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Manifolds of amphiphilic bilayers: Stability up to the boundary ☆

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Abstract

We consider the mass preserving L^2-gradient flow of the strong scaling of the functionalized Cahn Hilliard gradient flow and establish the nonlinear stability of a manifold comprised of quasi-equilibrium bilayer distributions up to the manifold's boundary. In the limit of thin but non-zero interfacial width, $\varepsilon \ll 1$, the bilayer manifold is parameterized by meandering modes that describe the interfacial evolution. The normal coercivity of the manifold is limited by "pearling" modes that control the structure of the profile near the interface, these are weakly damped and can lead to the dynamic rupture of the interface. Amphiphilic interfaces may decrease energy by either lengthening or shortening, depending upon the amphiphilic mass distributed in the bulk. We introduce an implicitly defined parameterization of the interfacial shape that uncouples the length change from the parameters describing the shape and introduce a nonlinear projection onto the manifold from a surrounding neighborhood. The bilayer manifold has asymptotically large but finite dimension tuned to maximize normal coercivity while preserving the wave-number gap between the meandering and the pearling modes. Modulo a pearling stability assumption, we show that the manifold attracts nearby orbits into a tubular neighborhood about itself so long as the interfacial shape remains sufficiently smooth and far from self-intersection. In a companion paper, [8], we identify open sets of initial data whose orbits converge to circular equilibrium after a significant transient, and derive a singularly perturbed interfacial evolution comprised of motion against curvature regularized by an asymptotically weak Willmore term.

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1. Introduction

The functionalized Cahn-Hilliard (FCH) free energy models the free energy of mixtures of amphiphilic molecules and solvent. Amphiphilic molecules are formed by chemically bonding two components whose individual interactions with the solvent are energetically favorable and unfavorable, respectively. When blended with the solvent, amphiphilic molecules have a propensity to phase separate, forming thin amphiphilic rich domains that are generically the thickness of two molecules in at least one direction. On a periodic domain $\Omega \subset \mathbb{R}^2$ the FCH free energy is given in terms of the volume fraction $u - b_-$ of the amphiphilic molecule

$$\mathcal{F}(u) := \int_\Omega \frac{\varepsilon}{2}\left(\Delta u - \frac{1}{\varepsilon^2} W'(u)\right)^2 - \varepsilon^{p-1}\left(\frac{\eta_1}{2}|\nabla u|^2 + \frac{\eta_2}{\varepsilon^2} W(u)\right) \mathrm{d}x, \tag{1.1}$$

where $W : \mathbb{R} \mapsto \mathbb{R}$ is a smooth tilted double well potential with local minima at $u = b_\pm$ with $b_- < b_+$, $W(b_-) = 0 > W(b_+)$, and $W''(b_-) > 0$. The state $u \equiv b_-$ corresponds to pure solvent, while $u \equiv b_+$ denotes a maximum packing of amphiphilic molecules. The system parameters $\eta_1 > 0$ and η_2 characterize key structural properties of the amphiphilic molecules. The small positive parameter $\varepsilon \ll 1$ characterizes the ratio of the length of the molecule to the domain size, and $p = 1$ or 2 selects a balance between the Willmore-type residual of the dominant squared term and the amphiphilic structure terms. We select the strong scaling $p = 1$, in which the amphiphilic structure terms dominate the Willmore residual. The FCH energy was introduced in [16], motivated by the work of Gommper [17–19]. In particular the form of the energy in [20] corresponds to the FCH with $\eta_1 = 0$ and $\varepsilon^p \eta_2 = -f_0$, where f_0 is a key bifurcation parameter. A central feature of the functionalized Cahn-Hilliard energy (1.1) is that its approximate minima include vast families of saddle points of a Cahn-Hilliard type energy. Within the FCH the competitors for minima include codimension one bilayer, codimension two pores, and codimension three micelles that are the building blocks of many biologically relevant organelles [14,15,26].

The chemical potential of $\mathcal{F}$, denoted $\mathrm{F} = \mathrm{F}(u)$, is a rescaling of its variational derivative

$$\mathrm{F}(u) := \varepsilon^3 \frac{\delta \mathcal{F}}{\delta u} = (\varepsilon^2 \Delta - W''(u))(\varepsilon^2 \Delta u - W'(u)) + \varepsilon^p(\eta_1 \varepsilon^2 \Delta u - \eta_2 W'(u)). \tag{1.2}$$

We take the strong, $p = 1$, scaling of the FCH and consider the mass-preserving L^2 gradient flow

$$\partial_t u = -\Pi_0 \mathrm{F}(u), \tag{1.3}$$

subject to periodic boundary conditions on $\Omega \subset \mathbb{R}^2$. Here Π_0 is the zero-mass projection given by

$$\Pi_0 f := f - \langle f \rangle_{L^2}, \tag{1.4}$$

where we have introduced the averaging operator

$$\langle f \rangle_{L^2} := \frac{1}{|\Omega|} \int_\Omega f \, \mathrm{d}x. \tag{1.5}$$

We consider the mass-preserving L^2 gradient flow of the strong scaling of the FCH free energy, (1.2)-(1.3), and construct a bilayer manifold, $\mathcal{M}_b$, with boundary contained in $H^2(\Omega)$ that is comprised of quasi-equilibrium of the system, called bilayer distributions. Each bilayer distribution is associated to an immersed interface in Ω, and varies predominantly through the ε-scaled signed distance to that interface. The bilayer manifold has a nonlinear projection that maps an open neighborhood of the bilayer manifold onto itself and decomposes functions u in the open neighborhood of the bilayer manifold into a point on the manifold (a bilayer distribution) and a perturbation that is orthogonal to the tangent plane of $\mathcal{M}_b$. The bilayer distribution is parameterized by a finite but asymptotically large set of "meander modes" that characterize the shape of its associated interface and a single bulk density parameter that characterizes the excess amphiphilic mass in the bulk. The orthogonal perturbation is further decomposed, through a linear projection, into an asymptotically large but finite dimensional set of "pearling modes" and an infinite dimensional set of "fast modes." The pearling modes modify the internal structure of the bilayer distribution near its interface and are weakly damped, subject to a pearling stability condition. The fast modes are uniformly damped under the flow. The meander modes perturb the shape of a predefined base interface, so that $\mathcal{M}_b$ accommodates interfaces whose range of shapes is independent of ε.

The bilayer manifold is defined as a graph over a bounded domain of meander modes. We show that initial data that start asymptotically close to $\mathcal{M}_b$ will remain close unless the meander modes become sufficiently large that they hit the boundary of the domain. This domain is selected to insure that the associated interfaces do not self-intersect, that their curvatures remain uniformly bounded, independent of ε, and that the pearling stability condition holds uniformly. Establishing the stability of the manifold up to the meander mode boundary requires two classes of sharp bounds. The first are upper bounds on the coupling of the evolution of the interfacial geometry, characterized by the meander modes, upon the pearling and the fast modes. The second are lower bounds on the coercivity of the second variation of the energy evaluated at points on the manifold when restricted to act on the pearling and meander spaces.

In a companion paper, [8], we rigorously analyze the evolution of the interfaces. In particular we identify an open set of initial data of (1.3) whose projection defines interfaces that are sufficiently close to circular, and show that the evolving curvature of the interfaces satisfies the curvature bounds for all $t > 0$, and that after a transient in which the deviation of the interface from circularity may *grow* by an $o(1)$ amount, the interface ultimately converges to a nearly circular equilibrium. Together these results partially validate the formal results obtained in [7], where the authors applied multiscale analysis to the H^{-1} gradient flow of the strong FCH free energy. They considered the evolution of bilayer distribution with high density of amphiphilic material that separate bulk regions of low density via a codimension one interface Γ. On the ε^{-3} time scale they formally derived the evolution of the curvature κ of Γ

$$\partial_t \kappa = -(\Delta_s + \kappa^2) V, \tag{1.6}$$

in terms of the ε-scaled normal velocity

$$V = (\sigma(t) - \sigma_1^*)\kappa + \varepsilon \Delta_s \kappa + \text{H.O.T.}. \tag{1.7}$$

The normal velocity is proportional to the curvature κ through a time-dependent coefficient that can be positive or negative depending upon the initial data. Here Δ_s is the Laplace-Beltrami operator associated to Γ, and for simplicity we have omitted positive constants that are independent of time and ϵ. The bulk density parameter $\sigma = \sigma(t)$ controls the spatially constant density of amphiphilic material in the bulk. This couples strongly to the length of the interface in a relation that is determined by conservation of mass, which is particularly one of the key differences of the FCH model and the phase field model introduced in [11–13]. The critical value σ_1^*, is a constant depending only upon the system parameters η_1, η_2, and the well W. When the bulk density is above this critical value, $\sigma > \sigma_1^*$, the interface absorbs mass from the bulk, and moves *against* curvature, in a singularly perturbed meandering or buckling motion that is regularized by the higher order diffusion, Δ_s. This regime is called regularized curve lengthening (RCL), and the weak surface diffusion plays an essential role in the local existence. Conversely, when $\sigma < \sigma_1^*$, the interface releases mass to the bulk and contracts under a mean curvature driven flow (MCF). In both cases the flow drives σ towards σ_1^* and the curve attains an equilibrium length set by the mass of the initial data.

In the absence of a maximum principle for the fourth-order system (1.3), we use energy estimates and modulation methods. The modulation methods for extended manifolds, [27,5,21] consider the linearization of the flow about points on the manifold and establish lower bounds on decay rates based upon coercivity estimates of the linearization restricted to subspaces that are approximately tangent and approximately normal to the manifold. As a form of normal hyperbolicity they require a spectral gap between eigenvalues associated to the tangent plane of the manifold, the slow modes, and those associated to the normal direction, the fast modes. These results refine earlier estimates in [10,22,24], which introduced the slow space comprised of pearling and meander modes. In particular [24], conducted spectral analysis of the linearized operators restricted to the slow space and, modulo the pearling stability condition, used a slow space with dimension $O(\varepsilon^{-1/2})$ to establish a fast space coercivity that scales with $\sqrt{\varepsilon}$. However, these results are too rough to close our nonlinear estimates.

Our analysis requires two significant modifications. First, the ansatz that defines the manifold is constructed implicitly in the parameters that define the shape of the interface. A single parameter controls interfacial length, uncoupling those that control the shape and making the natural basis modes of the tangent plane of the ansatz substantially closer to orthogonal. This allows us to extend the size of the slow spaces while preserving the diagonal dominance of the correlation matrix obtained from restricting the linearization of the FCH equation to the slow space. We combine the implicit ansatz with higher-order corrections to the slow space to build the improved estimates for the slow-fast, and pearling-meander coupling. We enlarge the dimension of the slow space, to $O(\rho\varepsilon^{-1})$, where the spectral cut-off $\rho \ll 1$ is independent of ε. This yields a fast space coercivity that scales with ρ and is independent of ε. Indeed, the choice of the size of the slow space requires a delicate balance. A larger slow space allows for stronger coercivity of the fast space. However the pearling modes have asymptotically short in-plane wave-length, while the meander modes admit relatively long in-plane wave-length. *The asymptotically large gap between the in-plane wave lengths of the pearling and meander modes decreases the strength of their coupling by one order of magnitude in ε.*

It is illuminating to compare the estimates derived here for the bilayer manifold to the classic results that establish rigorous results for front evolution in the scalar Cahn-Hilliard (CH) equation, such as [1–4,25]. The bilayer distributions are not fronts. An immediate distinction is that the limiting curve motion for the FCH is singularly perturbed and ill-posed in the $\varepsilon \to 0^+$ limit, while the CH interfacial motion is locally well posed in this sharp interface limit. This requires us to fix $\varepsilon > 0$ small but nonzero, and to perform detailed analysis in the regions near the interface. A second distinction is that the bilayer manifold has asymptotically weak relaxation rates to perturbations that incite the pearling modes that regulate the width of the interface. The coercivity associated to fronts in CH is uniform with respect to ε. For the FCH the pearling modes have the capacity to destabilize the interfacial structure, modulating its width to the point that it can perforate. Their amplitude must be controlled through tight bounds on the coupling between the front evolution and the pearling modes. There is no analogue for these structural dynamics of the interface within the fronts of the scalar CH models.

The seminal result, [6], of Bates, Lu, and Zeng establishes the existence of a true invariant manifold for a general class of dynamical systems when they possess an approximately invariant manifold that is approximately hyperbolic under the flow. It is important to place the results here relative to the conditions of that work. The approximate normal hyperbolicity requires a decomposition into a stable, center, and unstable space, with the center space associated to the tangent plane of the manifold – here the meander modes. The rates of contractivity associated to the semiflow on the stable space must exceed those of the center space, associated to the tangent plane of the manifold, embodied in assumption (H3') eqn (2.7) of that work. In our application the stable space includes the pearling modes, whose linear semigroup generates weak contractivity, asymptotically weaker than the strongest contractive rates of the center space. We compensate for this difficulty by tuning the parameter ρ precisely to weaken the coupling between the meander modes and the pearling modes. It is plausible that the results of [6] can be adapted to this situation, but the details may be delicate.

The singular nature of the interface motion and the weak damping of the internal pearling modes generate significant technical obstacles whose resolution requires restrictions. The most striking of these is that the interfaces must be sufficiently close to a base point interface Γ_0 that is far from self intersection. Self intersection in the RCL regime is a real possibility. Numerical benchmark calculations have identified bulk parameter values for initial data that initiate the formation of defects within the bilayer distributions, [9]. These results show that an initial bulk density state σ that deviates from the scaled equilibrium σ_1^* by an $O(1)$ amount can lead to defect formation, suggesting that the restriction $|\sigma - \sigma_1^*| \ll 1$ we require here, see (5.54) and Lemma 3.4, is not far from optimal.

The remainder of this article is organized as follows. In Section 2, we present the local coordinates and estimates on the variation of the interface through the meander parameters. In particular we introduce the implicitly defined perturbed interfaces $\Gamma_{\mathbf{p}}$ in Definition 2.6 and show that they are well posed in Lemma 2.10. In Lemma 3.2 of Section 3 we construct the quasi-equilibrium bilayer distributions $\Phi_{\mathbf{p}}$ as the dressing (Definition 2.3) of the perturbed interface $\Gamma_{\mathbf{p}}$, and estimate their residual $\mathrm{F}(\Phi_{\mathbf{p}})$. The bilayer manifold $\mathcal{M}_b$ is presented in Definition 3.3 as the graph of the map $\mathbf{p} \mapsto \Phi_{\mathbf{p}}(\cdot; \sigma(\mathbf{p}))$ with the bulk density parameter σ slaved to constrain the mass of $\Phi_{\mathbf{p}}$. Section 4 introduces the slow space in Definition 4.1 and the spectral cut-off parameter ρ. The modified slow space is presented in Lemma 4.6, followed by a characterization of the spectrum of the operator $\Pi_0\mathbb{L}$ arising from the linearization of the flow (1.2) about $\Phi_{\mathbf{p}}$. In Lemma 4.9 we establish the $O(\varepsilon)$ weak coercivity of the pearling spaces modulo the pearling stability condition and in Theorem 4.11 we give sharp bounds on the pearling-meander and fast-slow coupling required for closure of the nonlinear estimates. In Theorem 4.13 we establish the strong coercivity for the fast modes in terms of the spectral cut-off parameter ρ. In Section 5, we define the bilayer manifold that includes the pearling modes, and the nonlinear projection onto the manifold. It concludes with the main result, Theorem 5.13, which establishes the nonlinear stability of the bilayer manifold up to its boundary. We emphasize that there are three small parameters used in this work, $\varepsilon_0, \rho, \delta > 0$. The first, ε_0 sets the upper bound on the size of the dominant small parameter ε. The spectral parameter $\rho > 0$ controls the dimension of slow spaces, while δ is a technical parameter used to close the nonlinear estimates. We first fix δ sufficiently small in Lemma 5.2, and then ρ sufficiently small in Theorem 4.11 and Lemma 5.12. The value ε_0 in set in terms of these fixed values in Theorem 5.13.

The companion paper [8] establishes the unconditional stability of the bilayer manifold built form a circular base point interface, and recovers the evolution of the meander modes and associated interfacial motion, as well as the scope of the transient and the rate of convergence to equilibrium.

1.1. Notation

We present some general notation.

(1) The symbol C generically denotes a positive constant whose value depends only on the system parameters η_1, η_2, the domain Ω, and geometric quantities of initial curve Γ_0. In particular its value is independent of ε, ρ, and δ, so long as they are sufficiently small. The value of C may vary line to line without remark. In addition, $A \lesssim B$ indicates that quantity A is less than quantity B up to a multiplicative constant C as above, and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. The notation $A \wedge B$ denotes the minimum of A and B. The expression $f = O(a)$ indicates the existence of a constant C, as above, and a norm $|\cdot|$ for which

$$|f| \leqslant C|a|.$$

We also use $f = O(a, b)$ for the case $f \leqslant C|a| + C|b|$.

(2) The quantity ν is a positive number, independent of ε, that denote an exponential decay rate. It may vary from line to line.

(3) If a function space $X(\Omega)$ is comprised of functions defined on the whole spatial domain Ω, we will drop the symbol Ω.

(4) We use $\mathbf{1}_E$ as the characteristic function of an index set $E \subset \mathbb{N}$, i.e. $\mathbf{1}_E(x) = 1$ if $x \in E$; $\mathbf{1}_E(x) = 0$ if $x \notin E$. We denote the usual Kronecker delta by

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

(5) For a vector $\mathbf{q} = (\mathrm{q}_j)_j$, we denote the norms

$$\|\mathbf{q}\|_{l^k} = \left(\sum_j |\mathrm{q}_j|^k\right)^{1/k}, \qquad \text{for } k \in \mathbb{N}^+,$$

and $\|\mathbf{q}\|_{l^\infty} = \max_j |\mathrm{q}_j|$. For a matrix $\mathbb{Q} = (\mathbb{Q}_{ij})_{ij}$ as a map from l^2 to l^2 has operator norm l^2_* defined by

$$\|\mathbb{Q}\|_{l^2_*} = \sup_{\{\|\mathbf{q}\|_{l^2}=1\}} \|\mathbb{Q}\mathbf{q}\|_{l^2}.$$

We write

$$\mathrm{q}_j = O(a)e_j, \quad \mathbb{Q}_{ij} = O(a)\mathbb{E}_{ij},$$

where $\boldsymbol{e} = (e_j)_j$ is a vector with $\|\boldsymbol{e}\|_{l^2} = 1$ or $\mathbb{E}$ is a matrix with operator norm $\|\mathbb{E}\|_{l^2_*} = 1$ to imply that $\|\mathbf{q}\|_{l^2} = O(a)$ or $\|\mathbb{Q}\|_{l^2_*} = O(a)$ respectively. See (2.47)-(2.48) of Notation 2.13 for usage.

(6) The matrix $e^{\theta\mathcal{R}}$ denotes rotation through the angle θ with the generator $\mathcal{R}$. More explicitly,

$$\mathcal{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e^{\theta\mathcal{R}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

2. Coordinates and preliminary estimates

In this section we recall the local coordinates associated to a general smooth interface and use them to define a finite dimensional family of perturbations of the interface. In particular we establish bounds controlling the variation of the interface in terms of the parameters.

2.1. The local coordinates

We consider a closed, smooth, non-intersecting curve $\Gamma \subset \mathbb{R}^2$ which divides $\Omega = \Omega_+ \cup \Omega_-$ into an exterior Ω_+ and an interior Ω_-. The interface Γ is given parametrically as

$$\Gamma = \{\boldsymbol{\gamma}(s) : s \in \mathscr{I} \subset \mathbb{R}\},$$

with tangent vector $\mathbf{T}(s) \in \mathbb{R}^2$

$$\mathbf{T}(s) := \boldsymbol{\gamma}'/|\boldsymbol{\gamma}'|. \tag{2.1}$$

Denoting the outer normal to Γ by $\mathbf{n}(s)$ we have the relations

$$\mathbf{T}' = \kappa\mathbf{n}, \quad \mathbf{n}' = -\kappa\mathbf{T}, \tag{2.2}$$

where $\kappa = \kappa(s)$ is the curvature of Γ at $\boldsymbol{\gamma}(s)$. By the implicit function theorem, there exists an open set $\mathcal{N}$ containing Γ such that for each $x \in \mathcal{N}$ we may write

$$x = \boldsymbol{\gamma}(s) + r\mathbf{n}(s) \tag{2.3}$$

where $r = r(x)$ is the well-known signed distance of the point x to the curve Γ and $s = s(x)$ is determined by the choice of parameterization $\boldsymbol{\gamma}$. In this neighborhood, we define the scaled signed distance $z = r/\varepsilon$ and "whiskers" of length ℓ:

$$w_\ell(s) := \{\boldsymbol{\gamma}(s) + \varepsilon z\mathbf{n}(s) : r \in [-\ell, \ell]\},$$

and the ℓ-reach of Γ,

$$\Gamma^\ell = \bigcup_{s \in \mathscr{I}} w_\ell(s). \tag{2.4}$$

In the following, (z, s) will be referred as the local coordinates near Γ.

2.1.1. Dressings

We say that a curve Γ is ℓ-far from self intersection if none of the whiskers of length ℓ intersect each other nor with $\partial\Omega$, and if the set Γ^ℓ contains all points of Ω whose distance to Γ is at most ℓ. We introduce the following class of curves and show they have a uniform distance from self-intersection.

Definition 2.1. Given $K, \ell > 0$ the class $\mathcal{G}^k_{K,\ell}$ consists of those closed embedding curves Γ whose parameterization $\boldsymbol{\gamma}$ satisfies (a) $\inf|\boldsymbol{\gamma}'(s)| \geqslant 1/4$, $\|\boldsymbol{\gamma}\|_{W^{k,\infty}(\mathscr{I})} \leqslant K$ and (b) if points two points on Γ satisfy $|s_1 - s_2|_{\mathscr{I}} > 1/(8K)$ then $|\boldsymbol{\gamma}(s_1) - \boldsymbol{\gamma}(s_2)| > \ell$. Here $|\cdot|_{\mathscr{I}}$ denotes the periodic distance $|s|_{\mathscr{I}} = \min\left\{\Big|s - |\mathscr{I}|k\Big| : k \in \mathbb{Z}\right\}$.

Lemma 2.2. *If $\tilde{\ell} < \ell$ then $\mathcal{G}^k_{K,\tilde{\ell}} \subset \mathcal{G}^k_{K,\ell}$. If $\ell \leqslant \pi/(4K)$ then every curve in $\mathcal{G}^2_{K,\ell}$ is ℓ-far from self-intersection.*

Proof. The first statement follows directly from the definition of $\mathcal{G}^k_{K,\ell}$. Pick $\Gamma \in \mathcal{G}^2_{K,\ell}$ with parameterization $\boldsymbol{\gamma}$. The reach Γ^ℓ contains the set of points whose distance to Γ is less than or equal to ℓ. Indeed, if $x \in \Omega$ lies within ℓ of Γ, then there exists a least one point $s \in \mathscr{I}$ such that $\boldsymbol{\gamma}(s)$ is the closest point on Γ to x. Since the tangent $\mathbf{T}$ has a smooth derivative, it follows that $(x - \boldsymbol{\gamma}(s)) \cdot \mathbf{T}(s) = 0$. If not, then $(|x - \boldsymbol{\gamma}(s)|^2)' \neq 0$, which contradicts $\boldsymbol{\gamma}(s)$ being the closest point on Γ to x. Consequently $x \in w_\ell(s) \subset \Gamma^\ell$. To see that the whiskers of length ℓ do not intersect consider two points $s_1, s_2 \in \mathscr{I}$. Since $|\mathbf{T}(s)| = 1$ and

$$|\mathbf{T}'(s)| = \left|\frac{\boldsymbol{\gamma}'' \cdot \mathbf{n}}{|\boldsymbol{\gamma}'|}\right| \leqslant 4K,$$

we have

$$\Big|\mathbf{T}'(s_1) \cdot \mathbf{T}(s_2)\Big| \leqslant 4K.$$

And we deduce that $|s_1 - s_2|_{\mathscr{I}} < 1/(8K)$ implies $\mathbf{T}(s_1) \cdot \mathbf{T}(s_2) > \frac{1}{2}$. Moreover if $\mathbf{S}(s_1, s_2)$ is the unit secant vector from $\boldsymbol{\gamma}(s_1)$ to $\boldsymbol{\gamma}(s_2)$, with $s_1 < s_2$, then the mean-value theorem implies that $\mathbf{T}(s_1) \cdot \mathbf{S}(s_1, s_2) > \frac{1}{2}$. This estimate and the lower bound on the rate of parameterization $|\boldsymbol{\gamma}'|$ yield

$$-\frac{\mathrm{d}}{\mathrm{d}s_1}|\boldsymbol{\gamma}(s_1) - \boldsymbol{\gamma}(s_2)| = \boldsymbol{\gamma}'(s_1) \cdot \mathbf{S}(s_1, s_2) = |\boldsymbol{\gamma}'(s_1)|\mathbf{T}(s_1) \cdot \mathbf{S}(s_1, s_2) \geqslant \frac{1}{8}.$$

Integrating this result with respect to s_1 over $[s_1, s_2]$ yields

$$|\boldsymbol{\gamma}(s_1) - \boldsymbol{\gamma}(s_2)| \geqslant \frac{|s_1 - s_2|_{\mathscr{I}}}{8}. \tag{2.5}$$

If the whiskers from $\boldsymbol{\gamma}(s_1)$ and $\boldsymbol{\gamma}(s_2)$ intersect at a distance ℓ from Γ, then $\boldsymbol{\gamma}(s_1)$ and $\boldsymbol{\gamma}(s_2)$ lie on a circle of radius ℓ. In particular the straight line distance between these two points must be less than the arc-length along that circle. If $\nu(s)$ denotes the angle of $\mathbf{T}(s)$ to the horizontal, then this distance inequality implies

$$\frac{\ell}{2\pi}|\nu(s_1)-\nu(s_2)| > |\boldsymbol{\gamma}(s_1)-\boldsymbol{\gamma}(s_2)|.$$

Since $\nu'=\kappa$, we have the Lipschitz estimate $|\nu(s_1)-\nu(s_2)| \leqslant K|s_1-s_2|_{\mathscr{I}}$, and combining these estimates and dividing by $|s_1-s_2|_{\mathscr{I}}$ yields the bound

$$\ell \geqslant \frac{\pi}{4K}.$$

This shows that the whiskers of two points whose arc-length distance is less than $1/(8K)$ can only intersect at a distance of at least $\frac{\pi}{4K}$ from Γ. However since $\Gamma \in \mathcal{G}^2_{K,\ell}$, if $|s_1-s_2|_{\mathscr{I}} > 1/(8K)$ then by condition (b) the whiskers through $\boldsymbol{\gamma}(s_1)$ and $\boldsymbol{\gamma}(s_2)$ of length ℓ cannot intersect as the points $\boldsymbol{\gamma}(s_1)$ and $\boldsymbol{\gamma}(s_2)$ are more than 2ℓ apart. By assumption $\ell \leqslant \min\{\frac{\pi}{4K},\ell\}$, and we deduce that Γ is ℓ-far from self intersection. □

Definition 2.3 *(Dressing).* Fix a smooth cut-off function $\chi:\mathbb{R}\to\mathbb{R}$ satisfying: $\chi(r)=1$ if $r\leqslant 1$ and $\chi(r)=0$ if $r\geqslant 2$. Given an interface Γ which is 2ℓ-far from self intersection and a smooth function $f(z):\mathbb{R}\to\mathbb{R}$ which tends to a constant f^∞ and whose derivatives tend to zero at an ε independent exponential rate as $z\to\pm\infty$, then we define the dressed function, $f^d:\Omega\mapsto\mathbb{R}$, of f with respect to Γ as

$$f^d(x)=f(z(x))\chi(\varepsilon|z(x)|/\ell)+f^\infty(1-\chi(\varepsilon|z(x)|/\ell)).$$

From this definition the dressed function satisfies

$$f^d(x)=\begin{cases} f(z(x)), & \text{if } |z(x)|\leqslant \ell/\varepsilon;\\ f^\infty, & \text{if } |z(x)|\geqslant 2\ell/\varepsilon.\end{cases}$$

Definition 2.4 *(Dressed operator).* Let $\mathrm{L}:\mathrm{D}\subset L^2(\mathbb{R})\mapsto L^2(\mathbb{R})$ be a self-adjoint differential operator with smooth coefficients whose derivatives of all order decay to zero at an exponential rate at ∞. We define the space $\mathcal{D}$ to consist of the functions $f:\mathbb{R}\mapsto\mathbb{R}$ as in Definition 2.3, and the dressed operator $\mathrm{L}_d:\mathrm{D}\cap\mathcal{D}\mapsto L^2(\Omega)$ and its r'th power, $r\in\mathbb{N}$,

$$\mathrm{L}_d^r f := (\mathrm{L}^r f)^d. \tag{2.6}$$

If $r<0$ then we assume that $f\in\mathcal{R}(\mathrm{L})$ and the inverse $\mathrm{L}_d^{-1}f$ decays exponentially to a constant at $\pm\infty$. For simplicity we abuse notation and drop the subscript 'd' in both the dressed operator and the dressed function where the context is clear.

A function $f=f(x)\in L^1(\Omega)$ is said to be *localized* near the interface Γ if there exists $\nu>0$ such that for all $x\in\Gamma^{2\ell}$,

$$|f(x(s,z))|\lesssim e^{-\nu|z|}.$$

2.1.2. The Jacobian

Let $\mathbb{J}(s,z)$ be the Jacobian matrix with respect to the change to the whiskered coordinates and denote the Jacobian by $\mathrm{J}(s,z) = \det \mathbb{J}(s,z)$. In two dimensions, $\mathbf{n}' \parallel \boldsymbol{\gamma}'$ so that

$$\mathbb{J} = \begin{pmatrix} \boldsymbol{\gamma}'(s) + \varepsilon z \mathbf{n}'(s) & \varepsilon \mathbf{n}(s) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\gamma}'(s) & \mathbf{n}(s) \end{pmatrix}^T \begin{pmatrix} 1 - \varepsilon z \kappa(s) & 0 \\ 0 & \varepsilon \end{pmatrix}$$

where the curvature

$$\kappa(s) = -\frac{\boldsymbol{\gamma}'(s) \cdot \mathbf{n}'(s)}{|\boldsymbol{\gamma}'(s)|^2}. \tag{2.7}$$

We decompose the Jacobian as: $\mathrm{J}(s,z) = \tilde{\mathrm{J}}(s,z)\mathrm{J}_0(s)$ where

$$\tilde{\mathrm{J}}(s,z) = \varepsilon(1 - \varepsilon z \kappa(s)), \quad \mathrm{J}_0(s) = |\boldsymbol{\gamma}'(s)|. \tag{2.8}$$

The metric tensor takes the form $\mathbb{G} = \mathbb{J}^T \mathbb{J}$.

If $f, g \in L^2(\Omega)$ have support in $\Gamma^{2\ell}$, then the usual $L^2(\Omega)$-inner product can be rewritten as

$$\langle f, g \rangle_{L^2} = \int\limits_{\mathbb{R}_{2\ell}} \int\limits_{\mathscr{I}} f(s,z) g(s,z) \mathrm{J}(s,z)\, \mathrm{d}s \mathrm{d}z, \tag{2.9}$$

where $\mathbb{R}_a := [-a, a]$ for $a \in \mathbb{R}^+$. If $\tilde{s}$ denotes the arc-length reparameterization of Γ over the interval $\mathscr{I}_\Gamma = [0, |\Gamma|]$, then $\mathrm{d}\tilde{s} = \mathrm{J}_0(s)\, \mathrm{d}s$ and the L^2-inner product becomes

$$\langle f, g \rangle_{L^2} = \int\limits_{\mathbb{R}_{2\ell}} \int\limits_{\mathscr{I}_\Gamma} f(s,z) g(s,z) \tilde{\mathrm{J}}(s,z)\, \mathrm{d}\tilde{s} \mathrm{d}z. \tag{2.10}$$

Moreover, if $f \in L^2$ is localized near the interface Γ, then

$$\int\limits_{\Omega} f \, \mathrm{d}x = \int\limits_{\mathbb{R}_{2\ell}} \int\limits_{\mathscr{I}} f(x(s,z)) \mathrm{J}\, \mathrm{d}s \mathrm{d}z + O(e^{-\nu\ell/\varepsilon}).$$

2.1.3. Laplacian

The ε-scaled Laplacian can be expressed in the local coordinates near Γ as

$$\varepsilon^2 \Delta_x = \mathrm{J}^{-1} \partial_z (\mathrm{J}\, \partial_z) + \varepsilon^2 \Delta_g = \partial_z^2 + \varepsilon \mathrm{H} \partial_z + \varepsilon^2 \Delta_g. \tag{2.11}$$

Here H is the extended curvature

$$\mathrm{H}(s,z) := -\frac{\kappa(s)}{1 - \varepsilon z \kappa(s)} = \frac{\partial_z \mathrm{J}}{\varepsilon \mathrm{J}}, \tag{2.12}$$

and Δ_g is the induced Laplacian under metric tensor $\mathbb{G}$, which can be decomposed into

$$\Delta_g := \frac{1}{\sqrt{\det \mathbb{G}}} \partial_s (\mathbb{G}^{11} \sqrt{\det \mathbb{G}}\, \partial_s) = \Delta_s + \varepsilon z D_{s,2}, \quad \mathbb{G} = \begin{pmatrix} |\boldsymbol{\gamma}' + \varepsilon z \mathbf{n}'|^2 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}.$$

Here $\mathbb{G}^{ij}$ denotes the (i, j)-component of the inverse matrix $\mathbb{G}^{-1}$; Δ_s is the Laplace-Beltrami operator on the surface Γ and $D_{s,2}$ is a relatively bounded perturbation of Δ_s. In particular, since $|\boldsymbol{\gamma}' + \varepsilon z \mathbf{n}'| = |\boldsymbol{\gamma}'||1 - \varepsilon z \kappa|$, we have

$$\Delta_s = \frac{1}{|\boldsymbol{\gamma}'(s)|} \partial_s \left(\frac{1}{|\boldsymbol{\gamma}'(s)|} \partial_s \right), \quad D_{s,2} = a(s, z)\Delta_s + b(s, z)\partial_s, \tag{2.13}$$

where the coefficients a, b have the explicit formulae

$$a(s, z) = (\varepsilon z)^{-1} \left(\frac{1}{|1 - \varepsilon z \kappa|^2} - 1 \right), \qquad b(s, z) = \frac{(\varepsilon z)^{-1}}{2|\boldsymbol{\gamma}'|^2} \partial_s a(s, z). \tag{2.14}$$

2.2. Perturbed interfaces

We construct families of interfaces by perturbation from a fixed base curve which we label Γ_0 with parameterization $\boldsymbol{\gamma}_0$, curvature $\kappa_0(s)$ and length $|\Gamma_0|$. Without loss of generality we assume that s is the arc length parameterization of Γ_0 and takes values in $\mathscr{I} = [0, |\Gamma_0|]$, and that Γ_0 is centered at the origin in the sense that the average value of $\boldsymbol{\gamma}_0$ is $(0, 0)$. The effective radius

$$R_0 := \frac{|\Gamma_0|}{2\pi},$$

forms a natural scaling parameter. The Laplace-Beltrami operator $-\Delta_s : H^2(\mathscr{I}) \to L^2(\mathscr{I})$ has in-plane wave numbers $\{\beta_i\}_{i=0}^{\infty}$ whose squares are the scaled eigenvalues of the $|\Gamma_0|$-periodic eigenfunctions $\{\Theta_i\}_{i=0}^{\infty}$

$$-\Delta_s \Theta_i = \beta_i^2 \Theta_i \big/ R_0^2. \tag{2.15}$$

The ground state eigenmode is spatially constant:

$$\Theta_0 = 1/\sqrt{2\pi R_0}, \qquad \beta_0 = 0; \tag{2.16}$$

and for $k \geqslant 1$, we normalize the eigenmodes in $L^2(\mathscr{I})$,

$$\Theta_{2k-1} = \frac{1}{\sqrt{2\pi R_0}} \cos\left(\frac{ks}{R_0}\right), \quad \Theta_{2k} = \frac{1}{\sqrt{2\pi R_0}} \sin\left(\frac{ks}{R_0}\right); \quad \text{and } \beta_{2k-1} = \beta_{2k} = k. \tag{2.17}$$

To control the smoothness of the perturbed interface we introduce the weighted $\mathbf{p}$-norms.

Definition 2.5 *(Weighted* $\mathbf{p}$*-norms).* Given $N_1, k > 0$, the weighted space $\mathbb{V}_k^r = \mathbb{V}_k^r(N_1)$ is defined on the N_1-vectors $\mathbf{p} = (\mathrm{p}_0, \cdots, \mathrm{p}_{N_1-1})^T \in \mathbb{R}^{N_1}$ as

$$\|\mathbf{p}\|_{\mathbb{V}_k^r(N_1)}^r := \sum_{j=0}^{N_1-1} \beta_j^{kr} |\mathrm{p}_j|^r < \infty.$$

The elements of $\mathbf{p}$ starting with p_3 that control the shape of the interface are denoted

$$\hat{\mathbf{p}} := (\mathrm{p}_3, \cdots, \mathrm{p}_{N_1-1})^T, \tag{2.18}$$

and by abuse of notation we apply the same norm to $\hat{\mathbf{p}}$, starting the sum with $j = 3$. When $r = 1$, we omit the superscript r and denote the space by $\mathbb{V}_k$.

