



Mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ and entropy covariance control: a convex optimisation approach

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ABSTRACT

In this paper, we develop a covariance control problem to address a tradeoff between $\mathcal{H}_2$ performance and $\mathcal{H}_\infty$ disturbance attenuation. In particular, we formulate a mixed-norm $\mathcal{H}_2/\mathcal{H}_\infty$ and entropy covariance control problem that guarantees that the state covariance of an uncertain dynamical system driven by white noise is upper bounded in the sense of the cone of nonnegative definite matrices by a given threshold matrix via state feedback as well as output feedback control. This is accomplished by combining $\mathcal{H}_2$ covariance control theory and mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ control theory. By using suitable transformations involving dynamic weighting on the complementary sensitivity system transfer function, the proposed formulation is applicable to robustness problems involving nominal performance subject to a robust stability requirement. The proposed formulation allows for solutions via semidefinite programming. Finally, two illustrative numerical examples are provided to show the efficacy of the proposed approach.

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1. Introduction

The fundamental differences between Wiener-Hopf-Kalman control design (e.g. LQG theory (Kwakernaak & Sivan, 1972)) and $\mathcal{H}_\infty$ control theory (Zames, 1981; Zhou et al., 1996) can be traced back to the modelling and treatment of exogenous disturbances. In particular, LQG design is based upon a stochastic white noise disturbance model possessing a fixed covariance (power spectral density), whereas $\mathcal{H}_\infty$ theory is predicated on a deterministic disturbance model consisting of bounded energy (square-integrable) signals. Since LQG design utilises a quadratic cost criterion, it follows from Plancherel's theorem that Wiener-Hopf-Kalman theory strives to minimise the $\mathcal{H}_2$ norm of the closed-loop frequency response, while $\mathcal{H}_\infty$ control addresses the problem of disturbance attenuation with $\mathcal{L}_2$ signal norms on the disturbances and performance variables and with the induced $\mathcal{H}_\infty$ closed-loop transfer function norm corresponding to the worst-case disturbance attenuation.

For systems with poorly modelled disturbances possessing significant power within arbitrarily small bandwidths, $\mathcal{H}_\infty$ is clearly appropriate, whereas for systems with well-known disturbance power spectral densities, Wiener-Hopf-Kalman theory may be less conservative. However, $\mathcal{H}_\infty$ control theory offers two main advantages over $\mathcal{H}_2$ (Wiener-Hopf-Kalman) theory. First, for the disturbance attenuation problem, $\mathcal{H}_\infty$ theory seeks to minimise the worst-case disturbance transmission (sensitivity) over the disturbance frequency band. And, secondly, by introducing suitable dynamic weightings, $\mathcal{H}_\infty$ design provides the means for loop shaping, and hence, stability robustness. In particular, by applying the small gain theorem, $\mathcal{H}_\infty$ design yields robustness with respect to unstructured plant perturbations. In contrast, the $\mathcal{H}_2$ topology has been shown

to be too weak for addressing a practical robustness theory (Francis, 1980).

In Bernstein and Haddad (1989a) and Haddad and Bernstein (1990a, 1990b), the authors unify the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control design problems to address design tradeoffs between $\mathcal{H}_2$ performance and $\mathcal{H}_\infty$ disturbance rejection. This mixed norm unification provides the means to tradeoff rms (quadratic performance) and disturbance rejection as well as stability robustness. In particular, the goal of the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ control problem is to minimise an $\mathcal{H}_2$ performance criterion subject to a prespecified $\mathcal{H}_\infty$ constraint on the closed-loop transfer function. The $\mathcal{H}_\infty$ constraint is embedded within the optimisation process by replacing the closed-loop system covariance Lyapunov equation by a Riccati equation whose solution leads to an upper bound on the $\mathcal{H}_2$ system performance. The key idea to the approach is to view this upper bound as an auxiliary cost and, for a given controller architecture, seek the controller gains that minimise the $\mathcal{H}_2$ bound and guarantee that the disturbance attenuation constraint is enforced. An added advantage of the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ problem is that the upper bound on the $\mathcal{H}_2$ system performance corresponds to the closed-loop system entropy giving a measure of the distance from the $\mathcal{H}_\infty$ norm of the closed-loop system transfer function to the achievable disturbance attenuation constraint and is further shown to be equivalent to the exponential-of-quadratic cost of the risk-sensitive LQG control problem (Mustafa, 1989; Mustafa & Glover, 1990).

While the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ control design problem has been extensively studied in the literature (Bernstein & Haddad, 1989a; Haddad & Bernstein, 1990a, 1990b; Khargonekar & Rotea, 1991; Limebeer et al., 1994; Rotea

& Khargonekar, 1991; Zhou et al., 1994), with Khargonekar and Rotea (1991) reformulating the full-state and output feedback control problems into a finite-dimensional convex optimisation problem over a bounded subset of real matrices, the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ and entropy covariance control problem has not been addressed in the literature. The covariance control problem was first addressed in Hotz and Skelton (1987) and involves optimal steering of the covariance of a controlled linear stochastic system over a finite or infinite horizon to a reachable covariance matrix corresponding to a final probability distribution. This problem has been extensively studied within the Wiener-Hopf-Kalman setting (Bakolas, 2016; Chen et al., 2015a, 2016a, 2016b, 2018; Skelton & Ikeda, 1989; Skelton et al., 1998; Yasuda et al., 1993), with Chen et al. (2016a, 2016b) providing fundamental connections to the theory of Schrödinger bridges. In this paper, we merge mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ control theory with $\mathcal{H}_2$ covariance control theory to develop a mixed-norm $\mathcal{H}_2/\mathcal{H}_\infty$ and entropy covariance control framework.

The contents of this paper are as follows. In Section 2, we give a few definitions and present the problem formulation for the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ full-state feedback control problem using a Riccati equation approach (Haddad & Bernstein, 1990b). In Section 3, we provide a brief review of the $\mathcal{H}_2$ covariance control problem. Next, in Section 4, using the results of Khargonekar and Rotea (1991) we provide a convex formulation of the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ full-state feedback control problem using a quadratic matrix inequality leading to a finite-dimensional convex optimisation problem over a bounded subset of real matrices involving the control input and system state dimensions. In Section 5, we merge the results of Sections 3 and 4 to provide a convex formulation of the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ full information covariance control problem. In Sections 6 and 7, we extend the results of the first part of the paper to the dynamic output feedback control problem. In particular, we show that the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ dynamic output feedback covariance control problem reduces to a specific state-feedback control problem involving an auxiliary system whose solution is obtained from the auxiliary system by solving a $\mathcal{H}_\infty$ filtering Riccati equation along with a mixed-norm $\mathcal{H}_2/\mathcal{H}_\infty$ state-feedback covariance control problem. Thus, as in the state-feedback case, the solution of the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ dynamic output feedback covariance control problem reduces to a convex optimisation problem over a finite-dimensional space. In Section 8, we provide two illustrative numerical examples that highlight the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ covariance control framework. Finally, in Section 9, we present conclusions and highlight some future research directions.

2. Mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ and entropy full-state feedback control

In this section, we provide a review of the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ full-state feedback control problem (Bernstein & Haddad, 1989a; Haddad & Bernstein, 1990a, 1990b). Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $D_1 \in \mathbb{R}^{n \times d}$, and consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t), \quad (1)$$

where $t \geq 0$, $x(0) = x_0 \in \mathbb{R}^n$ is the initial condition, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, and $w(t) \in \mathbb{R}^d$ is the exogenous signal. Here we assume that the control signal $u(\cdot)$ is restricted to the class of admissible controls consisting of measurable functions $u(\cdot)$. In addition, within the context of $\mathcal{H}_2$ optimality the disturbances $w(\cdot)$ are interpreted as white noise signals while, simultaneously, for the purpose of $\mathcal{H}_\infty$ attenuation the very same disturbance signals have the alternative interpretation of deterministic $\mathcal{L}_2$ functions. For further details see Bernstein and Haddad (1989a). Furthermore, let $K \in \mathbb{R}^{m \times n}$, and consider the full-state feedback control law

$$u(t) = Kx(t). \quad (2)$$

Using (1) and (2), it follows that the closed-loop system can be written as

$$\dot{x}(t) = \tilde{A}x(t) + D_1w(t), \quad (3)$$

where

$$\tilde{A} \triangleq A + BK. \quad (4)$$

Consider the $\mathcal{H}_2$ performance variable $z_2(t) \in \mathbb{R}^p$, which is given by

$$z_2(t) = E_1x(t) + E_2u(t), \quad (5)$$

where $E_1 \in \mathbb{R}^{p \times n}$ and $E_2 \in \mathbb{R}^{p \times m}$. Next, defining

$$\tilde{E} \triangleq E_1 + E_2K, \quad (6)$$

it follows that (5) can be expressed as

$$z_2(t) = \tilde{E}x(t). \quad (7)$$

Thus, the closed-loop transfer function from disturbances w to $\mathcal{H}_2$ performance variables z_2 is given by

$$\tilde{G}_{z_2w}(s) \triangleq \tilde{E}(sI_n - \tilde{A})^{-1}D_1. \quad (8)$$

Furthermore, consider the $\mathcal{H}_\infty$ performance variable $z_\infty(t) \in \mathbb{R}^{p_\infty}$, which is given by

$$z_\infty(t) = E_{1\infty}x(t) + E_{2\infty}u(t), \quad (9)$$

where $E_{1\infty} \in \mathbb{R}^{p_\infty \times n}$ and $E_{2\infty} \in \mathbb{R}^{p_\infty \times m}$. Next, defining

$$\tilde{E}_\infty \triangleq E_{1\infty} + E_{2\infty}K, \quad (10)$$

it follows that (9) can be expressed as

$$z_\infty(t) = \tilde{E}_\infty x(t). \quad (11)$$

Thus, the closed-loop transfer function from disturbances w to $\mathcal{H}_\infty$ performance variables z_∞ is given by

$$\tilde{G}_{z_\infty w}(s) \triangleq \tilde{E}_\infty(sI_n - \tilde{A})^{-1}D_1. \quad (12)$$

Next, define the set of stabilising feedback controller gains by

$$\mathcal{S} \triangleq \{K \in \mathbb{R}^{m \times n} : A + BK \text{ is asymptotically stable}\} \quad (13)$$

and note that $\mathcal{S}$ is nonempty if and only if (A, B) is stabilisable. Let γ be a positive number. Then, the objective of the *mixed*

norm $\mathcal{H}_2/\mathcal{H}_\infty$ control problem is to determine the controller (2) such that the following design criteria are satisfied:

- (i) $K \in \mathcal{S}$.
- (ii) The $\mathcal{H}_\infty$ constraint

$$\begin{aligned} \|\tilde{G}_{z_\infty w}\|_{\mathcal{H}_\infty} &\triangleq \sup_{w(\cdot) \in \mathcal{L}_2} \frac{\|z_\infty\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}} \\ &= \sup_{\omega \in \mathbb{R}} \sigma_{\max}[\tilde{G}_{z_\infty w}(j\omega)] \leq \gamma, \end{aligned} \quad (14)$$

where $\sigma_{\max}(\cdot)$ denotes the maximum singular value, is satisfied.

- (iii) The $\mathcal{H}_2$ cost

$$J(K) \triangleq \|\tilde{G}_{z_2 w}\|_{\mathcal{H}_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\tilde{G}_{z_2 w}(j\omega)\|_{\mathbb{F}}^2 d\omega, \quad (15)$$

where $\|\cdot\|_{\mathbb{F}}$ denotes the Frobenius matrix norm, is minimised subject to the constraint (14).

The mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ control problem involves both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control objectives. More specifically, the objective is to minimise the $\mathcal{H}_2$ cost (15) subject to the $\mathcal{H}_\infty$ constraint (14). However, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance variables z_2 and z_∞ are not necessarily equal to one another. Thus, the closed-loop transfer functions $\tilde{G}_{z_2 w}$ and $\tilde{G}_{z_\infty w}$ associated with the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control objectives need not be equal to one another.

Note that if $K \in \mathcal{S}$, then it follows from standard $\mathcal{H}_2$ theory that the $\mathcal{H}_2$ cost (15) is given by

$$J(K) = \text{tr } \tilde{E}Q\tilde{E}^T, \quad (16)$$

where $Q \in \mathbb{R}^{n \times n}$ is the nonnegative definite solution to the algebraic Lyapunov equation

$$\tilde{A}Q + Q\tilde{A}^T + D_1D_1^T = 0. \quad (17)$$

Next, it is convenient to define the $\mathcal{H}_2$ weights

$$\begin{aligned} R_1 &\triangleq E_1^T E_1 \in \mathbb{R}^{n \times n}, \quad R_{12} \triangleq E_1^T E_2 \in \mathbb{R}^{n \times m}, \\ R_2 &\triangleq E_2^T E_2 \in \mathbb{R}^{m \times m}, \end{aligned} \quad (18)$$

and the $\mathcal{H}_\infty$ weights

$$\begin{aligned} R_{1\infty} &\triangleq E_{1\infty}^T E_{1\infty} \in \mathbb{R}^{n \times n}, \quad R_{12\infty} \triangleq E_{1\infty}^T E_{2\infty} \in \mathbb{R}^{n \times m}, \\ R_{2\infty} &\triangleq E_{2\infty}^T E_{2\infty} \in \mathbb{R}^{m \times m}, \end{aligned} \quad (19)$$

and we assume that at least one of R_2 and $R_{2\infty}$ is positive definite.

To enforce the $\mathcal{H}_\infty$ constraint (14), we replace the algebraic Lyapunov equation (17) by an algebraic Riccati equation, whose solution over bounds (in the sense of the cone of nonnegative definite matrices) the solution Q to the algebraic Lyapunov equation (17). Note that if w is Gaussian white noise, then Q is the steady-state covariance matrix $\lim_{t \rightarrow \infty} \mathbb{E}[x(t)x^T(t)]$, where $\mathbb{E}[\cdot]$ denotes the expectation operator. Alternatively, if w is a deterministic signal, then Q is the state covariance matrix due to impulsive disturbances (Skelton, 1988).

For the statement of the next result define $V \triangleq D_1D_1^T$, $\tilde{R} \triangleq \tilde{E}^T \tilde{E}$, $\tilde{R}_\infty \triangleq \tilde{E}_\infty^T \tilde{E}_\infty$, and the entropy functional for the $\mathcal{H}_\infty$ performance variables z_∞ by

$$\begin{aligned} I(\tilde{G}_{z_\infty w}, \gamma) &\triangleq - \lim_{s_0 \rightarrow \infty} \left[\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln |\det(I_n - \gamma^{-2} \tilde{G}_{z_\infty w}(j\omega) \right. \\ &\quad \times \left. G_{z_\infty w}^*(j\omega))| \left[\frac{s_0^2}{s_0^2 + \omega^2} \right] d\omega \right]. \end{aligned} \quad (20)$$

As discussed in Mustafa and Glover (1990), the entropy functional (20) can be viewed as a measure of the distance from $\|\tilde{G}_{z_\infty w}\|_\infty$ to γ . Like the $\mathcal{H}_2$ norm, but unlike the $\mathcal{H}_\infty$ norm, however, the entropy $I(\tilde{G}_{z_\infty w}, \gamma)$ accounts for $\tilde{G}_{z_\infty w}(j\omega)$ at all frequencies. Furthermore, it can be shown (Mustafa & Glover, 1990) that the entropy functional at infinity is equivalent to the exponential-of-quadratic cost of the risk-sensitive LQG control problem (Whittle, 1981).

