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Minimal Resolutions Over Codimension 2 Complete Intersections

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Abstract

We construct an explicit free resolution $\mathbf{T}$ for a maximal Cohen-Macaulay module M over a local complete intersection of codimension 2 with infinite residue field. The resolution is minimal when the module M is a sufficiently high syzygy. Our starting point is a layered free resolution $\mathbf{L}$, described in [7], of length 2 over a regular local ring. We provide explicit formulas for the differential in $\mathbf{T}$ in terms of the differential and homotopies on the finite resolution $\mathbf{L}$. One application of our construction is to describe Ulrich modules over a codimension 2 quadratic complete intersection.

Keywords Free resolutions · Complete intersections · CI operators · Eisenbud operators · Maximal Cohen-Macaulay modules

Mathematics Subject Classification (2010) Primary 13D02

Introduction

Minimal free resolutions of finitely generated modules over a local complete intersection have attracted attention ever since the work of Tate [10], who provided an elegant construction of the resolution of the residue field of any such ring. A generalization of Tate's construction due to Shamash and Eisenbud [5, 9] provides a non-minimal resolution of any module. However, much more information can be derived from minimal resolutions, so it is natural to ask about these.

Taking a cue from the Auslander-Buchsbaum-Theorem maximal Cohen-Macaulay modules over regular local rings are free, one might hope that at least the minimal free

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resolutions of maximal Cohen-Macaulay modules over complete intersections would be tractable. For hypersurface rings—the case of codimension 1—Eisenbud [5] showed that minimal free resolutions of finitely generated Cohen-Macaulay modules without free summands are periodic of period at most 2, and that these periodic resolutions are described by matrix factorizations, which he introduced for this purpose.

In this paper, we treat the next case. We give closed formulas for differentials in the free resolution of a maximal Cohen-Macaulay module M over a complete intersection $R = S/(f_1, f_2)$ of codimension 2 in a regular local ring S. Our description is based on the notion of a higher matrix factorization introduced in [6] and [7]. In case the higher matrix factorization is *minimal*, the resulting resolution is also minimal, and this is the case when M is a sufficiently high syzygy and f_1, f_2 are chosen generally over an infinite field. The description is in terms of the homotopies for f_1, f_2 on a special S-free resolution of M, also defined in terms of higher matrix factorizations.

An interesting feature of our description is the presence in the differentials of higher and higher powers of certain maps X and Y that appear as components in the homotopies for the special finite resolution.

Avramov and Buchweitz [1, Section 5.5] have given a different description of the minimal free resolutions of modules over a local complete intersection R of codimension 2 that are sufficiently high syzygies of a given module (such modules are automatically maximal Cohen-Macaulay modules). Their description consists of giving a non-minimal resolution $\mathbf{C}_\bullet$ of M and an explicit R-free summand D_i of each term C_i of $\mathbf{C}_\bullet$ in such a way that applying the differential of $\mathbf{C}_\bullet$ to a basis of C_i complementary to D_i, and then projecting the result back to a complement of D_{i-1} gives the minimal resolution of M. However, it is not clear how to give a closed formula for the differentials of the minimal free resolution from this description or, for example, how to see the presence of power of maps like X and Y.

In Section 1, we review the notions around higher matrix factorizations. The following section contains the application of this notion to the explicit construction. The last section gives our description of Ulrich modules over a codimension 2 quadratic complete intersection.

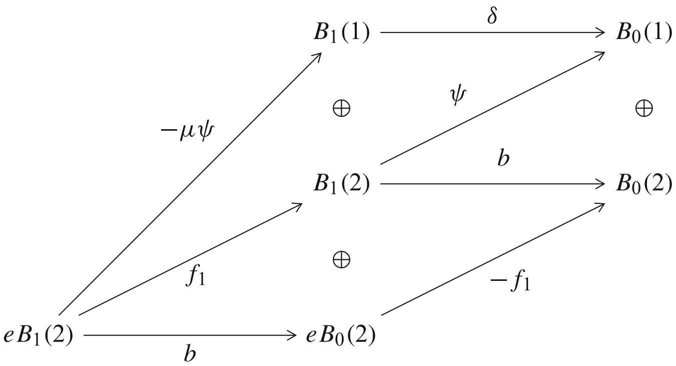
Using our methods, Mastroeni [8] has constructed a functor from codimension two higher matrix factorizations to the singularity category of the corresponding complete intersection.

Throughout the paper, f_1, f_2 will denote a regular sequence in a regular local ring S, and we consider the codimension 2 complete intersection $R = S/(f_1, f_2)$. We will denote by M the module of a higher matrix factorization (d, h) in the sense of [6] and [7].

1 The Finite Free Resolution of a Higher Matrix Factorization Module

Assumptions 1.1 Throughout the paper, f_1, f_2 is a regular sequence in a regular local ring S, and we consider the complete intersection $R = S/(f_1, f_2)$. By [7, Theorem 10.3], the following conditions are equivalent for a finitely generated R-module M:

(1) M is maximal Cohen-Macaulay over R.
(2) M is the module of a higher matrix factorization (d, h) in the sense of [6].
(3) M has a (possibly non-minimal) S-free resolution $\mathbf{L}$ of the form



where:

(1) $B_0(1),\ B_1(1),\ B_0(2),\ B_1(2)$ are finitely generated free S-modules;

(2) The module denoted eB_i is identified with B_s (the symbol e (denoted e_1 in [6]) serves only to distinguish the two copies of B_s notationally);

(3) There is a map $\mu : B_0(1) \to B_1(1)$ satisfying $\delta\mu = \mu\delta = f_1\mathrm{Id}$; that is, μ is a homotopy for f_1 on $B_1(1) \xrightarrow{\delta} B_0(1)$.

By [6, Theorem 3.1.4], the resolution $\mathbf{L}$ is minimal if M is the module of a minimal higher matrix factorization.

The notation above is consistent with that in the diagram in [7, Definition 10.2], but for simplicity we use ψ instead of ψ_2, we use b instead of b_2, we use e instead of e_1, and we use μ instead of h_1'.

If φ is a map of modules $\oplus_{1\le i\le s} P_i \longleftarrow \oplus_{1\le j\le s} Q_j$, then we write $\varphi_{P_i\leftarrow Q_j}$ for the component of φ with target P_i and source Q_j.

Lemma 1.2 *Under the assumptions* 1.1*, let h be a homotopy for multiplication by* f_2 *on* $\mathbf{L}$*. We consider two of its components and denote them by*

$$v := -h_{eB_1(2)\leftarrow B_1(1)}$$
$$\sigma := h_{B_1(2)\leftarrow B_0(2)}.$$

(1) *We have* $h_{B_1(2)\leftarrow B_0(1)} = v\mu$.

(2) *The component* $h_{eB_1(2)\leftarrow eB_0(2)}$ *can be chosen to be* σ.

Remark When considering maps in this paper, we often identify $eB_1(2) \cong B_1(2)$ and $eB_0(2) \cong B_0(2)$. For example, in the above lemma, the map $v\mu$ is from $B_0(1)$ to $eB_1(2)$, but we also identify it with a map from $B_0(1)$ to $B_1(2)$ when we write $h_{B_1(2)\leftarrow B_0(1)} = v\mu$.