The following definition introduces $\Gamma_{\mathbf{p}}$ the $\mathbf{p}$-variation of Γ_0, through an implicit construction that incorporates perturbations so that the change in length of $\Gamma_{\mathbf{p}}$ is controlled solely by p_0 which scales the effective radius R_0. This definition is shown to be well-posed in Lemma 2.10.

Definition 2.6 *(Perturbed interfaces).* Fix a smooth interface Γ_0.

(a) Given $\mathbf{p} \in \mathbb{V}_2$, we define the $\mathbf{p}$-variation of Γ_0, denoted by $\Gamma_{\mathbf{p}}$, through the parametric form:

$$\boldsymbol{\gamma}_{\mathbf{p}}(s) := \frac{(1+\mathrm{p}_0)}{A(\mathbf{p})}\boldsymbol{\gamma}_{\bar{p}}(s) + \mathrm{p}_1\Theta_0\mathbf{E}_1 + \mathrm{p}_2\Theta_0\mathbf{E}_2, \quad \text{for} \quad s \in \mathscr{I}, \tag{2.19}$$

where $\{\mathbf{E}_1, \mathbf{E}_2\}$ are the canonical basis for $\mathbb{R}^2$, the scaling constant $A(\mathbf{p})$ normalizes the length of $\boldsymbol{\gamma}_{\bar{p}}$

$$A(\mathbf{p}) := |\Gamma_0|^{-1} \int_{\mathscr{I}} |\boldsymbol{\gamma}'_{\bar{p}}(s)|\, \mathrm{d}s, \tag{2.20}$$

and the perturbed curve $\boldsymbol{\gamma}_{\bar{p}}$ is

$$\boldsymbol{\gamma}_{\bar{p}}(s) = \boldsymbol{\gamma}_0 + \bar{p}(\tilde{s})\mathbf{n}_0(s) \tag{2.21}$$

where the vector $\mathbf{n}_0(s)$ denotes the outer normal vector of Γ_0 parameterized by s. The definition is made implicit through the relations

$$\bar{p}(\tilde{s}) := \sum_{i=3}^{N_1-1} \mathrm{p}_i \tilde{\Theta}_i(\tilde{s}), \qquad \tilde{\Theta}_i(\tilde{s}) := \Theta_i\left(\frac{2\pi R_0 \tilde{s}}{|\Gamma_{\mathbf{p}}|}\right), \tag{2.22}$$

where $\tilde{s} = \tilde{s}(s; \mathbf{p}) \in \mathscr{I}_{\mathbf{p}} = [0, |\Gamma_{\mathbf{p}}|]$ is the arc length parametrization of the perturbed curve $\boldsymbol{\gamma}_{\mathbf{p}}$ solving

$$\frac{\mathrm{d}\tilde{s}}{\mathrm{d}s} = |\boldsymbol{\gamma}'_{\mathbf{p}}|, \qquad \tilde{s}(0) = 0. \tag{2.23}$$

Remark 2.7. The parameters p_1 and p_2 rigidly translate the interface, and are the only terms that contribute motion along the tangent to Γ_0. In particular their normal components recover $\{\Theta_1, \Theta_2\}$,

$$\Theta_0\mathbf{E}_1 \cdot \mathbf{n}_0 = \frac{1}{\sqrt{2\pi R_0}} \cos\frac{s}{R_0} = \Theta_1, \quad \Theta_0\mathbf{E}_2 \cdot \mathbf{n}_0 = \frac{1}{\sqrt{2\pi R_0}} \sin\frac{s}{R_0} = \Theta_2. \tag{2.24}$$

As will be shown in Lemma 2.10 the curve $|\Gamma_{\mathbf{p}}|$ has length $(1+\mathrm{p}_0)|\Gamma_0|$, and the role of p_0 is to scale the effective radius $R_{\mathbf{p}} := (1+\mathrm{p}_0)R_0$ of $\Gamma_{\mathbf{p}}$. Indeed it follows from (2.15) and (2.22) that the arc-length scaled Laplace-Beltrami eigenmodes $\{\tilde{\Theta}_j\}_{j\geqslant 0}$ of Γ_{p} satisfy

$$-\tilde{\Theta}_j''(\tilde{s}) = \beta_{\mathbf{p},j}^2 \tilde{\Theta}_j(\tilde{s}), \qquad \beta_{\mathbf{p},j} = \frac{\beta_j}{(1+\mathrm{p}_0)R_0}, \tag{2.25}$$

where we address $'$ of $\tilde{\Theta}_j$ always denotes differentiation with respect to $\tilde{s}$. The most significant contribution of the rescaling is that it renders the perturbed eigenmodes mutually orthogonal in $L^2(\mathscr{I}_{\mathbf{p}})$, satisfying

$$\int_{\mathscr{I}_{\mathbf{p}}} \tilde{\Theta}_j \tilde{\Theta}_k |\boldsymbol{\gamma}_{\mathbf{p}}'| \,\mathrm{d}s = \int_{\mathscr{I}_{\mathbf{p}}} \tilde{\Theta}_j \tilde{\Theta}_k \,\mathrm{d}\tilde{s} = (1+\mathrm{p}_0)\delta_{jk}. \tag{2.26}$$

The weighted norms are equivalent to usual Sobolev norms of $\bar{p}$. Indeed the orthogonality (2.26) implies

$$\|\hat{\mathbf{p}}\|_{\mathbb{V}_k^2} \sim \|\bar{p}\|_{H^k(\mathscr{I}_{\mathbf{p}})}, \qquad \|\bar{p}^{(k)}\|_{L^\infty(\mathscr{I}_{\mathbf{p}})} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_k}, \tag{2.27}$$

where the constants depend only upon the $|\Gamma_{\mathbf{p}}.|$ The following embeddings are direct results of Hölder's inequality and the bound $\beta_j \leqslant j$ for $j \geqslant 3$, details are omitted.

Lemma 2.8. *It holds that*

$$\|\hat{\mathbf{p}}\|_{\mathbb{V}_k} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_{k+1}^2}, \qquad \|\hat{\mathbf{p}}\|_{\mathbb{V}_k} \lesssim N_1^{1/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_k^2}, \qquad \|\hat{\mathbf{p}}\|_{\mathbb{V}_{k+1}^r} \lesssim N_1\|\hat{\mathbf{p}}\|_{\mathbb{V}_k^r}.$$

In addition, for any vector $\mathbf{a} \in l^2(\mathbb{R}^m)(m \in \mathbb{Z}^+)$ *we have the dimension dependent bound*

$$\|\mathbf{a}\|_{l^1} \leqslant m^{1/2}\|\mathbf{a}\|_{l^2}. \tag{2.28}$$

We assume that the base interface $\Gamma_0 \in \mathcal{G}^4_{K_0,\ell_0}$ with ℓ_0 sufficiently small that Γ_0 is ℓ_0-far from self intersection. To insure that the implicit construction of $\Gamma_{\mathbf{p}}$ is well posed and the resultant curves are uniformly far from self-intersection we assume that the meander parameters $\mathbf{p}$ lie in the set

$$\mathcal{D}_\delta := \left\{\mathbf{p} \in \mathbb{R}^{N_1} \,\middle|\, \|\hat{\mathbf{p}}\|_{\mathbb{V}_2} \leqslant C,\ \|\hat{\mathbf{p}}\|_{\mathbb{V}_1} \leqslant C\delta,\ \mathrm{p}_0 > -1/2\right\} \tag{2.29}$$

for some positive constant $C \lesssim 1$. The quantity $\delta > 0$ will be chosen sufficiently small, depending only upon the system parameters and the choice of ℓ in $\mathcal{G}^4_{K,\ell}$. The lower bound on p_0 is chosen to prevent the curve being scaled to a point, any fixed value greater than -1 is sufficient. We assume that $\mathbf{p} \in \mathcal{D}_\delta$ throughout the sequel.

Remark 2.9. Dimension N_1 is asymptotically large in this article, in fact, $N_1 \leqslant \varepsilon^{-1}$. The uniform $\mathbb{V}_2$ bound on $\hat{\mathbf{p}} \in \mathcal{D}_\delta$ implies that

$$\|\hat{\mathbf{p}}\|_{\mathbb{V}_3} \lesssim \varepsilon^{-1}, \qquad \|\hat{\mathbf{p}}\|_{\mathbb{V}_4} \lesssim \varepsilon^{-2}.$$

This affords finite but asymptotically large bounds on the third and fourth derivatives of $\boldsymbol{\gamma}_{\mathbf{p}}$ in Lemma 2.11.

Lemma 2.10. *Suppose that $\Gamma_0 \in \mathcal{G}^4_{K_0,\ell_0}$ for some $K_0, \ell_0 > 0$. Then for all $\mathbf{p} \in \mathcal{D}_\delta$ the system (2.23) defined through (2.19) has a unique solution and the resulting interface $\Gamma_{\mathbf{p}}$ is well defined provided that δ is suitably small in terms of K_0, ℓ_0. Moreover, the length of the curve $\Gamma_{\mathbf{p}}$ is*

$$|\Gamma_{\mathbf{p}}| = (1 + \mathrm{p}_0)|\Gamma_0|. \tag{2.30}$$

Proof. The construction of $\boldsymbol{\gamma}_{\mathbf{p}}$ given in Definition 2.6 requires only that the ODE (2.23) is well posed. The issue is that the right-hand side of this expression is implicit in $\tilde{s}$. To apply the general ODE existence theory we must establish a Lipschitz estimate on $|\boldsymbol{\gamma}'_{\mathbf{p}}|$. From the definition of $\boldsymbol{\gamma}_{\mathbf{p}}$ in (2.19), we take the derivative with respect to s, obtaining

$$\boldsymbol{\gamma}'_{\mathbf{p}} = \frac{1+\mathrm{p}_0}{A(\mathbf{p})}\left(\boldsymbol{\gamma}'_0 + \bar{p}'(\tilde{s})\frac{\mathrm{d}\tilde{s}}{\mathrm{d}s}\mathbf{n}_0 + \bar{p}(\tilde{s})\mathbf{n}'_0(s)\right). \tag{2.31}$$

Here and below, primes of $\bar{p}$ denote derivatives with respect to $\tilde{s}$. Recalling

$$\mathbf{n}'_0(s) = -\kappa_0(s)\boldsymbol{\gamma}'_0(s), \tag{2.32}$$

and combining the identity above (2.32) and (2.23) with (2.31) implies

$$\boldsymbol{\gamma}'_{\mathbf{p}} = \frac{1+\mathrm{p}_0}{A(\mathbf{p})}\Big((1-\kappa_0(s)\bar{p}(\tilde{s}))\boldsymbol{\gamma}'_0(s) + \bar{p}'(\tilde{s})|\boldsymbol{\gamma}'_{\mathbf{p}}|\mathbf{n}_0(s)\Big). \tag{2.33}$$

Since $\boldsymbol{\gamma}'_0$ is a tangent to Γ_0 while $\mathbf{n}_0$ is the unit outer normal, we have $\boldsymbol{\gamma}'_0 \cdot \mathbf{n}_0 = 0$. Taking the norm of (2.33), squaring, expanding, and solving for $|\boldsymbol{\gamma}'_{\mathbf{p}}|$, we find the equality

$$|\boldsymbol{\gamma}'_{\mathbf{p}}| = \frac{1+\mathrm{p}_0}{A(\mathbf{p})}(1-\kappa_0\bar{p}(\tilde{s}))\left(1 - \left(\frac{1+\mathrm{p}_0}{A(\mathbf{p})}\right)^2 |\bar{p}'(\tilde{s})|^2\right)^{-1/2}. \tag{2.34}$$

Taking derivative with respect to $\tilde{s}$, and using (2.27) we bound the L^∞ norms of $\bar{p}$ and its derivatives, to deduce

$$\left|\partial_{\tilde{s}}|\boldsymbol{\gamma}'_{\mathbf{p}}|\right| \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_1}(1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2}) \tag{2.35}$$

for $1+\mathrm{p}_0 > 1/2$, $\|\hat{\mathbf{p}}\|_{\mathbb{V}_1} < A(\mathbf{p})/(1+\mathrm{p}_0)$. Here we note $A(\mathbf{p})$ can be bounded as in the Appendix Lemma 6.1 which implies the second condition is correct for $\|\hat{\mathbf{p}}\|_{\mathbb{V}_1}$ small enough. That is, $|\boldsymbol{\gamma}'_{\mathbf{p}}|$ is globally(uniformly) Lipschitz with respect to $\tilde{s}$ provided that $\hat{\mathbf{p}} \in \mathbb{V}_2$ satisfying $1 + \mathrm{p}_0 > 1/2$ and $\|\hat{\mathbf{p}}\|_{\mathbb{V}_1}$ small enough. Hence by classical Picard–Lindelöf existence theory, the system (2.23) is solvable on a small interval with smallness depending on the Lipschitz constant only. In addition, by construction the length of $\Gamma_{\mathbf{p}}$ satisfies (2.30) which implies that $\tilde{s}$ is uniformly bounded independent of $\hat{\mathbf{p}}$, and the solution is extendable to the whole finite interval $\mathscr{I}$ for all $\mathbf{p} \in \mathcal{D}_\delta$ with δ suitably small. $\square$

The following Lemma establishes uniform bounds on the smoothness and distance from self-intersection of the interfaces $\Gamma_{\mathbf{p}}$.

Lemma 2.11 *(Smoothness of $\Gamma_{\mathbf{p}}$). Suppose that $\Gamma_0 \in \mathcal{G}^4_{K_0,\ell_0}$ for some $K_0 > 0$ and $\ell_0 \in (0, \pi/(8K_0))$. Then there exist $K, \ell > 0$ and δ suitably small depending on Γ_0, independent of $\varepsilon > 0$ such that for all $\mathbf{p} \in \mathcal{D}_\delta$ the associated $\Gamma_{\mathbf{p}}$ resides in $\mathcal{G}^2_{K,\ell}$ and is ℓ-far from self-intersection. Moreover the perturbed curves $\boldsymbol{\gamma}_{\mathbf{p}}$ satisfy the bounds*

$$|\boldsymbol{\gamma}'_{\mathbf{p}}| \geqslant \frac{1}{4}; \qquad |\boldsymbol{\gamma}^{(k)}_{\mathbf{p}}| \lesssim 1 + \sum_{l=1}^{k} |\bar{p}^{(l)}(\tilde{s})| \lesssim 1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_k}, \qquad k = 1, 2, \cdots 4. \tag{2.36}$$

The curvature and normal of $\Gamma_{\mathbf{p}}$, defined by

$$\kappa_{\mathbf{p}} := \boldsymbol{\gamma}''_{\mathbf{p}} \cdot \mathbf{n}_{\mathbf{p}} / |\boldsymbol{\gamma}'_{\mathbf{p}}|^2, \qquad \mathbf{n}_{\mathbf{p}} = e^{-\pi\mathcal{R}/2} \boldsymbol{\gamma}'_{\mathbf{p}} / |\boldsymbol{\gamma}'_{\mathbf{p}}|, \tag{2.37}$$

admit the bounds

$$|\mathbf{n}_{\mathbf{p}}| \lesssim 1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_1}; \qquad |\kappa_{\mathbf{p}}| \lesssim 1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2}; \qquad \|\kappa_{\mathbf{p}}\|_{H^k(\mathscr{I}_{\mathbf{p}})} \lesssim 1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}^2_{k+2}} \tag{2.38}$$

for $k = 1, 2$.

Proof. We first establish the bounds on $\boldsymbol{\gamma}_{\mathbf{p}}$ and its curvatures. From identity (2.34) and relation (2.27), we have upper and lower bound of the metric $|\boldsymbol{\gamma}'_{\mathbf{p}}|$

$$\frac{1}{4} \leqslant |\boldsymbol{\gamma}'_{\mathbf{p}}| \leqslant 2(1 + |\mathrm{p}_0|), \tag{2.39}$$

for $\mathbf{p} \in \mathcal{D}_\delta$ with δ suitably small. This further implies that the first derivative of the metric has the bound

$$||\boldsymbol{\gamma}'_{\mathbf{p}}|'| = \frac{|\boldsymbol{\gamma}'_{\mathbf{p}} \cdot \boldsymbol{\gamma}''_{\mathbf{p}}|}{|\boldsymbol{\gamma}'_{\mathbf{p}}|} \lesssim |\boldsymbol{\gamma}''_{\mathbf{p}}|. \tag{2.40}$$

The higher derivatives of the metric $|\boldsymbol{\gamma}'_{\mathbf{p}}|$ enjoy the bounds

$$\left| |\boldsymbol{\gamma}'_{\mathbf{p}}|'' \right| \lesssim |\boldsymbol{\gamma}''_{\mathbf{p}}|^2 + |\boldsymbol{\gamma}'''_{\mathbf{p}}|, \qquad \left| |\boldsymbol{\gamma}'_{\mathbf{p}}|''' \right| \lesssim |\boldsymbol{\gamma}^{(4)}_{\mathbf{p}}| + |\boldsymbol{\gamma}''_{\mathbf{p}}||\boldsymbol{\gamma}'''_{\mathbf{p}}|. \tag{2.41}$$

Moreover the definition (2.19) of $\boldsymbol{\gamma}_{\mathbf{p}}(s)$ with $\bar{p} = \bar{p}(\tilde{s})$ and $\tilde{s} = \tilde{s}(s)$ given in (2.22)-(2.23), and the smoothness of Γ_0 imply

$$\begin{aligned} &|\boldsymbol{\gamma}''_{\mathbf{p}}| \lesssim 1 + |\bar{p}''| + |\bar{p}'| \cdot ||\boldsymbol{\gamma}'_{\mathbf{p}}|'|; \qquad |\boldsymbol{\gamma}'''_{\mathbf{p}}| \lesssim 1 + |\bar{p}'''| + |\bar{p}'| \cdot ||\boldsymbol{\gamma}'_{\mathbf{p}}|''| + |\bar{p}''| \cdot ||\boldsymbol{\gamma}'_{\mathbf{p}}|'|; \\ &|\boldsymbol{\gamma}^{(4)}_{\mathbf{p}}| \lesssim 1 + |\bar{p}'''| + |\bar{p}^{(4)}| + |\bar{p}'|||\boldsymbol{\gamma}'_{\mathbf{p}}|'''| + |\bar{p}''| \left(||\boldsymbol{\gamma}'_{\mathbf{p}}|'|^2 + ||\boldsymbol{\gamma}'_{\mathbf{p}}|''| \right) + |\bar{p}'''| \cdot ||\boldsymbol{\gamma}'_{\mathbf{p}}|'|, \end{aligned} \tag{2.42}$$

provided that $\|\hat{\mathbf{p}}\|_{\mathbb{V}_2}$ uniformly bounded independent of ε. Combining the first estimate in (2.42) with the estimate (2.40) yields the bound

$$|\boldsymbol{\gamma}_{\mathbf{p}}''| \lesssim 1 + |\bar{p}''(\tilde{s})| \tag{2.43}$$

for $\|\bar{p}'\| \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_1} \lesssim \delta$ suitably small. In a similar manner, combining the last two estimates of (2.42) with (2.40)-(2.41) and (2.43) yields

$$|\boldsymbol{\gamma}_{\mathbf{p}}'''| \lesssim 1 + |\bar{p}'''(\tilde{s})|, \qquad |\boldsymbol{\gamma}_{\mathbf{p}}^{(4)}| \lesssim 1 + |\bar{p}^{(4)}(\tilde{s})| + |\bar{p}'''(\tilde{s})| \tag{2.44}$$

since $\|\bar{p}''\|_{L^\infty} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2} \lesssim 1$ for $\mathbf{p} \in \mathcal{D}_\delta$. Now the curvature $\kappa_{\mathbf{p}}$ in (2.37) admits bound

$$|\kappa_{\mathbf{p}}| \lesssim |\boldsymbol{\gamma}_{\mathbf{p}}''| \lesssim 1 + |\bar{p}''(\tilde{s})|,$$

so that the L^∞ and $L^2(\mathscr{I}_{\mathbf{p}})$ bounds of the curvature follow from (2.27). Taking the derivative of the curvature and using the bounds (2.39), (2.43) and (2.44) with $\|\hat{\mathbf{p}}\|_{\mathbb{V}_2} \lesssim 1$ implies

$$|\kappa_{\mathbf{p}}'| = \left| \frac{e^{-\pi\mathcal{R}/2}\boldsymbol{\gamma}_{\mathbf{p}}'' \cdot \boldsymbol{\gamma}_{\mathbf{p}}'' + e^{-\pi\mathcal{R}/2}\boldsymbol{\gamma}_{\mathbf{p}}'' \cdot \boldsymbol{\gamma}_{\mathbf{p}}'''}{|\boldsymbol{\gamma}_{\mathbf{p}}'|^3} - 3\frac{e^{-\pi\mathcal{R}/2}\boldsymbol{\gamma}_{\mathbf{p}}'' \cdot \boldsymbol{\gamma}_{\mathbf{p}}''}{|\boldsymbol{\gamma}_{\mathbf{p}}'|^5}\boldsymbol{\gamma}_{\mathbf{p}}' \cdot \boldsymbol{\gamma}_{\mathbf{p}}'' \right| \lesssim 1 + |\bar{p}'''|.$$

The $L^2(\mathscr{I}_{\mathbf{p}})$-bound of $\kappa_{\mathbf{p}}'$ now follows from (2.27). The $L^2(\mathscr{I}_{\mathbf{p}})$ bound of $\kappa_{\mathbf{p}}''$ is obtained from similar calculations, the details are omitted.

To see that $\Gamma_{\mathbf{p}} \in \mathcal{G}^2_{K,\ell}$, we remark from (2.36) that $\boldsymbol{\gamma}_{\mathbf{p}}$ is uniformly bounded in $W^{2,\infty}(\mathscr{I})$ by some $K > K_0$. In particular $\|\kappa_{\mathbf{p}}\|_{L^\infty}$ inherits this uniform bound. To establish condition (b) of Definition 2.1 we first establish that $\Gamma_0 \in \mathcal{G}^2_{K,\tilde{\ell}}$ for some $\tilde{\ell} > 0$. We have condition (b) for Γ_0 with K_0 and ℓ_0. If $1/(8K) < |s_1 - s_2|_{\mathscr{I}} < 1/(8K_0)$ then by (2.5) we have $|\boldsymbol{\gamma}_0(s_1) - \boldsymbol{\gamma}_0(s_2)| > 1/(64K)$. Combining these cases we have $\Gamma_0 \in \mathcal{G}^2_{K,\tilde{\ell}}$ with $\tilde{\ell} = \min\{\ell_0, 1/(64K)\}$. For $|s_1 - s_2|_{\mathscr{I}} > 1/(8K)$, by (2.19) with Θ_0 independent of s and (2.22) we derive

$$|\boldsymbol{\gamma}_{\mathbf{p}}(s_1) - \boldsymbol{\gamma}_{\mathbf{p}}(s_2)| = \frac{1+\mathrm{p}_0}{A(\mathbf{p})}|\boldsymbol{\gamma}_{\bar{p}}(s_1) - \boldsymbol{\gamma}_{\bar{p}}(s_2)| \geqslant \frac{1+\mathrm{p}_0}{A(\mathbf{p})}\left(\tilde{\ell} - 2\|\hat{\mathbf{p}}\|_{\mathbb{V}_1}\right).$$

Here we used (2.27) to bound the L^∞-norm $\bar{p}$ and $\|\hat{\mathbf{p}}\|_{\mathbb{V}_0} \leqslant \|\hat{\mathbf{p}}\|_{\mathbb{V}_1}$. Lemma 6.1 affords the bound $A(\mathbf{p}) = O(1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_1})$, and we deduce that $\Gamma_{\mathbf{p}} \in \mathcal{G}^2_{K,\ell}$ for ℓ less than $\tilde{\ell}/4$ for all $\mathbf{p} \in \mathcal{D}_\delta$ by choosing δ suitably small. We deduce from Lemma 2.1 that each $\Gamma_{\mathbf{p}}$ is ℓ-far from self-intersection. □

The dressing of interfaces requires a 2ℓ reach. From Lemma 2.11 we may choose $\ell > 0$ such that the collection of perturbed interfaces belong to $\mathcal{G}^2_{K,2\ell}$ with associated reach $\Gamma^{2\ell}_{\mathbf{p}}$. To each $\mathbf{p} \in \mathcal{D}_\delta$ this allows us to introduce the local whiskered coordinates $(s_{\mathbf{p}}, z_{\mathbf{p}})$ associated to $\Gamma_{\mathbf{p}}$. Similarly, the geometric structures $\mathbf{n}_{\mathbf{p}}, \boldsymbol{\gamma}_{\mathbf{p}}$ and $\kappa_{\mathbf{p}}$ associated to $\Gamma_{\mathbf{p}}$ have natural extensions to $\Gamma^{2\ell}_{\mathbf{p}}$. The domain $\Gamma^{2\ell}_{\mathbf{p}}$ of $(s_{\mathbf{p}}, z_{\mathbf{p}})$ overlaps with the domain $\Gamma^{2\ell_0}_0$ of the local coordinates (s, z) associated to the base point Γ_0. On the interface $\Gamma_{\mathbf{p}}$, corresponding to $z_{\mathbf{p}} = 0$, the whiskered variable $s_{\mathbf{p}}$ reduces to s, that is $s_{\mathbf{p}}\big|_{z_{\mathbf{p}}=0} = s$. The quantity $\tilde{s}$, and not s, corresponds to arc-length on $\Gamma_{\mathbf{p}}$. In the sequel the term "local coordinates of $\Gamma_{\mathbf{p}}$" refers to $(s_{\mathbf{p}}, z_{\mathbf{p}})$ on $\Gamma^{2\ell}_{\mathbf{p}}$, however it is convenient to introduce $\tilde{s}_{\mathbf{p}}$, the extension of $\tilde{s}$ to $\Gamma^{2\ell}_{\mathbf{p}}$, as this is the natural variable for the Laplace-Beltrami eigenmodes $\{\tilde{\Theta}_j\}_{j\geqslant 0}$ of $\Delta_{s_{\mathbf{p}}}$, and of their integrals.

Notation 2.1. To simplify the presentation of the subsequent calculations, we will use the blanket notation $h(z_{\mathbf{p}}, \boldsymbol{\gamma}_{\mathbf{p}}^{(k)})$ for any smooth function defined in $\Gamma_{\mathbf{p}}^{2\ell}$ that depends upon $s_{\mathbf{p}}$ only through the first k derivatives of $\boldsymbol{\gamma}_{\mathbf{p}}$. If the function is independent of $z_{\mathbf{p}}$ we will denote it by $h(\boldsymbol{\gamma}_{\mathbf{p}}^{(k)})$.

The following Lemma presents a common use of Notation 2.1.

Lemma 2.12. *If function $f = f(s_{\mathbf{p}})$ defined on $\Gamma_{\mathbf{p}}^{2\ell}$ depends upon $s_{\mathbf{p}}$ only through $|\boldsymbol{\gamma}'_{\mathbf{p}}|, \kappa_{\mathbf{p}}, \mathbf{n}_{\mathbf{p}} \cdot \mathbf{n}_0$, $\varepsilon^k \nabla_{s_{\mathbf{p}}}^k$, and their derivatives, then under the assumptions (2.29) there exists $h = h(\boldsymbol{\gamma}''_{\mathbf{p}})$ in the sense of Notation 2.1 such that $f(s_{\mathbf{p}}) = h(\boldsymbol{\gamma}''_{\mathbf{p}})$, where h satisfies*

$$\|h(\boldsymbol{\gamma}''_{\mathbf{p}})\|_{L^2(\mathscr{I}_{\mathbf{p}})} + \|h(\boldsymbol{\gamma}''_{\mathbf{p}})\|_{L^\infty} \lesssim 1; \tag{2.45}$$

and for $l \geqslant 1$,

$$\left\|\varepsilon^{l-1}\nabla_{s_{\mathbf{p}}}^l h(\boldsymbol{\gamma}''_{\mathbf{p}})\right\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim 1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2}, \qquad \|\varepsilon^{l-1}\nabla_{s_{\mathbf{p}}}^l h(\boldsymbol{\gamma}''_{\mathbf{p}})\|_{L^\infty} \lesssim 1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_3}. \tag{2.46}$$

Proof. The estimates (2.45)-(2.46) are direct results of Lemma 2.11. $\square$

The following lemma is used frequently to establish bounds on vector and operator norms in various error terms.

Lemma 2.13. *Recalling the notation of section 1.1, if $f \in L^2(\mathscr{I}_{\mathbf{p}})$, then there exists a unit vector $\boldsymbol{e} = (e_i) \in l^2$ such that*

$$\int_{\mathscr{I}_{\mathbf{p}}} f\tilde{\Theta}_i \, \mathrm{d}\tilde{s}_{\mathbf{p}} = O(\|f\|_{L^2(\mathscr{I}_{\mathbf{p}})}) e_i. \tag{2.47}$$

If in addition $f \in L^\infty$ on $\mathscr{I}_{\mathbf{p}}$, then for any vector $\mathbf{a} = (\mathrm{a}_j) \in l^2$, we have

$$\left|\sum_j \int_{\mathscr{I}_{\mathbf{p}}} f\tilde{\Theta}_i \mathrm{a}_j \tilde{\Theta}_j \, \mathrm{d}\tilde{s}_{\mathbf{p}}\right| \lesssim \|\mathbf{a}\|_{l^2} \|f\|_{L^\infty} e_i, \tag{2.48}$$

and there exists a matrix $\mathbb{E} = (\mathbb{E}_{ij})$ with operator norm l^2_* norm equal to one, such that

$$\int_{\mathscr{I}_{\mathbf{p}}} f\tilde{\Theta}_i\tilde{\Theta}_j \, \mathrm{d}\tilde{s}_{\mathbf{p}} = O(\|f\|_{L^\infty}) \mathbb{E}_{ij}. \tag{2.49}$$

Proof. The estimates follow from Plancherel and classic applications of Fourier theory. $\square$

3. Quasi-equilibrium profiles and the bilayer manifold

Fix $K_0, \ell_0 > 0$ and a base interface $\Gamma_0 \in \mathcal{G}^4_{K_0,2\ell_0}$. We associate the collection of perturbed interfaces $\{\Gamma_{\mathbf{p}}\}_{\mathbf{p}\in\mathcal{D}_\delta}$ and construct the bilayer manifold as the graph of the quasi-equilibrium bilayer distribution $\Phi_{\mathbf{p}}$ over the set $\mathcal{D}_\delta$. The bulk density parameter σ is slaved to the meander parameters to enforce a prescribed total mass constraint. The construction of the quasi-equilibrium bilayer distribution begins with ϕ_0 defined on $L^2(\mathbb{R})$ as the nontrivial solution of

$$\partial_z^2\phi_0 - W'(\phi_0) = 0, \tag{3.1}$$

that is homoclinic to the left well b_- of W. In particular ϕ_0 is unique up to translation, even about its maximum, and $\phi_0 - b_-$ converges to 0 as z tends to $\pm\infty$ at the exponential rate $\sqrt{W''(b_-)} > 0$.

The linearization L_0 of (3.1) about ϕ_0,

$$\mathrm{L}_0 := -\partial_z^2 + W''(\phi_0(z)), \tag{3.2}$$

is a Sturm-Liouville operator on the real line whose coefficients decay exponentially fast to constants at $z = \infty$. The following Lemma follows from classic results and direct calculations, see for example Chapter 2.3.2 of [23].

Lemma 3.1. *The spectrum of* L_0 *is real, and uniformly positive except for two point spectra:* $\lambda_0 < 0$ *and* $\lambda_1 = 0$ *and associated ortho-normal eigenmodes* ψ_0 *and* ψ_1*. Moreover, it holds that*

$$\mathrm{L}_0\phi_0' = 0, \quad \mathrm{L}_0\phi_0'' = -W'''(\phi_0)\left|\phi_0'\right|^2, \quad \mathrm{L}_0\left(z\phi_0'\right) = -2\phi_0''.$$

The ground state eigenmode ψ_0 *is even and positive. The kernel of* L_0 *is spanned by* $\psi_1 = \phi_0'/\|\phi_0'\|_{L^2}$*. The operator* L_0 *is invertible on the* L^2 *perp of its kernel, and both* L_0 *and its inverse preserve parity.*

Some care must be taken to distinguish between functions in $L^2(\mathbb{R})$ and their dressings that reside in $L^2(\Omega)$. As an example, since 1 is $L^2(\mathbb{R})$ orthogonal to ϕ_0' we may define $B_k = \mathrm{L}_0^{-k}1 \in L^\infty(\mathbb{R})$ and its dressing subject to $\Gamma_{\mathbf{p}}$,

$$B^d_{\mathbf{p},k}(x) := (\mathrm{L}_0^{-k}1)^d \in L^\infty, \tag{3.3}$$

defined on all of Ω. Recalling the averaging operator, (1.5) we introduce

$$\overline{B}^d_{\mathbf{p},k} := |\Omega|\left\langle B^d_{\mathbf{p},k}\right\rangle_{L^2}. \tag{3.4}$$

Here and below, we drop the d superscript on the dressed function to simplify notation when no ambiguity arises. Introducing $\eta_d := \eta_1 - \eta_2$, we define the first dressed correction ϕ_1 to the pulse profile

$$\phi_1(\sigma) = \phi_1(z_{\mathbf{p}};\sigma) := \sigma B_{\mathbf{p},2} + \frac{\eta_d}{2}\mathrm{L}_0^{-1}\left(z_{\mathbf{p}}\phi_0'\right), \tag{3.5}$$

which depends upon the bulk density and meander parameters, $\sigma \in \mathbb{R}$ and $\mathbf{p}$. The bulk density parameter controls the value of ϕ_1 outside of $\Gamma_{\mathbf{p}}^{2\ell}$, where the profile is constant. In the construction of the bilayer manifold $\sigma = \sigma(\mathbf{p})$, adjusting the bulk density state to make bilayer mass $|\Omega|\langle \Phi_{\mathbf{p}} - b_- \rangle_{L^2}$ independent of $\mathbf{p}$. Viewed as a function on $\mathbb{R}$, ϕ_1 is smooth and is even with respect to z, while as a function on Ω it is smooth and even in $z_{\mathbf{p}}$ to leading order.

The second order correction ϕ_2 is composed of products of whisker independent dressed functions and the whisker dependent curvature $\kappa_{\mathbf{p}} = \kappa_{\mathbf{p}}(s_{\mathbf{p}})$. As such ϕ_2 is not strictly the dressing of a function of one variable, indeed for each fixed value of $s_{\mathbf{p}}$, we define it as the $L^2(\mathbb{R})$ solution of

$$\begin{aligned} \mathrm{L}_0^2\phi_2(z, s_{\mathbf{p}}) = g_2(z, s_{\mathbf{p}}) := -\mathrm{L}_0\left(z\kappa_{\mathbf{p}}^2\phi_0' + \frac{W'''(\phi_0)}{2}\phi_1^2\right) - \Big(\kappa_{\mathbf{p}}^2\phi_0'' + (-\eta_1 + W'''(\phi_0)\phi_1)\mathrm{L}_0\phi_1 \\ + \eta_d W''(\phi_0)\phi_1\Big) - \kappa_{\mathbf{p}}\Big(2\mathrm{L}_0\phi_1'(\sigma_1^*) + (-\eta_1 + 2W'''(\phi_0)\phi_1(\sigma_1^*))\phi_0'\Big). \end{aligned} \tag{3.6}$$

The constant σ_1^* determined below to insure the right-hand side of (3.6) is in the range of L_0 on each whisker. Since each $s_{\mathbf{p}}$ dependent term decays exponentially to zero in $z_{\mathbf{p}}$, the resulting whisker-dependent function extends to a smooth dressing ϕ_2 around $\Gamma_{\mathbf{p}}$. We denote this extension by

$$\begin{aligned} \phi_2 := -\mathrm{L}_0^{-1}\left(z_{\mathbf{p}}\kappa_{\mathbf{p}}^2\phi_0' + \frac{W'''(\phi_0)}{2}\phi_1^2\right) - \mathrm{L}_0^{-2}\Big(\kappa_{\mathbf{p}}^2\phi_0'' + (-\eta_1 + W'''(\phi_0)\phi_1)\mathrm{L}_0\phi_1 \\ + \eta_d W''(\phi_0)\phi_1\Big) - \kappa_{\mathbf{p}}\mathrm{L}_0^{-2}\Big(2\mathrm{L}_0\phi_1'(\sigma_1^*) + (-\eta_1 + 2W'''(\phi_0)\phi_1(\sigma_1^*))\phi_0'\Big). \end{aligned} \tag{3.7}$$

To verify that the inverses of L_0 are well defined we observe that the first two applications of L_0^{-1} in the right-hand side of (3.7) are to functions that are even in z, and hence orthogonal in $L^2(\mathbb{R})$ to ϕ_0'. The third application is to a function that is odd in z, which we denote by $(g_2)^{\text{odd}}$. In $L^2(\mathbb{R})$ we calculate the projection of g_2^{odd} onto the kernel of L_0,

$$\int_{\mathbb{R}} g_2^{\text{odd}}\phi_0'\,\mathrm{d}z = -\eta_1 m_1^2 + 2\int_{\mathbb{R}} W'''(\phi_0)|\phi_0'|^2\phi_1(\sigma_1^*)\mathrm{d}z, \tag{3.8}$$

where m_1 is defined as

$$m_1 := \|\phi_0'\|_{L^2(\mathbb{R})}. \tag{3.9}$$

In light of Lemma 3.1, we have $\mathrm{L}_0^2(z\phi_0') = -2\mathrm{L}_0(\phi_0'') = 2W'''(\phi_0)|\phi_0'|^2$. Using the definition of ϕ_1 and integration by parts, we have

$$2\int_{\mathbb{R}} W'''(\phi_0)|\phi_0'|^2\phi_1(\sigma_1^*)\,\mathrm{d}z = \int_{\mathbb{R}} \mathrm{L}_0^2(z\phi_0')\phi_1(\sigma_1^*)\,\mathrm{d}z = -m_0\sigma_1^* + \frac{\eta_d}{2}m_1^2. \tag{3.10}$$

For σ_1^* given by

$$\sigma_1^* := -\frac{(\eta_1+\eta_2)m_1^2}{2m_0} \qquad \text{with} \quad m_0 := \int_{\mathbb{R}} (\phi_0(z) - b_-)\,\mathrm{d}z, \tag{3.11}$$

we see that the terms on the right-hand side of (3.8) cancel, and we deduce the bounded invertibility of L_0 in (3.7). We are in position to introduce the profile.

