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Zero modes of local operators in 2d CFT on a cylinder

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ABSTRACT: Studies of Eigenstate Thermalization Hypothesis (ETH) in two-dimensional CFTs call for calculation of the expectation values of local operators in highly excited energy eigenstates. This can be done efficiently by representing zero modes of these operators in terms of the Virasoro algebra generators. In this paper we present a pedagogical introduction explaining how this calculation can be performed analytically or using computer algebra. We illustrate the computation of zero modes by a number of examples and list explicit expressions for all local operators from the vacuum family with the dimension of less or equal than eight. Finally, we derive an explicit expression for the quantum KdV generator Q_7 in terms of the Virasoro algebra generators. The obtained results can be used for quantitative studies of ETH at finite value of central charge.

KEYWORDS: Conformal Field Theory, Conformal and W Symmetry, Integrable Field Theories

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1 Introduction

Emergence of statistical thermodynamics from quantum mechanics is one of the exciting open questions of theoretical physics. It has became an accepted paradigm that universality of thermal equilibrium can be explained in terms of Eigenstate Thermalization [1–3]. The latter is an expectation that local properties of individual microstates — energy eigenstates of spatially extended sufficiently complex quantum system — may only depend on thermodynamically relevant quantities, i.e. in most cases only on energy density of the microstate. Technically, this means that the diagonal matrix elements of local observables

$$A_{ii} \equiv \langle E_i | A | E_i \rangle = f_A(E_i), \tag{1.1}$$

are smooth functions of energy density E_i/V .

When the system, besides energy, possesses a number of additional local (or quasi-local [4]) conserved quantities, as is normally the case for integrable systems, densities of these additional conserved charges are also thermodynamically relevant. This simply means emerging equilibrium state will be dependent on these additional quantities, or, more accurately their densities. Further assuming local properties of individual microstates (energy eigenstates) will be dependent on these charges, but otherwise will be physically equivalent, we arrive at the notation of generalized Eigenstate Thermalization Hypothesis [5–7],

$$A_{ii} = f_A(Q_k(E_i)), \tag{1.2}$$

where f_A is a smooth function of its arguments, Q_k are the conserved charges, and $Q_k(E_i)$ are the charge values associated with an individual energy eigenstate $|E_i\rangle$.

Eigenstate Thermalization Hypothesis has been established numerically for a range of lattice models [8]. It is therefore natural to ask if Eigenstate Thermalization also applies to conformal field theories [9-11, 11-23], which can be thought of as a continuous limit of lattice systems at criticality. This assumes conformal field theory is quantized on a cylinder of finite spatial size. The object of interest is the diagonal matrix element $\langle E|A|E\rangle$, where $|E\rangle$ is the eigenstate of the CFT Hamiltonian and A is a local observable. In case of twodimensional theories, which is a focus of this work, there is a number of specifics which need to be taken into account. First, the theory is split into "left" and "right" non-interacting sectors, such that in conventional units the Hamiltonian H is a sum of $L_0 - c/24$ and $L_0 - c/24$. We can assume that $|E\rangle$ is a tensor product of two eigenstates in each sector, while A is a product of local holomorphic and anti-holomorphic components. Then the calculation factorizes and in the rest of the paper we focus on the left-moving sector only. Second, when A is a global descendant, its expectation value in the eigenstate of $L_0 - c/24$ is zero, thus in the following we assume A is a quasi-primary. Third, and final point, we have two distinct cases to consider: when A is a Virasoro descendant of identity or when A is not from the vacuum family. In two dimensions thermal expectation values of all operators outside of the vacuum family vanish (this also applies to KdV Generalized Gibbs Ensemble). Thus a prediction of ETH in d=2 for such operators would be that $\langle E|A|E\rangle$ vanish when $|E\rangle$ is sufficiently heavy [12]. This is a physically straightforward but challenging question to verify, as the behavior of individual OPE coefficients which govern this matrix element is currently outside of our theoretical control. To summarize, we are interested in a special set of observables — quasi-primary operators from the vacuum family, for which matrix elements A_{ii} are non-zero, universal and fixed by the Virasoro algebra.

The spectrum of 2d theories is highly degenerate and therefore the choice of $|E\rangle$ is not unique. This is related to the fact that 2d theories possess an infinite number of local conserved quantum KdV charges [24–26], which must be taken into account in the context of eigenstate thermalization. In recent papers [18–20] we have demonstrated that organizing descendant states in the eigenstates of the KdV hierarchy allows one to formulate the diagonal part of the generalized Eigenstate Thermalization Hypothesis (1.2).

Any descendant state, including the KdV eigenstates $|E\rangle$ can be written as a linear combination of Virasoro algebra generators acting on some primary state Δ ,

$$|E\rangle = \sum_{k_1,\dots,k_m} c_{k_1,\dots,k_m} L_{-k_1} \dots L_{-k_m} |\Delta\rangle, \qquad k_1 + \dots + k_m = n, \qquad E = \Delta + n, \quad k_i > 0.$$

Even if coefficients $c_{k_1,...,k_m}$ are known explicitly, to calculate the diagonal matrix element $\langle E|A|E\rangle$, one needs to know an explicit form of the zero mode of the local operator A in terms of the Virasoro algebra generators,

$$A_0 = \oint \frac{dw}{2\pi} A(w) \equiv \frac{1}{2\pi} \int_0^{2\pi} du \, A(w). \tag{1.3}$$

Here $w = u + i\tau$ is a holomorphic coordinate on the cylinder, and u is a periodic coordinate on a spacial circle of length 2π . The coordinate w is related to the conventional coordinate

on the plane by $z = e^{-iw}$. Local operator A admits the following mode expansion

$$A(w) = \sum_{n = -\infty}^{\infty} A_n e^{-inw}.$$
 (1.4)

Since we assumed A is from the vacuum family, it is "built" out of energy-momentum tensor T(w) whose mode expansion is

$$T(w) = -\frac{c}{24} + \sum_{n = -\infty}^{+\infty} L_n e^{-inw}.$$
 (1.5)

The operators L_n satisfy conventional commutation relations of Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}.$$
 (1.6)

In practical terms A are appropriately regularized polynomials in stress-energy tensor T(w) and its holomorphic derivatives.