Proof (1) Consider the homotopy equation $h\delta - f_1 v = 0 : B_1(1) \longrightarrow B_1(2)$. Applying μ on the right-hand side, we get $h\delta\mu = f_1 v\mu$, so $h_{B_1(2)\leftarrow B_0(1)} f_1 = f_1 v\mu$. As f_1 is a non-zero divisor, it follows that $h_{B_1(2)\leftarrow B_0(1)} = v\mu$.

(2) Set $\varsigma := h_{eB_1(2)\leftarrow eB_0(2)}$. The equations that ς needs to satisfy are

$$\begin{aligned}(b\varsigma - hf_1)_{eB_0(2)\leftarrow eB_0(2)} &= f_2\mathrm{Id}\\(f_1\varsigma - \sigma f_1)_{B_1(2)\leftarrow eB_0(2)} &= 0\\(\mu\psi\varsigma - hf_1)_{B_1(1)\leftarrow eB_0(2)} &= 0\\(\varsigma b + hf_1 + v\mu\psi - f_2\mathrm{Id})_{eB_1(2)\leftarrow eB_1(2)} &= 0.\end{aligned}$$

Clearly, the second equation is satisfied by σ. The first equation is satisfied by σ since $(b\sigma - f_1h)_{B_0(2)\leftarrow B_0(2)} = f_2\mathrm{Id}$. For the third equation, note that

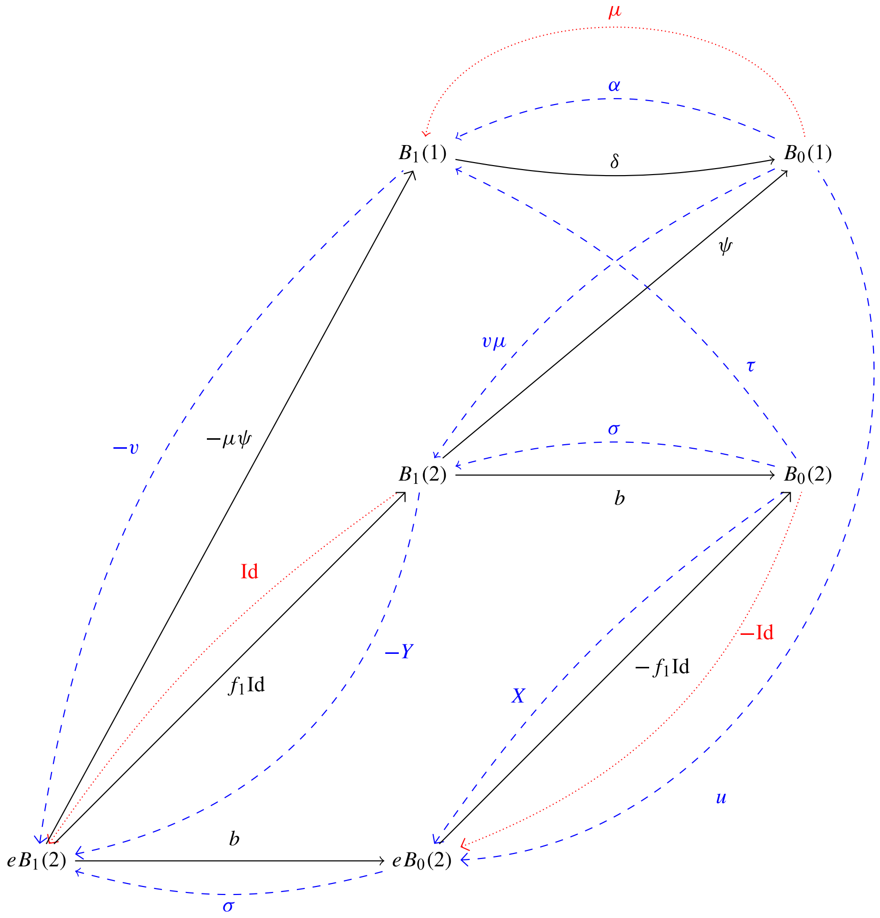
$$(\mu\psi\sigma - f_1h)_{B_1(1)\leftarrow B_0(2)} = (\mu\psi\sigma - \mu\delta h)_{B_1(1)\leftarrow B_0(2)} = \mu(\psi\sigma - \delta h)_{B_0(1)\leftarrow B_0(2)} = 0.$$

The last equation is satisfied by σ since

$$(\sigma b + hf_1 + v\mu\psi - f_2\mathrm{Id})_{B_1(2)\leftarrow B_1(2)} = 0.$$

□

Notation 1.3 The above lemma shows that we can relabel the diagram of **L** in 1.1 as follows. The homotopy for f_1 is shown with dotted arrows, and the homotopy for f_2 is shown with dashed arrows. In the diagram, we introduce names for the different components of the homotopies.

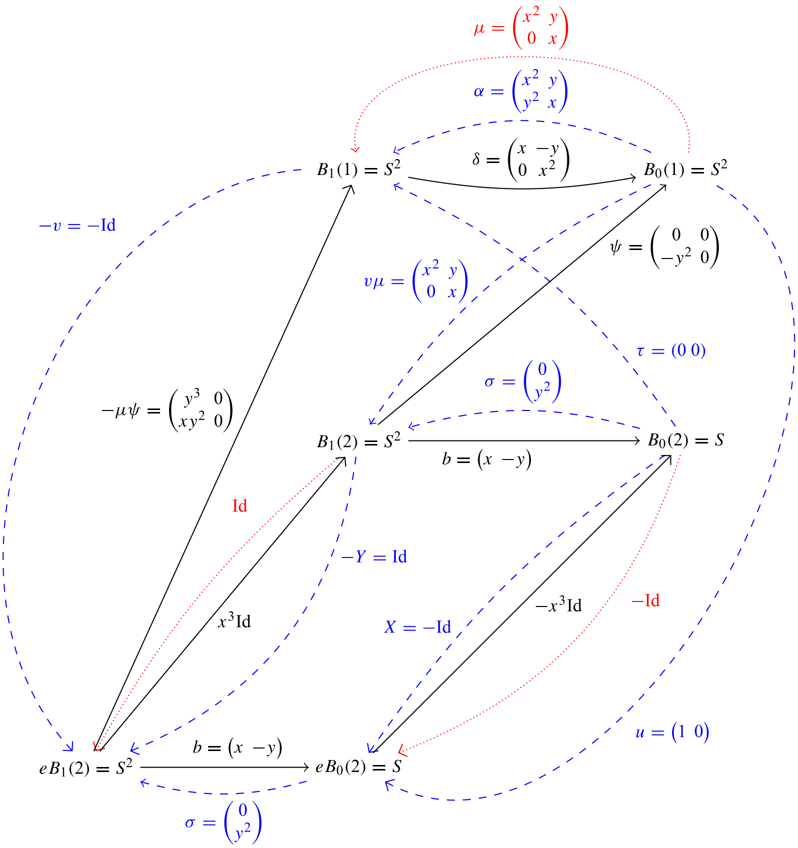


The Macaulay2 package `CompleteIntersectionResolutions` contains a routine called `makeFiniteResolution2` that produces the displayed maps in the diagram, with notation matching that is given here, in a hash table.

