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Equations and Syzygies of K3 Carpets and Unions of Scrolls

David Eisenbud[1,2] · **Frank-Olaf Schreyer**[3]

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Abstract

We describe the equations and Gröbner bases of some degenerate K3 surfaces associated to rational normal scrolls. These K3 surfaces are members of a class of interesting singular projective varieties we call correspondence scrolls. The ideals of these surfaces are nested in a simple way that allows us to analyze them inductively. We describe explicit Gröbner bases and syzygies for these objects over the integers and this lets us treat them in all characteristics simultaneously.

Keywords K3 surfaces · Green's conjecture in positive characteristic · Canonical curves · Canonical ribbons · K3 carpets

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1 Introduction

Let $S(a, b)$ be the rational normal surface scroll of degree $a + b$ in $\mathbb{P}^{a+b+1}$ over an arbitrary field $\mathbb{F}$, that is, the embedding of the projectivized vector bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b))$ by the line bundle $\mathcal{O}(1)$ (see [15] for an exposition). A striking theorem of Gallego and Purnaprajna [21, Theorem 1.3] asserts that there is a unique *K3 Carpet* that is a double structure on $S(a, b)$; that is, a unique scheme $X(a, b) \subset \mathbb{P}^{a+b+1}$ whose reduced scheme $X(a, b)_{\mathrm{red}}$ is $S(a, b)$ such that $X(a, b)$ has degree $2(a + b)$ with $\mathrm{H}^1(\mathcal{O}_{X(a,b)}) = 0$ and $\omega_{X(a,b)} \cong \mathcal{O}_{X(a,b)}$ (or, equivalently, with homogeneous coordinate ring Gorenstein of

✉ David Eisenbud
de@msri.org

Frank-Olaf Schreyer
schreyer@math.uni-sb.de

1 Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720, USA

2 Mathematical Sciences Research Institute, 17 Gauss Way, Berkeley, CA 94720, USA

3 Fachbereich Mathematik, Universität des Saarlandes, Campus E2 4, 66123 Saarbrücken, Germany

a-invariant 0.) Gallego and Purnaprajna prove that $X(a, b)$ can be written as a limit of smooth K3 surfaces [20, Corollary 2.8] whose general hyperplane sections are canonical curves of genus $a + b - 1$ and gonality $\min(a, b) + 2$.

A quick description of the homogeneous ideal of $X(a, b)$ is that, for $a, b \geq 2$, it is generated by the rank 3 quadrics in the ideal of $S(a, b)$ (Theorem 4.5). The goal of this paper is to elucidate the generators of this ideal, and those of certain related varieties, in a much more explicit way, similar to the well-known description of the ideal of $S(a, b)$ as an ideal of 2×2 minors. This enables us to compute explicit Gröbner bases and even resolutions over the integers.

One of our motivations has to do with Green's conjecture relating the Clifford index of a smooth projective curve to the length of the linear strand of its free resolution. Deopurkar [12] has recently proven that all canonical ribbons satisfy Green's conjecture. Since every canonical ribbon of genus g and Clifford index c is the hyperplane section of the K3 carpet $X(c, g - 1 - c)$ [5, Section 8], this implies that all K3 carpets satisfy the analogue of Green's conjecture. One can also hope that K3 carpets could shed some light on the questions of the stability of syzygies raised in [13].

Deopurkar's argument relies on Voisin's theorem [27] that canonical curves lying on sufficiently general K3 surfaces over $\mathbb{C}$ satisfy Green's conjecture, and this is also the case of the recent result of Farkas and Aprodu [2] showing that every curve lying on a smooth K3 surface over $\mathbb{C}$. In very recent work, Aprodu, Farkas, Papadima, Raicu and Weyman [3] have given a far simpler proof of Voisin's theorem based on the degeneration of K3 surfaces to tangent developable surfaces of rational normal curves.

It seems natural to hope that there might also be a proof based on K3 carpets, and this would have the advantage that it would automatically treat curves of every Clifford index: indeed, the analogue of Green's conjecture for $X(a, a)$ (which corresponds to Green's conjecture for general curves) directly implies Green's conjecture for all $X(a, b)$ with $b \leq a$, and thus for some curves of each Clifford index. This is because a Gröbner basis for the ideal of each $X(a, b)$ with $b < a$ is a subset of that of $X(a, a)$.

Green's conjecture is known to fail in some finite characteristics [7, 10]. Because the Gröbner bases we construct are valid over the integers, we are able to tabulate the characteristics of the fields over which the conjecture fails for K3 carpets of sectional genus up to 15 and thus for canonical ribbons of these genera. The data lead us to conjecture:

Conjecture 1.1 *Green's conjecture is true for general curves of genus g over fields of characteristic $p > 0$ whenever $p \geq (g - 1)/2$.*

The evidence for this conjecture is presented in more detail in the last section.

Three Examples of K3 Carpets **1)** $S(1, 1) \subset X(1, 1)$: Any quartic equation in four variables defines a scheme that has the characteristics of a K3 surface. The scroll $S(1, 1)$ is a smooth quadric surface in $\mathbb{P}^3$. The unique double structure $X(1, 1)$ is defined by the square of the form defining the quadric.

2) $S(2, 1) \subset X(2, 1)$: In suitable coordinates $S(2, 1)$ is defined by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & y_0 \\ x_1 & x_2 & y_1 \end{pmatrix}.$$

The carpet $X(2, 1)$ supported on this scroll is the complete intersection defined by the 2×2 minor in the upper left corner, together with the determinant, of the symmetric matrix

$$\begin{pmatrix} x_0 & x_1 & y_0 \\ x_1 & x_2 & y_1 \\ y_0 & y_1 & 0 \end{pmatrix}.$$

3) $S(2, 2) \subset X(2, 2)$: For a more typical example, we take $S(2, 2)$ to be the scroll defined by the 2×2 minors of

$$\begin{pmatrix} x_0 & x_1 & y_0 & y_1 \\ x_1 & x_2 & y_1 & y_2 \end{pmatrix}$$

then $X(2, 2)$ is defined by the complete intersection of the three quadrics

$$\det\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix}, \det\begin{pmatrix} y_0 & y_1 \\ y_1 & y_2 \end{pmatrix}, \det\begin{pmatrix} x_0 + y_0 & x_1 + y_1 \\ x_1 + y_1 & x_2 + y_2 \end{pmatrix}.$$

We shall see other useful representations as well.

What's in this Paper In Section 2, below we describe a family of projective schemes, we call *correspondence scrolls* that includes the rational normal scrolls, and the degenerate K3 surfaces treated in the rest of this paper. In Section 3, we give an informal description of the family of degenerate K3 surfaces that depend on a pair of automorphisms of $\mathbb{P}^1$, and describe their degeneration to a K3 carpet.

Our main results are in Sections 4 and 5. In Section 4, we give various descriptions of the minimal generators of the ideals of the K3 carpets and certain reducible K3 surfaces, and prove that these generators form a Gröbner basis for a suitable term order.

In Section 5, we study nonminimal free resolutions of these surfaces that have simple descriptions valid over the ring of integers. Explicit computation then yields information about the characteristics in which Green's conjecture might fail.

Finally, in Section 6, we formulate two conjectures about the minimal free resolutions of these surfaces, and present the data which give the evidence. In particular, we proof Conjecture 1.1 for curves of genus $g \leq 15$.

2 Correspondence Scrolls

Consider disjoint projective spaces $\mathbb{P}^{a_i} = \mathbb{P}(V_i)$, for $i = 1, \dots, m$, embedded in

$$\mathbb{P}^N = \mathbb{P}(\oplus_i V_i),$$

and a *correspondence*, that is a subscheme $\Gamma \subset \prod_i \mathbb{P}^{a_i}$ (or more generally a multi-homogeneous subscheme of $\prod_i \mathbb{A}^{1+a_i}$). The *correspondence scroll* S_Γ defined by Γ may be described set-theoretically as the union of the planes in $\mathbb{P}^N$ spanned by the sets of points $\{p_1, \dots, p_m\}$ with $(p_1, \dots, p_m) \in \Gamma$. To S_Γ scheme-theoretically, we first consider the set of planes of dimension $m-1$ in $\mathbb{P}^N$ that are spanned by all sets of points $\{p_1, \dots, p_m\}$ with $p_i \in \mathbb{P}^a_i \subset \mathbb{P}^N$. We consider this set as a subvariety of the Grassmannian. As such, it is the image of the product $\prod_i \mathbb{P}^{a_i}$. We pull back the tautological bundle of $m-1$-planes on the Grassmannian to $\Gamma \subset \prod_i \mathbb{P}^{a_i}$, and we define S_Γ to be the image in $\mathbb{P}^N$ of this bundle over Γ.

For example, the ordinary surface scroll $S(a, b)$ is the result of taking

$$m = 2,\ a_1 = a,\ a_2 = b$$

and taking Γ to be the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^a \times \mathbb{P}^b$ as the product of the rational normal curves of degrees a and b. The K3 carpet $X(a, b)$ described below is obtained by taking Γ to be the image of twice the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$, and the other degenerate K3 surfaces, we consider correspond to other divisors of type (2, 2) on $\mathbb{P}^1 \times \mathbb{P}^1$.

It is not hard to describe correspondence scrolls that have the properties of Calabi-Yau varieties of other dimensions, and to give other interesting singular models. This is the subject a paper in preparation by the first author and Allessio Sammartano [16].

In the next section, we concentrate on the family of degenerate K3 surfaces.

3 Degenerate K3 Surfaces from Rational Normal Scrolls: Geometry

In this section, we sketch the geometry of the reducible surfaces whose equations we will study.

Fix positive integers a, b, and consider two-dimensional rational normal scrolls of type (a, b) in $\mathbb{P}^{a+b+1}$. Recall that such a scroll may be described geometrically by fixing disjoint subspaces $\mathbb{P}^a, \mathbb{P}^b \subset \mathbb{P}^{a+b+1}$, rational normal curves $C_a \subset \mathbb{P}^a$ and $C_b \subset \mathbb{P}^b$ of degrees a and b respectively, and a one-to-one correspondence $\phi \subset C_a \times C_b$. We write $S = S_\phi$ for the correspondence scroll, which is the union of the lines $\overline{(x, y)}$ for $(x, y) \in \phi$. When $a, b \geq 1$ the surface S is a smooth rational surface of degree $a + b$, isomorphic to

$$\mathrm{Proj}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(a - b) \oplus \mathcal{O}_{\mathbb{P}^1}).$$

In addition to the double structure on S that is the $K3$ carpet $X(a, b)$, we will also study the equations of a family of reducible K3 surfaces, the union of two scrolls $S_1 \cup S_2$ that degenerates to $X(a, b)$. We take $S_1 = S = S_\phi$ and define $S_2 = S_{\phi\tau}$ as the scroll corresponding to the correspondence $\phi \circ (\tau \times 1) \subset C_a \times C_b$, where τ is an automorphism of $C_a \cong \mathbb{P}^1$. Finally, we set

$$X_{\phi,\tau} = S_1 \cup S_2.$$

Now suppose that τ has two distinct fixed points, which we take to be 0 and ∞. In this case, we may identify τ as multiplication by a scalar $t \neq 1$. Had we reversed the roles of 0 and ∞ (or of C_a and C_b), we would replace t by t^{-1}, but up to these changes t is well-defined by the (abstract) surface $X_{\phi,\tau}$ as the ratio of the points of $C_a \setminus \{0, \infty\}$ corresponding to a given point of $C_b \setminus \{\phi(0), \phi(\infty)\}$.

The intersection $S_\phi \cup S_{\phi\tau}$ is a curve of degree $a+b+2$ and arithmetic genus 1 consisting of $C_a \cup C_b \cup L_0 \cup L_\infty$, where L_0, L_∞ are the rulings of either scroll through the points 0 and ∞ on C_a.