Lemma 3.2. *Let meander parameters* $\mathbf{p}$ *satisfy* (2.29)*. Then for* ϕ_0*,* ϕ_1*, and* ϕ_2 *defined in* (3.1)*,* (3.5)*, and* (3.7) *respectively, we define the bilayer distribution*

$$\Phi_{\mathbf{p}}(x;\sigma) := \phi_0(z_{\mathbf{p}}) + \varepsilon\phi_1(z_{\mathbf{p}};\sigma) + \varepsilon^2\phi_2(s_{\mathbf{p}}, z_{\mathbf{p}};\sigma,\sigma^*), \tag{3.12}$$

which has the following residual

$$\mathrm{F}(\Phi_{\mathbf{p}}) = \varepsilon\sigma + \varepsilon^2\mathrm{F}_2 + \varepsilon^3\mathrm{F}_3 + \varepsilon^4\mathrm{F}_{\geqslant 4}. \tag{3.13}$$

Here the expansion terms in the main residual F_m *have the form*

$$\begin{aligned} \mathrm{F}_2 &= \kappa_{\mathbf{p}}(\sigma - \sigma_1^*) f_2(z_{\mathbf{p}}); \qquad\qquad \mathrm{F}_3 = -\phi_0' \Delta_{s_{\mathbf{p}}}\kappa_{\mathbf{p}} + f_3(z_{\mathbf{p}}, \boldsymbol{\gamma}_{\mathbf{p}}''), \\ \mathrm{F}_{\geqslant 4} &= f_{4,1}(z_{\mathbf{p}}, \boldsymbol{\gamma}_{\mathbf{p}}'')\Delta_g f_{4,2}(z_{\mathbf{p}}, \boldsymbol{\gamma}_{\mathbf{p}}'') + f_{4,2}(z_{\mathbf{p}}, \boldsymbol{\gamma}_{\mathbf{p}}''), \end{aligned} \tag{3.14}$$

where $f = f(z, \boldsymbol{\gamma}_{\mathbf{p}}'')$ *with various subscripts are smooth functions which decay exponentially fast to a constant as* $|z| \to \infty$*. In particular,* $f_2(z)$ *is odd with respect to* z *and decays to zero as* $|z| \to \infty$*. In addition,* $\mathrm{F}_2, \mathrm{F}_3$ *satisfy the following projection properties:*

$$\begin{aligned} \int_{\mathbb{R}_{2\ell}} \mathrm{F}_2\,\phi_0'\,\mathrm{d}z_{\mathbf{p}} &= m_0(\sigma_1^* - \sigma)\kappa_{\mathbf{p}} + O(e^{-\ell\nu/\varepsilon}); \\ \int_{\mathbb{R}_{2\ell}} \mathrm{F}_3\,\phi_0'\,\mathrm{d}z_{\mathbf{p}} &= m_1^2\left(-\Delta_{s_{\mathbf{p}}}\kappa_{\mathbf{p}} - \frac{\kappa_{\mathbf{p}}^3}{2} + \alpha\kappa_{\mathbf{p}}\right) + O(e^{-\ell\nu/\varepsilon}). \end{aligned} \tag{3.15}$$

Here $\alpha = \alpha(\sigma;\eta_1,\eta_2)$ *is a smooth function of* σ*.*

Proof. For brevity of notation, we drop the subscript $\mathbf{p}$ in the proof. The variational derivative $\mathrm{F}(\Phi)$ can be written as

$$\begin{aligned} \mathrm{F}(\Phi) = &\left[\partial_z^2 + \varepsilon \mathrm{H}\partial_z + \varepsilon^2\Delta_g - W''(\Phi) + \varepsilon\eta_1\right]\left[\partial_z^2\Phi + \varepsilon\mathrm{H}\partial_z\Phi + \varepsilon^2\Delta_g\Phi - W'(\Phi)\right] \\ &+ \varepsilon\eta_d W'(\Phi). \end{aligned} \tag{3.16}$$

The components of the profile Φ were chosen to make the residual $\Pi_0\mathrm{F}(\Phi)$ small to $O(\varepsilon^2)$. We expand $\mathrm{F}(\Phi)$ in powers of ε, and introduce $\phi_{\geqslant 1} := \phi_1 + \varepsilon\phi_2$. Taylor expanding the k-th derivative of $W(\Phi)$ around ϕ_0 for $k = 1, 2$ and keeping terms up to third order we find,

$$
\begin{aligned}
W^{(k)}(\Phi) = W^{(k)}(\phi_0) + \varepsilon W^{(k+1)}(\phi_0)\phi_1 + \varepsilon^2 \left(W^{(k+1)}(\phi_0)\phi_2 + \frac{W^{k+2}(h_k)}{2}\phi_1^2 \right) + \\
+ \frac{\varepsilon^3}{2} W^{(k+2)}(\phi_0)(2\phi_1 + \varepsilon\phi_2)\phi_2 + \varepsilon^3 \frac{W^{(k+3)}(\phi_0)}{3!}\phi_{\geqslant 1}^3.
\end{aligned}
\tag{3.17}
$$

Similarly the expansion of the extended curvature H to third order takes the form

$$
\mathrm{H} = -\frac{\kappa}{1-\varepsilon z\kappa} = -\kappa - \varepsilon z \frac{\kappa^2}{1-\varepsilon z\kappa} = -\kappa - \varepsilon z\kappa^2 - \varepsilon^2 z^2\kappa^3 - \varepsilon^3 z^3 \frac{\kappa^4}{1-\varepsilon z\kappa}.
$$

The whiskered coordinate expression (3.16) of $\mathrm{F}(\Phi)$ admits the expansion

$$
\mathrm{F}(\Phi) = \varepsilon \left(\mathrm{L}_0^2\phi_1 + \eta_d W'(\phi_0) \right) + \varepsilon^2 \mathrm{F}_2 + \varepsilon^3 \mathrm{F}_3 + \varepsilon^4 \mathrm{F}_{\geqslant 4}. \tag{3.18}
$$

Using the identities from Lemma 3.1, F_2 and F_3 reduce to

$$
\begin{aligned}
\mathrm{F}_2 &= \mathrm{L}_0 \left(\kappa\phi_1' + \mathrm{L}_0\phi_2 + z\kappa^2\phi_0' + \frac{W'''(\phi_0)}{2}\phi_1^2 \right) + \left(\kappa\partial_z - \eta_1 + W'''(\phi_0)\phi_1 \right) \\
&\quad \times (\kappa\phi_0' + \mathrm{L}_0\phi_1) + \eta_d W''(\phi_0)\phi_1; \\
\mathrm{F}_3 &= \mathrm{L}_0 \left(\kappa\partial_z\phi_2 + W'''(\phi_0)\phi_1\phi_2 + z^2\kappa^3\phi_0' + z\kappa^2\phi_1' + \frac{W^{(4)}(\phi_0)}{3!}\phi_1^3 \right) \\
&\quad + (\kappa\partial_z - \eta_1 + W'''(\phi_0)\phi_1)\left(\mathrm{L}_0\phi_2 + \kappa\phi_1' + \kappa^2 z\phi_0' + \frac{W'''(\phi_0)}{2}\phi_1^2 \right) \\
&\quad - \Delta_s\kappa\phi_0' + \left(\frac{W^{(4)}(\phi_0)}{2}\phi_1^2 + W'''(\phi_0)\phi_2 \right)(\kappa\phi_0' + \mathrm{L}_0\phi_1) \\
&\quad + \kappa^3 z\phi_0'' + z\kappa^2\partial_z\mathrm{L}_0\phi_1 + \eta_d \left(W''(\phi_0)\phi_2 + \frac{W'''(\phi_0)}{2}\phi_1^2 \right).
\end{aligned}
$$

Within $\Gamma_{\mathbf{p}}^{\ell}$ using the expressions for ϕ_1, ϕ_2 in (3.5) and (3.7) we see that the $O(\varepsilon)$ term in (3.18) reduces to the constant σ. Using the definition of ϕ_2 given in (3.7), the term F_2 further reduces to

$$
\mathrm{F}_2 = \kappa\mathrm{L}_0(\phi_1 - \phi_1(\sigma_1^*))' + \kappa\partial_z\mathrm{L}_0(\phi_1 - \phi_1(\sigma_1^*)) + W'''(\phi_0)(\phi_1 - \phi_1(\sigma_1^*))\kappa\phi_0',
$$

and the final expression for F_2 in (3.14) follows from (3.5) with the reductions for F_3 and F_4 obtained from similar calculations. In particular $\mathrm{F}_{\geqslant 4}$ takes the exact form:

$$
\begin{aligned}
\mathrm{F}_{\geqslant 4} = & -(\partial_z^2 + \varepsilon \mathrm{H}\partial_z + \varepsilon^2 \Delta_g - W''(\Phi) + \varepsilon\eta_1)\Bigg(\frac{W'''(\phi_0)}{2}\phi_2^2 + \Delta_g \phi_2 \\
& + \frac{W^{(4)}(h)}{3!}(3\phi_1^2\phi_2 + 3\varepsilon\phi_1\phi_2^2 + \varepsilon^2\phi_2^3) + \frac{(z^2\kappa^3\phi_0' + z\kappa^2\phi_1' + \kappa\partial_z\phi_2)z\kappa}{1-\varepsilon z\kappa}\Bigg) \\
& + \left(\frac{z^2\kappa^3}{1-\varepsilon z\kappa}\partial_z + \frac{W^{(4)}(h)}{2}(2\phi_1 + \varepsilon\phi_2)\phi_2\right)(\kappa\phi_0' + \mathrm{L}_0\phi_1) - (\mathrm{H}\partial_z + \varepsilon\Delta_g \\
& - W'''(h)\phi_{\geqslant 1} + \eta_1)\left(W'''(\phi_0)\phi_1\phi_2 + z^2\kappa^3\phi_0' + z\kappa^2\phi_1' + \kappa\partial_z\phi_2 + \frac{W^{(4)}(\phi_0)}{3!}\phi_1^3\right) \\
& - \left(\Delta_g - z\kappa^2\partial_z - \frac{W^{(4)}(h)}{2}\phi_{\geqslant 1}^2 - W'''(\phi_0)\phi_2\right)\left(\kappa\phi_1' + z\kappa^2\phi_0' + \mathrm{L}_0\phi_2 + \frac{W'''(\phi_0)}{2}\phi_1^2\right),
\end{aligned}
$$

where the h terms denote remainders from Taylor expansion. The highest derivative with respect to s arises from $\Delta_g\phi_2$ where $\phi_2 = \phi_2(s,z)$ through its definition in (3.7).

The projection of F_2 onto ϕ_0' is similar to the calculation of (3.8) and (3.10) and omitted. $L^2(\mathbb{R}_\ell)$ to $z\phi_0'$ is zero. To estimate the projection of F_3 in $\mathbb{L}^2(\mathbb{R}_\ell)$ to the function $\phi_0' = \phi_0'(z)$, it suffices to consider the odd part of F_3. Indeed, since ϕ_0, ϕ_1 are all even functions with respect to z, we have

$$
\begin{aligned}
\mathrm{F}_3^{\mathrm{odd}} = & \mathrm{L}_0\left(\kappa\partial_z\phi_2^{\mathrm{even}} + z^2\kappa^3\phi_0' + W'''(\phi_0)\phi_1\phi_2^{\mathrm{odd}}\right) - \Delta_s\kappa\phi_0' + \left(-\eta_1 + W'''(\phi_0)\phi_1\right) \\
& \times (\mathrm{L}_0\phi_2^{\mathrm{odd}} + \kappa\phi_1') + \kappa\partial_z\left(\mathrm{L}_0\phi_2^{\mathrm{even}} + +z\kappa^2\phi_0' + \frac{W'''(\phi_0)}{2}\phi_1^2\right) \\
& + \kappa\frac{W^{(4)}(\phi_0)}{2}\phi_1^2\phi_0' + \kappa W'''(\phi_0)\phi_2^{\mathrm{even}}\phi_0' + \kappa^3 z\phi_0'' + \eta_d W''(\phi_0)\phi_2^{\mathrm{odd}}.
\end{aligned}
$$

Integrating by parts, using properties of L_0 from Lemma 3.1 and re-organizing, we obtain

$$
\int_{\mathbb{R}_\ell} \mathrm{F}_3\phi_0'\,\mathrm{d}z = -\Delta_s\kappa m_1^2 - \frac{\eta_1\kappa}{2}\int_{\mathbb{R}_\ell}\mathrm{L}_0\phi_1\phi_0' z\,\mathrm{d}z + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + O(e^{-\frac{\ell\nu}{\varepsilon}})
$$

where

$$
\mathcal{I}_1 := \kappa\int_{\mathbb{R}_\ell}\mathrm{L}_0^2\phi_2\phi_0' z\,\mathrm{d}z; \qquad \mathcal{I}_2 := \int_{\mathbb{R}_\ell} W'''(\phi_0)\phi_0'\phi_1\mathrm{L}_0\phi_2^{odd}\,\mathrm{d}z;
$$

$$
\mathcal{I}_3 := \eta_d\int_{\mathbb{R}_\ell} W''(\phi_0)\phi_0'\phi_2^{odd}\,\mathrm{d}z.
$$

For $\phi_1 = \phi_1(\sigma)$ and some smooth function $\alpha = \alpha(\sigma)$, the projection of F_3 in (3.15) follows from the identities:

$$
\begin{aligned}
\mathcal{I}_1 = &-\frac{\kappa^3}{2}m_1^2 + \eta_1\kappa\int_{\mathbb{R}_\ell} \mathrm{L}_0\phi_1\phi_0' z\,\mathrm{d}z - \kappa\int_{\mathbb{R}_\ell} W'''(\phi_0)\phi_1\mathrm{L}_0\phi_1\phi_0' z\,\mathrm{d}z \\
&-\eta_d\kappa\int_{\mathbb{R}_\ell} W''(\phi_0)\phi_1\phi_0' z\,\mathrm{d}z + \kappa\int_{\mathbb{R}_\ell} W'''(\phi_0)\phi_1^2\phi_0''\,\mathrm{d}z; \\
\mathcal{I}_2 = &-\kappa\int_{\mathbb{R}_\ell} W'''(\phi_0)\phi_0'\phi_1\mathrm{L}_0^{-1}((-\eta_1+2W'''(\phi_0)\phi_1(\sigma_1^*))\phi_0') \\
&-2\kappa\int_{\mathbb{R}_\ell} W'''(\phi_0)\phi_0'\phi_1\phi_1'(\sigma_1^*)\,\mathrm{d}z; \\
\mathcal{I}_3 = &-\eta_d\kappa\int_{\mathbb{R}_\ell} W''(\phi_0)\phi_0'\mathrm{L}_0^{-2}\left(2\mathrm{L}_0\phi_1'(\sigma_1^*) + (-\eta_1+2W'''(\phi_0)\phi_1(\sigma_1^*))\phi_0'\right)\mathrm{d}z. \quad \square
\end{aligned}
$$

Outside of $\Gamma_{\mathbf{p}}^{2\ell}$, the profile $\Phi_{\mathbf{p}}$ reduces to a constant value that admits the expansion

$$
\Phi = b_- + \varepsilon\phi_1^\infty + \varepsilon^2\phi_2^\infty, \tag{3.19}
$$

where the leading order correction relates to the bulk density parameter

$$
\phi_1^\infty = B_2^\infty\sigma.
$$

The flow (1.3) conserves the system mass, making it a key parameter that is fixed by the initial data. As we study bilayers of length $O(1)$ it is natural to scale the mass

$$
\int_\Omega (u-b_-)\,\mathrm{d}x = \varepsilon M_0. \tag{3.20}
$$

We adjust the bulk density parameter so that $\Phi_{\mathbf{p}}$ has mass εM_0, and a solution u of (1.3) satisfies

$$
0 = \langle u(t) - \Phi_{\mathbf{p}}\rangle_{L^2} = \frac{\varepsilon M_0}{|\Omega|} - \langle \Phi_{\mathbf{p}} - b_-\rangle_{L^2}. \tag{3.21}
$$

The exact relation of σ required to guarantee (3.21) is determined from the expansion (3.12) of $\Phi_{\mathbf{p}}$ with $\phi_1 = \phi_1(\sigma)$ given by (3.5),

$$
\sigma(\mathbf{p}) = \frac{1}{\overline{B}_{\mathbf{p},2}}\left\{M_0 - \int_\Omega \left[\frac{1}{\varepsilon}\Big(\phi_0(z_{\mathbf{p}}) - b_- + \varepsilon^2\phi_2(s_{\mathbf{p}}, z_{\mathbf{p}})\Big) + \frac{\eta_d}{2}\mathrm{L}_{\mathbf{p},0}^{-1}(z_{\mathbf{p}}\phi_0')\right]\mathrm{d}x\right\}. \tag{3.22}
$$

The bilayer manifold of perturbations from Φ_0 is constructed as the graph of $\Phi_{\mathbf{p}}$ over the domain $\mathcal{D}_\delta$ subject to the mass condition $\langle \Phi_{\mathbf{p}} - b_-\rangle_{L^2} = \varepsilon M_0/|\Omega|$.

Definition 3.3 *(Bilayer Manifold).* Fix $K_0, \ell_0, \delta > 0$. Given a base point interface $\Gamma_0 \in \mathcal{G}^4_{K_0,2\ell_0}$ and system mass M_0, we define the bilayer manifold $\mathcal{M}_b(\Gamma_0, M_0)$ to be the graph of the map $\mathbf{p} \mapsto \Phi_{\mathbf{p}}(\sigma)$ over the domain $\mathcal{D}_\delta$ with $\sigma = \sigma(\mathbf{p})$ given by (3.22).

From Lemma 2.11 for each fixed K_0, ℓ_0 there exists $K, \ell > 0$ such that for all $\mathbf{p} \in \mathcal{D}_\delta$ the interfaces $\Gamma_{\mathbf{p}} \in \mathcal{G}^2_{K,2\ell}$ are 2ℓ far from self-intersection and each bilayer distribution $\Phi_{\mathbf{p}}$ has the mass $(b_-|\Omega| + \varepsilon M_0)$.

Lemma 3.4. *For a given bilayer manifold, the map $\sigma = \sigma(\mathbf{p})$ over $\mathcal{D}_\delta$ can be approximated by*

$$\sigma(\mathbf{p}) = \frac{M_0 - m_0|\Gamma_0|}{B_2^\infty|\Omega|} - \frac{m_0|\Gamma_0|}{B_2^\infty|\Omega|}\mathrm{p}_0 + O(\varepsilon),$$

where B_2^∞ is the nonzero far field of B_2 introduced in (3.3) and p_0 is the first component of $\mathbf{p}$ that scales the length of $\Gamma_{\mathbf{p}}$.

Proof. At leading order, the mass per unit length of interface associated to $\Phi_{\mathbf{p}}$ is independent of $\mathbf{p}$ and given by m_0, defined in (3.11). The mass of $\Phi_{\mathbf{p}}$ satisfies

$$M_0 = \frac{|\Omega|\langle \Phi_{\mathbf{p}} - b_-\rangle_{L^2}}{\varepsilon} = m_0|\Gamma_{\mathbf{p}}| + B_2^\infty|\Omega|\sigma + O(\varepsilon). \tag{3.23}$$

Combining this with (2.30) yields the result. $\square$

Remark 3.5. In the companion paper [8] we present a refinement of $\Phi_{\mathbf{p}}$ which reduces to an equilibrium of the system for $\mathbf{p} = (\mathrm{p}_0, \mathrm{p}_1, \mathrm{p}_2, 0, \dots, 0)$, e.g., when $\hat{\mathbf{p}} = \mathbf{0}$, and Γ_0 is a circle.

At leading order the residual of $\Phi_{\mathbf{p}}$ is controlled by the deviation of the bulk parameter σ from σ_1^*.

Lemma 3.6. *Under assumption (2.29), the residual satisfies*

$$\|\Pi_0 \mathrm{F}(\Phi_{\mathbf{p}})\|_{L^2} \lesssim \varepsilon^{5/2}|\sigma - \sigma_1^*| + \varepsilon^{7/2}(1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}).$$

Proof. The second estimate results directly from the form of F_2 and F_3 in (3.14) and the use of the estimates

$$\|\kappa_{\mathbf{p}}\|_{L^\infty} \lesssim 1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_2} \lesssim 1, \qquad \|\Delta_{s_{\mathbf{p}}}\kappa_{\mathbf{p}}\|_{L^2(\mathscr{I}_{\mathbf{p}})} \lesssim 1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}. \quad \square$$

4. Fast and slow spaces and coercivity

The nonlinear stability of the bilayer manifold hinges upon the properties of the linearization of the flow (1.3) about each fixed quasi-steady bilayer distribution $\Phi_{\mathbf{p}}$ constructed in Lemma 3.2. In this section we establish the coercivity properties of the linearized operators that allows the nonlinear control established in Section 5.

We fix K_0, ℓ_0 and a base point interface $\Gamma_0 \in \mathcal{G}^4_{K_0,2\ell_0}$ and choose $K, \ell > 0$ such that $\Gamma_{\mathbf{p}} \in \mathcal{G}^2_{K,2\ell}$ for all $\mathbf{p} \in \mathcal{D}_\delta$. The linearization of (1.3) about $\Phi_{\mathbf{p}}$ takes the form $\Pi_0\mathbb{L}$ where

$$\mathbb{L} := \frac{\delta^2 \mathcal{F}}{\delta u^2}\Big|_{u=\Phi_{\mathbf{p}}} = (\varepsilon^2\Delta - W''(\Phi_{\mathbf{p}}) + \varepsilon\eta_1)(\varepsilon^2\Delta - W''(\Phi_{\mathbf{p}})) \\ - (\varepsilon^2\Delta\Phi_{\mathbf{p}} - W'(\Phi_{\mathbf{p}}))W'''(\Phi_{\mathbf{p}}) + \varepsilon\eta_d W''(\Phi_{\mathbf{p}}), \tag{4.1}$$

denotes the second variational derivative of $\mathcal{F}$ at $\Phi_{\mathbf{p}}$ and recall that $\eta_d = \eta_1 - \eta_2$. When restricted to functions with support within the reach $\Gamma_{\mathbf{p}}^{2\ell}$, the Cartesian Laplacian admits the local coordinate expression (2.11) in terms of $(s_{\mathbf{p}}, z_{\mathbf{p}})$ which induces the expansion

$$\mathbb{L} = \mathbb{L}_0 + \varepsilon\mathbb{L}_1 + \varepsilon^2\mathbb{L}_{\geqslant 2}. \tag{4.2}$$

The leading order operator takes the form

$$\mathbb{L}_0 := \left(\mathrm{L}_0 - \varepsilon^2\Delta_{s_{\mathbf{p}}}\right)^2 = \mathcal{L}^2, \tag{4.3}$$

where we have introduced $\mathcal{L} := \mathrm{L}_0 - \varepsilon^2\Delta_{s_{\mathbf{p}}}$. Much of the structure of the FCH flow stems from $\mathbb{L}_0$, particularly its balancing of the $\Gamma_{\mathbf{p}}$-dressed operator L_0, defined in (3.2), against the Laplace-Beltrami operator associated to $\Gamma_{\mathbf{p}}$. The next correction to $\mathbb{L}$ takes the form

$$\mathbb{L}_1 = (\kappa_{\mathbf{p}}\partial_{z_{\mathbf{p}}} + W'''(\phi_0)\phi_1 - z_{\mathbf{p}}\varepsilon^2 D_{s_{\mathbf{p}},2} - \eta_1)\mathcal{L} + \mathcal{L}(\kappa_{\mathbf{p}}\partial_{z_{\mathbf{p}}} + W'''(\phi_0)\phi_1 \\ - z_{\mathbf{p}}\varepsilon^2 D_{s_{\mathbf{p}},2}) + W'''(\phi_0)\left(\kappa_{\mathbf{p}}\left(\phi_0'\right) + \mathrm{L}_0\phi_1\right) + \eta_d W''(\phi_0). \tag{4.4}$$

The second and higher order correction term, $\mathbb{L}_{\geqslant 2}$, is relatively compact with respect to $\mathbb{L}_0$ and its precise form is not material. We use the expansion (4.2) to construct approximate slow spaces, the meander and pearling spaces, that characterize the small spectrum of $\mathbb{L}$ in the sense that the operator is uniformly coercive on their complement. We tune the spectral cut-off parameter ρ that controls the size of the pearling and meander spaces and to preserve the asymptotically large gap in their in-plane wave numbers while obtaining optimal coercivity.

4.1. Approximate slow spaces

Up to exponentially small terms, the approximate slow space $\mathcal{Z}$ is a product of functions of $z_{\mathbf{p}}$ and $s_{\mathbf{p}}$ that exploit the balance of the operator $\mathbb{L}_0$ viewed as acting on the tensor product space $L^2(\mathbb{R}) \times L^2(\mathscr{I}_{\mathbf{p}})$. As both $\mathbb{L}_0$ and $\mathcal{L}$ are self-adjoint, it is sufficient to establish coercivity of $\mathcal{L}$. The spectrum of L_0, in particular its first two eigenmodes $\{\psi_k\}_{k=0,1}$ of L_0 are introduced in Lemma 3.1. The Laplace-Beltrami eigenvalues $\{\beta_{\mathbf{p},j}^2\}_{j\geqslant 0}$ of $\Delta_{s_{\mathbf{p}}} = \partial_{\tilde{s}_{\mathbf{p}}}^2$ are discussed in (2.25).

Definition 4.1. For $k = 0, 1$, we introduce the disjoint index sets:

$$\Sigma_k = \Sigma_k(\mathbf{p}, \rho) = \left\{ j \,\middle|\, \Lambda_{kj}^2 := (\lambda_k + \varepsilon^2\beta_{\mathbf{p},j}^2)^2 \leqslant \rho \right\}, \tag{4.5}$$

their union, $\Sigma := \Sigma_0 \cup \Sigma_1$, and the index function $I : \Sigma \mapsto \{0, 1\}$ which takes the value $I(j) = k$ if $j \in \Sigma_k$.

The preliminary pearling and meander spaces, denoted by $\mathcal{Z}^0(\mathbf{p}, \rho)$ and $\mathcal{Z}^1(\mathbf{p}, \rho)$ respectively, are defined in terms of their basis functions

$$Z_{\mathbf{p}}^{I(j)j} := \tilde{\psi}_{I(j)}(z_{\mathbf{p}}(x))\tilde{\Theta}_j(\tilde{s}_{\mathbf{p}}(x)), \quad j \in \Sigma, \tag{4.6}$$

where the dressed and scaled versions of the eigenmodes of L_0 are defined by

$$\tilde{\psi}_k(z_{\mathbf{p}}) := \varepsilon^{-1/2}\psi_k(z_{\mathbf{p}}) \quad k = 0, 1.$$

In particular,

$$\mathcal{Z}^k(\mathbf{p}, \rho) = \mathrm{span}\left\{ \mathcal{Z}_{\mathbf{p}}^{kj} \mid j \in \Sigma_k \right\},$$

and the preliminary slow space $\mathcal{Z}(\mathbf{p}, \rho) := \mathcal{Z}^0(\mathbf{p}, \rho) \cup \mathcal{Z}^1(\mathbf{p}, \rho) \subset L^2(\Omega)$, is their union. For simplicity of notation, we use $\mathcal{Z}^k$, $\mathcal{Z}$ to denote $\mathcal{Z}^k(\mathbf{p}, \rho)$ and $\mathcal{Z}(\mathbf{p}, \rho)$, respectively when there is no ambiguity.

The exponential decay of $\tilde{\psi}_k$ to zero away from the interface implies that the corrections arising from dressing are exponentially small, in particular there exist $\nu > 0$ such that

$$\mathbb{L}_0 Z_{\mathbf{p}}^{I(i)i} = \Lambda_{I(i)i}^2 Z_{\mathbf{p}}^{I(i)i} + O(e^{-\ell\nu/\varepsilon}), \tag{4.7}$$

for all $i \in \Sigma$. Since the set $\{\Lambda_{I(i)i}^2\}_{i\in\Sigma}$ lies in the interval $(0, \rho)$ the functions in the slow space are compressed by a factor of ρ under the action of $\mathbb{L}_0$.

To estimate the sizes N_0 and N_1 of Σ_0 and Σ_1, we remark from (2.25) and (2.17) that $\beta_{\mathbf{p},j}^2 \sim Cj^2$. The ground-state eigenvalue $\lambda_0 < 0$, hence k lies in $\Sigma_0(\rho)$ if and only if

$$\varepsilon^{-1}\sqrt{-\lambda_0 - \rho^{1/2}} \lesssim j \lesssim \varepsilon^{-1}\sqrt{-\lambda_0 + \rho^{1/2}}, \quad \Longrightarrow \quad N_0 := |\Sigma_0(\rho)| \sim \varepsilon^{-1}\rho^{1/2}. \tag{4.8}$$

On the other hand $\lambda_1 = 0$, so j lies in $\Sigma_1(\rho)$ if and only if

$$0 \leqslant j \lesssim \varepsilon^{-1}\rho^{1/4}, \quad \Longrightarrow \quad N_1 := |\Sigma_1(\rho)| \sim \varepsilon^{-1}\rho^{1/4}. \tag{4.9}$$

The lower bound of elements in Σ_0, $\varepsilon^{-1}\sqrt{-\lambda_0 - \rho^{1/2}}$, is of order ε^{-1} while the upper bound of Σ_1 is of order $\varepsilon^{-1}\rho^{1/4}$. We deduce that Σ_0 and Σ_1 are disjoint for ρ suitably small. Indeed there exists $\rho_0, c > 0$ such that for $\rho < \rho_0$ the associated in-plane wave numbers $\{\beta_i\}$ satisfy

$$|\beta_i - \beta_j| \geqslant c\varepsilon^{-1}, \quad \forall i \in \Sigma_0, j \in \Sigma_1. \tag{4.10}$$

The slow space $\mathcal{Z}$ has dimension

$$N := |\Sigma(\rho)| = |\Sigma_0(\rho)| + |\Sigma_1(\rho)| \sim \varepsilon^{-1}\rho^{1/4}.$$

Remark 4.2. The wave-number gap (4.10) plays an important role in bounding interactions between meander and pearling modes. In particular it yields the factor of ε in the upper bound of (4.12). This is used in Proposition 4.8 and is required to close the nonlinear estimates in the follow-on paper [8].

Using the formalism of Notation 2.1 we have the following estimates.