Example 1.4 This example is computed with Macaulay2. Let $S = k[x, y]$ and consider the regular sequence $f_1 = x^3$, $f_2 = x^3 - y^3$. Consider the S-module M with presentation:

$$S^5 \xrightarrow{\begin{pmatrix} x & -y & 0 & 0 & 0 \\ 0 & x^2 & -y^2 & 0 & 0 \\ 0 & 0 & x & -y & -x^3 \end{pmatrix}} S^3 \to M \to 0.$$

In the notation above, we have the following minimal free resolution of M with homotopies:



We will use the identities in Lemma 1.5 in the computations in Section 2.

Lemma 1.5 *We have the following identities on the diagram in 1.3*

(1) $\delta\alpha + \psi v\mu = f_2\mathrm{Id}$ *at* $B_0(1)$.
(2) $b\sigma - f_1 X = f_2\mathrm{Id}$ *at* $B_0(2)$.
(3) $bv\mu - f_1 u = 0$ *mapping* $B_0(1) \to B_0(2)$.
(4) $\delta\tau + \psi\sigma = 0$ *mapping* $B_0(2) \to B_0(1)$.

(5) $\alpha\delta + \mu\psi v = f_2\mathrm{Id}$ *at* $B_1(1)$.
(6) $\sigma b + v\mu\psi - f_1 Y = f_2\mathrm{Id}$ *at* $B_1(2)$ *and at* $eB_1(2)$.
(7) $-f_1 X + b\sigma = f_2\mathrm{Id}$ *at* $eB_0(2)$.
(8) $v\mu\delta - f_1 v = 0$ *mapping* $B_1(1) \to B_1(2)$.
(9) $\tau b + \alpha\psi + \mu\psi Y = 0$ *mapping* $B_1(2) \to B_1(1)$.
(10) $u\delta - bv = 0$ *mapping* $B_1(1) \to eB_0(2)$.
(11) $-f_1\tau - \mu\psi\sigma = 0$ *mapping* $eB_0(2) \to B_1(1)$.
(12) $Xb + u\psi - bY = 0$ *mapping* $B_1(2) \to eB_0(2)$.

Proof The identities above express the fact that we have a homotopy for f_2 on the resolution. □

2 The Infinite Free Resolution of a Higher Matrix Factorization Module

In this section, we obtain formulas for the differentials in an infinite R-free resolution $\mathbf{T}$ of the module M of a higher matrix factorization. We work under the assumptions 1.1 and use the notation 1.3. We will use the identities in Lemma 1.5.

Given a homomorphism $\varphi : N \to L$ of S-modules, we abuse notation and denote the induced homomorphism $R \otimes_S N \to R \otimes_S L$ by φ as well. Throughout this section, I stands for an identity map of appropriate size, and O stands for a zero map of appropriate size.

When considering maps in this paper, we often identify $eB_1(2) \cong B_1(2)$ and $eB_0(2) \cong B_0(2)$; see the remark after Lemma 1.2.

Theorem 2.1 *There is a free resolution* $\mathbf{T}$ *of* M *whose differentials are given as follows: For* $i \geq 0$, *the odd differentials* d_{2i+1} *of* $\mathbf{T}$ *have the form*

$$d_{2i+1} = \begin{pmatrix} \delta & \psi Y^0 v & \psi Y^1 v & \cdots & \psi Y^{i-2} v & \psi Y^{i-1} v & \psi Y^i \\ O & \delta & \psi Y^0 v & \cdots & \psi Y^{i-3} v & \psi Y^{i-2} v & \psi Y^{i-1} \\ O & O & \delta & \cdots & \psi Y^{i-4} v & \psi Y^{i-3} v & \psi Y^{i-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & \delta & \psi Y^0 v & \psi Y^1 \\ O & O & O & \cdots & O & \delta & \psi Y^0 \\ O & O & O & \cdots & O & O & b \end{pmatrix}.$$

For $i \geq 1$, *the even differentials* d_{2i} *of* $\mathbf{T}$ *have the form*

$$d_{2i} = \begin{pmatrix} \mu & \alpha & \tau X^0 u & \cdots & \tau X^{i-3} u & \tau X^{i-2} u & \tau X^{i-1} \\ O & \mu & \alpha & \cdots & \tau X^{i-4} u & \tau X^{i-3} u & \tau X^{i-2} \\ O & O & \mu & \cdots & \tau X^{i-5} u & \tau X^{i-4} u & \tau X^{i-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & \alpha & \tau X^0 u & \tau X^1 \\ O & O & O & \cdots & \mu & \alpha & \tau X^0 \\ O & O & O & \cdots & O & v\mu & \sigma \end{pmatrix}.$$

If the higher matrix factorization (d, h) *is minimal, then the resolution* $\mathbf{T}$ *is minimal as well.*

The minimality of $\mathbf{T}$ is achieved for high syzygies by the following result:

Theorem 2.2 [6, Theorem 1.3.1] *Let* S *be a regular local ring with infinite residue field, and let* $I \subset S$ *be an ideal generated by a regular sequence of length* c. *Set* $R = S/I$, *and suppose that N is a finitely generated R-module. Let $f_1, \dots, f_c$ be a generic choice of elements minimally generating I. If M is a sufficiently high syzygy of N over R, then M is the module of a minimal higher matrix factorization (d, h) with respect to $f_1, \dots, f_c$.*

Example 2.3 We continue Example 1.4. By Theorem 2.1, the even differentials in the infinite minimal free resolution of M over $R = k[x, y]/(x^3, y^3)$ have the form

$$\begin{pmatrix} \mu & \alpha & & & & \\ & \mu & \alpha & & & \\ & & \ddots & \ddots & & \\ & & & \mu & \alpha & \\ & & & & \mu & \sigma \end{pmatrix}$$

since $\tau = (0\ 0)$ and $\upsilon = \mathrm{Id}$. The odd differentials have the form

$$\begin{pmatrix} \delta & \psi & -\psi & \psi & -\psi & \dots & (-1)^{i-2}\psi & (-1)^{i-1}\psi & (-1)^{i}\psi \\ & \delta & \psi & -\psi & \psi & \dots & (-1)^{i-3}\psi & (-1)^{i-2}\psi & (-1)^{i-1}\psi \\ & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & \ddots & & & \\ & & & & & & \delta & \psi & -\psi \\ & & & & & & & \delta & \psi \\ & & & & & & & & b \end{pmatrix}$$

since $Y = -\mathrm{Id}$ and $\upsilon = \mathrm{Id}$.

Construction 2.4 We consider the free resolution $(\mathbf{T}, d)$ given in [6, Theorem 5.1.2]. We choose CI operators t_1, t_2 as in [6, Theorem 5.1.4] so that they commute on $\mathbf{T}$; we remark that it was proved in [1] that there exists a choice of commuting CI operators, but our computations use the particular choice of CI operators that we have constructed. We dualize and consider $\mathrm{Hom}_R(\mathbf{T}, R)$.

In the rest of this section, we write $\widehat{-}$ for $\mathrm{Hom}_R(R \otimes -, R)$.

