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Duality and socle generators for residual intersections

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Abstract. We prove duality results for residual intersections that unify and complete results of van Straten, Huneke–Ulrich and Ulrich, and settle conjectures of van Straten and Warmt. Suppose that I is an ideal of codimension g in a Gorenstein ring, and $J \subset I$ is an ideal with $s = g + t$ generators such that $K := J : I$ has codimension s. Let $\overline{I}$ be the image of I in $\overline{R} := R/K$. In the first part of the paper we prove, among other things, that under suitable hypotheses on I, the truncated Rees ring $\overline{R} \oplus \overline{I} \oplus \cdots \oplus \overline{I}^{t+1}$ is a Gorenstein ring, and that the modules $\overline{I}^u$ and $\overline{I}^{t+1-u}$ are dual to one another via the multiplication pairing into $\overline{I}^{t+1} \cong \omega_{\overline{R}}$. In the second part of the paper we study the analogue of residue theory, and prove that, when R/K is a finite-dimensional algebra over a field of characteristic 0 and certain other hypotheses are satisfied, the socle of $I^{t+1}/JI^t \cong \omega_{R/K}$ is generated by a Jacobian determinant.

Introduction

There are two important aspects of duality for local complete intersections. We write $T = k[[x_1, \dots, x_n]]/(a_1, \dots, a_{n-d})$ for a power series ring over a field k modulo an ideal generated by the regular sequence $a_1, \dots, a_{n-d}$. The first aspect is so central that it has become a definition: such a ring T is *Gorenstein* – that is, $T \cong \omega_T$, the canonical module of T. In the case where T is 0-dimensional, this means that $T \cong \operatorname{Hom}_k(T, k)$ as a T-module; and more generally that $T \cong \operatorname{Hom}_A(T, A)$ as T-module, where A is a Noether normalization of T.

The second important aspect is the theory of residues, which we think of as the explicit identification of the canonical module. Suppose that $(T, \mathfrak{m})$ is a reduced, equidimensional complete local k-algebra of dimension d, where k is a perfect field, and let L be its total ring of quotients. Let $A = k[\![x_1, \dots, x_d]\!] \subset T$ be a separable Noether normalization, that is, T is module finite over A and L is a product of separable field extensions of K, the quotient field of A. We think of the canonical module ω_T as $\operatorname{Hom}_A(T, A)$, which, after tensoring with K,

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is generated by the trace map $\mathrm{Tr}_{L/K}$. Thus there is a fractional ideal $\mathfrak{C}(T/A) \subset L$, called the *Dedekind complementary module*, such that

$$\mathrm{Hom}_A(T, A) = \mathfrak{C}(T/A)\ \mathrm{Tr}_{L/K}\,.$$

The resulting representation of ω_T as

$$\mathfrak{C}(T/A)\, dx_1 \wedge \cdots \wedge dx_d \subset L\, dx_1 \wedge \cdots \wedge dx_d = L \otimes_T \bigwedge^d \Omega_{T/k},$$

where $\Omega_{T/k}$ is the universally finite module of differentials, is independent of the choice of A. The usual residue map $\mathrm{H}^d_{\mathfrak{m}}(\omega_T) \to k$, which serves to make local duality explicit, is then defined by representing an element $\alpha \in \mathrm{H}^d_{\mathfrak{m}}(\omega_T)$ as a Čech class

$$\alpha = \begin{bmatrix} f dx_1 \wedge \cdots \wedge dx_d \\ x_1 \cdots x_d \end{bmatrix},$$

for suitable $f \in \mathfrak{C}(T/A)$ and suitable A, and mapping α to $\mathrm{Tr}_{L/K}(f)(0)$. For all this, see for example Kunz [22, Chapter 10].

A goal of the theory is thus to compute $\mathfrak{C}(T/A)$. When T is a complete intersection, the classical theory says that

$$\mathfrak{C}(T/A) = \Delta^{-1}T,$$

where Δ is the Jacobian determinant of T over A. Equivalently, $\mathrm{Tr}_{T/A}$ is Δ times a generator σ of $\mathrm{Hom}_A(T, A) \cong T$.

Thus, if k has characteristic 0 and T is a complete intersection, then $\Delta\overline{T}$ is the socle of the ring $\overline{T} := T/(x_1, \ldots, x_d)$. The well-known argument goes as follows: Since T is Cohen–Macaulay, the fact that the trace is Δ times σ is preserved if we factor out $x_1, \ldots, x_d$ to get a 0-dimensional ring $\overline{T}$. Since the maximal ideal $\mathfrak{m}\overline{T}$ is nilpotent, the trace $\mathrm{Tr}_{\overline{T}/k}$ annihilates $\mathfrak{m}\overline{T}$, but, because the characteristic is 0, the trace is not zero. It follows that $\mathrm{Tr}_{\overline{T}/k} = \Delta\overline{\sigma}$ generates the socle of $\mathrm{Hom}_k(\overline{T}, k) = \overline{\sigma}\overline{T}$. Thus $\Delta\overline{T}$ is the socle of $\overline{T}$. In Section A we give the classical proof for complete intersections.

In this paper we provide analogous duality results for residual intersections. We recall the definition: Let I be an ideal of codimension g in a local Gorenstein ring R, and let $s \geq g$. A *residual intersection* (or *s-residual intersection*) of I is a proper ideal K of codimension at least s that can be written in the form $K = J : I$, where $J \subset I$ is an ideal generated by s elements. We set $t = s - g$. We will use this notation for the rest of this introduction. We think of t as measuring how far J is from being a complete intersection. The case when I is unmixed and $t = 0$ is the case of linkage ([27]). The class of residual intersections contains the ideals of maximal minors of sufficiently general matrices and many other examples. Our general results have technical hypotheses, so we begin with an example.

Duality. Suppose that I is generated by a regular sequence of length g in a local Gorenstein ring R with infinite residue field, and J is generated by $s = g+t < \dim R$ elements chosen generally inside the maximal ideal times I. The ideal $K = J : I$ is then an s-residual intersection (even a geometric s-residual intersection, as defined in Section 1). We write $\overline{I}$ for the image of I in $\overline{R} := R/K$. By a result of Huneke and Ulrich [20] (see Theorem 3.1), the canonical module of $\overline{R}$ is $\overline{I}^{t+1}$; in particular, when $t = 0$, the truncated Rees algebra $\overline{R} \oplus \overline{I}$ is Gorenstein. We show for arbitrary t that the truncated Rees algebra $\overline{R} \oplus \overline{I} \oplus \overline{I}^2 \oplus \cdots \oplus \overline{I}^{t+1}$ is Gorenstein, which implies that the complementary intermediate powers $\overline{I}^u$ and $\overline{I}^{t+1-u}$

are dual to each other via the multiplication pairing into $\overline{I}^{t+1}$. We also prove corresponding results for the truncated associated graded ring $\overline{R}/\overline{I} \oplus \overline{I}/\overline{I}^2 \oplus \cdots \oplus \overline{I}^t/\overline{I}^{t+1}$ (Theorem 2.5 and Proposition 5.2).

Residues. To illustrate the second main result of this paper, again in the case where I is a complete intersection, we suppose in addition to the above that R is a power series ring in d variables x_i over a field of characteristic 0. Let $A = k[\![x_{s+1}, \ldots, x_d]\!]$ be a general Noether normalization of $\overline{R}$. Write $J = (a_1, \ldots, a_s)$ and set

$$\Delta = \det \begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \cdots & \frac{\partial a_1}{\partial x_s} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_s}{\partial x_1} & \cdots & \frac{\partial a_s}{\partial x_s} \end{pmatrix}.$$

We strengthen the statement $\omega_{\overline{R}} \cong \overline{I}^{t+1}$ by proving in Theorem 7.4 that $\mathfrak{C}(\overline{R}/A) = \Delta^{-1}\overline{I}^{t+1}$ if $\overline{R}$ is reduced. This gives an explicit description of the complementary module of residual intersections.

As an application, in Corollary 7.6, we give a formula for the complementary module of any reduced ring defined by an ideal of maximal minors of generic codimension.

We also apply Theorem 7.4 to certain 0-dimensional residual intersections, with the goal of identifying the socles of their canonical modules as Jacobian determinants. For example, when R/K is 0-dimensional, we obtain a formula for the socle of $I^{t+1}/JI^t \cong \omega_{R/K}$: it is generated by the image of an element of the form $\Delta + p$, where $p \in (a_1, \ldots, a_{s-1})$ (Theorem 7.8). In general, Δ itself is not even in I^{t+1}, but, when it is, it generates the socle.

We show that $\Delta \in I^{t+1}$ when the generators a_j of J are forms of the same degree and I is radical (Theorem 7.10). In Proposition 7.12 we prove this without the radical condition when I is principal – already a nontrivial computation. In general, we do not know whether the radical condition is necessary.

When the generators of J have different degrees, the ideal ΔR depends on the choice of generators, and in this case Δ may not be in I^{t+1} (Example 7.14). We show that this can even happen when J is generated by the partial derivatives of a quasi-homogeneous polynomial, and thus have the same degrees with respect to an appropriate weighting (Example 7.15).

Our results are much more general than the setting above. We assume that R is Gorenstein and that I satisfies two sorts of conditions: one on the local numbers of generators and the other that $\operatorname{depth}(R/I^u) \geq \dim R/I - u + 1$ for some range of values of u. We assume that $K = J : I$ is an s-residual intersection of I and we set $t = s - \operatorname{codim} I$.

Our main results on duality are Theorems 2.2 and 2.6, which unify and complete a number of results of Huneke, Ulrich and van Straten. Theorem 2.2 says that

$$\frac{I^u}{JI^{u-1}} \quad \text{is dual to} \quad \frac{I^{t+1-u}}{JI^{t-u}} \quad \text{for } u = 0, \ldots, t+1,$$

where, in the case $u = 0$, we interpret JI^{-1} as $J : I$. In fact, we show that the duality is given in the most natural way, by multiplication,

$$\frac{I^u}{JI^{u-1}} \otimes \frac{I^{t+1-u}}{JI^{t-u}} \xrightarrow{\text{mult}} \frac{I^{t+1}}{JI^t} \cong \omega_{R/K}.$$

On the other hand, Example 6.4 shows that the duality statement above can hold even when the multiplication maps are not perfect pairings.

Theorem 2.6 gives a deformation condition under which such dualities hold that is in many cases more general than the condition of Theorem 2.2. In Section 6 we present examples showing the necessity of some of the hypotheses.

In Theorems 7.4, 7.8, and 7.10 we prove theorems about $\mathfrak{C}(\overline{R}/A)$ and the socle extending the results described above to the more general case as well.

History. Residual intersections have a long history in Algebraic Geometry, perhaps beginning with Chasles' Theorem that there are 3,264 conics in the complex projective plane that are tangent to five general conics [8]. The theory became part of commutative algebra with the work of Artin and Nagata [1]. They asserted the Cohen–Macaulay property of residual intersections, but stated it more generally than it is true. The error was corrected by Huneke [19], and a series of papers, culminating in [31], gave stronger and stronger results in this direction (see also [7, 13, 14]).

The first duality results for residual intersections were proven by Peskine and Szpiro [27] in the case $t = 0$, the theory of liaison: if R is a local Gorenstein ring and $J \subsetneq I$ are ideals of the same codimension with R/I Cohen–Macaulay and J generated by a regular sequence, then I/J is the canonical module of $\overline{R} = R/K = R/(J : I)$. The formula for $\mathfrak{C}(\overline{R}/A)$ in this case can be found in [23, 3.5 (a)].

For $t > 0$, such results were considered in two separate lines of work, starting about 25 years ago. In one, Duco van Straten showed that if J is 1-dimensional and $t = 1$, then the module I/J is self-dual. Around the same time work of Huneke and Ulrich [20], generalizing the corresponding statement in the theory of linkage [27], showed that, for any s and t, under suitable hypotheses on I, the modules R/K and I^{t+1}/JI^t are dual to one another; in particular, $I^{t+1}/JI^t \cong \omega_{R/K}$. The paper [7] gives another version of the duality, to which we will return in Section 6.

Comparing our Theorem 2.2, we see that the result of Huneke and Ulrich is the case $u = 0$, while the result of van Straten is included in our result for $t = 1$.

Van Straten's result, cited above, appears with geometric applications in the papers of van Straten and Warmt [32, 33]. Sernesi [30] gives further geometric applications.

Conjectures of van Straten and Warmt. The paper of van Straten and Warmt contains interesting conjectures, which we were able to settle in much generalized form. The conjectures [32, Conjecture 7.1 (1)–(3)] are essentially as follows:

Conjecture 0.1. Suppose that J is an ideal of codimension g and dimension 1, with $s = g + 1 = d$ generators, in a power series ring $R = k[\![x_1, \dots, x_d]\!]$ over a field k of characteristic 0, and I is the unmixed part of J, so that I/J has finite length. (Note that in this case van Straten's original result shows that I/J is self-dual.) If I is a radical ideal and $I \neq J$, then:

(1) The module I/J is self-dual by a pairing that factors through the multiplication map $I/J \otimes I/J \to I^2/JI$.

(2) The R-module I^2/JI has a 1-dimensional socle.

(3) The socle of I^2/JI is generated by the Jacobian determinant of the generators of J.

Van Straten and Warmt were particularly interested in the case when J is generated by the partial derivatives of a given power series f.

In our terms (see Section 1), the ideal I in the conjecture satisfies the Strong Hypothesis (G_s because it is reduced and the depth conditions because it is Cohen–Macaulay of dimension 1). We give a proof of Conjecture (1) (Theorem 2.2) in a more general setting. Conjecture (2) was in fact already known [31, 2.9], also in a more general setting.

As stated, Conjecture (3) is false, even for the case when the ideal J is generated by the partial derivatives of a quasi-homogeneous polynomial, and we give a counter-example in Example 7.15. However, we prove Conjecture (3) in Theorem 7.10, again in a more general setting, under the additional hypothesis that J is generated by homogeneous polynomials of the same degree.

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1. Definitions, hypotheses and notation

Let I be an ideal of codimension g in a Noetherian local ring R. Recall that a proper ideal of the form $K = J : I$ is called an s-residual intersection (of I with respect to J), for some integer $s \geq g$, if $J \subset I$ is generated by s elements and K has codimension at least s. The ideal K is said to be a *geometric* s-residual intersection if in addition $\operatorname{codim}(I + K) \geq s + 1$.

In order for an s-residual intersection of I to exist, it is clearly necessary that I be generated by s elements locally at every prime of codimension $< s$, and for a geometric s-residual intersection to exist, this condition must also be satisfied at primes of codimension s containing I. For inductive purposes, the proofs of most results in the theory require a slightly stronger hypothesis: *The ideal I is said to satisfy the condition G_s if $\mu(I_P) \leq \operatorname{codim} P$ for all prime ideals $P \supset I$ with* $\operatorname{codim} P \leq s - 1$.

For example, the homogeneous ideal of any smooth variety in $\mathbb{P}^n$ satisfies G_{n+1}.

The significance of the condition G_s is in the following result, which allows an induction that we will use often.

Lemma 1.1. *Let R be a Noetherian local ring with infinite residue field, and let $I \subset R$ be an ideal that satisfies G_s. Let $\mathfrak{a} \subsetneq I$ be any ideal with* $\operatorname{codim}(\mathfrak{a} : I) \geq s$. *Let $a_1, \ldots, a_s$ be general elements of $\mathfrak{a}$, and set $J_u = (a_1, \ldots, a_u)$, $K_u = J_u : I$. Write $R_u = R/K_u$. For $g \leq u \leq s$ the ideal K_u is a u-residual intersection, and this residual intersection is geometric if $u < s$.*

Here, and in the rest of the paper, the notion of a set of general elements may be defined as follows. Let R be a Noetherian local ring with infinite residue field k, and let $\mathfrak{a}$ be an ideal. We say that the elements $a_1, \ldots, a_s \subset \mathfrak{a}$ are general in $\mathfrak{a}$ if the image of the point $(a_1, \ldots, a_s) \in \mathfrak{a}^s$ in $(k \otimes_R \mathfrak{a})^s$ is general.

Proof. The result follows from the theory of basic elements [10]. For a detailed treatment, see [31, Section 1], and in particular [31, 1.5 (ii)–(iii)]. □

Now suppose in addition that R is Gorenstein. We say that I satisfies the *Standard Hypothesis* (respectively *Weak* or *Strong Hypothesis*) with respect to $s = g + t$ if I satisfies G_s and, in addition, the *Depth Conditions*

$$\operatorname{depth}(R/I^j) \geq \dim(R/I) - j + 1$$

for $j \leq t$ (respectively $j \leq t-1$ or $j \leq t+1$).

For example, if $t = 1$, then the Standard Hypothesis is equivalent to the condition that R/I is Cohen–Macaulay and I is generically a complete intersection. Also note that if s is equal to $\dim R$, then the Strong Hypothesis is the same as the weak hypothesis, since the extra requirement is that the depth of R/I^{t+1} is ≥ 0.

Assuming that the ideal I satisfies G_s, the Strong Hypothesis holds, for example, if the Koszul homology modules $H_i(I)$ of some generating sequence of I are Cohen–Macaulay in the range $0 \leq i \leq t$ (see [31, 2.10]); in particular it holds for strongly Cohen–Macaulay ideals; thus it is satisfied by Cohen–Macaulay almost complete intersection ideals, Cohen–Macaulay ideals of deviation 2 (see [2, p. 259]), and ideals in the linkage class of a complete intersection (see [18, 1.11]). Standard examples include perfect ideals of codimension 2 and perfect Gorenstein ideals of codimension 3 (see [34, proof of the only theorem]).

The ideal of the Veronese surface in $\mathbb{P}^5$ satisfies the Standard hypothesis with $s = 4$ and the Weak Hypothesis with $s = 5$ – this is the ideal that appears in the five conics problem of Chasles [8]. (It also satisfies "sliding depth" for the Koszul homology, so the general residual intersection $K := (a_1, \ldots, a_5) : I$ is unmixed – see [17, 2.3 and 3.3]. By a Bertini argument as in the proof of Proposition 3.4, the ideal K is the homogeneous ideal of a set of reduced points.)

2. Duality results

We will assume throughout this section that I is an ideal of codimension g in the local Gorenstein ring R, and $K = J : I$ is an s-residual intersection for some $s \geq g$. We set $t = s - g$. When we refer to the Standard, Weak, or Strong Hypothesis, it will always be with respect to s.

In this section we give precise statements of our main duality results. Proofs will be found in Section 5.

Huneke gave a simple proof of van Straten's $t = 1$ result in a more general context. We include it with his gracious permission:

Theorem 2.1. *Suppose that the ring R/I is Cohen–Macaulay of codimension g and $J = (a_1, \ldots, a_{g+1}) \subset I$ is such that $K = J : I$ has codimension $g+1$. Then the R/K-module I/J is self-dual; that is, $I/J \cong \operatorname{Hom}_R(I/J, \omega_{R/K})$.*

Assuming the Standard Hypothesis allows us to extend the result to higher values of t, and to prove a statement that is stronger even in the case $t = 1$:

Theorem 2.2. *Under the Standard Hypothesis, Theorem* 4.1 *applies to give an injective map $\mu_t : I^{t+1}/JI^t \to \omega_{R/K}$. For $1 \leq u \leq t$, both the multiplication map*

$$m(I,u,t) : I^u/JI^{u-1} \otimes I^{t+1-u}/JI^{t-u} \xrightarrow{\text{mult}} I^{t+1}/JI^t$$

and the composition $\mu_t \circ m(I,u,t)$ are perfect pairings.

