Commun. Math. Phys. 385, 1835–1870 (2021)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-021-04011-3

Communications in
Mathematical Physics

Evaluating Thin Flat Surfaces

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Received: 8 September 2020 / Accepted: 15 January 2021
Published online: 25 February 2021 –

Abstract: We consider recognizable evaluations for a suitable category of oriented two-dimensional cobordisms with corners between finite unions of intervals. We call such cobordisms thin flat surfaces. An evaluation is given by a power series in two variables. Recognizable evaluations correspond to series that are ratios of a two-variable polynomial by the product of two one-variable polynomials, one for each variable. They are also in a bijection with isomorphism classes of commutative Frobenius algebras on two generators with a nondegenerate trace fixed. The latter algebras of dimension n correspond to points on the dual tautological bundle on the Hilbert scheme of n points on the affine plane, with a certain divisor removed from the bundle. A recognizable evaluation gives rise to a functor from the above cobordism category of thin flat surfaces to the category of finite-dimensional vector spaces. These functors may be non-monoidal in interesting cases. To a recognizable evaluation we also assign an analogue of the Deligne category and of its quotient by the ideal of negligible morphisms.

Contents

1. Introduction

Universal constructions of topological theories [1,13,21] that are not necessarily multiplicative [9] are interesting even in dimension two [14,16], providing examples somewhat different from commutative Frobenius algebras for the invariants of two-dimensional cobordisms. In this note we consider the analogue of the latter construction for oriented two-dimensional cobordisms with corners. For simplicity we restrict to cobordisms between finite unions of intervals; boundary points of the intervals give rise to corners of cobordisms. Furthermore, we require that each connected component of a cobordism has non-empty boundary, which is a natural condition when excluding cobordisms with corners that have circles as some boundary components.

Cobordisms that we consider can be "thinned" to consist of ribbons glued to disks and can be depicted in the plane as regular neighbourhoods of immersed graphs, see Fig. 2 below for an example. For this reason we refer to these cobordisms as *thin flat surfaces* or *tf-surfaces* throughout the paper. When viewed as a morphism in the appropriate category TFS of thin flat cobordisms, a particular immersion of the surface into the plane is inessential, and morphisms are equivalence classes of such cobordisms modulo diffeomorphisms that fix the boundary.

The category TFS admits an analogue of α-evaluations from [14–16]. This time closed connected morphisms S (connected endomorphisms of the unit object 0, the empty union of intervals) are parametrized by two non-negative integers (ℓ, g), where $\ell + 1$ is the number of boundary components of S and g is the genus. Assigning an element $\alpha_{\ell,g}$ of the ground field $\mathbf{k}$ (or a ground commutative ring R) to such a component and extending multiplicatively to disjoint unions gives an evaluation α on endomorphisms of the unit object 0. Evaluation α can be conveniently encoded as power series

$$Z_\alpha(T_1, T_2) = \sum_{k,g\geq 0} \alpha_{k,g} T_1^k T_2^g, \quad \alpha = (\alpha_{k,g})_{k,g\in\mathbb{Z}_+}, \quad \alpha_{k,g} \in \mathbf{k}, \tag{1}$$

where the degree of the first variable T_1 counts "holes" in a cobordism (a disk has no holes and an annulus has one hole) and T_2 keeps track of the genus.

With α as above and $n \geq 0$, one can define a bilinear form on a $\mathbf{k}$-vector space with a basis given by equivalence classes of thin flat surfaces with n boundary intervals. The quotient by the kernel of the bilinear form is a vector space $A_\alpha(n)$. The collection of quotient spaces $\{A_\alpha(n)\}_{n\geq 0}$ is what we refer to as *the universal construction* for the category TFS, given α.

The spaces $A_\alpha(n)$ rarely satisfy the Atiyah factorization axiom, that is, the relation

$$A_\alpha(m+n) \cong A_\alpha(m) \otimes A_\alpha(n)$$

does not hold. From the quantum field theory (QFT) perspective, this violation may happen if the 2-dimensional QFT is embedded as a 2-dimensional defect inside a higher-dimensional QFT.

It is straightforward to see that $A_\alpha(n)$ is finite-dimensional for all n iff $A_\alpha(1)$ is finite-dimensional iff the series (1) is *recognizable* or *rational* (terms from the control theory and the theory of noncommutative power series). Recognizable power series in this case have the form

$$Z_\alpha(T_1,T_2) = \frac{P(T_1,T_2)}{Q_1(T_1)Q_2(T_2)}, \tag{2}$$

that is, Z_α can be written as a ratio of a polynomial in T_1, T_2 and two one-variable polynomials, see Proposition 3.1 and Fliess [7].