We may let t go to 1, and when this happens, the union of the two scrolls approaches $X(a, b)$ (Theorem 4.2).

4 Equations and Gröbner Bases

4.1 Notation

Let $a \geq b \geq 1$ be integers, consider a projective space $\mathbb{P}^{a+b+1}_{\mathbb{F}}$ over an arbitary field $\mathbb{F}$, and let

$$P = \mathbb{F}[x_0, x_1, \ldots, x_a,\ y_0, y_1, \ldots, y_b]$$

be its homogeneous coordinate ring. Define matrices

$$MX := \begin{pmatrix} x_0 & x_1 & \ldots & x_{a-1} \\ x_1 & x_2 & \ldots & x_a \end{pmatrix}, \qquad MY_t := \begin{pmatrix} y_0 & y_1 & \ldots & y_{b-1} \\ ty_1 & ty_2 & \ldots & ty_b \end{pmatrix}$$

and let

$$M_t = \begin{pmatrix} x_0 & x_1 & \dots & x_{a-1} & y_0 & y_1 & \dots & y_{b-1} \\ x_1 & x_1 & \dots & x_a & ty_1 & ty_2 & \dots & ty_b \end{pmatrix}$$

be their concatenation.

We omit the subscript and write MY or M for MY_1 or M_1. We will use the symbol | to denote concatenation, for example, $M = MX|MY$.

Let $I_2(MX)$, $I_2(MY)$, and $I_2(M)$ be the ideals in P generated by the 2×2 minors of these matrices. In the case $b = 1$, we will also use the 2×2 matrix

$$MY2 := \begin{pmatrix} y_0^2 & y_0 y_1 \\ y_0 y_1 & y_1^2 \end{pmatrix}.$$

Write $R := R(a, b) = P/I_2(M)$ for the homogeneous coordinate ring of the scroll $S_t \cong S(a, b)$ defined by $I_2(M_t)$. The line bundle corresponding to the ruling of the scroll S_t is the cokernel of the matrix M_t, and the elements x_0, x_1 may be identified with the sections of this bundle.

4.2 The K3 Carpets

Now let $M = M_1 = MX|MY$. The minimal free resolution of $I_2(M)$ is an Eagon-Northcott complex. From the form of this complex [5], we see that the canonical module ω_R of R is isomorphic to the ideal

$$(x_0, x_1)^{a+b-2} R,$$

shifted so that the generators are in degree 2, that is,

$$\omega_R \cong (x_0, x_1)^q R(q-2),$$

where $q = a + b - 2$.

By [21, Theorem 1.3] there exists a unique surjection $I \to \omega_R$. We begin by making this explicit:

Theorem 4.1 *Set $q = a+b-2$. The unique surjection $\alpha : I(S) \to \omega_R$ from the ideal $I(S)$ of S to the module ω_R annihilates $I_2(MX) + I_2(MY)$ and sends*

$$\det \begin{pmatrix} x_i & y_j \\ x_{i+1} & y_{j+1} \end{pmatrix}$$

to the monomial $x_0^{q-i-j} x_1^{i+j}$.

Proof The given formula for α defines a surjection from the vector space generated by the quadrics in $I(S)$ to the vector space generated by the forms $p_\ell = x_0^{q-\ell} x_1^\ell \in R$. To see that this defines a homomorphism of P-modules, we must show that the relations on the quadrics go to 0.

In the case $a = b = 1$, the ideal $I(S)$ is principal, the canonical module is isomorphic to R, and the result is trivial. Thus, we may assume that $a \geq 2$.

The exactness of the Eagon-Northcott complex shows that the relations on the quadrics are generated by the relations on the minors of the 2×3 submatrices M' of M. Such a submatrix must involve either two columns from MX or two columns from MY. Since the two cases are similar, we may as well suppose that the submatrix is

$$\begin{matrix} & 0 & 1 & 2 \end{matrix}$$

$$M' = \begin{pmatrix} x_i & x_j & y_s \\ x_{i+1} & x_{j+1} & y_{s+1} \end{pmatrix}$$

with $0 \le i < j \le a-1$ and $0 \le s \le b-1$. The relations on the minors of M' are generated by

$$x_i \Delta_{1,2} - x_j \Delta_{0,2} + y_s \Delta_{0,1} = 0$$
$$x_{i+1} \Delta_{1,2} - x_{j+1} \Delta_{0,2} + y_{s+1} \Delta_{0,1} = 0,$$

where $\Delta_{u,v}$ denotes the determinant of the 2×2 submatrix of M' involving the uth and vth columns.

The map α sends $\Delta_{0,1}$ to 0, so these relations go to

$$-x_j p_{i+s} + x_i p_{j+s}$$
$$-x_{j+1} p_{i+s} + x_{i+1} p_{j+s}.$$

In the fraction field of R, we have

$$x_1/x_0 \equiv x_2/x_1 \equiv \cdots \equiv y_1/y_0 = \cdots \quad \mod I(S).$$

In particular, for $j = 0, \dots a$, we have

$$x_j \equiv \left(\frac{x_1}{x_0}\right)^j x_0 \quad \mod I(S).$$

Thus, the two binomials above are both congruent mod $I(S)$ to

$$-\left(\frac{x_1}{x_0}\right)^j x_0 x_0^{q-i-s} x_1^{i+s} + \left(\frac{x_1}{x_0}\right)^i x_0 x_0^{q-j-s} x_1^{j+s} = 0,$$

as required. □

Some Reducible K3 Surfaces We now turn to the ideal of the K3 surfaces $X_{\phi,\tau}$ in the case where τ is multiplication by a scalar t. It turns out that it is convenient to write down generators in some cases where t is not defined over the ground field $\mathbb{F}$, but is the ratio $t = t_1/t_2$ of two the roots $t_1, t_2 \neq 0$ of a quadratic equation $p(z) = z^2 - e_1 z + e_2 \in \mathbb{F}[z]$. We include the possibility $\mathbb{F} = \mathbb{Z}$ as well—this will be important in Section 5. We write e for the pair (e_1, e_2). As we shall see, if $(e_1, e_2) \in \mathbb{F}$, then the scheme $X_{\phi,\tau}$ has a model X_e defined over $\mathbb{F}$.

We think of the t_i as being in a fixed algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$, and set $\overline{P} := \overline{\mathbb{F}}[x_0, \dots, x_a, y_0, \dots, y_b]$. If $t_1 = t_2$, so that $t = 1$ then, for simplicity, we will suppose that $t_1 = t_2 = 1$.

Other than the minors of MX and MY, the forms that will enter into our description are defined as follows:

(1) In the case $a, b \ge 2$, we let $J_e \subset S$ be the ideal generated by the bilinear forms

$$Q_{i,j} := x_{i+2} y_j - e_1 x_{i+1} y_{j+1} + e_2 x_i y_{j+2},$$

for $0 \le i \le a-2$ and $0 \le j \le b-2$. The ideal J_e can be perhaps more conveniently specified as the ideal generated by the entries of the $(a-1) \times (b-1)$ matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ \vdots & \vdots & \vdots \\ x_{a-2} & x_{a-1} & x_a \end{pmatrix} \begin{pmatrix} 0 & 0 & e_2 \\ 0 & -e_1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_0 & y_1 & \dots & y_{b-2} \\ y_1 & y_2 & \dots & y_{b-1} \\ y_2 & y_3 & \dots & y_b \end{pmatrix}. \tag{4.1}$$

(2) In the case $a \ge 2$, $b = 1$, we let J_e be the ideal generated by the cubic forms

$$Q_{i,0} := x_{i+2} y_0^2 - e_1 x_{i+1} y_0 y_1 + e_2 x_i y_1^2$$

for $0 \le i \le a-2$, i.e., the entries of the $(a-1)\times 1$ matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ \vdots & \vdots & \vdots \\ x_{a-2} & x_{a-1} & x_a \end{pmatrix} \begin{pmatrix} 0 & 0 & e_2 \\ 0 & -e_1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_0^2 \\ y_0 y_1 \\ y_1^2 \end{pmatrix}.$$

(3) Finally, in case $a = b = 1$, we let J_e be the ideal generated by the quartic form

$$Q_{0,0} := x_1^2 y_0^2 - e_1 x_0 x_1 y_0 y_1 + e_2 x_0^2 y_1^2 = (x_1 y_0 - t_1 x_0 y_1)(x_1 y_0 - t_2 x_0 y_1)\,.$$

Set $I_e := I_2(MX) + I_2(MY) + J_e$. We will show that I_e is the ideal of forms vanishing on $X_{\phi,\tau}$ and that P/I_e is a Gorenstein ring with $\omega_{P/I_e} \cong P/I_e$ as graded modules. Let X_e be the scheme defined by I_e, so that X_e is a degenerate K3 surface.

Theorem 4.2 *Let $\mathbb{F}$ be any field. $I_e := I_2(MX) + I_2(MY) + J_e$ is a saturated ideal.*

(1) *If $t_1 = t_2 = 1$, hence $e = (2, 1)$, then I_e is the kernel of the map α of Theorem 4.1, and thus I_e is the saturated ideal of $X_e = X(a, b)$.*

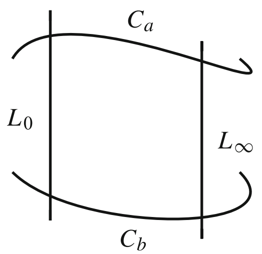
(2) *Suppose that $t_1 \neq t_2$. Define $2 \times (a+b)$ matrices over $\overline{P}$ by*

$$m_\ell := M_{t_\ell} = \begin{pmatrix} x_0 & x_1 & \dots & x_{a-1} & y_0 & y_1 & \dots & y_{b-1} \\ x_1 & x_2 & \dots & x_a & t_\ell y_1 & t_\ell y_2 & \dots & t_\ell y_b \end{pmatrix}$$

for $\ell = 1, 2$. We have

$$I_e = I_2(m_1) \cap I_2(m_2) \subset \overline{P}$$

and thus I_e is the saturated ideal of an $\mathbb{F}$-scheme X_e that becomes isomorphic over $\overline{\mathbb{F}}$ to $X_{\phi,\tau}$, which is the union of the two scrolls defined by $I_2(m_1)$ and $I_2(m_2)$. These two scrolls meet along a reduced curve



where the L_0, L_∞ are the lines in $\mathbb{P}^{a+b+1}_{\overline{k}}$ defined by the vanishing of the first and second rows of the matrix m_ℓ, while the curves C_a and C_b are rational normal curves of degrees a, b defined by the minors of MX and MY in the subspaces defined by the vanishing of the y_j and the x_i respectively.

(3) *The $Q_{i,j}$, together with the 2×2 minors of MX and the 2×2 minors of MY, form a Gröbner basis for I_e with respect to the reverse lexicographic order with*

$$x_0 > \dots > x_a > y_0 > \dots > y_b.$$

(4) *The ring P/I_e is Gorenstein, with $\omega_{P/I_e} \cong P/I_e$ as graded modules.*

We will make use of some identities whose proofs are immediate:

Lemma 4.3 *Suppose that t_1, t_2 are nonzero scalars, and let*

$$e_1 = t_1 + t_2 \quad e_2 = t_1 t_2$$

be the elementary symmetric functions.