If in addition I satisfies the Strong Hypothesis, then the duality of Theorem 2.2 holds in the full range $0 \le u \le t+1$. Here, when $u = 0$, we interpret I^u/JI^{u-1} as $R/(J : I) = R/K$, and the statement is simply that $I^{t+1}/JI^t \cong \omega_{R/K}$ and $R/K \cong \mathrm{End}(\omega_{R/K})$, which holds by Theorem 3.1.

Note that the hypothesis of Theorem 2.1 does not include the condition G_s; on the other hand, Example 6.4 shows that the duality asserted in Theorem 2.1 does not necessarily come from the multiplication map as in Theorem 2.2. Examples suggest that the weaker result should also be true with a condition weaker than G_s:

Conjecture 2.3. The duality

$$I^u/JI^{u-1} \cong \mathrm{Hom}_R(I^{t+1-u}/JI^{t-u}, \omega_{R/K})$$

holds for $1 \le u \le t$ if $K = J : I$ is an s-residual intersection and I satisfies a weakened Standard Hypothesis with G_s replaced by G_{s-1}.

The conjecture is immediate in the case where R is regular and $g = 1$: then $I = (G)$ is principal, and $J = (GF)$, where F is a regular sequence (of length s). In this case the pairings all reduce to the usual isomorphisms $R/(F) \to \mathrm{Hom}(R/(F), R/(F))$ induced by multiplication. We will prove the conjecture under an additional assumption in Corollary 2.7 of Theorem 2.6 below.

The condition G_s in the Strong, Standard, and Weak Hypotheses is used in the inductive proof of many theorems about residual intersections, but it is not clear why it should be necessary. Recent work [7, 13, 14] has aimed at removing this hypothesis, and has had success in the case when I is strongly Cohen–Macaulay. In particular, Chardin, Naéliton and Tran [7] have proved an analogue of Theorem 2.2 in this setting, replacing the modules I^u/JI^{u-1} with the modules $\mathrm{Sym}_u(I/J)$. In Section 6 we will see that this statement does not extend too far beyond the strongly Cohen–Macaulay case; see Examples 6.6 and 6.7.

Under the Strong Hypothesis we can combine all the dualities of Theorem 2.2 in the statement that a certain quotient of the Rees algebra $R[Iz]$ of I is Gorenstein:

Corollary 2.4. *Under the Strong Hypothesis, the following ring is Gorenstein:*

$$\mathcal{R} := R/K \oplus I/J \oplus I^2/JI \oplus \cdots \oplus I^{t+1}/JI^t = R[Iz]/(K, Jz, (Iz)^{t+2}).$$

As an application of Theorem 2.2 and Corollary 2.4 we will deduce:

Theorem 2.5. *In addition to the Strong Hypothesis, suppose that $K = J : I$ is a geometric s-residual intersection.*

(1) *Let $\overline{I} \subset \overline{R} := R/K$ be the image of I. The truncated Rees algebra*

$$\overline{R} \oplus \overline{I} \oplus \overline{I}^2 \oplus \cdots \oplus \overline{I}^{t+1}$$

is Gorenstein. In particular, $\overline{I}^{t+1} \cong \omega_{\overline{R}}$ and the multiplication maps

$$\overline{I}^u \otimes \overline{I}^{t+1-u} \to \overline{I}^{t+1}$$

are perfect pairings.

(2) *Let $I' \subset R' := R/(K + I^{t+1})$ be the image of I. The associated graded ring $\mathrm{gr}_{I'}(R')$ is Gorenstein.*

Sometimes the duality statements of Theorem 2.2 hold only for a restricted range of values of u. Our most general result involves another definition: We say that a pair $(\tilde{R}, \tilde{I})$ consisting of a Noetherian local ring $\tilde{R}$ and an ideal $\tilde{I}$ is a *deformation* of the pair (R, I) if $\tilde{R}$ contains a regular sequence $x_1, \dots, x_n$ whose image in $\tilde{R}/\tilde{I}$ is also a regular sequence such that R is isomorphic to $\tilde{R}/(x_1, \dots, x_n)$ and $I = \tilde{I}R$.

Theorem 2.6. *Suppose that (R, I) has a deformation $(\tilde{R}, \tilde{I})$ such that $\tilde{I}$ satisfies the condition G_s and the Koszul homology $H_i(\tilde{I})$ is Cohen–Macaulay for $0 \le i \le t = s - g$. Assume further that I satisfies the condition G_{g+v} for some $\frac{t-1}{2} \le v \le t$. Let $\tilde{J}$ be a lifting of J to an ideal with s generators contained in $\tilde{I}$. The ideal $\tilde{K} = \tilde{J} : \tilde{I}$ is an s-residual intersection of $\tilde{I}$. Our hypothesis implies that Theorem* 4.1 *holds with $\tilde{K}$ in place of K and gives an isomorphism μ_t. The inverse $\phi : \omega_{\tilde{R}/\tilde{K}} \to \tilde{I}^{t+1}/\tilde{J}\tilde{I}^t$ of μ_t induces a map*

$$\phi' : \omega_{R/K} \to I^{t+1}/JI^t.$$

We have:

(1) *ϕ' is a surjection, and is an isomorphism if K is a geometric s-residual intersection.*

(2) *There are perfect pairings*

$$m : I^u/JI^{u-1} \otimes I^{t+1-u}/JI^{t-u} \to \omega_{R/K}$$

for

$$t - v \le u \le v + 1$$

or, equivalently, for

$$\frac{t+1}{2} - \varepsilon \le u \le \frac{t+1}{2} + \varepsilon,$$

where $\varepsilon = v - \frac{t-1}{2}$.

(3) *If the perfect pairing m is chosen as in Figure* 1 *in the proof, then $\phi' \circ m$ is the map induced by multiplication $I^u \otimes I^{t+1-u} \to I^{t+1}$.*

Under the hypotheses of Theorem 2.6, the ideal $\tilde{I}$ satisfies the Strong Hypothesis (see [31, 2.10]). Thus the ring $\tilde{R}/\tilde{K}$ is Cohen–Macaulay with canonical module $\tilde{I}^{t+1}/\tilde{J}\tilde{I}^t$ by Theorem 3.1. From the proofs below it follows that the map ϕ' can also be described as a composition

$$\omega_{R/K} \hookrightarrow R \otimes_{\tilde{R}} \omega_{\tilde{R}/\tilde{K}} \xrightarrow{R\otimes_{\tilde{R}}\phi} R \otimes_{\tilde{R}} \tilde{I}^{t+1}/\tilde{J}\tilde{I}^t \twoheadrightarrow I^{t+1}/JI^t.$$

We remark that all the hypotheses of Theorem 2.6 are satisfied when I is licci and satisfies G_{g+v} (see [18, 1.11] and [20, proof of 5.3]). We will see that the G_{g+v} assumption cannot be weakened to G_{g+v-1}, even when I is a codimension 2 perfect ideal (Example 6.3), and also that the deformation assumption cannot be dropped, even when I satisfies G_s (Examples 6.5 and 6.6).

Applying Theorem 2.6 with $g + v = s - 1$, we obtain a result extending Theorem 2.1 under the additional hypothesis that the pair (R, I) admits a "good" deformation:

Corollary 2.7. *Assume that (R, I) has a deformation $(\tilde{R}, \tilde{I})$ such that $\tilde{I}$ satisfies the condition G_s and the Koszul homology $H_i(\tilde{I})$ is Cohen–Macaulay for $0 \le i \le t$. If I satisfies G_{s-1}, then*

$$I^u/JI^{u-1} \cong \operatorname{Hom}_R(I^{t+1-u}/JI^{t-u}, \omega_{R/K})$$

for $1 \le u \le t$.

3. Preliminary results

We will repeatedly use a number of results from [31]. For the reader's convenience we gather them here. Unless stated otherwise, we will assume that R is a local Gorenstein ring of dimension d, I is an ideal of codimension g, and $K = J : I$ is an s-residual intersection of I for some $s \geq g$. As before, we write $t = s - g$, and when referring to the Weak, Standard, or Strong Hypotheses, we mean that these hypotheses hold with respect to s.

Theorem 3.1. *If I satisfies the Strong Hypothesis, then R/K is Cohen–Macaulay of codimension s with $\omega_{R/K} \cong I^{t+1}/JI^t$.*

Proof. This is [31, 2.9]. □

Proposition 3.2. *If I satisfies the Standard Hypothesis, then the ideal K is unmixed of codimension exactly s.*

Proof. One uses Theorem 3.1 and [31, 1.7 (a)]. □

Proposition 3.3. *Let $J = (a_1, \ldots, a_s)$. For u with $0 \leq u \leq s$ write $J_u = (a_1, \ldots, a_u)$, $K_u = J_u : I$, $R_u = R/K_u$, and assume that K_u is a geometric u-residual intersection of I whenever $g \leq u < s$.*

(1) *If I satisfies the Weak Hypothesis and $u \geq 1$, then the element a_u is regular on R_{u-1} and $K_u R_{u-1} = a_u R_{u-1} : IR_{u-1}$.*

(2) *If I satisfies the Standard Hypothesis and $2 \leq j \leq t+1$, or if I satisfies the Strong Hypothesis and $2 \leq j \leq t+2$, then there are exact sequences*

$$0 \to \frac{I^{j-1}}{J_{u-1}I^{j-2}} \xrightarrow{a_u} \frac{R}{J_{u-1}I^{j-1}} \to \frac{R}{J_u I^{j-1}} \to 0$$

for $u \geq 1$.

(3) *If I satisfies the Standard Hypothesis and $1 \leq j \leq t$, or if I satisfies the Strong Hypothesis and $1 \leq j \leq t+1$, then*

$$\operatorname{depth} I^j/J_u I^{j-1} \geq \min\{d-u, \dim R/I - j + 2\}.$$

In particular, $I^j/J_u I^{j-1}$ is a maximal Cohen–Macaulay R_u-module if in addition $j \leq u - g + 2$.

(4) *If I satisfies the Standard Hypothesis and $1 \leq j \leq t+1$, or if I satisfies the Strong Hypothesis and $1 \leq j \leq t+2$, then*

$$\operatorname{depth} R/J_u I^{j-1} \geq \min\{d-u, \dim R/I - j + 2\}.$$

(5) *If I satisfies the Standard Hypothesis, $u < s$, and $1 \leq j \leq t+1$, or if I satisfies the Strong Hypothesis, K is a geometric s-residual intersection, and $1 \leq j \leq t+2$, then*

$$I^j \cap K_u = J_u I^{j-1}.$$

Proof. First notice that if $u < g$, then $K_u = J_u$ is generated by the regular sequence $a_1, \ldots, a_u$. Part (1) follows from Theorem 3.1 and [31, 1.7 (a), (c)], part (2) is a consequence of Theorem 3.1 and [31, 2.7 (a)], and item (3) follows from Theorem 3.1 and [31, 2.7 (b)].

We now prove (4). The assertion for $j = 1$ follows from Theorem 3.1 and [31, 1.7 (b)]. Thus we may assume that $j \geq 2$. We show part (4) by induction on u. The assertion is obvious for $u = 0$. If $1 \leq u \leq s$, we apply the exact sequence of part (2), the depth estimate of part (3), and the induction hypothesis.

If the ideal I satisfies the Strong Hypothesis, K is a geometric residual intersection and $1 \leq j \leq t + 2$, then part (5) follows from Theorem 3.1 and [31, 2.7 (c)]. If on the other hand I satisfies the Standard Hypothesis, $u < s$ and $1 \leq j \leq t + 1$, then I satisfies the Strong Hypothesis with respect to $s-1$, K_{s-1} is a geometric residual intersection and $1 \leq j \leq (s-1-g)+2$, so the assertion follows from the previous case. □

Proposition 3.4. *Suppose that R satisfies Serre's condition (R_{s-1}) and contains a field of characteristic 0. Let $a_1, \ldots, a_{s-1}$ be general elements of J, and set $J_u = (a_1, \ldots, a_u)$, $K_u = J_u : I$, $R_u = R/K_u$. If R is reduced and I satisfies the Weak Hypothesis with respect to s, then the factor ring R_u is reduced and equidimensional of codimension u for every $u < s$.*

Proof. Again, if $u < g$, then $K_u = J_u$ is generated by the regular sequence $a_1, \ldots, a_u$. If $g \leq u < s$, then K_u is a geometric u-residual intersection by Lemma 1.1 and hence this ideal is unmixed of codimension u according to Proposition 3.2 because I satisfies the Standard Hypothesis with respect to u. In either case, K_u is unmixed of codimension u and I is not contained in any of the minimal primes of K_u.

Let P be any of these minimal primes. To show that R_u is reduced, it suffices to prove that the ring $(R_u)_P$ is regular. Since $\operatorname{codim}(J : I) \geq s > u = \operatorname{codim} P$ and P does not contain I, it follows that P cannot contain J either. Since the elements $a_1, \ldots, a_u$ are general in J, the local Bertini theorems [12, 4.6] show that $(R/(a_1, \ldots, a_u))_P$ is regular. But this ring is $(R_u)_P$, again since P does not contain I. □

4. Connecting the canonical module with powers of I

We next explain the maps that connect powers of I with the canonical module, refining Theorem 3.1. As we shall see, these maps are defined under a certain assumption that is satisfied under the Standard Hypothesis, but also in some cases of geometric residual intersections. Unless stated otherwise, the general assumptions of Section 3 are still in effect.

Theorem 4.1. *Let $a_1, \ldots, a_s$ be generators of J and, for every u with $0 \leq u \leq s$, let $J_u = (a_1, \ldots, a_u)$ and $K_u = J_u : I$. Assume that* $\operatorname{codim}((K_{u-1}, a_u) : I) = u$ *whenever $1 \leq u \leq s$ and that* $\operatorname{codim}(I + K_u) \geq u + 1$ *whenever $0 \leq u < s$. For every u with $0 \leq u \leq s$ one has* $\operatorname{codim} K_u = u$. *Set $R_u = R/K_u$ and $R'_u = R/K'_u$, where K'_u denotes the unmixed part of K_u of codimension u. For every u with $g \leq u \leq s$, there are maps*

$$\frac{I^{u-g+1}}{J_u I^{u-g}} \xrightarrow{\mu_{u-g}} \omega_{R_u}$$

defined inductively:

(1) *For $u = g$, μ_0 is the map induced by the inclusion of I into the double annihilator*

$$I/J_g \hookrightarrow \frac{(J_g : (J_g : I))}{J_g} = \omega_{R_g}.$$

(2) *For $s \geq u > g$, μ_{u-g} is the map obtained from μ_{u-g-1} and an embedding*

$$I\omega_{R_{u-1}}/a_u\omega_{R_{u-1}} \hookrightarrow \omega_{R_u}$$

obtained from the diagram of homomorphisms of rings

$$\begin{array}{ccccccc} R_{u-1} & \xrightarrow{\pi_1} & R'_{u-1} & \xrightarrow{\pi_2} & R'_{u-1}/(a_u) & & \\ & & & & \uparrow{\scriptstyle \pi_3} & & \\ & & & & R_{u-1}/(a_u) & \xrightarrow{\pi_4} & R_{u-1}/((a_u) : IR_{u-1}) \\ & & & & & & \uparrow{\scriptstyle \pi_5} \\ & & & & & & R_u \end{array}$$

as explained in the proof.

If I satisfies the Standard Hypothesis with respect to s, then the map μ_{s-g} is an injection, while if I satisfies the Strong Hypothesis with respect to s, then μ_{s-g} is an isomorphism.

Proof. We first show that $\operatorname{codim} K_u = u$ for $0 \leq u \leq s$ and we compute the codimensions of all the rings in the diagram.

Since the codimension of the ideal K_0 is obviously 0, we assume that $1 \leq u \leq s$. By Lemma 4.2, the codimension of K_u is at least u. As $K_u \subset (K_{u-1}, a_u) : I$ and the second ideal has codimension u by hypothesis, we see that the codimensions of the two ideals are exactly u. Thus, the rings R_{u-1} and R'_{u-1} have codimension $u-1$ in R, and the rings R_u and $R_{u-1}/((a_u) : IR_{u-1})$ have codimension u.

We now claim that a_u is not in any codimension $u-1$ prime P containing K_{u-1}. Since $I + K_{u-1}$ has codimension $\geq u$ by hypothesis, we have $I \not\subset P$, and since K_u has codimension u, we see that $I_P = (J_u)_P$ and therefore $J_u \not\subset P$. As $J_{u-1} \subset P$, it follows that $a_u \notin P$. From this we see that the rings $R_{u-1}/(a_u)$ and $R'_{u-1}/(a_u)$ have codimension u, and moreover a_u is a nonzerodivisor on R'_{u-1}.

We take the map μ_0 to be the natural inclusion. Moreover, this map is an isomorphism if the Strong Hypothesis holds since then R/I is Cohen–Macaulay. Therefore we assume from now on that $u > g$.