We consider $\widehat{\mathbf{T}} = \mathrm{Hom}_R(\mathbf{T}, R)$ with differential $\widehat{d}$. Set

$$r_1 = \widehat{t_1} \quad \text{and} \quad r_2 = \widehat{t_2}$$

acting on $\widehat{\mathbf{T}}$. By [6, Theorem 5.1.2], the free module $\widehat{T_{2i}}$ is the direct sum of the modules

$$r_1^i \widehat{B}_0(1),\ r_1^{i-1} r_2 \widehat{B}_0(1),\ r_1^{i-2} r_2^2 \widehat{B}_0(1),\ \dots, r_2^i \widehat{B}_0(1),\ r_2^i \widehat{B}_0(2),$$

and similarly $\widehat{T_{2i+1}}$ is the direct sum of the modules

$$r_1^i \widehat{B}_1(1),\ r_1^{i-1} r_2 \widehat{B}_1(1),\ r_1^{i-2} r_2^2 \widehat{B}_1(1),\ \dots, r_2^i \widehat{B}_1(1),\ r_2^i \widehat{B}_1(2).$$

Furthermore, the first two differentials of $\widehat{\mathbf{T}}$ in that decomposition are

$$\widehat{d_1} = \begin{matrix} & \widehat{B}_0(1) & \widehat{B}_0(2) \\ \widehat{B}_1(1) & \widehat{\delta} & O \\ \widehat{B}_1(2) & \widehat{\psi} & \widehat{b} \end{matrix} \tag{2.1}$$

$$\widehat{d}_2 = \begin{array}{c} \\ r_1\widehat{B}_0(1) \\ r_2\widehat{B}_0(1) \\ r_2\widehat{B}_0(2) \end{array} \begin{array}{c} \begin{array}{cc} \widehat{B}_1(1) & \widehat{B}_1(2) \end{array} \\ \begin{pmatrix} \widehat{\mu} & O \\ \widehat{\alpha} & \widehat{\mu}\widehat{v} \\ \widehat{\tau} & \widehat{\sigma} \end{pmatrix} \end{array}.$$

In this notation, we will prove the following formulas for Theorem 2.1:
For $i \geq 0$, the odd differentials $\widehat{d}_{2i+1}$ of $\widehat{\mathbf{T}}$ have the form

$$\begin{array}{c|cccccc} & r_1^i\widehat{B}_0(1) & r_1^{i-1}r_2\widehat{B}_0(1) & \dots & r_1r_2^{i-1}\widehat{B}_0(1) & r_2^i\widehat{B}_0(1) & r_2^i\widehat{B}_0(2) \\ \hline r_1^i\widehat{B}_1(1) & \widehat{\delta} & O & \dots & O & O & O \\ r_1^{i-1}r_2\widehat{B}_1(1) & \widehat{v}\widehat{Y}^0\widehat{\psi} & \widehat{\delta} & \dots & O & O & O \\ r_1^{i-2}r_2^2\widehat{B}_1(1) & \widehat{v}\widehat{Y}^1\widehat{\psi} & \widehat{v}\widehat{Y}^0\widehat{\psi} & \dots & O & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ r_1r_2^{i-1}\widehat{B}_1(1) & \widehat{v}\widehat{Y}^{i-2}\widehat{\psi} & \widehat{v}\widehat{Y}^{i-3}\widehat{\psi} & \dots & \widehat{\delta} & O & O \\ r_2^i\widehat{B}_1(1) & \widehat{v}\widehat{Y}^{i-1}\widehat{\psi} & \widehat{v}\widehat{Y}^{i-2}\widehat{\psi} & \dots & \widehat{v}\widehat{Y}^0\widehat{\psi} & \widehat{\delta} & O \\ r_2^i\widehat{B}_1(2) & \widehat{Y}^i\widehat{\psi} & \widehat{Y}^{i-1}\widehat{\psi} & \dots & \widehat{Y}^1\widehat{\psi} & \widehat{Y}^0\widehat{\psi} & \widehat{b} \end{array}.$$

For $i \geq 1$, the even differentials $\widehat{d}_{2i}$ of $\widehat{\mathbf{T}}$ have the form

$$\begin{array}{c|cccccc} & r_1^{i-1}\widehat{B}_1(1) & r_1^{i-2}r_2\widehat{B}_1(1) & \dots & r_1r_2^{i-2}\widehat{B}_1(1) & r_2^{i-1}\widehat{B}_1(1) & r_2^{i-1}\widehat{B}_1(2) \\ \hline r_1^i\widehat{B}_0(1) & \widehat{\mu} & O & \dots & O & O & O \\ r_1^{i-1}r_2\widehat{B}_0(1) & \widehat{\alpha} & \widehat{\mu} & \dots & O & O & O \\ r_1^{i-2}r_2^2\widehat{B}_0(1) & \widehat{u}\widehat{X}^0\tau & \widehat{\alpha} & \dots & O & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ r_1r_2^{i-1}\widehat{B}_0(1) & \widehat{u}\widehat{X}^{i-3}\widehat{\tau} & \widehat{u}\widehat{X}^{i-4}\widehat{\tau} & \dots & \widehat{\alpha} & \widehat{\mu} & O \\ r_2^i\widehat{B}_0(1) & \widehat{u}\widehat{X}^{i-2}\widehat{\tau} & \widehat{u}\widehat{X}^{i-3}\widehat{\tau} & \dots & \widehat{u}\widehat{X}^0\widehat{\tau} & \widehat{\alpha} & \widehat{\mu}\widehat{v} \\ r_2^i\widehat{B}_0(2) & \widehat{X}^{i-1}\widehat{\tau} & \widehat{X}^{i-2}\widehat{\tau} & \dots & \widehat{X}^1\widehat{\tau} & \widehat{X}^0\widehat{\tau} & \widehat{\sigma} \end{array}.$$

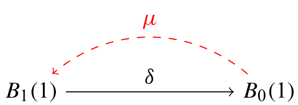
Remark The Shamash construction is a generalization of Tate's construction [10]. It is due to Shamash [9] for one element and to Eisenbud [5] for several elements. It provides a resolution of any module over a complete intersection, but the resolution is usually non-minimal. We will make use of the Shamash construction. We will use it in the form and with the notation in [6, Construction 4.1.3 and Proposition 4.1.4].

The next lemma will be used in the proof of Theorem 2.1.