(1) *If $a, b \geq 2$ then:*

$$\begin{aligned} Q_{i,j} &:= x_{i+2}y_j - e_1x_{i+1}y_{j+1} + e_2x_iy_{j+2} \\ &= t_2 \det\begin{pmatrix} x_i & y_{j+1} \\ x_{i+1} & t_1y_{j+2} \end{pmatrix} - \det\begin{pmatrix} x_{i+1} & y_j \\ x_{i+2} & t_1y_{j+1} \end{pmatrix} \\ &= t_1 \det\begin{pmatrix} x_i & y_{j+1} \\ x_{i+1} & t_2y_{j+2} \end{pmatrix} - \det\begin{pmatrix} x_{i+1} & y_j \\ x_{i+2} & t_2y_{j+1} \end{pmatrix} \\ &\equiv \det\begin{pmatrix} x_i + t_2y_j & x_{i+1} + y_{j+1} \\ t_2x_{i+1} + t_1y_{j+1} & t_2x_{i+2} + y_{j+2} \end{pmatrix} \quad \bmod (I_2(MX) + I_2(MY)). \end{aligned}$$

(2) *If, on the other hand, $a \geq 2$ but $b = 1$ then:*

$$\begin{aligned} Q_{i,0} &:= x_{i+2}y_0^2 - e_1x_{i+1}y_0y_1 + e_2x_iy_1^2 \\ &= t_2 \det\begin{pmatrix} x_i & y_0y_1 \\ x_{i+1} & t_1y_1^2 \end{pmatrix} - \det\begin{pmatrix} x_{i+1} & y_0^2 \\ x_{i+2} & t_1y_0y_1 \end{pmatrix} \\ &= t_1 \det\begin{pmatrix} x_i & y_0y_1 \\ x_{i+1} & t_2y_1^2 \end{pmatrix} - \det\begin{pmatrix} x_{i+1} & y_0y_1 \\ x_{i+2} & t_2y_0^2 \end{pmatrix}. \end{aligned}$$

We will use also use the following result, which is a transposition of a well-known result on multiplicity into the context of Gröbner bases:

Lemma 4.4 *Let $P = \mathbb{F}[x_0 \ldots, x_n]$ be a standard graded polynomial ring, with a monomial order $>$, and let $I \subset P$ be a homogeneous ideal of dimension d. If $g_1, \ldots, g_m$ are forms in I and $\ell_1, \ldots, \ell_d$ are linear forms such that*

$$\text{length}(P/(\text{in}_< g_1, \ldots, \text{in}_< g_m, \ell_1, \ldots, \ell_d)) \leq \deg P/I$$

then $g_1, \ldots, g_m$ is a Gröbner basis for I, the rings P/I and $P/\text{in}_< I$ are Cohen-Macaulay, and $\ell_1, \ldots, \ell_d$ is a regular sequence modulo $\text{in}_< I$. *Moreover, if σ_t, for $t \in \mathbb{A}^1 \setminus \{0\}$, is the one-parameter family of transformations of $\mathbb{P}^n$ corresponding to the Gröbner degeneration associated to the monomial order $<$ then, for general values of t, the elements $\ell_1, \ldots, \ell_d$ form a regular sequence modulo I_t.*

Proof For $t \neq 0$ we have $\deg P/\sigma_t I = \deg P/I$ because the transformation σ_t is an automorphism of $\mathbb{P}^n$. Moreover, by the semi-continuity of fiber dimension, $\ell_1, \ldots, \ell_d$ is a system of parameters modulo $\sigma_t I$ for general t. The degree is also semi-continuous, and $\text{in}_< \sigma_t g_i = \text{in}_< g_i$, so for general t, we have the following:

$$\begin{aligned} \deg P/I &= \deg P/\sigma_t I \\ &\leq \text{length } P/\sigma_t I + (\ell_1, \ldots, \ell_d) \\ &\leq \text{length } P/(\sigma_t g_1, \ldots, \sigma_t g_m, \ell_1, \ldots, \ell_d) \\ &\leq \text{length } P/(\text{in}_< g_1, \ldots, \text{in}_< g_m, \ell_1, \ldots, \ell_d). \end{aligned}$$

Our hypothesis implies that all the inequalities are equalities, so by [4, Theorem 5.10] the rings P/I and $P/\text{in}_< I$ are Cohen-Macaulay, and $\ell_1, \ldots, \ell_d$ is a regular sequence modulo $\text{in}_< I$. Since any proper factor ring of a Cohen-Macaulay ring must have smaller degree, and since in any case $\deg \text{in}_< I = \deg I$, we see that $\text{in}_< I = (\text{in}_< g_1, \ldots, \text{in}_< g_m)$, so $g_1, \ldots, g_m$ is a Gröbner basis for I. □

Proof of Theorem 4.2 It follows at once from the identities in Lemma 4.3 that I_e is contained in the ideal of $X_{\phi,\tau}$.

We next show that the generators of I_e form a Gröbner basis. In the case $a = b = 1$, the ideal I_e is generated by one equation, and the assertions are easy, so we assume not only that $a \geq b \geq 1$ but also that $a > 1$.

Let I' be the ideal generated by the initial forms of the generators, that is, by:

(1) The initial forms of the 2×2 minors of MX, namely $x_i x_j$ for $1 \leq i \leq j \leq a-1$.
(2) The initial forms of the 2×2 minors of MY, namely $y_i y_j$ for $1 \leq i \leq j \leq b-1$.
(3) The initial forms of the $Q_{i,j}$, namely $x_{i+2} y_j$ with $0 \leq i \leq a-2$ and $0 \leq j \leq b-2$ if $b \geq 2$, or $x_{i+2} y_0^2$ with $0 \leq i \leq a-2$ if $b = 1$.

Since $I' \subset \mathrm{in}_< I$, we see that $\dim S/I' \geq 3$. Set

$$P' = \mathbb{F}[x_1, \ldots, x_a, y_1, \ldots, y_{b-1}] \cong P/(x_0,\ x_a - y_0,\ y_b).$$

The image of I' in P' contains every monomial of degree 2 except

$$\{x_1 y_j \mid 1 \leq j \leq b-1\} \cup \{x_i y_{b-1} \mid 1 \leq i \leq a\},$$

every monomial of degree 3 except $x_1 x_a y_{b-1}$, (or $x_1 x_a^2$ in case $b = 1$), and every monomial of degree ≥ 4. Thus, $x_0, x_a - y_0, y_b$ is a system of parameters modulo I' and $P'/I'P'$ has Hilbert function $\{1,\ a+b-1,\ a+b-1,\ 1\}$. In particular,

$$\dim_k(P'/I') = 2a + 2b.$$

By Lemma 4.4, this implies that $x_0, x_a - y_0, y_b$ is a regular sequence modulo I' and modulo I, that $I' = \mathrm{in}_< I$; and that P/I and P/I' are Cohen-Macaulay rings of degree $2(a+b)$. In particular, I_e is the saturated homogeneous ideal of X_e. This completes the proof of parts (1)–(3).

To complete the proof of part (4), we must show that $\omega_{P/I} \cong P/I$, and for this, we may harmlessly assume that $\mathbb{F} = \overline{\mathbb{F}}$. In the case $t_1 = t_2$, this is implied by the result of Gallego and Purnaprajna [21, Theorem 1.3], so we need only treat the case $t_1 \neq t_2$, where $X_e = S_1 \cup S_2$ is the union of two scrolls.

From the fact that P/I is Cohen-Macaulay, together with Hilbert function of P/I', we know that the Hilbert function of ω_{P/I_e} is equal to the Hilbert function of P/I_e, and it suffices to show that the annihilator of the element of degree 0 is precisely $I_e = I_2(m_1) \cap I_2(m_2)$. Since ω_{P/I_e} is a Cohen-Macaulay module, no element can have annihilator of dimension $< \dim I_e$; thus, the annihilator of the element of degree 0 is either I_e or $I_2(m_\ell)$ for $\ell = 1$ or $\ell = 2$.

Now, the annihilator of $I_2(m_\ell)$ in ω_{P/I_e} is equal to $\omega_{P/I_2(m_\ell)}$. Since $S(a, b)$ is rational its canonical divisor is ineffective, so the nonzero global section of ω_{X_e} cannot come from either of the scrolls, and we are done. □

Theorem 4.5 *The ideal $I(a, b)$ of the K3 carpet $X(a, b)$ contains all the rank 3 quadrics vanishing on the scroll $S(a, b)$, and if $a, b \geq 2$ then $I(a, b)$ is generated by them.*

The projective variety of rank 3 quadrics in $I(a, b)$ is the Veronese embedding of

$$\nu_2 : \mathbb{P}\big(\mathrm{Sym}_{(a-2)}(\mathbb{F}^2) \oplus \mathrm{Sym}_{(b-2)}(\mathbb{F}^2)\big)$$

in the subspace of

$$\mathbb{P}\big(\wedge^2 \mathrm{Sym}_{a-1}(\mathbb{F}^2) \oplus \wedge^2 \mathrm{Sym}_{a-1}(\mathbb{F}^2)\big)$$

spanned by the $\binom{a+b-1}{2}$ rank 3 quadrics described in part (3) of Theorem 4.2.

Proof If we identify $x_0, \dots, x_a$ with the dual basis to the monomial basis of $\mathrm{Sym}_a(\mathbb{F}^2)$, then we may regard MX as a map from $\mathrm{Sym}_{a-1}(\mathbb{F}^2)$ to $(\mathbb{F}^2)^*$. With this identification, writing s, t for the basis of $\mathbb{F}^2$, some of the rank 3 quadrics in $I_2(MX)$ correspond to the 2×2 submatrices of MX involving the pair of generalized columns sf, tf for arbitrary $f \in \mathrm{Sym}_{a-2}(\mathbb{F}^2)$. We first prove by induction on a that these rank 3 quadrics in $I_2(MX)$ generate all of $I_2(MX)$. This is obvious when $a = 1$. By induction, we may assume that the rank 3 quadrics generate all the minors in the first $a-1$ columns of MX. But for $i+1 \leq a-2$ we have the following:

$$\det\begin{pmatrix} x_i & x_{a-1} \\ x_{i+1} & x_a \end{pmatrix} = \det\begin{pmatrix} x_i + x_{a-2} & x_{i+1} + x_{a-1} \\ x_{i+1} + x_{a-1} & x_{i+2} + x_a \end{pmatrix} - \det\begin{pmatrix} x_i & x_{i+1} \\ x_{i+1} & x_{i+2} \end{pmatrix}$$
$$- \det\begin{pmatrix} x_{a-2} & x_{a-1} \\ x_{a-1} & x_a \end{pmatrix} + \det\begin{pmatrix} x_{i+1} & x_{a-2} \\ x_{i+2} & x_{a-1} \end{pmatrix}.$$

All the terms on the right except the last have rank 3 and are of the given form, and the last is a minor from the first $a - 1$ columns, proving the claim.

The map from this $a + 1$-dimensional space of matrices to the $\binom{a}{2}$-dimensional space of quadrics in $I_2(MX)$ is quadratic, and since the image spans $I_2(MX)$, the map must be the quadratic Veronese embedding.

The same consideration holds for the rank 3 quadrics of MY. As in part (3) of Theorem 4.2, we may obtain a further rank three quadric by adding the submatrix corresponding to $f \in \mathrm{Sym}_{a-2}(\mathbb{F}^2)$ to one corresponding to $g \in \mathrm{Sym}_{b-2}(\mathbb{F}^2)$, thus giving us a vector space $\mathrm{Sym}_{a-2}(\mathbb{F}^2) \oplus \mathrm{Sym}_{b-2}(\mathbb{F}^2)$ of 2×2 matrices whose determinants are rank 3 quadrics. The determinant map from this vector space to the space of quadrics is also quadratic. Since the dimension of the space of quadrics in $I(X(a, b))$ is $\binom{a+b-1}{2}$, and this space is spanned by the image of the determinant map, we see that the determinant map must be the quadratic Veronese map.

To see that $I(X(a, b))$ contains all rank 3 quadrics in $I(S(a, b))$, we do induction on $a + b$. If $a = b = 1$, then $I(X(a, b))$ contains no quadrics, and if $a = 2, b = 1$ or $a = 1, b = 2$, there is a unique quadric, and it does have rank 3 (Example 2 in the introduction), so the result is trivial in these cases. We now suppose that $a, b \geq 2$.