The map π_1 induces an isomorphism

$$(\pi_1^\vee)^{-1} : \omega_{R_{u-1}} \xrightarrow{\sim} \omega_{R'_{u-1}}.$$

Since a_u is a nonzerodivisor on R'_{u-1}, the connecting homomorphism of $\operatorname{Ext}_R(-, \omega_R)$ applied to the exact sequence

$$0 \longrightarrow R'_{u-1} \xrightarrow{a_u} R'_{u-1} \xrightarrow{\pi_2} R'_{u-1}/(a_u) \longrightarrow 0$$

yields an embedding $\sigma_2 : \omega_{R'_{u-1}}/a_u\omega_{R'_{u-1}} \hookrightarrow \omega_{R'_{u-1}/(a_u)}$. The map π_3 induces an embedding

$$\pi_3^\vee : \omega_{R'_{u-1}/(a_u)} \hookrightarrow \omega_{R_{u-1}/(a_u)}.$$

For simplicity of notation we set

$$\omega := \omega_{R_{u-1}/(a_u)} \quad \text{and} \quad H := (a_u) : IR_{u-1} \subset R_{u-1}.$$

Multiplying by I, we see that the maps $(\pi_1^\vee)^{-1}$, σ_2 and $\pi_3^\vee$ together induce an embedding

$$I\omega_{R_{u-1}}/a_u\omega_{R_{u-1}} \hookrightarrow I\omega.$$

On the other hand,

$$I\omega \subset 0 :_\omega H = \omega_{R_{u-1}/H},$$

and combining these two embeddings, we obtain

$$I\omega_{R_{u-1}}/a_u\omega_{R_{u-1}} \hookrightarrow \omega_{R_{u-1}/H}. \tag{1}$$

Finally, the map π_5 induces an embedding $\pi_5^\vee : \omega_{R_{u-1}/H} \hookrightarrow \omega_{R_u}$, which together with the map in (1) gives an embedding

$$I\omega_{R_{u-1}}/a_u\omega_{R_{u-1}} \hookrightarrow \omega_{R_u}. \tag{2}$$

By induction, we may assume that the process just explained induces a map

$$I^{u-g}/J_{u-1}I^{u-g-1} \to \omega_{R_{u-1}},$$

and thus we obtain a map

$$\frac{I^{u-g+1} + J_{u-1}I^{u-g-1}}{a_uI^{u-g} + J_{u-1}I^{u-g-1}} \to \omega_{R_u}.$$

The left-hand side is obviously a homomorphic image of I^{u-g+1}/J_uI^{u-g}, and this gives the desired homomorphism

$$\mu_{u-g} : I^{u-g+1}/J_uI^{u-g} \to \omega_{R_u}.$$

We now show by induction on $u > g$ that if I satisfies the Standard Hypothesis or the Strong Hypothesis, then μ_{u-g} is an injection or an isomorphism, respectively. In either case, I satisfies the Strong Hypothesis with respect to $u-1$, so in particular

$$\mu_{u-g-1} : I^{u-g}/J_{u-1}I^{u-g-1} \to \omega_{R_{u-1}}$$

is an isomorphism by the induction hypothesis. Multiplying by I and factoring out a_uI^{u-g}, we get an induced isomorphism

$$\frac{I^{u-g+1}}{a_uI^{u-g} + (J_{u-1}I^{u-g-1} \cap I^{u-g+1})} \xrightarrow{\sim} \frac{I\omega_{R_{u-1}}}{a_u\omega_{R_{u-1}}}.$$

By (2), the right-hand side embeds in ω_{R_u}. So to prove the injectivity of μ_{u-g} it suffices to show that

$$a_uI^{u-g} + (J_{u-1}I^{u-g-1} \cap I^{u-g+1}) = J_uI^{u-g}.$$

The right-hand side is obviously contained in the left-hand side, so it remains to prove the opposite inclusion. We trivially have

$$J_{u-1}I^{u-g-1} \cap I^{u-g+1} \subset I^{u-g+1} \cap K_{u-1},$$

and Proposition 3.3 (5) gives $I^{u-g+1} \cap K_{u-1} = J_{u-1}I^{u-g}$. This concludes the proof that μ_{u-g} is an injection.

We now show that if I satisfies the Strong Hypothesis, then μ_{u-g} is a surjection. To this end, it suffices to prove that the map in (2) is a surjection. Since π_5 is an isomorphism according to Proposition 3.3 (1), it remains to show that the map in (1) is surjective. We summarize the argument in the proof of [31, 2.9 (b)]. Recall that R_{u-1} is Cohen–Macaulay by Theorem 3.1.

We first prove that $I\omega_{R_{u-1}}$ is $\omega_{R_{u-1}}$-reflexive. By induction,

$$\omega_{R_{u-1}} \cong I^{u-g}/J_{u-1}I^{u-g-1}.$$

Proposition 3.3 (5) shows that $J_{u-1}I^{u-g-1} = I^{u-g} \cap K_{u-1}$ and therefore

$$I^{u-g}/J_{u-1}I^{u-g-1} \cong I^{u-g}R_{u-1}.$$

It follows that $I\omega_{R_{u-1}} \cong I^{u-g+1}R_{u-1}$. But again by Proposition 3.3 (5),

$$I^{u-g+1}R_{u-1} \cong I^{u-g+1}/J_{u-1}I^{u-g}.$$

Putting this together, we obtain

$$I\omega_{R_{u-1}} \cong I^{u-g+1}/J_{u-1}I^{u-g}.$$

By Proposition 3.3 (3), $I^{u-g+1}/J_{u-1}I^{u-g}$ is a maximal Cohen–Macaulay R_{u-1}-module and thus $I\omega_{R_{u-1}}$ is $\omega_{R_{u-1}}$-reflexive, which we write as $I\omega_{R_{u-1}} = (I\omega_{R_{u-1}})^{\vee\vee}$.

We deduce that

$$I\omega_{R_{u-1}}/a_u\omega_{R_{u-1}} = (I\omega_{R_{u-1}})^{\vee\vee}/a_u\omega_{R_{u-1}} = \omega_{R_{u-1}/H},$$

where the last identification holds according to [31, 2.1 (a)]. Therefore the map in (1) is surjective. This concludes the proof. □

Lemma 4.2. *Let R be a Noetherian ring, let $J \subset I$ be ideals, and let $a \in R$ be an element. If*

$$\operatorname{codim}(J : I, a) : I \geq u \quad \text{and} \quad \operatorname{codim}\big(I + ((J, a) : I)\big) \geq u,$$

then

$$\operatorname{codim}(J, a) : I \geq u.$$

Proof. One sees that

$$\big((J : I, a) : I\big)\big(I + ((J, a) : I)\big) \subset (J : I, a) + ((J, a) : I) \subset (J, a) : I$$

as desired. □

For future use we record the following statements, proved in the course of the proof of Theorem 4.1.

Corollary 4.3. *With the notation and assumptions of Theorem* 4.1*, assume that I satisfies the Strong Hypothesis. For $0 < g \leq u \leq s$ the rings $R_{u-1}/(a_u)$ and R_u are Cohen–Macaulay of dimension $d - u$, and the surjection $R_{u-1}/(a_u) \twoheadrightarrow R_u$ induces an inclusion of canonical modules*

$$I^{u-g+1}/J_uI^{u-g} \cong \omega_{R_u} \hookrightarrow \omega_{R_{u-1}/(a_u)} \cong I^{u-g}R_{u-1}/(a_uI^{u-g}R_{u-1})$$

that is compatible with the natural inclusion $I^{u-g+1} \subset I^{u-g}$.

Remark 4.4 (The graded case). Suppose that R is a standard graded polynomial ring $k[x_1, \dots, x_d]$, that the ideal I is homogeneous and that the generators $a_1, \dots, a_s$ of J are homogeneous of degrees $\delta_1, \dots, \delta_s$. In this setting the construction of Theorem 4.1 yields a homogeneous map

$$I^{u-g+1}/J_u I^{u-g} \xrightarrow{\mu_{u-g}} \omega_{R_u}\left(d - \sum_{j=1}^{u} \delta_j\right).$$

5. Proofs of the duality theorems from Section 2

We follow a suggestion of the referee, and include the statement of each theorem from Section 2 before its proof. Theorems from Section 2 retain the numbering that they had there. Unless specified otherwise, I will again denote an ideal of codimension g in a local Gorenstein ring R, the ideal $K = J : I$ is assumed to be an s-residual intersection, and we set $t = s - g$.

Theorem 2.1. *Suppose that the ring R/I is Cohen–Macaulay of codimension g and $J = (a_1, \dots, a_{g+1}) \subset I$ is such that $K := J : I$ has codimension $g+1$. Then the R/K-module I/J is self-dual, that is,*

$$I/J \cong \operatorname{Hom}_R(I/J, \omega_{R/K}).$$

Proof (Huneke). We may suppose that $J = (a_1, \dots, a_g, b)$, where $a_1, \dots, a_g$ form a regular sequence. Factoring out $a_1, \dots, a_g$, we may assume $g = 0$.

Let $L = 0 : b$, and consider the short exact sequence

$$0 \to R/L \xrightarrow{b} R \to R/(b) \to 0.$$

Dualizing into R, we obtain an exact sequence

$$R \xrightarrow{\beta} 0 : L \to \operatorname{Ext}^1_R(R/(b), R) \to 0.$$

The image of β is the ideal generated by b. Also, we claim that $0 : L = I$. Because R is Gorenstein, the ideal $0 : L = 0 : (0 : b)$ is the unmixed part of (b), which is equal to I because I is unmixed of codimension 0 and $(b) : I$ has positive codimension in R. Putting these two observations together, we get

$$I/(b) \cong \operatorname{Ext}^1_R(R/(b), R).$$

On the other hand, because R/I is a maximal Cohen–Macaulay R-module, and R is Gorenstein, we have

$$\operatorname{Ext}^1_R(R/I, R) = \operatorname{Ext}^2_R(R/I, R) = 0,$$

so from the short exact sequence

$$0 \to I/(b) \to R/(b) \to R/I \to 0$$

we get

$$\operatorname{Ext}^1_R(R/(b), R) \cong \operatorname{Ext}^1_R(I/(b), R).$$

Since K is an ideal of codimension 1 in the Gorenstein ring R and K annihilates $I/(b)$, it follows that $\operatorname{Ext}^1_R(I/(b), R) \cong \operatorname{Hom}_R(I/(b), \omega_{R/K})$, and since we already showed that $I/(b) \cong \operatorname{Ext}^1_R(R/(b), R)$, we conclude that $I/(b) \cong \operatorname{Hom}_R(I/(b), \omega_{R/K})$ as required. □

For the proof of Theorem 2.2 we will need the following.

Lemma 5.1. *In addition to the Standard Hypothesis assume that the residue field k is infinite. Write d for the dimension of R and let $x_1, \dots, x_{d-s}$ be general elements in the maximal ideal. For any $1 \le u \le t$ one has:*

(1) *The elements $x_1, \dots, x_{d-s}$ form a regular sequence on R and on R/I^u.*

(2) *The image $\overline{I}$ of I in $R/(x_1, \dots, x_{d-s})$ satisfies the condition G_s.*

(3) *We adopt the notation of Theorem* 2.2. *The image $\overline{J}$ defines an s-residual intersection $\overline{J} : \overline{I}$ in $R/(x_1, \dots, x_{d-s})$. If $m(\overline{I}, u, t)$ is a perfect pairing, then so are $m(I, u, t)$ and $\mu_t \circ m(I, u, t)$.*

Proof. (1) By the Standard Hypothesis,

$$\operatorname{depth}(R/I^u) \ge \dim(R/I) - u + 1 \ge \dim(R/I) - t + 1 = d - s + 1.$$

In particular, the elements $x_1, \dots, x_{d-s}$ form a regular sequence on R and on R/I^u.

(2) The condition G_s is equivalent to the condition that the codimension of $I + \operatorname{Fitt}_{i-1}(I)$ is at least i for $1 \le i \le s$. The Fitting ideals of the image $\overline{I}$ of I in $R/(x_1, \dots, x_{d-s})$ contain the image of the Fitting ideals, and so the codimensions of $\overline{I} + \operatorname{Fitt}_{i-1}(\overline{I})$ satisfy the same inequalities because the elements $x_1, \dots, x_{d-s}$ are general and $\dim R/(x_1, \dots, x_{d-s}) = s$.

(3) By Proposition 3.2, the codimension of K is exactly s. Let $y_1, \dots, y_s$ be a regular sequence inside K, and set $A = R/(y_1, \dots, y_s)$. Note that $x_1, \dots, x_{d-s}$ is a regular sequence on A.

We recall the map μ_t of Theorem 4.1, which in an embedding under the present assumptions. The maps

$$I^u/JI^{u-1} \otimes_R I^{t+1-u}/JI^{t-u} \xrightarrow{m(I,u,t)} I^{t+1}/JI^t \xhookrightarrow{\mu_t} \omega_{R/K} \hookrightarrow \omega_A$$

induce maps

$$\begin{aligned} I^{t+1-u}/JI^{t-u} &\xrightarrow{\alpha} \operatorname{Hom}_R(I^u/JI^{u-1}, I^{t+1}/JI^t) \\ &\xhookrightarrow{\beta} \operatorname{Hom}_R(I^u/JI^{u-1}, \omega_{R/K}) \\ &\xrightarrow{\cong} \operatorname{Hom}_R(I^u/JI^{u-1}, \omega_A), \end{aligned}$$

where the last map is an isomorphism by Hom-tensor adjointness. We must show that under our hypothesis α and β are both isomorphisms.

By Proposition 3.3 (3), the module I^u/JI^{u-1} is a maximal Cohen–Macaulay A-module. As A is Cohen–Macaulay, we infer that $\operatorname{Hom}_R(I^u/JI^{u-1}, \omega_A)$ is a maximal Cohen–Macaulay A-module too. Thus $x_1, \dots, x_{d-s}$ form a regular sequence on this module.

Let $\overline{A} = A/(x_1, \dots, x_{d-s})$ and $\overline{R} = R/(x_1, \dots, x_{d-s})$; we write $\overline{I}, \overline{J}, \overline{K}$ for the images of I, J, K in $\overline{R}$, respectively. We will show that $(\beta\alpha) \otimes_R \overline{R}$ is an isomorphism. This implies that $\beta\alpha$ is surjective, and thus β is surjective and consequently β is an isomorphism. It follows that α is also surjective, and $x_1, \dots, x_{d-s}$ is a regular sequence on the image of α. Because $\alpha \otimes_R \overline{R}$ is an injection and $x_1, \dots, x_{d-s}$ is a regular sequence on the image of α, it follows that α is a monomorphism.

It remains to show that $(\beta\alpha) \otimes_R \overline{R}$ is an isomorphism. The ideal $\overline{I}$ satisfies the Standard Hypothesis by items (1) and (2). Since $s = \dim \overline{R}$, the Standard Hypothesis is the same as the

Strong Hypothesis in this case. The ideal $\overline{K}$ has codimension s and is contained in $\overline{J} : \overline{I}$, hence $\overline{J} : \overline{I}$ is an s-residual intersection. Arguing as above, there are maps

$$\begin{aligned}\overline{I}^{t+1-u}/\overline{JI}^{t-u} &\xrightarrow{\overline{\alpha}} \operatorname{Hom}_{\overline{R}}(\overline{I}^u/\overline{JI}^{u-1}, \overline{I}^{t+1}/\overline{JI}^t)\\ &\xrightarrow{\overline{\beta}} \operatorname{Hom}_{\overline{R}}(\overline{I}^u/\overline{JI}^{u-1}, \omega_{\overline{R}/(\overline{J}:\overline{I})})\\ &\xrightarrow{\cong} \operatorname{Hom}_{\overline{R}}(\overline{I}^u/\overline{JI}^{u-1}, \omega_{\overline{A}}),\end{aligned}$$

induced by $m(\overline{I}, u, t)$ and μ_t. By assumption, $\overline{\alpha}$ is an isomorphism. Moreover, since $\overline{I}$ satisfies the Strong Hypothesis, $\overline{\beta}$ is an isomorphism by Theorem 4.1.

Because $x_1, \dots, x_{d-s}$ form a regular sequence on R/I^u by item (1), it follows that $\overline{I}^u/\overline{JI}^{u-1} \cong I^u/JI^{u-1} \otimes_R \overline{R}$. Further,

$$\begin{aligned}\operatorname{Hom}_{\overline{R}}(\overline{I}^u/\overline{JI}^{u-1}, \omega_{\overline{A}}) &\cong \operatorname{Hom}_{\overline{R}}(I^u/JI^{u-1} \otimes_R \overline{R}, \omega_A \otimes_R \overline{R})\\ &\cong \operatorname{Hom}_R(I^u/JI^{u-1}, \omega_A) \otimes_R \overline{R},\end{aligned}$$

where the second isomorphism holds because I^u/JI^{u-1} is a maximal Cohen–Macaulay A-module.

In the commutative diagram

$$\begin{array}{ccc} I^{t+1-u}/JI^{t-u} \otimes_R \overline{R} & \xrightarrow{(\beta\alpha)\otimes_R \overline{R}} & \operatorname{Hom}_R(I^u/JI^{u-1}, \omega_A) \otimes_R \overline{R} \\ \cong \big\downarrow & & \big\downarrow \cong \\ \overline{I}^{t+1-u}/\overline{JI}^{t-u} & \xrightarrow{\cong} & \operatorname{Hom}_{\overline{R}}(\overline{I}^u/\overline{JI}^{u-1}, \omega_{\overline{A}}) \end{array}$$

we can take the vertical maps and the bottom horizontal map to be the isomorphisms established above. Thus $(\beta\alpha) \otimes_R \overline{R}$ is an isomorphism as required. □

Theorem 2.3. *Under the Standard Hypothesis, Theorem* 4.1 *applies to give an injective map* $\mu_t : I^{t+1}/JI^t \to \omega_{R/K}$. *For* $1 \le u \le t$, *both the multiplication map*

$$m(I, u, t) : I^u/JI^{u-1} \otimes I^{t+1-u}/JI^{t-u} \xrightarrow{\text{mult}} I^{t+1}/JI^t$$

and the composition $\mu_t \circ m(I, u, t)$ *are perfect pairings.*

Proof. The injectivity of μ_t was proven in Theorem 4.1, so it suffices to prove the duality statements. We proceed by induction on $t \ge 0$, the case $t = 0$ being vacuous.

We may assume that the residue field k is infinite. We may harmlessly replace R by $R[\![x]\!]$ and replace I, J by $(I, x), (J, x)$. In this new setting we have $g > 0$. After proving the result in this new setting, the original result is recovered by taking the degree 0 part with respect to x. By Lemma 5.1, we may further assume that $d = s$.

In this case the extra strength of the Strong Hypothesis is vacuous. Thus we may apply Theorem 3.1 to deduce that $I^{t+1}/JI^t \cong \omega_{R/K}$. Further, I/J has finite length, and it follows that the lengths of the modules I^u/JI^{u-1} and $\operatorname{Hom}(I^u/JI^{u-1}, I^{t+1}/JI^t)$ are equal. We will prove that

$$JI^t : I^u \subset JI^{t-u} \tag{3}$$

for all $1 \le u \le t$. It will follow that the map

$$I^{t+1-u}/JI^{t-u} \to \operatorname{Hom}(I^u/JI^{u-1}, I^{t+1}/JI^t)$$

induced by multiplication is injective, and thus

$$\begin{aligned}\operatorname{length}(I^{t+1-u}/JI^{t-u}) &\le \operatorname{length}\operatorname{Hom}(I^u/JI^{u-1}, I^{t+1}/JI^t)\\ &= \operatorname{length}(I^u/JI^{u-1}).\end{aligned}$$

Since this set of inequalities is symmetric under interchanging u and $t+1-u$, it follows that $\operatorname{length}(I^u/JI^{u-1}) = \operatorname{length}(I^{t+1-u}/JI^{t-u})$, and thus the injective map above is an isomorphism.