Theorem 2.1 (Haddad & Bernstein, 1990b): *Let $K \in \mathbb{R}^{m \times n}$ be given and assume there exists a nonnegative-definite matrix $\mathcal{Q} \in \mathbb{R}^{n \times n}$ satisfying*

$$\tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \gamma^{-2} \mathcal{Q}\tilde{R}_\infty \mathcal{Q} + V = 0. \quad (21)$$

Then

$$(\tilde{A}, D_1) \text{ is stabilisable} \quad (22)$$

if and only if

$$\tilde{A} \text{ is Hurwitz.} \quad (23)$$

In this case, the following statements hold:

- (i) *The transfer function $\tilde{G}_{z_\infty w}$ satisfies*

$$\|\tilde{G}_{z_\infty w}\|_{\mathcal{H}_\infty} \leq \gamma. \quad (24)$$

- (ii) *If $\|\tilde{G}_{z_\infty w}\|_{\mathcal{H}_\infty} < \gamma$, then*

$$I(\tilde{G}_{z_\infty w}, \gamma) \leq \text{tr } \mathcal{Q}\tilde{R}_\infty. \quad (25)$$

- (iii) *The transfer function $\tilde{G}_{z_2 w}$ is given by*

$$\|\tilde{G}_{z_2 w}\|_{\mathcal{H}_2}^2 = \text{tr } Q\tilde{R}, \quad (26)$$

where the $n \times n$ matrix Q satisfies

$$\tilde{A}Q + Q\tilde{A}^T + V = 0. \quad (27)$$

- (iv) *The solution Q to (27) satisfies the bound*

$$Q \leq \mathcal{Q}, \quad (28)$$

and hence,

$$\|\tilde{G}_{z_2 w}\|_{\mathcal{H}_2}^2 \leq \text{tr } \mathcal{Q}\tilde{R}. \quad (29)$$

- (v) *All real symmetric solutions to (21) are nonnegative definite.*
- (vi) *There exists a (unique) minimal solution to (21) in the class of real symmetric solutions.*

(vii) $\mathcal{Q}$ is the minimal solution to (21) if and only if

$$\operatorname{Re} \lambda \leq 0, \lambda \in \operatorname{spec}(\tilde{A} + \gamma^{-2} \mathcal{Q} \tilde{R}_\infty), \quad (30)$$

where $\operatorname{spec}(\cdot)$ denotes spectrum.

(viii) $\|\tilde{G}_{z_\infty w}\|_{\mathcal{H}_\infty} < \gamma$ if and only if $\tilde{A} + \gamma^{-2} \mathcal{Q} \tilde{R}_\infty$ is Hurwitz, where $\mathcal{Q}$ is the minimal solution to (21).

(ix) If $\mathcal{Q}$ is the minimal solution to (21) and $\|\tilde{G}_{z_\infty w}\|_{\mathcal{H}_\infty} < \gamma$, then

$$I(\tilde{G}_{z_\infty w}, \gamma) = \operatorname{tr} \mathcal{Q} \tilde{R}_\infty. \quad (31)$$

Remark 2.1: Consider the equalised weight case $z_2 = z_\infty$ so that $\tilde{G}_{z_2 w} = \tilde{G}_{z_\infty w}$. In this case, it follows from (29) and (31) that

$$\|\tilde{G}_{z_2 w}\|_{\mathcal{H}_2}^2 \leq I(\tilde{G}_{z_2 w}, \gamma), \quad (32)$$

that is, the entropy is an upper bound for the $\mathcal{H}_2$ cost (see also Mustafa, 1989; Mustafa & Glover, 1990). If the $\mathcal{H}_\infty$ disturbance attenuation constraint is sufficiently relaxed, that is, $\gamma \rightarrow \infty$, then it can be shown (Mustafa, 1989; Mustafa & Glover, 1990) that the entropy functional (20) coincides with the $\mathcal{H}_2$ cost, and hence,

$$I(\tilde{G}_{z_2 w}, \infty) = \|\tilde{G}_{z_2 w}\|_{\mathcal{H}_2}^2 = \operatorname{tr} \tilde{Q} \tilde{R}. \quad (33)$$

The goal of the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ control problem is to minimise an $\mathcal{H}_2$ performance criterion subject to a pre-specified $\mathcal{H}_\infty$ constraint on the closed-loop transfer function. However, finding a gain matrix $K \in \mathbb{R}^{m \times n}$ such that the design criteria i)–iii) hold, that is, including both $\mathcal{H}_2$ and entropy performance measures within the context of constrained $\mathcal{H}_\infty$ design, is quite challenging. This was addressed in Haddad and Bernstein (1990b) as a multiobjective control problem by forming a convex combination of both performance measures. This approach is reminiscent of scalarisation techniques for Pareto optimisation (Khargonekar & Rotea, 1989) and results in a formidable set of four coupled Lyapunov and Riccati equations for characterising the optimal full-state feedback controller gain.

Alternatively, rather than minimising the $\mathcal{H}_2$ cost directly, Bernstein and Haddad (1989a) and Haddad and Bernstein (1990a) embed the $\mathcal{H}_\infty$ constraint within the optimisation process by replacing the closed-loop covariance Lyapunov equation (27) by the Riccati equation (21) whose solution leads to an upper bound on the $\mathcal{H}_2$ performance. The key idea of this approach is to view the upper bound (29) as an *auxiliary cost* and, for a fixed controller architecture, design controller gains that minimise the $\mathcal{H}_2$ bound (29) and guarantee that the disturbance attenuation constraint (14) is enforced.

We formulate this auxiliary minimisation problem as follows.

Auxiliary Optimisation Problem. Determine admissible $K \in \mathbb{R}^{m \times n}$ that minimises

$$\mathcal{J}(K, \mathcal{Q}) \triangleq \operatorname{tr}[\mathcal{Q}(E_1 + E_2 K)^T (E_1 + E_2 K)] \quad (34)$$

subject to

$$\begin{aligned} (A + BK) \mathcal{Q} + \mathcal{Q} (A + BK)^T + V \\ + \gamma^{-2} \mathcal{Q} (E_{1\infty} + E_{2\infty} K)^T (E_{1\infty} + E_{2\infty} K) \mathcal{Q} = 0. \end{aligned} \quad (35)$$

Now, since, by (28), $Q \leq \mathcal{Q}$, it follows that

$$J(K) = \|\tilde{G}_{z_2 w}\|_{\mathcal{H}_2}^2 \leq \mathcal{J}(K) \triangleq \operatorname{tr} \tilde{E} \mathcal{Q} \tilde{E}^T. \quad (36)$$

3. $\mathcal{H}_2$ covariance control problem

In this section, we provide a brief review of the $\mathcal{H}_2$ covariance control problem (Bakolas, 2016; Chen et al., 2015a, 2016a, 2016b, 2018; Hotz & Skelton, 1987; Skelton & Ikeda, 1989; Skelton et al., 1998; Yasuda et al., 1993). Recall that in covariance control the objective is to control the covariance matrix Q . Specifically, given a positive-definite matrix $X > 0$, the goal of the full-state feedback infinite horizon covariance control problem is to design a feedback control strategy (2) such that $Q = X$ or $Q \leq X$. In this paper, we consider the latter constraint. In particular, among all possible stabilising feedback controllers that achieve $Q \leq X$, we seek the controller that minimises the quadratic cost criterion

$$\begin{aligned} J(K) &= \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \left\{ \int_0^t \|z_2(s)\|_2^2 ds \right\} \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \left\{ \int_0^t [x^T(s) R_1 x(s) + 2x^T(s) R_{12} u(s) \right. \\ &\quad \left. + u^T(s) R_2 u(s)] ds \right\} \\ &= \|\tilde{G}_{z_2 w}\|_{\mathcal{H}_2}^2. \end{aligned} \quad (37)$$

In view of (16) and (26), we obtain the following optimisation problem for covariance control:

$$\min_{K, Q} \operatorname{tr}[Q(E_1 + E_2 K)^T (E_1 + E_2 K)] \quad (38)$$

subject to

$$(A + BK)Q + Q(A + BK)^T + V = 0, \quad (39)$$

$$0 < Q \leq X. \quad (40)$$

If B and D_1 have the same range space and (A, B) is controllable, then, for any given $X > 0$, the minimisation problem (38)–(40) is feasible (Chen et al., 2016b). And if we relax (40), then the covariance control problem reduces to the standard $\mathcal{H}_2$ optimal linear-quadratic regulator problem. Note that (38)–(40) can be reformulated as a finite-dimensional convex optimisation problem. For further connections between the optimal linear-quadratic regulator problem and the $\mathcal{H}_2$ covariance control problem, see Chen et al. (2016b).