Let Q be a rank 3 quadric hypersurface containing $S(a, b)$. The vertex of Q, which is a codimension 3 linear space, is set-theoretically the intersection of Q with a general linear space of codimension 2 containing it, as one can see by diagonalizing the equation of Q. Such a codimension 2 space must intersect the two-dimensional surface $S(a, b)$, necessarily in a point p lying in the vertex. Let $\pi : \mathbb{P}^{a+b+1} \to \mathbb{P}^{a+b}$ be the projection from this point.

We may choose variables within the spaces $(x_0, \dots, x_a)$ and $(y_0, \dots, y_b)$ so that (possibly after reversing the roles of x, y) the point p has homogeneous coordinates $(1, 0, \dots, 0)$, and thus lies on the rational normal curve $C_a \subset S(a, b)$. It follows that $\pi(S(a, b)) = S(a - 1, b)$.

The variety $\pi(X(a, b))$ is defined by the ideal

$$I' := I(X(a, b)) \cap \mathbb{F}[x_1, \dots, x_a, y_0, \dots, y_b],$$

and (after renumbering the variables) this ideal contains all the quadrics in the ideal $I(X(a - 1, b))$ described in Theorem 4.2. Thus, $\pi(X(a, b)) \subset X(a - 1, b)$. Since the general codimension 2 plane through p meets $X(a, b)$ in a double point at p, we have $\deg \pi(X(a, b)) = \deg(X(a, b)) - 2 = \deg X(a - 1, b)$. Since $\pi(X(a, b))$ also has the same dimension as $X(a - 1, b)$, and the latter is Cohen-Macaulay, we have $\pi(X(a, b) = X(a - 1, b)$.

By induction, $X(a-1,b)$ lies on all the rank 3 quadric hypersurfaces containing $S(a-1,b)$; in particular, it lies on $\pi(Q)$. Thus, $X(a,b)$ lies on Q. □

Proposition 4.6 *Suppose that $t_1 \neq t_2$. The scheme $X_e = S_\phi \cup S_{\phi\tau}$ has a transverse A_1 singularity along the intersection of the two scrolls away from the four double points of the curve $E = L_0 \cup L_\infty \cup C_a \cup C_b$.*

Proof We may harmlessly assume $\mathbb{F} = \overline{\mathbb{F}}$ and $a \geq b \geq 1$. Consider the affine chart $U \cong \mathbb{A}^{a+b+1}$ of $\mathbb{P}^{a+b+1}$ defined by $\{x_0 = 1\}$. This open set misses the curves L_∞ and C_b that are defined by the vanishing of the first row of the matrix $MX|MY$ and the vanishing of all the variables of MX, respectively.

The variables x_1, y_0 restrict to global coordinates both on $S_\phi \cap U \cong \mathbb{A}^2$ and $S_{\phi\tau} \cap U \cong \mathbb{A}^2$. Because $0 \neq e_2 \in K$, we can eliminate $x_2, \ldots, x_a$ from the coordinate ring of $X_e \cap \mathbb{A}^{a+b+1}$ using the minors of MX and, if $b \geq 2$, we can eliminate $y_2, \ldots, y_b$ using the equations

$$Q_{0,j}\ |_U = x_2 y_j - e_1 x_1 y_{j+1} + e_2 y_{i+2} \text{ for } j = 0, \ldots, b-2.$$

It follows that x_1, y_0 and y_1 generate the coordinate ring of the affine scheme $X_e \cap U$.

One remaining equation of $X_e \cap \mathbb{A}^{a+b+1}$ in these generators is obtained from $y_1^2 - y_0 y_2$, which, after substitution, corresponds to the equation

$$e_2 y_1^2 - (e_1 x_1 y_1 - x_1^2 y_0) y_0 = (t_1 y_1 - x_1 y_0)(t_2 y_1 - x_1 y_0).$$

All other generators reduce to zero modulo this one, since otherwise X_e would have a component of dimension < 2.

Thus, the intersection of the two components of $X_e \cap U$ in $\mathbb{A}^3$ defined by the following:

$$y_1 - \frac{1}{t_1} x_1 y_0 \quad \text{and} \quad \left(\frac{t_2}{t_1} - 1\right) x_1 y_0.$$

This set has components $x_1 = y_1 = 0$ corresponding to L_∞ and $y_0 = y_1 = 0$ corresponding to C_a, and the intersection is transverse away from the point $x_0 = x_1 = y_1 = 0$.

The arguments for the three charts $\{x_a = 1\}$, $\{y_0 = 1\}$ and $\{y_b = 1\}$ are similar. □

5 Syzygyies over $\mathbb{Z}$ and $\mathbb{Z}/p$

In this section, we investigate the question: for which prime numbers p does the carpet $X(a,b)$ satisfy Green's conjecture over a field of characteristic p? We begin by unpacking this question.

Let R denote a field or $\mathbb{Z}$. If F is a graded free complex over a graded R-algebra with $R = P_0 \cong P/P_+$ a domain, then we set the following:

$$\beta_{i,j}(F) := (\operatorname{rank}_R F_i \otimes_P R)_j.$$

Following the convention used in Macaulay2, we display the $\beta_{i,j}$ in a *Betti table* with whose ith column and jth row contains the value $\beta_{i,i+j}(F)$. If R is a field or $\mathbb{Z}$ we write $X^R(a,b)$ to denote the subscheme of $\mathbb{P}_R^{a+b+1}$ that is defined by the ideal $I_{2,1}$ of Theorem 4.2, and we write $X_e^R(a,b)$ for the subscheme defined by $I_e(a,b)$ more generally. We write $P^R(a,b)$ and P_e^R for the corresponding homogeneous coordinate rings.

If F is the minimal free resolution of $P^{\mathbb{F}}(a,b)$ as a module over

$$\mathbb{F}[x_0, \ldots, x_a, y_0, \ldots, y_b],$$

where $\mathbb{F}$ is a field of characteristic p, we say that Green's conjecture holds for $X^{\mathbb{F}}(a, b)$ if $\beta_{i,i+1}(F) = 0$ for $i \geq \max(a, b)$, and similarly for $X_e^{\mathbb{F}}(a, b)$. Note that the presence of the ideal of the rational normal curves of degree a and b inside the ideal of $X(a, b)$ implies that $\beta_{i,i+1}(F) \neq 0$ for $0 < i < \max(a, b)$, so that when Green's conjecture holds, it is sharp.

We have already shown that $P^{\mathbb{F}}(a, b)$ is Cohen-Macaulay. The hyperplane section, which is a ribbon canonical curve, thus has minimal free resolution with the same Betti numbers [11, Proposition 1.1.5]. Since the hyperplane is a ribbon of genus $g = a+b+1$ and Clifford index b by [5, p. 730], this is what Green's conjecture predicts for ribbons [5, Corollary 7.3]. Since ribbons do satisfy Green's conjecture in characteristic 0 [12], it follows that this is true for K3 carpets as well.

Returning to the general setting of a graded free complex F over a graded R-algebra P with $R = P_0 \cong P/P_+$, we define the kth *constant strand* of F, denoted $F^{(k)}$, to be the submodule of elements of internal degree k of the complex $F \otimes_P R$. Thus, $F^{(k)}$ has the form:

$$F^{(k)} : \cdots \leftarrow R^{\beta_{k-2,k}(F)} \leftarrow R^{\beta_{k-1,k}(F)} \leftarrow R^{\beta_{k,k}(F)} \leftarrow \cdots .$$

We write $H_i(F^{(k)})$ for the homology of this subcomplex at the term $R^{\beta_{i,k}(F)}$. If R is a field, F is any graded P-free resolution of a module M, and F' is the minimal free resolution of M, then since the minimal free resolution is a summand of any free resolution, we have $\beta_{i,k}(F') = H_i(F^{(k)})$.

To survey what happens for all primes p at once, we work over $\mathbb{Z}$. We have shown that the homogenous ideal of $X(a, b) \subset \mathbb{P}_{\mathbb{Z}}^{a+b+1}$ is minimally generated by a Gröbner basis consisting of forms with integer coefficients, and the coefficients of the lead terms are ± 1. Thus, the homogeneous coordinate ring $P^{\mathbb{Z}}(a, b)$ of $X^{\mathbb{Z}}(a, b)$ is a free $\mathbb{Z}$-algebra, and any free resolution over $P^{\mathbb{Z}}(a, b)$ reduces, modulo a prime p, to a free resolution of $P^{\mathbb{Z}/p}(a, b)$ over in characteristic p.

This means that we can deduce properties in all characteristics from properties of a free resolution over $\mathbb{Z}$. We will use the (not necessarily minimal) free resolution introduced (in a slightly different form) in [26], called the *Schreyer resolution* in Singular. See [6] for a mathematical exposition, and [18] for an efficient algorithm. We have implemented a Macaulay2 package K3Carpets.m2 [17] for exploration of these questions.

The definition of the Schreyer resolution of an ideal I, described in [6], starts with a normalized Gröbner basis

$$f_1, \ldots, f_n$$

of I, sorted first by degree and then by the reverse lexicographic order of the initial terms. Each minimal monomial generator of the monomial ideal

$$M_i = (\mathrm{in}(f_1), \ldots, \mathrm{in}(f_{i-1})) : \mathrm{in}(f_i) \text{ for } i = 2, \ldots, n$$

determines a syzygy. One shows that these syzygies form a Gröbner basis for the first syzygy module of $f_1, \ldots, f_n$ with respect to the induced monomial order. Their lead terms are $m_j u_i$ for generators u_i of F_1 mapping to f_i and $m_j \in M_i$ a minimal monomial generator. Continuing with the algorithm, we get the finite free resolution F whose terms F_i are free modules with chosen bases.

It will be useful in the proof of Theorem 5.4 to give each of the chosen basis elements of F_p a name, which is a sequence $m_1, \ldots, m_p$ of monomials:

Definition 5.1 The basis element u_i of F_1 gets as a name the monomial $\mathrm{in}(f_i)$. If the minimal generator $u_j \in F_p$ is mapped to a syzygy with lead term $mu_k \in F_{p-1}$, then the name of a generator u_j of F_p is as follows:

$$\mathrm{name}(u_j) = \mathrm{name}(u_k), m.$$

We define the *name product* of a generator F_p to be the product of the monomials in its name. The total (internal, as opposed to homological) degree of a generator is thus the degree of its name product.

For simplicity, when we write $X(a, b)$, we will henceforward assume that $a \geq b$. To check whether Green's conjecture holds, we only need to check a single homology group of a constant strand in an arbitrary free resolution:

Proposition 5.2 *The K3 carpet $X^{\mathbb{F}}(a, b)$ over a field $\mathbb{F}$ satisfies Green's conjecture if and only if, for any graded free resolution F of the homogeneous coordinate ring of $P^{\mathbb{Z}}(a, b)$, the constant strand $F^{(a+1)}$ satisfies $H_a(F^{(a+1)} \otimes_{\mathbb{Z}} \mathbb{F}) = 0$.*

Proof We must show that in the minimal free resolution F' of $P^{\mathbb{F}}(a, b)$, the term F'_k, for $k \geq a$, has no generators of degree $\leq k + 1$. The construction of the Schreyer resolution F of $P^{\mathbb{Z}}(a, b)$ shows that F has no generators of degree $\leq k$, and since F' is a summand of $F \otimes_{\mathbb{Z}} \mathbb{F}$, the same is true for F. The hypothesis that $H_a(F^{(a+1)} \otimes_{\mathbb{Z}} \mathbb{F}) = 0$ (for any resolution F over the integers) implies that F'_a does not have any generators of degree $a+1$, either, proving the assertion for $k = a$. We complete the proof by induction on $k \geq a$.