It remains to prove equation (3). We use Lemma 1.1 with $\mathfrak{a} = J$, and we adopt the notations J_{s-1}, a_s and K_{s-1} from that lemma. We write $\overline{R} = R/K_{s-1}$ and $\overline{I} = I\overline{R}$. By Proposition 3.3 (5), $\overline{I}^u \cong I^u/J_{s-1}I^{u-1}$. By the induction hypothesis, $m(I,u,t-1)$ is a perfect pairing. That is, for $1 \le u \le t-1$ the natural maps

$$\overline{I}^{t-u} \to \operatorname{Hom}_{\overline{R}}(\overline{I}^u, \overline{I}^t)$$

are isomorphisms, and this condition holds also for $u = t$ because $\overline{R}$ is Cohen–Macaulay with canonical module $\overline{I}^t$, by Theorem 3.1 and Proposition 3.3 (5). Recall that $J\overline{R} = a_s\overline{R} \subset \overline{I}$. By Proposition 3.3 (1), a_s is regular on $\overline{R}$. Since the ideal $\overline{I}$ contains a nonzerodivisor, there is a natural isomorphism $\operatorname{Hom}_{\overline{R}}(\overline{I}^u, \overline{I}^t) \cong \overline{I}^t :_{Q(\overline{R})} \overline{I}^u$, where $Q(\overline{R})$ denotes the total ring of quotients of $\overline{R}$. Therefore

$$\overline{I}^t :_{Q(\overline{R})} \overline{I}^u = \overline{I}^{t-u}.$$

Since $J\overline{R}$ is generated by the nonzerodivisor a_s, it follows that

$$(J\overline{I}^t) :_{Q(\overline{R})} \overline{I}^u = (a_s\overline{I}^t) :_{Q(\overline{R})} \overline{I}^u = a_s(\overline{I}^t :_{Q(\overline{R})} \overline{I}^u) = a_s\overline{I}^{t-u}.$$

In particular,

$$(J\overline{I}^t) :_{\overline{R}} \overline{I}^u \subset J\overline{I}^{t-u},$$

and hence

$$(JI^t) :_R I^u \subset JI^{t-u} + K_{s-1}. \tag{4}$$

On the other hand, our assumptions on I imply that locally on the punctured spectrum of R, the associated graded ring $\operatorname{gr}_I(R)$ is Cohen–Macaulay (Theorem 3.1, [17, 3.4], and [16, 6.1]). Since $g > 0$, it follows that locally on the punctured spectrum of R, the irrelevant ideal of $\operatorname{gr}_I(R)$ has positive grade and therefore $I^{t+1} : I^u = I^{t+1-u}$. Since by the Standard Hypothesis the maximal ideal is not an associated prime of R/I^{t+1-u}, we conclude that $I^{t+1} : I^u = I^{t+1-u}$ globally in R. In particular, $JI^t : I^u \subset I^{t+1-u}$, so equation (4) gives

$$JI^t : I^u \subset JI^{t-u} + K_{s-1} \cap I^{t+1-u}.$$

Finally, by Proposition 3.3 (5),

$$K_{s-1} \cap I^{t+1-u} = J_{s-1}I^{t-u},$$

which completes the proof of (3). □

Corollary 2.4. *Under the Strong Hypothesis, the following ring is Gorenstein:*

$$\mathcal{R} := R/K \oplus I/J \oplus I^2/JI \oplus \cdots \oplus I^{t+1}/JI^t = R[Iz]/(K, Jz, (Iz)^{t+2}).$$

Proof. As a graded R-algebra $\mathcal{R}$ is generated in degree 1 and concentrated in degrees $0, \dots, t+1$, so the Gorenstein property is equivalent to the statements:

(1) $\mathcal{R}$ is a maximal Cohen–Macaulay module over $\mathcal{R}_0$.

(2) $\mathcal{R}_{t+1} = \omega_{\mathcal{R}_0}$.

(3) For $u = 1, \dots, t$, the pairings $\mathcal{R}_u \otimes \mathcal{R}_{t+1-u} \to \mathcal{R}_{t+1}$ induced by multiplication are perfect.

Here items (2) and (3) are equivalent to the existence of an isomorphism of graded $\mathcal{R}$-modules

$$\operatorname{Hom}_{\mathcal{R}_0}(\mathcal{R}, \omega_{\mathcal{R}_0})(-t-1) \cong \mathcal{R}.$$

Item (1) follows from Theorem 3.1 and Proposition 3.3 (3). Item (2) follows from Theorem 3.1, while item (3) is the conclusion of Theorem 2.2. □

For the proof of Theorem 2.5 we will use the following general result:

Proposition 5.2. *Let R be a local Cohen–Macaulay ring, let $I \subset R$ be an ideal of positive codimension, and let $t \geq 0$ be an integer. If the truncated Rees ring $\mathcal{R}(I)/\mathcal{R}(I)_{\geq t+2}$ is Gorenstein, then so is the truncated associated graded ring $\operatorname{gr}_I(R)/\operatorname{gr}_I(R)_{\geq t+1}$ and the ring R/I^{t+1}.*

Proof. We may assume that $I \neq R$. Write $d = \dim R$ and set

$$A := \mathcal{R}(I)/\mathcal{R}(I)_{\geq t+2} = R \oplus I \oplus \cdots \oplus I^{t+1},$$
$$B := \operatorname{gr}_I(R)/\operatorname{gr}_I(R)_{\geq t+1} = R/I \oplus I/I^2 \oplus \cdots \oplus I^t/I^{t+1}.$$

Since A is a Cohen–Macaulay ring, finite over R, the ideal I^j is a maximal Cohen–Macaulay module for $j \leq t+1$, and it follows that R/I^j is a Cohen–Macaulay ring of dimension $d-1$ for $j \leq t+1$. From this we see that I^j/I^{j+1} is a maximal Cohen–Macaulay R/I-module for $j \leq t$. Thus B is a Cohen–Macaulay ring of dimension $d-1$.

To prove that the ring B is Gorenstein, we will show that $\omega_B = \operatorname{Ext}^1_R(B, \omega_R)$ is cyclic as a B-module by showing that there is a surjection of A-modules from the cyclic A-module $\omega_A(-1)$ to ω_B. The exact sequence of A-modules

$$0 \to A_{\geq 1} \to A \to R \to 0$$

is split as a sequence of R-modules, so there is a surjection of A-modules

$$\omega_A = \operatorname{Hom}_R(A, \omega_R) \twoheadrightarrow \operatorname{Hom}_R(A_{\geq 1}, \omega_R).$$

On the other hand, from the exact sequence of A-modules

$$0 \to A_{\geq 1}(1) \to A/A_{t+1} \to B \to 0$$

we get a map

$$\operatorname{Hom}_R(A_{\geq 1}, \omega_R)(-1) \to \operatorname{Ext}^1_R(B, \omega_R) = \omega_B$$

that is surjective because A/A_{t+1} is a maximal Cohen–Macaulay R-module.

Finally, since B is Gorenstein and B is the associated graded ring of R/I^{t+1} with respect to the ideal I/I^{t+1}, it follows that R/I^{t+1} is Gorenstein as well. □

Theorem 2.5. *In addition to the Strong Hypothesis, suppose that $K = J : I$ is a geometric s-residual intersection.*

(1) *Let $\overline{I} \subset \overline{R} := R/K$ be the image of I. The truncated Rees algebra*

$$\overline{R} \oplus \overline{I} \oplus \overline{I}^2 \oplus \cdots \oplus \overline{I}^{t+1}$$

is Gorenstein. In particular, $\overline{I}^{t+1} \cong \omega_{\overline{R}}$ and the multiplication maps

$$\overline{I}^u \otimes \overline{I}^{t+1-u} \to \overline{I}^{t+1}$$

are perfect pairings for $0 \le u \le t+1$.

(2) *Let $I' \subset R' := R/(K + I^{t+1})$ be the image of I. The associated graded ring $\mathrm{gr}_{I'}(R')$ is Gorenstein.*

Proof. Recall that $\overline{R}$ is Cohen–Macaulay according to Theorem 3.1. By assumption, the residual intersection is geometric, so $\overline{I}$ has positive codimension in $\overline{R}$ by Proposition 3.2. The truncated Rees algebra $\mathcal{R}(\overline{I})/\mathcal{R}(\overline{I})_{\ge t+2}$ is equal to $R[Iz]/(K, Jz, (Iz)^{t+2})$ by Proposition 3.3 (5). From Corollary 2.4 we see that this ring is Gorenstein. Thus, by Proposition 5.2, the truncated associated graded ring $\mathrm{gr}_{\overline{I}}(\overline{R})/\mathrm{gr}_{\overline{I}}(\overline{R})_{\ge t+1}$ is Gorenstein. Since $R' = \overline{R}/\overline{I}^{t+1}$, the associated graded ring $\mathrm{gr}_{I'}(R')$ is equal to $\mathrm{gr}_{\overline{I}}(\overline{R})/\mathrm{gr}_{\overline{I}}(\overline{R})_{\ge t+1}$, completing the argument. □

Theorem 2.6. *Suppose that (R, I) has a deformation $(\tilde{R}, \tilde{I})$ such that $\tilde{I}$ satisfies the condition G_s and the Koszul homology $H_i(\tilde{I})$ is Cohen–Macaulay for $0 \le i \le t = s - g$. Assume further that I satisfies the condition G_{g+v} for some $\frac{t-1}{2} \le v \le t$. Let $\tilde{J}$ be a lifting of J to an ideal with s generators contained in $\tilde{I}$. The ideal $\tilde{K} = \tilde{J} : \tilde{I}$ is an s-residual intersection of $\tilde{I}$. Our hypothesis implies that Theorem* 4.1 *holds with $\tilde{K}$ in place of K and gives an isomorphism μ_t. The inverse $\phi : \omega_{\tilde{R}/\tilde{K}} \to \tilde{I}^{t+1}/\tilde{J}\tilde{I}^t$ of μ_t induces a map*

$$\phi' : \omega_{R/K} \to I^{t+1}/JI^t.$$

We have:

(1) *ϕ' is a surjection, and is an isomorphism if K is a geometric s-residual intersection.*

(2) *There are perfect pairings*

$$m : I^u/JI^{u-1} \otimes I^{t+1-u}/JI^{t-u} \to \omega_{R/K}$$

for

$$t - v \le u \le v + 1$$

or, equivalently, for

$$\frac{t+1}{2} - \varepsilon \le u \le \frac{t+1}{2} + \varepsilon,$$

where $\varepsilon = v - \frac{t-1}{2}$.

(3) *If the perfect pairing m is chosen as in Figure* 1 *in the proof below, then $\phi' \circ m$ is the map induced by multiplication $I^u \otimes I^{t+1-u} \to I^{t+1}$.*

Proof. We first show that $\tilde{K} = \tilde{J} : \tilde{I}$ is an s-residual intersection of $\tilde{I}$, that is, we have $\operatorname{codim} \tilde{K} \ge s$. To this end, note that $\tilde{K}R \subset K$ and, by [20, 4.1], $K \subset \sqrt{\tilde{K}R}$. Thus we obtain $\operatorname{codim} \tilde{K}R = \operatorname{codim} K \ge s$. Since $\operatorname{codim} \tilde{K} \ge \operatorname{codim} \tilde{K}R$, we see that $\operatorname{codim} \tilde{K} \ge s$ as required.

The ideal $\tilde{I}$ satisfies the Strong Hypothesis, by the discussion in Section 1. If we had assumed that the residue field was infinite, Lemma 1.1 would give the appropriate lower bounds for codimensions of ideals in the assumptions of Theorem 4.1. The lower bounds follow even without an infinite residue field from the references in the proof of Lemma 1.1. On the other hand, the necessary upper bounds follow from Proposition 3.3 (1) and Proposition 3.2. Hence Theorem 4.1 gives an isomorphism $\mu_t : \tilde{I}^{t+1}/\tilde{J}\tilde{I}^t \to \omega_{\tilde{R}/\tilde{K}}$.

Since $\tilde{I}$ satisfies the Strong Hypothesis, we also know from Theorem 3.1 and Proposition 3.2 that $\tilde{R}/\tilde{K}$ is Cohen–Macaulay and $\tilde{K}$ has codimension exactly s. It follows that

$$\operatorname{codim} \tilde{K} = \operatorname{codim} \tilde{K}R = \operatorname{codim} K.$$

Thus $(\tilde{R}, \tilde{K})$ is a deformation of $(R, \tilde{K}R)$, and $\omega_{R/\tilde{K}R} \cong \omega_{\tilde{R}/\tilde{K}} \otimes_{\tilde{R}} R$. From the surjection $R/\tilde{K}R \to R/K$ and the equality of dimensions, we get an inclusion

$$\omega_{R/K} \hookrightarrow \omega_{R/\tilde{K}R} \cong \omega_{\tilde{R}/\tilde{K}} \otimes_{\tilde{R}} R$$

that identifies $\omega_{R/K}$ with the set of elements of $\omega_{R/\tilde{K}R}$ that are annihilated by K. From this inclusion and the isomorphism

$$\phi : \omega_{\tilde{R}/\tilde{K}} \to \tilde{I}^{t+1}/\tilde{J}\tilde{I}^t$$

of Theorem 4.1 we derive a map $\phi' : \omega_{R/K} \to I^{t+1}/JI^t$.

Next we will show that $\tilde{I}^u \otimes_{\tilde{R}} R \cong I^u$ for all $u \le v+1$. Because $(\tilde{R}, \tilde{I})$ is a deformation of (R, I), we may write $R = \tilde{R}/(\underline{x})$, where $\underline{x}$ is a regular sequence on $\tilde{R}$ and on $\tilde{R}/\tilde{I}$. It suffices to show $\underline{x}$ is a regular sequence modulo $\tilde{I}^u$ for u in the given range. Since we know this for $u = 1$, we may do induction on u, and it is enough to show that $\underline{x}$ is a regular sequence on $\tilde{I}^{u-1}/\tilde{I}^u$.

Fix a set of generators of $\tilde{I}$, and their images in I. Using these generators, we define surjective maps from free modules $\tilde{F} \to \tilde{I}$ and $F := \tilde{F} \otimes_{\tilde{R}} R \to I$ and compute Koszul homology modules $\tilde{H}_i := H_i(\tilde{I})$ and $H_i := H_i(I)$.

We now form the approximation complexes (see [15, p. 470])

$$0 \to \tilde{H}_{u-1} \otimes \operatorname{Sym}_0 \tilde{F} \to \cdots \to \tilde{H}_0 \otimes \operatorname{Sym}_{u-1} \tilde{F} \to \tilde{I}^{u-1}/\tilde{I}^u \to 0,$$
$$0 \to H_{u-1} \otimes \operatorname{Sym}_0 F \to \cdots \to H_0 \otimes \operatorname{Sym}_{u-1} F \to I^{u-1}/I^u \to 0.$$

Since $u - 1 \le v \le t$, our hypothesis shows that the modules $\tilde{H}_i$ are either 0 or are maximal Cohen–Macaulay $\tilde{R}/\tilde{I}$-modules whenever $0 \le i \le u-1$. This implies, in the given range, that $\underline{x}$ is a regular sequence on the nonzero $\tilde{H}_i$, that $H_i \cong \tilde{H}_i \otimes_{\tilde{R}} R$, and that the latter modules are Cohen–Macaulay R/I-modules.

Since $u - 1 \le v$, both $\tilde{I}$ and I satisfy G_{g+u-1}, and it follows from [15, the proofs of 2.5 and 2.3] that both approximation complexes are exact. Since $\underline{x}$ is a regular sequence on all the nontrivial $\tilde{H}_i$ that appear, and $H_i \cong \tilde{H}_i \otimes_{\tilde{R}} R$, the exactness of the complexes implies that $\underline{x}$ is a regular sequence on $\tilde{I}^{u-1}/\tilde{I}^u$.

This completes the argument that $\tilde{I}^u \otimes_{\tilde{R}} R \cong I^u$ for all $u \le v+1$. From this isomorphism, we see that

$$\tilde{I}^u/\tilde{J}\tilde{I}^{u-1} \otimes_{\tilde{R}} R \cong I^u/JI^{u-1}.$$

Now let

$$\frac{t+1}{2} - \varepsilon \le u \le \frac{t+1}{2} + \varepsilon.$$

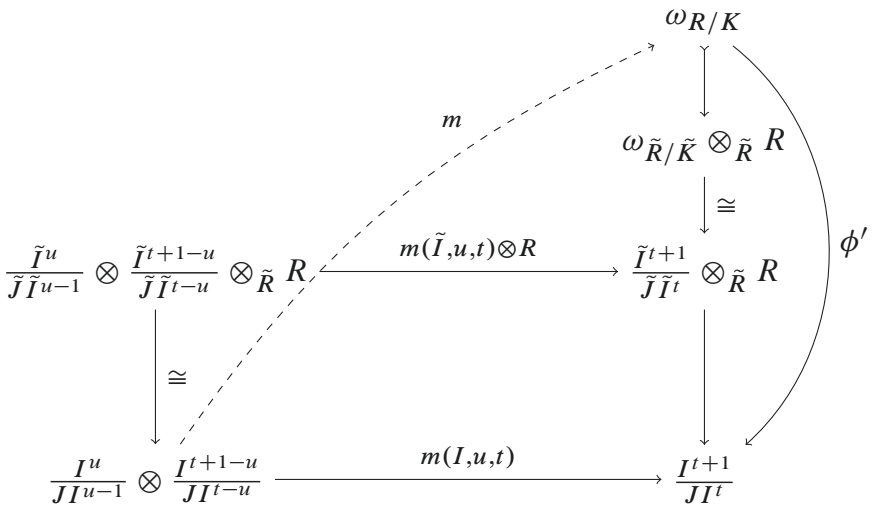

Figure 1. Definition of m.

Note that $u \le v+1$ and $t+1-u \le v+1$ so, by what we have just proven,

$$\tilde{I}^u/\tilde{J}\tilde{I}^{u-1} \otimes_{\tilde{R}} R \cong I^u/JI^{u-1},$$
$$\tilde{I}^{t+1-u}/\tilde{J}\tilde{I}^{t-u} \otimes_{\tilde{R}} R \cong I^{t+1-u}/JI^{t-u}.$$

Theorem 3.1 shows that $\omega_{\tilde{R}/\tilde{K}} \cong \tilde{I}^{t+1}/\tilde{J}\tilde{I}^t$. By the argument at the beginning of this proof, $\omega_{R/K}$ can be identified with the submodule of $\omega_{\tilde{R}/\tilde{K}} \otimes_{\tilde{R}} R$ consisting of all elements annihilated by K. Thus we obtain the commutative diagram of solid arrows as shown in Figure 1. From the left-hand vertical isomorphism we see that the source of the map $m(\tilde{I},u,t) \otimes_{\tilde{R}} R$ is annihilated by K. Hence its image in $\tilde{I}^{t+1}/\tilde{J}\tilde{I}^t \otimes_{\tilde{R}} R \cong \omega_{\tilde{R}/\tilde{K}} \otimes_{\tilde{R}} R$ is contained in $\omega_{R/K}$, yielding a map m indicated by the dotted arrow in the diagram.

By our assumption on v, there exists u with $\frac{t+1}{2} - \varepsilon \le u \le \frac{t+1}{2} + \varepsilon$, and then the surjectivity of $m(I,u,t)$ implies that ϕ' is surjective. To prove that the surjection ϕ' is an isomorphism if K is a geometric s-residual intersection, it suffices to verify that the source and target of ϕ' are isomorphic locally at every associated prime P of the R-module $\omega_{R/K}$. But we have seen before that K has codimension s, hence every such P has codimension s, and therefore cannot contain I. It follows that the source and target of $(\phi')_P$ are both isomorphic to the Gorenstein ring $(R/J)_P$.