Lemma 2.5 *The third differential of* $\widehat{\mathbf{T}}$ *is*

$$d_3^* = \begin{array}{c} \\ r_1\widehat{B}_1(1) \\ r_2\widehat{B}_1(1) \\ r_2\widehat{B}_1(2) \end{array} \begin{array}{c} \begin{array}{ccc} r_1\widehat{B}_0(1) & r_2\widehat{B}_0(1) & r_2\widehat{B}_0(2) \end{array} \\ \begin{pmatrix} \widehat{\delta} & O & O \\ \widehat{v}\widehat{\psi} & \widehat{\delta} & O \\ \widehat{Y}\widehat{\psi} & \widehat{\psi} & \widehat{b} \end{pmatrix} \end{array}.$$

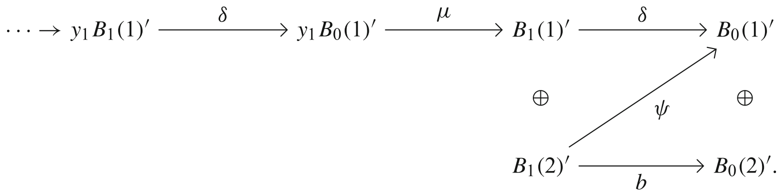
Proof We will follow [6, Construction 5.1.1]. We start with the matrix factorization



of f_1. Applying the Shamash construction, we get the periodic (of period 2) free resolution

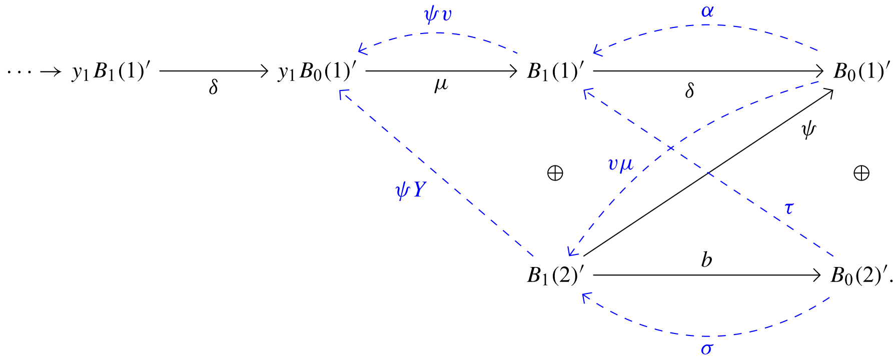
$$\cdots \to y_1 B_1(1)' \xrightarrow{\delta} y_1 B_0(1)' \xrightarrow{\mu} B_1(1)' \xrightarrow{\delta} B_0(1)',$$

where $-'$ stands for $- \otimes R'$ and $R' := S/(f_1)$, and y_1 is a variable. Then, we write the Box complex



(2.2)

The free resolution **T** over R is obtained by applying the Shamash construction to that Box complex. For this purpose, we need a homotopy for f_2 on the Box complex. Since our goal here is to compute only the third differential, it suffices to construct only the first 2 steps of the homotopy. Straightforward computation using the homotopy relations on the diagram in 1.3 shows that the homotopy can be chosen to be



(2.3)

Applying the Shamash construction, we conclude that

$$d_3 = \begin{pmatrix} \delta & \psi v & \psi Y \\ O & \delta & \psi \\ O & O & b \end{pmatrix}.$$

□

Proposition 2.6 *The first CI operators are:*

$$r_1\big|_{T_0^*} = \begin{matrix} & \widehat{B}_0(1) & \widehat{B}_0(2) \\ r_1\widehat{B}_0(1) & I & O \\ r_2\widehat{B}_0(1) & O & \widehat{u} \\ r_2\widehat{B}_0(2) & O & \widehat{X} \end{matrix}, \quad r_1\big|_{T_1^*} = \begin{matrix} & \widehat{B}_1(1) & \widehat{B}_1(2) \\ r_1\widehat{B}_1(1) & I & O \\ r_2\widehat{B}_1(1) & O & \widehat{v} \\ r_2\widehat{B}_1(2) & O & \widehat{Y} \end{matrix},$$

and

$$r_2\big|_{T_0^*} = \begin{matrix} & \widehat{B}_0(1) & \widehat{B}_0(2) \\ r_1\widehat{B}_0(1) & O & O \\ r_2\widehat{B}_0(1) & I & O \\ r_2\widehat{B}_0(2) & O & I \end{matrix}, \quad r_2\big|_{T_1^*} = \begin{matrix} & \widehat{B}_1(1) & \widehat{B}_1(2) \\ r_1\widehat{B}_1(1) & O & O \\ r_2\widehat{B}_1(1) & I & O \\ r_2\widehat{B}_1(2) & O & I \end{matrix}.$$

Proof Throughout this proof, we write $\overline{}$ for $\mathrm{Hom}_S(S \otimes -, S)$.

We lift the differentials of $\widehat{\mathbf{T}}$ to S and denote the lifting of $\widehat{d_i}$ by $\widetilde{d_i}$. Then,

$$\begin{aligned}
\widetilde{d_2}\widetilde{d_1} &= \begin{pmatrix} \overline{\mu} & O \\ \overline{\alpha} & \overline{\mu v} \\ \overline{\tau} & \overline{\sigma} \end{pmatrix} \begin{pmatrix} \overline{\delta} & O \\ \overline{\psi} & \overline{b} \end{pmatrix} \\
&= \begin{pmatrix} \overline{\mu}\,\overline{\delta} & O \\ \overline{\alpha}\,\overline{\delta} + \overline{\mu v}\,\overline{\psi} & \overline{\mu v}\,\overline{b} \\ \overline{\tau}\,\overline{\delta} + \overline{\sigma}\,\overline{\psi} & \overline{\sigma}\,\overline{b} \end{pmatrix} = \begin{pmatrix} \overline{\delta\mu} & \overline{O} \\ \overline{\delta\alpha + \psi v \mu} & \overline{b v \mu} \\ \overline{\delta\tau + \psi\sigma} & \overline{b\sigma} \end{pmatrix} \\
&= \begin{pmatrix} f_1 I & O \\ f_2 I & \overline{u} f_1 I \\ O & f_2 I + f_1 \overline{X} \end{pmatrix} = f_1 \begin{pmatrix} I & O \\ O & \overline{u} \\ O & \overline{X} \end{pmatrix} + f_2 \begin{pmatrix} O & O \\ I & O \\ O & I \end{pmatrix}.
\end{aligned}$$

This proves the desired formulas for the CI operators acting on $\widehat{T_0}$.

Furthermore, we compute

$$\begin{aligned}
\widetilde{d_3}\widetilde{d_2} &= \begin{pmatrix} \overline{\delta} & O & O \\ \overline{v}\,\overline{\psi} & \overline{\delta} & O \\ \overline{Y}\,\overline{\psi} & \overline{\psi} & \overline{b} \end{pmatrix} \begin{pmatrix} \overline{\mu} & O \\ \overline{\alpha} & \overline{\mu v} \\ \overline{\tau} & \overline{\sigma} \end{pmatrix} \\
&= \begin{pmatrix} \overline{\delta}\,\overline{\mu} & O \\ \overline{v}\,\overline{\psi}\,\overline{\mu} + \overline{\delta}\,\overline{\alpha} & \overline{\delta}\,\overline{\mu v} \\ \overline{Y}\,\overline{\psi}\,\overline{\mu} + \overline{\psi}\,\overline{\alpha} + \overline{b}\,\overline{\tau} & \overline{\psi}\,\overline{\mu v} + \overline{b}\,\overline{\sigma} \end{pmatrix} \\
&= \begin{pmatrix} \overline{\mu\delta} & \overline{O} \\ \overline{\mu\psi v + \alpha\delta} & \overline{v\mu\delta} \\ \overline{\mu\psi Y + \alpha\psi + \tau b} & \overline{v\mu\psi + \sigma b} \end{pmatrix} \\
&= \begin{pmatrix} f_1 I & O \\ f_2 I & \overline{v} f_1 I \\ O & f_2 I + f_1 \overline{Y} \end{pmatrix} \\
&= f_1 \begin{pmatrix} I & O \\ O & \overline{v} \\ O & \overline{Y} \end{pmatrix} + f_2 \begin{pmatrix} O & O \\ I & O \\ O & I \end{pmatrix}.
\end{aligned}$$

