ON STRUCTURE-PRESERVING HIGH ORDER METHODS FOR CONSERVATION LAWS

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Abstract. In this paper we review the algorithm development in high order methods for some conservation laws. The emphasis is on our recent contribution in the study of two model classes: Fokker-Planck-type equations and hyperbolic conservation law systems. For the former we will review free-energy-satisfying and positivity-preserving schemes. For the later we will review the general invariant-region-preserving (IRP) limiter, and its application to high order methods for multi-dimensional hyperbolic systems of conservation laws.

1. **Introduction.** Systems of conservation laws for field quantities arise in diverse applications. Their solutions may be visualized as evolving observables or propagating waves. When the system is nonlinear, solution profiles can become steeper as shocks or even concentrated as measures, propagation of these profiles cause mathematical and numerical challenges in solving systems of conservation laws.

We are interested in building structure-preserving high order numerical methods for time-dependent conservation laws through model refinement. In this paper we restrict to two model classes: Fokker-Planck-type equations and hyperbolic conservation law systems. By structure preserving algorithms we mean algorithms that can preserve certain intrinsic solution properties at the discrete level.

For Fokker-Planck-type equations, the three main solution properties are mass conservation, non-negativity, and the free energy/entropy dissipation law. We present a second order explicit-implicit scheme that satisfies all three properties at the discrete level, without a strict time step restriction [15], and discuss how to incorporate these solution properties into a high order discontinuous Galerkin (DG) method of arbitrary order [19]. For multi-dimensional hyperbolic conservation law systems endowed with a convex invariant region in the phase space, main solution properties are also in three aspects: solution conservation, invariant region preservation, and the entropy dissipation law. Here we only review the invariant-region-preserving (IRP) limiter designed in [10], and has been tested in [9, 11] for systems of Euler equations.

The organization of this paper is as follows. Section 2 is devoted to two models and their main mathematical properties. Section 3 gives a short account of the direct DG discretization techniques. Section 4 contains a review of the entropy satisfying methods for Fokker-Planck type equations. In section 5 we address the

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invariant-region-preserving limiter and its applications to multi-D hyperbolic systems of conservation laws, and finally in section 6 we give some concluding remarks.

2. **PDE models and solution properties.** We begin with the fundamental transport equation

$$\partial_t \rho(t,x) + \nabla_x \cdot (\rho(t,x)u) = 0, \tag{1}$$

for which the probability density space

$$\mathbb{P} = \{\rho, \quad \rho \geq 0, \quad \int \rho = 1\}$$

is invariant. This transport equation alone is not closed, unless u can be related to ρ or governed by further equations.

In dynamics driven by an entropy/ free energy functional $E = E[\rho]$, a direct verification (assuming zero-flux boundary condition) shows that fast decay of E along the transport dynamics (1) can be ensured if $u = -\nabla_x (\delta_\rho E)$, where δ_ρ denotes the usual L^2 variational derivative. We are led to the Fokker-Planck type equation

$$\partial_t \rho = \nabla_x \cdot (\rho \nabla_x \delta_\rho E[\rho]). \tag{2}$$

Dictated by different forms of E, this class includes many equations such as the heat equation, the Fokker-Planck equation [28], the aggregation equation [12, 33] with

$$E = \int \left[\rho \log \rho + V(x)\rho + \frac{1}{2} W * \rho\rho\right] dx,$$

as well as drift-diffusion models such as the Poisson-Nernst-Planck equation [7] and the Keller-Segel system [26]. Equation (2) is a natural gradient flow generated by functionals $E[\rho]$ in Wasserstein distance, directly linked to the minimization problem $\min_{\rho\in\mathbb{P}} E[\rho]$, and has received ample attention in multiple contexts. Solutions to (2) are usually not sensitive to initial distributions, but often to the critical mass, some patterns will emerge as time evolves leading to rich solution structures when coupled with notrivial forces. In order for a numerical method to generate solutions with satisfying long time behavior, it is crucial to preserve some intrinsic solution properties. The main solution properties are

- nonnegativity principle, $\rho_0 \geq 0 \Longrightarrow \rho(t,x) \geq 0 \quad \forall t > 0.$
- mass conservation $\int \rho(t,x)\, dx = \int \rho_0(x)\, dx \quad \forall t > 0.$
- the entropy/energy dissipation inequality

$$\frac{d}{dt} E = -\int \rho |\nabla_x \delta_\rho E|^2 dx \leq 0.$$

These properties are naturally desired for high order numerical schemes.