To prove that m is a perfect pairing, recall that $m(\tilde{I},u,t)$ is a perfect pairing by Theorem 2.2. According to Theorem 3.1 and Proposition 3.3 (3), the module $\tilde{I}^u/\tilde{J}\tilde{I}^{u-1}$ is a maximal Cohen–Macaulay module over the Cohen–Macaulay ring $\tilde{R}/\tilde{K}$. We proved above that $\underline{x}$ is a regular sequence on $\tilde{R}/\tilde{K}$, so it is also a regular sequence on $\tilde{I}^u/\tilde{J}\tilde{I}^{u-1}$. It follows that

$$\operatorname{Hom}_{\tilde{R}}(\tilde{I}^u/\tilde{J}\tilde{I}^{u-1}, \omega_{\tilde{R}/\tilde{K}}) \otimes_{\tilde{R}} R \cong \operatorname{Hom}_R(\tilde{I}^u/\tilde{J}\tilde{I}^{u-1} \otimes_{\tilde{R}} R, \omega_{\tilde{R}/\tilde{K}} \otimes_{\tilde{R}} R).$$

The right-hand module is isomorphic to

$$\operatorname{Hom}_R(I^u/JI^{u-1}, \omega_{\tilde{R}/\tilde{K}} \otimes_{\tilde{R}} R),$$

and because I^u/JI^{u-1} is annihilated by K, this is isomorphic to

$$\operatorname{Hom}_R(I^u/JI^{u-1}, \omega_{R/K}).$$

Since

$$\operatorname{Hom}_{\tilde{R}}(\tilde{I}^u/\tilde{J}\tilde{I}^{u-1}, \omega_{\tilde{R}/\tilde{K}}) \otimes_{\tilde{R}} R \cong \tilde{I}^{t+1-u}/\tilde{J}\tilde{I}^{t-u} \otimes_{\tilde{R}} R \cong I^{t+1-u}/JI^{t-u},$$

there is a composite isomorphism

$$I^{t+1-u}/JI^{t-u} \xrightarrow{\cong} \operatorname{Hom}_R(I^u/JI^{u-1}, \omega_{R/K}).$$

The commutativity of the diagram in Figure 1 shows that this isomorphism is induced by m, so we are done. □

6. Examples and counterexamples on duality

Residual intersections of codimension 2 ideals.

Example 6.1 (Explicit duality). Let R be a local Gorenstein ring and suppose that C is an $(n+1) \times (n+s)$ matrix with entries in R, where $n \geq 1$ and $s \geq 2$. Suppose that the maximal minors of C generate an ideal K of codimension s, the generic value. Set $t := s-2$ and $M := \operatorname{coker} C$. Buchsbaum and Eisenbud [4] (see also [9, Appendix A.2.6]) computed minimal free R-resolutions of the first $t+1$ symmetric powers of M, and observed that, for $0 \leq u \leq t+1$, these are perfect R-modules of codimension s, and that the resolutions of $\operatorname{Sym}_u(M)$ and $\operatorname{Sym}_{t+1-u}(M)$ are dual to one another; that is,

$$\operatorname{Sym}_{t+1-u}(M) \cong \operatorname{Ext}^s_R(\operatorname{Sym}_u(M), R) \cong \operatorname{Hom}_R(\operatorname{Sym}_u(M), \omega_{R/K}).$$

If we assume that the entries of C are in the maximal ideal and the residue field of R is infinite, then, possibly after column operations, we may suppose the $(n+1) \times n$ submatrix A consisting of the first n columns of C has the property that the $n \times n$ minors of A generate an ideal I of codimension 2. (Reason: Since K has codimension s, we see that $N := \operatorname{coker}(C^*)$ is locally free of rank $s-1$ in codimension $< s$ in R. It follows from the theory of basic elements that after factoring out s general generators of N we obtain a module of codimension ≥ 2. This is the module presented by A^*.)

In this situation, the ideal I is strongly Cohen–Macaulay. Huneke [19] showed that K is an s-residual intersection of I, see also Theorem 6.2. In [7] the duality statement above is generalized to residual intersections of any strongly Cohen–Macaulay ideal.

In addition, Andy Kustin and the second author observed (unpublished) that for geometric residual intersections of codimension 2 perfect ideals, the symmetric power $\operatorname{Sym}_u(I/J)$ is isomorphic to I^u/JI^{u-1} in the range of u that we consider, and we reprove this in Theorem 6.2 below. This gives a concrete example of our theory.

Let B be the $(n+1) \times s$ matrix made from the last s columns of C, so that $C = (A|B)$. Let J be the image of the composite map

$$R^s \xrightarrow{B} R^{n+1} \cong \bigwedge^n R^{n+1*} \xrightarrow{\bigwedge^n A^*} \bigwedge^n R^{n*} \cong R.$$

By the Hilbert–Burch Theorem,

$$\operatorname{coker} A \cong \operatorname{image} \bigwedge^n A^* \cong I,$$

and thus

$$M = \operatorname{coker}(A|B) \cong I/J.$$

Theorem 6.2. *With notation and hypotheses as above, $K = J : I$ and thus K is an s-residual intersection of I. Let $\overline{I}$ be the image of I in the ring $\overline{R} := R/K$. If $I + K$ has codimension $\geq s + 1$ (so that K is a geometric s-residual intersection of I), then*

$$I^u/JI^{u-1} \cong \overline{I}^u \cong \mathrm{Sym}_u(\mathrm{coker}\, C)$$

for $0 \leq u \leq t + 1$ (interpreting I^u/JI^{u-1} as $\overline{R}$ when $u = 0$). In particular, I^u/JI^{u-1} and I^{t+1-u}/JI^{t-u} have dual, finite free R-resolutions.

Note that this does not require the condition G_s.

Proof. By assumption, the codimension of the ideal K of $(n + 1) \times (n + 1)$ minors of C is s, so by [5], $\mathrm{ann}(\mathrm{coker}\, C) = K$. But $\mathrm{ann}(\mathrm{coker}\, C) = \mathrm{ann}(I/J) = J : I$.

There are natural surjections

$$\mathrm{Sym}_u(I/J) \twoheadrightarrow I^u/JI^{u-1} \twoheadrightarrow \overline{I}^u.$$

Recall that the determinantal ideal K is perfect of codimension s. Thus, if K is a geometric s-residual intersection, then $\overline{I}^u$ has grade ≥ 1, and both $\overline{I}^u$ and $\mathrm{Sym}_u(I/J)$ are locally free of rank 1 at the associated primes of $\overline{R}$. Since $\mathrm{Sym}_u(I/J)$ is a maximal Cohen–Macaulay $\overline{R}$-module, it is torsion free, and thus the two epimorphisms are isomorphisms. □

There are two kinds of hypotheses on the ideal I in Theorem 2.6: the condition G_{g+v} on I itself and the existence of a good deformation $\tilde{I}$. We will show in Example 6.3 that the first cannot be weakened and, in Examples 6.5 and 6.6, that the second cannot be dropped. Here we write $w := g + v$.

Example 6.3 (A codimension 2 perfect ideal satisfying G_{w-1} but not G_w). The following examples show that, even for licci ideals, the condition G_w in Theorem 2.6 cannot be replaced by the condition G_{w-1}. They are based on the construction explained in Example 6.1.

By the Hilbert–Burch Theorem, any perfect codimension 2 ideal I with $n + 1$ generators is the ideal of $n \times n$ minors of an $(n + 1) \times n$ matrix. Such ideals satisfy the deformation assumption: they are specializations of the generic ideal of minors, which satisfies the condition G_s for every s, and all their Koszul homology modules are Cohen–Macaulay ([2]). (These are the original examples of the licci ideals mentioned in the introduction.)

Let $2 \leq w \leq s$, let R be a power series ring $k[\![x_1, \ldots, x_s]\!]$, and let M_s be the $s \times (2s - 1)$ "Macaulay matrix", where the i-th principal diagonal entries are x_i and the other entries are 0 (we illustrate with the case $s = 5$):

$$M_s = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & 0 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & x_3 & x_4 & x_5 & 0 & 0 & 0 \\ 0 & 0 & x_1 & x_2 & x_3 & x_4 & x_5 & 0 & 0 \\ 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 & x_5 & 0 \\ 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}.$$

We define the ideal $I_{s,w}$ to be the ideal of $(s - 1) \times (s - 1)$ minors of the matrix $N_{s,w}$ made from columns 2 through s of M_s by replacing the entry x_w of the $(s - w + 1)$ row with 0; for

example $I_{5,3}$ is the ideal of 4×4 minors of

$$N_{5,3} = \begin{pmatrix} x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 \\ 0 & x_1 & x_2 & 0 \\ 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & x_1 \end{pmatrix}.$$

It is easy to see that $I_{s,w}$ is a perfect ideal of codimension 2, and by computing the codimensions of the ideals of lower order minors of $N_{s,w}$ one sees that $I_{s,w}$ satisfies G_{w-1} but not G_w.

We consider the cases $4 \leq s \leq 7$, and we construct an s-residual intersection

$$K_{s,w} = J_{s,w} : I_{s,w}$$

of $I_{s,w}$ as follows:

Let $M'_{s,w}$ be the matrix constructed from M_s by replacing columns 2 up to s with the matrix $N_{s,w}$, and adding the variable x_w to the entries in the $(s - w + 1)$ row and both the first and $(2s - w + 1)$ columns. Thus

$$M'_{5,3} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & 0 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & x_3 & x_4 & x_5 & 0 & 0 & 0 \\ x_3 & 0 & x_1 & x_2 & 0 & x_4 & x_5 & x_3 & 0 \\ 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 & x_5 & 0 \\ 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}.$$

Macaulay2 computations show that for $s \leq 7$ and any $2 \leq w \leq s$, the ideal $K_{s,w}$ generated by the maximal minors of $M'_{s,w}$ has the generic codimension, s, and we conjecture that this is true in general.

Assuming that $K_{s,w}$ has codimension s, we can use Theorem 6.2 to show that $K_{s,w}$ is an s-residual intersection of $I_{s,w}$. Explicitly, let $P_{s,w}$ be the submatrix of $M'_{s,w}$ consisting of the columns not in $N_{s,w}$; for example if $s = 5$, $w = 3$, then

$$P_{s,w} = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 \\ 0 & x_5 & 0 & 0 & 0 \\ x_3 & x_4 & x_5 & x_3 & 0 \\ 0 & x_3 & x_4 & x_5 & 0 \\ 0 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}.$$

After rearranging the columns of $M'_{s,w}$ we may write $M'_{s,w} = (N_{s,w} | P_{s,w})$. Thus we may apply Theorem 6.2 to conclude that, if we take $J_{s,w}$ to be the ideal generated by the entries of the matrix obtained by multiplying $P_{s,w}$ by the row of signed maximal minors of $N_{s,w}$, we will have $K_{s,w} = J_{s,w} : I_{s,w}$, an s-residual intersection of $I_{s,w}$. For example $J_{5,3}$ is generated by the entries of the row vector

$$\begin{pmatrix} \Delta_1 & -\Delta_2 & \Delta_3 & -\Delta_4 & \Delta_5 \end{pmatrix} \cdot P_{5,3},$$

where Δ_i is the determinant of the matrix obtained from $N_{5,3}$ by omitting the i-th row.

We now consider Theorem 2.6 in the cases of the ideals $I_{s,w}$ and $J_{s,w}$ with $4 \le s \le 7$. We have $g = 2$, and we consider values of v in the range specified in the theorem, so that $w = g + v = 2 + v$ lies in the range $\frac{s+1}{2} \le w \le s$. As explained above, the ideal $I_{s,w}$ satisfies the deformation hypothesis of the theorem, and satisfies $G_{g+v-1} = G_{w-1}$ but not G_{g+v}. This has the effect of adding 1 to the lower bound, and subtracting 1 from the upper bound, for u in Theorem 2.6. Thus for the triples (s, w, u) in the list

$$(4,3,1), (5,3,2), (5,4,1), (6,4,2), (6,5,1), (7,4,3), (7,5,2), (7,6,1),$$

the theorem does not guarantee duality. Of course the same goes for the "dual" triples $(s, w, s-1-u)$.

Computations in Macaulay2 show that, indeed, duality does not hold in these cases. To check this, we compute resolutions of I^u/JI^{u-1} and I^{s-1-u}/JI^{s-2-u}. When the total Betti numbers in the minimal resolutions over R of these two modules are not dual to one another, the duality clearly does not hold. It turns out that this occurs in each case. (We note that in other cases, where these have the same graded Betti numbers as in the generic case, they must be reductions from the generic case, and thus dual to one another.)

Consider, as an example, the case $(s, w, u) = (5, 4, 1)$: According to Macaulay2, the Betti table of the minimal graded free resolution of $I^u/JI^{u-1} = I/J$ is

```
 total: 5 9 84 180 135 35
     4: 5 9  .   .   .  .
     5: . .  .   .   .  .
     6: . .  .   .   .  .
     7: . .  .   .   .  .
     8: . . 84 180 135 35
```

while the Betti table of the minimal graded free resolution of $I^{s-1-u}/JI^{s-2-u} = I^3/JI^2$ is

```
total: 35 136 188 106 28 9
   12: 35 136 183  87  1 .
   13:  .   .   .   .  . .
   14:  .   .   .   .  . .
   15:  .   .   5  19 27 8
   16:  .   .   .   .  . 1
```

By local duality, the dual, $\mathrm{Hom}(I/J, \omega_{R/(J:I)})$ of I/J is isomorphic, up to a shift in grading, to $\mathrm{Ext}^5_R(I/J, R)$. From the first resolution we see that the presentation of this module (as a graded module or over the power series ring) has 35 generators and 135 relations, whereas from the second Betti table we see that the minimal presentation of I^3/JI^2 has 35 generators and 136 relations; thus I/J is not dual to I^3/JI^2.

Example 6.4 (Duality not given by multiplication). Let $R = k[\![x, y, z]\!] \supset I = (x, y)^2$, where k is an infinite field. The pair (R, I) admits a deformation $(\tilde{R}, \tilde{I})$, where

$$\tilde{R} = k[\![z_{1,1}, \ldots, z_{2,3}, z]\!],$$

the ideal $\tilde{I}$ is generated by the 2×2 minors of the generic matrix

$$Z := \begin{pmatrix} z_{1,1} & z_{1,2} & z_{1,3} \\ z_{2,1} & z_{2,2} & z_{2,3} \end{pmatrix},$$

and the specialization $\tilde{R} \to R$ sends Z to the matrix

$$\begin{pmatrix} x & y & 0 \\ 0 & x & y \end{pmatrix}.$$

If J is generated by three sufficiently general homogeneous polynomials of degree 3 in I, then the ideal $K = J : I$ is a 3-residual intersection, so by Theorem 2.1 or Theorem 2.6, I/J is self-dual.

Computation shows that $K = (x, y, z)^3$. Thus $\omega_{R/K} = \mathrm{Hom}_k(R/K, k)$ has Hilbert function $6, 3, 1$. The surjection $\phi' : \omega_{R/K} \to I^2/JI$ described in Theorem 2.6 is, in this case, the dual of the inclusion $(x, y)(R/K) \hookrightarrow R/K$. Thus the Hilbert function of I^2/JI is $5, 2$. We see that, unlike in Theorem 2.2, there is no injection $I^2/JI \to \omega_{R/K}$ because the socle of the first module is 2-dimensional.

We also claim that, unlike in the situation of Theorem 2.2, the self-duality map of I/J is not given by multiplication. Indeed, there can be no perfect pairing $I/J \otimes I/J \to I^2/JI$ because the target is annihilated by $(x, y, z)^2$ while I/J is not.

By Theorem 2.6, there is a duality map $I/J \otimes I/J \to \omega_{R/K}$, and the multiplication map $I/J \otimes I/J \to I^2/JI$ is the composite of this map with the surjection $\phi' : \omega_{R/K} \to I^2/JI$ described in the same Theorem. Moreover, the duality map is a symmetric surjection, induced by the corresponding duality map in the generic case. Thus $R/K \oplus I/J \oplus \omega_{R/K}$ is a commutative standard graded Gorenstein algebra over R/K and $R/K \oplus I/J \oplus I^2/JI$ is a proper homomorphic image.

It is shown in [7] that, for residual intersections of strongly Cohen–Macaulay ideals, such as the one in this example, the duality between symmetric powers is always induced by multiplication.

Residual intersections of codimension 3 ideals. Even when I itself satisfies the condition G_s, the conclusion of Theorem 2.6 may fail if I does not have a deformation whose Koszul homology modules are Cohen–Macaulay.

Example 6.5 (No surjection $\omega_{R/K} \twoheadrightarrow I^2/JI$). Let $R = k[\![x_1, \ldots, x_5]\!]$, where k is an infinite field, and let I be the ideal of 2×2 minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{pmatrix}.$$

If we take J to be the ideal generated by four sufficiently general cubic forms in I, then by Theorem 2.2, the multiplication map $I/J \otimes I/J \to I^2/JI$ is a perfect pairing. We claim that, unlike in the situation of Theorem 2.6, there is *no* surjection $\omega_{R/K} \twoheadrightarrow I^2/JI$: computation shows that I^2/JI requires twenty generators, whereas $\omega_{R/K}$ requires only sixteen. Of course by Theorem 2.2, there is a natural injection $I^2/JI \hookrightarrow \omega_{R/K}$ such that the composite pairing is also a perfect pairing. However, unlike the situation in [7], the multiplication $I/J \otimes I/J \to \mathrm{Sym}_2(I/J)$ is not a perfect pairing.

Could there be some "mystery module" X and maps

$$\omega_{R/K} \longleftarrow X \longrightarrow I^2/JI$$

that explains both Examples 6.4 and 6.5?

Example 6.6 (No perfect pairing). Let $s = 5$ and take I to be the ideal of the nondegenerate rational quartic curve in $\mathbb{P}^4$ or of the Veronese surface in $\mathbb{P}^5$ that is, the ideal of 2×2 minors of either

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}.$$

These ideals satisfy G_5 and admit a 5-residual intersection $K = J : I$, where J is generated by five general cubic forms in I. For each of the two ideals I above, all Koszul homology modules are Cohen–Macaulay except the first, and they satisfy the sliding depth condition for Koszul homology. Nevertheless, Macaulay2 computation shows that the modules I/J and I^2/JI are not dual to one another.

Computation shows that there is no useful duality among the first three symmetric powers either: $\mathrm{Sym}_3(I/J) \ncong \omega_{R/K}$,

$$\mathrm{Hom}(\mathrm{Sym}_2(I/J), \omega_{R/K}) \ncong I/J, \quad \mathrm{Hom}(I/J, \omega_{R/K}) \ncong \mathrm{Sym}_2(I/J),$$

and likewise for dualizing into $\mathrm{Sym}_3(I/J)$.

Residual intersections of a codimension 5 ideal.

Example 6.7. Let $R = k[\![x_1, \dots, x_{10}, y_1, \dots, y_5]\!]$, where k is an infinite field, and let I be the ideal generated by the 4×4 Pfaffians of the generic skew symmetric matrix

$$M = \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 \\ -x_1 & 0 & x_5 & x_6 & x_7 \\ -x_2 & -x_5 & 0 & x_8 & x_9 \\ -x_3 & -x_6 & -x_8 & 0 & x_{10} \\ -x_4 & -x_7 & -x_9 & -x_{10} & 0 \end{pmatrix}$$

together with the entries of the vector

$$\begin{pmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \end{pmatrix} M.$$

This is a prime ideal of codimension 5, and is a complete intersection locally on the punctured spectrum, so I satisfied G_{15}. Mark Johnson found this ideal as an example where each of $R/I, R/I^2$ and R/I^3 is Cohen–Macaulay (and thus of depth 10), while R/I^4 has depth 6. The ideal I thus satisfies the Strong Hypothesis with $s = 7$, but not $s = 8$.