Lemma 4.3. *Assume* $\mathbf{p} \in \mathcal{D}_\delta$ *with* $\mathcal{D}_\delta$ *introduced in* (2.29)*,* ρ *suitably small and* $h = h(\boldsymbol{\gamma}_{\mathbf{p}}^{(k)})$ *is a function satisfying Notation* 2.1*. Then there exists a matrix* $\mathbb{E} = (\mathbb{E}_{ij})$ *which is bounded in* l_*^2 *as a map from* $l^2(\mathbb{R}^N)$ *to* $l^2(\mathbb{R}^N)$ *such that*

$$\left| \int_{\mathscr{I}_{\mathbf{p}}} h(\boldsymbol{\gamma}_{\mathbf{p}}^{(k)}) \tilde{\Theta}_i \tilde{\Theta}_j \, \mathrm{d}\tilde{s}_{\mathbf{p}} \right| \lesssim \mathbb{E}_{ij} \tag{4.11}$$

hold for $i, j \in \Sigma = \Sigma_0 \cup \Sigma_1$*, and* $k = 1, 2$*. Moreover for all* i, j *such that* $I(i) \neq I(j)$ *we have*

$$\left| \int_{\mathscr{I}_{\mathbf{p}}} h(\boldsymbol{\gamma}_{\mathbf{p}}^{(k)}) \tilde{\Theta}_i \tilde{\Theta}_j \, \mathrm{d}\tilde{s}_{\mathbf{p}} \right| \lesssim \varepsilon (1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}) \mathbb{E}_{ij}. \tag{4.12}$$

Proof. With h satisfying Notation 2.1, we bound the L^∞-norm from Lemma 2.12 as

$$|h(\boldsymbol{\gamma}_{\mathbf{p}}^{(k)})| \lesssim 1, \qquad k = 1, 2 \tag{4.13}$$

and deduce from Lemma 2.13 that these terms are $O(1)\mathbb{E}_{ij}$ for a matrix $\mathbb{E}$ as above; the estimate (4.11) follows. For (4.12), when $I(i) \neq I(j)$ we have $\beta_i \neq \beta_j$. Integrating by parts twice we transfer the highest derivative of $\tilde{\Theta}_i$ to $\tilde{\Theta}_j$, which generates lower derivative terms from the product rule with h. Noting $\partial_{\tilde{s}_{\mathbf{p}}} = \nabla_{s_{\mathbf{p}}}$, we write the result in the form

$$\begin{aligned} \int_{\mathscr{I}_{\mathbf{p}}} h\left(\boldsymbol{\gamma}_{\mathbf{p}}''\right) \tilde{\Theta}_i'' \tilde{\Theta}_j \, \mathrm{d}\tilde{s}_{\mathbf{p}} &= - \int_{\mathscr{I}_{\mathbf{p}}} h\left(\boldsymbol{\gamma}_{\mathbf{p}}''\right) \tilde{\Theta}_i' \tilde{\Theta}_j' \, \mathrm{d}\tilde{s}_{\mathbf{p}} - \int_{\mathscr{I}_{\mathbf{p}}} \nabla_{s_{\mathbf{p}}} h \tilde{\Theta}_i' \tilde{\Theta}_j \, \mathrm{d}\tilde{s}_{\mathbf{p}} \\ &= \int_{\mathscr{I}_{\mathbf{p}}} h\left(\boldsymbol{\gamma}_{\mathbf{p}}''\right) \tilde{\Theta}_i \tilde{\Theta}_j'' |\boldsymbol{\gamma}_{\mathbf{p}}'| \, \mathrm{d}s_{\mathbf{p}} + \int_{\mathscr{I}_{\mathbf{p}}} \nabla_{s_{\mathbf{p}}} h \left(\tilde{\Theta}_i \tilde{\Theta}_j' - \tilde{\Theta}_i' \tilde{\Theta}_j\right) \mathrm{d}\tilde{s}_{\mathbf{p}}. \end{aligned} \tag{4.14}$$

Applying identity (2.25) and after some algebraic rearrangement we obtain

$$(\beta_{\mathbf{p},j}^2 - \beta_{\mathbf{p},i}^2) \int_{\mathscr{I}_{\mathbf{p}}} h\left(\boldsymbol{\gamma}_{\mathbf{p}}''\right) \tilde{\Theta}_i \tilde{\Theta}_j \, \mathrm{d}\tilde{s}_{\mathbf{p}} = \int_{\mathscr{I}_{\mathbf{p}}} \nabla_{s_{\mathbf{p}}} h \left(\tilde{\Theta}_i \tilde{\Theta}_j' - \tilde{\Theta}_i' \tilde{\Theta}_j\right) \mathrm{d}\tilde{s}_{\mathbf{p}}. \tag{4.15}$$

By the relation (4.17), the right hand side can be rewritten as

$$\int_{\mathscr{I}_{\mathbf{p}}} \nabla_{s_{\mathbf{p}}} h \left(\tilde{\Theta}_i \tilde{\Theta}_j' - \tilde{\Theta}_i' \tilde{\Theta}_j\right) \mathrm{d}\tilde{s}_{\mathbf{p}} = \beta_{\mathbf{p},j} \int_{\mathscr{I}_{\mathbf{p}}} \nabla_{s_{\mathbf{p}}} h \tilde{\Theta}_i \tilde{\Theta}_j' / \beta_{\mathbf{p},j} \, \mathrm{d}\tilde{s}_{\mathbf{p}} - \beta_{\mathbf{p},i} \int_{\mathscr{I}_{\mathbf{p}}} \nabla_{s_{\mathbf{p}}} h \tilde{\Theta}_i' / \beta_{\mathbf{p},i} \tilde{\Theta}_j \, \mathrm{d}\tilde{s}_{\mathbf{p}}.$$

Note that from (4.17) that $\{\pm \tilde{\Theta}_j' / \beta_{\mathbf{p},j}, j \in \Sigma\}$ is equivalent to the set $\{\pm \tilde{\Theta}_j, j \in \Sigma\}$. Hence Lemma 2.13 applies and there exists a matrix $\mathbb{E} = (\mathbb{E}_{ij})$ with l_2^* norm one such that

$$\left| \int\limits_{\mathscr{I}_{\mathbf{p}}} \nabla_{s_{\mathbf{p}}} h \left(\tilde{\Theta}_i \tilde{\Theta}_j' - \tilde{\Theta}_i' \tilde{\Theta}_j \right) \mathrm{d}\tilde{s}_{\mathbf{p}} \right| \lesssim (\beta_{\mathbf{p},i} + \beta_{\mathbf{p},j})(1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_3}) \mathbb{E}_{ij}.$$

Here we also used Lemma 2.12 to bound the L^∞-norm of $\nabla_{s_{\mathbf{p}}} h$, or $\nabla_{s_{\mathbf{p}}} h$. We divide both sides of the equality (4.15) by the quantity $\beta_{\mathbf{p},j}^2 - \beta_{\mathbf{p},i}^2$ to obtain

$$\int\limits_{\mathscr{I}_{\mathbf{p}}} h\left(\boldsymbol{\gamma}_{\mathbf{p}}''\right) \tilde{\Theta}_i \tilde{\Theta}_j \, \mathrm{d}\tilde{s}_{\mathbf{p}} \lesssim \frac{1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_3}}{|\beta_{\mathbf{p},i} - \beta_{\mathbf{p},j}|} \mathbb{E}_{ij} \lesssim \frac{1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}}{|\beta_{\mathbf{p},i} - \beta_{\mathbf{p},j}|} \mathbb{E}_{ij}. \tag{4.16}$$

Here we used embedding Lemma 2.8. Moreover, $\beta_{\mathbf{p},k} = \frac{\beta_k}{(1+\mathrm{p}_0)R_0}$ by (2.26), the bound (4.12) follows from (4.10) and (4.16). $\square$

The estimates of Lemma 4.3 are central to controlling the action of the operator $\mathbb{L}$ when restricted to the asymptotically large slow space $\mathcal{Z}$. A benefit of conducting our analysis in $\mathbb{R}^2$ is that single derivatives of the Laplace-Beltrami eigenmodes behave well. Indeed, from (2.17) we have

$$\tilde{\Theta}_i' = \begin{cases} -\beta_{\mathbf{p},i} \tilde{\Theta}_{i+1}, & i \quad \text{odd}, \\ \beta_{\mathbf{p},i} \tilde{\Theta}_{i-1}, & i \quad \text{even}, \end{cases} \tag{4.17}$$

which furthermore implies

$$\tilde{\Theta}_i^{(k)} \in \mathrm{span}\{\tilde{\Theta}_i, \tilde{\Theta}_i'\}. \tag{4.18}$$

The following lemma provides the asymptotic form of the restriction of $\mathbb{L}$. It uses shape parameter

$$S_1 := \int\limits_{\mathbb{R}} W'''(\phi_0(z)) B_1(z) \, |\psi_0(z)|^2 \, \mathrm{d}z, \tag{4.19}$$

where $B_{\mathbf{p},1}$ and $\phi_1 = \phi_1(z_{\mathbf{p}}; \sigma)$ are introduced in (3.3) and (3.5) respectively. This parameter is independent of choice of $\mathbf{p} \in \mathcal{D}_\delta$. In addition, for $k = 0, 1$, we have the σ dependent parameters

$$S_{2,k}(\sigma) = 2 \int\limits_{\mathbb{R}} W'''(\phi_0(z)) \phi_1(z; \sigma) |\psi_k(z)|^2 \, \mathrm{d}z - \eta_1. \tag{4.20}$$

Lemma 4.4. *Let $\mathbf{p} \in \mathcal{D}_\delta$ with $\mathcal{D}_\delta$ defined in (2.29) and ρ suitably small. The basis functions of the slow space $\left\{ Z_{\mathbf{p}}^{I(k)k}, k \in \Sigma \right\}$ are approximately orthonormal in L^2. More precisely there exists a matrix $\mathbb{E}$ with l_*^2-norm one for which*

$$\left\langle Z_{\mathbf{p}}^{I(i)i}, Z_{\mathbf{p}}^{I(j)j} \right\rangle_{L^2} = \begin{cases} (1 + \mathrm{p}_0)\, \delta_{ij}, & I(i) = I(j); \\ O\left(\varepsilon^2, \varepsilon^2 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\right) \mathbb{E}_{ij}, & I(i) \neq I(j). \end{cases} \tag{4.21}$$

Moreover, the action of the linear operator $\mathbb{L}$ *restricted to the preliminary slow space* $\mathcal{Z}(\mathbf{p},\rho)$ *is given by*

$$\mathbb{M}_{ij} := \left\langle \mathbb{L} Z_{\mathbf{p}}^{I(i)i}, Z_{\mathbf{p}}^{I(j)j} \right\rangle_{L^2},$$

whose entries have the leading order approximations

$$\mathbb{M}_{ij} = \begin{cases} (1+\mathrm{p}_0)\Big(\Lambda_{0i}^2 + \varepsilon(\sigma S_1 + \eta_d \lambda_0) + \varepsilon S_{2,0}(\sigma)\Lambda_{0i}\Big) + O(\varepsilon^2) & i = j \text{ and } I(i) = 0; \\ (1+\mathrm{p}_0)\Big(\Lambda_{1i}^2 + \varepsilon S_{2,1}(\sigma)\Lambda_{1i}\Big) + O(\varepsilon^2) & \text{if } i = j \text{ and } I(i) = 1; \\ O\Big(\varepsilon^2\Big)\,\mathbb{E}_{ij} & i \neq j \text{ and } I(i) = I(j); \\ O\Big(\varepsilon^2, \varepsilon^2 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\Big)\,\mathbb{E}_{ij} & I(i) \neq I(j). \end{cases}$$

Remark 4.5. In the absence of the asymptotic gap between Σ_0 and Σ_1, then the leading term in $\mathbb{M}_{ij}$ for $I(i) \neq I(j)$ generically increases to $O(\varepsilon)$.

Proof. Using the localization of the basis functions, we establish the approximate orthonormality (4.21) by integrating over $\Gamma_{\mathbf{p}}^{2\ell}$. Recalling that $\mathrm{d}x = \tilde{\mathrm{J}}\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}}$ with $\tilde{\mathrm{J}} = \varepsilon(1 - \varepsilon z_{\mathbf{p}}\kappa_{\mathbf{p}})$ in local coordinates, we write

$$\left\langle Z_{\mathbf{p}}^{I(i)i}, Z_{\mathbf{p}}^{I(j)j} \right\rangle_{L^2} = \int_{\mathbb{R}_{2\ell}} \int_{\mathscr{I}_{\mathbf{p}}} \psi_{I(i)}\psi_{I(j)}\tilde{\Theta}_i\tilde{\Theta}_j\,\mathrm{d}\tilde{s}_{\mathbf{p}}\mathrm{d}z_{\mathbf{p}} - \varepsilon \int_{\mathscr{I}_{\mathbf{p}}} \kappa_{\mathbf{p}}\tilde{\Theta}_i\tilde{\Theta}_j\,\mathrm{d}\tilde{s}_{\mathbf{p}} \int_{\mathbb{R}_{2\ell}} \psi_{I(i)}\psi_{I(j)} z_{\mathbf{p}}\mathrm{d}z_{\mathbf{p}}. \tag{4.22}$$

The orthogonality of $\{\tilde{\Theta}_i\}$ given in (2.26) shows that the first term on the right-hand side contributes the main δ_{ij} term in (4.21). We claim second term on the right hand side can be bounded by

$$\varepsilon \int_{\mathscr{I}_{\mathbf{p}}} \kappa_{\mathbf{p}}\tilde{\Theta}_i\tilde{\Theta}_j\,\mathrm{d}\tilde{s}_{\mathbf{p}} \int_{\mathbb{R}_{2\ell}} \psi_{I(i)}\psi_{I(j)} z_{\mathbf{p}}\mathrm{d}z_{\mathbf{p}} = \begin{cases} 0, & \text{if } I(i) = I(j) \\ O(\varepsilon^2, \varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2})\mathbb{E}_{ij}, & \text{if } I(i) \neq I(j). \end{cases} \tag{4.23}$$

Indeed, if $I(i) = I(j)$ (4.23) holds by parity since $|\psi_{I(i)}|^2 z_{\mathbf{p}}$ is odd. On the other hand, if $I(i) \neq I(j)$ we use estimate (4.12) from Lemma 4.3 to bound the projection of $\kappa_{\mathbf{p}}$ to $\tilde{\Theta}_i\tilde{\Theta}_j$ in $L^2(\mathscr{I}_{\mathbf{p}})$, and (4.23) follows. Returning back to (4.22) implies the approximate orthogonality (4.21).

To establish the estimates of $\mathbb{L}_{\mathbf{p}}$ on $\mathcal{Z}$ we apply the expansion (4.2) of $\mathbb{L}$ to the inner product:

$$\begin{aligned} \left\langle \mathbb{L} Z_{\mathbf{p}}^{I(i)i}, Z_{\mathbf{p}}^{I(j)j} \right\rangle_{L^2} = \left\langle \mathbb{L}_0 Z_{\mathbf{p}}^{I(i)i}, Z_{\mathbf{p}}^{I(j)j} \right\rangle_{L^2} + \varepsilon \left\langle \mathbb{L}_1 Z_{\mathbf{p}}^{I(i)i}, Z_{\mathbf{p}}^{I(j)j} \right\rangle_{L^2} \\ + \varepsilon^2 \left\langle \mathbb{L}_{\geqslant 2} Z_{\mathbf{p}}^{I(i)i}, Z_{\mathbf{p}}^{I(j)j} \right\rangle_{L^2}. \end{aligned} \tag{4.24}$$

Recalling (4.7) and employing the approximate orthogonality identity (4.21), we obtain the leading order approximation

$$\left\langle \mathbb{L}_0 Z_{\mathbf{p}}^{I(i)i}, Z_{\mathbf{p}}^{I(j)j} \right\rangle_{L^2} = \begin{cases} (1+\mathrm{p}_0)\,\Lambda_{I(j)j}^2 \delta_{ij} + O\left(e^{-\ell\nu/\varepsilon}\right)\mathbb{E}_{ij}, & I(i) = I(j); \\ O\left(\varepsilon^2, \varepsilon^2 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\right), & I(i) \neq I(j). \end{cases} \tag{4.25}$$

Estimates on $\mathbb{L}_1$ restricted to $\mathcal{Z}$ are more complicated. Recalling (4.4), direct calculations establish

$$\begin{aligned} \mathbb{L}_1(\psi_{I(i)}\tilde{\Theta}_i) = {} & \Lambda_{I(i)i}\Big(\kappa_{\mathbf{p}}\psi'_{I(i)}\tilde{\Theta}_i + W'''(\phi_0)\phi_1\psi_{I(i)}\tilde{\Theta}_i - z_{\mathbf{p}}\varepsilon^2 D_{s_{\mathbf{p}},2}\tilde{\Theta}_i\psi_{I(i)} \\ & - \eta_1\psi_{I(i)}\tilde{\Theta}_i\Big) + \mathcal{L}\Big(\kappa_{\mathbf{p}}\psi'_{I(i)}\tilde{\Theta}_i + W'''(\phi_0)\phi_1\psi_{I(i)}\tilde{\Theta}_i - z_{\mathbf{p}}\varepsilon^2 D_{s_{\mathbf{p}},2}\tilde{\Theta}_i\psi_{I(i)}\Big) \\ & + W'''(\phi_0)(\kappa_{\mathbf{p}}\phi'_0 + \mathrm{L}_0\phi_1)\psi_{I(i)}\tilde{\Theta}_i + \eta_d W''(\phi_0)\psi_{I(i)}\tilde{\Theta}_i. \end{aligned} \tag{4.26}$$

Since the operators $D_{s_{\mathbf{p}},2}$ and $\mathcal{L}$ incorporate derivatives with respect to $\tilde{s}_{\mathbf{p}}$ scaled with ε, we apply (4.17)-(4.18) and we separate into cases for $\tilde{\Theta}_i$ and $\tilde{\Theta}'_i$. We also exploit the even and odd parity of functions with respect to $z_{\mathbf{p}}$. We define functions $h_1(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})$ and $h_2(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})$, denoting higher order terms, that enjoy the properties of Notation 2.1. With these steps the identity (4.26) is rewritten as

$$\begin{aligned} \mathbb{L}_1(\psi_{I(i)}\tilde{\Theta}_i) = {} & \left(g_1^{\perp}(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}}) + \varepsilon h_1(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})\right)\tilde{\Theta}_i + \left(g_2^{\perp}(\boldsymbol{\gamma}''_{\mathbf{p}}, z_{\mathbf{p}}) + \varepsilon h_2(\boldsymbol{\gamma}''_{\mathbf{p}}, z_{\mathbf{p}})\right)\varepsilon\tilde{\Theta}'_i \\ & + g^*(z_{\mathbf{p}})\tilde{\Theta}_i \end{aligned} \tag{4.27}$$

where the functions $g_k^{\perp} = g_k^{\perp}(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})$ have opposite $z_{\mathbf{p}}$ parity of $\psi_{I(i)}$. Hence they satisfy

$$\int_{\mathbb{R}_{2\ell}} g_k^{\perp}(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})\psi_{I(i)}\,\mathrm{d}z_{\mathbf{p}} = 0, \qquad k = 1, 2. \tag{4.28}$$

The $z_{\mathbf{p}}$ dependent only function $g^* = g^*(z_{\mathbf{p}})$ is given explicitly by

$$\begin{aligned} g^*(z_{\mathbf{p}}) := {} & \Lambda_{I(i)i}\left(W'''(\phi_0)\phi_1\psi_{I(i)} - \eta_1\psi_{I(i)}\right) + \left(\mathrm{L}_0 + \varepsilon^2\beta_{\mathbf{p},i}^2\right)\left(W'''(\phi_0)\phi_1\psi_{I(i)}\right) \\ & + W'''(\phi_0)\mathrm{L}_0\phi_1\psi_{I(i)} + \eta_d W''(\phi_0)\psi_{I(i)}. \end{aligned} \tag{4.29}$$

From (4.27) we decompose the (i, j)-th component of the bilinear form of $\mathbb{L}_1$ restricted to $\mathcal{Z}$ as

$$\left\langle \mathbb{L}_1 Z_{\mathbf{p}}^{I(i)i}, Z_{\mathbf{p}}^{I(j)j} \right\rangle_{L^2} = \mathcal{I}_0 + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \tag{4.30}$$

where we have defined

$$
\begin{aligned}
\mathcal{I}_0 &:= \int_{\mathbb{R}_{2\ell}} g^*(z_{\mathbf{p}})\psi_{I(j)}\,\mathrm{d}z_{\mathbf{p}} \int_{\mathscr{I}_{\mathbf{p}}} \tilde{\Theta}_i\tilde{\Theta}_j\,\mathrm{d}\tilde{s}_{\mathbf{p}},\\
\mathcal{I}_1 &:= \int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}} \left(g_1^{\perp}(z_{\mathbf{p}},\boldsymbol{\gamma}_{\mathbf{p}}'') + \varepsilon h_1(\boldsymbol{\gamma}_{\mathbf{p}}'', z_{\mathbf{p}})\right)\psi_{I(j)}\tilde{\Theta}_i\tilde{\Theta}_j\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}},\\
\mathcal{I}_2 &:= \int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}} \left(g_2^{\perp}(z_{\mathbf{p}},\boldsymbol{\gamma}_{\mathbf{p}}'') + \varepsilon h_2(\boldsymbol{\gamma}_{\mathbf{p}}'', z_{\mathbf{p}})\right)\psi_{I(j)}\varepsilon\tilde{\Theta}_i'\tilde{\Theta}_j\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}},\\
\mathcal{I}_3 &:= -\varepsilon\int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}} \mathbb{L}_1(\psi_{I(i)}\tilde{\Theta}_i)\psi_{I(j)}\tilde{\Theta}_j z_{\mathbf{p}}\kappa_{\mathbf{p}}\,\mathrm{d}\tilde{s}_{\mathbf{p}}\mathrm{d}z_{\mathbf{p}}.
\end{aligned}
\tag{4.31}
$$

In light of orthogonality (4.28), we see that $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ are higher order terms. Indeed, with the aids of Lemma 4.3, (4.17)-(4.18), and uniform bounds on $\varepsilon\beta_{\mathbf{p},i}$, a direct calculation establishes

$$
\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 = O(\varepsilon)\mathbb{E}_{ij}. \tag{4.32}
$$

From the orthogonality of $\{\tilde{\Theta}_i\}$ given in (2.26), the term $\mathcal{I}_0$, is zero unless $i = j$. As $g^* = g^*(z_{\mathbf{p}})$ defined in (4.29) decays exponentially in $z_{\mathbf{p}}$, we may decompose

$$
\int_{\mathbb{R}_{2\ell}} g^*(z_{\mathbf{p}})\psi_{I(i)}\,\mathrm{d}z_{\mathbf{p}} = \mathcal{I}_{01} + \mathcal{I}_{02} + \mathcal{I}_{03} + Ce^{-\ell\nu/\varepsilon}, \tag{4.33}
$$

where we have introduced the sub-terms

$$
\begin{aligned}
\mathcal{I}_{01} &:= \Lambda_{I(i)i}\int_{\mathbb{R}} \left(W'''(\phi_0)\phi_1\psi_{I(i)} - \eta_1\psi_{I(i)}\right)\psi_{I(i)}\,\mathrm{d}z\\
&\quad + \int_{\mathbb{R}} \left(\mathrm{L}_0 + \varepsilon^2\beta_{\mathbf{p},i}^2\right)\left(W'''(\phi_0)\phi_1\psi_{I(i)}\right)\psi_{I(i)}\,\mathrm{d}z,\\
\mathcal{I}_{02} &:= \int_{\mathbb{R}} W'''(\phi_0)\mathrm{L}_0\phi_1\psi_{I(i)}\psi_{I(i)}\,\mathrm{d}z,\\
\mathcal{I}_{03} &:= \eta_d\int_{\mathbb{R}} W''(\phi_0)\psi_{I(i)}\psi_{I(i)}\,\mathrm{d}z.
\end{aligned}
$$

Proceeding term by term, we integrate by parts in the second integral of $\mathcal{I}_{01}$, rewriting it as

$$
\mathcal{I}_{01} = S_{2,I(i)}\Lambda_{I(i)i} \tag{4.34}
$$

where $S_{2,I(i)} = S_{2,I(i)}(\sigma)$ depending on σ is introduced in (4.20). Recalling the definition (3.5) of ϕ_1, we separate $\mathcal{I}_{02}$,

$$\mathcal{I}_{02} = \mathcal{I}_{02,1} + \mathcal{I}_{02,2} := \sigma \int_{\mathbb{R}} W'''(\phi_0) B_1 |\psi_{I(i)}|^2 \, \mathrm{d}z + \frac{\eta_d}{2} \int_{\mathbb{R}} W'''(\phi_0) z \phi_0' |\psi_{I(i)}|^2 \, \mathrm{d}z. \tag{4.35}$$

From the definition of L_0 we observe that

$$W'''(\phi_0)\phi_0'\psi_k = \lambda_k \psi_k' - \mathrm{L}_0 \psi_k', \quad \text{and} \quad \mathrm{L}_0(z\psi_k) = z\lambda_k \psi_k - 2\psi_k',$$

which together with the self-adjointness of L_0 on $L^2(\mathbb{R})$ yield

$$\begin{aligned} \mathcal{I}_{02,2} &= \frac{\eta_d}{2} \int_{\mathbb{R}} \Big(\lambda_{I(i)} \psi_{I(i)}' \psi_{I(i)} z - \psi_{I(i)}' \mathrm{L}_0(z\psi_{I(i)}) \Big) \, \mathrm{d}z \\ &= \eta_d \|\psi_{I(i)}'\|_{L^2(\mathbb{R})}^2. \end{aligned} \tag{4.36}$$

When $I(i) = 1$ we have $\psi_{I(i)} = \phi_0'/m_1$. Recalling the identify $W'''(\phi_0)|\phi_0'|^2 = -\mathrm{L}_0 \phi_0''$ from Lemma 3.1 yields

$$\int_{\mathbb{R}} W'''(\phi_0) B_1 |\psi_1|^2 \, \mathrm{d}z = -\frac{1}{m_1^2} \int_{\mathbb{R}} B_1 \mathrm{L}_0 \phi_0'' \, \mathrm{d}z = -\frac{1}{m_1^2} \int_{\mathbb{R}} \phi_0'' \, \mathrm{d}z = 0, \tag{4.37}$$

and hence $\mathcal{I}_{02,1} = 0$ when $I(i) = 1$. Combining the identity (4.36) with (4.35) we obtain

$$\mathcal{I}_{02} = \sigma S_1 \delta_{I(i)0} + \eta_d \|\psi_{I(i)}'\|_{L^2(\mathbb{R})}^2, \tag{4.38}$$

where S_1 was introduced in (4.19). Finally, from the definitions of L_0 and $\psi_{I(i)}$, $\mathcal{I}_{03}$ reduces to

$$\mathcal{I}_{03} = \eta_d \int_{\mathbb{R}} (\mathrm{L}_0 + \partial_z^2) \psi_{I(i)} \psi_{I(i)} \, \mathrm{d}z = \eta_d \lambda_{I(i)} - \eta_d \|\psi_{I(i)}'\|_{L^2(\mathbb{R})}^2, \tag{4.39}$$

where $\psi_{I(i)}$ has been normalized in $L^2(\mathbb{R})$. Combining estimates (4.34), (4.38) and (4.39) with (4.33) yields for some bounded $\mathbf{p}$-independent constant C,

$$\int_{\mathbb{R}_{2\ell}} g^*(z_{\mathbf{p}}) \psi_{I(i)}(z_{\mathbf{p}}) \, \mathrm{d}z_{\mathbf{p}} = (\sigma S_1 + \eta_d \lambda_0) \delta_{I(i)0} + S_{2,I(i)} \Lambda_{I(i)i} + C e^{-\ell\nu/\varepsilon}, \tag{4.40}$$

which combined with the orthogonality (2.26) and $\mathcal{I}_0$ defined in (4.31) furthermore implies

$$\mathcal{I}_0 = (1 + \mathrm{p}_0) \Big[(\sigma S_1 + \eta_d \lambda_0) \delta_{I(i)0} + S_{2,I(i)} \Lambda_{I(i)i} + O(e^{-\ell\nu/\varepsilon}) \Big] \delta_{ij}. \tag{4.41}$$

Combining estimates (4.41) and (4.32) with (4.30) imply

$$\left\langle \mathbb{L}_1 Z_{\mathbf{p}}^{I(i)i}, Z_{\mathbf{p}}^{I(j)j} \right\rangle_{L^2} = \begin{cases} (1 + \mathrm{p}_0) \left[(\sigma S_1 + \eta_d \lambda_0) \, \delta_{I(i)0} + S_{2,I(i)} \Lambda_{I(i)} \right], & i = j; \\ \quad O(\varepsilon) \mathbb{E}_{ij}, & i \neq j; \end{cases} \tag{4.42}$$

To address the bilinear form induced by $\mathbb{L}_{\geqslant 2}$ we employ (4.18) to arrive at the general form

$$\mathbb{L}_{\geqslant 2} Z_{\mathbf{p}}^{I(i)i} = \varepsilon^{-1/2}\left(h_1(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})\tilde{\Theta}_i + h_2(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})\varepsilon\tilde{\Theta}'_i\right),$$

where the functions h_1 and h_2 enjoy the properties of Notation 2.1 and are localized near $\Gamma_{\mathbf{p}}$. Integrating out $z_{\mathbf{p}}$, and employing Lemma 4.3, (4.17) and the uniform bounds on $\varepsilon\beta_i$ we deduce

$$\left|\left\langle \mathbb{L}_{\geqslant 2} Z_{\mathbf{p}}^{I(i)i}, Z_{\mathbf{p}}^{I(j)j}\right\rangle\right| \lesssim \mathbb{E}_{ij}. \tag{4.43}$$

The conclusion follows from (4.24), the estimates (4.25) and (4.42)-(4.43). $\square$

4.2. Modified approximate slow spaces

The modified spaces are corrections to the preliminary spaces that make them closer to being an invariant subspace of $\mathbb{L}$. This provides the better control required to close the nonlinear estimates of Section 5.

Lemma 4.6. *For $i \in \Sigma$, there exist functions $\varphi_{k,i} = \varphi_{k,i}(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})(k = 1, 2)$ localized near $\Gamma_{\mathbf{p}}$ that enjoy the properties of Notation 2.1 for which*

$$\int_{\mathbb{R}_{2\ell}} \varphi_{k,i}(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})\psi_{I(i)}(z_{\mathbf{p}})\,\mathrm{d}z_{\mathbf{p}} = 0, \qquad k = 1, 2. \tag{4.44}$$

The modified basis functions

$$Z_{\mathbf{p},*}^{I(i)i} := \left(\tilde{\psi}_{I(i)} + \varepsilon\tilde{\varphi}_{1,i}\right)\tilde{\Theta}_i + \varepsilon\tilde{\varphi}_{2,i}\varepsilon\tilde{\Theta}'_i = \varepsilon^{-1/2}\left[\left(\psi_{I(i)} + \varepsilon\varphi_{1,i}\right)\tilde{\Theta}_i + \varepsilon\varphi_{2,i}\varepsilon\tilde{\Theta}'_i\right], \tag{4.45}$$

are $\mathbb{L}$ is invariant up to order ε^2 in $L^2(\Omega)$, satisfying

$$\begin{aligned}\mathbb{L}Z_{\mathbf{p},*}^{I(i)i} &= \left(\Lambda_{I(i)i}^2 + \varepsilon\delta_{I(i)0}(\sigma S_1 + \eta_d\lambda_0) + S_{2,I(i)}(\sigma)\Lambda_{I(i)i}\right)Z_{\mathbf{p},*}^{I(i)i} + \varepsilon^{3/2}\left(h_1\tilde{\Theta}_i + h_2\varepsilon\tilde{\Theta}'_i\right)\\ &\quad + \varepsilon^{3/2}\sum_{k=1}^{4}\left(\varepsilon^{k-1}\partial_{s_{\mathbf{p}}}^k h_{3,k}\tilde{\Theta}_i + \varepsilon^{k-1}\partial_{s_{\mathbf{p}}}^k h_{4,k}\varepsilon\tilde{\Theta}'_i\right).\end{aligned} \tag{4.46}$$

Here the functions $h = h(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})$ are localized near $\Gamma_{\mathbf{p}}$, enjoy the properties of Notation 2.1, and have $L^2(\Omega)$ norm of $O(\sqrt{\varepsilon})$.

Proof. To establish the Lemma it suffices to construct $\varphi_{k,i}$ in the interior region as the dressing process incorporates only exponentially small errors. Using the expansion (4.2) of $\mathbb{L}$, we compute

$$\begin{aligned}\mathbb{L}Z_{\mathbf{p},*}^{I(i)i} &= \mathbb{L}_0 Z_{\mathbf{p}}^{I(i)i} + \varepsilon\cdot\varepsilon^{-1/2}\Big(\mathbb{L}_1(\psi_{I(i)}\tilde{\Theta}_i) + \mathbb{L}_0(\varphi_{1,i}\tilde{\Theta}_i) + \mathbb{L}_0(\varphi_{2,i}\varepsilon\tilde{\Theta}'_i)\Big)\\ &\quad + \varepsilon^2\cdot\varepsilon^{-1/2}\left(\mathbb{L}_1(\varphi_{1,i}\tilde{\Theta}_i) + \mathbb{L}_1(\varphi_{2,i}\varepsilon\tilde{\Theta}'_i) + \varepsilon^{1/2}\mathbb{L}_{\geqslant 2}Z_{\mathbf{p},*}^{I(i)i}\right).\end{aligned} \tag{4.47}$$

The first term is calculated as in (4.7). Since $\mathbb{L}_0 = \mathcal{L}^2$, from (2.25) we see that

$$\mathbb{L}_0(\varphi_{k,i}\tilde{\Theta}_i) = \left(\mathrm{L}_0 + \varepsilon^2\beta_{\mathbf{p},i}^2\right)^2 \varphi_{k,i}\tilde{\Theta}_i + \left(\mathcal{L}^2 - \left(\mathrm{L}_0 + \varepsilon^2\beta_{\mathbf{p},i}^2\right)^2\right)\left(\varphi_{k,i}\tilde{\Theta}_i\right). \tag{4.48}$$

We show that $\varphi_{k,i} = \varphi_{k,i}(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})$ in the sense of Notation 2.1, and consequently, in (4.51), bound the second term of (4.48).

It remains to determine $\varphi_{k,i}$ for which the ε-order term in (4.47) equals $\Lambda_{I(i)i}^2(\varphi_{1,i}\Theta_i + \varphi_{2,i}\varepsilon\Theta'_i)$ to leading order. From (4.27), we define $\varphi_{k,i}(\cdot, \boldsymbol{\gamma}''_{\mathbf{p}})$ as the $L^2(\mathbb{R})$ solutions to

$$\begin{aligned}&\left(\left(\mathrm{L}_0 + \varepsilon^2\beta_{\mathbf{p},i}^2\right)^2 - \Lambda_{I(i)i}^2\right)\varphi_{k,i}\\ &\quad = -g_k^{\perp}(z, \boldsymbol{\gamma}''_{\mathbf{p}}) + \delta_{1I(i)}\Big(\left(\delta_{I(i)0}(\sigma S_1 + \eta_d\lambda_0) + S_{2,I(i)}\Lambda_{I(i)i}\right)\psi_{I(i)} - g^*(z)\Big),\end{aligned} \tag{4.49}$$

in the subspace perpendicular to $\psi_{I(i)}$. The definition is well posed since (4.28) and (4.40) imply that the right-hand side of the identity is orthogonal to $\psi_{I(i)}$ in $L^2(\mathbb{R})$. Dressing these functions on $\Gamma_{\mathbf{p}}$, we extend $\varphi_{k,i}$ to Ω. Applying (4.48), identity (4.27) and (2.25) implies

$$\begin{aligned}&\mathbb{L}_1(\psi_{I(i)}\tilde{\Theta}_i) + \mathbb{L}_0(\varphi_{1,i}\tilde{\Theta}_i) + \mathbb{L}_0(\varphi_{2,i}\tilde{\Theta}_i)\\ &= \left(\delta_{I(i)0}(\sigma S_1 + \eta_d\lambda_0) + S_{2,I(i)}\Lambda_{I(i)i}\right)\psi_{I(i)}\tilde{\Theta}_i + \Lambda_{I(i)i}^2(\varphi_{1,i}\tilde{\Theta}_i + \varphi_{2,i}\tilde{\varepsilon}\Theta'_i)\\ &\quad + \left(\mathcal{L}^2 - \left(\mathrm{L}_0 + \varepsilon^2\beta_{\mathbf{p},i}^2\right)^2\right)(\varphi_{1,i}\tilde{\Theta}_i + \varphi_{2,i}\varepsilon\tilde{\Theta}'_i)\\ &\quad + \varepsilon\left(h_1(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})\tilde{\Theta}_i + h_2(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})\varepsilon\tilde{\Theta}'_i\right).\end{aligned}$$

Returning this expansion to (4.47), we obtain

$$\begin{aligned}\mathbb{L}Z_{\mathbf{p},*}^{I(i)i} &= \left(\Lambda_{I(i)i}^2 + \varepsilon\delta_{I(i)0}(\sigma S_1 + \eta_d\lambda_0) + S_{2,I(i)}\Lambda_{I(i)i}\right)Z_{\mathbf{p},*}^{I(i)i} + \varepsilon^2\mathbb{L}_1\Big(\tilde{\varphi}_{1,i}\tilde{\Theta}_i\\ &\quad + \tilde{\varphi}_{2,i}\varepsilon\tilde{\Theta}'_i\Big) + \varepsilon^2\mathbb{L}_{\geqslant 2}Z_{\mathbf{p},*}^{I(i)i} + \varepsilon^2\cdot\varepsilon^{-1/2}\left(h_1(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})\tilde{\Theta}_i + h_2(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})\varepsilon\tilde{\Theta}'_i\right)\\ &\quad + \varepsilon\left(\mathcal{L}^2 - \left(\mathrm{L}_0 + \varepsilon^2\beta_{\mathbf{p},i}^2\right)^2\right)(\tilde{\varphi}_{1,i}\tilde{\Theta}_i + \tilde{\varphi}_{2,i}\varepsilon\tilde{\Theta}'_i).\end{aligned} \tag{4.50}$$

Expanding the operators $\mathbb{L}_1$ and $\mathbb{L}_{\mathbf{p},\geqslant 2}$, and using (4.17), we write the second and third terms as

$$\mathbb{L}_1\left(\tilde{\varphi}_{1,i}\tilde{\Theta}_i + \tilde{\varphi}_{2,i}\varepsilon\tilde{\Theta}'_i\right) + \mathbb{L}_{\geqslant 2}Z_{\mathbf{p},*}^{I(i)i} = \varepsilon^{-1/2}\left(h_1(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})\tilde{\Theta}_i + h_2(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})\varepsilon\tilde{\Theta}'_i\right),$$

where h_1 and h_2 are new general functions. The conclusion follows from this identity, (4.50), and the relation

$$\begin{aligned}&\left(\mathcal{L}^2 - \left(\mathrm{L}_0 + \varepsilon^2\beta_{\mathbf{p},i}^2\right)^2\right)(\tilde{\varphi}_{1,i}\Theta_i + \tilde{\varphi}_{2,i}\varepsilon\Theta'_i)\\ &= \sum_{k=1}^{4}\left(\varepsilon^k\partial_{s_{\mathbf{p}}}^k h_{3,k}(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})\Theta_i + \varepsilon^k\partial_{s_{\mathbf{p}}}^k h_{4,k}(z_{\mathbf{p}}, \boldsymbol{\gamma}''_{\mathbf{p}})\varepsilon\Theta'_i\right).\end{aligned} \tag{4.51}$$

Here we note the dependence of $\varphi_{k,i}$ on $\tilde{s}_{\mathbf{p}}$ is uniform on i by its definition from (4.49) and hence we omit the dependence of hs on i by abusing notation. □

Note the dependence of $\varphi_{k,j}$ on $s_{\mathbf{p}}$ is uniform in j, we may use this fact without further mention.

The modified approximate slow spaces are defined as the spans of the modified basis functions of (4.45):

$$\mathcal{Z}_*(\mathbf{p},\rho) := \mathcal{Z}_*^0(\mathbf{p},\rho) \cup \mathcal{Z}_*^1(\mathbf{p},\rho) \qquad \text{with} \qquad \mathcal{Z}_*^k(\mathbf{p},\rho) = \operatorname{span}\left\{ Z_{\mathbf{p},*}^{I(i)i}, i \in \Sigma_k \right\}. \tag{4.52}$$

Similarly as we used for the leading order slow spaces, we utilize $\mathcal{Z}_*^k$, $\mathcal{Z}_*$ to simplify the notation when there is no ambiguity. When restricted to $\mathcal{Z}_*$ the bilinear form of the full linearized operator $\Pi_0 \mathbb{L}\big|_{\mathcal{Z}_*}$, induces an $N \times N$ matrix $\mathbb{M}^*$ with entries

$$\mathbb{M}_{ij}^* = \left\langle \Pi_0 \mathbb{L} Z_{\mathbf{p},*}^{I(i)i}, Z_{\mathbf{p},*}^{I(j)j} \right\rangle_{L^2}. \tag{4.53}$$

By construction, $\varphi_{I(i),i}$ are perpendicular to $\psi_{I(i)}$, see (4.44). Following the arguments that establish (4.21) it is easy to verify that under assumption (2.29)

$$\left\langle Z_{\mathbf{p},*}^{I(i)i}, Z_{\mathbf{p},*}^{I(j)j} \right\rangle_{L^2} = \begin{cases} (1+\mathrm{p}_0)\,\delta_{ij} + O\left(\varepsilon^2, \varepsilon^2 \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}\right) \mathbb{E}_{ij}, & I(i) = I(j); \\ O\left(\varepsilon^2, \varepsilon^2 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\right) \mathbb{E}_{ij}, & I(i) \neq I(j). \end{cases} \tag{4.54}$$

From the definition of the zero-mass projection Π_0, the identity (4.53) can be written as

$$\mathbb{M}_{ij}^* = \left\langle \mathbb{L} Z_{\mathbf{p},*}^{I(i)i}, Z_{\mathbf{p},*}^{I(j)j} \right\rangle_{L^2} - \frac{1}{|\Omega|} \int_\Omega \mathbb{L} Z_{\mathbf{p},*}^{I(i)i}\, \mathrm{d}x \int_\Omega Z_{\mathbf{p},*}^{I(j)j}\, \mathrm{d}x. \tag{4.55}$$

To estimate $\mathbb{M}_{ij}^*$, use the following Corollary to control the mass of $Z_{\mathbf{p},*}^{I(j)j}$ and its image under $\mathbb{L}$.