Among the local (not necessarily quasi-primary) operators there is a family of special densities which, upon integration, give rise to the tower of mutually commuting quantum KdV charges Q_{2k-1} . There is one such density J_{2k} for each even dimension 2k such that

$$Q_{2k-1} = \oint \frac{dw}{2\pi} J_{2k}(w). \tag{1.7}$$

The general expression for J_{2k} is not known. However, they are uniquely determined up to normalization by the requirement of commutativity $[Q_{2k-1}, Q_{2l-1}] = 0$ and scaling homogeneity. First few densities $J_{2k}(w)$ are

$$J_2(w) = T, (1.8)$$

$$J_4(w) = (TT), \tag{1.9}$$

$$J_6(w) = (T(TT)) + \frac{c+2}{12}(\partial T \partial T), \tag{1.10}$$

$$J_8(w) = (T(T(TT))) + \frac{c+2}{3}(T(\partial T\partial T)) + \frac{2c^2 - 17c - 42}{360}(\partial^2 T\partial^2 T).$$
 (1.11)

Here the parentheses (·) denote normal ordering which will be defined in the next section. Explicit form of Q_{2k-1} , i.e. the zero modes of J_{2k} , in terms of the Virasoro algebra generators would allow computing the spectrum of Q_{2k-1} at finite c in terms of computer algebra.

In this paper we pedagogically develop the machinery of calculating zero modes of local operators from the vacuum family and calculate explicit expressions for all quasi-primaries of the dimension less or equal than eight. We also calculate the explicit expression for Q_7 . Some of these results, albeit in a simplified thermodynamic limit, when certain terms can be neglected, were already used by us in previous studies of ETH in [20]. We plan to use the explicit results obtained in this paper to further elucidate ETH in 2d CFT in the subsequent works.

The paper is organized as follows. In section 2 we develop general technique for calculating the zero modes of the normal ordered product of local operators. In section 3 we

show in detail how this technique applies in the simplest case and calculate first non-trivial qKdV charge Q_3 . In section 4 we list the explicit expressions of the zero modes for all quasi-primary operators with the dimension of less or equal to eight. Then, in section 5 we give the explicit expression for Q_7 in terms of the Virasoro algebra generators and verify its consistency in the large c limit and also using the constraints coming from the (9,2) minimal model. We conclude in section 6.

2 Zero modes of the operator product

In this section we will describe general technique of finding zero modes of the product of arbitrary local operators A(w) and B(w) on a cylinder worldsheet, namely

$$(AB)_0 = \oint \frac{dw}{2\pi} (AB)(w), \tag{2.1}$$

where the contour is taken over a spacial circle. The operator A(w) and B(w) are assumed to have analytic mode expansion

$$A(w) = \sum_{n = -\infty}^{+\infty} A_n e^{-inw}, \qquad (2.2)$$

and similarly for B(w). The parentheses in (AB)(w) denote normal ordering of the operators and is defined as

$$(AB)(w) = \oint_{w} \frac{dz}{2\pi i} \frac{1}{z - w} \mathcal{T}(A(z)B(w)), \qquad (2.3)$$

where the integration is performed over the circle around w and the symbol \mathcal{T} stands for "chronological ordering," i.e.

$$\mathcal{T}(A(z)B(w)) = \begin{cases} A(z)B(w), & \text{if } \operatorname{Im} z < \operatorname{Im} w, \\ B(w)A(z), & \text{if } \operatorname{Im} z > \operatorname{Im} w. \end{cases}$$
 (2.4)

To perform the integration (2.3) we will split the contour into two pieces as showed in the fugure 1, where we deform the contour in such a way that we can deal with two chronologically ordered expressions separately, 1 namely

$$(AB)(w) = \int_{-i\epsilon}^{2\pi - i\epsilon} \frac{dz}{2\pi i} \frac{A(z)B(w)}{z - w} - \int_{i\epsilon}^{2\pi + i\epsilon} \frac{dz}{2\pi i} \frac{B(w)A(z)}{z - w}.$$
 (2.5)

The strategy of calculating (2.5) is to express the integrand in terms of mode expansion e^{iw} and e^{iz} . Let us introduce two axillary variables u and v,

$$u = e^{iz}, \quad v = e^{iw}, \tag{2.6}$$

¹We thank Pavlo Gavrilenko for help with the following calculation.

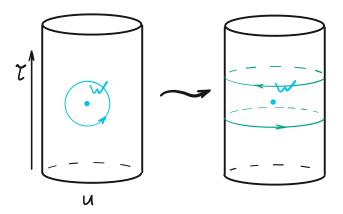


Figure 1. Deformation of the blue contour (2.3) into the green one (2.5).

and make the following transformation,

$$\frac{1}{2\pi i} \frac{dz}{z - w} = \frac{1}{2\pi} \frac{dz}{\log(u) - \log(w)} = \frac{1}{2\pi} \frac{dz}{\log\left(1 + \frac{u - v}{v}\right)}$$

$$= \frac{dz}{2\pi} \left(\frac{v}{u - v} + \sum_{k=0}^{\infty} c_k \left(\frac{u - v}{v}\right)^k\right), \tag{2.7}$$

where we formally expanded one over logarithm and c_k denote the coefficients of this expansion. The combination v/(u-v) can be easily represented in terms of the mode expansion, namely

$$\frac{v}{u-v} = \begin{cases}
\sum_{k=1}^{+\infty} e^{ik(w-z)}, & \text{Im}(z) < \text{Im}(w), \\
\sum_{k=0}^{+\infty} e^{ik(z-w)}, & \text{Im}(z) > \text{Im}(w).
\end{cases}$$
(2.8)

Plugging (2.7) and (2.8) into (2.5) we obtain

$$(AB)(w) = \int_{-i\epsilon}^{2\pi - i\epsilon} \frac{dz}{2\pi} \left(\sum_{k=1}^{\infty} e^{ik(w-z)} + \sum_{k=0}^{\infty} c_k \left(e^{i(z-w)} - 1 \right)^k \right) A(z) B(w)$$

$$- \int_{+i\epsilon}^{2\pi + i\epsilon} \frac{dz}{2\pi} \left(-\sum_{k=0}^{\infty} e^{ik(z-w)} + \sum_{k=0}^{\infty} c_k \left(e^{i(z-w)} - 1 \right)^k \right) B(w) A(z).$$
(2.9)

In each of the two integrals in (2.9) we can integrate the first term assuming (2.2) and combine the rest into an integral of the commutator [A(z), B(w)], i.e.

$$(AB)(w) = A_{-}(w)B(w) + B(w)A_{+}(w) + \int_{0}^{2\pi} \frac{dz}{2\pi} \sum_{k=0}^{\infty} c_{k} \left(e^{i(z-w)} - 1\right)^{k} [A(z), B(w)],$$
(2.10)

where

$$A_{-}(w) = \sum_{n=1}^{\infty} A_{-n}e^{inw}, \quad A_{+}(w) = \sum_{n=0}^{\infty} A_{n}e^{-inw}.$$
 (2.11)