Let $J \subset I$ be generated by seven general quadrics in I, and let $K = J : I$. As in the previous example, computation shows that there is no useful duality among the first three symmetric powers: $\mathrm{Sym}_3(I/J) \ncong \omega_{R/K}$,

$$\mathrm{Hom}(\mathrm{Sym}_2(I/J), \omega_{R/K}) \ncong I/J, \quad \mathrm{Hom}(I/J, \omega_{R/K}) \ncong \mathrm{Sym}_2(I/J),$$

and likewise for dualizing into $\mathrm{Sym}_3(I/J)$.

However, by Theorems 2.2 and 3.1, the multiplication map does give a perfect pairing

$$I/J \otimes I^2/JI \to I^3/JI^2 \cong \omega_{R/K}.$$

7. Complementary module and socle

We begin by reminding the reader of the classic description of the socle of a complete intersection of equicharacteristic 0. Recall that if k is a field and R is a complete local k-algebra, then the *Kähler different* $\mathfrak{D}_K(R/k) \subset R$ is the 0-th Fitting ideal of the universally finite module of differentials $\Omega_{R/k}$; for example, if $R = k[\![x_1, \dots, x_d]\!]/(a_1, \dots, a_d)$, then $\mathfrak{D}_K(R/k)$ is the ideal generated by the Jacobian determinant

$$\Delta = \det \begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \cdots & \frac{\partial a_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_d}{\partial x_1} & \cdots & \frac{\partial a_d}{\partial x_d} \end{pmatrix}.$$

Theorem 7.1. *If k is a field of characteristic 0 and R is a complete local k-algebra, then $\mathfrak{D}_K(R/k)$ is nonzero if and only if R is a 0-dimensional complete intersection, and in this case $\mathfrak{D}_K(R/k)$ is the socle of R.*

This result was proven by Scheja and Storch [29] (see also Kunz [21]). The basic ideas are due to Tate [25, Appendix]. For the reader's convenience we give the classic arguments in Appendix A.

Throughout this section we suppose that R is a local Gorenstein ring of dimension d with maximal ideal $\mathfrak{m}$, that $I \subset R$ is an ideal of codimension g, and $K = J : I$ is an s-residual intersection, and we set $t = s - g$. If T is any ring we write $Q(T)$ for the total ring of quotients obtained by inverting every nonzerodivisor in T.

We want to identify the socle of $\omega_{R/K}$ in the case $\dim R/K = d - s = 0$. We will show that, under suitable hypotheses, the socle of $\omega_{R/K} \cong I^{t+1}/JI^t$ is generated by the image of the Jacobian determinant of generators of J (Theorems 7.8 and 7.10).

We begin with the following general result about the socle of the local cohomology module $\mathrm{H}^{d-s}_{\mathfrak{m}}(R/JI^t)$.

Theorem 7.2. *If I satisfies the Strong Hypothesis with respect to s, then $\mathrm{H}^{d-s}_{\mathfrak{m}}(R/JI^t)$ has a simple socle and the natural map*

$$\mathrm{H}^{d-s}_{\mathfrak{m}}(\omega_{R/K}) \cong \mathrm{H}^{d-s}_{\mathfrak{m}}(I^{t+1}/JI^t) \to \mathrm{H}^{d-s}_{\mathfrak{m}}(R/JI^t)$$

is injective. In particular, the two modules have the same socle.

Proof. Recall that $I^{t+1}/JI^t \cong \omega_{R/K}$ has dimension $d - s$ by Theorem 3.1. Hence $\mathrm{H}^{d-s}_{\mathfrak{m}}(I^{t+1}/JI^t) \neq 0$. This module embeds into $\mathrm{H}^{d-s}_{\mathfrak{m}}(R/JI^t)$ since depth $R/I^{t+1} \geq d - s$. Thus it remains to show that the socle of $\mathrm{H}^{d-s}_{\mathfrak{m}}(R/JI^t)$ is simple.

If $t = 0$, the result is the usual duality for complete intersections, so we assume that $t > 0$. We may harmlessly suppose that k is infinite and that the generators $a_1, \dots, a_s$ of J are general. Set $J_i = (a_1, \dots, a_i)$ and $K_i = J_i : I$. By Lemma 1.1, the ideal $J_i : I$ is a geometric i-residual intersection for $g \leq i \leq s - 1$.

From Proposition 3.3 (2) we have an exact sequence

$$0 \longrightarrow \frac{I^t}{J_{s-1}I^{t-1}} \xrightarrow{a_s} \frac{R}{J_{s-1}I^t} \longrightarrow \frac{R}{J_sI^t} \longrightarrow 0.$$

The module in the middle has depth at least $d - s + 1$ according to Proposition 3.3 (4). Hence

the long exact sequence of local cohomology gives an embedding

$$\mathrm{H}^{d-s}_{\mathfrak{m}}(R/JI^t) \subset \mathrm{H}^{d-s+1}_{\mathfrak{m}}(I^t/J_{s-1}I^{t-1}).$$

Now the theorem follows because by Theorem 3.1,

$$I^t/J_{s-1}I^{t-1} \cong \omega_{R/K_{s-1}}$$

and R/K_{s-1} is Cohen–Macaulay, thus $\mathrm{H}^{d-s+1}_{\mathfrak{m}}(I^t/J_{s-1}I^{t-1})$ has simple socle. □

If k is a field of characteristic 0 and T is a local finite-dimensional k-algebra, then the trace homomorphism $\mathrm{Tr}_{T/k}$ is nonzero and annihilates the maximal ideal, since the maximal ideal consists of nilpotent elements. Thus $\mathrm{Tr}_{T/k}$ generates the socle of $\omega_T = \mathrm{Hom}_k(T,k)$.

From this point on we will assume that $R = k[\![x_1,\dots,x_d]\!]$ is a power series ring in d variables over a field k of characteristic 0. To identify the socle of I^{t+1}/JI^t with the Jacobian determinant Δ of a given set of generators of J in the case $s = d$ (Theorem 7.8), we begin by making explicit the composite isomorphism

$$I^{t+1}/JI^t \cong \omega_{R/K} \cong \mathrm{Hom}_k(R/K,k).$$

We show that if Δ is in I^{t+1}, then, under this isomorphism, Δ corresponds to the trace homomorphism $\mathrm{Tr}_{(R/K)/k}$. This is accomplished in Theorem 7.4. In order to do this, we establish a result about the Dedekind complementary module of R/K that requires R/K to be reduced, and holds for $s < d$.

Lemma 7.3. *Let k be a field of characteristic* 0 *and $R = k[\![x_1,\dots,x_d]\!]$. Let $T = R/L$ be a reduced Cohen–Macaulay factor ring of R with* $\operatorname{codim} L = s-1$. *Let $a_1,\dots,a_{s-1}$ be elements of L that generate L generically, and let $a_s \in \mathfrak{m}$ be a nonzerodivisor modulo L. Let $x_1,\dots,x_d$ be general variables of R. Set*

$$\begin{aligned} A_i &= k[\![x_i,\dots,x_d]\!],\\ A' &= k[\![a_s,x_{s+1},\dots,x_d]\!],\\ \Delta_i &= \det\begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \cdots & \frac{\partial a_1}{\partial x_i}\\ \vdots & \ddots & \vdots\\ \frac{\partial a_i}{\partial x_1} & \cdots & \frac{\partial a_i}{\partial x_i}\end{pmatrix}. \end{aligned}$$

We have:

(1) $\Omega_{Q(T)/A_{s+1}} := Q(T)\otimes_T \Omega_{T/A_{s+1}}$ *is a free $Q(T)$-module of rank 1 generated by dx_s. For any differential form df we write $\frac{df}{dx_s}$ for the ratio as elements of $\Omega_{Q(T)/A_{s+1}}$.*

(2) $\Delta_s = \frac{da_s}{dx_s}\Delta_{s-1}$.

(3) $\mathfrak{C}(T/A_s) = \frac{da_s}{dx_s}\mathfrak{C}(T/A')$.

Proof. The $Q(T)$-module $\Omega_{Q(T)/A_{s+1}}$ is presented by the transpose of the $(s-1)\times s$ matrix

$$\Theta = \begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \cdots & \frac{\partial a_1}{\partial x_s}\\ \vdots & \ddots & \vdots\\ \frac{\partial a_{s-1}}{\partial x_1} & \cdots & \frac{\partial a_{s-1}}{\partial x_s}\end{pmatrix}.$$

We write Θ_i for $(-1)^i$ times the determinant of the $(s-1)\times(s-1)$ submatrix of Θ omitting the i-th column. Note that $\Delta_{s-1} = (-1)^s\Theta_s$.

(1) Because $x_1,\dots,x_d$ are general, the ring $A_s = k[\![x_s,\dots,x_d]\!]$ is a Noether normalization of the reduced, equidimensional ring T, so $\Omega_{Q(T)/k} := Q(T)\otimes_T \Omega_{T/k}$ is free of rank $d-s+1$ with basis $dx_s,\dots,dx_d$. Thus $\Omega_{Q(T)/A_{s+1}}$ is free of rank 1 with basis dx_s as claimed.

(2) It follows that Θ has rank $s-1$. Moreover, the vector

$$\begin{pmatrix} \frac{dx_1}{dx_s} \\ \vdots \\ \frac{dx_s}{dx_s} \end{pmatrix}$$

is in $\ker\Theta$. Of course, Θ also annihilates the vector

$$\begin{pmatrix} \Theta_1 \\ \vdots \\ \Theta_s \end{pmatrix},$$

and so, because the entries of either vector generate the unit ideal in $Q(T)$,

$$\begin{pmatrix} \Theta_1 \\ \vdots \\ \Theta_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{dx_1}{dx_s} \\ \vdots \\ \frac{dx_s}{dx_s} \end{pmatrix}$$

are proportional. Since $\frac{dx_s}{dx_s} = 1$, we get

$$\Theta_i = \Theta_s \frac{dx_i}{dx_s}$$

for $i = 1,\dots,s$. By the chain rule,

$$\frac{da_s}{dx_s} = \sum_{i=1}^{s} \frac{\partial a_s}{\partial x_i}\frac{dx_i}{dx_s},$$

so

$$\Theta_s \frac{da_s}{dx_s} = \sum_{i=1}^{s} \frac{\partial a_s}{\partial x_i}\Theta_i$$

in $Q(T)$. Expanding Δ_s along the last row we get

$$\Delta_s = (-1)^s \sum_{i=1}^{s} \Theta_i \frac{\partial a_s}{\partial x_i} = (-1)^s\Theta_s\frac{da_s}{dx_s} = \Delta_{s-1}\frac{da_s}{dx_s}$$

as required.

(3) Since $a_s \in \mathfrak{m}$ is regular on T and $x_1,\dots,x_d$ are general, the ring A' is another Noether normalization of T. By [22, 9.2],

$$\mathfrak{C}(T/A_s)\, dx_s \wedge\cdots\wedge dx_d = \mathfrak{C}(T/A')\, da_s \wedge\cdots\wedge dx_d$$

in $\bigwedge^{d-s+1}\Omega_{Q(T)/k}$. Since $\Omega_{Q(T)/k}$ is a free $Q(T)$-module with basis $dx_s,\dots,dx_d$, we have

$$da_s \wedge dx_{s+1}\wedge\cdots\wedge dx_d = \frac{da_s}{dx_s}\, dx_s \wedge dx_{s+1}\wedge\cdots\wedge dx_d$$

as desired. □

The next theorem is one of our main results. It gives an explicit description of the complementary module of residual intersections.

Theorem 7.4. *Let k be a field of characteristic 0, and let $R = k[\![x_1, \dots, x_d]\!]$. Let $I \subset R$ be an ideal satisfying the Strong Hypothesis for some $s < d$, and let $J : I$ be a geometric s-residual intersection such that $\overline{R} = R/(J : I)$ is reduced. Let $a_1, \dots, a_s$ be general elements in J. Let $x_1, \dots, x_d$ be general variables in R and write $A = k[\![x_{s+1}, \dots, x_d]\!]$. We have*

$$I^{t+1}\overline{R} = \Delta\, \mathfrak{C}(\overline{R}/A),$$

where Δ is the Jacobian determinant

$$\Delta = \det \begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \cdots & \frac{\partial a_1}{\partial x_s} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_s}{\partial x_1} & \cdots & \frac{\partial a_s}{\partial x_s} \end{pmatrix}.$$

Proof. Since R is a domain we must have $g > 0$. For $i \geq g - 1$ we set

$$\begin{aligned} J_i &:= (a_1, \dots, a_i) \subset I, \\ R_i &:= R/(J_i : I), \\ A_i &:= k[\![x_{i+1}, \dots, x_d]\!], \\ A_i' &:= k[\![a_{i+1}, x_{i+2}, \dots, x_d]\!], \end{aligned}$$

and

$$\Delta_i := \det \begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \cdots & \frac{\partial a_1}{\partial x_i} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_i}{\partial x_1} & \cdots & \frac{\partial a_i}{\partial x_i} \end{pmatrix}.$$

If $i = g - 1$, then $J_i : I = J_i$ is generated by the regular sequence $a_1, \dots, a_i$. If $i \geq g$, then by Lemma 1.1, the ideal $J_i : I$ is a geometric i-residual intersection. By Theorem 3.1, the ring R_i is Cohen–Macaulay of dimension $d - i$. It follows that the geometric i-residual intersection $J_i : I$ is generically generated by $a_1, \dots, a_i$. Moreover, by Proposition 3.3 (1), the element a_{i+1} is regular on R_i for $i \leq s - 1$. Proposition 3.4 shows that the ring R_i is reduced. Finally, Theorem 3.1 and Proposition 3.3 (5) give $\omega_{R_i} \cong I^{i-g+1} R_i$ for any i.

Since the $x_1, \dots, x_d$ are general, the ring A_i is a Noether normalization of R_i. Since $a_{i+1} \in \mathfrak{m}$ is a nonzerodivisor on R_i, the ring A_i' is also a Noether normalization of R_i.

By induction on $i = g - 1, \dots, s$, we prove that

$$I^{i-g+1} R_i = \Delta_i \mathfrak{C}(R_i/A_i).$$

The case $i = s$ is the statement of the theorem.

If $i = g - 1$, the assertion is that

$$R_i = \Delta_i \mathfrak{C}(R_i/A_i),$$

or equivalently that $\mathfrak{C}(R_i/A_i) = \Delta_i^{-1} R_i$. This is classically known since $R_i = R/(a_1, \dots, a_i)$ and $a_1, \dots, a_i$ is a regular sequence; we will give a self-contained proof of this fact in Section A, see Corollary A.4.

Now take $i \geq g$ and assume the result is known for $i-1$. Consider the following diagram that will be explained below:

$$\begin{array}{ccccccc}
I^{i-g}R_{i-1} & \overset{\text{ind. hyp.}}{=\!=} & \Delta_{i-1}\mathfrak{C}(R_{i-1}/A_{i-1}) & \overset{7.3}{=\!=} & \Delta_i\mathfrak{C}(R_{i-1}/A'_{i-1}) & \xrightarrow{\cong} & \Delta_i\mathfrak{C}(R_{i-1}/A'_{i-1})\operatorname{Tr}_{R_{i-1}/A'_{i-1}} \\
 & & & & & & \| \\
 & & & & & & \Delta_i \operatorname{Hom}_{A'_{i-1}}(R_{i-1}, A'_{i-1}) \\
\text{mod } a_i & & & & & & \uparrow \cong \\
 & & & & & & \operatorname{Hom}_{A'_{i-1}}(R_{i-1}, A'_{i-1}) \\
 & & & & & & \downarrow \text{mod } a_i \\
I^{i-g}R_{i-1}/a_i I^{i-g}R_{i-1} & & & \overset{\phi}{\dashrightarrow} & & & \operatorname{Hom}_{A_i}(R_{i-1}/(a_i), A_i) \\
\uparrow & & & & & & \uparrow \varepsilon \\
 & & & & & & \operatorname{Hom}_{A_i}(R_i, A_i) \\
 & & & & & & \downarrow \cong \\
I^{i-g+1}R_i & \overset{\psi}{\dashrightarrow} & \Delta_i\mathfrak{C}(R_i/A_i) & \xrightarrow{\cong} & \Delta_i\mathfrak{C}(R_i/A_i)\operatorname{Tr}_{R_i/A_i} & =\!= & \Delta_i \operatorname{Hom}_{A_i}(R_i, A_i).
\end{array}$$

By the induction hypothesis and Lemma 7.3, we have

$$I^{i-g}R_{i-1} = \Delta_{i-1}\mathfrak{C}(R_{i-1}/A_{i-1}) = \Delta_i\mathfrak{C}(R_{i-1}/A'_{i-1}).$$

By Proposition 3.3 (1), the ideal $I^{i-g}R_{i-1}$ has positive grade, so Δ_i is a nonzerodivisor in R_{i-1}. The arrow marked mod a_i on the right in the diagram is surjective because R_{i-1} is a free A'_{i-1}-module.

The isomorphism ϕ is induced by the first row. In the inclusion of $I^{i-g+1}R_i \cong \omega_{R_i}$ in $I^{i-g}R_{i-1}/a_i I^{i-g}R_{i-1} \cong \omega_{R_{i-1}/(a_i)}$ the first module is the annihilator of

$$L := \ker(R_{i-1}/(a_i) \twoheadrightarrow R_i);$$

see Corollary 4.3. Similarly, we take

$$\varepsilon : \operatorname{Hom}_{A_i}(R_i, A_i) \hookrightarrow \operatorname{Hom}_{A_i}(R_{i-1}/(a_i), A_i)$$

to be the map induced by the surjection $R_{i-1}/(a_i) \twoheadrightarrow R_i$, so the source of ε is the annihilator of L in the target of ε. Since R_i is generically a finite separable extension of A_i defined by the vanishing of $a_1, \ldots, a_i$, the element Δ_i is a nonzerodivisor of R_i. Thus ϕ induces an isomorphism ψ in the diagram.

We next will show that, regarded as a map of subsets of $Q(R_i)$, the map ψ is the identity. The source of ψ contains a nonzerodivisor by Proposition 3.3 (1). We may write it as the image of an element $v \in I^{i-g+1}$. Since Δ_i is a nonzerodivisor on R_i, both source and target of ψ are fractional ideals containing nonzerodivisors, so ψ is multiplication by some element in $Q(R_i)$. To show that ψ is the identity, it suffices to show that $\psi(u) = u$ for some nonzerodivisor $u \in I^{i-g+1}R_i$. We take u to be the image of $\Delta_i v$ in $I^{i-g+1}R_i$.