Corollary 4.7. *Under assumption (2.29) there exists a unit vector $\boldsymbol{e} = (e_j)_{j\in\Sigma}$ such that for $j \in \Sigma$,*

$$\int_\Omega Z_{\mathbf{p},*}^{I(j)j}\, \mathrm{d}x = O(\varepsilon^{3/2})\, e_j. \tag{4.56}$$

Furthermore,

$$\int_\Omega \mathbb{L} Z_{\mathbf{p},*}^{I(j)j}\, \mathrm{d}x = O\left(\varepsilon^{3/2}(1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2})\right) e_j. \tag{4.57}$$

Proof. With $Z^{I(j)j}_{\mathbf{p},*}$ introduced in (4.45), we have

$$\int_\Omega Z^{0j}_{\mathbf{p},*}\,\mathrm{d}x = \varepsilon^{1/2}\int_{\mathbb{R}_{2\ell}} \psi_{I(j)}(z_\mathbf{p})\,\mathrm{d}z_\mathbf{p}\int_{\mathscr{I}_\mathbf{p}} \tilde\Theta_j\,\mathrm{d}\tilde s_\mathbf{p}$$
$$+\varepsilon^{3/2}\int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_\mathbf{p}}\left(\varphi_{1,j}\tilde\Theta_j+\varphi_{2,j}\varepsilon\tilde\Theta_j'\right)(1-\varepsilon z_\mathbf{p}\kappa_\mathbf{p})\,\mathrm{d}\tilde s_\mathbf{p}\mathrm{d}z_\mathbf{p}.$$

The integration of $\tilde\Theta_j$ with respect to $\tilde s_\mathbf{p}$ is zero for $j\in\Sigma\setminus\{0\}$ while for $j=0$, $\psi_{I(j)}=\psi_1$ has odd parity in $z_\mathbf{p}$. We deduce that the first term on the right-hand side is zero. After integrating with respect to $z_\mathbf{p}$, the second integral takes the form

$$\varepsilon^{3/2}\left(\int_{\mathscr{I}_\mathbf{p}} h_1(\boldsymbol{\gamma}''_\mathbf{p})\tilde\Theta_j\,\mathrm{d}\tilde s_\mathbf{p}+\int_{\mathscr{I}_\mathbf{p}} h_2(\boldsymbol{\gamma}''_\mathbf{p})\varepsilon\tilde\Theta_j'\,\mathrm{d}\tilde s_\mathbf{p}\right). \tag{4.58}$$

The estimate (4.56) follows from (2.47) in Lemma 2.13 and (2.45). To derive (4.57) we employ Lemma 4.6 and the estimate (4.56). The error bound involves the $\mathbb{V}^2_3$-norm instead of the $\mathbb{V}^2_2$-norm of $\hat{\mathbf{p}}$ because there is an additional higher derivative acting on $h=h(\boldsymbol{\gamma}''_\mathbf{p})$ as shown in (4.46). □

Applying the orthogonality and mass estimates (4.54) and (4.56) to (4.55) yields the expansion of $\mathbb{M}^*_{ij}$. This principle result gives a sharp characterization of the behavior of the linearized operator on the modified slow space, which we summarize below.

Proposition 4.8. *For $i,j\in\Sigma$, the $\mathbb{M}^*$ with components $\mathbb{M}^*_{ij}$ defined in (4.53) can be approximated by*

$$\mathbb{M}^*_{ij}=\begin{cases}(1+\mathrm{p}_0)\left(\Lambda^2_{0i}+\varepsilon(\sigma S_1+\eta_d\lambda_0)+\varepsilon S_{2,0}\Lambda_{0i}\right)+O(\varepsilon^2) & \text{if } i=j,\ I(i)=0;\\ (1+\mathrm{p}_0)\left(\Lambda^2_{1i}+\varepsilon S_{2,1}\Lambda_{1i}\right)+O(\varepsilon^2) & \text{if } i=j,\ I(i)=1;\\ O\left(\varepsilon^2\right)\mathbb{E}_{ij} & \text{if } i\neq j,\ I(i)=I(j);\\ O\left(\varepsilon^2,\varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4}\right)\mathbb{E}_{ij} & \text{if } I(i)\neq I(j),\end{cases} \tag{4.59}$$

where the matrix $\mathbb{E}$ is norm-one as an operator from $l^2(\mathbb{R}^N)$ to $l^2(\mathbb{R}^N)$.

We decompose $\mathbb{M}^*$ into a block structure corresponding to the pearling and meandering spaces,

$$\mathbb{M}^*=\begin{pmatrix}\mathbb{M}^*(0,0) & \mathbb{M}^*(0,1)\\ \mathbb{M}^*(1,0) & \mathbb{M}^*(1,1)\end{pmatrix},\qquad \mathbb{M}^*_{ij}(k,l)=\mathbb{M}^*_{ij}\quad\text{for } i\in\Sigma_k,\ j\in\Sigma_l. \tag{4.60}$$

Since matrix $\mathbb{E}$ is norm-one, the $N_0 \times N_0$ subblock matrix $\mathbb{M}^*(0,0)$ is diagonally dominant. In particular, under the *pearling stability condition*

$$\textbf{(PSC)} \qquad \sigma S_1 + \eta_d \lambda_0 > 0, \tag{4.61}$$

$\mathbb{M}^*(0,0)$ is positive definite. This pearling-mode coercivity is formulated in the following Lemma.

Lemma 4.9. *Assume $\sigma(\mathbf{p})$ given by (3.22) is uniformly bounded, independent of $\varepsilon > 0$, for all $\mathbf{p} \in \mathcal{D}_\delta$. Then there exists ε_0 sufficiently small such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $\mathbf{p} \in \mathcal{D}_\delta$ for which the pearling stability condition (4.61) holds, we have*

$$\mathbf{q}^T \mathbb{M}^*(0,0)\mathbf{q} \geqslant \frac{\varepsilon}{4}(\sigma S_1 + \eta_d \lambda_0)\|\mathbf{q}\|_{l^2}^2, \quad \forall \mathbf{q} \in l^2(\mathbb{R}^{N_0}).$$

Proof. The constants $S_{2,0}$ and $S_{2,1}$ in (4.59) depend upon σ, but are uniformly bounded, independent of ε since σ is bounded by assumption. In view of the expansion of $\mathbb{M}^*_{ij}$ from (4.59) for $i, j \in \Sigma_0$, the Lemma follows for $\varepsilon < \varepsilon_0$ sufficiently small, by completing the square in $\Lambda_{I(i)i}$ in the diagonal terms and using the pearling stability condition (**PSC**), the uniform bounds on $S_{2,0}, S_{2,1}$, and the diagonal dominance of $\mathbb{M}^*(0,0)$. □

We denote by the L^2 projections to the finite-dimensional slow spaces $\mathcal{Z}^0_*, \mathcal{Z}^1_*$ and $\mathcal{Z}_*$ by $\Pi_{\mathcal{Z}^0_*}, \Pi_{\mathcal{Z}^1_*}, \Pi_{\mathcal{Z}_*}$ respectively. Introducing the H^2 inner norm

$$\|u\|_{H^2_{\rm in}} := \|u\|_{L^2} + \varepsilon^2 \|u\|_{H^2}, \tag{4.62}$$

then we have the following result with regards to these projections. We state it for $\mathcal{Z}^0_*$, similar statements hold for $\mathcal{Z}_*$ and $\mathcal{Z}^1_*$.

Lemma 4.10. *Suppose $\mathbf{p} \in \mathcal{D}_\delta$ satisfying $\varepsilon^2 \|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4} \leqslant \delta$. If δ is sufficiently small then for any $u \in L^2$ there exists a unique N_0-vector $\mathbf{q} = (\mathrm{q}_j)_j \in l^2$ such that $Q := \Pi_{\mathcal{Z}^0_*} u$, can be expressed as*

$$Q := \sum_{j \in \Sigma_0} \mathrm{q}_j Z^{0j}_{\mathbf{p},*}. \tag{4.63}$$

Moreover, there exists ε_0 suitably small such that for all $\varepsilon \in (0, \varepsilon_0)$ we have the relations

$$\|\mathbf{q}\|_{l^2} \lesssim \|u\|_{L^2}; \qquad \|Q\|_{H^2_{\rm in}} \sim \|Q\|_{L^2} \sim \|\mathbf{q}\|_{l^2}.$$

The parameters δ, ε_0 depend only upon the domain, the system parameters, and the choice of K_0, ℓ_0.

Proof. For any $u \in L^2(\Omega)$, the L^2 linear projection $Q := \Pi_{\mathcal{Z}^0_*} u \in \mathcal{Z}^0_*$ is well-defined by the Projection theorem, and hence there exists $\mathbf{q} = (\mathrm{q}_j) \in l^2$ satisfying (4.63). In particular, the vector $\mathbf{q} = (\mathrm{q}_j)$ satisfies the linear algebraic system

$$\sum_{j\in\Sigma_0} \mathrm{q}_j \left\langle Z^{0j}_{\mathbf{p},*}, Z^{0k}_{\mathbf{p},*}\right\rangle_{L^2} = \left\langle u, Z^{0k}_{\mathbf{p},*}\right\rangle_{L^2}, \qquad \forall k \in \Sigma_0.$$

Due to the approximate orthogonality afforded by (4.54) and the bound $\varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2} \leqslant \delta$ with δ suitably small, there exists a unique $\mathbf{q}$ solving the system and $\mathbf{q}$ can be bounded in terms of L^2-norm of u as

$$\|\mathbf{q}\|_{l^2} \lesssim \|u\|_{L^2}.$$

It remains to show the norm equivalences among Q and $\mathbf{q}$. First, the equivalence of the L^2-norm of Q and l^2-norm of $\mathbf{q}$ follows directly from the orthogonality relation (4.54) which requires the condition $\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2} \lesssim 1$ and ε_0 suitably small. The set $\{\varepsilon^2\Delta Z^{0j}_{\mathbf{p},*}\}_{j\in N_0}$ is approximately $L^2(\Omega)$ orthogonal due to the local coordinates Laplacian expansion (2.11), the form of $Z^{0j}_{\mathbf{p},*}$ and Lemma 2.13. Combining these implies

$$\|\varepsilon^2\Delta Q\|_{L^2} \sim \|\mathbf{q}\|_{l^2},$$

and the Lemma follows. □

We call $Q = \Pi_{\mathcal{Z}^0_*} u$ and the associated vector $\mathbf{q} = (\mathrm{q}_j)_j \in l^2$ defined through (4.63) the pearling mode component and pearling parameters of u, respectively. The relations (4.56) and (4.57) imply

$$\int_\Omega Q\,\mathrm{d}x = O\left(\varepsilon^3\|\mathbf{q}\|_{l^2}\right), \qquad \int_\Omega \mathbb{L}_{\mathbf{p}} Q\,\mathrm{d}x = O\left(\varepsilon^3(1+\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2})\|\mathbf{q}\|_{l^2}\right). \tag{4.64}$$

We present our principal results on the linear coupling between the pearling-meander and slow-fast modes.

Theorem 4.11. *Assume that ρ, δ are suitably small depending on the domain Ω, the system parameters, and the choice of K_0, ℓ_0. Then the following results hold uniformly for all $\mathbf{p} \in \mathcal{D}_\delta$, defined in (2.29).*

(1) All Q in the pearling slow space $\mathcal{Z}^0_*$ take the form (4.63) and satisfy $\|Q\|_{H^2_{\mathrm{in}}} \sim \|\mathbf{q}\|_{l^2}$. Moreover the pearling-meander coupling satisfies the bound

$$\|\Pi_{\mathcal{Z}^1_*}\Pi_0\mathbb{L}Q\|_{L^2} \lesssim \left(\varepsilon^2 + \varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\right)\|\mathbf{q}\|_{l^2}.$$

(2) For any function $v \in H^2$, the slow-fast coupling satisfies the bound

$$\|\Pi^{\perp}_{\mathcal{Z}_*}\mathbb{L}\Pi_{\mathcal{Z}_*}v\|_{L^2} + \|\Pi_{\mathcal{Z}_*}\mathbb{L}\Pi^{\perp}_{\mathcal{Z}_*}v\|_{L^2} \lesssim \left(\varepsilon^2 + \varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\right)\|v\|_{L^2}.$$

Proof. We address the bounds in term.

(1) The equivalence of the H^2_{in} and l^2 norms, $\|Q\|_{H^2_{\text{in}}} \sim \|\mathbf{q}\|_{l^2}$ is established in Lemma 4.10. For the pearling-translation coupling estimate we remark that

$$\|\Pi_{\mathcal{Z}^1_*}\Pi_0\mathbb{L}Q\|_{L^2} = \left(\sum_{i\in\Sigma_1}\left\langle \Pi_0\mathbb{L}Q, Z^{1i}_{\mathbf{p},*}\right\rangle^2_{L^2}\right)^{1/2} = \|\mathbb{M}^*(0,1)\mathbf{q}\|_{l^2}.$$

Applying (4.59) for the case $I(i)\neq I(j)$ yields the first bound.

(2) To establish the second bound it suffices to show for any $v\in\mathcal{Z}_*$, $w\in\mathcal{Z}_*^{\perp}$, we have

$$\langle \mathbb{L}v, w\rangle_{L^2} \lesssim \left(\varepsilon^2+\varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4}\right)\|v\|_{L^2}\|w\|_{L^2}. \tag{4.65}$$

Writing $v\in\mathcal{Z}_*$ in the form $v=\sum_{i\in\Sigma} v_i Z^{I(i)i}_{\mathbf{p},*}$ for $\{v_i\}\in\mathbb{R}^N$, we obtain

$$\langle \mathbb{L}v, w\rangle_{L^2} = \sum_i v_i\left\langle \mathbb{L}_{\mathbf{p}} Z^{I(i)i}_{\mathbf{p},*}, w\right\rangle_{L^2}. \tag{4.66}$$

We consider each component in the summation. Utilizing Lemma 4.6 and the orthogonality of w and $\mathcal{Z}_*$ implies

$$\begin{aligned}\left\langle w, \mathbb{L}Z^{I(j)j}_{\mathbf{p},*}\right\rangle_{L^2} &= \varepsilon^2\left\langle w, \varepsilon^{-1/2}(h_1\tilde{\Theta}_j + h_2\varepsilon\tilde{\Theta}'_j)\right\rangle_{L^2}\\ &\quad+\varepsilon^{3/2}\sum_{k=1}^4\left\langle w, \varepsilon^{k-1}\partial^k_{s_{\mathbf{p}}}h_{3,k}\tilde{\Theta}_i + \varepsilon^{k-1}\partial^k_{s_{\mathbf{p}}}h_{4,k}\varepsilon\tilde{\Theta}'_i\right\rangle_{L^2},\end{aligned} \tag{4.67}$$

where the functions $h=h(z_{\mathbf{p}},\boldsymbol{\gamma}''_{\mathbf{p}})$ enjoy the properties of Notation 2.1, localized in $\Gamma^{2\ell}_{\mathbf{p}}$ and can be bounded in two ways,

$$\|\varepsilon^k\partial^k_{s_{\mathbf{p}}}h\|_{L^\infty}\lesssim 1+\|\hat{\mathbf{p}}\|_{\mathbb{V}_2}\lesssim 1, \qquad \|\varepsilon^{k-1}\partial^k_{s_{\mathbf{p}}}h\|_{L^\infty}\lesssim 1+\|\hat{\mathbf{p}}\|_{\mathbb{V}_3}\lesssim 1+\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4}. \tag{4.68}$$

Inserting (4.66)-(4.68) into (4.65) and using Lemma 2.13 completes the proof. □

We extend these results to the full linearization $\Pi_0\mathbb{L}_{\mathbf{p}}$ of the mass preserving flow (1.3) at $\Phi_{\mathbf{p}}$.

Corollary 4.12. *Under the same assumptions as Theorem 4.11, if $w\in\mathcal{Z}_*^{\perp}$ and $\mathbf{q}$ is such that $w+Q$ is mass free, with Q as in (4.63), then*

$$\|\Pi_{\mathcal{Z}_*}\Pi_0\mathbb{L}w\|_{L^2}\lesssim\left(\varepsilon^2+\varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4}\right)\|w\|_{L^2}+\varepsilon^3\|\mathbf{q}\|_{l^2}.$$

Proof. As in the slow-fast coupling estimate of Theorem 4.11, we need show for any $v\in\mathcal{Z}_*$, $w\in\mathcal{Z}_*^{\perp}$, and $\mathbf{q}$ as above, that

$$\langle\Pi_0\mathbb{L}w, v\rangle_{L^2}\lesssim\left[\left(\varepsilon^2+\varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4}\right)\|w\|_{L^2}+\varepsilon^3\|\mathbf{q}\|_{l^2}\right]\|v\|_{L^2}. \tag{4.69}$$

Writing as $v = \sum_{i \in \Sigma} v_i Z_{\mathbf{p},*}^{I(i)i}$, we have the equality

$$\langle \Pi_0 \mathbb{L} w, v \rangle_{L^2} = \sum_{i \in \Sigma} v_i \left\langle \Pi_0 \mathbb{L} Z_{\mathbf{p},*}^{I(i)i}, w \right\rangle_{L^2}.$$

We use the definition of Π_0

$$\left\langle \Pi_0 \mathbb{L} w, Z_{\mathbf{p},*}^{I(j)j} \right\rangle_{L^2} = \left\langle \mathbb{L} w, Z_{\mathbf{p},*}^{I(j)j} \right\rangle_{L^2} - \frac{1}{|\Omega|} \int_\Omega Z_{\mathbf{p},*}^{I(j)j} \, \mathrm{d}x \int_\Omega \mathbb{L} w \, \mathrm{d}x, \tag{4.70}$$

and apply the estimate (4.56) and identity (4.73) from Lemma 4.13 below to deduce

$$\left\| \Pi_{\mathcal{Z}_*} \Pi_0 \mathbb{L} w \right\| \lesssim \| \Pi_{\mathcal{Z}_*} \mathbb{L} w \|_{L^2} + \varepsilon^2 (\|w\|_{L^2} + \varepsilon \|\mathbf{q}\|_{l^2}). \tag{4.71}$$

The corollary follows from Theorem 4.11 by noting $w \in \mathcal{Z}_*^\perp$. $\square$

4.3. Coercivity

The coercivity estimates on the operator $\mathbb{L}$ restricted to the orthogonal complement of the modified slow space, $\mathcal{Z}_*^\perp$, are essential to the orbital stability of the underlying manifold. Coercivity estimates for the constrained bilinear form $\mathbb{L}\big|_{\mathcal{Z}}$ for the preliminary slow space were derived in [10,24] and Theorem 2.5 of [22], for the weak functionalization under the restriction $\rho \sim \sqrt{\varepsilon}$. However, these results lead to an ε dependent coercivity estimate. Our main coercivity result, requires only $\rho = o(1)$, independent of ε, and exploits the improved orthogonality of the modified slow spaces. In this subsection we establish this enhanced coercivity of the linearized operator $\mathbb{L}$ on the space orthogonal to the modified approximate slow space $\mathcal{Z}_*$.

Theorem 4.13. *Suppose $\rho > 0$ is suitably small. Then there exists $\varepsilon_0 > 0$, dependent upon ρ, and a coercivity constant C independent of ρ, such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $w \in \mathcal{Z}_*^\perp$,*

$$\langle \mathbb{L} w, w \rangle_{L^2} \geqslant C \rho^2 \|w\|_{H^2_{\mathrm{in}}}^2 \qquad \text{and} \quad \|\mathbb{L} w\|_{L^2}^2 \geqslant C \rho^2 \langle \mathbb{L} w, w \rangle_{L^2}. \tag{4.72}$$

Moreover, if for any $\mathbf{q}$ the associated $Q \in \mathcal{Z}_*^0$ satisfies $\langle w + Q \rangle_{L^2} = 0$, then we have the average estimate

$$\left| \langle \mathbb{L} w \rangle_{L^2} \right| \lesssim \varepsilon^{1/2} \|w\|_{L^2} + \varepsilon^{3/2} \|\mathbf{q}\|_{l^2}, \tag{4.73}$$

and in addition

$$C \varepsilon^3 \|\mathbf{q}\|_{l^2}^2 + \langle \Pi_0 \mathbb{L} w, \mathbb{L} w \rangle_{L^2} \geqslant \|\mathbb{L} w\|_{L^2}^2. \tag{4.74}$$

Proof. To establish (4.72), we introduce

$$\mathrm{L}_1 := -\varepsilon^2 \Delta + W''(\Phi_{\mathbf{p}}) - \frac{1}{2} \varepsilon \eta_1,$$

and rewrite the linearized operator $\mathbb{L}$ defined by (4.1) in the form $\mathbb{L} = (\mathrm{L}_1)^2 + \varepsilon \mathrm{R}$ where

$$\mathrm{R} = -\frac{\varepsilon\eta_1^2}{4} - \frac{W'''(\Phi_{\mathbf{p}})}{\varepsilon}\Big(\varepsilon^2\Delta\Phi_{\mathbf{p}} - W'(\Phi_{\mathbf{p}})\Big) + \eta_d W''(\Phi_{\mathbf{p}}).$$

Since R is a multiplier operator with a finite L^∞-norm, it follows that

$$\langle \mathbb{L}w, w\rangle_{L^2} \geqslant \Big\langle (\mathrm{L}_1)^2\, w, w\Big\rangle_{L^2} - \varepsilon\|\mathrm{R}\|_{L^\infty}\|w\|_{L^2}^2,$$

and moreover for some $C > 0$ independent of ε,

$$\|\mathbb{L}w\|_{L^2}^2 \geqslant \Big\|(\mathrm{L}_1)^2\, w\Big\|_{L^2}^2 - \varepsilon^2 C\|\mathrm{R}\|_{L^\infty}^2\|w\|_{L^2}^2.$$

Imposing the condition $\varepsilon_0 \ll \rho^2 = o(1)$, then the coercivity estimates (4.72) for $\mathbb{L}$ follow from Theorem 2.5 of [22] by replacing the preliminary approximate slow space $\mathcal{Z}$ with the modified approximation $\mathcal{Z}_*$. It remains to obtain estimates (4.73) and (4.74). From the definition of Π_0,

$$\langle \Pi_0\mathbb{L}w, \mathbb{L}w\rangle_{L^2} = \|\mathbb{L}w\|_{L^2}^2 - \frac{1}{|\Omega|}\left(\int_\Omega \mathbb{L}w\,\mathrm{d}x\right)^2. \tag{4.75}$$

To estimate the averaged term we turn to the definition, (4.1), of $\mathbb{L}$ which implies

$$\begin{aligned}\int_\Omega \mathbb{L}w\,\mathrm{d}x = \int_\Omega \Big[&\Big(\varepsilon^2\Delta - W''(\Phi_{\mathbf{p}}) + \varepsilon\eta_1\Big)\Big(\varepsilon^2\Delta - W''(\Phi_{\mathbf{p}})\Big)w \\ &- \Big(\varepsilon^2\Delta\Phi_{\mathbf{p}} - W'(\Phi_{\mathbf{p}})\Big)W'''(\Phi_{\mathbf{p}})w + \varepsilon\eta_d W''(\Phi_{\mathbf{p}})w\Big]\,\mathrm{d}x.\end{aligned} \tag{4.76}$$

Since w satisfies periodic boundary conditions, both Δw and $\Delta^2 w$ has no mass which allows us to rewrite (4.76) as

$$\int_\Omega \mathbb{L}w\,\mathrm{d}x = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

where the terms $\mathcal{I}_k (k = 1, 2, 3)$ are defined by

$$\mathcal{I}_1 := -2\varepsilon^2\int_\Omega W''(\Phi_{\mathbf{p}})\Delta w\,\mathrm{d}x, \qquad \mathcal{I}_2 := \int_\Omega \Big(W''(\Phi_{\mathbf{p}})\Big)^2 w\,\mathrm{d}x,$$

$$\mathcal{I}_3 := -\int_\Omega \Big[\Big(\varepsilon^2\Delta\Phi_{\mathbf{p}} - W'(\Phi_{\mathbf{p}})\Big)W'''(\Phi_{\mathbf{p}}) - \varepsilon(\eta_d - \eta_1)W''(\Phi_{\mathbf{p}})\Big]w\,\mathrm{d}x.$$

We address these terms one by one. For the first term we integrate by parts and add a zero term

$$\mathcal{I}_1 = -2\varepsilon^2 \int\limits_\Omega \left(\Delta W''(\Phi_{\mathbf{p}})\right) w \,\mathrm{d}x = -2\varepsilon^2 \int\limits_\Omega \Delta\left(W''(\Phi_{\mathbf{p}}) - W''(\phi_0^\infty)\right) w \,\mathrm{d}x.$$

Since $\varepsilon^2\Delta(W''(\Phi_{\mathbf{p}}) - W''(\phi_0^\infty))$ is bounded in L^∞ and exponentially localized near the interface $\Gamma_{\mathbf{p}}$ we obtain

$$|\mathcal{I}_1| \lesssim \varepsilon^{1/2}\|w\|_{L^2}. \tag{4.77}$$

By the definition of $\Phi_{\mathbf{p}}$, the quantity $\varepsilon^2\Delta\Phi_{\mathbf{p}} - W'(\Phi_{\mathbf{p}})$ is order of ε in L^∞, we deduce that the part of the integrand in the brackets in $\mathcal{I}_3$ is of order of ε in L^∞, hence

$$|\mathcal{I}_3| \lesssim \varepsilon\|w\|_{L^2}. \tag{4.78}$$

Finally, to bound $\mathcal{I}_2$ we decompose it into near and far-field parts

$$\mathcal{I}_2 = \int\limits_\Omega \left[\left(W''(\Phi_{\mathbf{p}})\right)^2 - \left(W''(\phi_0^\infty)\right)^2\right] w \,\mathrm{d}x + \left(W''(\phi_0^\infty)\right)^2 \int\limits_\Omega w \,\mathrm{d}x.$$

The mass of w balances with the mass of Q, that is, $\langle w\rangle_{L^2} = -\langle Q\rangle_{L^2}$. From (4.64) we deduce that

$$|\mathcal{I}_2| \lesssim \varepsilon^{1/2}\|w\|_{L^2} + \varepsilon^{3/2}\|\mathbf{q}\|_{l^2}. \tag{4.79}$$

Combining estimates for $\mathcal{I}_k (k = 1, 2, 3)$ in (4.77)-(4.79) yields (4.73). We deduce (4.74) from these results together with (4.75). □

5. Orbital stability of the bilayer manifold

The tangent plane of the bilayer manifold $\mathcal{M}_b$ lies approximately in the meander space $\mathcal{Z}_*^1$. In this section we construct a nonlinear projection that maps a tubular projection neighborhood of the bilayer manifold onto the bilayer manifold. The projection uniquely decomposes each u in the projection neighborhood into a bilayer distribution parameterized by the meander modes $\mathbf{p}$ plus an orthogonal perturbation $v^\perp \in (\mathcal{Z}_*^1)^\perp$. The FCH gradient flow (1.6) weakly excites the pearling modes, which from the coercivity estimates of Lemma 4.9 are weakly damped when the pearling stability condition (4.61) holds. Accommodating the weak damping necessitates extracting the pearling modes from the remainder and tracking their evolution dynamically. This is accomplished by further decomposing the orthogonal perturbation $v^\perp$ in its components in the $Q = Q(\mathbf{q})$ in the pearling slow space $\mathcal{Z}_*^0$ and the fast modes $w \in \mathcal{Z}_*^\perp$.

We rewrite the flow as an evolution in these variables, and show that for initial data sufficiently close to the bilayer manifold whose projected meander parameters lie within a set $\mathcal{O}_\delta \subset \mathcal{D}_\delta$, then the solution $u = u(t)$ remains close to $\mathcal{M}_b$ so long as $\mathbf{p}$ remains inside of a slightly bigger set $\mathcal{O}_{2,\delta} \subset \mathcal{D}_\delta$. In a companion paper, [8], we consider a circular base point interface associated to an equilibrium of the flow and construct classes of initial data for which $\mathbf{p}$ remains inside of $\mathcal{O}_{2,\delta}$ for all time and derive a curvature driven flow that captures the leading order evolution of the meander parameters.

5.1. Decomposition of the flow

We say that a base interface Γ_0 and a scaled system mass M_0 introduced in (3.20) are an *admissible base-point pair* if $\Gamma_0 \in \mathcal{G}^4_{K_0,2\ell_0}$ and the system mass balances with the length of Γ_0 in the sense that

$$|M_0 - m_0|\Gamma_0|| \lesssim 1,$$

where m_0 is the mass per unit length of bilayer, defined in (3.11). The collection of admissible pairs, the admissible set, is denoted $\mathcal{A}(K_0, \ell_0)$. This condition enforces that the far-field value of $\Phi_{\mathbf{p}}$ lies within $O(\varepsilon)$ of b_-, and hence that the bulk parameter $|\sigma| \lesssim 1$, see (3.21)-(3.23).

For each admissible pair (Γ_0, M_0), we introduce an N_1-dimensional bilayer manifold $\mathcal{M}_b = \mathcal{M}_b(\Gamma_0, M_0; \rho)$ as given in Definition 3.3, where the ρ dependence arises through $N_1 = N_1(\rho)$, see Definition 4.1. With the H^2 inner norm defined in (4.62), we construct a projection onto the bilayer manifold $\mathcal{M}_b$ defined on the tubular projection neighborhood $\mathcal{U}$ of the bilayer manifold $\mathcal{M}_b$,

$$\mathcal{U}(\mathcal{M}_b) := \left\{ u \in H^2(\Omega) \,\middle|\, \inf_{\mathbf{p}\in\mathcal{D}_\delta} \|u - \Phi_{\mathbf{p}}(\sigma)\|_{H^2_{\rm in}} \leqslant \delta\varepsilon,\ \langle u - b_-\rangle_{L^2} = \frac{\varepsilon M_0}{|\Omega|} \right\}, \tag{5.1}$$

where $\Phi_{\mathbf{p}}(\sigma)$ is defined in Lemma 3.2 with $\sigma = \sigma(\mathbf{p})$ given by (3.22).

Definition 5.1. For $u \in \mathcal{U}(\mathcal{M}_b)$, we say $\Pi_{\mathcal{M}_b} u := \Phi_{\mathbf{p}}(\sigma)$ is the projection onto $\mathcal{M}_b$ and $\Pi^{\perp}_{\mathcal{M}_b} u := v^{\perp}$ is its complement if there exist unique $\mathbf{p} \in \mathcal{D}_\delta$ and mass-free orthogonal perturbation $v^{\perp} \in (\mathcal{Z}^1_*)^{\perp}$ such that

$$u = \Phi_{\mathbf{p}} + v^{\perp}. \tag{5.2}$$

In this case we introduce $Q(\mathbf{q}) := \Pi_{\mathcal{Z}^0_*} v^{\perp}$, the projection of the orthogonal perturbation onto $\mathcal{Z}^0_*$ and $w := \Pi^{\perp}_{\mathcal{Z}^0_*} v^{\perp}$, the projection onto the fast modes. We call $(\mathbf{p}, \mathbf{q})$ the projected parameters of u.

The following lemma establishes the existence of a projection of $\mathcal{U}$ to $\mathcal{M}_b$ and $\mathcal{Z}^0_*$.

Lemma 5.2. *Let $\mathcal{M}_b = \mathcal{M}_b(\Gamma_0, M_0)$ be the bilayer manifold as defined in Definition 3.3. Then for $\delta, \varepsilon_0 > 0$ sufficiently small the projection $\Pi_{\mathcal{M}_b}$ is well posed on $\mathcal{U}$ for all $\varepsilon \in (0, \varepsilon_0)$. Moreover, for $u \in \mathcal{U}$ of the form $u = \Phi_{\mathbf{p}_0} + v$ with $\mathbf{p}_0 \in \mathcal{D}_\delta$ and massless perturbation $v \in H^2$ satisfying $\|v\|_{H^2_{\rm in}} \leqslant \delta\varepsilon$, then u's projected parameters $(\mathbf{p}, \mathbf{q})$ and its orthogonal and fast perturbations, $v^{\perp}$ and w, satisfy*

$$\|\mathbf{q}\|_{l^2} + \varepsilon^{-1/2}\|\mathbf{p} - \mathbf{p}_0\|_{l^2} \lesssim \|v\|_{L^2}; \qquad \|v^{\perp}\|_{H^2_{\rm in}} \lesssim \|w\|_{H^2_{\rm in}} + \|Q\|_{H^2_{\rm in}} \lesssim \|v\|_{H^2_{\rm in}}.$$

The proof of this Lemma is postponed to the appendix.

Let $u = u(t)$ be a solution of the flow (1.3) corresponding to initial data $u_0 \in \mathcal{U}(\mathcal{M}_b)$. So long as $u(t) \in \mathcal{U}(\mathcal{M}_b)$ then u admits the decomposition

$$u(x,t)=\Phi_{\mathbf{p}}(x;\sigma)+v^{\perp}(x,t;\mathbf{q}),\quad v^{\perp}\in(\mathcal{Z}_*^1)^{\perp},\quad \int_\Omega v^{\perp}\,\mathrm{d}x=0, \tag{5.3}$$

where the projected parameters $(\mathbf{p},\mathbf{q})=(\mathbf{p}(t),\mathbf{q}(t))$ and the bulk density parameter $\sigma=\sigma(\mathbf{p}(t))$ defined by (3.22) are all time dependent. Substituting the ansatz (5.3) into the equation (1.3) leads to an equation for $\Phi_{\mathbf{p}}$ and $v^{\perp}$:

$$\partial_t\Phi_{\mathbf{p}}+\partial_t v^{\perp}=-\Pi_0\mathrm{F}(\Phi_{\mathbf{p}})-\Pi_0\mathbb{L}v^{\perp}-\Pi_0\mathrm{N}(v^{\perp}), \tag{5.4}$$

where $\mathbb{L}$ is the linearization of F about $\Phi_{\mathbf{p}}$ introduced in (4.1), and $\mathrm{N}(v^{\perp})$ is the nonlinear term defined by

$$\mathrm{N}(v^{\perp}):=\mathrm{F}(\Phi_{\mathbf{p}}+v^{\perp})-\mathrm{F}(\Phi_{\mathbf{p}})-\mathbb{L}v^{\perp}. \tag{5.5}$$

To exploit the strong coercivity of $\Pi_0\mathbb{L}_{\mathbf{p}}$ on $\mathcal{Z}_*^{\perp}$ and its $O(\varepsilon)$-weak coercivity on $(\mathcal{Z}_*^1)^{\perp}$ we follow Definition 5.1 and decompose the orthogonal perturbation $v^{\perp}$ into its pearling and fast mode sub-components

$$v^{\perp}=Q(x,t)+w(x,t),\quad w\in\mathcal{Z}_*^{\perp}(\mathbf{p},\rho). \tag{5.6}$$

In the following, we make a priori assumptions that bound the rate of change of $\mathbf{p}$ induced by the flow and norm estimates on $\mathbf{p}$ that subsume those of in $\mathcal{D}_\delta$, defined in (2.29):

$$|\mathrm{p}_0(t)|+\|\hat{\mathbf{p}}\|_{\mathbb{V}_1}\leqslant\delta,\qquad \|\hat{\mathbf{p}}\|_{\mathbb{V}_2}+\varepsilon\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\leqslant 1,\qquad \|\dot{\mathbf{p}}\|_{l^2}\leqslant\varepsilon^3. \tag{5.7}$$

Here δ is as prescribed in the definition of $\mathcal{D}_\delta$ given in (2.29). The first two assumptions in (5.7) ensure the existence and smoothness of the perturbed interface $\Gamma_{\mathbf{p}}$. The third assumption, on the l^2-norm of $\dot{\mathbf{p}}$, controls the flow of the FCH energy in the absence of prior estimates developed in sub-section 5.5. Once in place, these estimates allow the condition on $\dot{\mathbf{p}}$ to be dropped.

In the remainder of this section we develop bounds on w and $\mathbf{q}$, which require an L^2-bound on the nonlinear term $\mathrm{N}(v^{\perp})$. The projection of the solution u onto the manifold involves the approximate tangent spaces $\mathcal{Z}_*(\mathbf{p})$. Although the flow of $\mathbf{p}$ is slow in the sense of (5.7), it induces temporal variation of the tangent plane that must be accounted for. We emphasize that the linear operator $\mathbb{L}=\mathbb{L}_{\mathbf{p}}$ and the spaces $\mathcal{Z}_*^0(\mathbf{p})$ and $\mathcal{Z}_*^1(\mathbf{p})$ are independent of $\mathbf{q}$. The linearization and tangent spaces are defined along the bilayer manifold $\mathcal{M}_b$. More significant is the fact that the space $\mathcal{Z}_*(\mathbf{p})$ is only approximately invariant under the action of the linearized operator. This produces terms whose control is crucial to the closure of the estimates. Indeed these terms motivate the introduction of the modified approximate slow space $\mathcal{Z}_*(\mathbf{p})$.