This proves the desired formulas for the CI operators acting on $\widehat{T_1}$. □

Proposition 2.7 *The CI operator r_2 acts on $\widehat{\mathbf{T}}$ as an identity embedding that is dual to the projection t_2 (by construction). The CI operator r_1 acts on $\widehat{\mathbf{T}}$ as follows:*

For $i \geq 0$, the CI operator $r_{1,2i+2} : \widehat{T_{2i}} \to \widehat{T_{2i+2}}$ is given by

$$\begin{array}{c|cccccc}
 & r_1^i \widehat{B}_0(1) & r_1^{i-1} r_2 \widehat{B}_0(1) & \cdots & r_2^i \widehat{B}_0(1) & r_2^i \widehat{B}_0(2) \\
\hline
r_1^{i+1} \widehat{B}_0(1) & I & & & & \\
r_1^i r_2 \widehat{B}_0(1) & & I & & & \\
\vdots & & & \ddots & & \\
r_1 r_2^i \widehat{B}_0(1) & & & & I & \\
r_2^{i+1} \widehat{B}_0(1) & & & & & \widehat{u} \\
r_2^{i+1} \widehat{B}_0(2) & & & & & \widehat{X}
\end{array}$$

and the CI operator $r_{1,2i+3}: \widehat{T}_{2i+1} \to \widehat{T}_{2i+3}$ *is given by*

$$r_{1,2i+3} = \begin{array}{c} \\ r_1^{i+1}\widehat{B}_1(1) \\ r_1^i r_2 \widehat{B}_1(1) \\ \vdots \\ r_1 r_2^i \widehat{B}_1(1) \\ r_2^{i+1}\widehat{B}_1(1) \\ r_2^{i+1}\widehat{B}_1(2) \end{array} \begin{array}{c} \begin{array}{ccccc} r_1^i\widehat{B}_1(1) & r_1^{i-1}r_2\widehat{B}_1(1) & \cdots & r_2^i\widehat{B}_1(1) & r_2^i\widehat{B}_1(2) \end{array} \\ \begin{pmatrix} I & & & & \\ & I & & & \\ & & \ddots & & \\ & & & I & \\ & & & & \widehat{v} \\ & & & & \widehat{Y} \end{pmatrix} \end{array}.$$

For $i \geq 1$*, we have:*

$$r_1^i|_{\widehat{T}_2} = \begin{pmatrix} I & O & O \\ O & I & O \\ O & O & \widehat{u} \\ O & O & \widehat{u}\widehat{X} \\ O & O & \widehat{u}\widehat{X}^2 \\ \vdots & \vdots & \vdots \\ O & O & \widehat{u}\widehat{X}^{i-1} \\ O & O & \widehat{X}^i \end{pmatrix}$$

and

$$r_1^i|_{\widehat{T}_1} = \begin{pmatrix} I & O \\ O & \widehat{v} \\ O & \widehat{v}\widehat{Y} \\ O & \widehat{v}\widehat{Y}^2 \\ \vdots & \vdots \\ O & \widehat{v}\widehat{Y}^{i-1} \\ O & \widehat{Y}^i \end{pmatrix}.$$

Proof For $i \geq 1$, the formulas for $r_{2,2i+2},\ r_{1,2i+2},\ r_{2,2i+3},\ r_{1,2i+3}$ follow from the fact that r_1 and r_2 commute on $\widehat{\mathbf{T}}$.

Straightforward computation shows that

$$r_1^2|_{\widehat{T}_2} = \begin{pmatrix} I & O & O \\ O & I & O \\ O & O & \widehat{u} \\ O & O & \widehat{u}\widehat{X} \\ O & O & \widehat{X}^2 \end{pmatrix}$$

and

$$r_1^2|_{\widehat{T}_1} = \begin{pmatrix} I & O \\ O & \widehat{v} \\ O & \widehat{v}\widehat{Y} \\ O & \widehat{Y}^2 \end{pmatrix}.$$

By induction, we get

$$r_1^i|_{\widehat{T}_2}= \begin{pmatrix} I & O & O \\ O & I & O \\ O & O & \widehat{u} \\ O & O & \widehat{u}\widehat{X} \\ O & O & \widehat{u}\widehat{X}^2 \\ \vdots & \vdots & \vdots \\ O & O & \widehat{u}\widehat{X}^{i-1} \\ O & O & \widehat{X}^i \end{pmatrix}$$

and

$$r_1^i|_{\widehat{T}_1}= \begin{pmatrix} I & O \\ O & \widehat{v} \\ O & \widehat{v}\widehat{Y} \\ O & \widehat{v}\widehat{Y}^2 \\ \vdots & \vdots \\ O & \widehat{v}\widehat{Y}^{i-1} \\ O & \widehat{Y}^i \end{pmatrix}.$$

□

Lemma 2.8 *For $i \geq 1$, we have*

$$\widehat{d}\left(r_1^i\widehat{B}_0(1)\right) = \widehat{\delta}r_1^i\widehat{B}_1(1) + \left(\sum_{1\leq q\leq i}(\widehat{v}\widehat{Y}^{q-1}\widehat{\psi})\, r_1^{i-q}r_2^q\widehat{B}_1(1)\right) + \left(\widehat{Y}^i\widehat{\psi}\right) r_2^i\widehat{B}_1(2)$$

and

$$\begin{aligned} &\widehat{d}\left(r_1^{i-1}\widehat{B}_1(1)\right) \\ &= \widehat{\mu}\, r_1^i\widehat{B}_0(1) + \widehat{\alpha}r_1^{i-1}r_2\widehat{B}_0(1) + \left(\sum_{1=q\leq i-1}\left(\widehat{u}\widehat{X}^{q-1}\widehat{\tau}\right) r_1^{i-q-1}r_2^{q+1}\widehat{B}_0(1)\right) + \left(\widehat{X}^{i-1}\widehat{\tau}\right) r_2^i\widehat{B}_0(2). \end{aligned}$$

Proof The second equality below uses the formula for $\widehat{d_1}$ in (2.1). The last equality uses a formula in Proposition 2.7. The following computation proves the first formula in our lemma:

$$\widehat{d}(r_1^i\widehat{B}_0(1)) = r_1^i\widehat{d}_1|_{\widehat{B}_0(1)} = r_1^i\begin{pmatrix} \widehat{\delta} & O \\ \widehat{\psi} & \widehat{b} \end{pmatrix}\begin{pmatrix} I \\ O \end{pmatrix} = r_1^i|_{\widehat{T}_1}\begin{pmatrix} \widehat{\delta} \\ \widehat{\psi} \end{pmatrix} = \begin{pmatrix} \widehat{\delta} \\ \widehat{v}\widehat{\psi} \\ \widehat{v}\widehat{Y}\widehat{\psi} \\ \widehat{v}\widehat{Y}^2\widehat{\psi} \\ \vdots \\ \widehat{v}\widehat{Y}^{i-1}\widehat{\psi} \\ \widehat{Y}^i\widehat{\psi} \end{pmatrix}.$$