Constructions of [16] go through for the category of thin flat surfaces and any recognizable α as above. We define the category STFS_α (skein thin flat surfaces) where homs are finite linear combinations of cobordisms, closed components evaluate to coefficients of α, and there are skein relations given by adding holes and handles to a component of a cobordism and equating to zero linear combinations corresponding to elements of the kernel ideal $I_\alpha \subset \mathbf{k}[T_1, T_2]$ associated to α and also known as *the syntactic ideal* of rational series α. A two variable polynomial $z = z(T_1, T_2)$ is in I_α iff $\alpha(zf) = 0$ for any polynomial $f \in \mathbf{k}[T_1, T_2]$, with $\alpha(T_1^\ell T_2^g) = \alpha_{\ell,g}$ extended to a linear map $\mathbf{k}[T_1, T_2] \xrightarrow{\alpha} \mathbf{k}$.

For the rest of the paper we change our terminology and call connected components of a thin flat surface that have neither top nor bottom boundary intervals *floating* components instead of *closed* components, since they otherwise have boundary, what we call *side* boundary, that is present inside the cobordism but not at its top or bottom. This avoid possible confusion with the usual notion of a closed surface. A non-empty thin flat surface is never closed in the latter sense.

The category STFS_α has finite-dimensional hom spaces. Taking the additive Karoubi envelope of this category to form

$$\mathrm{DTFS}_\alpha \ := \ \mathrm{Kar}(\mathrm{STFS}_\alpha^{\oplus})$$

gives an idempotent-complete $\mathbf{k}$-linear rigid symmetric monoidal category DTFS_α which is the analogue of the Deligne category [3,4,6] for TFS and recognizable series α in two variables.

Once we pass to $\mathbf{k}$-linear combinations of cobordisms, and α is available to evaluate floating cobordisms, there is a trace map on endomorphisms of any object n. It is given by closing each term in the linear combination of tf-surfaces describing the endomorphism via n strips into a floating tf-surface and evaluating it via α. Consequently, one can form the ideal J_α of negligible morphisms [3,4,6,16] and quotient the category by that ideal.

We call the quotient category *gligible quotient* to avoid the awkward-sounding word "non-negligible quotient" and mirroring the terminology from [15]. The gligible quotient TFS_α of the skein category STFS_α carries non-degenerate bilinear forms on its hom spaces and otherwise shares key properties of STFS_α: objects are non-negative integers, category TFS_α is rigid symmetric tensor, and the hom spaces are finite-dimensional over $\mathbf{k}$.

Likewise, the Deligne category DTFS_α has the gligible quotient $\underline{\mathrm{DTFS}}_\alpha$ by the ideal of negligible morphisms. The same category can be recovered as the additive Karoubi closure of TFS_α.

Section 3.4 and diagram (12) contain a summary of these categories and key functors relating them.

Similar to [5,6,15], it is natural to ask under what conditions will $\underline{\mathrm{DTFS}}_\alpha$ be semisimple. Unlike [14,15], we do not work out any specific examples of these categories here and leave that to an interested reader or another time.

Our evaluation α is encoded by a power series Z_α in two variables (1), and the recognizable series Z_α gives rise to a finite-codimension ideal I_α in $\mathbf{k}[T_1, T_2]$, the largest ideal contained in the hyperplane $\ker(\alpha)$. Such an ideal defines a point on the Hilbert scheme of the affine plane $\mathbb{A}^2$. We discuss the relation to the Hilbert schemes in Section 4 and explain a bijection between recognizable power series with the ideal I_α of codimension k and points in the complement $\mathcal{T}_k^\vee \backslash D_k$ of the dual tautological bundle $\mathcal{T}_k^\vee$ on the Hilbert scheme and a suitable divisor D_k on it.

It is not clear whether the appearance of the Hilbert scheme of $\mathbb{A}^2$ is a bug or a feature. In Section 5 we explain two generalizations of our construction. One of them involves "coloring" side boundary components of a thin flat surface into r colors. For the resulting category, recognizable series depends on $r+1$ parameters (generalizing from 2 parameters for $r = 1$), and one would get a generalization of our construction from the Hilbert scheme of $\mathbb{A}^2$ to that of $\mathbb{A}^{r+1}$, with the appropriate divisor removed from the dual tautological bundle in both cases. Of course, the Hilbert scheme has vastly different properties and uses in the case of algebraic surfaces versus higher dimensional varieties.

The other generalization considered in that section is given by extending the TFS (thin flat surfaces) category by allowing closed components and circles as boundaries. This corresponds to the usual category of two-dimensional oriented cobordisms with boundary and corners studied in [2,17,19,23] and other papers. Objects of that category are finite disjoint unions of intervals and circles. We briefly touch on this generalization and explain encoding of recognizable series via certain rational power series in this case as well.

Relations between Frobenius algebras, recognizable power series, codes and two-dimensional TFTs are considered in Friedrich [8], which is quite close in spirit to this paper.