5.2. *Energy estimate for w*

We derive an H^2-bound on w under the flow induced by (1.3) assuming the a priori estimates (5.7) on $\mathbf{p}$ and $\dot{\mathbf{p}}$. We decompose $v^{\perp}$ as in (5.6) to rewrite (5.4) as an evolution for the fast modes w,

$$\partial_t w+\Pi_0\mathbb{L}w=-\partial_t\Phi_{\mathbf{p}}-\partial_t Q-\Pi_0\mathrm{F}(\Phi_{\mathbf{p}})-\Pi_0\mathbb{L}Q-\Pi_0\mathrm{N}(v^{\perp}). \tag{5.8}$$

Lemma 5.3. *Let $\varepsilon \in (0, \varepsilon_0)$ and the a priori assumptions (5.7) hold, the function $w \in \mathcal{Z}_*^{\perp}$, obeys*

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbb{L}w, w\rangle_{L^2} + \|\mathbb{L}w\|_{L^2}^2 \lesssim \varepsilon^{-1}\|\dot{\mathbf{p}}\|_{l^2}^2 + \varepsilon^2\rho^{-4}(\|\mathbf{q}\|_{l^2}^2 + \|\dot{\mathbf{q}}\|_{l^2}^2) + \varepsilon^5|\sigma - \sigma^*|^2 + \varepsilon^7(1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}^2) + \|\mathrm{N}(v^\perp)\|_{L^2}^2, \tag{5.9}$$

provided that ε_0 small enough depending on ρ.

Proof. Since the linearized operator $\mathbb{L}$ depends on time through $\mathbf{p}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbb{L}w, w\rangle_{L^2} = 2\langle \partial_t w, \mathbb{L}w\rangle_{L^2} + \langle \partial_t(\mathbb{L})w, w\rangle_{L^2}. \tag{5.10}$$

Considering the last term on the right-hand side, the definition (4.1) of $\mathbb{L}$ provides the expansion

$$\begin{aligned}\partial_t(\mathbb{L}) = &-\left(\varepsilon^2\Delta - W'' + \varepsilon\eta_1\right)\left(W'''\partial_t\Phi_{\mathbf{p}}\right) - W'''\partial_t\Phi_{\mathbf{p}}\left(\varepsilon^2\Delta - W''\right)\\ &-\left(\varepsilon^2\Delta\Phi_{\mathbf{p}} - W'\right)W^{(4)}\partial_t\Phi_{\mathbf{p}} - W'''\left(\varepsilon^2\Delta - W''\right)\partial_t\Phi_{\mathbf{p}} + \varepsilon\eta_d W'''\partial_t\Phi_{\mathbf{p}},\end{aligned}$$

where the potential well W is evaluated at $\Phi_{\mathbf{p}}$. Since $\Phi_{\mathbf{p}}$ is uniformly bounded in L^∞ and in L^2 after action by powers of $\varepsilon^2\Delta$, we identify the upper bound on the bilinear form generated by $\partial_t(\mathbb{L})$

$$\langle \partial_t(\mathbb{L})w, w\rangle_{L^2} \lesssim \left(\left\|\dot{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\Phi_{\mathbf{p}}\right\|_{L^\infty} + \|\dot{\mathbf{p}}\cdot\nabla_{\mathbf{p}}(\varepsilon^2\Delta\Phi_{\mathbf{p}})\|_{L^\infty}\right)\left(\|w\|_{L^2}^2 + \|\varepsilon^2\Delta w\|_{L^2}^2\right). \tag{5.11}$$

Utilizing the bounds of $\Phi_{\mathbf{p}}$ established in the Appendix Lemma 6.4, assumption (5.7) and the coercivity estimate (4.72), we obtain the upper bound on the bilinear term

$$\langle \partial_t(\mathbb{L})w, w\rangle_{L^2} \lesssim \varepsilon\rho^{-4}\|\mathbb{L}w\|_{L^2}^2 \leqslant \varepsilon^{1/2}\|\mathbb{L}w\|_{L^2}^2$$

by choosing ε_0 small enough depending on ρ. Returning to (5.10), substituting (5.8) for $\partial_t w$, using the coercivity estimate (4.74) and bounding the second term via the bilinear estimate above, leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbb{L}w, w\rangle_{L^2} + \|\mathbb{L}w\|_{L^2}^2 \leqslant -2\left\langle \partial_t\Phi_{\mathbf{p}} + \partial_t Q + \Pi_0\mathbb{L}Q, \mathbb{L}w\right\rangle_{L^2} - 2\left\langle \Pi_0\mathrm{F}(\Phi_{\mathbf{p}}) + \Pi_0\mathrm{N}(v^\perp), \mathbb{L}w\right\rangle_{L^2}. \tag{5.12}$$

Here we also used $\varepsilon \in (0, \varepsilon_0)$ with ε_0 small enough depending on domain, system parameters and (Γ_0, M_0). Considering the terms on the right-hand side of (5.12), we apply Hölder's inequality to the last term

$$\left|\left\langle \Pi_0\mathrm{N}(v^\perp), \mathbb{L}w\right\rangle_{L^2}\right| \lesssim \|\mathrm{N}(v^\perp)\|_{L^2}\|\mathbb{L}w\|_{L^2}. \tag{5.13}$$

Utilizing Hölder's inequality and L^2-bound of $\partial_t\Phi_{\mathbf{p}}$, we establish the bound

$$\left|\left\langle \partial_t \Phi_{\mathbf{p}}, \mathbb{L} w\right\rangle_{L^2}\right| \lesssim \varepsilon^{-1/2} \|\dot{\mathbf{p}}\|_{l^2} \|\mathbb{L} w\|_{L^2}. \tag{5.14}$$

For the $\Pi_0 \mathbb{L} Q$ term we project onto $\mathcal{Z}_*$ and its complement, use $Q \in \mathcal{Z}_*$, Theorem 4.11, and finally the coercivity of Lemma 4.13 to establish

$$\begin{aligned} \left|\langle \Pi_0 \mathbb{L} Q, \mathbb{L} w\rangle_{L^2}\right| &= \left|\left\langle \Pi_{\mathcal{Z}_*^{\perp}} \Pi_0 \mathbb{L} Q, \mathbb{L} w\right\rangle_{L^2} + \left\langle \Pi_{\mathcal{Z}_*^{\perp}} \mathbb{L} \Pi_{\mathcal{Z}_*} \Pi_0 \mathbb{L} Q, w\right\rangle_{L^2}\right| \\ &\lesssim (\varepsilon^2 + \varepsilon^2 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}) \|\mathbf{q}\|_{l^2} (\|w\|_{L^2} + \|\mathbb{L} w\|_{L^2}) \\ &\lesssim \varepsilon \rho^{-2} \|\mathbf{q}\|_{l^2} \|\mathbb{L} w\|_{L^2}. \end{aligned} \tag{5.15}$$

For the second term on right-hand side of (5.12) requires an investigation of $\partial_t Q$. By the definition of Q we calculate

$$\partial_t Q = \sum_{j \in \Sigma_0} \dot{\mathrm{q}}_j Z^{0j}_{\mathbf{p},*} + \sum_{j \in \Sigma_0} \mathrm{q}_j \partial_t Z^{0j}_{\mathbf{p},*}. \tag{5.16}$$

Note that the second term can be written as

$$\langle \partial_t Q, \mathbb{L} w\rangle_{L^2} = \left\langle \Pi_{\mathcal{Z}_*^{\perp}} \mathbb{L} \partial_t Q, w\right\rangle_{L^2}. \tag{5.17}$$

By employing relation (5.16), statement (2) of Theorem 4.11 and estimate (6.14) we may bound

$$\|\Pi_{\mathcal{Z}_*^{\perp}} \mathbb{L} \partial_t Q\|_{L^2} \lesssim (\varepsilon^2 + \varepsilon^2 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}) \|\dot{\mathbf{q}}\|_{l^2} + \varepsilon^2 \|\mathbf{q}\|_{l^1}. \tag{5.18}$$

Here by the l^2-l^1 estimate and scaling of N_0 from (4.8), we have

$$\|\mathbf{q}\|_{l^1} \leqslant \varepsilon^{-1/2} \|\mathbf{q}\|_{l^2}. \tag{5.19}$$

Using Holder's inequality, the a priori assumptions (5.7) and the coercivity to bound $\|w\|_{L^2}$ by $\|\mathbb{L} w\|_{L^2}$ we deduce from (5.17) - (5.19)

$$\left|\langle \partial_t Q, \mathbb{L} w\rangle_{L^2}\right| \lesssim \varepsilon \rho^{-2} \|\mathbb{L} w\|_{L^2} (\|\dot{\mathbf{q}}\|_{l^2} + \|\mathbf{q}\|_{l^2}). \tag{5.20}$$

Combining the estimates (5.13) and (5.14)-(5.20) with (5.12) and using Young's inequality, yields the estimate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbb{L} w, w\rangle_{L^2} + \|\mathbb{L} w\|_{L^2}^2 \leqslant & C\left(\varepsilon^{-1/2} \|\dot{\mathbf{p}}\|_{L^2} + \varepsilon \rho^{-2} (\|\mathbf{q}\|_{l^2} + \|\dot{\mathbf{q}}\|_{l^2}) + \|\mathrm{N}(v^{\perp})\|_{L^2}\right) \|\mathbb{L} w\|_{L^2} \\ & - 2\left\langle \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}), \mathbb{L} w\right\rangle_{L^2}. \end{aligned} \tag{5.21}$$

It remains to bound the $\mathrm{F}(\Phi_{\mathbf{p}})$ term on the right-hand side of the above inequality. Using Lemma 3.6 to bound the L^2-norm of $\Pi_0 \mathrm{F}(\Phi_{\mathbf{p}})$ terms yields

$$\left|\left\langle \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}), \mathbb{L} w\right\rangle_{L^2}\right| \lesssim \varepsilon^{5/2} |\sigma - \sigma^*| \|\mathbb{L} w\|_{L^2} + \varepsilon^{7/2} (\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2} + 1) \|\mathbb{L} w\|_{L^2}. \tag{5.22}$$

Combining the above estimate (5.22) with (5.21) and using Young's inequality, yields the estimate (5.9). The proof is complete. □

5.3. Estimates on the pearling parameters $\mathbf{q}(t)$

We derive l^2 estimates of $\mathbf{q}$ and $\dot{\mathbf{q}}$ subject to the a priori assumptions (5.7). We rewrite (5.8) as an evolution for Q,

$$\partial_t Q + \Pi_0 \mathbb{L} Q = \mathscr{R}[\mathbf{p}, w, \mathrm{N}], \tag{5.23}$$

where $\mathscr{R}[\mathbf{p}, w, \mathrm{N}]$ is the pearling remainder contributed by $\mathbf{p}$, w, and the nonlinear terms $\mathrm{N}(v^\perp)$, specifically

$$\mathscr{R}[\mathbf{p}, w, \mathrm{N}] := -\partial_t \Phi_{\mathbf{p}} - \partial_t w - \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}) - \Pi_0 \mathbb{L} w - \Pi_0 \mathrm{N}(v^\perp). \tag{5.24}$$

We derive the evolution of $\dot{\mathbf{q}}$ by projecting this system onto the slowly evolving space $\mathcal{Z}^1_*(\mathbf{p}(t))$.

Lemma 5.4. *Assuming the a priori estimates (5.7) and the pearling stability condition (4.61) hold, $\varepsilon \in (0, \varepsilon_0)$ with ε_0 suitably small, then there exists $C > 0$ independent of ε, ρ such that the pearling parameters $\mathbf{q} = (\mathrm{q}_k(t))_{k \in \Sigma_0}$ obey*

$$\begin{aligned} &\|\dot{\mathbf{q}}\|_{l^2}^2 \lesssim \|\mathbf{q}\|_{l^2}^2 + \varepsilon^2 \|w\|_{L^2}^2 + \varepsilon \|\dot{\mathbf{p}}\|_{l^2}^2 + \|\mathrm{N}(v^\perp)\|_{L^2}^2 + \varepsilon^9 + \varepsilon^9 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}^2; \\ &\partial_t \|\mathbf{q}\|_{l^2}^2 + C\varepsilon \|\mathbf{q}\|_{l^2}^2 \lesssim \varepsilon \|w\|_{L^2}^2 + \|\dot{\mathbf{p}}\|_{l^2}^2 + \varepsilon^{-1} \|\mathrm{N}(v^\perp)\|_{L^2}^2 + \varepsilon^8 + \varepsilon^8 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}^2. \end{aligned}$$

Proof. Taking the L^2-inner product of equation (5.23) with Q yields

$$\langle \partial_t Q, Q \rangle_{L^2} + \langle \Pi_0 \mathbb{L} Q, Q \rangle_{L^2} = \langle \mathscr{R}[\mathbf{p}, w], Q \rangle_{L^2}. \tag{5.25}$$

Using (5.16) we rewrite the first term on the left-hand side as

$$\langle \partial_t Q, Q \rangle_{L^2} = \sum_{i,j \in \Sigma_0} \dot{\mathrm{q}}_i \mathrm{q}_j \left\langle Z^{0i}_{\mathbf{p},*}, Z^{0j}_{\mathbf{p},*} \right\rangle_{L^2} + \sum_{i,j \in \Sigma_0} \mathrm{q}_i \mathrm{q}_j \left\langle \partial_t Z^{0i}_{\mathbf{p},*}, Z^{0j}_{\mathbf{p},*} \right\rangle_{L^2}. \tag{5.26}$$

The partial orthogonality of the basis $\{Z^{0j}_{\mathbf{p},*}\}_{j \in \Sigma_0}$, from (4.54), and the a priori estimate $\|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2} \lesssim 1$ yield

$$\sum_{i,j \in \Sigma_0} \dot{\mathrm{q}}_i \mathrm{q}_j \left\langle Z^{0i}_{\mathbf{p},*}, Z^{0j}_{\mathbf{p},*} \right\rangle_{L^2} \geqslant (1 + \mathrm{p}_0) \frac{\partial_t \|\mathbf{q}\|_{l^2}^2}{2} - C\varepsilon^2 \|\mathbf{q}\|_{l^2} \|\dot{\mathbf{q}}\|_{l^2}. \tag{5.27}$$

Applying Hölder's inequality to the second term on the right-hand side of (5.26), the estimate (6.14) on $\partial_t Z^{0j}_{\mathbf{p},*}$ and the l^1-l^2 estimate (5.19) yields the bound

$$\sum_{i,j \in \Sigma_0} \mathrm{q}_i \mathrm{q}_j \left\langle \partial_t Z^{0i}_{\mathbf{p},*}, Z^{0j}_{\mathbf{p},*} \right\rangle_{L^2} \lesssim \varepsilon^2 \|\mathbf{q}\|_{l^2} \|\mathbf{q}\|_{l^1} \lesssim \varepsilon^{3/2} \|\mathbf{q}\|_{l^2}^2. \tag{5.28}$$

Combining estimates (5.27)-(5.28) with (5.26) and applying Young's inequality yield

$$\frac{1+\mathrm{p}_0}{2}\partial_t\|\mathbf{q}\|_{l^2}^2 - C\varepsilon^{3/2}\|\mathbf{q}\|_{l^2}^2 - C\varepsilon^{3/2}\|\dot{\mathbf{q}}\|_{l^2}^2 \leqslant \langle \partial_t Q, Q\rangle_{L^2},$$

which when substituted into (5.25) after multiplying by 2 implies

$$(1+\mathrm{p}_0)\partial_t\|\mathbf{q}\|_{l^2}^2 + 2\langle \Pi_0\mathbb{L}Q, Q\rangle_{L^2} \leqslant C\varepsilon^{3/2}\|\mathbf{q}\|_{l^2}^2 + C\varepsilon^{3/2}\|\dot{\mathbf{q}}\|_{l^2}^2 + 2\langle \mathscr{R}[\mathbf{p}, w, \mathrm{N}], Q\rangle_{L^2}. \tag{5.29}$$

The last term on the right hand side can be bounded by Hölder's inequality

$$\langle \mathscr{R}[\mathbf{p}, w, \mathrm{N}], Q\rangle_{L^2} \leqslant \|\Pi_{\mathcal{Z}_*^0}\mathscr{R}[\mathbf{p}, w, \mathrm{N}]\|_{L^2}\|\mathbf{q}\|_{l^2}.$$

With the projection of the remainder $\mathscr{R}[\mathbf{p}, w, \mathrm{N}]$ bounded in next Lemma 5.5, we derive

$$\begin{aligned}(1+\mathrm{p}_0)\partial_t\|\mathbf{q}\|_{l^2}^2 + 2\langle \Pi_0\mathbb{L}Q, Q\rangle_{L^2} &\lesssim \varepsilon^{3/2}\|\dot{\mathbf{q}}\|_{l^2}^2 + \varepsilon^{3/2}\|\mathbf{q}\|_{l^2}^2 + \varepsilon\|w\|_{L^2}\|\mathbf{q}\|_{l^2} \\ &\quad + \left(\varepsilon^{1/2}\|\dot{\mathbf{p}}\|_{l^2} + \|\mathrm{N}(v^\perp)\|_{L^2} + \varepsilon^{9/2} + \varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\right)\|\mathbf{q}\|_{l^2}.\end{aligned} \tag{5.30}$$

Since the system is pearling stable, Lemma 4.9 implies the existence of $C>0$ independent of ε for which,

$$\langle \Pi_0\mathbb{L}Q, Q\rangle_{L^2} = \mathbf{q}^T\mathbb{M}^*(0,0)\mathbf{q} \geqslant C\varepsilon\|\mathbf{q}\|_{l^2}^2.$$

The a priori estimates imply that $|\mathrm{p}_0|$ is small and hence $(1+\mathrm{p}_0)$ is bounded away from zero. Dividing both sides of (5.30) by $(1+\mathrm{p}_0)>0$ and applying Young's inequality to the right-hand side of the resulting inequality yields

$$\partial_t\|\mathbf{q}\|_{l^2}^2 + C\varepsilon\|\mathbf{q}\|_{l^2}^2 \lesssim \varepsilon^{3/2}\|\dot{\mathbf{q}}\|_{l^2}^2 + \varepsilon\|w\|_{L^2}^2 + \|\dot{\mathbf{p}}\|_{l^2}^2 + \varepsilon^{-1}\|\mathrm{N}(v^\perp)\|_{L^2}^2 + \varepsilon^8 + \varepsilon^8\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}^2 \tag{5.31}$$

for $\varepsilon \in (0, \varepsilon_0)$ provided that ε_0 small enough depending on domain, system parameters and K_0, ℓ_0.

It remains to bound $\|\dot{\mathbf{q}}\|_{l^2}$. Taking the L^2 inner product of equation (5.23) with $\sum_{j\in\Sigma_0}\dot{\mathrm{q}}_j Z_{\mathbf{p},*}^{0j}$ implies

$$\left\langle \partial_t Q, \sum_{j\in\Sigma_0}\dot{\mathrm{q}}_j Z_{\mathbf{p},*}^{0j}\right\rangle_{L^2} = -\left\langle \Pi_0\mathbb{L}Q, \sum_{j\in\Sigma_0}\dot{\mathrm{q}}_j Z_{\mathbf{p},*}^{0j}\right\rangle_{L^2} + \left\langle \mathscr{R}[\mathbf{p}, w, \mathrm{N}], \sum_{j\in\Sigma_0}\dot{\mathrm{q}}_j Z_{\mathbf{p},*}^{0j}\right\rangle_{L^2}. \tag{5.32}$$

The term on the left can be dealt with similarly as we deal with $\langle \partial_t Q, Q\rangle_{L^2}$ in (5.26)-(5.28), which gives us

$$(1+\mathrm{p}_0)\|\dot{\mathbf{q}}\|_{l^2}^2 \leqslant \left\langle \partial_t Q, \sum_{j\in\Sigma_0}\dot{\mathrm{q}}_j Z_{\mathbf{p},*}^{1j}\right\rangle_{L^2} + C\varepsilon^{3/2}\|\dot{\mathbf{q}}\|_{l^2}\|\mathbf{q}\|_{l^2}$$

for some numerical constant C independent of ε and ρ. In light of Lemma 5.5, the last term of (5.32) which includes the remainder projection, can be bounded by

$$\left\langle \mathscr{R}[\mathbf{p}, w, \mathrm{N}], \sum_{j\in\Sigma_0} \dot{\mathrm{q}}_j Z^{1j}_{\mathbf{p},*} \right\rangle_{L^2} \leqslant \|\Pi_{\mathcal{Z}^0_*}\mathscr{R}[\mathbf{p}, w, \mathrm{N}]\|_{L^2}\|\dot{\mathbf{q}}\|_{l^2}.$$

Adding the two estimates above with (5.32), and applying Lemma 5.5 we obtain

$$(1+\mathrm{p}_0)\|\dot{\mathbf{q}}\|^2_{l^2} \leqslant -\left\langle \Pi_0\mathbb{L}Q, \sum_{j\in\Sigma_0} \dot{\mathrm{q}}_j Z^{0j}_{\mathbf{p},*} \right\rangle_{L^2} + C\Big(\varepsilon^{3/2}\|\mathbf{q}\|_{l^2} + \varepsilon\|w\|_{L^2} + \|\mathrm{N}(v^\perp)\|_{L^2} + \varepsilon^{9/2} \\ + \varepsilon^{1/2}\|\dot{\mathbf{p}}\|_{l^2} + \varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4}\Big)\|\dot{\mathbf{q}}\|_{l^2}. \tag{5.33}$$

From the definition (4.63) of Q and the estimate (4.59) on $\mathbb{M}^*$ we rewrite the term involving $\Pi_0\mathbb{L}Q$ as

$$\left|\left\langle \Pi_0\mathbb{L}Q, \sum_{j\in\Sigma_0} \dot{\mathrm{q}}_j Z^{0j}_{\mathbf{p},*} \right\rangle_{L^2}\right| = \left|\sum_{i,j\in\Sigma_0} \mathrm{q}_i\dot{\mathrm{q}}_j \mathbb{M}^*_{ij}(0,0)\right| \lesssim \|\mathbf{q}\|_{l^2}\|\dot{\mathbf{q}}\|_{l^2}. \tag{5.34}$$

We establish the l^2-estimate of $\dot{\mathbf{q}}$ by returning this bound to (5.33), applying Young's inequality and assumption (5.7) on p_0. The estimate on $\partial_t\|\mathbf{q}\|^2_{l^2}$ follows from (5.31). $\square$

The proof of Lemma 5.4 requires the following estimate on the projection of the remainder to $\mathcal{Z}^0_*$.

Lemma 5.5. *Under the assumptions of* (5.7)*, the projection of the remainder, defined in* (5.24)*, to the pearling slow space can be bounded by*

$$\|\Pi_{\mathcal{Z}^0_*}\mathscr{R}[\mathbf{p}, w, \mathrm{N}]\|_{L^2} \lesssim \varepsilon\|w\|_{L^2} + \varepsilon^3\|\mathbf{q}\|_{l^2} + \varepsilon^{1/2}\|\dot{\mathbf{p}}\| + \varepsilon^{9/2} + \varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4} + \|\mathrm{N}(v^\perp)\|_{L^2}.$$

Proof. Since $\|Q\|_{L^2} \sim \|\mathbf{q}\|_{l^2}$, see Theorem 4.11, it suffices to establish the following inequality for any $Q = \sum_{j\in\Sigma_0} \mathrm{q}_j Z^{0j}_{\mathbf{p},*} \in \mathcal{Z}^0_*$,

$$\langle \mathscr{R}[\mathbf{p}, w, \mathrm{N}], Q\rangle_{L^2} \lesssim \left(\varepsilon\|w\|_{L^2} + \varepsilon^3\|\mathbf{q}\|_{l^2} + \varepsilon^{1/2}\|\dot{\mathbf{p}}\| + \varepsilon^{9/2}\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4} + \|\mathrm{N}(v^\perp)\|_{L^2}\right)\|\mathbf{q}\|_{l^2}. \tag{5.35}$$

By the definition of $\mathscr{R}[\mathbf{p}, w, \mathrm{N}]$, we expand

$$\langle \mathscr{R}[\mathbf{p}, w, \mathrm{N}], Q\rangle_{L^2} = -\big\langle \partial_t\Phi_{\mathbf{p}}, Q\big\rangle_{L^2} - \big\langle \Pi_0\mathrm{F}(\Phi_{\mathbf{p}}), Q\big\rangle_{L^2} - \langle \partial_t w, Q\rangle_{L^2} - \langle \Pi_0\mathbb{L}w, Q\rangle_{L^2} \\ - \left\langle \Pi_0\mathrm{N}(v^\perp), Q\right\rangle_{L^2}.$$

We deal with these terms one by one in the following. First, in order to bound the term involving $\partial_t\Phi_{\mathbf{p}}$ we use its L^2 projection estimate to $\mathcal{Z}^0_*$. We use Lemma 6.4 from the appendix. Since $Q \in \mathcal{Z}^0_*$, we have

$$\left|\big\langle \partial_t\Phi_{\mathbf{p}}, Q\big\rangle_{L^2}\right| \lesssim \|\Pi_{\mathcal{Z}^0_*}\partial_t\Phi_{\mathbf{p}}\|_{L^2}\|Q\|_{L^2} \lesssim \varepsilon^{1/2}\|\dot{\mathbf{p}}\|_{l^2}\|\mathbf{q}\|_{l^2}. \tag{5.36}$$

Second, since $w \in \mathcal{Z}_*^\perp$ we may apply the expansion (5.16) of $\partial_t Q$ to deduce

$$\langle \partial_t w, Q\rangle_{L^2} = -\langle w, \partial_t Q\rangle_{L^2} = \sum_{j\in\Sigma_0} \left\langle w, \mathrm{q}_j \partial_t Z^{0j}_{\mathbf{p},*}\right\rangle_{L^2}.$$

From the estimate (6.14), Hölder's inequality, and the l^2-l^1 estimate of $\mathbf{q}$ (5.19) we obtain

$$\left|\langle \partial_t w, Q\rangle_{L^2}\right| \lesssim \varepsilon^{-1}\|\dot{\mathbf{p}}\|_{l^2}\|w\|_{L^2}\|\mathbf{q}\|_{l^1} \lesssim \varepsilon^{3/2}\|w\|_{L^2}\|\mathbf{q}\|_{l^2}. \tag{5.37}$$

For the inner product with the nonlinear term, so Hölder's inequality yields

$$\left|\left\langle \Pi_0 \mathrm{N}(v^\perp), Q\right\rangle_{L^2}\right| \lesssim \|\mathrm{N}(v^\perp)\|_{L^2}\|\mathbf{q}\|_{l^2}. \tag{5.38}$$

For the $\mathbb{L}w$ term, Corollary 4.12 and a priori assumptions (5.7) yield

$$\begin{aligned}\left|\langle \Pi_0 \mathbb{L} w, Q\rangle_{L^2}\right| &\lesssim \left(\varepsilon^2 + \varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\right)\|w\|_{L^2}\|\mathbf{q}\|_{l^2} + \varepsilon^3\|\mathbf{q}\|_{l^2}^2 \\ &\lesssim \varepsilon\|w\|_{L^2}\|\mathbf{q}\|_{l^2} + \varepsilon^3\|\mathbf{q}\|_{l^2}^2.\end{aligned} \tag{5.39}$$

The estimate of the term involving the residual, $\mathrm{F}(\Phi_{\mathbf{p}})$, is deferred to (5.40) of Lemma 5.7. As a consequence of (5.37)–(5.39) and (5.40), the estimate (5.35) follows, and hence the proof is complete. □

To complete the estimation of the projection of Q and N to the pearling space we require the following simple lemma which exploits the high in-plane wave number of the pearling modes. This affords better bounds on the coupling of the residual to the pearling modes that compensates for the weaker coercivity they experience.

Lemma 5.6 *(High pearling wavenumber). Let $h = h(\boldsymbol{\gamma}''_{\mathbf{p}})$ in the sense of Notation 2.1, then there exists a unit vector (e_j) such that*

$$\int h(\boldsymbol{\gamma}''_{\mathbf{p}})\tilde{\Theta}_j \, \mathrm{d}\tilde{s}_{\mathbf{p}} = O(\varepsilon, \varepsilon\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2})e_j, \qquad j \in \Sigma_0.$$

Proof. The proof closely follows that of Lemma 4.3 for the case $i = 0$, $j \in \Sigma_0$, we omit the details. □

With Lemma 5.6 we obtain an improved bound on the coupling of the residual with the pearling modes.

Lemma 5.7. *Assuming (5.7), the projection of the residual to pearling space satisfies the estimate*

$$\left|\left\langle \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}), Q\right\rangle_{L^2}\right| \lesssim \varepsilon^{9/2}(1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2})\|\mathbf{q}\|_{l^2}. \tag{5.40}$$

Proof. Subtracting off the far-field value F^∞ of the residual and using the definition of Π_0, we have

$$\left\langle \Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}), Q \right\rangle_{L^2} = \left\langle \mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}^\infty, Q \right\rangle_{L^2} - \frac{1}{|\Omega|} \int_\Omega \left(\mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}^\infty \right) \mathrm{d}x \int_\Omega Q \, \mathrm{d}x. \tag{5.41}$$

Using Lemma 6.6 and the estimate (4.64), the second term on the right-hand side satisfies

$$\frac{1}{|\Omega|} \int_\Omega \left(\mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}_m^\infty \right) \mathrm{d}x \int_\Omega Q \, \mathrm{d}x = O\left(\varepsilon^{11/2} \|\mathbf{q}\|_{l^2} \right). \tag{5.42}$$

We use the expansion of $\mathrm{F}(\Phi_{\mathbf{p}})$ given in Lemma 3.2 to estimate the first term on the right-hand side of (5.41). Examining the L^2-inner product of F_2 and Q, since $\mathrm{F}_2 = (\sigma_1^* - \sigma)\kappa_{\mathbf{p}} f_2(z_{\mathbf{p}})$ with f_2 odd, the leading order vanishes since ψ_0 has even parity in $z_{\mathbf{p}}$. Integrating out $z_{\mathbf{p}}$ we then deduce from Lemma 5.6 that

$$\left| \langle \mathrm{F}_2, Q \rangle_{L^2} \right| = \varepsilon^{3/2} \left| (\sigma_1^* - \sigma) \sum_{j \in \Sigma_0} \int_{\mathscr{I}_{\mathbf{p}}} h(\boldsymbol{\gamma}_{\mathbf{p}}'') \mathrm{q}_j \tilde{\Theta}_j \, \mathrm{d}\tilde{s}_{\mathbf{p}} \right| \lesssim \varepsilon^{5/2} |\sigma_1^* - \sigma| (1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}) \|\mathbf{q}\|_{l^2}. \tag{5.43}$$

Using the form of F_3 for Lemma 3.2 we rewrite

$$\left\langle \mathrm{F}_3 - \mathrm{F}_3^\infty, Q \right\rangle_{L^2} = - \left\langle \phi_0' \Delta_{s_{\mathbf{p}}} \kappa_{\mathbf{p}}, Q \right\rangle_{L^2} + \left\langle f_3(z_{\mathbf{p}}, \boldsymbol{\gamma}_{\mathbf{p}}'') - f_3^\infty, Q \right\rangle_{L^2}. \tag{5.44}$$

Applying Lemma 5.6 again, we bound the lower order term on the right

$$\left| \left\langle f_3(z_{\mathbf{p}}, \boldsymbol{\gamma}_{\mathbf{p}}'') - f_3^\infty, Q \right\rangle_{L^2} \right| \lesssim \varepsilon^{3/2} \|\mathbf{q}\|_{l^2} (1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}). \tag{5.45}$$

For the higher order term including the second derivative of curvature $\Delta_{s_{\mathbf{p}}} \kappa_{\mathbf{p}}$, since ϕ_0' is perpendicular to ψ_0 in $L^2(\mathbb{R}_{2\ell})$, the leading order vanishes yielding

$$\left\langle \phi_0' \Delta_{s_{\mathbf{p}}} \kappa_{\mathbf{p}}, Q \right\rangle_{L^2} = \varepsilon^{3/2} \sum_{j \in \Sigma_0} \mathrm{q}_j \int_{\mathscr{I}_{\mathbf{p}}} \left(h_1(\boldsymbol{\gamma}_{\mathbf{p}}'') \Delta_{s_{\mathbf{p}}} \kappa_{\mathbf{p}} \tilde{\Theta}_j + h_2(\boldsymbol{\gamma}_{\mathbf{p}}'') \Delta_{s_{\mathbf{p}}} \kappa_{\mathbf{p}} \varepsilon \tilde{\Theta}_j' \right) \mathrm{d}\tilde{s}_{\mathbf{p}} \tag{5.46}$$

for some h_1, h_2 satisfying Notation 2.1. Note that $h_k(\boldsymbol{\gamma}_{\mathbf{p}}'')$ for $k = 1, 2$ lies in L^∞ since $\hat{\mathbf{p}} \in \mathbb{V}_2$, and utilizing the curvature $H^2(\mathscr{I}_{\mathbf{p}})$ bound in Lemma 2.11 yields

$$\left| \left\langle \phi_0' \Delta_{s_{\mathbf{p}}} \kappa_{\mathbf{p}}, Q \right\rangle_{L^2} \right| \lesssim \varepsilon^{3/2} (1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}) \|\mathbf{q}\|_{l^2}. \tag{5.47}$$

Combining the above estimate and (5.45) with (5.44) and multiplying by ε^3 implies

$$\left| \left\langle \varepsilon^3 (\mathrm{F}_3 - \mathrm{F}_3^\infty), Q \right\rangle_{L^2} \right| \lesssim \varepsilon^{9/2} (1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}) \|\mathbf{q}\|_{l^2}. \tag{5.48}$$

In a similar manner we have

$$\left|\left\langle \varepsilon^4(\mathrm{F}_4 - \mathrm{F}_4^\infty), Q\right\rangle_{L^2}\right| \lesssim \varepsilon^{9/2}(1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2})\|\mathbf{q}\|_{l^2},$$

which combined with the estimates on F_2 given by (5.43) and F_3 from (5.48) yields

$$\left|\langle \mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}_m^\infty, Q\rangle_{L^2}\right| \lesssim \varepsilon^{9/2}(1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2})\|\mathbf{q}\|_{l^2}. \tag{5.49}$$

Combining estimates (5.49) and (5.42) with (5.41) completes the Lemma. $\square$

Remark 5.8. It is essential to separate the pearling modes Q from the fast modes, w. The linear operator has a weaker coercivity on the pearling slow space, which is compensated for by the high-wave number estimates available for the pearling modes in Lemma 5.6. These decrease the coupling of the residual to the pearling modes. It is instructive to compare (5.22) with (5.40).

5.4. *Estimates on the nonlinearity*

The estimates of Lemmas 5.3, 5.4, incorporate L^2-bounds of the nonlinear term $\mathrm{N}(v^\perp)$. The following lemma affords these bounds on $\mathrm{N}(v^\perp)$ in terms of w and $\mathbf{q}$.