Similarly, we prove the second formula in this lemma. The second equality below uses the formula for $\widehat{d_2}$ in (2.1). The last equality uses a formula in Proposition 2.7.

$$\widehat{d}\left(r_1^{i-1}B_1(1)\right) = r_1^{i-1}\widehat{d_2}|_{B_1(1)} = r_1^{i-1}\begin{pmatrix}\widehat{\mu} & O\\ \widehat{\alpha} & \widehat{\mu}\widehat{\upsilon}\\ \widehat{\tau} & \widehat{\sigma}\end{pmatrix}\begin{pmatrix}I\\ O\end{pmatrix}$$
$$= r_1^{i-1}|_{\widehat{T}_2}\begin{pmatrix}\widehat{\mu}\\ \widehat{\alpha}\\ \widehat{\tau}\end{pmatrix} = \begin{pmatrix}\widehat{\mu}\\ \widehat{\alpha}\\ \widehat{u}\widehat{\tau}\\ \widehat{u}\widehat{X}\widehat{\tau}\\ \widehat{u}\widehat{X}^2\widehat{\tau}\\ \vdots\\ \widehat{u}\widehat{X}^{i-2}\widehat{\tau}\\ \widehat{X}^{i-1}\widehat{\tau}\end{pmatrix}.$$

□

Proof of Theorem 2.1 We will prove the formulas for the differential given at the end of Construction 2.4.

The first column follows from Lemma 2.8.

The computations below use the fact that the CI-operator t_2 is a projection and so r_2 is an identity embedding; see Proposition 2.7.

First, we compute the last two columns in the formula for the differential. The formula for the last column follows from

$$\widehat{d}\Big|_{r_2^i\widehat{B}_0(2)} = r_2^i\widehat{d_1}\Big|_{\widehat{B}_0(2)} = \widehat{b}\Big|_{r_2^i\widehat{B}_0(2)}$$
$$\widehat{d}\Big|_{r_2^{i-1}\widehat{B}_1(2)} = r_2^{i-1}\widehat{d_2}\Big|_{\widehat{B}_1(2)} = \widehat{\mu}\widehat{\upsilon}\Big|_{r_2^{i-1}\widehat{B}_1(2)} + \widehat{\sigma}\Big|_{r_2^{i-1}\widehat{B}_1(2)},$$

for $i \geq 1$. The formula for the column preceding the last column follows from

$$\widehat{d}\Big|_{r_2^i\widehat{B}_0(1)} = r_2^i\widehat{d_1}\Big|_{\widehat{B}_0(1)} = \widehat{\delta}\Big|_{r_2^i\widehat{B}_0(1)} + \widehat{\psi}\Big|_{r_2^i\widehat{B}_0(1)}$$
$$\widehat{d}\Big|_{r_2^{i-1}\widehat{B}_1(1)} = r_2^{i-1}\widehat{d_2}\Big|_{\widehat{B}_1(1)} = \widehat{\mu}\Big|_{r_2^{i-1}\widehat{B}_1(1)} + \widehat{\alpha}\Big|_{r_2^{i-1}\widehat{B}_1(1)} + \widehat{\tau}\Big|_{r_2^{i-1}\widehat{B}_1(1)},$$

for $i \geq 1$.

Now, we compute the formulas for the rest of the columns. For $i \geq 1$ and $1 \leq j \leq i-1$ we have:

$$\begin{aligned}
&\widehat{d}\left(r_1^{i-j}r_2^j\widehat{B}_0(1)\right)\\
&= r_2^j\left(\widehat{d}r_1^{i-j}\widehat{B}_0(1)\right)\\
&= r_2^j\left(\widehat{\delta}\,r_1^{i-j}\widehat{B}_1(1) + \sum_{1\leq q\leq i-j}\left(\widehat{\upsilon}\widehat{Y}^{q-1}\widehat{\psi}\right)r_1^{i-q-j}r_2^q\widehat{B}_1(1) + \left(\widehat{Y}^{i-j}\widehat{\psi}\right)r_2^{i-j}\widehat{B}_1(2)\right)\\
&= \widehat{\delta}\,r_1^{i-j}r_2^j\widehat{B}_1(1) + \sum_{1\leq q\leq i-j}\left(\widehat{\upsilon}\widehat{Y}^{q-1}\widehat{\psi}\right)r_1^{i-q-j}r_2^{q+j}\widehat{B}_1(1) + \left(\widehat{Y}^{i-j}\widehat{\psi}\right)r_2^i\widehat{B}_1(2)
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{d}\left(r_1^{i-j-1} r_2^j \widehat{B}_1(1)\right)\\
&= r_2^j r_1^{i-j-1} \widehat{d}_2 \Big|_{\widehat{B}_1(1)}\\
&= r_2^j \Bigg(\widehat{\mu}\, r_1^{i-j} \widehat{B}_0(1) + \widehat{\alpha}\, r_1^{i-j-1} r_2 \widehat{B}_0(1) + \sum_{2\le q\le i-j} \left(\widehat{u}\widehat{X}^{q-2}\widehat{\tau}\right) r_1^{i-j-q} r_2^q \widehat{B}_0(1)\\
&\quad + (X^{i-j-1}\tau) r_2^{i-j} \widehat{B}_0(2)\Bigg)\\
&= \widehat{\mu}\, r_1^{i-j} r_2^j \widehat{B}_0(1) + \widehat{\alpha}\, r_1^{i-j-1} r_2^{j+1} \widehat{B}_0(1) + \sum_{2\le q\le i-j} \left(\widehat{u}\widehat{X}^{q-2}\widehat{\tau}\right) r_1^{i-j-q} r_2^{j+q} \widehat{B}_0(1)\\
&\quad + \left(\widehat{X}^{i-j-1}\widehat{\tau}\right) r_2^i \widehat{B}_0(2).
\end{aligned}$$

□

3 Ulrich Modules

In this section, f_1 and f_2 are a quadratic regular sequence in a standard graded polynomial ring S. We consider the graded complete intersection $R = S/(f_1, f_2)$. An *Ulrich* module over R may be defined as a maximal Cohen-Macaulay module M over R, generated in a single degree such that the entries of the matrices in the minimal S-free resolution of M are all linear forms. The equivalence of this definition to the original definition (which states that a maximal Cohen-Macaulay module M is an Ulrich module if the multiplicity of M is equal to the minimal number of generators) is proved in [3, Proposition 1.4], where they are called MGMCM modules. The existence of such modules is proved in [2]. We give a description of the Ulrich modules over R.

A *matrix factorization* of an element $f \in S$ is a pair of square matrices (δ, μ) such that $\delta\mu = \mu\delta = f\,\mathrm{Id}$. It is *graded* if the entries in the matrices are homogeneous.

Theorem 3.1 *A finitely generated graded R-module M is an Ulrich module if and only if there exists a graded minimal matrix factorization (δ, μ) of f_1 with size w and a number $p \le w$ so that:*

(1) *The matrix δ' that consists of the first p rows of δ is linear.*
(2) *The matrix μ' that consists of the last $w - p$ columns of μ is linear.*
(3) *There exists a $(w \times p)$-matrix g with $\delta' g = f_2\,\mathrm{Id}$.*
(4) $M = \mathrm{Coker}(\delta')$.