Substituting (2.10) into (2.1) we obtain

$$(AB)_{0} = \sum_{n=1}^{\infty} A_{-n}B_{n} + \sum_{n=0}^{\infty} B_{-n}A_{n} + \int_{0}^{2\pi} \frac{dw}{2\pi} \int_{0}^{2\pi} \frac{dz}{2\pi} \sum_{k=0}^{\infty} c_{k} \left(e^{i(z-w)} - 1\right)^{k} [A(z), B(w)].$$

$$(2.12)$$

To move further one should calculate the commutator [A(z), B(w)] and perform the integration of an infinite sum over k. This seems to be a difficult task, but we will see shortly that only finite number of terms in this sum give non-zero contributions to the integral

$$\int_0^{2\pi} \frac{dz}{2\pi} \sum_{k=0}^{\infty} c_k \left(e^{i(z-w)} - 1 \right)^k [A(z), B(w)]$$
 (2.13)

from (2.10). Indeed, if we fix k and expand $(e^{i(z-w)}-1)^k$ we immediately see that the only modes of A(z) which contribute are those from 0 to k. More precisely for a given fixed k we have

$$\sum_{n=0}^{k} c_k \frac{k!}{n!(n-k)!} (-1)^n e^{-inw} [A_n, B(w)]$$
 (2.14)

If we now rewrite the commutator in terms of the mode expansion $e^{-inw}[A_n, B(w)] = \sum_m [A_n, B_m] e^{-(n+m)w}$ and keep in mind that both operators A and B are "built" out of stress-energy tensor, then both A_n and B_n will be some normal ordered polynomials in L_i such that the total sum of indexes is equal to n. The commutator $[A_n, B_m]$ is therefore also a polynomial in L_i with a coefficient which is a polynomial in n. Therefore the sum in (2.14) will be a linear combination of the terms

$$S(a,k) = \sum_{m=0}^{k} \frac{1}{n!(n-k)!} (-1)^{k-n} n^a,$$
(2.15)

where a is some non-zero integer. The expression above is the Stirling number of the second kind, which vanishes unless $k \leq a$. This immediately confirms that only finite number of terms in (2.14) with $k \leq a$ contribute.

We illustrate the emergence of the polynomial expression in n explicitly in the case A = B = T(w) in the next section. Here we only note that the finite number of terms contributing in the sum over k in (2.10) provide a crucial simplification. It allows computing (2.10) efficiently using computer algebra, which we use extensively in the computation of the zero modes in section 4.2 as well as of Q_7 in section 5.

3 Warm-up: computation of Q_3

In this section we apply the machinery devised in the previous section to the simplest non-trivial example $Q_3 = \oint \frac{dz}{2\pi} (TT)(z)$ and show explicitly how to perform the integration (2.12). It's convenient to introduce shifted Virasoro generators,

$$\tilde{L}_n = L_n - \frac{c}{24} \delta_{n,0}. \tag{3.1}$$

In terms of these operators the stress-energy tensor T(z) has the following mode expansion

$$T(z) = \sum_{n = -\infty}^{+\infty} \tilde{L}_n e^{-inz},$$
(3.2)

and the Virasoro algebra in terms of the shifted generators is modified as

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m} + \frac{c}{12}n^3\delta_{n+m,0}.$$
 (3.3)

Substituting A(z) = T(z) and B(w) = T(w) into (2.12) we obtain

$$Q_{3} = (TT)_{0} = 2 \sum_{n=1}^{+\infty} \tilde{L}_{-n} \tilde{L}_{n} + \tilde{L}_{0}^{2}$$

$$+ \int_{0}^{2\pi} \frac{dw}{2\pi} \int_{0}^{2\pi} \frac{dz}{2\pi} \sum_{k=0}^{\infty} c_{k} \left(e^{i(z-w)} - 1 \right)^{k} [T(z), T(w)].$$
(3.4)

The calculation of the commutator is a bit tedious but straightforward:

$$[T(z), T(w)] = \sum_{m,n \in \mathbb{Z}} e^{-inz} e^{-imw} [\tilde{L}_n, \tilde{L}_m]$$

$$= \sum_{m,n \in \mathbb{Z}} e^{-imz} e^{-inw} \left((n-m)\tilde{L}_{n+m} + \frac{c}{12} n^3 \delta_{n+m,0} \right)$$

$$= \sum_{m,n \in \mathbb{Z}} e^{-in(z-w)} (2n - (m+n)) e^{-i(m+n)w} \tilde{L}_{m+n} + \frac{c}{12} \sum_{n \in \mathbb{Z}} n^3 e^{-in(z-w)}.$$
(3.5)

As was anticipated in the end of previous section, the commutator of two local operators gives rise to a polynomial expression in n. Namely, using the notations of the previous section $(A_n$ is equal to \tilde{L}_n and B(w) is equal to T(w),

$$e^{-inw}[A_n, B(w)] = \sum_{m \in \mathbb{Z}} (2n - m)e^{-imw}\tilde{L}_m + \frac{c}{12}n^3,$$

and therefore only terms with $k \leq 3$ will contribute in the sum over k in (2.10). Here we would like to illustrate that by combing (3.5) into a local expression. We continue,

$$[T(z), T(w)] = 2(i\partial_z) \sum_{n \in \mathbb{Z}} e^{-in(z-w)} T(w) + i \sum_{n \in \mathbb{Z}} e^{-in(z-w)} \partial_w T(w) + \frac{c}{12} (i\partial_z)^3 \sum_{n \in \mathbb{Z}} e^{-in(z-w)}$$
$$= 4\pi i \partial_z \delta(z-w) T(w) + 2\pi i \delta(z-w) \partial_w T(w) + \frac{c}{12} (i\partial_z)^3 2\pi \delta(z-w), \tag{3.6}$$

where on the last line we have used the following representation of delta-function,

$$2\pi\delta(z-w) = \sum_{n\in\mathbb{Z}} e^{-in(z-w)}.$$
 (3.7)

Note, that we have represented the commutator of stress-energy tensors [T(z), T(w)] in such a way that every term in the final expression contains delta-function $\delta(z-w)$ or its

derivatives. This representation helps us easily perform the integration over z in (3.4), namely

$$\oint \frac{dz}{2\pi} \sum_{k=0}^{\infty} c_k \left(e^{i(z-w)} - 1 \right)^k [T(z), T(w)] = ic_0 \partial_w T(w)
-2i\partial_z \sum_{k=0}^{+\infty} c_k \left(e^{i(z-w)} - 1 \right)^k \bigg|_{z=w} T(w) + \frac{c}{12} \left(-i\partial_z \right)^3 \sum_{k=0}^{+\infty} c_k \left(e^{i(z-w)-1} \right)^k \bigg|_{z=w} .$$
(3.8)