A possible relation between moduli spaces of $SU(m)$ instantons on $\mathbb{R}^4$ (the Hilbert scheme of $\mathbb{C}^2$ corresponds to $U(1)$ case) and control theory is explored in [12,22] and the follow-up papers. We do not know how to connect it to the constructions in the present paper.

2. The Category of Thin Flat Surfaces

2.1. Category TFS. We introduce the category TFS of *thin flat surfaces*. Its objects are non-negative integers $n \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$. An object is represented by n intervals $I_1, \dots, I_n$ placed along the x-axis in the xy-plane. A morphism from n to m is a "thin" surface S immersed in $\mathbb{R} \times [0, 1]$ connecting n intervals on the line $\mathbb{R}^2 \times \{0\}$ with m intervals on the line $\mathbb{R}^2 \times \{1\}$. The immersion map $S \longrightarrow \mathbb{R}^2 \times [0, 1]$ is a local diffeomorphism, but the image of S may have overlaps, that can be thought of as virtual overlaps and ignored. The surface S inherits an orientation from its immersion into $\mathbb{R} \times [0.1]$. Restricting to the complement of the boundary of S, the immersion is open.

Alternatively, the immersion can be perturbed to an embedding of S into $\mathbb{R}^2 \times [0, 1]$ by turning overlaps into over- and under-crossings of strips of a surface. This can be done just for aesthetic purposes, and whether one chooses an over- or an under-crossing does not matter for the morphism associated to the surface.

The boundary of S consists of several circles (at least one circle unless S is the empty surface) and decomposes into $n+m$ disjoint intervals that constitute *horizontal* boundary $\partial_h S$ and $n + m$ intervals and some number of circles that constitute *side*, or *vertical*, or *inner* boundary $\partial_v S$ of S:

$$\partial S = \partial_h S \cup \partial_v S.$$

Horizontal intervals that constitute $\partial_h S$ are the intersections of S with $\mathbb{R} \times \{0, 1\} \subset \mathbb{R} \times [0, 1]$. Vertical boundary $\partial_v S$ is the closure of the intersection of ∂S with $\mathbb{R} \times (0, 1)$.

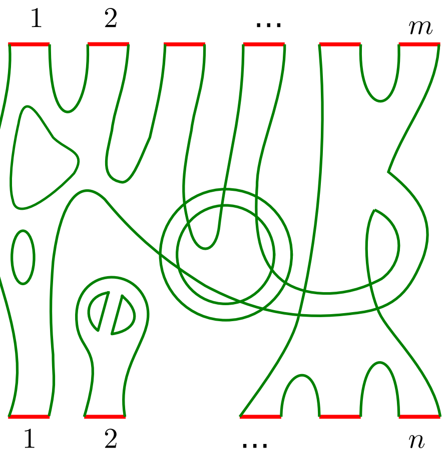

Fig. 1. A thin flat surface in $\mathbb{R} \times [0, 1]$

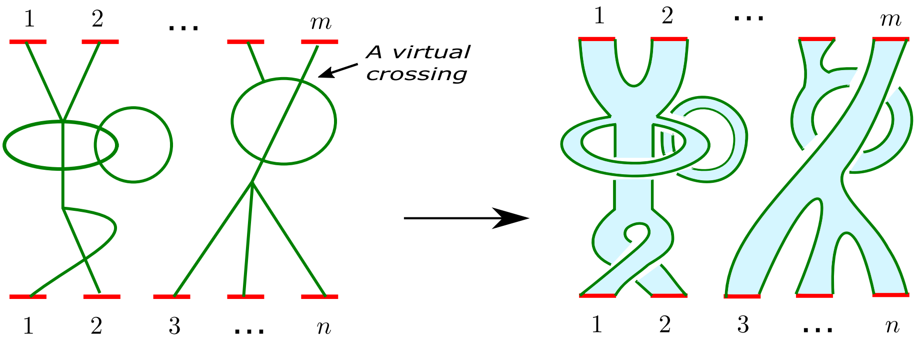

Fig. 2. An immersed graph Γ in $\mathbb{R} \times [0, 1]$ and associated thin flat surface $N(\Gamma, \jmath)$

The intersection $\partial_h S \cap \partial_v S$ consists of $2(n + m)$ boundary points of the horizontal intervals. These are also the *corners* of the surface S.

In the graphical depicitions of thin flat surfaces below, we will draw horizontal boundary segments as red intervals, and vertical boundary components as green arcs for better visualization (Figs. 1 and 2 right), but the figures can also be viewed and carry full information in greyscale. Starting from Fig. 2, we depict tf-surfaces in light aquamarine.