Assuming that F'_k has no generators of internal degree $\leq k + 1$, the differential of F' would map any generators of F_{k+1} having internal degree $k+2$ to scalar linear combinations of generators of F_k having internal degree $k+2$. Because F' is minimal, this cannot happen. □

Example 5.3 Here is the Betti table of the Schreyer resolution F of $P^{\mathbb{Z}}(6, 6)$ computed with Macaulay2:

$j\backslash i$	0	1	2	3	4	5	6	7	8	9	10	11
0:	1	.	.	.	.	.	.	.	.	.	.	.
1:	.	55	320	930	1688	2060	1728	987	368	81	8	.
2:	.	.	39	280	906	1736	2170	1832	1042	384	83	8
3:	.	.	.	1	8	28	56	70	56	28	8	1

In this case, Proposition 5.2 shows that Green's conjecture over $\mathbb{F}$ depends only on a property of the 7th constant strand $F^{(a+1)} = F^{(7)}$. In our example, this has the form as follows:

$$0 \leftarrow \mathbb{Z}^8 \leftarrow \mathbb{Z}^{1736} \leftarrow \mathbb{Z}^{1728} \leftarrow 0.$$

It has a surjective first map, so the vanishing of $H_a(F^{(7)} \otimes_{\mathbb{Z}} \mathbb{F})$ is equivalent to the nondivisibility by p of the determinant of a certain 1728×1728 matrix M over $\mathbb{Z}$. Computationally we find that

$$\det M = 2^{1312}\, 3^{72}\, 5^{120}.$$

Thus, in characteristic 0 or characteristic $p \neq 2, 3, 5$, this carpet satisfies Green's conjecture with Betti tables.

	0	1	2	3	4	5	6	7	8	9	10	11
0:	1	.	.	.	.	.	.	.	.	.	.	.
1:	.	55	320	891	1408	1155	.	.	.	.	.	.
2:	.	.	.	.	.	.	1155	1408	891	320	55	.
3:	.	.	.	.	.	.	.	.	.	.	.	1

For the exceptional primes p, we can determine the Betti tables by computing the Smith normal form of M and the other matrices in the constant strands of the nonminimal resolution. They are as follows:

$p = 2:$

0:	1	.	.	.	.	.	.	.	.	.	.	.
1:	.	55	320	900	1488	1470	720	315	80	9	.	.
2:	.	.	9	80	315	720	1470	1488	900	320	55	.,
3:	.	.	.	.	.	.	.	.	.	.	.	1

$p = 3:$

0:	1	.	.	.	.	.	.	.	.	.	.	.
1:	.	55	320	891	1408	1162	48	7	.	.	.	.
2:	.	.	.	.	7	48	1162	1408	891	320	55	.
3:	.	.	.	.	.	.	.	.	.	.	.	1

$p = 5:$

0:	1	.	.	.	.	.	.	.	.	.	.	.
1:	.	55	320	891	1408	1155	120	.	.	.	.	.
2:	.	.	.	.	.	120	1155	1408	891	320	55	..
3:	.	.	.	.	.	.	.	.	.	.	.	1

Experimentally, we have strong evidence that $p = 2$ and $p = 5$ are also exceptional primes for the general curve of genus 13, while a general curve of this genus in characteristic 3 satisfies Green's Conjecture (see [7] and Remark 6.2 below). For characteristic $p = 2$, the experiments support the conjecture that a general smooth curve of genus 13 has the following the Betti table with much smaller numbers

0:	1	.	.	.	.	.	.	.	.	.	.	.
1:	.	55	320	891	1408	1155	64	.	.	.	.	.
2:	.	.	.	.	.	64	1155	1408	891	320	55	.,
3:	.	.	.	.	.	.	.	.	.	.	.	1

than the carpet, while, for $p = 5$, the experimental findings suggest that the Betti table of the carpet coincides with the conjectural Betti table of a general smooth curve of genus 13.

The Schreyer resolution is rarely minimal, even for monomial ideals. Thus, the following surprised us:

Theorem 5.4 *Let $a, b \geq 2$, and write $I = I_{(2,1)}$ for the saturated ideal defining $X^{\mathbb{Z}}(a, b)$, as exhibited in Theorem 4.2. The Schreyer resolution of* $\text{in}(I)$ *is minimal.*

Proof In our case, the minimal generators of I form a Gröbner basis (Theorem 4.2), which is thus automatically normalized. Let F denote the Schreyer resolution of $J = \text{in}(I)$. Defining the M_i as above, we see from the construction that the Schreyer resolution G

of $\mathrm{in}(f_1), \ldots, \mathrm{in}(f_{n-1})$ is a subcomplex of F, and the quotient complex is the Schreyer resolution of M_n, appropriately twisted and shifted.

There are $n = \binom{a+b-1}{2}$ generators of J, which we sort by degree refined by the reverse lexicographic order as follows:
$x_1^2, x_1x_2, x_2^2, \ldots, x_{a-1}^2, x_2y_0, x_3y_0, \ldots, x_ay_0, x_2y_1, x_3y_1, \ldots, x_ay_1, y_1^2, x_2y_2, x_3y_2, \ldots, x_ay_2, y_1y_2, y_2^2, \ldots\ldots, x_2y_{b-2}, x_3y_{b-2}, \ldots, x_ay_{b-2}, y_1y_{b-2}, y_2y_{b-2}, \ldots, y_{b-2}^2, y_1y_{b-1}, \ldots, y_{b-2}y_{b-1}, y_{b-1}^2$.

Thus, for $1 \le k \le n-1$ we have the following:

$\mathrm{in}(f_k)$	Range	M_k
x_ix_j	$1 \le i \le j \le a-1$	$(x_1, \ldots, x_{j-1})$
x_iy_j	$2 \le i \le a-1, 0 \le j \le b-2$	$(x_1, \ldots, x_{a-1}, y_0, \ldots, y_{j-1})$
x_ay_j	$0 \le j \le b-2$	$(x_2, \ldots, x_{a-1}, y_0, \ldots, y_{j-1}, x_1^2)$
y_iy_j	$1 \le i \le j \le b-2$	$(x_2, \ldots, x_{a-1}, y_1, \ldots, y_{j-1}, x_1^2)$
y_iy_{b-1}	$1 \le i < b-1$	$(x_2, \ldots, x_{a-1}, y_1, \ldots, y_{b-2}, x_1^2)$

The monomial ideal M_n is more complicated. The initial term of f_n is $\mathrm{in}(f_n) = y_{b-1}^2$, and we get the following:

$$M_n = (y_1, \ldots, y_{b-2}, x_1^2, x_1x_2, \ldots, x_{a-1}^2, x_2y_0, \ldots, x_ay_0).$$

Lemma 5.5 *The Schreyer resolution G of the ideal $(\mathrm{in}(f_1), \ldots, \mathrm{in}(f_{n-1}))$ is the minimal free resolution of this ideal.*

Proof For $k < n$, each M_k is generated by a regular sequence of monomials.The name of each generator of G_p is thus an initial monomial of an f_k, followed by a decreasing sequence of distinct elements of M_k of length $p-1$.

We must show that there are no constant terms in the differential $G_{p+1} \to G_p$ for each $p > 0$. The generators of G_p have degrees $p+1$ and $p+2$. The $\mathbb{Z}^{a+b+2}$-grading of the monomial ideal induces a $\mathbb{Z}^{a+b+2}$-grading on G. Again, in this grading, a generator of G_p has same total degree as its name product.

Each name product of a generator of G_p of degree $p+2$ is divisible by x_1^2 and some y_j. However, the only name products of generators of G_{p+1} of degree $p+2$ that are divisible by x_1^2 are monomials in $\mathbb{F}[x_1, \ldots, x_{a-1}]$, and the conclusion follows. □

To treat the case of M_n, we first study a smaller resolution:

Lemma 5.6 *The Schreyer resolution H of the monomial ideal*

$$J_H = (x_1^2, x_1x_2, \ldots, x_{a-1}^2, x_2y_0, \ldots, x_ay_0)$$

is the minimal free resolution of this ideal.

Proof We order the monomial generators m_k of J_H as indicated above, and obtain this time

m_k	Range	$(m_1, \ldots, m_{k-1}) : m_k$
x_ix_j	$1 \le i \le j \le a-1$	$(x_1, \ldots, x_{j-1})$
x_iy_0	$2 \le i \le a-1$	$(x_1, \ldots, x_{a-1})$
x_ay_0		$(x_2, \ldots, x_{a-1}, x_1^2)$

As in the proof of Lemma 5.5, the generators of H_p for $p \geq 1$ are in degree $p+1$ and $p+2$, and only the name products of those in degree $p+2$ are divisible by $x_1^2 y_0$, so no constant terms can occur in the differential by the $\mathbb{Z}^{a+b+2}$-grading. □

We continue now with the proof of Theorem 5.4. The resolution of M_n is the tensor product of the resolution H from Lemma 5.6 with the Koszul complex $\mathbb{K} = \mathbb{K}(y_1, \ldots, y_{b-2})$. Thus, the terms of the complex F resolving $\mathrm{in}(I)$ are built from the terms of G and terms of the tensor product complex $\mathbb{K} \otimes H$ shifted and twisted:

$$F_p = G_p \oplus \bigoplus_{q=0}^{\min(b-2,p-1)} \mathbb{K}_q \otimes H_{p-1-q}(-2).$$

Since G is a subcomplex of F, the only possibly nonminimal parts of the differentials in F have source in the subquotient complex $\mathbb{K}(y_1, \ldots, y_{b-2}) \otimes S[-1](-2)$ and target in G.

The Schreyer resolution FY of $(y_1, \ldots, y_{b-1})^2$ is a subcomplex of F of which $\mathbb{K}(y_1, \ldots, y_{b-2}) \otimes S[-1](-2)$ is a subquotient. Since FY has only generators of degree $p+1$ in homological degree $p \geq 1$, all maps of FY and hence F are minimal. This completes the proof of Theorem 5.4. □

Corollary 5.7 *The minimal free resolution of* $\mathrm{in}(I)$ *and the Schreyer resolution of* I *have length* $a+b-1$ *and their nonzero Betti numbers are as follows:*

$$\begin{aligned}
\beta_{0,0}(F) &= 1, \\
\beta_{p,p+1}(F) &= p\binom{a}{p+1} + \sum_{j=0}^{b-2}\left((a-2)\binom{a+j-1}{p-1} + \binom{a+j-2}{p-1}\right) \\
&\quad + \sum_{j=1}^{b-2} j\binom{a+j-2}{p-1} + (b-2)\binom{a-2+b-1}{p-1} + \binom{b-2}{p-1} \\
&\quad \textit{for } 1 \leq p \leq a+b-2,
\end{aligned}$$

and

$$\begin{aligned}
&\beta_{p,p+2}(F) \\
&= \sum_{j=0}^{b-2}\binom{a+j-2}{p-2} + \sum_{j=1}^{b-2} j\binom{a+j-2}{p-2} + (b-2)\binom{a-2+b-1}{p-2} \\
&+ \sum_{q=0}^{p-2}\binom{b-2}{q}\left((p-q-1)\binom{a}{p-q}\right. \\
&\left. +(a-p+q+1)\binom{a}{p-q-2} + \binom{a-2}{p-q-4}\right) \\
&\textit{for } 2 \leq p \leq a+b-1
\end{aligned}$$

and

$$\beta_{p,p+3}(F) = \binom{a+b-4}{p-3} \qquad \textit{for } 3 \leq p \leq a+b-1.$$