Here in the first term we have used $0^0 = 1$. Let us denote

$$a_k = -i\partial_z \left(e^{i(z-w)} - 1 \right)^k \Big|_{z=w}, \quad b_k = (-i\partial_z)^3 \left(e^{i(z-w)} - 1 \right)^k \Big|_{z=w}.$$
 (3.9)

The coefficients a_k and b_k are non-zero only for the first few values of $k \leq 3$. Specifically,

$$a_1 = 1, \quad a_{2,3,4,\dots} = 0,$$

 $b_1 = 1, \quad b_2 = 6, \quad b_3 = 6, \quad b_{4,5,6,\dots} = 0.$ (3.10)

That means that the sums $\sum c_k a_k$ and $\sum c_k b_k$ contain only finite number of terms and only four first coefficients c_k contribute to these sums. From the definition we find

$$c_0 = \frac{1}{2}, \quad c_1 = -\frac{1}{12}, \quad c_2 = \frac{1}{24}, \quad c_3 = -\frac{19}{720}.$$
 (3.11)

Combining (3.8) and (2.10) we get the normal ordered expression

$$(TT)(w) = T_{-}(w)T(w) + TT_{+}(w) - \frac{1}{6}T(w) + \frac{i}{2}\partial_{w}T(w) + \frac{c}{1440},$$
(3.12)

where $T_{-}(w) = \sum_{k=1} e^{ikw} \tilde{L}_{-k}$ and $T_{+}(w) = \sum_{k=0} e^{-ikw} \tilde{L}_{k}$. And, finally, for zero mode

$$Q_{3} = (TT)_{0} = \oint \frac{dw}{2\pi} (TT)(w) = 2 \sum_{n=1}^{+\infty} \tilde{L}_{-n} \tilde{L}_{n} + \tilde{L}_{0}^{2} - \frac{1}{6} \tilde{L}_{0} + \frac{c}{1440}$$

$$= 2 \sum_{n=1}^{+\infty} L_{-n} L_{n} + L_{0}^{2} - \frac{c+2}{12} L_{0} + \frac{c(5c+22)}{2880}.$$
(3.13)

The result matches that one of [24]. The technique we have described here in principle can be applied to a calculation of zero modes of any product of local operators.

4 Quasi-primaries

In this section we list the explicit expressions for the zero modes of all quasi-primaries from the vacuum family with the dimension less or equal to eight. Up to dimension nine all quasi-primaries have even dimension. There is a unique operator of dimension zero—the identity operator, which is a primary. There is also a unique quasi-primary at level

two, the stress-energy tensor T. Its zero mode is the CFT Hamiltonian — first KdV charge $Q_1 = L_0 - c/24$. At level two there is also a unique quasi-primary

$$T_2 = (TT) - \frac{3}{10}\partial^2 T.$$
 (4.1)

Its zero mode is the first non-trivial KdV charge Q_3 given by (3.13). At all other levels the quasi-primaries are not unique and we organize them by dimension, nested powers of T and orthogonality of Zamolodchikov metric.

4.1 Quasi-primaries of dimension 6

At level four there are two quasi-primaries,

$$\mathcal{B} = (\partial T \partial T) - \frac{4}{5} (\partial^2 T T) - \frac{1}{42} \partial^4 T, \tag{4.2}$$

and

$$\mathcal{D} = (T(TT)) - \frac{9}{10}(\partial^2 TT) - \frac{1}{28}\partial^4 T + \frac{93}{70c + 29}\mathcal{B}.$$
 (4.3)

Zero mode of their combination $\mathcal{D} - \frac{5(43+14c)}{2(29+70c)}\mathcal{B}$ is the KdV charge Q_5 , which was found explicitly in [24]. To find the explicit form of \mathcal{B}_0 and \mathcal{D}_0 we introduce the "building block"

$$(\partial T \partial T)_0 = -(\partial^2 T T)_0 = 2 \sum_{n=1}^{\infty} n^2 \tilde{L}_{-n} \tilde{L}_n + \frac{\tilde{L}_0}{60} - \frac{c}{3024}, \tag{4.4}$$

which is different from the quasi-primary \mathcal{B} by a total derivative. Therefore

$$\mathcal{B}_0 = \frac{9}{5} (\partial T \partial T)_0. \tag{4.5}$$

Similarly, to calculate \mathcal{D}_0 we introduce

$$(T(TT))_{0} = \sum_{k,l=0}^{\infty} \tilde{L}_{-k-l} \tilde{L}_{k} \tilde{L}_{l} + 2 \sum_{k=1,l=0}^{\infty} \tilde{L}_{-k} \tilde{L}_{k-l} \tilde{L}_{l} + \sum_{k,l=1}^{\infty} \tilde{L}_{-k} \tilde{L}_{-l} \tilde{L}_{k+l} - \sum_{n=1}^{\infty} \tilde{L}_{-n} \tilde{L}_{n} - \frac{\tilde{L}_{0}^{2}}{2} + \frac{c\tilde{L}_{0}}{480} + \frac{\tilde{L}_{0}}{15} - \frac{c}{3024},$$

$$(4.6)$$

such that

$$\mathcal{D}_0 = (T(TT))_0 + \frac{9}{10}(\partial T \partial T)_0. \tag{4.7}$$

4.2 Quasi-primaries of dimension 8

At level eight there are three quasi-primaries,

$$\mathcal{E} = (\partial^2 T \partial^2 T) - \frac{10}{9} (\partial^3 T \partial T) + \frac{10}{63} (\partial^4 T T) - \frac{1}{324} \partial^6 T, \tag{4.8}$$

$$\mathcal{H} = (\partial T(\partial TT)) - \frac{4}{5}(\partial^2 T(TT)) + \frac{2}{15}(\partial^3 T\partial T) - \frac{3}{70}(\partial^4 TT) + \frac{9(140c + 83)}{50(105c + 11)}\mathcal{E}, \quad (4.9)$$

and

$$\mathcal{I} = (T(T(TT))) - \frac{9}{5}(\partial^2 T(TT)) + \frac{3}{10}(\partial^3 T \partial T) + \frac{81(35c - 51)}{100(105c + 11)}\mathcal{E} + \frac{12(465c - 127)}{5c(210c + 661) - 251}\mathcal{H}.$$
(4.10)

The "building block" for calculating \mathcal{E}_0 is

$$(\partial^2 T \partial^2 T)_0 = -(\partial^3 T \partial T)_0 = (\partial^4 T T)_0 = 2 \sum_{n=1}^{\infty} n^4 \tilde{L}_{-n} \tilde{L}_n - \frac{\tilde{L}_0}{126} + \frac{c}{2880}, \tag{4.11}$$

