin{array}{l}
\sin\phi\dfrac{\partial g}{\partial\eta}+F(\eta,\psi)\cos\phi\dfrac{\partial g}{\partial\phi}+g=S(\eta,\phi,\psi),\\[2ex]
g(0,\phi,\psi)=h(\phi,\psi)\quad\text{for}\quad\sin\phi>0,\\[2ex]
g(L,\phi,\psi)=g(L,\mathscr{R}[\phi],\psi).
\end{array}
\right.\tag{4.465}
$$

Since there is no non-local term $\bar{g}$, the estimates is even simpler. We can always track along the characteristics back to the boundary data. Here S is not arbitrary and in all applications, it is always delicately chosen such that the solution g decays exponentially to a constant g_L.

Equivalently, $\mathscr{W}=g-g_L$ satisfies

$$
\left\{
\begin{array}{l}
\sin\phi\dfrac{\partial\mathscr{W}}{\partial\eta}+F(\eta,\psi)\cos\phi\dfrac{\partial\mathscr{W}}{\partial\phi}+\mathscr{W}=S(\eta,\phi,\psi),\\[2ex]
\mathscr{W}(0,\phi,\psi)=h(\phi,\psi)\quad\text{for}\quad\sin\phi>0,\\[2ex]
\mathscr{W}(L,\phi,\psi)=\mathscr{W}(L,\mathscr{R}[\phi],\psi).
\end{array}
\right.\tag{4.466}
$$

Theorem 4.6. *Assume S and h satisfy* (4.459) *and* (4.460)*. Then there exists $K_0>0$ such that the solution $g(\eta,\phi,\psi)$ to the equation* (4.465) *satisfies*

$$
\left\Vert\mathrm{e}^{K_0\eta}\mathscr{W}\right\Vert_{L^2_{\eta}L^2_{\phi,\psi}}+\left\Vert\mathrm{e}^{K_0\eta}\mathscr{W}\right\Vert_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\leq C.\tag{4.467}
$$

Theorem 4.7. *For $K_0>0$ sufficiently small, the solution $\mathscr{V}$ to the difference equation (4.466) satisfies*

$$\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial\mathscr{W}}{\partial\eta}\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}F(\eta,\psi)\cos\phi\frac{\partial\mathscr{W}}{\partial\phi}\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}} \tag{4.468}$$
$$\leq C\left|\ln(\epsilon)\right|^8\left(\left\|p\right\|_{L^{\infty}_{+,\phi,\psi}}+\left\|\epsilon\frac{\partial p}{\partial\phi}\right\|_{L^{\infty}_{+,\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}S\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial S}{\partial\eta}\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}+\left\|\mathrm{e}^{K_0\eta}\mathscr{W}\right\|_{L^{\infty}_{\eta}L^{\infty}_{\phi,\psi}}\right).$$

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(L. Wu)
Department of Mathematics, Lehigh University
Bethlehem, PA 18015, USA
E-mail address: `lew218@lehigh.edu`