Lemma 5.9. *If $\|v^\perp\|_{L^\infty(\Omega)}$ is bounded independent of ε, then*

$$\|\mathrm{N}(v^\perp)\|_{L^2} \lesssim \varepsilon^{-1}\Big(\rho^{-2}\,\langle \mathbb{L}w, w\rangle_{L^2} + \|\mathbf{q}\|_{l^2}^2\Big). \tag{5.50}$$

Moreover, decomposing $v^\perp = w + Q$ as in (5.6), we have the bound

$$\|v^\perp\|_{L^\infty} \lesssim \varepsilon^{-1}\left(\rho^{-1}\,\langle \mathbb{L}w, w\rangle_{L^2}^{1/2} + \|\mathbf{q}(t)\|_{l^2}\right).$$

Proof. From the definition (5.5) of the nonlinear term $\mathrm{N}(v^\perp)$, with F given by (1.2) and $\mathbb{L}$ given by (4.1), some rearrangements lead to the equality

$$\begin{aligned}\mathrm{N}(v^\perp) = &-\Big(W''(u) - W''\Big)\Big(\varepsilon^2\Delta v^\perp - W''v^\perp\Big) - (\varepsilon^2\Delta - W'' + \varepsilon\eta_2)\Big(W'(u) - W' - W''v^\perp\Big)\\ &-\Big(W''(u) - W'' - W'''v^\perp\Big)\Big(\varepsilon^2\Delta\Phi_{\mathbf{p}} - W'(\Phi_{\mathbf{p}})\Big),\end{aligned}$$

where W', W'', W''' are evaluated at $\Phi_{\mathbf{p}}$ unless otherwise specified and $u = \Phi_{\mathbf{p}} + v^\perp$. The function u is uniformly bounded in L^∞ since $v^\perp$ is by assumption and $\varepsilon^k\nabla^k\Phi_{\mathbf{p}} \in L^\infty$ is uniformly bounded for $k = 1, \ldots 4$, since $\Phi_{\mathbf{p}}$ is smooth in the inner variables. We deduce that the nonlinear term N satisfies the pointwise bound

$$|\mathrm{N}(v^\perp)| \lesssim \|W\|_{C_c^6}\Big(\varepsilon^2|\nabla v^\perp|^2 + \varepsilon^2|\Delta v^\perp||v^\perp| + |v^\perp|^2\Big),$$

which yields the L^2 estimate

$$\|\mathrm{N}(v^\perp)\|_{L^2} \lesssim \varepsilon^2\|v^\perp\|_{L^4}^2 + \|v^\perp\|_{L^\infty}\varepsilon^2\|\Delta v^\perp\|_{L^2} + \|v^\perp\|_{L^4}^2.$$

In two space dimensions the Gargliardo-Nirenberg inequalities imply

$$\|\nabla v^\perp\|_{L^4}^2 \lesssim \|\nabla^2 v^\perp\|_{L^2}\|v^\perp\|_{L^\infty} \quad \text{and} \quad \|v^\perp\|_{L^\infty} \lesssim \|v^\perp\|_{L^2}^{1/2}\|v^\perp\|_{H^2}^{1/2}, \tag{5.51}$$

and the L^2-estimate of $\mathrm{N}(v^\perp)$ reduces to

$$\begin{aligned} \|\mathrm{N}(v^\perp)\|_{L^2} &\leqslant C\|v^\perp\|_{L^\infty}\left(\varepsilon^2\|\nabla^2 v^\perp\|_{L^2} + \|v^\perp\|_{L^2}\right) \\ &\leqslant C\|v^\perp\|_{L^2}^{1/2}\varepsilon^2\|\Delta v^\perp\|_{L^2}^{3/2} + C\|v^\perp\|_{L^2}^{3/2}\|v^\perp\|_{H^2}^{1/2} \\ &\leqslant C\varepsilon^{-1}\left(\|v^\perp\|_{L^2} + \varepsilon^2\|\Delta v^\perp\|_{L^2}\right)^2. \end{aligned} \tag{5.52}$$

From the decomposition $v^\perp = w + Q$, we have

$$\|v^\perp\|_{L^2} \lesssim \|w\|_{L^2} + \|\mathbf{q}(t)\|_{l^2}, \quad \varepsilon^2\|\Delta v^\perp\|_{L^2} \lesssim \varepsilon^2\|\Delta w\|_{L^2} + \|\mathbf{q}(t)\|_{l^2},$$

where we used the fact that $\varepsilon^2\Delta$ is a uniformly bounded operator on $\mathcal{Z}_*^0$ in L^2 and hence on Q, see (2.11) and (4.5). The estimate (5.50) follows from the coercivity Lemma 4.13. Applying the estimate (5.51) leads to

$$\begin{aligned} \|v^\perp\|_{L^\infty} &\leqslant \varepsilon^{-1}(\|w\|_{L^2} + \|\mathbf{q}\|_{l^2})^{1/2}(\varepsilon^2\|\Delta w\|_{L^2} + \|\mathbf{q}\|_{l^2})^{1/2} \\ &\lesssim \varepsilon^{-1}\left(\rho^{-1}\langle \mathbb{L}w, w\rangle_{L^2}^{1/2} + \|\mathbf{q}\|_{l^2}\right). \end{aligned}$$

The proof is complete. □

5.5. *Main theorem*

In this sub-section we introduce thinner tubular neighborhoods $\mathcal{V}_R(\mathcal{M}_b, \mathcal{O}_\delta) \subset \mathcal{U}(\mathcal{M}_b)$ of thickness R defined over the open base $\mathcal{O}_\delta \subset \mathcal{D}_\delta$. We show that solutions of the gradient flow (1.3) that start inside of $\mathcal{V}_{R_1}(\mathcal{M}_b, \mathcal{O}_\delta)$ remain in a slightly thicker neighborhood $\mathcal{V}_{R_2}(\mathcal{M}_b, \mathcal{O}_{2,\delta})$ so long as $\mathbf{p}$ remains in the slightly larger base $\mathcal{O}_{2,\delta}$. For $R \in (0, \varepsilon^2]$, the tubular neighborhood with width R and domain $\mathcal{O}_\delta$ is defined as

$$\mathcal{V}_R(\mathcal{M}_b, \mathcal{O}_\delta) = \left\{ u \in H^2(\Omega) \,\middle|\, \inf_{\mathbf{p}\in\mathcal{O}_\delta} \|u - \Phi_{\mathbf{p}}(\sigma)\|_{H^2_{\mathrm{in}}} < R,\ \langle u - b_-\rangle_{L^2} = \frac{\varepsilon M_0}{|\Omega|} \right\}. \tag{5.53}$$

We introduce the nested base domains $\mathcal{O}_{m,\delta}$ as the subsets of $\mathcal{D}_\delta$ that satisfy

$$\mathcal{O}_{m,\delta} := \left\{ \mathbf{p} \in \mathbb{R}^{N_1} \,\middle|\, |\mathrm{p}_0| + \|\hat{\mathbf{p}}\|_{\mathbb{V}_1} < m\delta; \quad \|\hat{\mathbf{p}}\|_{\mathbb{V}_2} + \varepsilon\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2} < m \right\}, \qquad m = 1, 2. \tag{5.54}$$

When $m = 1$, we denote $\mathcal{O}_{1,\delta}$ by $\mathcal{O}_\delta$. The parameter δ will be chosen sufficiently small that Lemma 5.10 holds. The condition on p_0 insures that the pearling stability condition (**PSC**) holds uniformly, see Lemma 5.10; the uniform bound on $\varepsilon\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}$ insures the smoothness of the perturbed curve $\Gamma_{\mathbf{p}}$.

From Lemma 2.8, each of the a priori bounds on $\hat{\mathbf{p}}$ in (5.54) are inferred from the single, stronger bound $\|\hat{\mathbf{p}}\|_{\mathbb{V}_3^2} \leqslant m\delta$. Hence we introduce a parallel set of smaller but more easily defined domains,

$$\mathcal{O}^{\circ}_{m,\delta} := \left\{ \mathbf{p} \in \mathbb{R}^{N_1} \;\middle|\; |\mathrm{p}_0| + \|\hat{\mathbf{p}}\|_{\mathbb{V}^2_3} < m\delta \right\} \subset \mathcal{O}_{m,\delta}. \tag{5.55}$$

The equilibrium pearling stability condition arises from replacing σ in the pearling stability condition (4.61) with its leading order equilibrium value σ_1^*, defined in (3.11),

$$(\mathbf{PSC}^*) \qquad \sigma_1^* S_1 + \eta_d \lambda_0 > 0. \tag{5.56}$$

The next lemma shows that if ($\mathbf{PSC}^*$) holds, then for a suitable admissible pair (Γ_0, M_0) the (**PSC**) holds uniformly for all $\mathbf{p} \in \mathcal{O}_{2,\delta}$ provided that δ is sufficiently small.

Lemma 5.10. *Suppose that the equilibrium pearling stability condition* (5.56) *holds and that* (Γ_0, M_0) *is a admissible pair satisfying*

$$\left| M_0 - m_0|\Gamma_0| - B_2^{\infty}|\Omega|\sigma_1^* \right| \leqslant \delta. \tag{5.57}$$

Then for $\mathbf{p} \in \mathcal{O}_{2,\delta}$*, the bulk parameter* $\sigma = \sigma(\mathbf{p})$ *defined in* (3.22) *is uniformly bounded, i.e.* $|\sigma| \lesssim 1$*, and the pearling stability condition* (**PSC**) *from* (4.61) *holds uniformly for all* $\mathbf{p} \in \mathcal{O}_{2,\delta}$ *provided that* δ, ε_0 *is sufficiently small, in terms of the domain, the system parameters, and* K_0, ℓ_0.

Proof. From the bound of Lemma 3.4, we estimate

$$\left| \sigma(\mathbf{p}) - \frac{M_0 - m_0|\Gamma_0|}{B_2^{\infty}|\Omega|} \right| \lesssim |\mathrm{p}_0| + \varepsilon.$$

The uniform bound on σ follows from the assumption on p_0 since $\mathbf{p} \in \mathcal{O}_{2,\delta}$. By assumption (5.57), $|\sigma(\mathbf{p}) - \sigma_1^*| \leqslant \delta$ and the pearling stability condition (4.61) holds uniformly. □

Lemma 5.11. *For* $\mathbf{p} \in \mathcal{O}_{2,\delta}$*, the temporal derivative of* $\mathbf{p}$ *satisfies the bound*

$$\|\dot{\mathbf{p}}\|_{l^2} \lesssim \varepsilon^3 + \varepsilon^{3/2}\|v^{\perp}\|_{L^2} + \varepsilon^{1/2}\|\mathrm{N}(v^{\perp})\|_{L^2} + \varepsilon^{-1}\|v^{\perp}\|_{L^2}\|\dot{\mathbf{p}}\|_{l^2}.$$

Proof. We rewrite the equation (5.4) as

$$\partial_t \Phi_{\mathbf{p}} = -\Pi_0 \mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{Re}(v^{\perp}), \qquad \mathrm{Re}[v^{\perp}] := \partial_t v^{\perp} + \Pi_0 \mathbb{L} v^{\perp} + \Pi_0 \mathrm{N}(v^{\perp}).$$

With a use of Lemmas 6.4, 3.6 and the a priori assumption on $\|\mathbf{p}\|_{\mathbb{V}^2_4}$ we derive

$$\begin{aligned} \|\dot{\mathbf{p}}\|_{l^2} &\lesssim \varepsilon^{1/2}\|\Pi_{\mathcal{Z}^1_*}\partial_t \Phi_{\mathbf{p}}\|_{L^2} \\ &\lesssim \varepsilon^{1/2}\|\Pi_0 \mathrm{F}(\Phi_{\mathbf{p}})\|_{L^2} + \varepsilon^{1/2}\|\Pi_{\mathcal{Z}^1_*}\mathrm{Re}[v^{\perp}]\|_{L^2} \\ &\lesssim \varepsilon^3 + \varepsilon^{1/2}\|\Pi_{\mathcal{Z}^1_*}\mathrm{Re}[v^{\perp}]\|_{L^2}. \end{aligned} \tag{5.58}$$

To estimate the projection of the remainder $\mathrm{Re}[v^{\perp}]$, we deal with its terms one by one. First, we rewrite the projection of $\partial_t v^{\perp}$ as

$$\left\langle \partial_t v^\perp, Z^{1k}_{\mathbf{p},*}\right\rangle_{L^2} = \partial_t \left\langle v^\perp, Z^{1k}_{\mathbf{p},*}\right\rangle_{L^2} - \left\langle v^\perp, \partial_t Z^{1k}_{\mathbf{p},*}\right\rangle_{L^2}.$$

The first term on the right hand side is zero since $v^\perp$ is perpendicular to the meandering slow space $\mathcal{Z}^1_*$; and the second term can be bounded with the aid of (6.14). Combining these, we deduce

$$|\left\langle \partial_t v^\perp, Z^{1k}_{\mathbf{p},*}\right\rangle_{L^2}| \lesssim \varepsilon^{-1}\|\dot{\mathbf{p}}\|_{l^2}\|v^\perp\|_{L^2} \qquad \forall k \in \Sigma_1,$$

which combined with a typical l^2-l^∞ estimate and the $N_1 \lesssim \varepsilon^{-1}$ implies

$$\|\Pi_{\mathcal{Z}^1_*}\partial_t v^\perp\|_{L^2} \lesssim \varepsilon^{-1} N_1^{1/2}\|\dot{\mathbf{p}}\|_{l^2}\|v^\perp\|_{L^2} \lesssim \varepsilon^{-3/2}\|\dot{\mathbf{p}}\|_{l^2}\|v^\perp\|_{L^2}.$$

Second, we apply Lemma 4.11 to bound the projection of the linear term $\Pi_0\mathbb{L}_{\mathbf{p}}v^\perp$,

$$\left\|\Pi_{\mathcal{Z}^1_*}\Pi_0\mathbb{L}_{\mathbf{p}}v^\perp\right\| \lesssim (\varepsilon^2 + \varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4})\|v^\perp\|_{L^2}.$$

Finally, the projection of the nonlinear term can be estimated trivially,

$$\|\Pi_{\mathcal{Z}^1_*}\Pi_0\mathrm{N}(v^\perp)\|_{L^2} \lesssim \|\mathrm{N}(v^\perp)\|_{L^2}.$$

These three estimates imply

$$\|\Pi_{\mathcal{Z}^1_*}\mathrm{Re}[v^\perp]\|_{L^2} \lesssim \varepsilon^{-3/2}\|v^\perp\|_{L^2}\|\dot{\mathbf{p}}\|_{l^2} + (\varepsilon^2 + \varepsilon^2\|\hat{\mathbf{p}}\|_{\mathbb{V}^2_4})\|v^\perp\|_{L^2} + \|\mathrm{N}(v^\perp)\|_{L^2},$$

which combined with (5.58) completes the proof. □

Lemma 5.12. *Fix K_0, ℓ_0, and assume (Γ_0, M_0) is a admissible pair from $\mathcal{A}(K_0, \ell_0)$. Then there exists ε_0 sufficiently small depending on δ, and a positive T_0 independent of δ, ρ and ε_0 such that for all initial data $u_0 \in \mathcal{V}_{\varepsilon^{5/2}}(\mathcal{M}_b, \mathcal{O}_\delta)$, the projection parameter $\mathbf{p}(t)$ corresponding to the solution $u = u(t)$ remains in the open set $\mathcal{O}_{2,\delta}$ for all $t \in [0, T_0\rho^{-1}]$ so long as u remains in the tubular projection neighborhood $\mathcal{U}(\mathcal{M}_b)$ for which the projection $\Pi_{\mathcal{M}_b}$ is well-defined.*

Proof. Since $u_0 \in \mathcal{V}_{\varepsilon^{5/2}}(\mathcal{M}_b, \mathcal{O}_\delta)$, then there exists $\mathbf{p}_0 \in \mathcal{O}_\delta$ and $v_0 \in L^2$ satisfying $\|v_0\|_{H^2_{\mathrm{in}}} \leqslant \varepsilon^{5/2}$ such that $u_0 = \Phi_{\mathbf{p}_0} + v_0$. We first note that for ε_0 small enough, Lemma 5.2 applies to u_0, and hence there exists $\mathbf{p}(0) \in \mathcal{D}_\delta$ such that $\Phi_{\mathbf{p}(0)} = \Pi_{\mathcal{M}_b}u_0$ satisfying

$$\|\mathbf{p}(0) - \mathbf{p}_0\|_{l^2} \lesssim \varepsilon^3. \tag{5.59}$$

By the Fundamental Theorem of Calculus we bound the difference

$$|\mathrm{p}_k(t) - \mathrm{p}_k(0)| \leqslant \int_0^t |\dot{\mathrm{p}}_k|\,\mathrm{d}\tau \qquad t > 0,$$

for any $k \in \Sigma_1$, which together with the Hölder's inequality and the a priori assumption (5.7) implies

$$\|\mathbf{p}(t) - \mathbf{p}(0)\|_{l^2} \lesssim t\|\dot{\mathbf{p}}\|_{l^2} \lesssim \varepsilon^3 t.$$

Combining with (5.59) with the aid of triangle inequality implies

$$\begin{aligned}\|\mathbf{p}(t) - \mathbf{p}_0\|_{l^2} &\leqslant \|\mathbf{p}(t) - \mathbf{p}(0)\|_{l^2} + \|\mathbf{p}_0 - \mathbf{p}(0)\|_{l^2} \\ &\lesssim \varepsilon^3 + \varepsilon^3 t.\end{aligned}$$

Note that $\mathbf{p}_0 \in \mathcal{O}_\delta$, from the estimate above the length parameter $\mathrm{p}_0(t)$, as the first component of $\mathbf{p}(t)$, satisfies $|\mathrm{p}_0(t)| < 2\delta$ for ε_0 small enough. It suffices to bound the difference of $\hat{\mathbf{p}}$ and $\hat{\mathbf{p}}_0$ in $\mathbb{V}_1$, $\mathbb{V}_2$ and $\mathbb{V}_4^2$. By the embedding Lemma 2.8 with $N_1 \lesssim \varepsilon^{-1}\rho^{1/4}$ from (4.9), we derive

$$\|\hat{\mathbf{p}}(t) - \hat{\mathbf{p}}_0\|_{\mathbb{V}_1} \lesssim \varepsilon^{-1}\rho^{1/4}\|\hat{\mathbf{p}}(t) - \hat{\mathbf{p}}_0\|_{l^1} \lesssim \varepsilon^{-3/2}\rho^{1/4}\|\mathbf{p}(t) - \mathbf{p}_0\|_{l^2} \lesssim \varepsilon^{3/2} + \varepsilon^{3/2}t,$$

and the following higher weighted estimate

$$\|\hat{\mathbf{p}} - \hat{\mathbf{p}}_0\|_{\mathbb{V}_2} + \varepsilon\|\hat{\mathbf{p}} - \hat{\mathbf{p}}_0\|_{\mathbb{V}_4^2} \lesssim \varepsilon^{-3}\rho\|\hat{\mathbf{p}} - \hat{\mathbf{p}}_0\|_{l^2} \lesssim \rho + \rho t.$$

Noting $\mathbf{p}_0 \in \mathcal{O}_\delta$, there exists $T_0 > 0$, independent of ε, ρ, such that $\mathbf{p} \in \mathcal{O}_{2,\delta}$ for any $t \in [0, T_0/\rho]$. $\square$

The following theorem presents the stability of the bilayer manifold up to its boundary. We recall that (Γ_0, M_0) is a admissible pair with associated $N_1(\rho)$-dimensional bilayer manifold $\mathcal{M}_b(\Gamma_0, M_0; \rho)$ defined in Definition 3.3, ρ is the spectral cut-off introduced in Definition 4.1 and is sufficiently small as required by Theorem 4.11 and Lemma 5.12. The slow spaces $\mathcal{Z}_*^k$, $\mathcal{Z}_*$ are defined in (4.52) and $\mathcal{V}_R(\mathcal{M}_b, \mathcal{O}_\delta)$ are the tubular neighborhoods with H^2_{in}-width R and base $\mathcal{O}_\delta$ defined in (5.53)-(5.54). The parameter $\delta > 0$ is a fixed sufficiently small as required by Lemma 5.2.

Theorem 5.13. *Consider the mass-preserving flow (1.3) subject to periodic boundary conditions on the domain $\Omega = [-L, L]^2$. Assume that the equilibrium pearling stability condition* (**PSC***)*–(5.56), holds for the given system parameters. Fix K_0, ℓ_0, then there exists an ε_0 and a $C > 0$ such that for each admissible pair (Γ_0, M_0) from $\mathcal{A}(K_0, \ell_0)$, and for all $\varepsilon \in (0, \varepsilon_0)$, the bilayer manifold $\mathcal{M}_b(\Gamma_0, M_0)$ has the following properties. Each solution $u = u(t)$ corresponding to initial data $u_0 \in \mathcal{V}_{\varepsilon^{5/2}}(\mathcal{M}_b, \mathcal{O}_\delta)$ remains in the slightly larger tubular neighborhood $\mathcal{V}_{C\varepsilon^{5/2}}(\mathcal{M}_b, \mathcal{O}_{2,\delta}) \subset \mathcal{U}(\mathcal{M}_b)$ so long as its projected meander parameters $\mathbf{p}$ remain in $\mathcal{O}_{2,\delta}$. Denoting this interval of residency as $[0, T]$, then $T > 0$ and during this interval u admits the dynamic decomposition*

$$u(t) = \Phi_{\mathbf{p}}(t; \sigma) + v^\perp, \qquad v^\perp = Q(t; \mathbf{q}) + w(t), \qquad \forall t \in [0, T],$$

where $Q = \Pi_{\mathcal{Z}_*^0} v^\perp \in \mathcal{Z}_*^0(\mathbf{p}, \rho)$, $w \in \mathcal{Z}_*^\perp(\mathbf{p}, \rho)$. In particular, the orthogonal perturbation $v^\perp$ and its fast and pearling decomposition satisfy

$$\|v^\perp\|_{H^2_{\mathrm{in}}} \lesssim \|w\|_{H^2_{\mathrm{in}}} + \|Q\|_{H^2_{\mathrm{in}}} \leqslant C\varepsilon^{5/2}\rho^{-2}. \tag{5.60}$$

Proof. Since $u_0 \in \mathcal{V}_{\varepsilon^{5/2}}(\mathcal{M}_b, \mathcal{O}_\delta) \subset \mathcal{U}(\mathcal{M}_b)$, Lemma 5.2 implies the existence of the decomposition $u_0 = \Phi_{\mathbf{p}(0)} + v_0^\perp = \Phi_{\mathbf{p}(0)} + Q_0(\mathbf{q}(0)) + w_0$ with $Q_0 \in \mathcal{Z}_*^0(\Gamma_{\mathbf{p}(0)})$ and $w_0 \in \mathcal{Z}_*^\perp(\Gamma_{\mathbf{p}(0)})$ satisfying,

$$\|w_0\|_{H^2_{\mathrm{in}}} \lesssim \varepsilon^{5/2}, \qquad \|\mathbf{q}(0)\|_{l^2} \lesssim \varepsilon^{5/2}. \tag{5.61}$$

We establish the existence of positive constants K_1, K_2 independent of $\varepsilon, \rho, \delta$ and $T > 0$ for which the bounds

$$(\mathbf{A}) \qquad \langle \mathbb{L}w, w\rangle_{L^2} \leqslant K_1 \varepsilon^5 \rho^{-2}, \qquad \|\mathbf{q}\|_{l^2}^2 \leqslant K_2 \varepsilon^5 \rho^{-4}, \tag{5.62}$$

hold uniformly for all $t \in [0, T]$ as long as $\mathbf{p}(t) \in \mathcal{O}_{2,\delta}$ on the interval. In the argument below we modify the notation of Section 1.1 writing '$A \lesssim B$' to denote '$A \leqslant CB$' for a constant C that is independent of K_1, K_2 as well as the small parameters $\varepsilon, \rho, \delta$. The existence of a $T > 0$ is assured by (5.61) and Lemma 5.12.

First, applying the coercivity Theorem 4.13 and assumption (**A**) implies

$$\|w\|_{H^2_{\mathrm{in}}}^2 \lesssim \rho^{-2} \langle \mathbb{L}w, w\rangle_{L^2} \lesssim K_1 \varepsilon^5 \rho^{-4}. \tag{5.63}$$

Then from the relation $\|Q\|_{H^2_{\mathrm{in}}} \sim \|\mathbf{q}\|_{l^2}$, Lemma 5.9 and assumption (**A**) we bound the L^2-norm of $v^\perp = Q + w$ and nonlinear term $\mathrm{N}(v^\perp)$ as

$$\|v^\perp\|_{H^2_{\mathrm{in}}}^2 \lesssim \|w\|_{H^2_{\mathrm{in}}}^2 + \|\mathbf{q}\|_{l^2}^2, \qquad \|\mathrm{N}(v^\perp)\|_{L^2}^2 \lesssim (K_1^2 + K_2^2)\varepsilon^8 \rho^{-8}. \tag{5.64}$$

It suffices to verify the assumption (**A**) for all $t \in [0, T]$, on which $\mathrm{p}(t) \in \mathcal{O}_{2,\delta}$, in order to establish the main estimate (5.60) in the Theorem. We first note from Lemma 5.11 for $K_1, K_2 > 1$,

$$\|\dot{\mathbf{p}}\|_{l^2}^2 \lesssim \varepsilon^6 + (K_1^2 + K_2^2)\varepsilon^8 \rho^{-8} + (K_1 + K_2)\varepsilon^3 \rho^{-4} \|\dot{\mathbf{p}}\|_{l^2}^2. \tag{5.65}$$

Since K_1, K_2 are independent of ε, for $\varepsilon \in (0, \varepsilon_0)$ with ε_0 small enough depending on ρ we have

$$\|\dot{\mathbf{p}}\|_{l^2}^2 \lesssim \varepsilon^6, \tag{5.66}$$

independent of K_1, K_2, and the a priori assumption (5.7) holds for $\mathbf{p} \in \mathcal{O}_{2,\delta}$. The pearling stability condition (**PSC**) (4.61) holds uniformly for all $t \in [0, T]$ by Lemma 5.10. In particular Lemma 5.4 applies. We restate the key estimates of Lemmas 5.3 and 5.4 as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbb{L}w, w\rangle_{L^2} + \|\mathbb{L}w\|_{L^2}^2 &\lesssim \varepsilon^{-1} \|\dot{\mathbf{p}}\|_{l^2}^2 + \varepsilon^5 + \varepsilon^2 \rho^{-4} (\|\mathbf{q}\|_{l^2}^2 + \|\dot{\mathbf{q}}\|_{l^2}^2) \\ &\quad + \varepsilon^7 (1 + \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}^2) + \|\mathrm{N}(v^\perp)\|_{L^2}^2; \end{aligned} \tag{5.67}$$

$$\partial_t \|\mathbf{q}\|_{l^2}^2 + C\varepsilon \|\mathbf{q}\|_{l^2}^2 \lesssim \varepsilon \|w\|_{L^2}^2 + \|\dot{\mathbf{p}}\|_{l^2}^2 + \varepsilon^{-1} \|\mathrm{N}(v^\perp)\|_{L^2}^2 + \varepsilon^8 + \varepsilon^8 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}^2,$$

and $\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2} \lesssim \varepsilon^{-1}$ for $\mathbf{p} \in \mathcal{O}_{2,\delta}$. From the l^2-bound of the pearling modes $\mathbf{q}$ from (**A**), the estimates of the fast modes w and the nonlinear terms N from (5.63)-(5.64), and $\mathbf{p} \in \mathcal{O}_{2,\delta}$ we reduce the l^2 bound of $\dot{\mathbf{q}}$ in Lemma 5.4 to

$$\|\dot{\mathbf{q}}\|_{l^2}^2 \lesssim \varepsilon^7 + (K_1 + K_2)\varepsilon^5\rho^{-4} + (K_1^2 + K_2^2)\varepsilon^8\rho^{-8}, \tag{5.68}$$

where the first term on the right-hand side comes from the a priori assumptions on $\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}^2$ and estimate of $\|\dot{\mathbf{p}}\|_{l^2}$ in (5.66). Combining this with the first inequality in (5.67), and reusing (5.62)-(5.64) and $\mathbf{p} \in \mathcal{O}_{2,\delta}$ yields for $K_1, K_2 > 1$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbb{L}w, w\rangle_{L^2} + C\rho^2\langle \mathbb{L}w, w\rangle_{L^2} \lesssim \varepsilon^5 + (K_1^2 + K_2^2)\varepsilon^7\rho^{-12}, \tag{5.69}$$

where the term with the dominant power of ε arises from the inhomogeneous term on the right-hand side of (5.67). Integrating this estimate in time we obtain

$$\begin{aligned}\langle \mathbb{L}w, w\rangle_{L^2} &\leqslant \big\langle \mathbb{L}_{\mathbf{p}(0)}w_0, w_0\big\rangle_{L^2} e^{-C\rho^2 t} + C\varepsilon^5\rho^{-2} + C(K_1^2 + K_2^2)\varepsilon^7\rho^{-14},\\ &\lesssim \|w_0\|_{H_{\mathrm{in}}^2}^2 + \varepsilon^5\rho^{-2} + (K_1^2 + K_2^2)\varepsilon^7\rho^{-14}\\ &\leqslant C_1(\varepsilon^5\rho^{-2} + (K_1^2 + K_2^2)\varepsilon^7\rho^{-14}),\end{aligned} \tag{5.70}$$

for some positive constant C_1 independent of ε, ρ and K_1, K_2. Here we used (5.61) to bound w_0.

Turning to the $\mathbf{q}$ estimate in (5.67) and utilizing (5.63)-(5.64), (5.66) to bound the first three terms on the right-hand side, we obtain the bound, valid for $\mathbf{p} \in \mathcal{O}_{2,\delta}$,

$$\partial_t\|\mathbf{q}\|_{l^2}^2 + C\varepsilon\|\mathbf{q}\|_{l^2}^2 \lesssim K_1\varepsilon^6\rho^{-4} + (K_1^2 + K_2^2)\varepsilon^7,$$

where the dominant ε-term arises from $\|w\|_{L^2}$. We integrate this inequality and apply the initial value estimates (5.61) to $\|\mathbf{q}_0\|_{l^2}$ to obtain

$$\|\mathbf{q}\|_{l^2}^2 \leqslant e^{-C\varepsilon t}\|\mathbf{q}(0)\|_{l^2}^2 + CK_1\varepsilon^5\rho^{-4} + C(K_1^2 + K_2^2)\varepsilon^6 \leqslant C_2\Big(K_1\varepsilon^5\rho^{-4} + (K_1^2 + K_2^2)\varepsilon^6\Big). \tag{5.71}$$

Here C_2 is a positive constant independent of ε, ρ and K_1, K_2. Taking $K_1 = 2C_1$, $K_2 = 2C_2K_1$ and ε_0 small enough, we combine (5.70) and (5.71) to establish (**A**). Together with the first inequality in (5.64), this completes the proof. □

6. Appendix

We present some technical and intermediate results.

6.1. The variation of the local coordinates with respect to p

Lemma 6.1. *Assume $\|\hat{\mathbf{p}}\|_{\mathbb{V}_1} \ll 1$ and $\Gamma_0 \in \mathcal{G}_{K_0,2\ell_0}^4$, the normalized length constant $A(\mathbf{p})$ defined in* (2.20) *admits the approximation*

$$A(\mathbf{p}) = 1 + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_1})$$

Furthermore, the rate of change of $A(\mathbf{p})$ *with respect to* $\mathbf{p}$ *can be bounded by*

$$\|\nabla_{\mathbf{p}} A(\mathbf{p})\|_{l^2} \lesssim \|\hat{\mathbf{p}}\|_{\mathbb{V}_2^2}.$$

If Γ_0 *is a circle then we have the isoperimetric bound*

$$A(\mathbf{p}) = 1 + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_1}^2).$$

Proof. The function $A(\mathbf{p})$ is the proportional change in length of $\boldsymbol{\gamma}_0$ to the perturbation $\boldsymbol{\gamma}_{\bar{p}}$ given in (2.21) that excluded the radial perturbation. In light of (2.23), taking the derivative of (2.21) we find

$$\boldsymbol{\gamma}'_{\bar{p}} = (1 - \kappa_0 \bar{p}(\tilde{s}))\boldsymbol{\gamma}'_0 + \bar{p}'(\tilde{s})|\boldsymbol{\gamma}'_{\mathbf{p}}|\mathbf{n}_0(s). \tag{6.1}$$

Taking absolute value, and using the orthogonality between and tangent $\boldsymbol{\gamma}'_0$ and normal $\mathbf{n}_0$, we deduce

$$|\boldsymbol{\gamma}'_{\bar{p}}| = \left((1 - \kappa_0 \bar{p}(\tilde{s}))^2 + |\bar{p}'(\tilde{s})|^2|\boldsymbol{\gamma}'_{\mathbf{p}}|^2\right)^{1/2} \tag{6.2}$$

Considering terms including $\bar{p}$ as small, the right-hand side has leading order term $1 - \kappa_0 \bar{p}(\tilde{s})$, and hence

$$A(\mathbf{p}) = \frac{1}{|\Gamma_0|}\int_{\mathscr{I}} (1 - \kappa_0(s)\bar{p}(\tilde{s}))\,\mathrm{d}s + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_1}^2). \tag{6.3}$$

The approximation of $A(\mathbf{p})$ follows if Γ_0 is a general smooth curve.

When Γ_0 is a circle we return to (6.3) and remark that the curvature $\kappa_0(s) = \kappa_0$ is a constant while $\bar{p}(\tilde{s}) = \sum_{i=3}^{N_1-1} \mathrm{p}_i \tilde{\Theta}_i(\tilde{s})$ inherits a zero-integral with respect to $\mathrm{d}\tilde{s} = |\boldsymbol{\gamma}'_{\mathbf{p}}|\,\mathrm{d}s$ from the Laplace-Beltrami eigenmodes $\{\tilde{\Theta}_i(\tilde{s})\}_{i\geqslant 3}$. From the definition of $\boldsymbol{\gamma}_{\mathbf{p}}$ given in (2.19), identity (6.2), and the general form expansion (6.3) of $A(\mathbf{p})$ we find

$$|\boldsymbol{\gamma}'_{\mathbf{p}}| = \frac{1 + \mathrm{p}_0}{A(\mathbf{p})}|\boldsymbol{\gamma}'_{\bar{p}}| = 1 + O(\|\hat{\mathbf{p}}\|_{\mathbb{V}_1}).$$

Changing variables from $\mathrm{d}s$ to $\mathrm{d}\tilde{s}$ and using the zero average of $\bar{p}$ with respect to $\mathrm{d}\tilde{s}$ we derive the isoperimetric bound which shows that circles are critical points of perturbations $\bar{p}$ that do not change the effective radius.

It remains to estimate the rate of change of A with respect to p_j. By plugging $|\boldsymbol{\gamma}'_{\mathbf{p}}| = (1 + \mathrm{p}_0)|\boldsymbol{\gamma}'_{\bar{p}}|$ into the right hand side of (6.2) and solving for $|\boldsymbol{\gamma}'_{\mathbf{p}}|$ we find

$$|\boldsymbol{\gamma}'_{\bar{p}}|^2 = \frac{(1 - \kappa_0 \bar{p})^2}{1 - (1 + \mathrm{p}_0)^2|\bar{p}'|^2}.$$

To calculate the derivative with respect to p_j, we need the derivative of $\bar{p}$ and $\bar{p}'$. In fact,

$$2|\boldsymbol{\gamma}'_{\bar{p}}|\partial_{\mathrm{p}_j}|\boldsymbol{\gamma}'_{\bar{p}}| = \frac{(1-\kappa_0\bar{p})\partial_{\mathrm{p}_j}\bar{p}}{1-(1+\mathrm{p}_0)^2|\bar{p}'|^2} + \frac{(1-\kappa_0\bar{p})^2}{(1-(1+\mathrm{p}_0)^2|\bar{p}'|^2)^2}\Big[2\delta_{j0}(1+\mathrm{p}_0)|\bar{p}'|^2 + (1+\mathrm{p}_0)^2 2\bar{p}'\partial_{\mathrm{p}_j}\bar{p}'\Big]. \tag{6.4}$$

On the other hand, by the definition of $\bar{p}$,

$$\partial_{\mathrm{p}_j}\bar{p} = \tilde{\Theta}_j\mathbf{1}_{\{j\geqslant 3\}} + \bar{p}'(\partial_{\mathrm{p}_j}\tilde{s} - \delta_{j0}\tilde{s}/(1+\mathrm{p}_0));$$
$$\partial_{\mathrm{p}_j}\bar{p}' = \tilde{\Theta}'_j\mathbf{1}_{\{j\geqslant 3\}} + \bar{p}''(\partial_{\mathrm{p}_j}\tilde{s} - \delta_{j0}\tilde{s}/(1+\mathrm{p}_0)).$$

We note $\|\nabla_{\mathbf{p}}\tilde{s}\|_{l^2} \lesssim 1$ by its definition in (2.23). The gradient estimate of A follows by combining these identities with (6.4), integrating by parts, the a priori bound on $\hat{\mathbf{p}}$, and Lemma 2.13. □

The following lemma estimates the $\mathbf{p}$-variation of the whiskered coordinates associated to $\Gamma_{\mathbf{p}}$. It controls the difference between the local coordinates $(s_{\mathbf{p}}, z_{\mathbf{p}})$ for $\Gamma_{\mathbf{p}}$ and (s, z) for Γ_0 in terms of the perturbation $\mathbf{p}$.