And therefore,

$$\mathcal{E}_0 = \frac{143}{63} (\partial^2 T \partial^2 T)_0. \tag{4.12}$$

There are two "building blocks" for \mathcal{H} :

$$(\partial T(\partial TT))_{0} = -\sum_{k=0,l=0}^{\infty} kl\tilde{L}_{-k-l}\tilde{L}_{k}\tilde{L}_{l} + 2\sum_{k=1,l=0}^{\infty} kl\tilde{L}_{-k}\tilde{L}_{k-l}\tilde{L}_{l}$$

$$-\sum_{k,l=1}^{\infty} kl\tilde{L}_{-k}\tilde{L}_{-l}\tilde{L}_{k+l} + \frac{1}{12}(\partial T\partial T)_{0} + \frac{1}{30}\sum_{n=1}^{\infty} \tilde{L}_{-n}\tilde{L}_{n} + \frac{\tilde{L}_{0}^{2}}{60} - \frac{(5c+93)\tilde{L}_{0}}{15120} + \frac{113c}{1814400}$$

$$(4.13)$$

and

$$(\partial^{2}T(TT))_{0} = -\sum_{k,l=0}^{\infty} l^{2}\tilde{L}_{-k-l}\tilde{L}_{k}\tilde{L}_{l} - \sum_{k=1,l=0}^{\infty} (k^{2} + l^{2})\tilde{L}_{-k}\tilde{L}_{k-l}\tilde{L}_{l}$$

$$-\sum_{k=1,l=1}^{\infty} k^{2}\tilde{L}_{-k}\tilde{L}_{-l}\tilde{L}_{k+l} + \frac{7}{120}\sum_{n=1}^{\infty} \tilde{L}_{-n}\tilde{L}_{n} - \frac{\tilde{L}_{0}^{2}}{30} + \frac{c\tilde{L}_{0}}{1512} + \frac{61\tilde{L}_{0}}{7560} - \frac{131c}{1814400}.$$

$$(4.14)$$

The zero mode of \mathcal{H} is

$$\mathcal{H}_0 = (\partial T(\partial TT))_0 - \frac{4}{5}(\partial^2 T(TT))_0 + \frac{7035c + 13652}{110250c + 11550}(\partial^2 T\partial^2 T)_0. \tag{4.15}$$

The expression for $(T(T(TT)))_0$ is too bulky to write it twice. We do not write it explicitly here, but simply mention that it can be obtained from the Virasoro algebra expression for Q_7 , which we give explicitly in the next section, by subtracting $(\partial^2 T \partial^2 T)_0$ and $(T(\partial T \partial T))_0$ with proper coefficients, see equation (5.1). Thus, in lieu of $(T(T(TT)))_0$ we give explicitly the expression for $(T(\partial T \partial T))_0$,

$$(T(\partial T\partial T))_{0} = \sum_{k,l=1}^{\infty} (k+l)l\tilde{L}_{-k}\tilde{L}_{-l}\tilde{L}_{k+l} + \sum_{l=0,k=1}^{\infty} (k-l)k\tilde{L}_{-k}\tilde{L}_{k-l}\tilde{L}_{l}$$

$$+ \sum_{k,l=0}^{\infty} (k+l)k\tilde{L}_{-k-l}\tilde{L}_{k}\tilde{L}_{l} + \sum_{k=0,l=1}^{\infty} (k-l)k\tilde{L}_{-l}\tilde{L}_{l-k}\tilde{L}_{k}$$

$$- \frac{7}{6} \sum_{n=1}^{\infty} n^{2}\tilde{L}_{-n}\tilde{L}_{n} + \frac{1}{30} \sum_{n=1}^{\infty} \tilde{L}_{-n}\tilde{L}_{n} + \frac{\tilde{L}_{0}^{2}}{60} + \frac{c\tilde{L}_{0}}{3024} - \frac{\tilde{L}_{0}}{135} + \frac{13c}{86400}.$$

$$(4.16)$$

Finally, for the last quasi-primary at level 8 we have

$$\mathcal{I}_{0} = (T(T(TT)))_{0} - \frac{9}{5}(\partial^{2}T(TT))_{0} - \frac{3}{10}(\partial^{2}T\partial^{2}T)_{0} + \frac{81(35c - 51)}{100(105c + 11)}\mathcal{E}_{0} + \frac{12(465c - 127)}{5c(210c + 661) - 251}\mathcal{H}_{0}.$$
(4.17)

This concludes the list of zero modes of the quasi-primary operators at level 8.2

5 Expression for Q_7

In this section we present the expression for Q_7 in terms of Virasoro generators, which we calculated by applying the technique described above. Then we test our result by showing that it is consistent with the known spectrum Q_7 at the leading 1/c order. We further check at that our expression for Q_7 vanishes as an operator for the (9,2) minimal model at the first dozen of descendant levels, as is predicted in [24, 27]. Finally, we will show how to use commutativity of qKdV charges, known results about spectrum in 1/c expansion and the constraints from the minimal models to get a shortcut for the expression (5.2).

5.1 The result

 Q_7 is the zero mode of the operator

$$J_8 = (T(T(TT))) + \frac{c+2}{3}(T(\partial T \partial T)) + \frac{2c^2 - 17c - 42}{360}(\partial^2 T \partial^2 T). \tag{5.1}$$

In principle J_8 (and higher densities J_{2n}) can be determined by requiring commutativity of Q_7 with Q_5 and Q_3 , however, the calculation is quite involved. Alternatively, one can use the expression for the thermal correlation function $\langle Q_{2n-1} \rangle$, which can be calculated using other means and fix the coefficients of J_{2n} in this way. This was done for J_8 , J_{10} and J_{12} in [27]. However, for J_{14} and higher densities the number of independent coefficients becomes too large to be uniquely fixed from the form of $\langle Q_{2n-1} \rangle$ alone.