In Eulerian dynamics of 'fluids', velocity field is governed by the moment equation

$$\partial_t u + u \cdot \nabla u = F.$$

Dictated by different forcing F, examples of such system include the Euler equation, the Navier-Stokes equation, the Euler-Poisson equation, etc. For such Eulerian balance laws the solution is often sensitive to the initial velocity field, leading to the so called critical threshold (CT) phenomena! [16]. We note that gradient flows (2) can be seen as describing the long-time response of an Euler equation with friction [4, system (2.1)].

The simplest hyperbolic balance laws is the system of compressible Euler equations, which belongs to the following model class:

$$\partial_t \mathbf{w} + \sum_{j=1}^{d} \partial_{x_j} F_j(\mathbf{w}) = 0, \quad x \in \mathbb{R}^d, \quad t > 0; \quad \mathbf{w}(0,x) = \mathbf{w}_0, \tag{3}$$

where $\mathbf{w} \in \mathbb{R}^l$ with $l > 1$, and the flux function $F_j(\mathbf{w}) \in \mathbb{R}^l$. It is known that discontinuities can develop at finite time even for smooth initial data [13], hence entropy inequalities should be used to single out the physically relevant solution among many weak solutions. In application problems, the pointwise range of solutions (invariant region) may be known from physical considerations.

The main solution properties, also desired at discrete level, are

- Invariant region $\quad \mathbf{w}_0 \in \Sigma \Longrightarrow \mathbf{w}(t,\cdot) \in \Sigma \quad \forall t > 0.$
- Conservation $\quad \int \mathbf{w}(t,x)dx = \int \mathbf{w}_0(x)\,dx \quad \forall t > 0.$
- Entropy inequality. $\quad \partial_t \eta(\mathbf{w}(t,x)) + \nabla_x \cdot \Psi(\mathbf{w}(t,x)) \le 0, a.e,$
  where (η, Ψ) is an admissible entropy-pair.

In the construction of structure-preserving algorithms for the above two model classes, we have adopted the following strategy:

- *Direct DG (DDG) discretization* of the PDE weak formulation, *choosing proper numerical fluxes* to preserve solution conservation and certain entropy dissipation law, together with Runge–Kutta methods [3] for time discretizaiton.
- *Limiting numerical solutions* to weakly enforce the point-wise solution bounds.

3. **Discretization techniques.** For solutions with either concentrations or discontinuities, the finite volume method as a natural choice can lead to the conservation form of a scheme which is a main ingredient of shock capturing methods for hyperbolic conservation laws. Its high order extension is the Discontinuous Galerkin (DG) method, which is also a class of finite element methods, using a completely discontinuous piecewise polynomial space for the numerical solution and the test functions [8, 29, 30].

For DG methods to conservative PDEs, the key is to design suitable numerical fluxes so that the resulting scheme satisfies the desired properties.

Taking $\partial_t u + \partial_x \cdot J = 0$ as an example, a simple integration by parts over any computational cell I gives

$$\int_I \partial_t u v dx - \int_I J v_x dx + J v|_{\partial I} = 0.$$

Here ∂I denotes the boundary of I. To complete the DG method, a single valued numerical flux $\hat{J}$ is needed to replace J, and values from inside I for the test function v. For first order scalar conservation laws $J = f(u)$, it is simple to take a monotone flux

$$\hat{J} = \hat{f}(u^-, u^+),$$

including the celebrated Lax-Friedrichs flux and Godunov flux, see [30].