Proof The complex H has length a and its nonzero Betti numbers are as follows:

$$\begin{aligned}\beta_{0,0}(H) &= 1,\\ \beta_{p,p+1}(H) &= p\binom{a}{p+1} + (a-p)\binom{a}{p-1} + \binom{a-2}{p-3} \text{ for } 1 \le p \le a\\ &\text{and}\\ \beta_{p,p+2}(H) &= \binom{a-2}{p-2} \text{ for } 2 \le p \le a.\end{aligned}$$

The complex G has length $a+b-1$ and its non-zero Betti numbers are as follows:

$$\begin{aligned}\beta_{0,0}(G) &= 1,\\ \beta_{p,p+1}(G) &= p\binom{a}{p+1} + \sum_{j=0}^{b-2}\left((a-2)\binom{a+j-1}{p-1} + \binom{a+j-2}{p-1}\right)\\ &\quad + \sum_{j=1}^{b-2} j\binom{a-2+j}{p-1} + (b-2)\binom{a-2+b-1}{p-1}\\ &\quad \text{for } 1 \le p \le a+b-2\end{aligned}$$

and

$$\begin{aligned}\beta_{p,p+2}(G) &= \sum_{j=0}^{b-2}\binom{a+j-2}{p-2} + \sum_{j=1}^{b-2} j\binom{a+j-2}{p-2} + (b-2)\binom{a-2+b-1}{p-2}\\ &\quad \text{for } 2 \le p \le a+b-1.\end{aligned}$$

The formula now follows from the following:

$$F_p = G_p \oplus \bigoplus_{q=0}^{\min(b-2,p-1)} \mathbb{K}_q \otimes H_{p-1-q}(-2).$$

□

Remark 5.8 The formula for $\beta_{p,p+1}(F)$ can be simplified:

$$\beta_{p,p+1}(F) = \binom{a-2}{p-1} + \binom{b-2}{p-1} + p\binom{a+b-1}{p+1} - 2\binom{a+b-3}{p-1}.$$

Using this and $\beta_{p-2,p+1}(F) = \binom{a+b-4}{p-1}$, we can also obtain a simplified formula for the $\beta_{p,p+2}(F)$'s by using the identities:

$$\begin{aligned}&\beta_{p,p+1}(F) - \beta_{p-1,p+1}(F) + \beta_{p-2,p+1}(F)\\ &= p\binom{a+b-3}{p+1} - (a+b-2-p)\binom{a+b-3}{a+b-1-p}\\ &= \frac{a+b-2-p}{p+1}\binom{a+b-2}{p-1}(a+b-2p-2).\end{aligned}$$

Remark 5.9 Eliminating y_0 from the equations of $X_e(a,b) \subset \mathbb{P}^{a+b+1}$ gives the equations of an $X_e(a,b-1) \subset \mathbb{P}^{a+b}$, and it follows that the Schreyer resolution of $X_e(a,b-1)$ is a subcomplex of the Schreyer resolution of $X_e(a,b)$. Indeed, the generators derived from

$\mathrm{in}(f_k)$	Range	M'_k
$x_i x_j$	$1 \le i \le j \le a-1$	$(x_1, \dots, x_{j-1})$
$x_i y_j$	$2 \le i \le a-1, 1 \le j \le b-2$	$(x_1, \dots, x_{a-1}, y_1, \dots, y_{j-1})$
$x_a y_j$	$1 \le j \le b-2$	$(x_2, \dots, x_{a-1}, y_1, \dots, y_{j-1}, x_1^2)$
$y_i y_j$	$2 \le i \le j \le b-2$	$(x_2, \dots, x_{a-1}, y_2, \dots, y_{j-1}, x_1^2)$
$y_i y_{b-1}$	$2 \le i \le b-2$	$(x_2, \dots, x_{a-1}, y_2, \dots, y_{b-2}, x_1^2)$

belong to this subcomplex. For the last equation with lead term $\mathrm{in}(f_{n'}) = y_{b-1}^2$, we get

$$M'_{n'} = (y_2, \dots, y_{b-2}, x_1^2, x_1x_2, \dots, x_{a-1}^2, x_2y_1, \dots, x_a y_1)$$

which is not a subset of the corresponding M_n. Hence, some generators of the Schreyer resolution for $X_e(a, b-1)$ are not mapped to generators of the Schreyer resolution of $X_e(a, b)$ but rather to linear combinations.

Remark 5.10 The equations of $X_e(a, b)$ allow a $\mathbb{Z}^3$-grading. The equations and the whole resolution is homogenous for $\deg x_i = (1, 0, i)$ and $\deg y_j = (0, 1, j)$. The nonminimal maps in the nonminimal resolution decompose into blocks with respect to this fine grading.

We can also compute the Betti table for the minimal resolutions of the K3 carpets $X^{\mathbb{F}}(a, b)$ over a field $\mathbb{F}$ of characteristic 2. Note that, because e_1, e_2 are elements of $\mathbb{F}$, the degenerate K3 surface $X^{\mathbb{F}}_{(0,1)}(a, b)$ coincides with the carpet $X^{\mathbb{F}}(a, b) = X^{\mathbb{F}}_{(2,1)}(a, b)$.

Theorem 5.11 *Let $a, b \ge 2$ and let $\mathbb{F}$ be an arbitrary field. The minimal free resolution of the homogeneous coordinate ring of $X := X_e(a, b) \subset \mathbb{P}^{a+b+1}$ for $e = (0, 1)$ has Betti numbers as follows:*

$$\beta_{i,i+1} = i\binom{a+b-2}{i+1} + (\max(a-i, 0) + \max(b-i, 0))\binom{a+b-2}{i-1}$$

for $i \ge 1$ and $\beta_{i,i+2} = \beta_{a+b-1-i,a+b-i}$ for $1 \le i \le a+b-2$. (These Betti numbers coincide with the Betti numbers of a 4-gonal canonical curve of genus $g = a+b+1$ with relative canonical resolution invariants $a-2$ and $b-2$ (see [24], Example (6.2)]).

Proof The 2×2 minors of the matrix

$$m = \begin{pmatrix} x_0 & x_1 & \dots & x_{a-2} & y_0 & y_1 & \dots & y_{b-2} \\ x_2 & x_3 & \dots & x_a & -y_2 & -y_3 & \dots & -y_b \end{pmatrix}$$

are contained in I_X. Thus, X is contained in a four-dimensional rational normal scroll of type

$$Y = S(\lfloor a/2 \rfloor, \lceil a/2 \rceil - 1, \lfloor b/2 \rfloor, \lceil b/2 \rceil - 1)$$

of degree $f = a-1+b-1$. As a subscheme of the scroll, X is the complete intersection of two divisors, whose classes are of class $2H-(a-2)R$ and $2H-(b-2)R$, where $H, R \in \mathrm{Pic}\, Y$ denote the hyperplane class and the ruling of Y. These are defined by the vanishing of

$$x_1^2 - x_0x_2, x_2^2 - x_1x_3, \dots, x_{a-1}^2 - x_{a-2}x_a$$

and

$$y_1^2 - y_0y_2, y_2^2 - y_1y_3, \dots, y_{b-1}^2 - y_{b-2}y_b,$$

respectively. In terms of the Cox ring $\mathbb{F}[s, t, u_0, u_1, v_0, v_1]$ of Y they are given by relative quadrics

$$\begin{cases} u_1^2 - stu_0^2 & \text{if } a \equiv 0 \mod 2 \\ su_1^2 - tu_0^2 & \text{if } a \equiv 1 \mod 2 \end{cases}$$

and

$$\begin{cases} v_1^2 - stv_0^2 & \text{if } b \equiv 0 \mod 2 \\ sv_1^2 - tv_0^2 & \text{if } b \equiv 1 \mod 2. \end{cases}$$

Thus by [24, Examples (3.6) and (6.2)], the minimal free resolution of I_X is given by an iterated mapping cone

$$\mathcal{C}^0 \leftarrow [\mathcal{C}^{a-2}(-2) \oplus \mathcal{C}^{b-2}(-2) \leftarrow \mathcal{C}^{f-2}(-4)]$$

where $\mathcal{C}^j$ denotes the jth Buchsbaum-Eisenbud complex associated to m. (The complexes $\mathcal{C}^0, \mathcal{C}^1$ are also known as Eagon-Northcott complex and Buchsbaum-Rim complex of m.) □

Part of Theorem 5.11 generalizes as follows.

Theorem 5.12 (Resonance) *Suppose $p(z) = z^2 - e_1 z + e_2$ has distinct nonzero roots $t_1, t_2 \in \mathbb{F}$ such that t_2/t_1 is a primitive kth root of unity and $a, b \geq k+1$, and set $X := X_e^{\mathbb{F}}(a, b)$.*

(1) *X is contained in a rational normal scroll of type*

$$Y = S(a_0, \ldots, a_{k-1}, b_0, \ldots, b_{k-1})$$

with

$$a_i = |\{0 \leq j \leq a \mid j \equiv i \mod k\}| - 1$$

and

$$b_i = |\{0 \leq j \leq b \mid j \equiv i \mod k\}| - 1.$$

(2) *The map $Y \to \mathbb{P}^1$ induces a fibration of X into $2k$-gons.*

(3) *If $a, b \geq 2k^2$, then X has graded Betti numbers $\beta_{\ell,\ell+1} = 0$ for $\ell > a+b-1+2-2k$ and $\beta_{\ell,\ell+2} = 0$ for $\ell < 2k-2$. In particular, the range of nonzero Betti numbers coincides with range predicted by Green's conjecture for a general $2k$-gonal curve of genus $g = a+b+1$.*

We speak of the phenomenon in the Theorem as *resonance* because it comes from a periodicity induced by $(t_2/t_1)^k = 1$. In characteristic 0, Green's conjecture is known to hold for general d-gonal curves of every genus by [1], and it is known in every characteristic for some d-gonal curve of genus g if $g > (d-1)(d-2)$ by [25]. However, we do not know that the family of curves of genus g and gonality d is irreducible. Indeed, the Hurwitz scheme could be reducible in positive characteristics (see [19, Example 10.3]).

Proof of Parts (1) *and* (2) By Theorem 4.2, X is the union of the two scrolls defined by the minors of the matrices

$$m_\ell = \begin{pmatrix} x_0 & x_1 & \ldots & x_{a-1} & | & y_0 & y_1 & \ldots & y_{b-1} \\ x_1 & x_2 & \ldots & x_a & | & t_\ell y_1 & t_\ell y_2 & \ldots & t_\ell y_b \end{pmatrix} \quad \text{for } \ell = 1, 2$$

respectively.

Applying an automorphism of $\mathbb{P}^{a+b-1}$, we may assume that $t_1 = 1$ and thus that $t = t_2$ is a $k$the root of unity. The minors of the matrix

$$m = \begin{pmatrix} x_0 & x_1 & \ldots & x_{a-k} & | & y_0 & y_1 & \ldots & y_{b-k} \\ x_k & x_{k+1} & \ldots & x_a & | & y_k & y_{k+1} & \ldots & y_b \end{pmatrix}$$

lie in the intersection of the ideals of minors of m_1 and m_2, as one sees from the formulas

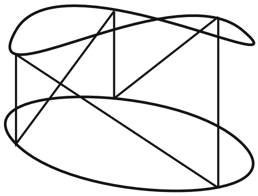
$$\sum_{\ell=0}^{k-1} t^{k-\ell-1} \begin{vmatrix} x_{i+\ell} & y_{j-\ell-1} \\ x_{i+\ell+1} & t y_{j-\ell} \end{vmatrix} = \begin{vmatrix} x_i & y_{j-k} \\ x_{i+k} & t^k y_j \end{vmatrix},$$

which hold for $0 \le i \le a-k$ and $k \le j \le b$. Thus, the scheme X is contained in a $2k$-dimensional scroll of the type claimed (for example

$$\begin{pmatrix} x_0 & x_k & \dots & x_{(a_0-1)k} \\ x_k & x_{2k} & \dots & x_{a_0 k} \end{pmatrix}$$

is a submatrix of m).