As noted in the Introduction, an alternative formulation of the $\mathcal{H}_2$ covariance control problem can involve optimal steering of a linear stochastic system to a final probability distribution. In this case, the goal of the covariance control problem is to steer the system state from an initial Gaussian distribution to a target Gaussian distribution over a finite-time interval via state-feedback. For details of this problem, see Chen et al. (2016a).

4. Convex formulation of the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ full-state feedback control problem

The mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ control problem as formulated in (34)–(36) is nonconvex due to the presence of bilinear terms. A convex formulation of the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ control problem over a finite-dimensional space for both the state feedback and output feedback cases was given in Khargonekar

and Rotea (1991). In particular, Khargonekar and Rotea (1991) reduce the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ control problem to convex optimisation problem over a bounded set of real matrices involving the control input and system state dimensions. This formulation lends itself to the use of well established techniques from convex programming to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem.

To reformulate the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem (34) and (35) into a convex optimisation problem, we first convert the equality constraint (35) into an inequality constraint and consider

$$\min_{K, \mathcal{Q} > 0} \mathcal{J}(K, \mathcal{Q}) = \text{tr}[\mathcal{Q}(E_1 + E_2 K)^T(E_1 + E_2 K)] \quad (41)$$

subject to

$$(A + BK)\mathcal{Q} + \mathcal{Q}(A + BK)^T + V + \gamma^{-2}\mathcal{Q}(E_{1\infty} + E_{2\infty}K)^T(E_{1\infty} + E_{2\infty}K)\mathcal{Q} \leq 0. \quad (42)$$

Next, let $K = W\mathcal{Q}^{-1}$, where $W \in \mathbb{R}^{m \times n}$, so that with this parametrisation (41) can be written as

$$\begin{aligned} \mathcal{J}(W, \mathcal{Q}) &= \text{tr}[\mathcal{Q}(E_1 + E_2 W\mathcal{Q}^{-1})^T(E_1 + E_2 W\mathcal{Q}^{-1})] \\ &= \text{tr}[(E_1\mathcal{Q} + E_2 W)\mathcal{Q}^{-1}(E_1\mathcal{Q} + E_2 W)^T] \end{aligned} \quad (43)$$

and (42) becomes

$$\begin{aligned} 0 &\geq (A + BW\mathcal{Q}^{-1})\mathcal{Q} + \mathcal{Q}(A + BW\mathcal{Q}^{-1})^T + V \\ &\quad + \gamma^{-2}\mathcal{Q}(E_{1\infty} + E_{2\infty}W\mathcal{Q}^{-1})^T(E_{1\infty} + E_{2\infty}W\mathcal{Q}^{-1})\mathcal{Q} \\ &= A\mathcal{Q} + \mathcal{Q}A^T + BW + W^TB^T + V \\ &\quad + \gamma^{-2}(E_{1\infty}\mathcal{Q} + E_{2\infty}W)^T(E_{1\infty}\mathcal{Q} + E_{2\infty}W). \end{aligned} \quad (44)$$

This leads to the optimisation problem

$$\min_{W, \mathcal{Q} > 0} \text{tr}[(E_1\mathcal{Q} + E_2 W)\mathcal{Q}^{-1}(E_1\mathcal{Q} + E_2 W)^T] \quad (45)$$

subject to

$$\begin{aligned} A\mathcal{Q} + \mathcal{Q}A^T + BW + W^TB^T + V \\ + \gamma^{-2}(E_{1\infty}\mathcal{Q} + E_{2\infty}W)^T(E_{1\infty}\mathcal{Q} + E_{2\infty}W) \leq 0. \end{aligned} \quad (46)$$

Now, expanding (45) yields three terms; the first two terms are linear, and hence, convex, whereas the convexity of the last term follows from xxvi) of Proposition 8.6.17 of Bernstein (2011). The convexity of (46) is immediate and follows from the fact that (46) is a convex ellipsoid over a bounded subset of $n \times n$ and $m \times n$ real matrices. Consequently, the optimisation problem (45) and (46) is convex.

Finally, using Schur complements (Cottle, 1974), we obtain an equivalent semidefinite programming (SDP) problem for

(45) and (46). Namely, we consider

$$\min_{W, \mathcal{Q}, G} \text{tr } G \quad (47)$$

subject to

$$\begin{bmatrix} -A\mathcal{Q} - \mathcal{Q}A^T - BW - W^TB^T - V & (E_{1\infty}\mathcal{Q} + E_{2\infty}W)^T \\ E_{1\infty}\mathcal{Q} + E_{2\infty}W & \gamma^2 I_n \end{bmatrix} \geq 0, \quad (48)$$

$$\begin{bmatrix} G & E_1\mathcal{Q} + E_2 W \\ (E_1\mathcal{Q} + E_2 W)^T & \mathcal{Q} \end{bmatrix} \geq 0. \quad (49)$$

To see this, first note that (46) is equivalent to (48), whereas (49) implies that

$$G \geq (E_1\mathcal{Q} + E_2 W)\mathcal{Q}^{-1}(E_1\mathcal{Q} + E_2 W)^T. \quad (50)$$

Now, minimising $\text{tr } G$ over $(W, \mathcal{Q}, G)$ is equivalent to minimising $\text{tr}[(E_1\mathcal{Q} + E_2 W)\mathcal{Q}^{-1}(E_1\mathcal{Q} + E_2 W)^T]$ over $(W, \mathcal{Q})$. The SDP problem (47)–(49) can be solved using standard solvers (see Vandenberghe & Boyd, 1996). The optimal gain K is then recovered by $K = W\mathcal{Q}^{-1}$.

5. Mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ and entropy covariance control problem

In this section, we merge the $\mathcal{H}_2$ covariance control and mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ and entropy covariance control problem formulations. In particular, we merge our results in Sections 3 and 4 to formulate a convex optimisation problem involving the trade-off between disturbance rejection, robustness, and covariance steering. To formulate this problem, note that if $\tilde{A}$ is Hurwitz, then it follows from (28) that $Q \leq \mathcal{Q}$. Thus, a sufficient condition that ensures $Q \leq X$ is $\mathcal{Q} \leq X$. This implies that $J(K)$ given by (37) is bounded above by (34).

This yields the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ covariance control problem

$$\min_{K, \mathcal{Q}} \text{tr}[\mathcal{Q}(E_1 + E_2 K)^T(E_1 + E_2 K)] \quad (51)$$

subject to

$$(A + BK)\mathcal{Q} + \mathcal{Q}(A + BK)^T + V + \gamma^{-2}\mathcal{Q}(E_{1\infty} + E_{2\infty}K)^T(E_{1\infty} + E_{2\infty}K)\mathcal{Q} \leq 0, \quad (52)$$

$$0 < \mathcal{Q} \leq X. \quad (53)$$

This optimisation problem is essentially (41) and (42) with the extra constraint $\mathcal{Q} \leq X$ imposed on the steady state covariance. Using identical steps as in Section 4, (51)–(53) gives the SDP problem

$$\min_{W, \mathcal{Q}, G} \text{tr } G \quad (54)$$

subject to

$$\begin{bmatrix} -A\mathcal{Q} - \mathcal{Q}A^T - BW - W^TB^T - V & (E_{1\infty}\mathcal{Q} + E_{2\infty}W)^T \\ E_{1\infty}\mathcal{Q} + E_{2\infty}W & \gamma^2 I_n \end{bmatrix} \geq 0, \quad (55)$$

$$\begin{bmatrix} G & E_1\mathcal{Q} + E_2 W \\ (E_1\mathcal{Q} + E_2 W)^T & \mathcal{Q} \end{bmatrix} \geq 0, \quad (56)$$

$$X - \mathcal{Q} \geq 0. \quad (57)$$

Note that the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ covariance control formulation (54)–(57) reduces to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem when the constraint on the state covariance is relaxed, that is, $X = +\infty$. Alternatively, relaxing the $\mathcal{H}_\infty$ constraint, that is, $\gamma \rightarrow \infty$, (54)–(57) reduce to the standard $\mathcal{H}_2$ covariance control problem. In particular, letting $\gamma \rightarrow \infty$, (52) reduces to

$$(A + BK)\mathcal{Q} + \mathcal{Q}(A + BK)^T + V \leq 0, \quad (58)$$

and hence, the optimisation problem (51)–(53) reduces to

$$\min_{K, \mathcal{Q}} \text{tr}[\mathcal{Q}(E_1 + E_2K)^T(E_1 + E_2K)] \quad (59)$$

subject to

$$(A + BK)\mathcal{Q} + \mathcal{Q}(A + BK)^T + V \leq 0, \quad (60)$$

$$0 < \mathcal{Q} \leq X, \quad (61)$$

which recovers the $\mathcal{H}_2$ covariance control problem (38)–(40).

6. Mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ dynamic output feedback covariance control

In this section, we formulate the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ dynamic output feedback covariance control problem. To address this problem, let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{l \times n}$, $D_1 \in \mathbb{R}^{n \times d}$, and $D_2 \in \mathbb{R}^{l \times d}$, and consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t), \quad (62)$$

$$y(t) = Cx(t) + D_2w(t), \quad (63)$$

where $t \geq 0$, $x(0) = x_0 \in \mathbb{R}^n$ is the initial condition, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, $y(t) \in \mathbb{R}^l$ is the measurement, and $w(t) \in \mathbb{R}^d$ is the exogenous signal. Furthermore, consider the n th-order dynamic output-feedback controller

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \quad (64)$$

$$u(t) = C_c x_c(t), \quad (65)$$

where $t \geq 0$, $x_c(0) = x_{c0} \in \mathbb{R}^n$, $A_c \in \mathbb{R}^{n \times n}$, $B_c \in \mathbb{R}^{n \times l}$, and $C_c \in \mathbb{R}^{m \times n}$. The closed-loop system (62)–(65) is given by

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t), \quad (66)$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \quad \tilde{x}_0 \triangleq \begin{bmatrix} x_0 \\ x_{c0} \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A & B C_c \\ B_c C & A_c \end{bmatrix}, \quad (67)$$

$$\tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix}.$$

The $\mathcal{H}_2$ performance variable $z_2(t) \in \mathbb{R}^p$ is given by (5), whereas the $\mathcal{H}_\infty$ performance variable $z_\infty(t) \in \mathbb{R}^{p_\infty}$ is given by (9). In this case, the transfer function matrices $\tilde{G}_{z_2 w}(s)$ and $\tilde{G}_{z_\infty w}(s)$ are defined as in (8) and (12) with $\tilde{A}$ and D_1 replaced by $\tilde{A}$ and $\tilde{D}$ given by (67), I_n replaced by $I_{\tilde{n}}$, where $\tilde{n} = 2n$, and $\tilde{E}$ and $\tilde{E}_\infty$ given by

$$\tilde{E} \triangleq \begin{bmatrix} E_1 & E_2 C_c \end{bmatrix}, \quad \tilde{E}_\infty \triangleq \begin{bmatrix} E_{1\infty} & E_{2\infty} C_c \end{bmatrix}. \quad (68)$$

Thus, the objective of the *mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ dynamic output feedback control problem* is to determine the dynamic controller (64) and (65) such that the following are satisfied:

- (i) $\tilde{A}$ is asymptotically stable.
- (ii) $\tilde{G}_{z_\infty w}$ satisfies the $\mathcal{H}_\infty$ constraint

$$\|\tilde{G}_{z_\infty w}\|_{\mathcal{H}_\infty} \leq \gamma, \quad (69)$$

where $\gamma > 0$ is a given constant.

- (iii) The $\mathcal{H}_2$ performance measure

$$J(A_c, B_c, C_c) \triangleq \|\tilde{G}_{z_2 w}\|_{\mathcal{H}_2}^2 \quad (70)$$

is minimised.

In this case, Theorem 2.1 holds with K replaced by (A_c, B_c, C_c) , $\mathcal{Q} \in \mathbb{R}^{n \times n}$ by $\tilde{\mathcal{Q}} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$, $Q \in \mathbb{R}^{n \times n}$ by $\tilde{Q} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$, V by $\tilde{V} \triangleq \tilde{D}\tilde{D}^T$, $\tilde{A}$ given by (67), and with $\tilde{R} \triangleq \tilde{E}^T \tilde{E}$ and $\tilde{R}_\infty \triangleq \tilde{E}_\infty^T \tilde{E}_\infty$, where $\tilde{E}$ and $\tilde{E}_\infty$ are given by (68). Furthermore, note that $\tilde{Q}$ and $\tilde{\mathcal{Q}}$ can be partitioned into the $n \times n$ blocks

$$\tilde{Q} \triangleq \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \quad \tilde{\mathcal{Q}} \triangleq \begin{bmatrix} \mathcal{Q}_1 & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^T & \mathcal{Q}_2 \end{bmatrix}. \quad (71)$$

Now, the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ dynamic output feedback covariance control problem can be cast as the following optimisation problem.

Auxiliary Minimisation Problem. Determine $(A_c, B_c, C_c, \tilde{\mathcal{Q}})$ that minimises

$$\mathcal{J}(A_c, B_c, C_c, \tilde{\mathcal{Q}}) \triangleq \text{tr } \tilde{\mathcal{Q}}\tilde{R} \quad (72)$$

subject to

$$\tilde{A}\tilde{\mathcal{Q}} + \tilde{\mathcal{Q}}\tilde{A}^T + \gamma^{-2}\tilde{\mathcal{Q}}\tilde{R}_\infty\tilde{\mathcal{Q}} + \tilde{V} = 0, \quad (73)$$

$$\mathcal{Q}_1 \leq X, \quad (74)$$

for a given $n \times n$ positive-definite matrix $X > 0$.

7. Convex formulation of the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ dynamic output feedback covariance control problem

In this section, we present a convex formulation of the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ dynamic output feedback covariance control problem. First, however, recall that the mixed norm output feedback control problem can be reduced to a mixed-norm state feedback control problem for an *auxiliary plant*, which can be obtained from the given plant (62) and (63) by solving an $\mathcal{H}_\infty$ filtering Riccati equation (Bernstein & Haddad, 1989b); for details, see Khargonekar and Rotea (1991). For the statement of the next result it is convenient to define

$$V_1 \triangleq D_1 D_1^T \in \mathbb{R}^{n \times n}, \quad V_{12} \triangleq D_1 D_2^T \in \mathbb{R}^{n \times l}, \quad (75)$$

$$V_2 \triangleq D_2 D_2^T \in \mathbb{R}^{l \times l},$$

and we assume that $V_{12} = 0$ and V_2 is positive definite.

Theorem 7.1 (Khargonekar & Rotea, 1991): *If $(A_c, B_c, C_c, \tilde{\mathcal{Q}}) \in \mathcal{S}$ solves the minimisation problem (72) and (73), then there*

exists $Q \in \mathbb{R}^{n \times n}$ such that

$$A_c = A - QC^T V_2^{-1} C + BK + \gamma^{-2} QR_{1\infty} + \gamma^{-2} QR_{12\infty} K, \quad (76)$$

$$B_c = QC^T V_2^{-1}, \quad (77)$$

$$C_c = K, \quad (78)$$

where Q satisfies

$$AQ + QA^T - QC^T V_2^{-1} CQ + \gamma^{-2} QR_{1\infty} Q + V_1 = 0, \quad (79)$$

and $K \in \mathbb{R}^{m \times n}$ minimises the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ full-state feedback control problem given in Section 2 with A , B , and D_1 replaced by

$$\begin{aligned} A_a &= A + \gamma^{-2} QR_{1\infty}, & B_a &= B + \gamma^{-2} QR_{12\infty}, \\ D_{1a} &= QC^T V_2^{-1/2}. \end{aligned} \quad (80)$$

In particular, K is given by

$$\min_{K, \hat{Q} > 0} \mathcal{J}_a(K, \hat{Q}) \triangleq \text{tr}[\hat{Q}(E_1 + E_2 K)^T (E_1 + E_2 K)] \quad (81)$$

subject to

$$\begin{aligned} (A_a + B_a K) \hat{Q} + \hat{Q} (A_a + B_a K)^T + D_{1a} D_{1a}^T \\ + \gamma^{-2} \hat{Q} (E_{1\infty} + E_{2\infty} K)^T (E_{1\infty} + E_{2\infty} K) \hat{Q} = 0. \end{aligned} \quad (82)$$

Moreover,

$$\mathcal{J}(A_c, B_c, C_c, \tilde{\mathcal{Q}}) = \text{tr} E_1 Q E_1^T + \mathcal{J}_a^*, \quad (83)$$

where $\mathcal{J}_a^*$ is the minimiser of (81) subject to (82).