However, for high order PDEs, it is subtle to define $\hat{J}$. For example, for $J = -\partial_x u$, there is a need to define a flux for $\partial_x u$. The average of $\partial_x u$ from traces of derivatives of two neighboring polynomials is known to give a wrong solution for P^1 polynomials! Indeed, various ideas have appeared in the literature to overcome such difficulty, see e.g. [29].

The solution of the heat equation $\partial_t u = \partial_x^2 u$ with initial data g which has only one discontinuity at $x = 0$ gives

$$u_x(t,0) = \frac{1}{\sqrt{4\pi t}}[g] + \{\partial_x g\} + \sqrt{\frac{t}{\pi}}[\partial_x^2 g] + \cdots,$$

where $[\cdot]$ denotes the jump and $\{\cdot\}$ the average. This led us to the flux formula in [23]

$$\hat{u}_x = \beta_0 \frac{[u]}{\Delta x} + \{u_x\} + \sum_{m=1}^{\lfloor k/2 \rfloor} \beta_m (\Delta x)^{2m-1} [\partial_x^{2m} u].$$

Motivated by such formula, we design a refined DDG for diffusion in [24] as

$$\int_{I_j} \partial_t u v dx + \int_{I_j} \partial_x u \partial_x v dx - \widehat{u_x} v\Big|_{\partial I_j} + (\{u\} - u) v_x\Big|_{\partial I_j} = 0,$$

where

$$\hat{u}_x = \beta_0 \frac{[u]}{\Delta x} + \{u_x\} + \beta_1 \Delta x [u_{xx}].$$

In [14], the DDG method is shown L^2 stable in the sense that

$$\int u^2(t,x)dx + \{\cdots\} \le \int u^2(0,x)dx,$$

with $\{\cdots\} \ge 0$ if

$$\beta_0 > \Gamma(\beta_1) := k^2 \left(1 - \beta_1(k^2-1) + \frac{\beta_1^2}{3}(k^2-1)^2\right).$$

The use of β_0, β_1 provides extra room for incorporating more desired solution properties. Sharp L^2 error estimates are obtained in [14] as

$$\|u_{exact}(t,\cdot) - u(t,\cdot)\|_{L_x^2} \le Ch^{(k+1)},$$

when using polynomial elements of degree k for $\partial_t u + \nabla_x \cdot f(u) = \Delta u$. Moreover, 3rd order maximum-principle-preserving DG scheme (P_2 polynomials) is possible, if

$$\frac{1}{8} < \beta_1 < \frac{1}{4}, \quad \beta_0 \ge 1;$$

as shown for linear Fokker-Planck equations [21], and for a class of convection-diffusion equations [34]. In addition, super-convergence rate of h^{2k} at nodes has been proved by Cao, Liu and Zhang [2] if

$$\beta_1 = \frac{1}{2k(k+1)}; \quad \beta_1 = \frac{1}{12} \quad \text{if } k = 2.$$

The results also include rate h^{k+1} for solution derivatives at Gauss points, h^{k+2} at Lobatto points, and h^{2k} at nodes.

4. **Fokker-Planck-type equations.** We begin with the aggregation model

$$\partial_t \rho = \nabla \cdot (\nabla \rho + \rho \nabla (V(\mathbf{x}) + W * \rho)),$$

where $V(x)$ is a given potential, and W is a symmetric, positive kernal with integral 1. Based on the reformulation of the form

$$\partial_t \rho = \nabla \cdot \left(M \nabla \left(\frac{\rho}{M}\right)\right), \quad M = e^{-V(x) - W*\rho},$$

we introduced in [15] an explicit-implicit scheme:

$$h_j \frac{\rho_j^{n+1}-\rho_j^n}{\Delta t} = h_{j+1/2}^{-1} M_{j+1/2}^n \left(\frac{\rho_{j+1}^{n+1}}{M_{j+1}^n} - \frac{\rho_j^{n+1}}{M_j^n} \right) - h_{j-1/2}^{-1} M_{j-1/2}^n \left(\frac{\rho_j^{n+1}}{M_j^n} - \frac{\rho_{j-1}^{n+1}}{M_{j-1}^n} \right),$$

where ρ_j^n approximates $\rho(t, x_j)$ at time $t = n\Delta t$. This scheme is easy to implement, and is shown to preserve all three desired properties without a strict time step restriction. This has extended and improved upon our earlier works [20, 17]. Extensions to multi-dimensional settings and/or the case when $W * \rho$ is replaced by Ψ solved by a Poisson equation are doable as shown in [15].