Proof By [7, Theorem 10.2], a module M is maximal Cohen-Macaulay over R if and only if it has a (possibly non-minimal) graded S-free resolution $\mathbf{L}$ of the form described in 1.3. From the definition above, it follows that M is an Ulrich module if and only if the maps

$$eB_0(2) \xrightarrow{-f_1} B_0(2) \quad \text{and} \quad eB_1(2) \xrightarrow{f_1} B_1(2)$$

in $\mathbf{L}$ disappear after the resolution is minimized. We will analyze when this happens. We use the notation in the diagram in 1.3.

The map b is minimal by [7, Proposition 3.5]. If $B_0(2) \neq 0$, then the map $eB_0(2) \xrightarrow{-f_1} B_0(2)$ would make the minimization non-linear. Hence, $B_0(2) = 0$.

The matrix factorization (δ, μ) of f_1 is a direct sum of a minimal matrix factorization and copies of the trivial matrix factorizations (Id, f_1) and (f_1, Id). We can assume that no copies of (Id, f_1) are present because the minimal free resolution $B_1(1) \xrightarrow{\delta} B_0(1)$ can be chosen to be minimal by [7, Theorem 4.1 and construction before it]. If we have any copies of (f_1, Id), then the $R' := S/(f_1)$-module $M' := \mathrm{Coker}(\delta)$ would have a direct R'-free summand which contradicts the construction of M' as the essential Cohen-Macaulay approximation of the module M over R' as in [7, Theorem 4.1] (the notation here is consistent with that in [7, Theorem 4.1]). Hence, no copies of (f_1, Id) appear. Thus, we can assume that the matrix factorization (δ, μ) is minimal.

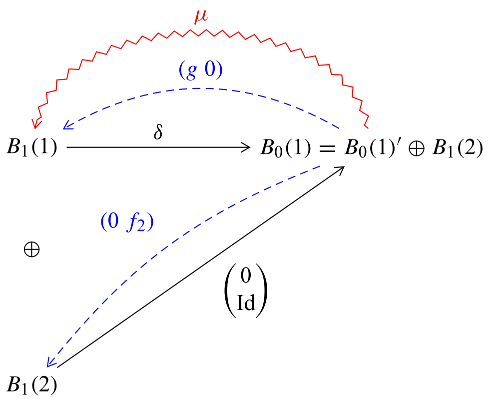
Since μ is minimal, the map $-\mu\psi$ is minimal as well. Recall that the map b is also minimal. Therefore, the map $eB_1(2) \xrightarrow{f_1} B_1(2)$ disappears after $\mathbf{L}$ is minimized if and only if $\psi = \mathrm{Id}$ maps $B_1(2)$ to a free summand of $B_0(1)$.

Thus we see that, after minimization of the non-minimal resolution $\mathbf{L}$, we get a minimal S-free resolution of M of the form

$$0 \to eB_1(2) \xrightarrow{\mu'} B_1(1) \xrightarrow{\delta'} B_0(1)',$$

where $B_0(1) \cong B_0(1)' \oplus \mathrm{Im}(\psi)$. Set $p = \mathrm{rank}\, B_0(1)'$ and $w = \mathrm{rank}\, B_0(1)$. Then, $\mathrm{rank}(eB_1(2)) = w - p$. Since the module M is annihilated by f_2, it follows that there exists a $(w \times p)$-matrix g with $\delta' g = f_2\,\mathrm{Id}$. Thus, conditions (1)–(4) are satisfied.

Conversely, suppose that M satisfies conditions (1)–(4). Denote by $B_0(1)'$ the free module spanned by the first p basis vectors of $B_0(1)$, and let $B_1(2)$ be the free module spanned by the last $w - p$ basis vectors; here, we consider the basis vectors with respect to which the matrices δ and μ are given. Then,



is a higher matrix factorization with respect to f_1, f_2. The module N of this higher matrix factorization has a non-minimal free resolution given by [6, Theorem 3.1.4]. We minimize it as above, and conclude that $M = N$ has a linear minimal S-free resolution. □

We close with an example illustrating the theorem above.

Example 3.2 Let $S = k[x_0, x_1, x_2, x_3]$ and set

$$f_1 = x_0x_1 - x_2x_3, \;\; f_2 = x_1^2 + x_0x_2 - x_3^2.$$

Consider

$$\delta = \begin{pmatrix} x_0 & x_0 & x_2 & 0 & 0 & x_1 & x_3 & x_2 \\ 0 & x_2 & 0 & -x_3 & -x_1 & 0 & x_1 & 0 \\ 0 & 0 & x_0 & x_3 & x_1 & x_3 & 0 & 0 \\ x_3 & 0 & 0 & x_0 & x_2 & 0 & 0 & x_1 \\ x_0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 \\ -x_0 & 0 & -x_2 & 0 & 0 & -x_1 & 0 & -x_2 \\ 0 & 0 & 0 & -x_3 & -x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & x_2 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mu = \begin{pmatrix} 0 & 0 & 0 & -x_2 & x_1 & 0 & 0 & x_2 \\ x_1 & -x_3 & 0 & 0 & 0 & x_1 & x_3 & 0 \\ 0 & 0 & x_1 & 0 & x_3 & x_3 & x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_2 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_0 & -x_3 \\ 0 & 0 & -x_2 & 0 & -x_0 & -x_0 & -x_2 & 0 \\ -x_2 & x_0 & 0 & 0 & 0 & -x_2 & -x_0 & 0 \\ 0 & 0 & 0 & x_0 & -x_3 & 0 & 0 & -x_0 \end{pmatrix}.$$

We have the matrix factorization

$$\mu\delta = \delta\mu = (x_0x_1 - x_2x_3)I,$$

where I is an 8×8 identity matrix. In the notation of Theorem 3.1, set $w = 8$, $p = 4$ and let δ' be the matrix consisting of the first 4 rows of δ, and let μ' be the matrix consisting of the last 4 columns of μ. The matrix

$$g = \begin{pmatrix} x_2 & -x_0 & 0 & -x_3 \\ 0 & x_0 & 0 & x_3 \\ 0 & 0 & x_2 & 0 \\ 0 & x_3 & 0 & x_2 \\ -x_3 & 0 & x_1 & 0 \\ x_1 & -x_3 & -x_3 & -x_2 \\ -x_3 & x_1 & x_1 & 0 \\ 0 & 0 & -x_2 & x_1 \end{pmatrix}$$

satisfies the condition $\delta' g = (x_1^2 + x_0x_2 - x_3^2)\,\mathrm{Id}$. Therefore, by Theorem 3.1, the module $M = \mathrm{Coker}(\delta')$ is an Ulrich module over

$$R = S/(x_0x_1 - x_2x_3,\ x_1^2 + x_0x_2 - x_3^2).$$

Indeed, one can verify using Macaulay2 that the minimal S-free resolution of M is linear and that M is a MCM R-module.

The ring R is the homogeneous coordinate ring of a smooth curve C of degree 4 and genus 1 in $\mathbb{P}^3$. In fact, M is the graded module associated to the coherent sheaf $\mathcal{O}_C(q - r)$, where r and q are the points $x_0 = x_2 = 0, x_1 = \pm x_3$.

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