Remark 7.1: Theorem 7.1 shows that the solution of the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ dynamic output feedback control problem can be obtained in two steps. Namely, the first step involves solving the $\mathcal{H}_\infty$ filtering algebraic Riccati equation (79) and the second step consists of solving for a state-feedback gain for the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ optimisation problem (81) and (82) involving the auxiliary system (80). Note that the solution to the second step can be obtained by solving a finite-dimensional convex programming problem over a bounded set of real matrices as outlined in Section 4.

Theorem 7.2 (Khargonekar & Rotea, 1991): Let $K \in \mathbb{R}^{m \times n}$ be any memoryless full-state feedback controller that stabilises (1) with A , B , and D_1 replaced by A_a , B_a , and D_{1a} , respectively, and satisfies the $\mathcal{H}_\infty$ constraint (14). Then, the dynamic output feedback controller (64) and (65) with (A_c, B_c, C_c) given by (76)–(78) stabilises the zero solution $\tilde{x}(t) \equiv 0$ to (66) with $w(t) \equiv 0$ and satisfies the $\mathcal{H}_\infty$ constraint (69). Moreover, the $\mathcal{H}_2$ performance bound (72) is given by

$$\begin{aligned} \mathcal{J}(A_c, B_c, C_c, \tilde{\mathcal{Q}}) \\ = \text{tr} E_1 Q E_1^T + \text{tr}[\hat{Q}(E_1 + E_2 K)^T (E_1 + E_2 K)], \end{aligned} \quad (84)$$

where $Q \in \mathbb{R}^{n \times n}$ and $\hat{Q} \in \mathbb{R}^{n \times n}$ satisfy (79) and (82).

Remark 7.2: If the state-feedback gain K in Theorem 7.2 is given by the finite-dimensional optimisation problem (81) and (82), then (84) recovers (83).

Proposition 7.1: Let $K \in \mathbb{R}^{m \times n}$ be any memoryless full-state feedback controller that stabilises (1) with A , B , and D_1 replaced by A_a , B_a , and D_{1a} , respectively. Suppose there exists $Q \in \mathbb{R}^{n \times n}$ and $\hat{Q} \in \mathbb{R}^{n \times n}$ satisfying (79) and (82), and let (A_c, B_c, C_c) be given by (76)–(78). Then, $(\tilde{A}, \tilde{D})$ is stabilisable if and only if $\tilde{A}$ is asymptotically stable. Furthermore,

$$\tilde{\mathcal{Q}} = \begin{bmatrix} Q + \hat{Q} & \hat{Q} \\ \hat{Q} & \hat{Q} \end{bmatrix} \quad (85)$$

satisfies (73).

Proof: It follows from Theorem 2.1 that the stabilisability of $(\tilde{A}, \tilde{D})$ is equivalent to the asymptotic stability of $\tilde{A}$. Next, expanding (73) and using the partition (71) yields

$$\begin{aligned} 0 &= A \mathcal{Q}_1 + \mathcal{Q}_1 A^T + B C_c \mathcal{Q}_{12}^T + \mathcal{Q}_{12} C_c^T B^T + \gamma^{-2} \mathcal{Q}_1 R_{1\infty} \mathcal{Q}_1 \\ &\quad + \gamma^{-2} \mathcal{Q}_{12} C_c^T R_{2\infty} C_c \mathcal{Q}_{12}^T + \gamma^{-2} \mathcal{Q}_{12} C_c^T R_{12\infty}^T \mathcal{Q}_1 \\ &\quad + \gamma^{-2} \mathcal{Q}_1 R_{12\infty} C_c \mathcal{Q}_{12}^T + V_1, \end{aligned} \quad (86)$$

$$\begin{aligned} 0 &= A \mathcal{Q}_{12} + \mathcal{Q}_{12} A_c^T + B C_c \mathcal{Q}_2 + \mathcal{Q}_1 C^T B_c^T + \gamma^{-2} \mathcal{Q}_1 R_{1\infty} \mathcal{Q}_{12} \\ &\quad + \gamma^{-2} \mathcal{Q}_{12} C_c^T R_{2\infty} C_c \mathcal{Q}_2 + \gamma^{-2} \mathcal{Q}_{12} C_c^T R_{12\infty}^T \mathcal{Q}_{12} \\ &\quad + \gamma^{-2} \mathcal{Q}_1 R_{12\infty} C_c \mathcal{Q}_2, \end{aligned} \quad (87)$$

$$\begin{aligned} 0 &= A_c \mathcal{Q}_2 + \mathcal{Q}_2 A_c^T + B_c C \mathcal{Q}_{12} + \mathcal{Q}_{12}^T C^T B_c^T \\ &\quad + \gamma^{-2} \mathcal{Q}_{12}^T R_{1\infty} \mathcal{Q}_{12} + \gamma^{-2} \mathcal{Q}_2 C_c^T R_{12\infty}^T \mathcal{Q}_{12} \\ &\quad + \gamma^{-2} \mathcal{Q}_{12}^T R_{12\infty} C_c \mathcal{Q}_2 \\ &\quad + \gamma^{-2} \mathcal{Q}_2 C_c^T R_{2\infty} C_c \mathcal{Q}_2 + B_c V_2 B_c^T. \end{aligned} \quad (88)$$

Now, substituting (A_c, B_c, C_c) given by (76)–(78) into (86)–(88) and defining

$$\mathcal{Q}_1 \triangleq Q + \hat{Q}, \quad \mathcal{Q}_{12} = \mathcal{Q}_2 \triangleq \hat{Q}, \quad (89)$$

where Q and $\hat{Q}$ satisfy (79) and (82), shows that (87) = (88), and hence, (88) is superfluous. Furthermore, it can be shown that (87) reduces to (82), whereas subtracting (87) from (86) yields (79). Thus, $\tilde{\mathcal{Q}}$ given by (85) satisfies (73). Finally, note that

$$\tilde{\mathcal{Q}} = \begin{bmatrix} Q & 0_n \\ 0_n & 0_n \end{bmatrix} + \begin{bmatrix} I_n \\ I_n \end{bmatrix} \hat{Q} \begin{bmatrix} I_n & I_n \end{bmatrix}, \quad (90)$$

which shows that $\tilde{\mathcal{Q}} \geq 0$. ■

For the statement of the main result of this section let $\mathcal{C}$ denote the set of dynamic output feedback controllers (A_c, B_c, C_c) given by (76)–(78), where Q satisfies (79) and $K \in \mathbb{R}^{m \times n}$ is any memoryless full-state feedback controller that stabilises (1) with A , B , and D_1 replaced by A_a , B_a , and D_{1a} , respectively, and satisfies the $\mathcal{H}_\infty$ constraint (14).

Theorem 7.3: Let $(A_c, B_c, C_c) \in \mathcal{C}$. If $K \in \mathbb{R}^{m \times n}$ solves the optimisation problem (81) and (82) with the additional constraint $Q + \hat{Q} \leq X$, where $Q \in \mathbb{R}^{n \times n}$ and $\hat{Q} \in \mathbb{R}^{n \times n}$ satisfy (79)

and (82), and $X \in \mathbb{R}^{n \times n}$ is a given positive-definite matrix, then the dynamic output-feedback controller (A_c, B_c, C_c) given by (76)–(78) solves the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ dynamic output feedback covariance control problem given by (72)–(74).