Since $X = S_1 \cup S_2$ is the union of two scrolls whose basic sections C_a and C_b coincide, we find a pencil of $2k$-gons (away from the ramification points at 0 and infinity of the k-power map from $\mathbb{P}^1$ to $\mathbb{P}^1$) as follows by alternating rulings from S_1 and S_2. Starting from a general point $(1 : s : s^2 : \dots : s^a : 0 \dots 0) \in C_a$, we have a ruling of the first scroll S_1 connecting it to the point $(0 : \dots : 0 : 1 : s : \dots : s^b) \in C_b$. The ruling of the second scroll S_2 joins this point on C_b with the point $(1 : ts : \dots : (ts)^a : 0 : \dots : 0)$.



3-resonance

Continuing with a ruling of the first scroll, and so on, this process closes with a $2k$-gon, since t is a primitive kth root of unity.

The map $Y \to \mathbb{P}^1$ sends a point of Y to the ratio of the two rows of m evaluated at that point, so the $2k$-gon is contained in the fiber defined by the following:

$$\begin{pmatrix} s^k, -1 \end{pmatrix} \begin{pmatrix} x_0 & x_1 & \dots & x_{a-k} & y_0 & y_1 & \dots & y_{b-k} \\ x_k & x_{k+1} & \dots & x_a & y_k & y_{k+1} & \dots & y_b \end{pmatrix} = 0.$$

Since $s^k = \tilde{s}$ has k distinct solutions for $\tilde{s} \neq 0$, the fiber of the composition $X = S_1 \cup S_2 \hookrightarrow Y \to \mathbb{P}^1$ over the point $(1 : \tilde{s})$ contains precisely k rulings of each of the two scrolls S_ℓ. Hence, the $2k$-gon is the complete fiber of $X \to \mathbb{P}^1$.

The last statement follows by resolving the relative resolution of X in the $2k$-dimensional scroll Y by an iterated mapping cone built from Buchsbaum-Eisenbud complexes following the strategy of [25]. Before we discuss details, we look at an example.

Example 5.13 We consider cases of 3-resonance, $k = 3$, and take $X = X_{(-1,1)}(a, b) \subset \mathbb{P}_{\mathbb{F}}^{a+b+1}$, since the polynomial $p(z) = z^2 + z + 1$ has as zeroes the primitive third roots of unity. Note that in characteristic 3, the union of scrolls $X_{(-1,1)}(a, b)$ coincides with the carpet $X(a, b) = X_{2,1}(a, b)$, so in characteristic 3, there is no 3-resonance, but the considerations of the free resolution below are the same. By Theorem 4.2, the scheme $X = X_{(-1,1)} \subset \mathbb{P}_{\mathbb{F}}^{a+b+1}$ is defined by the ideal $I_{(-1,1)}$ generated by the 2×2 minors of the two matrices

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{a-1} \\ x_1 & x_2 & \dots & x_a \end{pmatrix} \qquad \begin{pmatrix} y_0 & y_1 & \dots & y_{b-1} \\ y_1 & y_2 & \dots & y_b \end{pmatrix}$$

and the entries of the $(a-1)\times(b-1)$ matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ \vdots & \vdots & \vdots \\ x_{a-2} & x_{a-1} & x_a \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_0 & y_1 & \dots & y_{b-2} \\ y_1 & y_2 & \dots & y_{b-1} \\ y_2 & y_3 & \dots & y_b \end{pmatrix}.$$

We suppose for concreteness that $a, b \equiv 2 \mod 3$. Then, the scheme X is contained in a scroll Y of type

$$Y = S\left(\frac{a-2}{3}, \frac{a-2}{3}, \frac{a-2}{3}, \frac{b-2}{3}, \frac{b-2}{3}, \frac{b-2}{3}\right).$$

In terms of the Cox ring ($\equiv$ toric coordinate ring) $\mathbb{F}[s, t, u_0, u_1, u_2, v_0, v_1, v_2]$ of Y, the remaining equations reduce to an ideal sheaf $\mathcal{I}_{\text{Cox}}$ generated by nine relative quadrics that are the 2×2 minors of the matrices

$$\begin{pmatrix} u_0 & u_1 & su_2 \\ u_1 & u_2 & tu_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_0 & v_1 & sv_2 \\ v_1 & v_2 & tv_0 \end{pmatrix}$$

together with

$$u_2v_0 + u_1v_1 + u_0v_2,\ tu_0v_0 + su_2v_1 + su_1v_2,\ tu_1v_0 + tu_0v_1 + su_2v_2.$$

The relative resolution constructed in [24, Section 3] can be regarded as a complex of free modules over the Cox ring which sheafifies to a resolution of $\mathcal{O}_X$ by locally free $\mathcal{O}_Y$-modules. In our specific case, it has the Betti table:

	0	1	2	3	4
total:	1	9	16	9	1
0:	1	.	.	.	
1:	.	3	.	.	.
2:	.	6	16	6	.
3:	.	.	.	3	
4:	.	.	.	.	1

where we have given all the variables in the Cox ring degree 1.

We specialize further and take $a = b = 8$. Then,

$$Y = S(2, 2, 2, 2, 2, 2) \subset \mathbb{P}^{17}_{\mathbb{F}}$$

is a rational normal scroll of degree $f = 12$ isomorphic to $\mathbb{P}^1_{\mathbb{F}} \times \mathbb{P}^5_{\mathbb{F}}$.

The relative resolution of $\mathcal{O}_X = \mathcal{O}_{X_e(8,8)}$ as an $\mathcal{O}_Y$-module has shape

$$\mathcal{O}_X \leftarrow \mathcal{O}_Y \leftarrow \mathcal{O}_Y(-2H+3R)^6 \oplus \mathcal{O}_Y(-2H+4R)^3 \leftarrow \mathcal{O}_Y(-3H+5R)^{16} \leftarrow$$
$$\mathcal{O}_Y(-4H+6R)^3 \oplus \mathcal{O}_Y(-4H+7R)^3 \leftarrow \mathcal{O}_Y(-6H+10R) \leftarrow 0.$$

Here, H and R denote the hyperplane class and the ruling of Y.

Each term in the relative resolution is resolved by a Buchsbaum-Eisenbud complex $\mathcal{C}^j$ associated to the defining matrix m of Y regarded as a map $m\colon \mathcal{F} \to \mathcal{G}$ between vector bundles $\mathcal{F} \cong \mathcal{O}(-1)^f$ and $\mathcal{G} \cong \mathcal{O}^2$ on $\mathbb{P}^{a+b+1}$.

$$0 \leftarrow \mathcal{O}_Y(jR) \leftarrow S_j\mathcal{G} \leftarrow S_{j-1}\mathcal{G}\otimes\mathcal{F} \leftarrow \dots$$
$$\dots \leftarrow \Lambda^j\mathcal{F} \leftarrow \Lambda^{j+2}\mathcal{F}\otimes\Lambda^2\mathcal{G}^* \leftarrow \dots$$
$$\dots \leftarrow \Lambda^f\mathcal{F}\otimes\Lambda^2\mathcal{G}^*\otimes(S_{f-j-2}\mathcal{G})^* \leftarrow 0,$$

for $0 \le j \le f-2$ (see [24] and [14, Theorem A2.10 and Exercise A2.22]). Two further facts are important to us:

(1) The complexes $\mathcal{C}^j$ remain exact under the global section functor

$$\mathcal{E} \mapsto \Gamma_*(\mathcal{F}) = \oplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^{a+b+1}, \mathcal{E}(n)),$$

i.e., we obtain projective resolutions of $\Gamma_*(\mathcal{O}_Y(jR))$ over the polynomial ring $\mathbb{F}[x_0, \dots, x_a, y_0, \dots, y_b] = \Gamma_*(\mathcal{O}_{\mathbb{P}^{a+b+1}})$. (This holds because the complexes $\mathcal{C}^j$ have length $f-1 < \dim \mathbb{P}^{a+b+1}$.)

(2) The complex $\mathcal{C}^j$ has j linear maps followed by a quadratic map and further linear maps.

By (1), we can resolve the relative resolution by the iterated mapping cone of complex $\mathcal{C}^j(-d)$'s. In our specific example, this is the iterated mapping cone

$$\mathcal{C}^0 \leftarrow [\begin{matrix} \oplus^6\mathcal{C}^3(-2) \\ \oplus \\ \oplus^3\mathcal{C}^4(-2) \end{matrix} \leftarrow [\oplus^{16}\mathcal{C}^5(-3) \leftarrow [\begin{matrix} \oplus^3\mathcal{C}^7(-4) \\ \oplus \\ \oplus^6\mathcal{C}^6(-4) \end{matrix} \leftarrow \mathcal{C}^{10}(-6)]]] \ .$$

The iterated mapping cone F is not minimal. However, the complex $\mathcal{C}^j(-d)$ for $d \ge 2$ does not contribute to the linear strand in a range outside the contribution of the Eagon-Northcott complex $\mathcal{C}^0$, which proves assertion (3) of Theorem 5.12 in this specific case. Indeed, the additional contribution of maximal homological degree comes from the complex $\oplus^3\mathcal{C}^7(-4)[-3]$. It is a contribution to $\beta_{10,11}(F) = \dim(F_{10} \otimes_S \mathbb{F})_{11}$ to which also $\mathcal{C}^0$ contributes since

$$10 < \operatorname{length} \mathcal{C}^0 = f - 1 = 11.$$

The presence of $\mathcal{C}^0$ and its dual inside the minimal resolution gives a lower bound on the Betti numbers, which is realized for example in the case of $X_{(-1,1)}(6, 6)$ in characteristic 3 computed in Example 5.3, and therefore in characteristic 0 and all but finitely many other primes. Further computation shows that the only exceptional primes for $X_{(-1,1)}(6, 6)$ are 2 and 5.

Proof of Theorem 5.12 (3) We continue with the proof of Theorem 5.12 keeping the notation of the first part of the proof.

The Cox ring $\mathbb{F}[s, t, u_0, \dots, u_k, v_0, \dots, v_k]$ is $\mathbb{Z}^2$-graded with s, t of degree $(0, 1)$ and $\deg u_i = (1, -a_i)$ and $\deg v_i = (1, -b_i)$. The ideal I_{Cox} of $X = X_e(a, b)$ in the Cox ring is obtained by substituting

$$x_j = s^{a_i-\ell}t^\ell u_i \text{ if } j = \ell k + i \text{ with } 0 \le i < k$$

and

$$y_j = s^{b_i-\ell}s^\ell v_i \text{ if } j = \ell k + i \text{ with } 0 \le i < k$$

into the generators of the ideal I_e and saturating with the ideal (s, t).

We can alter and refine this grading to a $\mathbb{Z}^3$-grading by setting $\deg s = \deg t = (0, 0, 1)$, $\deg u_i = (1, 0, a_0 - a_i)$ and $\deg v_i = (0, 1, b_0 - b_i)$, since the substituted equations are homogeneous with respect to this grading. The last component of the degree of each variable of the Cox ring is now 0 or 1.

For the description of the generators of I_{Cox}, the residues $0 \le \alpha, \beta < k$ with $\alpha \equiv a, \beta \equiv b \mod k$ will play a role. Writing $j = \ell k + i$ as above the jth column of the matrix MX after substitution becomes

$$\begin{pmatrix} x_j \\ x_{j+1} \end{pmatrix} = \begin{pmatrix} s^{a_i-\ell}t^\ell u_i \\ s^{a_{i+1}-\ell}t^\ell u_{i+1} \end{pmatrix} \text{ or } \begin{pmatrix} s^{a_{k-1}-\ell}t^\ell u_{k-1} \\ s^{a_0-\ell-1}t^{\ell+1}u_0 \end{pmatrix}$$

in case $j + 1 \equiv 0 \mod k$. Thus, the minors of the $2 \times k$ matrix

$$A = \begin{pmatrix} u_0 & u_1 & \dots & su_\alpha & \dots & u_{k-1} \\ u_1 & u_2 & \dots & u_{\alpha+1} & \dots & tu_0 \end{pmatrix}$$

lie in I_{Cox}, where the factor s occurs only once in the first row, more precisely in front of u_α, and the factor t occurs once in the second row in front of u_0. Likewise, we get a $2 \times k$ matrix B involving the v's.