We find the following expression for Q_7 in terms of the Virasoro algebra generators

$$Q_{7} = \sum_{k,l,m=1}^{\infty} L_{-k}L_{-l}L_{-m}L_{k+l+m} + \sum_{k,l,m=0}^{\infty} L_{-k-l-m}L_{k}L_{l}L_{m}$$

$$+ 3\sum_{k,l=1}^{\infty} L_{-k}L_{-l}L_{k+l-m}L_{m} + 3\sum_{k=1}^{\infty} L_{-k}L_{k-l-m}L_{l}L_{m}$$

$$+ \frac{8+c}{3} \left[\sum_{k,l=1}^{\infty} (k+l)lL_{-k}L_{-l}L_{k+l} + \sum_{k=1}^{\infty} (k-l)kL_{-k}L_{k-l}L_{l} \right]$$

There are three quasi-primary operators at level 8, denoted \mathcal{E}, H, I , while we lilts five expressions. This is because two operators are linearly dependent with others, $(\partial T(\partial TT))_0 - (T(\partial T\partial T))_0 + \frac{1}{3}(\partial^2 T\partial^2 T)_0 + \frac{c}{1512}\tilde{L}_0 = 0$, $(\partial T(\partial TT))_0 + \frac{1}{2}(\partial^2 T(TT))_0 + \frac{1}{12}(\partial^2 T\partial^2 T)_0 + \frac{1}{12}\mathcal{B} - \frac{1}{32}Q_3 + \frac{1}{32}\tilde{L}_0^2 - \frac{\tilde{L}_0}{192} + \frac{c}{46080} = 0$. We thank A. Kakkar and S. Sugishita for discussing this point.

$$+\frac{8+c}{3}\left[\sum_{k,l=0}^{\infty}(k+l)kL_{-k-l}L_{k}L_{l}+\sum_{k=0}^{\infty}(k-l)kL_{-l}L_{l-k}L_{k}\right]$$

$$+\sum_{n=1}^{\infty}\left(\frac{c^{2}-c-141}{90}n^{4}-\frac{7c+59}{18}n^{2}\right)L_{-n}L_{n}-\left(\frac{1}{48}c^{2}+\frac{53}{360}c+\frac{19}{90}\right)\tilde{Q}_{3}$$

$$-\left(\frac{1}{6}c+1\right)\tilde{Q}_{5}-\frac{c+6}{6}L_{0}^{3}+\frac{15c^{2}+194c+568}{1440}L_{0}^{2}$$

$$-\frac{(c+2)(c+10)(3c+28)}{10368}L_{0}+\frac{c(3c+46)(25c^{2}+426c+1400)}{24883200}.$$
(5.2)

This is one of the main results of this paper. Here \tilde{Q}_3 and \tilde{Q}_5 are defined as parts of Q_3 and Q_5 which annihilate the primary states, $\tilde{Q}_3 |\Delta\rangle = 0$ and $\tilde{Q}_5 |\Delta\rangle = 0$, namely,

$$\tilde{Q}_3 = 2\sum L_{-n}L_n,\tag{5.3}$$

and

$$\tilde{Q}_{5} = \sum_{k,l=0}^{\infty} L_{-k-l} L_{k} L_{l} + 2 \sum_{k=1,l=0}^{\infty} L_{-k} L_{k-l} L_{l} + \sum_{k,l=1}^{\infty} L_{-k} L_{-l} L_{k+l} + \sum_{k=1}^{\infty} \left(\frac{c+2}{6} n^{2} - \frac{c}{4} - 1 \right) L_{-n} L_{n} - L_{0}^{3}.$$

$$(5.4)$$

The expression (5.4) can also be represented in a slightly different way [24] using the following identity,

$$\sum_{k,l=0}^{\infty} L_{-k-l}L_kL_l + 2\sum_{k=1,l=0}^{\infty} L_{-k}L_{k-l}L_l + \sum_{k,l=1}^{\infty} L_{-k}L_{-l}L_{k+l}$$

$$= \sum_{n_1+n_2+n_3=0} : L_{n_1}L_{n_2}L_{n_3} : +\frac{3}{2}\sum_{n=1}^{\infty} n^2L_{-n}L_n + \frac{3}{2}\sum_{r=1}^{\infty} L_{1-2r}L_{2r-1}.$$
(5.5)

5.2 The consistency check: 1/c expansion and the (9, 2) minimal model

In this section we perform two different consistency checks of the Virasoro algebra expression for Q_7 (5.2).

In an upcoming paper [28] we have calculated the spectrum of all KdV charges using semi-classical quantization at the first few orders in 1/c expansion. This calculation did not rely on the explicit form of Q_{2k-1} in terms of the Virasoro algebra generators, and hence can be used to cross-check our result.

The spectrum in 1/c expansion can be parametrized by "quantum numbers" of the conventional basis in the space of descendants, namely

$$|\{m_i\}, \Delta\rangle = L_{-m_1} \dots L_{-m_k} |\Delta\rangle. \tag{5.6}$$

Each set $\{m_i\}$ can be rewritten using the so-called boson representation. Namely, each set $\{m_i\}$ can be parametrized by the set of integers $\{n_k\}$, where n_k is the number of times

natural number k appears in $\{m_i\}$. Therefore, for any integer p,

$$\sum_{i} m_i^p = \sum_{k=1}^{\infty} k^p n_k, \tag{5.7}$$

Leading 1/c spectrum of Q_7 in these notations is given by $Q_7 |\lambda\rangle = \lambda |\lambda\rangle$, where

$$\begin{split} \lambda &= \Delta'^4 + \Delta'^3 \left(28 \sum_k n_k k - 1\right) + \Delta'^2 c \left(\frac{7}{3} \sum_k n_k k^3 + \frac{7}{720}\right) \\ &+ \Delta' c^2 \left(\frac{7}{90} \sum_k n_k k^5 - \frac{1}{6480}\right) + c^3 \left(\frac{1}{1080} \sum_k n_k k^7 + \frac{1}{518400}\right) \\ &+ \Delta'^2 \left(98 n^2 - 77 \sum_k n_k^2 k^2 + \frac{259}{3} \sum_k n_k k^3 - 77 \sum_k n_k k^2 - \frac{14}{3} \sum_k n_k k + \frac{71}{180}\right) \\ &+ \Delta' c \left(\frac{98}{15} \sum_{k,l} k^3 l n_k n_l - \frac{56}{15} \sum_k n_k^2 k^4 + \frac{63}{25} \sum_k n_k k^5 \right. \\ &- \frac{56}{15} \sum_k n_k k^4 - \frac{7}{90} \sum_k n_k k^3 + \frac{49}{1800} \sum_k n_k k - \frac{23}{4320}\right) \\ &+ c^2 \left(\frac{7}{180} \sum_{k,l} n_k n_l k^3 l^3 + \frac{7}{90} \sum_{k,l} n_k n_l k^5 l - \frac{7}{120} \sum_k n_k^2 k^6 + \frac{127}{5400} \sum_k n_k k^7 \right. \\ &- \frac{7}{120} \sum_k n_k k^6 + \frac{7}{21600} \sum_k n_k k^3 - \frac{1}{6480} \sum_k n_k k + \frac{103}{2073600}\right) \\ &- 504 \frac{\Delta'^3}{c} \left(\sum_k n_k^2 - 2 \sum_k n_k k + \sum_k n_k\right) + O(c) \end{split}$$

Using computer algebra one can check explicitly at first dozen of descendant levels that this spectrum matches with the one following from (5.2).