It is more challenging to design a high order scheme (3rd or higher order) while three properties remain preserved at the discrete level. Next we show how this can be achieved through a drift-diffusion system. A detailed account can be found in [19], also earlier works [18, 22].

In a mean field approximation of diffusive molecules or ions, one finds the Poisson–Nernst–Planck (PNP) system, $i = 1, \cdots, m$,

$$\partial_t c_i = \nabla \cdot (\nabla c_i + q_i c_i \nabla \psi) \quad x \in \Omega,\ t > 0 \tag{4a}$$

$$-\nabla \cdot (\epsilon(x) \nabla \psi) = \sum_{i=1}^m q_i c_i + \rho_0(x), \quad x \in \Omega,\ t > 0, \tag{4b}$$

$$c_i(0, x) = c_i^{\text{in}}(x), \quad x \in \Omega, \tag{4c}$$

$$\frac{\partial \psi}{\partial \mathbf{n}} = \sigma, \quad \frac{\partial c_i}{\partial \mathbf{n}} + q_i c_i \frac{\partial \psi}{\partial \mathbf{n}} = 0, \qquad x \in \partial\Omega,\ t > 0. \tag{4d}$$

Here $c_i = c_i(t, x)$ denotes density of i-th charged particle with charge q_i, at time t and position x, and $\psi = \psi(t, x)$ the electro-static potential. The PNP system has been widely accepted in applications in electrical engineering and electrokinetics, electrochemistry, and biophysics: for example in biological channels [7] or semiconductor devices [25].

Main mathematical features of the system include the conservation of ions, positivity of concentration, and dissipation of the free energy

$$\frac{d}{dt} F = -\sum_i^m \int_\Omega c_i^{-1} |\nabla c_i + c_i \nabla \psi|^2 dx \le 0$$

where

$$F = \int_\Omega \sum_{i=1}^m c_i \log c_i dx + \frac{1}{2} \int_\Omega |\nabla_x \psi|^2 dx.$$

In order to construct a DG scheme to incorporate these solution properties, we reformulate the PNP system (one dimensional case and $\epsilon = 1$, for notational simplicity) as

$$\begin{aligned} \partial_t c_i &= \partial_x (c_i \partial_x p_i),\ i = 1, \cdots, m, \\ p_i &= q_i \psi + \log c_i, \\ -\partial_x^2 \psi &= \sum_{i=1}^m q_i c_i + \rho_0(x). \end{aligned}$$

Let V_h denote a DG solution space (piecewise polynomials), then the DDG spatial discretization when coupled with the Euler forward time discretization gives us the

scheme: find $c_{ih}^n, p_{ih}^n, \psi_h^n \in V_h$, $\forall\ v_i, r_i, \eta \in V_h$, $i = 1, \cdots, m$,

$$\int_{I_j} \frac{c_{ih}^{n+1} - c_{ih}^n}{\Delta t} v_i dx = -\int_{I_j} c_{ih}^n \partial_x p_{ih}^n \partial_x v_i dx + \{c_{ih}^n\} \left(\widehat{\partial_x p_{ih}^n} v_i + (p_{ih}^n - \{p_{ih}^n\}) \partial_x v_i \right) \Big|_{\partial I_j},$$

$$\int_{I_j} p_{ih}^n r_i dx = \int_{I_j} (q_i \psi_h^n + \log c_{ih}^n) r_i dx,$$