Recall that $L \subset R_{i-1}/(a_i)$. Since $I^{i-g+1}L \subset IL = 0$, we have $vL = 0$. Since

$$L = \ker(R_{i-1}/(a_i) \twoheadrightarrow R_i)$$

and both $R_{i-1}/(a_i)$ and R_i are Cohen–Macaulay rings with Noether normalization A_i, it follows that they are free A_i-modules, and thus $R_{i-1}/(a_i) \cong R_i \oplus L$ as A_i-modules.

From $vL = 0$ one sees that

$$\varepsilon(v \operatorname{Tr}_{R_i/A_i}) = v \operatorname{Tr}_{(R_{i-1}/(a_i))/A_i}.$$

Following the maps in the diagram, we now see that $\psi(\Delta_i v) = \Delta_i v$ as required. □

Example 7.5. The following example illustrates a subtlety in the inductive proof above. The conclusion of Theorem 7.4 shows that the image of $\Delta_s := \Delta$ in $R_s := \overline{R}$ is contained in $I^{t+1}R_s$. The following example shows that Δ_s itself may not be contained in I^{t+1}, and, moreover, the image of Δ_s in $I^{t+1}R_s$ is not necessarily mapped under the inclusion

$$I^{t+1}R_s \hookrightarrow I^t R_{s-1}/a_s I^t R_{s-1}$$

to the image of Δ_s in the target. Take $s = 2$ and let $R = k[\![x, y, z]\!]$, $I = (z - x - y)$, and $J = (a_1, a_2)$, where $a_1 = xz - x^2 - xy$, $a_2 = yz - yx - y^2$. We have

$$\begin{aligned}(a_1) : I &= (x),\\ J : I &= (x, y) = K.\end{aligned}$$

Computation shows that $\Delta_2 \notin I^2$, and the map of canonical modules $I^2R_2 \hookrightarrow IR_1/a_2IR_1$ does not send the image of Δ_2 to the image of Δ_2.

From Theorem 7.4 we derive a formula for the Dedekind complementary module of certain determinantal rings:

Corollary 7.6. *Let k be a field of characteristic 0, and let $R = k[\![x_1, \dots, x_d]\!]$. Let C be an $(n+1) \times (n+s)$ matrix with entries in the maximal ideal of R, where $n \geq 1$ and $s \geq 2$, and assume that the maximal minors of C generate an ideal K of height s, the generic value. Suppose that the ring $\overline{R} = R/K$ is reduced. Let D be an $(n+1) \times n$ matrix consisting of n columns of C, let I be the ideal generated by the $n \times n$ minors of D, and let $a_1, \dots, a_s$ be the $(n+1) \times (n+1)$ minors of C that involve the n columns of D. Let $x_1, \dots, x_d$ be general variables in R, so that $\overline{R}$ is module finite over $A = k[\![x_{s+1}, \dots, x_d]\!]$. We have*

$$I^{s-1}\overline{R} = \Delta\, \mathfrak{C}(\overline{R}/A),$$

where Δ is the Jacobian determinant

$$\Delta = \det \begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \cdots & \frac{\partial a_1}{\partial x_s} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_s}{\partial x_1} & \cdots & \frac{\partial a_s}{\partial x_s} \end{pmatrix}.$$

Moreover, after suitable column operations on C, the submatrix D may be chosen so that the ideal $I\overline{R}$ has positive grade, and in this case Δ is a nonzerodivisor on $\overline{R}$.

Proof. Suppose that $\tilde{C} = (y_{i,j})$ is an $(n+1) \times (n+s)$ matrix of variables, and write $S = R[\![\{y_{i,j}\}]\!]$ and $B = A[\![\{y_{i,j}\}]\!]$. Let $\tilde{D}$, $\tilde{K}$, $\tilde{I}$, $\tilde{a}_1, \dots, \tilde{a}_s$, and $\tilde{\Delta}$ be the same objects as defined in the statement of the corollary, using the matrix $\tilde{C}$ instead of C. Write $\overline{S} = S/\tilde{K}$. Specializing $\tilde{C}$ to C, these objects specialize to the ones defined in the corollary. The ideal $\tilde{I}$

is perfect of codimension 2 and satisfies the Strong Hypothesis for s, $\tilde{a}_1, \dots, \tilde{a}_s$ are generic elements of $\tilde{I}$, and $\tilde{K} = (\tilde{a}_1, \dots, \tilde{a}_s) : \tilde{I}$ is a geometric s-residual intersection of $\tilde{I}$, by Theorem 6.2 or [19]. Theorem 7.4 and its proof show that

$$\tilde{I}^{s-1}\overline{S}\,\mathrm{Tr}_{\overline{S}/B} = \tilde{\Delta}\,\mathrm{Hom}_B(\overline{S}, B).$$

Since $\overline{S}$ is a free B-module of finite rank, we have $\mathrm{Hom}_B(\overline{S}, B) \otimes_B A = \mathrm{Hom}_A(\overline{R}, A)$. After taking images in this module, the equality above gives

$$I^{s-1}\overline{R}\,\mathrm{Tr}_{\overline{R}/A} = \Delta\,\mathrm{Hom}_A(\overline{R}, A),$$

and hence the main assertion of the corollary.

Since $\overline{R}$ is reduced, the ideal K is generically a complete intersection, so the $n \times n$ minors of C generate an ideal of positive grade in $\overline{R}$. It follows that after suitable column operations on C we may choose the submatrix D so that $I\overline{R}$ has positive grade in $\overline{R}$. (Reason: the column space of the matrix $\overline{C}$ over the ring $\overline{R}$ has rank n, and thus the same is true for a general choice of n columns.) □

In the next results we apply our theory to certain 0-dimensional residual intersections. Our goal is to give formulas for the socles of their canonical modules as Jacobian determinants.

Corollary 7.7. *Let k be a field of characteristic 0, and let $R = k[\![x_1, \dots, x_d]\!]$. Let $I \subset R$ be an ideal satisfying the Standard Hypothesis with respect to $s = d$, let $J : I$ be a d-residual intersection, and set $t = d - g$. Let $a_1, \dots, a_d$ be general elements in J, and let Δ be the Jacobian determinant of $a_1, \dots, a_d$. If $\overline{R} = R/((a_1, \dots, a_{d-1}) : I)$, then the image of Δ in $\overline{R}$ is in $I^t\overline{R}$. Further, the image of Δ generates the socle of*

$$I^t\overline{R}/(a_d I^t\overline{R}) \cong \omega_{\overline{R}/(a_d)}.$$

Proof. Lemma 1.1 and Proposition 3.4 show that $(a_1, \dots, a_{d-1}) : I$ is a geometric $(d-1)$-residual intersection and that $\overline{R}$ is reduced. Moreover, by Theorem 3.1 and Proposition 3.3, parts (5) and (1), the module $I^t\overline{R}/a_d I^t\overline{R}$ is isomorphic to $\omega_{\overline{R}/(a_d)}$.

The hypothesis of the corollary is sufficient to justify the upper half of the diagram in the proof of Theorem 7.4 for the case $i = d$. The first row of the diagram shows that the image of $\Delta = \Delta_d$ in $\overline{R}$ lies in $I^t\overline{R}$, and hence gives an element of $I^t\overline{R}/a_d I^t\overline{R}$. The isomorphism ϕ maps this element to $\mathrm{Tr}_{(\overline{R}/(a_d))/A_d}$, which generates the socle of the module $\mathrm{Hom}_{A_d}(\overline{R}/(a_d), A_d)$. □

Theorem 7.8. *Let k be a field of characteristic 0, and let $R = k[\![x_1, \dots, x_d]\!]$. Let $I \subset R$ be an ideal satisfying the Standard Hypothesis with respect to $s = d$, and let $J : I$ be a d-residual intersection. Let $a_1, \dots, a_d$ be general elements in J, and let Δ be the Jacobian determinant of $a_1, \dots, a_d$. There is an element $p \in (a_1, \dots, a_{d-1})$ such that*

$$\Delta' := \Delta + p \in I^{t+1},$$

and the image of Δ' generates the socle of

$$I^{t+1}/JI^t \cong \omega_{R/(J:I)}.$$

Moreover, if $\Delta \in I^{t+1}$, then the image of Δ generates this socle.

By Theorem 7.2, the socles of I^{t+1}/JI^t and R/JI^t are the same, so Theorem 7.8 can also be interpreted as a result on the socle of R/JI^t.

Proof. Recall that

$$I^{t+1}/JI^t \cong \omega_{R/(J:I)}$$

by Theorem 3.1. Let $K_{d-1} = (a_1, \dots, a_{d-1}) : I$ and write $\overline{R} = R/K_{d-1}$. We will first prove that we can take $p \in K_{d-1}$. As a consequence of the first statement of Corollary 7.7, there is an element $p_1 \in K_{d-1}$ such that $\Delta + p_1 \in I^t$. By Corollary 4.3, there is a natural inclusion

$$I^{t+1}/JI^t \hookrightarrow I^t\overline{R}/(a_d I^t\overline{R}).$$

From the second statement of Corollary 7.7 it follows that the image of $\Delta + p_1$ generates the socle of $I^t\overline{R}/a_d I^t\overline{R}$, and thus lies in the submodule I^{t+1}/JI^t and generates its socle. In particular, there are an element $p_2 \in K_{d-1}$ and an element $q \in a_d I^t$ so that

$$\Delta + p_1 + p_2 + q \in I^{t+1},$$

and the image of this element generates the socle of I^{t+1}/JI^t. Since $q \in JI^t$, we may take $p = p_1 + p_2 \in K_{d-1}$.

By Theorem 7.1, we have $\Delta \in J$ if $t > 0$, while $\Delta \in J : \mathfrak{m}$ if $t = 0$, in which case R/J is Gorenstein, and therefore in either case $\Delta \in I$. Thus $p \in I \cap K_{d-1}$. By Proposition 3.3 (5), $p \in (a_1, \dots, a_{d-1})$ as claimed.

If $\Delta \in I^{t+1}$ to begin with, we could take $p_1 = p_2 = 0$ proving the last statement. □

In the graded case, Remark 4.4 identifies the socle up to homogeneous isomorphism,

$$\operatorname{soc} \frac{I^{t+1}}{JI^t} \cong (\operatorname{soc} \omega_{R/(J:I)})\left(-\sum_{j=1}^{d}(\delta_j - 1)\right) \cong k\left(-\sum_{j=1}^{d}(\delta_j - 1)\right),$$

so the socle has the same degree as the Jacobian determinant of d homogeneous generators of J.

Motivated by Theorem 7.8, we try to find conditions when $\Delta \in I^{t+1}$.

Proposition 7.9. *Let k be a perfect field, let $R = k[x_1, \dots, x_d]$ be a standard graded polynomial ring in d variables, and let $J \subset R$ be an ideal. Set e equal to the maximum of the codimensions of the minimal primes of J. If J is generated by forms of the same degree > 1, then the $d \times d$ minors of the Jacobian matrix of these forms are contained in the symbolic power $(\sqrt{J})^{(d-e+1)}$.*

Proof. Set $t = d - e$. If $t = 0$, the result is trivial, so we may assume $t > 0$. Since k is perfect, we may assume that k is algebraically closed. In this case $\sqrt{J}$ is the intersection of the 1-dimensional linear ideals that contain it. By inverting a linear form not in any minimal prime of J and taking the degree 0 part, these become maximal ideals. By Zariski's Main Lemma on Holomorphic Functions (see for example [11, Corollary 1]), the $(t+1)$-st symbolic power of $\sqrt{J}$ in the dehomogenized ring is the intersection of the $(t+1)$-st powers of these maximal ideals, and thus in R the ideal $(\sqrt{J})^{(t+1)}$ contains, hence is equal to, the intersection of the $(t+1)$-st powers of the 1-dimensional linear ideals that contain it.

After changing notation, it is thus sufficient to prove that if J is contained in the ideal $L = (x_1, \dots, x_{d-1})$ and $f_1, \dots, f_d$ are forms in J of degree $\delta > 1$, then $\det \operatorname{Jac}(f_1, \dots, f_d)$

is in L^{t+1}, where Jac denotes the Jacobian matrix. Write

$$\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_{d-1} \end{pmatrix}$$

for some $d \times d-1$ matrix A with homogeneous entries of degree $\delta - 1$.

We may write A in the form $A = B + x_d^{\delta-1}C$, where $B = (b_{i,j})$ has entries in L and C is a matrix of scalars. By the product rule,

$$\begin{aligned} \operatorname{Jac}(f_1, \dots, f_d) = \sum_{j=1}^{d-1} x_j \operatorname{Jac}(b_{1,j}, \dots, b_{d,j}) + \begin{pmatrix} & & 0 \\ & B & \vdots \\ & & 0 \end{pmatrix} \\ + (\delta-1)x_d^{\delta-2} \sum_{j=1}^{d-1} x_j \begin{pmatrix} & c_{1,j} & \\ 0 & \vdots & \\ & c_{d,j} & \end{pmatrix} + x_d^{\delta-1} \begin{pmatrix} & & 0 \\ C & & \vdots \\ & & 0 \end{pmatrix}. \end{aligned}$$

Let D be the sum of the first two terms on the right-hand side of this expression, and let E be the sum of the two remaining terms. These matrices have the following properties:

(1) Each column of D has entries in L.

(2) The last column of D has entries in L^2. This is because the last column of the Jacobian matrix is defined by differentiating with respect to x_d.

On the other hand the rank of the scalar matrix C is at most the codimension of J localized at L, which is at most e. The last column of E is a linear combination of columns of C with coefficients in L. Thus:

(3) The rank of E is at most e.

(4) The last column of E has entries in L.

These properties of D and E imply that $\det(D + E) \in L^{t+1}$ as required. □

Theorem 7.10. *Let k be a field of characteristic 0, and let $R = k[x_1, \dots, x_d]$. Let $I \subset R$ be a homogeneous ideal satisfying the Standard Hypothesis with $s = d$, and let $J \subset I$ be an ideal generated by d forms of a single degree $\delta > 1$ such that $J : I$ is a d-residual intersection. If I is reduced and $\mu(I_P) \le \operatorname{codim} P - 1$ for all prime ideals $P \supset I$ with $g < \operatorname{codim} P < d$, then the Jacobian determinant of any d homogeneous generators of J of degree δ is in I^{t+1} and thus, by Theorem* 7.8*, generates the socle of $I^{t+1}/JI^t \cong \omega_{R/(J:I)}$.*

Proof. We may assume that $I \neq R$. In this case $\sqrt{J} = I$. Since I is Cohen–Macaulay, all minimal primes of I and hence of J have the same codimension g. By Proposition 7.9, the Jacobian determinant is contained in $I^{(t+1)}$. By the assumption on the $\mu(I_P)$, the powers and symbolic powers of I coincide on the punctured spectrum ([31, 4.9 (d)]). Therefore, $I^{(t+1)}/JI^t$ is contained in the finite length part of R/JI^t. The latter has a simple socle generated in the same degree $d(\delta - 1)$ as the Jacobian determinant by Theorem 7.2 and Remark 4.4.

Thus the image of the Jacobian determinant lies in $\operatorname{soc} I^{(t+1)}/JI^t = \operatorname{soc} I^{t+1}/JI^t$. In particular, the Jacobian determinant is in I^{t+1}. □

Theorem 7.10 can also be understood in terms of primary decompositions, rather than residual intersections; in this formulation, the result is a natural generalization of Theorem 7.1.

To explain this, let R be a local Gorenstein ring of dimension d and J any ideal of codimension g generated by d elements. For our purpose we may assume that R/J has depth 0. Consider a decomposition $J = I \cap L$, where L is the 0-dimensional primary component in any shortest primary decomposition of J and I is the intersection of the primary components of positive dimension. Notice that L is contained in the ideal $K = J : I$, which gives an embedding $\omega_{R/K} \hookrightarrow \omega_{R/L}$. Also observe that K is a d-residual intersection of I.

Now assume that I satisfies the Standard Hypothesis with $s = d$, set $t = d - g$, and let $E \supset I^{t+1}/JI^t$ be an injective envelope of I^{t+1}/JI^t as a module over R/L. Since I^{t+1}/JI^t is a canonical module of R/K, we may choose $\omega_{R/L}$ to be *equal* to E.

Corollary 7.11. *In addition to the assumptions of the preceding two paragraphs suppose that $R = k[\![x_1, \dots, x_d]\!]$ is a power series ring in d variables over a field of characteristic zero and that J is generated by homogeneous polynomials $f_1, \dots, f_d$ of a single degree > 1. If I is reduced and $\mu(I_P) \le \operatorname{codim} P - 1$ for all prime ideals $P \supset I$ with $g < \operatorname{codim} P < d$, then the socle of $\omega_{R/L}$ is generated by the image in I^{t+1}/JI^t of the Jacobian determinant of $f_1, \dots, f_d$.*

From examples it would seem that the formula for the socle as a Jacobian holds without the reduced hypothesis and without the assumptions on the local numbers of generators beyond the G_s condition of our Standard Hypothesis. We can at least prove this for $g = 1$.

Proposition 7.12. *Let k be a field, and let $R = k[x_1, \dots, x_d]$. Let $I = (G) \subset R$ be a principal ideal generated by a nonzero form of degree γ and let $F = f_1, \dots, f_d$ be a regular sequence of forms of the same degree δ. Assume that neither δ nor $\delta + \gamma$ is 0 in k, and let J be the ideal generated by the sequence of forms GF. The socle of R/JI^{d-1}, hence the socle of I^d/JI^{d-1}, is generated by the Jacobian determinant* $\det \operatorname{Jac}(GF)$.

Proof. By Theorem A.5, the socle of $R/(F)$ is generated by $\det \operatorname{Jac}(F)$, so the socle of $R/JI^{d-1} = R/(G^d F)$ is generated by $G^d \det \operatorname{Jac}(F)$. By Lemma 7.13, this is

$$\frac{\delta}{\delta + \gamma} \det \operatorname{Jac}(GF). \qquad \square$$

Lemma 7.13. *Let $R = k[x_1, \dots, x_d]$. If G is a form of degree γ and $F = f_1, \dots, f_d$ is a sequence of forms of the same degree δ, then*

$$\delta \det \operatorname{Jac}(GF) = (\delta + \gamma)\, G^d \det \operatorname{Jac}(F).$$

Proof. Write G_j for $\frac{\partial G}{\partial x_j}$ and $f_{i,j}$ for $\frac{\partial f_i}{\partial x_j}$. By the product rule,

$$\operatorname{Jac}(GF) = G \operatorname{Jac}(F) + \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \begin{pmatrix} G_1 & \cdots & G_d \end{pmatrix}.$$

The second summand has rank 1, so by the multilinearity of the determinant we have

$$\det \mathrm{Jac}(GF) = G^d \det \mathrm{Jac}(F) + G^{d-1} \sum_{i=1}^{d} \det J_i .$$

Here J_i is the matrix obtained from $\mathrm{Jac}(F)$ by replacing the i-th row by the row $f_i(G_1 \ \cdots \ G_d)$. Expansion along the first column shows that this sum is equal to

$$\det \begin{pmatrix} G^d & G^{d-1}G_1 & \cdots & G^{d-1}G_d \\ -f_1 & f_{1,1} & \cdots & f_{1,d} \\ \vdots & \vdots & & \vdots \\ -f_d & f_{d,1} & \cdots & f_{d,d} \end{pmatrix}.$$

We multiply the first column by δ, add x_i times the $(i+1)$-st column to the first column for all i, and use Euler's formula. From this we see that

$$\begin{aligned} \delta \det \mathrm{Jac}(GF) &= \det \begin{pmatrix} (\delta+\gamma)G^d & G^{d-1}G_1 & \cdots & G^{d-1}G_d \\ 0 & f_{1,1} & \cdots & f_{1,d} \\ \vdots & \vdots & & \vdots \\ 0 & f_{d,1} & \cdots & f_{d,d} \end{pmatrix} \\ &= (\delta+\gamma)G^d \det \mathrm{Jac}(F). \end{aligned}$$

□

Example 7.14. If we do not assume the forms generating J have the same degree, then the Jacobian need not be well-defined modulo JI^t, and in particular its image may not generate the socle, as the following example shows. Let k be a field of characteristic $\neq 2, 3$, and let $R = k[x, y]$. Let F be the regular sequence $x^2 + y^2, x + y$ and set $I = (G)$ with $G = x$, and $J = (GF)$. We have

$$JI : \det \mathrm{Jac}(GF) = (G^2F) : \det \mathrm{Jac}(GF) = (x),$$

so $\det \mathrm{Jac}(GF)$ is not in the socle modulo JI. Moreover, $\det \mathrm{Jac}(GF)$ is not even contained in I^2. However, we can replace F by a different sequence of generators $F' = x^2 - xy, x + y$ for (F), and then the Jacobian determinant of GF' *does* generate the socle modulo JI.