Proof: Since $K \in \mathbb{R}^{m \times n}$ stabilises (1) with A , B , and D_1 replaced by A_a, B_a , and D_{1a} , respectively, and satisfies the $\mathcal{H}_\infty$ constraint (14), and $(A_c, B_c, C_c) \in \mathcal{C}$, it follows from Theorem 7.2 that the dynamic output feedback controller (A_c, B_c, C_c) given by (76)–(78) stabilises $\tilde{A}$ given by (67) and satisfies the $\mathcal{H}_\infty$ constraint (69). Furthermore, since $(K, \hat{Q})$ minimises $\mathcal{J}_a(K, \hat{Q})$, (84) implies that $(A_c, B_c, C_c, \mathcal{Q})$ minimises $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$, where $\mathcal{Q}$ is the solution of (73). Now, by Proposition 7.1, $\mathcal{Q}_1 = Q + \hat{Q}$, and hence, $\mathcal{Q}_1 \leq X$. ■

Remark 7.3: Any feasible upper bound X for the controlled state covariance bound $\mathcal{Q}_1$ must satisfy $X \geq Q$, where Q is given by (79). Moreover, in the limit as $\gamma \rightarrow \infty$, the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ covariance control problem of Theorem 7.3 reduces to the $\mathcal{H}_2$ covariance control problem with partial information addressed in Chen et al. (2015b). In this case, Q given by (79) collapses to the state estimation error covariance associated with the Kalman filter for (62) and (63).

8. Illustrative numerical examples

In this section, we present two numerical examples to illustrate the proposed mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ covariance control problem. The first example involves a full-state feedback control problem consisting of a double pendulum, whereas the second example involves a system of coupled rotating disks with output feedback control. For both examples, we use standard solvers (Vandenberghe & Boyd, 1996; Vandenberghe et al., 2005) to solve the resulting SDP problems. Recall that for a standard SDP problem involving $n \times n$ variables, the worst case computational complexity per iteration is $\mathcal{O}(n^6)$ (Vandenberghe et al., 2005), where $\mathcal{O}(\cdot)$ denotes the Bachmann-Landau order. Since our SDP formulation (54)–(57) involves $2n$, $n + p$, and n variables, we have a worst-case computational complexity per iteration of $\mathcal{O}((4n + p)^6)$. Exploiting block sparsity this computational complexity can be further reduced.

8.1 Full-state feedback control

Consider the double pendulum shown in Figure 1, which has the nonlinear dynamics

$$(m_1 + m_2)l_1^2\ddot{q}_1 + m_2l_1l_2\ddot{q}_2 \cos(q_2 - q_1) - m_2l_1l_2\dot{q}_2^2 \sin(q_2 - q_1) + (m_1 + m_2)gl_1 \sin q_1 = u + \gamma_d d, \quad (91)$$

$$m_2l_2^2\ddot{q}_2 + m_2l_1l_2\ddot{q}_1 \cos(q_2 - q_1) + m_2l_1l_2\dot{q}_1^2 \sin(q_2 - q_1) + m_2gl_2 \sin q_2 = 0, \quad (92)$$

where $q_1 \in \mathbb{R}$ and $q_2 \in \mathbb{R}$ are the angles from the vertical position, $u \in \mathbb{R}$ is the control torque, $d \in \mathbb{R}$ is the disturbance, $m_1 > 0$ and $m_2 > 0$ are the masses, $l_1 > 0$ and $l_2 > 0$ are the pendulum lengths, g is the acceleration due to gravity, and $\gamma_d \in \mathbb{R}$.

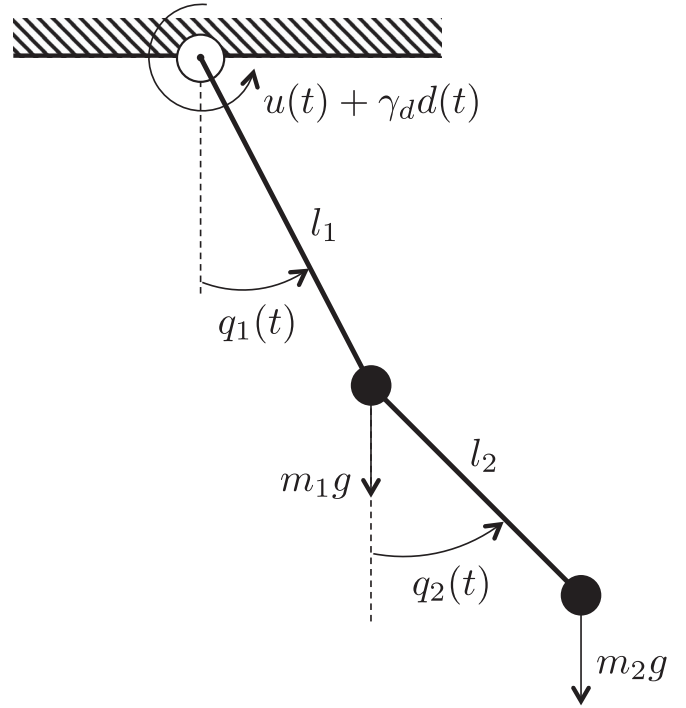


Figure 1. Double pendulum example.

Linearising (91) and (92) about the unforced inverted equilibrium, that is, $q_1 = \pi, \dot{q}_1 = 0, q_2 = \pi, \dot{q}_2 = 0$, yields (1) with $w(t) = d(t)$ and

$$x(t) = \begin{bmatrix} q_1(t) - \pi \\ \dot{q}_1(t) \\ q_2(t) - \pi \\ \dot{q}_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(m_1 + m_2)g}{m_1 l_1} & 0 & -\frac{m_2 g}{m_1 l_1} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(m_1 + m_2)g}{m_1 l_2} & 0 & \frac{(m_1 + m_2)g}{m_1 l_2} & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{m_1 l_1^2} \\ 0 \\ \frac{1}{m_1 l_1 l_2} \end{bmatrix}, \quad D_1 = \gamma_d B.$$

Here, we take

$$E_{1\infty} = E_1 = \begin{bmatrix} I_4 \\ 0_{1 \times 4} \end{bmatrix}, \quad E_{2\infty} = E_2 = \begin{bmatrix} 0_{4 \times 1} \\ 1 \end{bmatrix},$$

and let $m_1 = 2$ kg, $m_2 = 1$ kg, $l_1 = 0.4$ m, $l_2 = 0.2$ m, $g = 9.81$ m/s², and $\gamma_d = 1$.

Our goal is to achieve a stationary state covariance that is upper bounded by $X = 0.5I_4$. Setting $\gamma = 1.5$ for the $\mathcal{H}_\infty$ norm bound of the closed-loop system (4) and solving (54)–(57) yields

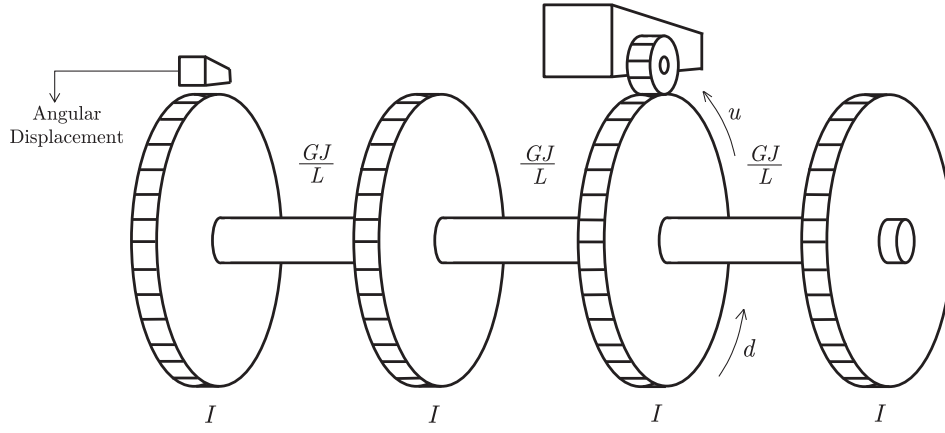


Figure 2. Coupled rotating disks example.

the feedback gain

$$K_1 = [1.402 \quad 16.57 \quad 140.9 \quad 19.87].$$

Now, if we perturb the system parameters by 5% so that $m_1 = 2.1$ kg, $m_2 = 1.05$ kg, $l_1 = 0.38$ m, and $l_2 = 0.21$ m, then the stationary covariance is given by

$$Q = \begin{bmatrix} 0.0194 & -0.0000 & -0.0026 & 0.0021 \\ -0.0000 & 0.1847 & -0.0021 & -0.2141 \\ -0.0026 & -0.0021 & 0.0010 & -0.0000 \\ 0.0021 & -0.2141 & -0.0000 & 0.3419 \end{bmatrix},$$

which remains bounded above by X .