$$\int_{I_j} \partial_x \psi_h^n \partial_x \eta dx - \left(\widehat{\partial_x \psi_h^n} \eta + (\psi_h^n - \{\psi_h^n\}) \partial_x \eta \right) \Big|_{\partial I_j} = \int_{I_j} \left[\sum_{i=1}^m q_i c_{ih}^n + \rho_0 \right] \eta dx,$$

with flux $\widehat{\partial_x p_{ih}} = Fl(p_{ih})$ and $\widehat{\partial_x \psi_h} = Fl(\psi_h)$, and

$$Fl(w) := \beta_0 \frac{[w]}{h} + \{\partial_x w\} + \beta_1 h [\partial_x^2 w].$$

The numerical solution is shown to have following properties.

Theorem 4.1. [19]

1. *The fully discrete scheme is conservative*

$$\sum_{j=1}^N \int_{I_j} c_{ih}^n dx = \sum_{j=1}^N \int_{I_j} c_{ih}^{n+1} dx, \quad i = 1, \cdots, m, \qquad n \in \mathbb{N}.$$

2. *Assuming $c_{ih}^n(x) > 0$, there exists $\mu^* > 0$ such that if the mesh ratio $\mu = \frac{\Delta t}{\Delta x^2} \in (0, \mu^*)$, then the fully discrete free energy*

$$F^n = \sum_{j=1}^N \int_{I_j} \left[\sum_{i=1}^m c_{ih}^n \log c_{ih}^n + \frac{1}{2} \left(\sum_{i=1}^m q_i c_{ih}^n + \rho_0 \right) \psi_h^n \right] dx + \frac{1}{2} \int_{\partial\Omega} \sigma \psi_h^n ds,$$

$$F^{n+1} \le F^n - \frac{\Delta t}{2} \sum_{i=1}^m A_{c_{ih}^n}(p_{ih}^n, p_{ih}^n).$$

where $A_c(\cdot,\cdot)$ is a weighted bilinear operator, which is coercive if β_0 is suitably large, and $\beta_1 = 0$ in $Fl(\psi_h)$.

The free energy dissipation law is also proved for high order strong stability preserving Runge-Kutta methods [3], which are convex combinations of several formal forward Euler steps.

As a result, steady states can well be preserved: if initial data c_{ih}^0 is already at steady states, i.e., $\log c_{ih}^0 + q_i \psi_h^0(x) = C_i$. By induction, it can be shown that the following holds:

$$\log c_{ih}^n + q_i \psi_h^n(x) = C_i \quad \forall n \in \mathbb{N}.$$

The scheme requires c_{ih} be positive, which is difficult to achieve for high order approximations. Inspired by the Zhang-Shu limiter [35] for scalar conservation laws, we impose a limiter. For approximation $w_h \in P^k(I_j)$ with cell averages $\bar{w}_j > \delta$, we reconstruct

$$w_h^\delta(x) = \bar{w}_j + \frac{\bar{w}_j - \delta}{\bar{w}_j - \min_{I_j} w_h(x)} (w_h(x) - \bar{w}_j), \qquad \text{if } \min_{I_j} w_h(x) < \delta.$$

This reconstruction maintains same cell averages, satisfies $\min_{I_j} w^\delta(x) \ge \delta$, and does not destroy accuracy when $\delta < h^{k+1}$.

The algorithm in [19] can be summarized in following steps.

1. (Initialization) Project $c_i^{\text{in}}(x)$ onto V_h to obtain $c_{ih}^0(x)$.

2. (Reconstruction) From $c_{ih}^n(x)$, apply, if necessary, the reconstruction limiter to update c_{ih}^n so that $c_{ih}^n > \delta$.
3. (Poisson solver) Solve the Poisson equation to obtain ψ_h^n.
4. (Projection) Obtain $p_{ih}^n \in V_h$ by projection of $q_i\psi_h^n + \log c_{ih}^n$.
5. (Update) Solve the NP equations to obtain c_{ih}^{n+1} with some Runge-Kutta solver.
   Repeat steps 2-5 until final time T.