Example 7.15. Over a field of characteristic 0, the polynomial

$$f = (x^2 - z)(xz - y^2)$$

is the product of two of the quasihomogeneous generators of the ideal of the space curve C with parametrization $t \mapsto (t^2, t^3, t^4)$. The Jacobian ideal J of f has codimension 2. The scheme defined by J has an isolated singularity, so J is generically reduced, and thus also its unmixed part I is reduced. In fact, $I = (x^2 - z, xz - y^2)$ is a prime complete intersection.

Nevertheless, one can compute that the Hessian determinant of f is not even contained in I^2. Thus f violates Conjecture (3) of van Straten and Warmt [32, 7.1].

In the case $s = g$ (so $t = 0$), when R is regular local and $s = d$, there is another famous (and easier) formula for the socle of R/J – it is generated by the image of the determinant of any "transition" matrix expressing the generators of J as linear combinations of the generators

of the maximal ideal of R. The following examples show that in Proposition 7.12 with $t > 0$ we cannot replace the Jacobian by such a transition matrix: the determinant could be outside the ideal $JI^t : \mathfrak{m}$ and could also be in JI^t (it could even be 0).

Example 7.16. (a) Let $R = k[\![x_1, \dots, x_d]\!]$, let $G \in (x_1, \dots, x_d)$ be nonzero, and let $F_1, \dots, F_d$ be a regular sequence in R. Writing $G = \sum_i a_i x_i$, we see that

$$\begin{pmatrix} GF_1 \\ \vdots \\ GF_d \end{pmatrix} = \begin{pmatrix} F_1 \\ \vdots \\ F_d \end{pmatrix} \begin{pmatrix} a_1 & \cdots & a_d \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}.$$

We may take the rank 1 matrix

$$A := \begin{pmatrix} F_1 \\ \vdots \\ F_d \end{pmatrix} \begin{pmatrix} a_1 & \cdots & a_d \end{pmatrix}$$

as transition matrix, and we have $\det A = 0$ as soon as $d \geq 2$.

(b) Let $R = k[x, y]$, where k is a field of characteristic $\neq 3$, and take $I = (G)$ with $G = x^2 + y^2$, and $J = (GF)$ with $F = x, y$. If we replace the Jacobian matrix $\mathrm{Jac}(GF)$ by

$$A := \frac{1}{3}\ \mathrm{Jac}(GF) + \begin{pmatrix} -y & x \\ -y & x \end{pmatrix},$$

then

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} GF_1 \\ GF_2 \end{pmatrix}$$

and $\det A$ is in I, but $\det A$ is not in the socle of I/JI.

(c) If in example (b) we change G to xy leaving everything else the same, then $\det A$ is not even in I.

(d) If F_1, F_2 is a regular sequence of forms of degree 2 in $k[x, y]$ and $G = a_1 x + a_2 y$ is a nonzero form, then there are examples with $\det A \neq 0$ but $\det A \in JI = G^2(F_1, F_2)$. For instance, take

$$A = \begin{pmatrix} a_1 F_1 - yG & a_2 F_1 + xG \\ a_1 F_2 - yG & a_2 F_2 + xG \end{pmatrix};$$

the determinant in this case is $G^2(F_1 - F_2)$.

A. Differents and socles for Gorenstein rings

In this section we provide self-contained expositions of the classical results on differents and socles that we have used, mostly for complete intersections in characteristic 0. More generally than is usually stated, these yield a formula for the socle of a 0-dimensional Gorenstein

ring. The results of this section are known, some in greater generality, but not easily available. Classic references are by Noether [26], Berger [3], Tate [25, Appendix], Scheja and Storch [28, 29], and Kunz [21, 22].

Let A be a Noetherian ring, let R be an A-algebra that is essentially of finite type, and write $R^e = R \otimes_A R$. Let $\mathbb{D}$ be the kernel of the multiplication map $\mu : R^e \to R$, so that we have an exact sequence

$$0 \to \mathbb{D} \to R^e \xrightarrow{\mu} R \to 0.$$

We want to compare three measures of ramification:

- The *Kähler different* $\mathfrak{D}_K(R/A)$, introduced in a different case in Section 7, is defined to be $\operatorname{Fitt}_0^R(\Omega_{R/A})$.
- The *Noether different* $\mathfrak{D}_N(R/A)$ is defined to be $\mu(\operatorname{ann}_{R^e} \mathbb{D})$.
- The *Dedekind different* $\mathfrak{D}_D(R/A)$ is defined, for instance, when $A \subset R$ is a ring extension, A is a Noetherian normal domain, R is reduced and a finitely generated torsion free A-module, and R/A is separable. The complementary module $\mathfrak{C}(R/A)$ is the fractional R-ideal such that
$$\operatorname{Hom}_A(R, A) = \mathfrak{C}(R/A) \operatorname{Tr}_{L/K},$$
where $K = Q(A)$ and $L = Q(R)$ are the total rings of quotients of A and R, respectively. The Dedekind different is defined to be the inverse of the complementary module, $\mathfrak{D}_D(R/A) = \mathfrak{C}(R/A)^{-1}$.

Because $\Omega_{R/A} \cong \mathbb{D} \otimes_{R^e} R$ and $\operatorname{Fitt}_0^{R^e}(\mathbb{D}) \subset \operatorname{ann}_{R^e} \mathbb{D}$, it follows that

$$\mathfrak{D}_K(R/A) \subset \mathfrak{D}_N(R/A).$$

The Dedekind different is an ideal because A is normal. We also have

$$\mathfrak{D}_N(R/A) \subset \mathfrak{D}_D(R/A),$$

which implies that

$$\mathfrak{D}_N(R/A) \operatorname{Hom}_A(R, A) \subset R \operatorname{Tr}_{R/A}.$$

For a short proof see [24, formula (3.3) proved in Lemma 3.4]. The last containment can be an equality even when the Dedekind different is not defined:

Theorem A.1. *Let A be a Noetherian ring and let R be an A-algebra that is finitely generated and free as an A-module. If $\operatorname{Hom}_A(R, A)$ is cyclic as an R-module, then*

$$\mathfrak{D}_N(R/A) \operatorname{Hom}_A(R, A) = R \operatorname{Tr}_{R/A}.$$

Proof. We will divide the proof into several parts:

Step (1). Because R is a free A-module, the natural map

$$\Phi : R \otimes_A R \to \operatorname{Hom}_A(\operatorname{Hom}_A(R, A), R)$$

given by

$$s \otimes t \mapsto \big(\varphi \mapsto \varphi(s)t\big)$$

is an isomorphism of R-R-bimodules. The annihilator of $\mathbb{D}$ is the unique largest R-R-submodule of $R \otimes_A R$ on which the left and the right R-module structures coincide, and the subset $\mathrm{Hom}_R(\mathrm{Hom}_A(R, A), R)$ has the same property in $\mathrm{Hom}_A(\mathrm{Hom}_A(R, A), R)$. It follows that Φ carries the annihilator of $\mathbb{D}$ onto $\mathrm{Hom}_R(\mathrm{Hom}_A(R, A), R)$.

Since R is a finitely generated free A-module and $\mathrm{Hom}_A(R, A)$ is cyclic as an R-module, we have $\mathrm{Hom}_A(R, A) \cong R$. It follows that $\mathrm{ann}_{R^e} \mathbb{D}$ is cyclic as an R-module.

Step (2). Let Γ be a generator of $\mathrm{ann}_{R^e} \mathbb{D}$. As $\Phi(\Gamma)$ generates $\mathrm{Hom}_R(\mathrm{Hom}_A(R, A), R)$ and $\mathrm{Hom}_A(R, A) \cong R$, we see that $\Phi(\Gamma)$ is an R-isomorphism. Let

$$\sigma = \Phi(\Gamma)^{-1}(1) \in \mathrm{Hom}_A(R, A).$$

It follows that $\sigma\mu : R \otimes_A R \to A$ is a symmetric, nonsingular A-bilinear form.

Step (3). Let $\{v_i\}$ be an A-basis of R, and suppose that $\Gamma = \sum_i v_i' \otimes v_i$. We claim that $\sigma(v_i' v_j) = \delta_{i,j}$ – that is, $\{v_i'\}$ is the dual basis of $\{v_i\}$ with respect to $\sigma\mu$. Indeed, since $\Phi(\Gamma)$ is R-linear, we have $\Phi(\Gamma)(r\sigma) = r$ for every $r \in R$. Thus, for each j,

$$v_j = \Phi(\Gamma)(v_j\sigma) = \Phi\Big(\sum_i v_i' \otimes v_i\Big)(v_j\sigma) = \sum_i (v_j\sigma)(v_i')v_i = \sum_i \sigma(v_j v_i')v_i.$$

Since the v_i form an A-basis, we see that $\sigma(v_i' v_j) = \delta_{i,j}$ as required.

Step (4). Finally, we claim that $\mathrm{Tr}_{R/A} = \mu(\Gamma)\sigma$. Let r be an element of R, regarded as an A-endomorphism of R by multiplication. We have

$$\mu(\Gamma)\sigma(r) = \sigma(\mu(\Gamma)r) = \sigma\Big(\sum_i v_i' r v_i\Big).$$

Since $\{v_i'\}$ and $\{v_i\}$ are dual bases with respect to $\sigma\mu$, this sum is equal to $\mathrm{Tr}_{R/A}(r)$. Since $\mathfrak{D}_N(R/A) = \mu(\mathrm{ann}_{R^e} \mathbb{D}) = R\mu(\Gamma)$, we see that

$$\mathfrak{D}_N(R/A)\,\mathrm{Hom}_A(R, A) = \mathfrak{D}_N(R/A)\,\sigma = R\mu(\Gamma)\,\sigma = R\,\mathrm{Tr}_{R/A}$$

as required. □

Theorem A.2. *In addition to the assumptions in the definition of the Dedekind different above, suppose that A is a regular local ring. If R is Gorenstein, then $\mathfrak{D}_D(R/A) = \mathfrak{D}_N(R/A)$.*

Proof. We first verify that the assumptions of Theorem A.1 are satisfied. Recall that $A \subset R$ and R is a finitely generated A-module. For all maximal ideals $\mathfrak{m}$ of R, the rings $R_{\mathfrak{m}}$ have the same dimension as A, as can be seen for instance by tensoring with the completion of A, so that R splits as a product of local rings, and using the torsion freeness of R over A. Thus, since the rings $R_{\mathfrak{m}}$ are Cohen–Macaulay, it follows that R is a maximal Cohen–Macaulay A-module, hence a free A-module. Moreover, as the rings $R_{\mathfrak{m}}$ are Gorenstein and have the same dimension as A, the R-module $\mathrm{Hom}_A(R, A)$ is locally free of rank 1. Therefore we have $\mathrm{Hom}_A(R, A) \cong R$ because R is semilocal.

Thus we may apply Theorem A.1. Since $\mathrm{Hom}_A(R, A) = \mathfrak{C}(R/A)\ \mathrm{Tr}_{R/A}$ by the definition of the complementary module, the theorem shows that $\mathfrak{D}_N(R/A)\,\mathfrak{C}(R/A) = R$, which gives $\mathfrak{D}_N(R/A) = \mathfrak{C}(R/A)^{-1} = \mathfrak{D}_D(R/A)$. □

Theorem A.3. *Let $A \subset R$ be a ring extension, where A is regular local and R is finitely generated and torsion free as an A-module. If R is locally a complete intersection, then*

$$R \cong A[x_1, \dots, x_n]_W/(F_1, \dots, F_n)$$

for some regular sequence $F_1, \dots, F_n$ of length n in the polynomial ring $A[x_1, \dots, x_n]$ and some multiplicatively closed subset W. Write Δ for the image in R of the Jacobian determinant of $F_1, \dots, F_n$ with respect to $x_1, \dots, x_n$. One has

$$\mathfrak{D}_N(R/A) = \mathfrak{D}_K(R/A) = R\Delta.$$

Proof. Write $R \cong A[x_1, \dots, x_n]/\mathcal{J}$. As R is a finitely generated torsion free A-module, it follows as in the previous proof that every maximal ideal $\mathfrak{m}$ of R has the same codimension $d := \dim A$. Since $\mathfrak{m}$ must contain the maximal ideal of A, its preimage $\mathfrak{M}$ in $A[x_1, \dots, x_n]$ has codimension $d + n$. Hence the ideal $\mathcal{J}_{\mathfrak{M}}$ has codimension n. Thus it is generated by n elements because R is locally a complete intersection. Write W for the complement in $A[x_1, \dots, x_n]$ of the union of the finitely many maximal ideals $\mathfrak{M}$. By basic element theory, $\mathcal{J}_W$ is again generated by n elements $F_1, \dots, F_n$ that can be chosen to form a regular sequence in $A[x_1, \dots, x_n]$.

To prove the claim about differents, first notice that R is a Cohen–Macaulay ring and hence a free A-module, as shown in the previous proof. As before, let $\mathbb{D}$ be the kernel of the multiplication map $\mu : R^e = R \otimes_A R \to R$. The preimage $\tilde{\mathbb{D}}$ of $\mathbb{D}$ in $A[x_1, \dots, x_n]_W \otimes_A R$ is the kernel of the natural map to R, so it is generated by a regular sequence $G = G_1, \dots, G_n$ of length n. The ideal $\tilde{\mathbb{D}}$ also contains the sequence $F \otimes 1 := F_1 \otimes 1, \dots, F_n \otimes 1$, which is still a regular sequence because R is flat over A.

Notice that $\mathbb{D} = \tilde{\mathbb{D}}/(F \otimes 1) = (G)/(F \otimes 1)$. The preimage in $A[x_1, \dots, x_n]_W \otimes_A R$ of the annihilator of $\mathbb{D}$ may thus be written as $(F \otimes 1) : (G)$. This ideal quotient is generated by $F \otimes 1$ and the determinant of any matrix Θ expressing the elements of $F \otimes 1$ as linear combinations of the elements of G, see [35] or [6]. It follows that $\mathfrak{D}_N(R/A)$ is generated by the image in R of $\det \Theta$.

On the other hand, since G is a regular sequence and since

$$\mathbb{D} \otimes_{R^e} R = \mathbb{D}/\mathbb{D}^2 \cong \Omega_{R/A},$$

the image in R of Θ is a presentation matrix of $\Omega_{R/A}$. Thus the image of $\det \Theta$ also generates the ideal $\mathfrak{D}_K(R/A) = R\Delta$. □

Corollary A.4. *If the assumptions in the definition of the Dedekind different and the hypotheses of Theorem* A.3 *are satisfied, then*

$$\mathfrak{C}(R/A) = \mathfrak{D}_K(R/A)^{-1} = R\Delta^{-1}.$$

Proof. One uses Theorem A.2, Theorem A.3, and the fact that the fractional ideal $\mathfrak{C}(R/A)$ is invertible, hence reflexive. □

Theorem A.5. *If R is a local Gorenstein algebra over a field k with $\dim_k R$ finite and not divisible by the characteristic of k, then $\mathfrak{D}_N(R/k)$ is equal to the socle of R. If, moreover, R is a complete intersection, then the socle of R is generated by the Jacobian determinant.*

Proof. Since the trace of any nilpotent element is 0, it follows that the trace lies in the socle of $\operatorname{Hom}_k(R,k)$ and generates it if the characteristic of k does not divide $\dim_k R$. Thus Proposition A.1 implies that $\mathfrak{D}_N(R/k)\operatorname{Hom}_k(R,k)$ is the socle of $\operatorname{Hom}_k(R,k)$. Therefore $\mathfrak{D}_N(R/k)$ is the socle of R since $\operatorname{Hom}_k(R,k)\cong R$ as R-modules.

Finally, if R is a complete intersection, then $\mathfrak{D}_N(R/k)=\mathfrak{D}_K(R/k)$ is generated by the Jacobian determinant, by Theorem A.3. □

Proof of Theorem 7.1. One implication is a special case of Theorem A.5. To prove the opposite implication, we must show that the Kähler different $\mathfrak{D}_K(R/k)$ is 0 when the ring $R=k[\![x_1,\dots,x_d]\!]/(a_1,\dots,a_s)$ is not a 0-dimensional complete intersection.

First suppose that R is 0-dimensional and not a complete intersection. Replacing the a_i by general linear combinations, we may assume that any d of the a_i form a regular sequence. By the previous theorem, the Jacobian determinant of $a_{i_1},\dots,a_{i_d}$ generates the socle modulo $(a_{i_1},\dots,a_{i_d})$ and is thus contained in $(a_1,\dots,a_s)$ as required.

Now suppose that R is not 0-dimensional. To simplify the notation, set $\mathfrak{m}=(x_1,\dots,x_d)$ and $\mathcal{J}=(a_1,\dots,a_s)$ and suppose that s is the minimal number of generators of $\mathcal{J}$. We may assume that R is not a complete intersection since otherwise $\mathfrak{D}_K(R/k)=0$. For any sufficiently large integer n, the Artin–Rees Lemma and the Principal Ideal Theorem together imply that $\mathcal{J}+\mathfrak{m}^n$ requires at least $s+\dim\mathcal{J}$ generators. Thus, $R/\mathfrak{m}^nR$ is not a complete intersection.

We conclude from the 0-dimensional argument that, for any $n\gg 0$,

$$\mathfrak{D}_K((R/\mathfrak{m}^nR)/k)=0.$$

In particular, $\mathfrak{D}_K(R/k)$ is in $\mathfrak{m}^nR$. By the Krull Intersection Theorem, $\mathfrak{D}_K(R/k)=0$. □

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