Lemma 6.2. *Let $(s_{\mathbf{p}}, z_{\mathbf{p}})$ be the local coordinates subject to $\Gamma_{\mathbf{p}}$ with whisker length 2ℓ. Under assumption (2.29) the tangent coordinate $s_{\mathbf{p}}$ has bounded variation on the domain $|\varepsilon z_{\mathbf{p}}| \ll 1$, satisfying*

$$\|\nabla_{\mathbf{p}} s_{\mathbf{p}}\|_{L^2(\mathscr{I})} \lesssim 1.$$

The normal local coordinate $z_{\mathbf{p}}$ varies quickly with respect to p_j satisfying

$$\begin{aligned}
\frac{\partial z_{\mathbf{p}}}{\partial \mathrm{p}_0} &= -\frac{1}{\varepsilon}\left(\frac{R_0+\bar{p}}{A}\big(1-(1+\mathrm{p}_0)\partial_{\mathrm{p}_0}\ln A\big) - \frac{\tilde{s}_{\mathbf{p}}\bar{p}'}{A(1+\mathrm{p}_0)}\right)\mathbf{n}_0\cdot\mathbf{n}_{\mathbf{p}}, && j=0,\\
\frac{\partial z_{\mathbf{p}}}{\partial \mathrm{p}_j} &= -\frac{\mathbf{E}_j\cdot\mathbf{n}_{\mathbf{p}}}{\varepsilon\sqrt{2\pi R_0}}, && j=1,2,\\
\frac{\partial z_{\mathbf{p}}}{\partial \mathrm{p}_j} &= -\frac{1}{\varepsilon}\left(\tilde{\Theta}_j - \frac{(1+\mathrm{p}_0)\partial_{\mathrm{p}_j}\ln A}{A}(R_0+\bar{p})\right)\mathbf{n}_0\cdot\mathbf{n}_{\mathbf{p}}, && j\geqslant 3.
\end{aligned}$$

Moreover, we have the estimates

$$\|s_{\mathbf{p}} - s\|_{L^\infty(\Gamma^{2\ell}_{\mathbf{p}})} \lesssim \|\mathbf{p}\|_{\mathbb{V}_0}, \quad \|z_{\mathbf{p}} - z\|_{L^\infty(\Gamma^{2\ell}_{\mathbf{p}})} \leqslant \varepsilon^{-1}\|\mathbf{p}\|_{\mathbb{V}_0}. \tag{6.5}$$

Proof. Any $x \in \Gamma^{2\ell}_{\mathbf{p}}$ can be expressed in the local coordinates of both $\Gamma_{\mathbf{p}}$ and Γ_0. Equating these yields

$$\boldsymbol{\gamma}_0(s) + \varepsilon z\mathbf{n}_0(s) = \boldsymbol{\gamma}_{\mathbf{p}}(s_{\mathbf{p}}) + \varepsilon z_{\mathbf{p}}\mathbf{n}_{\mathbf{p}}(s_{\mathbf{p}}). \tag{6.6}$$

Taking the derivative of (6.6) with respect to p_j, the j-th component of the vector $\mathbf{p}$, yields

$$0 = \frac{\partial\boldsymbol{\gamma}_{\mathbf{p}}}{\partial \mathrm{p}_j}(s_{\mathbf{p}}) + \boldsymbol{\gamma}'_{\mathbf{p}}\frac{\partial s_{\mathbf{p}}}{\partial \mathrm{p}_j} + \varepsilon\frac{\partial z_{\mathbf{p}}}{\partial \mathrm{p}_j}\mathbf{n}_{\mathbf{p}}(s_{\mathbf{p}}) + \varepsilon z_{\mathbf{p}}\frac{\partial \mathbf{n}_{\mathbf{p}}}{\partial \mathrm{p}_j}(s_{\mathbf{p}}) + \varepsilon z_{\mathbf{p}}\mathbf{n}'_{\mathbf{p}}\frac{\partial s_{\mathbf{p}}}{\partial \mathrm{p}_j}. \tag{6.7}$$

The vectors $\boldsymbol{\gamma}'_{\mathbf{p}}$ and $\mathbf{n}_{\mathbf{p}}$ are perpendicular to each other since $\boldsymbol{\gamma}'_{\mathbf{p}}$ lies in the tangent space while $\mathbf{n}_{\mathbf{p}}$ is the normal vector. Taking the dot product of (6.7)) with $\boldsymbol{\gamma}'_{\mathbf{p}}$ and rearranging, we obtain

$$\frac{\partial s_{\mathbf{p}}}{\partial \mathrm{p}_j} = \frac{1}{(1-\varepsilon z_{\mathbf{p}}\kappa_{\mathbf{p}})|\boldsymbol{\gamma}'_{\mathbf{p}}|^2}\left(-\frac{\partial \boldsymbol{\gamma}_{\mathbf{p}}}{\partial \mathrm{p}_j}\cdot \boldsymbol{\gamma}'_{\mathbf{p}} + \varepsilon z_{\mathbf{p}}\frac{\partial \boldsymbol{\gamma}'_{\mathbf{p}}}{\partial \mathrm{p}_j}\cdot \mathbf{n}_{\mathbf{p}}\right). \tag{6.8}$$

Here we used that $\boldsymbol{\gamma}'_{\mathbf{p}}\cdot\mathbf{n}_{\mathbf{p}}$ is zero so that $\partial_{\mathrm{p}_j}(\boldsymbol{\gamma}'_{\mathbf{p}}\cdot\mathbf{n}_{\mathbf{p}})=0$, and definition of $\kappa_{\mathbf{p}}$ given in Lemma 2.11. Taking the dot product of identity (6.7) with $\mathbf{n}_{\mathbf{p}}(s_{\mathbf{p}})$ and arguing as above we find

$$\frac{\partial z_{\mathbf{p}}}{\partial \mathrm{p}_j} = -\frac{1}{\varepsilon}\frac{\partial \boldsymbol{\gamma}_{\mathbf{p}}}{\partial \mathrm{p}_j}\cdot \mathbf{n}_{\mathbf{p}}. \tag{6.9}$$

From the definition of $\boldsymbol{\gamma}_{\mathbf{p}}$ in (2.19) and (2.22), for $j \geqslant 3$ it holds that

$$\begin{aligned}
\frac{\partial \boldsymbol{\gamma}_{\mathbf{p}}}{\partial \mathrm{p}_j} &= \tilde{\Theta}_j(\tilde{s}_{\mathbf{p}})\mathbf{n}_0 - \frac{\boldsymbol{\gamma}_0 + \bar{p}(\tilde{s}_{\mathbf{p}})\mathbf{n}_0}{A^2}(1+\mathrm{p}_0)\partial_{\mathrm{p}_j}A;\\
\frac{\partial \boldsymbol{\gamma}'_{\mathbf{p}}}{\partial \mathrm{p}_j} &= \left(\tilde{\Theta}'_j(\tilde{s}_{\mathbf{p}})|\boldsymbol{\gamma}'_{\mathbf{p}}| - \frac{\bar{p}'(\tilde{s}_{\mathbf{p}})}{A^2}(1+\mathrm{p}_0)\partial_{\mathrm{p}_j}A\right)\mathbf{n}_0\\
&\quad - \left(\kappa_0\tilde{\Theta}_j(\tilde{s}_{\mathbf{p}}) + \frac{1-\kappa_0\bar{p}(\tilde{s}_{\mathbf{p}})}{A^2}\right)(1+\mathrm{p}_0)\partial_{\mathrm{p}_j}A\boldsymbol{\gamma}'_0.
\end{aligned} \tag{6.10}$$

Here to obtain the second identity, we can take derivative with respect to $s_{\mathbf{p}}$ directly and $\mathbf{n}'_0 = -\kappa_0\mathbf{n}_0$. Secondly, it's a bit more complicate for the case $j=0$ due to the dependence of $\tilde{\Theta}_j$ on $|\Gamma_{\mathbf{p}}| = (1+\mathrm{p}_0)|\Gamma_0|$. Indeed, $\partial_{\mathrm{p}_0}\tilde{\Theta}_j = -\tilde{s}_{\mathbf{p}}\tilde{\Theta}'_j/(1+\mathrm{p}_0)^2$ by its definition in (2.22) which furthermore implies $\partial_{\mathrm{p}_0}\bar{p} = -\tilde{s}_{\mathbf{p}}\bar{p}'/(1+\mathrm{p}_0)^2$, and hence

$$\frac{\partial \boldsymbol{\gamma}_{\mathbf{p}}}{\partial \mathrm{p}_0} = (1-(1+\mathrm{p}_0)\partial_{\mathrm{p}_0}\ln A)\frac{\boldsymbol{\gamma}_0 + \bar{p}(\tilde{s}_{\mathbf{p}})\mathbf{n}_0}{A} - \frac{1+\mathrm{p}_0}{A}\frac{1}{(1+\mathrm{p}_0)^2}\tilde{s}_{\mathbf{p}}\bar{p}'\mathbf{n}_0. \tag{6.11}$$

Since $\boldsymbol{\gamma}'_{\mathbf{p}}$ is independent of p_1 and p_2, we have $\partial_{\mathrm{p}_j}\boldsymbol{\gamma}'_{\mathbf{p}} = 0$ for $j=1,2$. In addition, from (2.19) we have

$$\partial_{\mathrm{p}_j}\boldsymbol{\gamma}_{\mathbf{p}} = \mathbf{E}_j/\sqrt{2\pi R_0}, \qquad j=1,2. \tag{6.12}$$

The expressions for the derivatives of $s_{\mathbf{p}}$ and $z_{\mathbf{p}}$ with respect to p_j follow by plugging (6.10), (6.11) or (6.12) into (6.8)-(6.9) with the aid of $\boldsymbol{\gamma}_0 = R_0\mathbf{n}_0$ and $\kappa_0 = -1/R_0$. The estimates (6.5) follow directly from the Mean Value Theorem. □

Lemma 6.3. *For $\mathbf{p} \in \mathcal{D}_\delta$ introduced in (2.29), the sensitivity of σ defined in (3.22) to $\mathbf{p}$ can be bounded by*

$$\|\nabla_{\mathbf{p}}\sigma(\mathbf{p})\|_{l^2} \lesssim 1.$$

Proof. This is a direct result of the Lemma 6.2 and the definition of $\sigma(\mathbf{p})$, details are omitted. □

Lemma 6.4. *The bilayer distribution $\Phi_{\mathbf{p}}$ defined in Lemma 3.2, satisfies the expansion*

$$\frac{\partial \Phi_{\mathbf{p}}}{\partial \mathrm{p}_j} = \frac{1}{\varepsilon}(\phi_0' + \varepsilon\phi_1')\xi_j(s_{\mathbf{p}}) + \varepsilon \mathrm{R}_j,$$

where $\xi_j(s_{\mathbf{p}}) := \varepsilon \frac{\partial z_{\mathbf{p}}}{\partial \mathrm{p}_j}$ and the remainder $\mathrm{R} = (\mathrm{R}_j)$ can be bounded as a vector function in L^2

$$\|\mathrm{R}\|_{L^2} \lesssim 1; \qquad \|\Pi_{\mathcal{Z}_*^1}\mathrm{R}\|_{L^2} \lesssim \varepsilon^{1/2}.$$

Moreover, the quantity $\partial_t \Phi_{\mathbf{p}}$ satisfies the estimates L^2 and L^∞ estimates

$$\|\partial_t \Phi_{\mathbf{p}}\|_{L^2} \sim \|\Pi_{\mathcal{Z}_*^1}\partial_t\Phi_{\mathbf{p}}\| \sim \varepsilon^{-1/2}\|\dot{\mathbf{p}}\|_{l^2}; \qquad \|\Pi_{\mathcal{Z}_*^0}\partial_t\Phi_{\mathbf{p}}\|_{L^2} \lesssim \varepsilon^{1/2}\|\dot{\mathbf{p}}\|_{l^2},$$
$$\left\|\dot{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\Phi_{\mathbf{p}}\right\|_{L^\infty} + \|\dot{\mathbf{p}}\cdot\nabla_{\mathbf{p}}(\varepsilon^2\Delta\Phi_{\mathbf{p}})\|_{L^\infty} \lesssim \varepsilon^{-3/2}\|\dot{\mathbf{p}}\|_{l^2}.$$

Proof. From the definition of $\Phi_{\mathbf{p}}$ in Lemma 3.2, we calculate

$$\frac{\partial \Phi_{\mathbf{p}}}{\partial \mathrm{p}_j} = (\phi_0' + \varepsilon\phi_1' + \varepsilon^2\partial_{z_{\mathbf{p}}}\phi_2 + \varepsilon^3\phi_3')\frac{\partial z_{\mathbf{p}}}{\partial \mathrm{p}_j} + \varepsilon^2\partial_{s_{\mathbf{p}}}\phi_2\frac{\partial s_{\mathbf{p}}}{\partial \mathrm{p}_j} + \varepsilon\partial_\sigma(\phi_1 + \varepsilon\phi_2)\frac{\partial\sigma}{\partial \mathrm{p}_j}.$$

Combining with Lemmas 6.2 and 6.3 we obtain the expressions for derivatives of $\Phi_{\mathbf{p}}$ with respect to p_j,

$$\mathrm{R}_j := \varepsilon^2(\partial_{z_{\mathbf{p}}}\phi_2 + \varepsilon\phi_3')\frac{\partial z_{\mathbf{p}}}{\partial \mathrm{p}_j} + \varepsilon^2\partial_{s_{\mathbf{p}}}\phi_2\frac{\partial s_{\mathbf{p}}}{\partial \mathrm{p}_j} + \varepsilon\partial_\sigma(\phi_1 + \varepsilon\phi_2)\frac{\partial\sigma}{\partial \mathrm{p}_j}.$$

We remark that the leading order term comes from $\partial_\sigma\phi_1 = B_{\mathbf{p},2}$ which is nonzero as $|z_{\mathbf{p}}| \to \infty$. This fact combined with Lemma 6.3 and Lemma 6.2 yields the L^2 estimate of R. The estimate on the projection of R to the meandering slow space is similar with the exception that the functions in $\mathcal{Z}_*^1$ are localized, decaying exponentially fast to zero as $|z_{\mathbf{p}}| \to \infty$. This contributes an extra factor of $\varepsilon^{1/2}$ to the bound.

The time derivative of $\Phi_{\mathbf{p}}$ satisfies the chain rule

$$\partial_t\Phi_{\mathbf{p}} = \dot{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\Phi_{\mathbf{p}}. \tag{6.13}$$

The first L^2 estimate on this quantity follows directly from the expressions of $\frac{\partial\Phi_{\mathbf{p}}}{\partial \mathrm{p}_j}$, the orthogonality of $\{\tilde{\Theta}_j\}$, and l^2 bound of $\nabla_{\mathbf{p}}A$ in Lemma 6.1. Considering the L^2 estimate of the projection to the pearling space, the leading order of $\partial_{\mathrm{p}_j}\Phi_{\mathbf{p}}$ has odd parity in $z_{\mathbf{p}}$ while the leading order terms in $Z_{\mathbf{p},*}^{0j}$ for $j \in \Sigma_0$ have even parity. This renders the projection higher order of ε, and establishes the bound.

The final L^∞-norm bound is obtained from the form of $\Phi_{\mathbf{p}}$ and the fact that $\varepsilon^2\Delta_{s_{\mathbf{p}}}$ is a bounded operator when acting on $\Phi_{\mathbf{p}}$. More explicitly,

$$\left\|\dot{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\Phi_{\mathbf{p}}\right\|_{L^\infty} + \|\dot{\mathbf{p}}\cdot\nabla_{\mathbf{p}}(\varepsilon^2\Delta\Phi_{\mathbf{p}})\|_{L^\infty} \lesssim \varepsilon^{-1}\|\dot{\mathbf{p}}\|_{l^1}.$$

Using the l^1-l^2 estimate, $\|\dot{\mathbf{p}}\|_{l^1} \lesssim N_1^{1/2}\|\dot{\mathbf{p}}\|_{l^2}$ and bounding N_1 by ε^{-1} finishes the proof. □

Lemma 6.5. *The rate of change of the basis functions of the slow space $\mathcal{Z}_*$ with respect to* $\mathbf{p}$ *can be bounded by*

$$\|\nabla_{\mathbf{p}} Z^{I(j)j}_{\mathbf{p},*}\|_{L^2} \lesssim \varepsilon^{-1}.$$

Under the assumption $\|\dot{\mathbf{p}}\|_{l^2} \lesssim \varepsilon^3$ of (5.7) we have

$$\|\partial_t Z^{I(j)j}_{\mathbf{p},*}\|_{L^2} \lesssim \varepsilon^{-1}\|\dot{\mathbf{p}}\|_{l^2} \lesssim \varepsilon^2. \tag{6.14}$$

Proof. The basis functions of the tangent plane satisfy

$$\partial_{\mathrm{p}_i} Z^{I(j)j}_{\mathbf{p},*} = \varepsilon^{-1}\tilde{\Theta}_i \tilde{\psi}'_{I(j)}\tilde{\Theta}_j + O(1).$$

The result follows from the orthogonality of $\tilde{\Theta}_i$ on $L^2(\mathscr{I}_{\mathbf{p}})$ and l^1-l^2 estimate with $N \lesssim \varepsilon^{-1}$. □

Lemma 6.6. *For $\mathbf{p} \in \mathcal{D}_\delta$, the mass of residual can be estimated by*

$$\int_\Omega \left(\mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}^\infty\right) \mathrm{d}x = O\left(\varepsilon^4, \varepsilon^5\|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\right).$$

Proof. We expand $\mathrm{F}(\Phi_{\mathbf{p}})$ in Lemma 3.2, subtract F^∞_m and integrate,

$$\int_\Omega \left(\mathrm{F}(\Phi_{\mathbf{p}}) - \mathrm{F}^\infty\right) \mathrm{d}x = \varepsilon^2 \int_\Omega \mathrm{F}_2(\Phi_{\mathbf{p}})\,\mathrm{d}x + \varepsilon^3 \int_\Omega (\mathrm{F}_3(\Phi_{\mathbf{p}}) - \mathrm{F}_3^\infty)\,\mathrm{d}x + \varepsilon^4 \int_\Omega (\mathrm{F}_{\geqslant 4}(\Phi_{\mathbf{p}}) - \mathrm{F}^\infty_{\geqslant 4})\,\mathrm{d}x. \tag{6.15}$$

Recalling form of F_2 in (3.14), where $f_2(z_{\mathbf{p}})$ is odd in $z_{\mathbf{p}}$, we deduce

$$\begin{aligned}\varepsilon^2 \int_\Omega \mathrm{F}_2(\Phi_{\mathbf{p}})\,\mathrm{d}x &= -\varepsilon^4(\sigma - \sigma_1^*) \int_{\mathbb{R}_{2\ell}} \int_{\mathscr{I}} \kappa_{\mathbf{p}}^2 z_{\mathbf{p}} f_2(z_{\mathbf{p}})\,\mathrm{d}\tilde{s}_{\mathbf{p}}\mathrm{d}z_{\mathbf{p}},\\ &= C\varepsilon^4(\sigma - \sigma_1^*) \int_{\mathscr{I}} h(\boldsymbol{\gamma}''_{\mathbf{p}})\,\mathrm{d}\tilde{s}_{\mathbf{p}},\end{aligned} \tag{6.16}$$

where the function $h = h(\boldsymbol{\gamma}''_{\mathbf{p}})$ follows Notation 2.1. Since h is bounded in L^∞ by Lemma 2.12, we have

$$\varepsilon^2 \int_\Omega \mathrm{F}_2(\Phi_{\mathbf{p}})\,\mathrm{d}x = O(\varepsilon^4|\sigma - \sigma_1^*|). \tag{6.17}$$

From the form of F_3 given in (3.14), the odd parity of ϕ_0' with respect to $z_{\mathbf{p}}$ implies

$$\begin{aligned}\varepsilon^3 \int_\Omega (\mathrm{F}_3(\Phi_{\mathbf{p}}) - \mathrm{F}_3^\infty)\,\mathrm{d}x &= \varepsilon^4 \int_{\mathbb{R}_{2\ell}} \int_{\mathscr{I}} \Big(\phi_0' \Delta_{s_{\mathbf{p}}} \kappa_{\mathbf{p}} + (f_3(z_{\mathbf{p}}, \boldsymbol{\gamma}_{\mathbf{p}}'') - f_3^\infty)\Big)\, \mathrm{J}_{\mathbf{p}}\,\mathrm{d}s_{\mathbf{p}}\mathrm{d}z_{\mathbf{p}}, \\ &= \varepsilon^4 \left(\varepsilon \int_{\mathscr{I}} h(\boldsymbol{\gamma}_{\mathbf{p}}'') \Delta_{s_{\mathbf{p}}} \kappa_{\mathbf{p}}\,\mathrm{d}s_{\mathbf{p}} + \int_{\mathscr{I}} h(\boldsymbol{\gamma}_{\mathbf{p}}'')\,\mathrm{d}s_{\mathbf{p}}\right), \\ &= O\left(\varepsilon^5 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}\right) + O(\varepsilon^4).\end{aligned} \tag{6.18}$$

Here we used the $H^2(\mathscr{I}_{\mathbf{p}})$ bound of the curvature from Lemma 2.10. Similar estimates show that

$$\varepsilon^4 \int_\Omega (\mathrm{F}_4(\Phi_{\mathbf{p}}) - \mathrm{F}_4^\infty)\,\mathrm{d}x = O\left(\varepsilon^5 \|\hat{\mathbf{p}}\|_{\mathbb{V}_4^2}, \varepsilon^4\right). \tag{6.19}$$

Combining (6.17)-(6.19) with (6.16) yields (6.15). □

6.2. The decomposition

In the section, we prove Lemma 5.2. It suffices to consider $\mathbf{p}_0 = 0$. Define $\{\mathscr{F}_k\}_{k=0}^{N_1-1}$ to be real-valued functionals of v and the parameters $\mathbf{p}$, explicitly given by

$$\mathscr{F}_k(v, \mathbf{p}) = \int_\Omega \Big(\Phi_0 + v(x) - \Phi_{\mathbf{p}}\Big) Z_{\mathbf{p},*}^{1k}\,\mathrm{d}x,$$

for $i \in \Sigma_1(\Gamma_{\mathbf{p}}, \rho)$. Note that $\mathscr{F}_k(0, \mathbf{0}) = 0$ and from mean value theorem

$$\Phi_0 - \Phi_{\mathbf{p}} = -\mathbf{p} \cdot \nabla_{\mathbf{p}} \Phi_{\lambda \mathbf{p}}, \qquad \text{for some } \lambda = \lambda(\mathbf{p}) \in [0, 1].$$

If we introduce $\{\mathbf{e}_k\}$ as the canonical basis of $\mathbb{R}^{N_1}$ and the $\mathbf{p}$ dependent notation $\mathbb{T}_{kj}^\lambda = \mathbb{T}_{kj}^\lambda(\mathbf{p})$ as

$$\mathbb{T}_{kj}^\lambda(\mathbf{p}) := \int_\Omega \partial_{\mathrm{p}_j} \Phi_{\lambda \mathbf{p}}\, Z_{\mathbf{p},*}^{1k}\,\mathrm{d}x, \tag{6.20}$$

then the functional $\mathscr{F}_k$ can be rewritten as

$$\mathscr{F}_k(v, \mathbf{p}) = \int_\Omega v(x)\, Z_{\mathbf{p},*}^{1k}\,\mathrm{d}x - \left\langle \mathbb{T}^\lambda(\mathbf{p})\mathbf{p}, \mathbf{e}_k \right\rangle, \tag{6.21}$$

and the gradient of $\mathscr{F} := (\mathscr{F}_k)$ with respect to $\mathbf{p}$ at $(v, \mathbf{p}) = (0, \mathbf{0})$ can be represented by

$$\nabla_{\mathbf{p}} \mathscr{F}(0, \mathbf{0}) = -\mathbb{T}^T(\mathbf{0}).$$

Here $\mathbb{T}(\mathbf{0}) = \mathbb{T}^1(\mathbf{0})$ and the superscript T denotes matrix transpose. We will show in Lemma 6.7 that $\mathbb{T}(\mathbf{0})$ is invertible. We use the contraction mapping theorem to establish the existence of $\mathbf{p}$ such that $\mathscr{F}_k(v, \mathbf{p}) = 0$ for some v. Define the function

$$\mathscr{G}(\mathbf{p}; v) = (\mathbb{T}(\mathbf{0}))^{-1}\Big(\mathbb{T}(\mathbf{0})\mathbf{p} - \mathscr{F}(v; \mathbf{p})\Big),$$

we are going to show that $\mathscr{G}$ is a contraction of $\mathbf{p}$ near the origin when v is small in L^2. We observe $\mathscr{G}(\mathbf{p}; v) = \mathbf{p}$ is equivalent to $\mathscr{F}(v, \mathbf{p}) = \mathbf{0}$, that is, any fixed point of $\mathscr{G}(\mathbf{p})$ for a given v is a zero of $\mathscr{F}(v, \mathbf{p})$. With $\mathscr{F}$ written in terms of $\mathbb{T}$ as in (6.21), we rewrite

$$\mathscr{G}_k(\mathbf{p}; v) = \Big\langle (\mathbb{T}(\mathbf{0}))^{-1}\Big(\mathbb{T}(\mathbf{0}) - \mathbb{T}^\lambda(\mathbf{p})\Big)\mathbf{p}, \mathbf{e}_k\Big\rangle_{l^2} + (\mathbb{T}(\mathbf{0}))^{-1}\int_\Omega v(x)\, Z^{1k}_{\mathbf{p},*}\, \mathrm{d}x. \tag{6.22}$$

The following properties of $\mathbb{T}^\lambda(\mathbf{p})$ show that $\mathscr{G}(\mathbf{p}; v)$ is a contraction mapping of $\mathbf{p}$ near zero.

Lemma 6.7. $\mathbb{T}(\mathbf{0})$ *is invertible and satisfies the bound*

$$\|(\mathbb{T}(\mathbf{0}))^{-1}\|_{l^2_*} \lesssim \varepsilon^{1/2}.$$

Moreover the difference between $\mathbb{T}^{\lambda_1}(\mathbf{p}_1)$ *and* $\mathbb{T}^{\lambda_2}(\mathbf{p}_2)$*, for* $\lambda_k := \lambda(\mathbf{p}_k)$*, satisfies*

$$\|\left(\mathbb{T}^{\lambda_1}(\mathbf{p}_1) - \mathbb{T}^{\lambda_2}(\mathbf{p}_2)\right)\mathbf{p}_l\|_{l^2} \lesssim \varepsilon^{-2}\|\mathbf{p}_l\|_{l^2}\|\mathbf{p}_1 - \mathbf{p}_2\|_{l^2}, \quad l = 1, 2.$$

Proof. By the definition of $\mathbb{T}^\lambda_{kj}(\mathbf{p})$ in (6.20) with $Z^{1k}_{\mathbf{p},*}$ replaced by its definition and $\partial_{\mathrm{p}_j}\Phi_{\mathbf{p}}$ given in Lemma 6.4, there exists a matrix with l^2_*-norm one such that

$$\mathbb{T}^\lambda_{kj}(\mathbf{p}) = \frac{1}{\varepsilon^{1/2}}\int_{\mathbb{R}_{2\ell}}\int_{\mathscr{I}_{\mathbf{p}}} (\phi_0'(z_{\lambda\mathbf{p}}) + \varepsilon\phi_1'(z_{\lambda\mathbf{p}}))\frac{\phi_0'(z_{\mathbf{p}})}{m_1}\xi_j(s_{\lambda\mathbf{p}})\tilde{\Theta}_k(s_{\mathbf{p}})\,\mathrm{d}\tilde{s}_{\mathbf{p}}\,\mathrm{d}z_{\mathbf{p}} + \varepsilon^{3/2}\mathbb{E}_{ij}. \tag{6.23}$$

Here $(s_{\lambda\mathbf{p}}, z_{\lambda\mathbf{p}})$ denotes the scaled local coordinates near $\Gamma_{\lambda\mathbf{p}}$ for $\lambda \in [0, 1]$. Applying Lemma 6.2 and the l^1-l^2 estimate with $N_1 \lesssim \varepsilon^{-1}$ implies

$$|z_{\lambda\mathbf{p}} - z| + \varepsilon^{-1}|s_{\lambda\mathbf{p}} - s| \lesssim \varepsilon^{-1}\|\mathbf{p}\|_{\mathbb{V}_0} \lesssim \varepsilon^{-1}N_1^{-1/2}\|\mathbf{p}\|_{l^2} \lesssim \varepsilon^{-3/2}\|\mathbf{p}\|_{l^2},$$

where we recall $\mathbb{V}_0$ is equivalent to $l^1(\mathbb{R}^{N_1})$. Thus the estimate on the difference of $\mathbb{T}^{\lambda_1}(\mathbf{p}_1)$ and $\mathbb{T}^{\lambda_2}(\mathbf{p}_2)$ follows from standard calculations. We now estimate the inverse of $\mathbb{T}(\mathbf{0})$, i.e. $\mathbb{T}^\lambda(\mathbf{0})$ for $\lambda = 1$. From the definition of ϕ_0, ϕ_1 in (3.1), (3.5), we deduce the useful but straightforward identity

$$\int_{\mathbb{R}} (\phi_0' + \varepsilon\phi_1')\phi_0'\,\mathrm{d}z = m_1^2 + \varepsilon(\sigma m_2 + \eta_d m_3^2), \tag{6.24}$$

where m_1 is introduced in (3.9), and m_2, m_3 are constants defined by

$$m_2 = \frac{1}{2}\int_{\mathbb{R}} \mathrm{L}_0^{-1}(z\phi_0')\,\mathrm{d}z; \qquad m_3 = \frac{1}{2}\int_{\mathbb{R}} |z\phi_0'|^2\,\mathrm{d}z. \tag{6.25}$$

Applying (6.24) and taking $\mathbf{p} = \mathbf{0}$, (6.23) implies

$$\mathbb{T}_{kj}(\mathbf{0}) = \frac{m_1^2 + \varepsilon(\sigma m_2 + \eta_d m_3^2)}{\varepsilon^{1/2}} \int_{\mathscr{I}} \xi_j(s)\tilde{\Theta}_k(\tilde{s})\,\mathrm{d}\tilde{s} + O(\varepsilon^{3/2}\mathbb{E}_{kj}), \tag{6.26}$$

where $\xi_j(s) = -\Theta_j$ and $\tilde{\Theta}_k(\tilde{s}) = \Theta_k(s)$ since $\tilde{s} = s$. Hence the orthogonality of Θ_j in $L^2(\mathscr{I})$ implies

$$\mathbb{T}(\mathbf{0}) = -\frac{m_1^2 + \varepsilon(\sigma m_2 + \eta_d m_3^2)}{\varepsilon^{1/2}}\mathbb{I} + O(\varepsilon^{3/2})\mathbb{E}.$$

Since $\mathbb{E}$ has l^2_*-norm one, the first term dominates and hence $\mathbb{T}(\mathbf{0})$ is invertible. In fact, for some uniform constant C it holds that

$$\|\mathbb{T}(\mathbf{0})\mathbf{p}\|_{l^2} \geqslant C\varepsilon^{-1/2}\|\mathbf{p}\|_{l^2}.$$

The l^2_*-bound of the inverse $\mathbb{T}(\mathbf{0})^{-1}$ follows, which completes the proof. $\square$

Lemma 6.8. *Let $\Gamma_0 \in \mathcal{G}^4_{K_0,2\ell_0}$ with local coordinates (s,z) and $\varepsilon \in (0,\varepsilon_0)$. Let $0 \leqslant r \leqslant 1$ and $\|v\|_{L^2} \leqslant \delta\varepsilon$ for some δ, ε_0 small enough, then there exists $\mathbf{p} = \mathbf{p}(v) \in l^2$ such that $\mathscr{F}(v, \mathbf{p}(v)) = 0$ and*

$$\|\mathbf{p}(v)\|_{l^2} \lesssim \varepsilon^{1/2}\|v\|_{L^2}.$$

The smallness of δ, ε_0 depends on domain, system parameter and (Γ_0, M_0)

Proof. We check $\mathscr{G}(\mathbf{p}; v) = (\mathscr{G}_k(\mathbf{p}; v)) : l^2 \to l^2$ is a contraction mapping for $\mathbf{p} \in l^2$ satisfying $\|\mathbf{p}\|_{l^2} \leqslant \delta\varepsilon^{3/2}$ with δ small enough independent of ε. In fact, employing Lemma 6.7 yields

$$\|\mathscr{G}(\mathbf{p}; v)\|_{l^2} \lesssim \varepsilon^{1/2}\|v\|_{L^2} + \varepsilon^{-3/2}\|\mathbf{p}\|^2_{l^2}, \tag{6.27}$$

which lies in the ball $B_{\delta\varepsilon^{3/2}}(\mathbf{0}) \subset l^2$ provided that δ, ε are suitably small. Hence $\mathscr{G}$ is a closed map on the small ball $B_{\delta\varepsilon^{3/2}}(\mathbf{0})$. It remains to show the mapping is contractive. Indeed, for any $\mathbf{p}_1, \mathbf{p}_2 \in B_{\delta\varepsilon^{3/2}}(\mathbf{0})$ from (6.22) we have

$$\begin{aligned}\mathscr{G}_k(\mathbf{p}_1) - \mathscr{G}_k(\mathbf{p}_2) =& \Big\langle (\mathbb{T}(\mathbf{0}))^{-1}\big[-\big(\mathbb{T}^{\lambda_1}(\mathbf{p}_1) - \mathbb{T}(\mathbf{0})\big)(\mathbf{p}_1 - \mathbf{p}_2) + \big(\mathbb{T}^{\lambda_2}(\mathbf{p}_2) - \mathbb{T}^{\lambda_1}(\mathbf{p}_1)\big)\mathbf{p}_2\big], \mathbf{e}_k\Big\rangle_{l^2} \\ &+ (\mathbb{T}(\mathbf{0}))^{-1}\int_\Omega v(x)\left(Z^{1k}_{\mathbf{p}_1,*} - Z^{1k}_{\mathbf{p}_2,*}\right)\mathrm{d}x\end{aligned}$$

From the gradient estimate of $Z^{I(k)k}_{\mathbf{p},*}$ in Lemma 6.5, we bound

$$\|Z^{1k}_{\mathbf{p}_1,*} - Z^{1k}_{\mathbf{p}_2,*}\|_{L^2} \lesssim \varepsilon^{-1}\|\mathbf{p}_1 - \mathbf{p}_2\|_{l^2}, \qquad \forall k \in \Sigma_1.$$

Combining this with Lemma 6.7 and $\mathbf{p}_1, \mathbf{p}_2 \in B_{\delta\varepsilon^{3/2}}$ with δ suitably small yields

$$
\begin{aligned}
\|\mathscr{G}(\mathbf{p}_1)-\mathscr{G}(\mathbf{p}_2)\|_{l^2} &\lesssim \left(\varepsilon^{-3/2}\|\mathbf{p}_1\|_{l^2}+\varepsilon^{-3/2}\|\mathbf{p}_2\|_{l^2}+\varepsilon^{-1/2}N_1^{1/2}\|v\|_{L^2}\right)\|\mathbf{p}_1-\mathbf{p}_2\|_{l^2}\\
&\leqslant \frac{1}{2}\|\mathbf{p}_1-\mathbf{p}_2\|_{l^2}.
\end{aligned}
$$

Hence $\mathscr{G}$ is a contraction mapping on the space $B_{\delta\varepsilon^{3/2}}(\mathbf{0})\subset l^2$. The existence of $\mathbf{p}\in B_{\delta\varepsilon^{3/2}}(\mathbf{0})$ such that

$$\mathscr{G}(\mathbf{p};v)=\mathbf{p}$$

follows from contraction mapping principle. And then the bound of $\mathbf{p}$ in terms of L^2-norm of v follows from (6.27) provided with ε_0 suitably small. □

Now we prove the projection Lemma 5.2.

Proof of Lemma 5.2. Without loss of generality we assume $\mathbf{p}_0=\mathbf{0}$. With a use of Lemma 6.8, there exists $\mathbf{p}=\mathbf{p}(v)$ such that

$$u=\Phi_{\mathbf{p}}+v^{\perp},\qquad v^{\perp}\in(\mathcal{Z}_*^1)^{\perp},\qquad \int_\Omega v^{\perp}\,\mathrm{d}x=0$$

and $\mathbf{p}$ in l^2 can be bounded by

$$\|\mathbf{p}\|_{l^2}\lesssim\varepsilon^{1/2}\|v\|_{L^2}.$$

While $u=\Phi_0+v^{\perp}$, the mean value theorem and the bound of $\mathbf{p}$ afford the estimate

$$
\begin{aligned}
\|v^{\perp}\|_{H^{2k}} &\lesssim \sup_{\lambda\in[0,1]}\sum_j\left(\left\|\mathrm{p}_j\frac{\partial\Phi_{\mathbf{p}}}{\partial\mathrm{p}_j}\right\|_{H^{2k}}\right)\bigg|_{\mathbf{p}=\lambda\mathbf{p}}+\|v\|_{H^{2k}}\\
&\lesssim\varepsilon^{-2k-1/2}\|\mathbf{p}\|_{l^2}+\|v_0\|_{H^{2k}}\\
&\lesssim\varepsilon^{-2k}\|v_0\|_{L^2}+\|v_0\|_{H^{2k}}.
\end{aligned}
$$

By Lemma 4.10, $v^{\perp}$ can be further decomposed as

$$v^{\perp}=Q+w,\qquad Q=\sum_{j\in\Sigma_0}\mathrm{q}_j\in\mathcal{Z}_*^0,$$

where the coefficient vector $\mathbf{q}$ satisfies

$$\|\mathbf{q}\|_{l^2}\lesssim\|v^{\perp}\|_{L^2}.$$

Finally, with $w=v^{\perp}-Q$ we bound

$$\|w\|_{H^{2k}}\lesssim\|v^{\perp}\|_{H^{2k}}+\|Q\|_{H^{2k}}\lesssim\|v\|_{H^{2k}}+\varepsilon^{-2k}\|\mathbf{q}\|_{l^2}\lesssim\varepsilon^{-2k}\|v\|_{L^2}+\|v\|_{H^{2k}}.$$

The proof is completed by translating the base point from 0 to $\mathbf{p}_0$ and $\mathbf{p}$ to $\mathbf{p}-\mathbf{p}_0$. □

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