5. **IRP limiter for hyperbolic systems.** An invariant region to (3) is an open set in phase space $\mathbb{R}^l$ such that if the initial data is in this set, then the solution will remain in this set. It was proved by Hoff [6] that an invariant region for one dimensional hyperbolic conservation laws must be convex. For 2×2 systems such as the isentropic Euler system, an invariant region can be described by two Riemann invariants [31]. For general hyperbolic conservation law systems, it is a challenging task to identify a useful invariant region.

Shock capturing numerical methods have seen revolutionary developments over the past 40 years, with both conservation and entropy stability as two main ingredients in each scheme construction. However, it remains a difficult task to preserve an invariant region by a high order numerical method unless some nonlinear limiter is frequently imposed (Refs [1, 5] for first order IRP schemes). Indeed, recent efforts using limiting techniques have been made to construct high order maximum-principle-preserving schemes for scalar conservation laws (see [35]) and positivity-preserving schemes for hyperbolic systems including compressible Euler equations (see e.g. [27, 36, 38]). The work by Zhang and Shu in [37] introduced a limiter to preserve the minimum-entropy-principle [32] for high order schemes to the compressible Euler equation.

We now discuss the general explicit limiter introduced in [10]. Assume the multi-dimensional system of conservation laws admits an invariant region Σ, characterized by

$$\Sigma = \{\mathbf{w}\big| \quad U(\mathbf{w}) \le 0\},$$

where U is convex. Denote the interior of Σ by Σ_0. A key fact we have used is that for any bounded domain K, the averaging defined by

$$\bar{\mathbf{w}} = \frac{1}{|K|}\int_K \mathbf{w}(x)dx$$

is a contraction operator.

Lemma 5.1. *Let $\mathbf{w}(x)$ be non-trivial piecewise continuous vector functions. If $\mathbf{w}(x) \in \Sigma$ for all $x \in K \subset \mathbb{R}^d$ and U is strictly convex, then $\bar{\mathbf{w}} \in \Sigma_0$ for any bounded domain K.*

This lemma sets the foundation for using the domain average as a reference to limit the existing polynomials, through a linear convex combination as in [35, 37]. In the system case, the question of particular interest is whether the limited approximation is still high order accurate.

Let $\mathbf{w}_h(x)$ be a sequence of vector polynomials over K, a high order accurate approximation to the function $\mathbf{w}(x) \in \Sigma$. Assume $\bar{\mathbf{w}}_h \in \Sigma_0$, but $\mathbf{w}_h(x)$ is not entirely located in Σ. We construct

$$\tilde{\mathbf{w}}_h(x) = \theta\mathbf{w}_h(x) + (1-\theta)\bar{\mathbf{w}}_h,$$

where $\theta \in (0,1]$ is defined by $\theta = \min\{1, \theta_1\}$, where

$$\theta_1 = \frac{U(\bar{\mathbf{w}}_h)}{U(\bar{\mathbf{w}}_h) - U_h^{\max}}, \quad U_h^{\max} = \max_{x\in K} U(\mathbf{w}_h(x)) > 0.$$

If $\Sigma = \bigcap_{i=1}^{M} \{\mathbf{w} \big| \quad U_i(\mathbf{w}) \le 0\}$, then the limiter parameter needs to be modified as

$$\theta = \min\{1, \theta_1, \cdots, \theta_M\}.$$

This reconstruction has been shown to satisfy three desired properties.

Theorem 5.2. [10] *The reconstructed polynomial $\tilde{\mathbf{w}}_h(x)$ satisfies the following three properties:*

(i) *the average is preserved, i.e. $\bar{\mathbf{w}}_h = \bar{\tilde{\mathbf{w}}}_h$;*
(ii) *$\tilde{\mathbf{w}}_h(x)$ lies entirely within invariant region $\Sigma, \forall x \in K$;*
(iii) *order of accuracy is maintained, i.e., if $\|\mathbf{w}_h - \mathbf{w}\|_\infty \le 1$, then*

$$\|\tilde{\mathbf{w}}_h - \mathbf{w}\|_\infty \le \frac{C}{|U(\bar{\mathbf{w}}_h)|}\|\mathbf{w}_h - \mathbf{w}\|_\infty,$$

where $C > 0$ depends on $\mathbf{w}$ and Σ.