A similar pattern arises from the $(a-1) \times 3$ and $3 \times (b-1)$ Hankel matrices entering the definition of the bilinear equations (4.1) of $X_e(a, b)$. The Hankel matrix involving the x's becomes the $(k-1) \times 3$ matrix A' which is the transpose of

$$\begin{pmatrix} u_0 & u_1 & \dots & su_{\alpha-1} & su_\alpha & \dots & u_{k-2} \\ u_1 & u_2 & \dots & su_\alpha & u_{\alpha+1} & \dots & u_{k-1} \\ u_2 & u_3 & \dots & u_{\alpha+1} & u_{\alpha+2} & \dots & tu_0 \end{pmatrix}.$$

There are all together at most three factors s and one factor t. Similarly, we get a $3 \times (k-1)$ matrix B' involving the v's. The generators of $\mathcal{I}_{\mathrm{Cox}}$ of degree $(1, 1, *)$ are obtained from the entries of the $(k-1) \times (k-1)$ matrix

$$C = A' D B'$$

with D the 3×3 antidiagonal matrix with entries $1, -e_1, e_2$ from (4.1). The ideal generated by entries of C might be not saturated with respect to st. For example, the form

$$su_{\alpha+1}v_{\beta-1} - e_1 s^2 u_\alpha v_\beta + e_2 s u_{\alpha-1} v_{\beta+1}$$

is divisible by s.

By [24], there are exactly $\binom{2k-1}{2} - 1$ relative quadrics. From the calculation above, we see $\binom{k}{2}$ relative quadrics of each of types $(2, 0, *)$ and $(0, 2, *)$, and $(k-1)^2$ relative quadrics of type $(1, 1, *)$. Since

$$2\binom{k}{2} + (k-1)^2 = \binom{2k-1}{2},$$

we see that there is one superfluous relative quadric, and since the ones of type $(2, 0, *)$ and $(0, 2, *)$ are independent, it is of type $(1, 1, *)$. In summary, the ideal sheaf $\mathcal{I}_{\mathrm{Cox}}$ depends only on the residue classes α, β of a and $b \mod k$ and is generated by

$$2\binom{k}{2} + (k-1)^2 - 1 = \binom{2k-1}{2} - 1$$

relative quadratics of degrees $(2, 0, *)$, $(0, 2, *)$, $(1, 1, *)$ where $*$ represents values between 0 and 4.

The ℓth free module in our relative resolution E_ℓ has generators of degree (d_1, d_2, d_3) with $d_1 + d_2 = \ell + 1$ for $1 \le \ell \le 2k - 3$. The last module is cyclic with a generator of degree $(k, k, 2k - \alpha - \beta)$. Indeed, this is the sum of the degree of all variables of the Cox ring, which equals the degree of the generator of its canonical module. By adjunction, the relative resolution has to end with this term, since $X_e(a, b)$ has a trivial canonical bundle. The resolution is self-dual.

The sequences

$$\underline{d}_\ell = \min\{d_3 \mid \exists \text{ a generator of } E_\ell \text{ of degree } (d_1, d_2, d_3) \text{ with } d_1 + d_2 = \ell + 1\}$$

and

$$\overline{d}_\ell = \max\{d_3 \mid \exists \text{ a generator of } E_\ell \text{ of degree } (d_1, d_2, d_3) \text{ with } d_1 + d_2 = \ell + 1\}$$

are weakly increasing, because for each generator of the Cox ring the third component of its degree is nonnegative.

We write $\text{Pic}(Y) = \mathbb{Z}H \oplus \mathbb{Z}R$, where H denotes a hyperplane section and R a fiber of $Y \to \mathbb{P}^1$. In terms of the $\text{Pic}(Y)$-grading a generator of degree (d_1, d_2, d_3) corresponds to a summand

$$\mathcal{O}_Y(-(d_1 + d_2)H + (d_1 a_0 + d_2 b_0 - d_3)R).$$

To establish assertion (3) of Theorem 5.12, we must show that the multidegree (d_1, d_2, d_3) of every generator of E_ℓ for $1 \le \ell \le 2k - 3$ satisfies

$$d_1 + d_2 - 1 + d_1 a_0 + d_2 b_0 - d_3 \le \deg Y - 1 = f - 1.$$

Indeed, the left-hand side is the length of the contribution of

$$\mathcal{C}^{d_1 a_0 + d_2 b_0 - d_3}(-d_1 - d_2)$$

to the linear part of the iterated mapping cone, while the right-hand side is the length of the $\mathcal{C}^0$.

Note that $-d_3 \le -\underline{d}_\ell = -(2k - \alpha - \beta) + \overline{d}_{(2k-2-\ell)}$ holds by the self-duality of the relative resolution. Because $\omega_X \cong \mathcal{O}_X$, the last term in the relative resolution has to be $\mathcal{O}_Y(-2kH + (f-2)R) \cong \omega_Y$ so $f - 2 = ka_0 + kb_0 - (2k - \alpha - \beta)$.

Thus, utilizing $a_0 \ge b_0$, we see that the conditions

$$\ell - (2k - 1 - \ell)b_0 + \overline{d}_{2k-2-\ell} \le 1$$

suffice. We use the rough estimate $\overline{d}_{2k-2-\ell} \le 2k$, which holds since the maximal d_3 in the relative resolution is $2k - \alpha - \beta \le 2k$. The desired inequality holds for all ℓ with $1 \le \ell \le 2k - 3$ if

$$b_0 \ge 2k - 2 = \max\left\{\frac{2k + \ell - 1}{2k - 1 - \ell} \mid \ell = 1, \ldots, 2k - 3\right\}.$$

Since $b + 1 = kb_0 - (k - 1 - \beta) \le kb_0$, this follows from our assumption $a \ge b \ge 2k^2$. □

Remark 5.14 A proof of Theorem 5.12 (3) for $a, b \gg k$ can be deduced by substantially easier arguments, which do not rely on the description of $\mathcal{I}_{\text{Cox}}$ but only on the existence of a relative resolution proved in [24] and an analysis of how the numerical data change when we re-embed Y by $H' = H + jR$. Since

- (a, b) will be replaced by $(a + jk, b + jk)$ and thus f by $f + 2jk$ and
- $\mathcal{O}_Y(-dH + cR) = \mathcal{O}_Y(-dH' + (c + dj)R)$

the conclusion of (3) is obvious for j sufficiently large. Based on experiments, we conjecture that the optimal bound is $a \ge b \ge k^2 - k$. This is true for $k \le 5$.

For further information and conjectures about relative resolutions of canonical curves (see [8, 9]).

6 Conjectures and Computational Results

Remark 6.1 It follows from Proposition 5.2 that Green's conjecture is true for the balanced carpet $X(a,a)$ if and only if a certain $f(a) \times f(a)$ integer matrix has a nonzero determinant, where,

$$f(a) = a\binom{2a-1}{a+1} - 2\binom{2a-3}{a-1}$$

by Remark 5.8. By Theorem 5.11, we know that $\beta_{a,a+1}(X(a,a)) = a\binom{2a-2}{a+1}$ over fields of characteristic 2. Hence,

$$2^{a\binom{2a-2}{a+1}}$$

is a factor of this determinant. For small a, the relevant values are as follows:

a	2	3	4	5	6	7
$\lvert \det \rvert$	1	2^4	$2^{32}3^6$	$2^{266}3^{15}$	$2^{1312}3^{72}5^{120}$	$2^{6774}3^{1020}5^{315}$
$f(a)$	0	9	64	350	1728	8085
$a\binom{2a-2}{a+1}$	0	3	24	140	720	3465

One step in achieving a proof of Green's conjecture using K3 carpets might be to give an explanation of the prime power factorizations of the determinants in the table above.

The data in this table was produced by our Macaulay2 [22] package K3Carpets.m2 version 0.5 [17]. Here is how these determinants are actually computed. The first step is the computation of the Schreyer resolution of an carpet $X(a,a)$ over $\mathbb{F}[x_0, \ldots, x_a, y_0, \ldots, y_a]$ for a large finite prime field $\mathbb{F} = \mathbb{Z}/(p)$. In practice, we take $p = 32003$. The second step is to lift the matrices in the resolution to $P = \mathbb{Z}[x_0, \ldots, x_a, y_0, \ldots, y_a]$ by using the bijection of $\mathbb{Z}/32003$ with the integers in the interval $[-16001, 16001]$. The resulting matrices define the Schreyer resolution over P if and only if the lifted matrices form a complex. After checking this, we use the fine grading to find the blocks in the crucial constant strand. For the computation of the determinants of the blocks, we use their Smith normal forms. The final step is the factorization of the product of all determinants of all blocks.

Remark 6.2 The enormous size ot the determinants in Remark 6.1 must correspond to a combination of the resonance phenomenon with the exceptional behavior of Green's conjecture in positive characteristic.

Experimental data of [7], see also [10], suggests that a general canonical curve of odd genus $g = 2a + 1$ violates Green's conjecture in small characteristic in the following cases:

a	$g = 2a+1$	Primes	$\beta_{a-1,a+1} = \beta_{a,a+1}$
3	7	2	1
4	9	3	6
5	11	2, 3	28, 10
6	13	2, 5	64, 120
7	15	2, 3, 5	299, 390, 315

For genus $g = 7, 9$, this is rigorously proven by [24] and [23]. For genus $g = 11, 13, 15$, we know that the examples found in [7] violate the full Green conjecture; however, we do not know whether their Betti numbers coincide with the Betti numbers of the general curve of the given genus in these characteristics.

Computing a nonminimal resolution of the K3 union of scrolls $X_e(a,a)$ over the coefficient ring $\mathbb{Z}[e_1,e_2]$, we find the following values of the determinant of the crucial nonminimal part.

a	$\pm$ det
3	$2e_1^3e_2^3$
4	$3^6e_1^{32}e_2^{32}$
5	$2^{46}3^{10}e_1^{220}e_2^{235}(e_1^2-e_2)^5$
6	$2^{64}5^{120}e_1^{1248}e_2^{1464}(e_1^2-e_2)^{72}$
7	$2^{390}3^{390}5^{315}e_1^{6377}e_2^{8302}(e_1^2-e_2)^{630}(e_1^2-2e_2)^7$

Based on these values, we propose two conjectures:

Conjecture 6.3 *For $e=(e_1,e_2)\in\mathbb{F}^2$ with $e_2\neq 0$ the union of scrolls $X_e(a,a)$ has a pure resolution over an field $\mathbb{F}$ of characteristic 0 unless the polynomial $p(z)=z^2-e_1z+e_2=(z-t_1)(z-t_2)$ has roots such that $t_2/t_1\neq 1$ is a kth root of unity for some $k\leq\frac{a+1}{2}$.*

Conjecture 6.4 *For general $e=(e_1,e_2)\in\overline{\mathbb{F}}^2$, the union of scrolls $X_e(a,a)$ over an algebraically closed field $\overline{\mathbb{F}}$ of characteristic p has a pure resolution if $p\geq a$. In particular, Green's conjecture holds for the general curve over a field of characteristic p of genus g if $p\geq\frac{g-1}{2}$.*

By the table above and Remark 5.9, both conjectures hold for $g\leq 15$.

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