In contrast, when the $\mathcal{H}_\infty$ norm bound is relaxed (i.e. $\gamma \rightarrow \infty$), (59)–(61) yields the feedback gain

$$K_2 = [0.0424 \quad 10.04 \quad 93.65 \quad 13.91].$$

In this case, the stationary covariance for the perturbed dynamics is given by

$$Q = \begin{bmatrix} 0.0242 & -0.0000 & -0.0044 & 0.0021 \\ -0.0000 & 0.2761 & -0.0021 & -0.2923 \\ -0.0044 & -0.0021 & 0.0018 & 0.0000 \\ 0.0021 & -0.2923 & 0.0000 & 0.4447 \end{bmatrix},$$

which has a spectral norm of 0.6646, and hence, is no longer upper bounded by X . This comparison illustrates that the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ covariance control problem provides robustness to system parameter perturbations as compared to standard $\mathcal{H}_2$ covariance control.

8.2 Dynamic output feedback control

Consider the system involving four coupled rotating disks shown in Figure 2. This example was considered in Bernstein and Haddad (1989a) and involves a complex pair of nonminimum phase zeros because the system has noncollocated sensors and actuators. The plant is of eighth order and has two neutrally stable poles.

The problem data are $n = 8$, $m = l = 1$, $d = p = p_\infty = 2$,

$$A = \begin{bmatrix} -0.161 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6.004 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5822 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -9.9835 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.4073 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3.982 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0.0064 \\ 0.00235 \\ 0.0713 \\ 1.0002 \\ 0.1045 \\ 0.9955 \end{bmatrix}, \quad C = [1 \quad 0_{1 \times 7}],$$

$$D_1 = [B \quad 0_{8 \times 1}], \quad D_2 = [0 \quad 0.5], \quad E_1 = E_{1\infty}$$

$$= 10^{-3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0.55 & 11 & 1.32 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

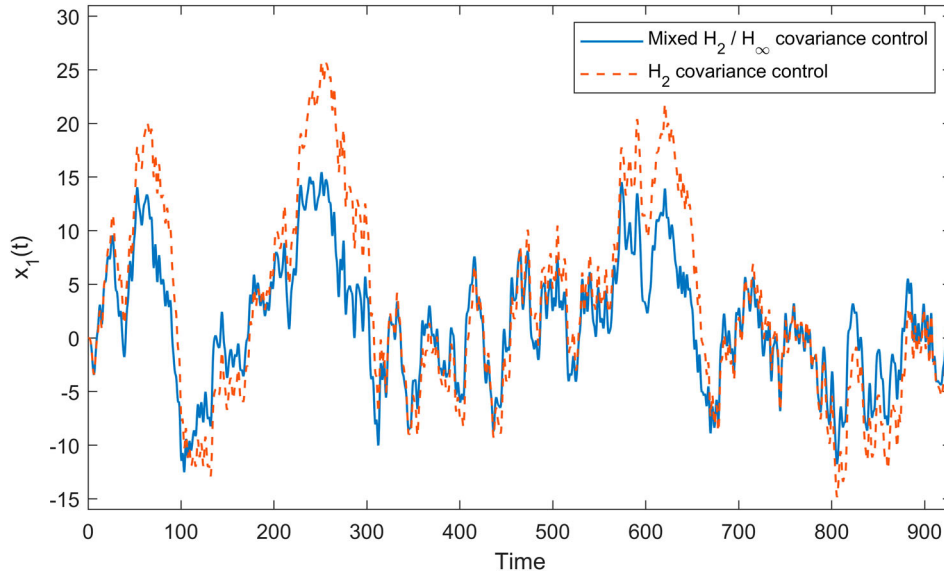
$$E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_{2\infty} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.$$

To set a constraint on the closed-loop state covariance matrix, we first compute the closed-loop state covariance matrix $Q_{1,LQG}$ for the LQG controller. Then, we set $X = 0.8Q_{1,LQG}$ as the bound for the controlled covariance. Furthermore, for the $\mathcal{H}_\infty$ constraint we set $\gamma = 1$. Table 1 shows the results obtained for the different controllers we consider.

Note that $\lambda_{\min}(X - Q_1) \geq 0$, where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue, when we enforce the covariance constraint, whereas $\lambda_{\min}(X - Q_1) < 0$ when the covariance bound is not enforced. Furthermore, comparing the spectral norm of the closed-loop state covariance matrix $\|Q_1\|_2$ for the $\mathcal{H}_2$ controller without the covariance control constraint and the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ controller without the covariance control constraint shows that by just enforcing the $\mathcal{H}_\infty$ bound yields a significant decrease in the controlled system state covariance. The mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ covariance controller achieves the best performance by simultaneously providing the lowest $\mathcal{H}_\infty$ norm bound

Table 1. Performance of different controllers.

	$\mathcal{J}(A_c, B_c, C_c)$	$\ \tilde{G}_{zw}\ _{\mathcal{H}_2}^2$	$\ \tilde{G}_{zw}\ _{\mathcal{H}_\infty}$	$\ Q_1\ _2$	$\lambda_{\min}(X - Q_1)$
$\mathcal{H}_2$ w/o Cov. Ctrl.	–	0.1366	1.3347	$2.276 \cdot 10^4$	$-4.5527 \cdot 10^3$
$\mathcal{H}_2$ w Cov. Ctrl.	–	0.1590	1.4364	$1.821 \cdot 10^4$	$2.8600 \cdot 10^{-9}$
$\mathcal{H}_2/\mathcal{H}_\infty$ w/o Cov. Ctrl.	0.1952	0.1577	0.8322	$1.022 \cdot 10^4$	-15.044
$\mathcal{H}_2/\mathcal{H}_\infty$ w Cov. Ctrl.	0.2327	0.2067	0.7050	$7.269 \cdot 10^3$	$5.5381 \cdot 10^{-5}$

**Figure 3.** (colour online) Time domain simulation of the $\mathcal{H}_2$ covariance controller and the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ covariance controller. Here, we show the evolution of the first state of the controlled system; a similar behaviour is observed for the remaining system states.

from disturbances w to $\mathcal{H}_\infty$ performance variables z_∞ and the lowest spectral norm of the closed-loop state covariance matrix.

To visualise the difference in system performance, Figure 3 shows a time domain simulation comparing the $\mathcal{H}_2$ covariance controller with the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ covariance controller. The closed-loop system is driven by a white noise process. The simulation shows that the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ covariance controller gives a better performance than the $\mathcal{H}_2$ covariance controller.

9. Conclusion

This paper develops a mixed-norm $\mathcal{H}_2/\mathcal{H}_\infty$ and entropy covariance control problem to address the steering of a controlled system state from an initial Gaussian distribution to a targeted steady state probability distribution while accounting for rms (quadratic performance), disturbance attenuation, and stability robustness. The proposed formulation allows for solutions via semidefinite programming and relies on several tools from covariance control, mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ control, and convex optimisation.

The problem of robust control design constitutes a significant challenge in dynamical systems theory which, at the same time, addresses a fundamental issue in practical implementation of feedback control systems, namely, modelling uncertainty. Modeling uncertainty must be accounted for within the control-design process. By utilising suitable weighting functions within a given system model, $\mathcal{H}_\infty$ norm bounds can be used to shape system transfer functions to guarantee robustness to unstructured plant uncertainty. Thus, the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ control

problem can capture system uncertainty by means of nonparametric neighbourhoods in the $\mathcal{H}_\infty$ topology, and hence, allows for the consideration of competing constraints involving mean-square error, disturbance attenuation, and stability robustness. To account for system robustness within the covariance control setting, in future work we will focus on extending the proposed mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ covariance control framework to nominal performance problems subject to robust stability requirements. Furthermore, we will address the mixed norm $\mathcal{H}_2/\mathcal{H}_\infty$ covariance control for linear stochastic discrete-time systems.

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