Let $\mathbf{w}_h^n$ be the numerical solution at n-th time step generated from a high order finite-volume-type scheme of an abstract form

$$\mathbf{w}_h^{n+1} = \mathcal{L}(\mathbf{w}_h^n), \quad \mathbf{w}_h^n = \mathbf{w}_h^n(x) \in V_h.$$

Provided that the scheme has the following property: there exists λ_0, and a test set S such that if

$$\lambda := \frac{\Delta t}{\Delta x} \le \lambda_0 \quad \text{and} \quad \mathbf{w}_h^n(x) \in \Sigma \text{ for } x \in S,$$

then

$$\bar{\mathbf{w}}_h^{n+1} \in \Sigma_0;$$

the limiter can then be applied with K replaced by $S_K : S \cap K$, i.e.,

$$U_h^{\max} = \max_{x\in S_K} U(\mathbf{w}_h(x)),$$

through the following algorithm:

Step 1. *Initialization:* take the piecewise L^2 projection of $\mathbf{w}_0$ onto V_h, such that

$$\langle \mathbf{w}_h^0 - \mathbf{w}_0, \phi\rangle = 0, \quad \forall \phi \in V_h.$$

Step 2. *Limiting:* Impose the modified limiter on $\mathbf{w}_h^n$ for $n = 0, 1, \cdots$ to obtain $\tilde{\mathbf{w}}_h^n$.

Step 3. *Update* by the scheme:

$$\mathbf{w}_h^{n+1} = \mathcal{L}(\tilde{\mathbf{w}}_h^n).$$

Return to Step 2.

A limiter as such was first reported in [11] for one-dimensional Euler equations, and in [9] for the isentropic gas dynamics. The limiter in [11] is explicit and simultaneously preserves the positivity of density and pressure and also a minimum principle for the specific entropy [32].

For multi-dimensional systems of conservation laws, there is a need to check whether the projected system shares the same invariant region as that for the full

multi-D system. For 2D compressible Euler equations with $\mathbf{w} = (\rho, m, n, E)^\top$, $\mathbf{F}(\mathbf{w}) = (F_1(\mathbf{w}), F_2(\mathbf{w}))$, where

$$F_1(\mathbf{w}) = (m, \rho u^2 + p, \rho uv, (E+p)u)^\top,$$
$$F_2(\mathbf{w}) = (n, \rho uv, \rho v^2 + p, (E+p)v)^\top$$
$$m = \rho u, \quad n = \rho v, \quad E = \frac{1}{2}\rho u^2 + \frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1},\ \gamma > 1,$$

this as been shown true with the invariant region expressed as

$$\Sigma = \{\mathbf{w} \big| \quad \rho > 0,\ p > 0,\ q < 0\},$$

where $s = \log\left(\frac{p}{\rho^\gamma}\right)$ and $s_0 = \inf_x \log\left(\frac{p_0(x)}{\rho_0^\gamma(x)}\right)$, and $q = (s_0 - s)\rho$ is convex in $\mathbf{w}$. Hence, a corresponding IRP algorithm can be well established, and has been tested in [10].

6. **Conclusions and outlook.** In this paper, we have reviewed some of our contributions to the development of structure-preserving algorithms for two model classes. It is clear from the works we have reviewed, and from related references in the literature, that these techniques are not limited to these model equations, it is interesting to check the algorithmic improvement with more complex systems. Interesting directions worth further investigation include: (1) Design of explicit-implicit structure-preserving algorithms for nonlinear Fokker-Planck-type equations so to enhance computational efficiency; (2) Design of local IRP algorithms for multi-dimensional systems of hyperbolic conservation laws, with more realistic applications.

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