Another check is provided by [24, 27], which shows that for the minimal models (2n + 3, 2), $n \leq 1$, the qKdV charges Q_k with k divisible by 2n + 1 vanish as operators. Hence Q_7 should vanish as an operator in the minimal model (9, 2), with n = 3 and central charge c = -46/3. This minimal model includes primaries with the following dimensions $\{\Delta_k\} = \{0, -1/3, -2/3, -5/9\}$. Using computer algebra we have verified that Q_7 vanishes for all non-zero states of this model up to the descendant level twelve.

5.3 A shortcut

In this subsection we show how to get the Virasoro algebra expression for Q_7 without full explicit calculation of all involved commutators by exploiting the restrictions from the commutativity of qKdV charges, 1/c expansion and the (9,2) minimal model. Our goal here will be to understand what kind of terms may appear in the Virasoro algebra expression of Q_7 and then fix the coefficients using the constraints.

We start from the last term of J_8 (5.1):

$$A_8 = (\partial^2 T \partial^2 T). (5.9)$$

Substiting A_8 into (2.10)–(2.12) we obtain

$$\oint \frac{dw}{2\pi} A_8 \sim \sum_{n=1}^{\infty} n^4 L_{-n} L_n + comm,$$
(5.10)

where *comm* comes from the integral of the commutator in (2.10). Expressing the commutator in terms of Virasoro generators we get $[\partial^2 T, \partial^2 T] \sim [L_n, L_m] \sim L_k$ due to Virasoro algebra. It is easy to see that the final answer for the zero mode should contain only such operators that map states of level k to the states of level k. That means that after integrating the commutator we can only get some function of L_0 . Therefore,

$$\oint \frac{dw}{2\pi} A_8 \sim \sum_{n=1}^{\infty} n^4 L_{-n} L_n + f(L_0).$$
(5.11)

Now we turn to the next term

$$B_8 = (T(\partial T \partial T)). \tag{5.12}$$

The term B_8 contains nested normal ordering. We will deal with it subsequently,

$$(\partial T \partial T)(w) = \partial T \partial T_{+} + \partial T_{-} \partial T + \oint \frac{dz}{2\pi} \sum_{k=0}^{\infty} c_{k} \left(e^{i(z-w)} - 1 \right)^{k} [\partial T(z), \partial T(w)]. \tag{5.13}$$

One can calculate the commutator and perform the integration (5.13) explicitly but we just notice that the result of the integration can only contain terms with one stress-energy tensor T or its derivatives and the terms like TT or $T\partial T$ are not present due to Virasoro algebra, which means

$$(\partial T \partial T)(w) = \partial T(w) \partial T_{+}(w) + \partial T_{-}(w) \partial T(w) + f(T(w), \partial T(w), \partial^{2} T(w)), \tag{5.14}$$

where $f(T, \partial T, \partial^2 T)$ is some linear function of its arguments. Substituting (5.13) into B_8 we get

$$(T(\partial T\partial T)) = T_{-}\partial T\partial T_{+} + \partial T\partial T_{+}T_{+} + T_{-}\partial T_{-}\partial T + \partial T_{-}\partial TT_{+} + comm.$$
 (5.15)

Here comm again denotes some expression associated with the commutators, which we will not calculate explicitly. But let us notice again that it contains at most two stress-tensors or its derivatives. The only expression that survives after integration of such terms is proportional to $\sim L_{-n}L_n$, namely

$$\oint dz \partial^{\alpha} T \partial^{\beta} T \sim \sum_{-\infty}^{+\infty} n^{\alpha+\beta} L_{-n} L_{n}.$$
(5.16)

Counting the amount of dervatives and integrating (5.15) we obtain

$$\oint \frac{dw}{2\pi} B_8 = \sum_{k,l=1}^{\infty} (k+l) l L_{-k} L_{-l} L_{k+l} + \sum_{\substack{k=1\\l=0}}^{\infty} (k-l) k L_{-k} L_{k-l} L_l
+ \sum_{k,l=0}^{\infty} (k+l) k L_{-k-l} L_k L_l + \sum_{\substack{k=0\\l=1}}^{\infty} (k-l) k L_{-l} L_{l-k} L_k
+ \sum_{n=1}^{\infty} (\alpha n^4 + \beta n^2 + \gamma) L_{-n} L_n + f(L_0),$$
(5.17)

where α , β , γ are some c-dependent coefficients and $f(L_0)$ is some function of c and L_0 . In the same manner one can deal with the term (T(T(TT))). As a result we get an expression for Q_7 with some coefficients to fix. The expression is

$$Q_{7} = \sum_{k,l,m=1}^{\infty} L_{-k}L_{-l}L_{-m}L_{k+l+m} + \sum_{k,l,m=0}^{\infty} L_{-k-l-m}L_{k}L_{l}L_{m}$$

$$+ 3 \sum_{k,l=1}^{\infty} L_{-k}L_{-l}L_{k+l-m}L_{m} + 3 \sum_{\substack{k=1\\l,m=0}}^{\infty} L_{-k}L_{k-l-m}L_{l}L_{m}$$

$$+ \frac{8+c}{3} \left[\sum_{k,l=1}^{\infty} (k+l)lL_{-k}L_{-l}L_{k+l} + \sum_{\substack{k=1\\l=0}}^{\infty} (k-l)kL_{-k}L_{k-l}L_{l} \right], \qquad (5.18)$$

$$+ \frac{8+c}{3} \left[\sum_{k,l=0}^{\infty} (k+l)kL_{-k-l}L_{k}L_{l} + \sum_{\substack{k=0\\l=1}}^{\infty} (k-l)kL_{-l}L_{l-k}L_{k} \right]$$

$$+ \sum_{l=0}^{\infty} (\alpha n^{4} + \beta n^{2}) L_{-n}L_{n} + \gamma \tilde{Q}_{3} + \delta \tilde{Q}_{5} + f(L_{0})$$

where α , β , γ and δ are the coefficients dependent on central charge and $f(L_0)$ is some polynomial of L_0 and central charge.

The term $f(L_0)$ determines the value of Q_7 on a primary state. That has been previously calculated in the appendix B of [29]. The coefficients α and β can be found from the commutativity constraint $[Q_3, Q_7] = 0$. From 1/c expansion of the spectrum (5.8) we see that δ is at most linear polynomial in c, $\delta = \delta_1 c + \delta_2$ and γ is at most quadratic polynomial in c, $\gamma = \gamma_1 c^2 + \gamma_2 c + \gamma_3$. The coefficients δ_1 , δ_2 and γ_1 , γ_2 can be extracted directly from (5.8) and, finally, γ_3 can be fixed by requiring that Q_7 vanishes acting on any descendant state of the (9,2) minimal model.

6 Conclusions

In this paper we pedagogically developed and presented the machinery of calculating the zero modes of local operators in a 2d CFT on a cylinder. We focused on the situation when the operators are from the vacuum family, i.e. are built from the powers of stress-energy tensor and its derivatives. We have calculated explicit expressions in terms of the Virasoro algebra generators for all quasi-primaries with the dimension of less or equal than eight. We have also calculated the explicit expression for the KdV charge Q_7 , thus bringing the number of explicitly known charges to three (excluding Q_1 , which is trivial). The explicit formulae obtained in this paper can be used to study spectral properties of the qKdV hierarchy and further investigate Eigenstate Thermalization Hypothesis in 2d CFTs, in particular by means of computer algebra.

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References

- [1] J.M. Deutsch, Quantum statistical mechanics in a closed system, Phys. Rev. A 43 (1991) 2046.
- [2] M. Srednicki, Chaos and quantum thermalization, Phys. Rev. E 50 (1994) 888.
- [3] M. Rigol, V. Dunjko and M. Olshanii, Thermalization and its mechanism for generic isolated quantum systems, Nature 452 (2008) 854.
- [4] E. Ilievski, M. Medenjak and T.z. Prosen, Quasilocal conserved operators in the isotropic Heisenberg spin-1/2 chain, Phys. Rev. Lett. 115 (2015) 120601 [arXiv:1506.05049] [INSPIRE].
- [5] M. Rigol, V. Dunjko, V. Yurovsky and M. Olshanii, Relaxation in a completely integrable many-body quantum system: an ab initio study of the dynamics of the highly excited states of 1d lattice hard-core bosons, Phys. Rev. Lett. 98 (2007) 050405.
- [6] A.C. Cassidy, C.W. Clark and M. Rigol, Generalized thermalization in an integrable lattice system, Phys. Rev. Lett. 106 (2011) 140405.
- [7] L. Vidmar and M. Rigol, Generalized Gibbs ensemble in integrable lattice models, J. Stat. Mech. 2016 (2016) 064007 [arXiv:1604.03990].
- [8] L. D'Alessio, Y. Kafri, A. Polkovnikov and M. Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics, Adv. Phys. 65 (2016) 239 [arXiv:1509.06411] [INSPIRE].
- [9] J. de Boer and D. Engelhardt, Remarks on thermalization in 2D CFT, Phys. Rev. D 94 (2016) 126019 [arXiv:1604.05327] [INSPIRE].
- [10] J. Sonner and M. Vielma, Eigenstate thermalization in the Sachdev-Ye-Kitaev model, JHEP 11 (2017) 149 [arXiv:1707.08013] [INSPIRE].

- [11] P. Basu, D. Das, S. Datta and S. Pal, Thermality of eigenstates in conformal field theories, Phys. Rev. E 96 (2017) 022149 [arXiv:1705.03001] [INSPIRE].
- [12] N. Lashkari, A. Dymarsky and H. Liu, Eigenstate thermalization hypothesis in conformal field theory, J. Stat. Mech. 1803 (2018) 033101 [arXiv:1610.00302] [INSPIRE].
- [13] N. Lashkari, A. Dymarsky and H. Liu, *Universality of quantum information in chaotic CFTs*, *JHEP* **03** (2018) 070 [arXiv:1710.10458] [INSPIRE].
- [14] T. Faulkner and H. Wang, Probing beyond ETH at large c, JHEP 06 (2018) 123 [arXiv:1712.03464] [INSPIRE].
- [15] F.-L. Lin, H. Wang and J.-J. Zhang, Thermality and excited state Rényi entropy in two-dimensional CFT, JHEP 11 (2016) 116 [arXiv:1610.01362] [INSPIRE].
- [16] W.-Z. Guo, F.-L. Lin and J. Zhang, Note on ETH of descendant states in 2D CFT, JHEP 01 (2019) 152 [arXiv:1810.01258] [INSPIRE].
- [17] S. He, F.-L. Lin and J.-J. Zhang, Dissimilarities of reduced density matrices and eigenstate thermalization hypothesis, JHEP 12 (2017) 073 [arXiv:1708.05090] [INSPIRE].
- [18] A. Dymarsky and K. Pavlenko, Generalized Gibbs ensemble of 2d CFTs at large central charge in the thermodynamic limit, JHEP 01 (2019) 098 [arXiv:1810.11025] [INSPIRE].
- [19] A. Dymarsky and K. Pavlenko, Exact generalized partition function of 2D CFTs at large central charge, JHEP 19 (2020) 077 [arXiv:1812.05108] [INSPIRE].
- [20] A. Dymarsky and K. Pavlenko, Generalized eigenstate thermalization hypothesis in 2D conformal field theories, Phys. Rev. Lett. 123 (2019) 111602 [arXiv:1903.03559] [INSPIRE].
- [21] A. Maloney, G.S. Ng, S.F. Ross and I. Tsiares, Generalized Gibbs ensemble and the statistics of KdV charges in 2D CFT, JHEP 03 (2019) 075 [arXiv:1810.11054] [INSPIRE].
- [22] S. Datta, P. Kraus and B. Michel, Typicality and thermality in 2d CFT, JHEP 07 (2019) 143 [arXiv:1904.00668] [INSPIRE].
- [23] M. Beşken, S. Datta and P. Kraus, Quantum thermalization and Virasoro symmetry, J. Stat. Mech. 2006 (2020) 063104 [arXiv:1907.06661] [INSPIRE].
- [24] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz, Commun. Math. Phys. 177 (1996) 381 [hep-th/9412229] [INSPIRE].
- [25] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, Integrable structure of conformal field theory. II. Q operator and DDV equation, Commun. Math. Phys. 190 (1997) 247 [hep-th/9604044] [INSPIRE].
- [26] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, Integrable structure of conformal field theory. III. The Yang-Baxter relation, Commun. Math. Phys. 200 (1999) 297 [hep-th/9805008] [INSPIRE].
- [27] A. Maloney, G.S. Ng, S.F. Ross and I. Tsiares, Thermal correlation functions of KdV charges in 2D CFT, JHEP 02 (2019) 044 [arXiv:1810.11053] [INSPIRE].
- [28] A. Dymarsky, K. Pavlenko and S. Sugishito, Spectrum of the quantum KdV hierarchy from the quasiclassical quantization, in preparation.
- [29] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, Integrable quantum field theories in finite volume: excited state energies, Nucl. Phys. B 489 (1997) 487 [hep-th/9607099] [INSPIRE].