Another way to describe S is to immerse a finite unoriented graph Γ, possibly with multiple edges and loops, into the strip $\mathbb{R} \times [0, 1]$, via the immersion $\jmath : \Gamma \longrightarrow \mathbb{R} \times [0, 1]$. The graph Γ may have several boundary vertices v of valency 1 such that $\jmath(v) \in \mathbb{R} \times \{0, 1\}$. The remaining vertices are mapped inside the strip. The immersion is disjoint on vertices. Edges of Γ may intersect in $\mathbb{R} \times [0, 1]$. We consider these virtual intersections and not vertices. An example of Γ and $\jmath$ is shown in Fig. 2 left.

Taking a regular neighbourhood $N(\Gamma, \jmath)$ of Γ under $\jmath$, locally in Γ, results in a thin flat surface $N(\Gamma)$. Vice versa, any thin flat surface S can be deformed to the surface $N(\Gamma)$ for some Γ.

Take a thin flat surface S and forget the embedding into $\mathbb{R} \times [0, 1]$, only remembering boundary intervals and their order, on both top and bottom lines. In this way we view S as a cobordism between ordered collections of oriented intervals (induced by the orientation

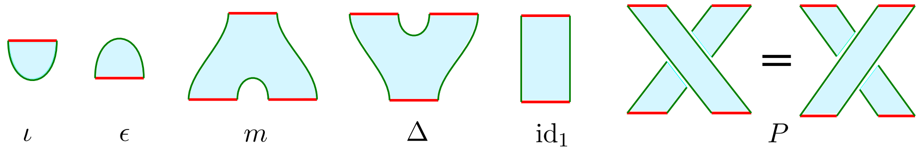

Fig. 3. A set of generating morphisms. From left to right: $\iota, \epsilon, m, \Delta$ are morphisms from 0 to 1, from 1 to 0, from 2 to 1 and from 1 to 2, respectively. The rightmost morphism P is the permutation morphism on $1 \otimes 1 = 2$. Identity morphism id_1 of object 1 is shown for completeness

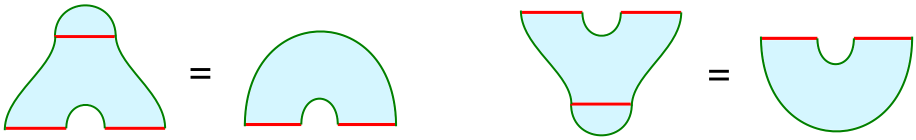

Fig. 4. Self-duality morphisms $\epsilon\, m : 1 \otimes 1 \longrightarrow 0$ and $\Delta\, \iota : 0 \longrightarrow 1 \otimes 1$ for the object 1

of $\mathbb{R}$, say from left to right). The cobordism S has corners (unless $n = m = 0$) and two types of boundary, as discussed. By definition, two cobordisms S_1, S_2 represent the same morphism if they are diffeomorphic rel horizontal boundary, that is, keeping all horizontal boundary points fixed.

The category TFS is symmetric monoidal, and a possible set of generating morphisms is shown in Fig. 3. We have included the identity morphism id_1 into the Figure to emphasize that the identity morphism id_n is represented by the surface which is the direct product of the disjoint union of n intervals (representing object n) and [0, 1]. The permutation morphism P of $2 = 1 \otimes 1$, shown on the right, is part of the symmetric monoidal structure on TFS and squares to the identity (Fig. 3).

The elements $\iota, \epsilon, m, \Delta, P$ constitute a set of monoidal generators of TFS. Together with the identity morphism id_1 they can be used to build any morphism in TFS, via horizontal and vertical compositions. In particular, from these generators we can build the self-duality morphisms for the object 1, see Fig. 4.

Some relations in the category TFS are shown in Fig. 5.

We call a surface S representing a morphism from n to m in TFS a *thin flat cobordism* from n to m. A thin flat cobordism S is a disjoint union of its connected components $S_1, \ldots, S_k$. Consider one such component S'. It necessarily has non-empty boundary, and we can assign to S' non-negative integers ℓ, g, where $\ell + 1$ is the number of boundary components and $g \geq 0$ is the genus of S'. The surface S' carries an orientation, inherited via an immersion from the orientation of the plane.

We will also call a thin flat surface a *tf-surface* and, when viewed as a cobordism, a *tf-cobordism*.

The morphisms

$$\iota : 0 \longrightarrow 1, \ \epsilon : 1 \longrightarrow 0, \ m : 1 \otimes 1 \longrightarrow 1, \ \Delta : 1 \longrightarrow 1 \otimes 1$$

and relations on them show that the object 1 is a symmetric Frobenius algebra object in TFS (top left relation in Fig. 5 shows that the trace map is symmetric). It's not a commutative algebra object, since the two morphisms in Fig. 6 left are not equal in TFS.

2.2. Classification of thin flat surfaces. By a *closed* or *floating* tf-surface S we mean one without horizontal boundary. A floating tf-surface necessarily has side boundary, unless

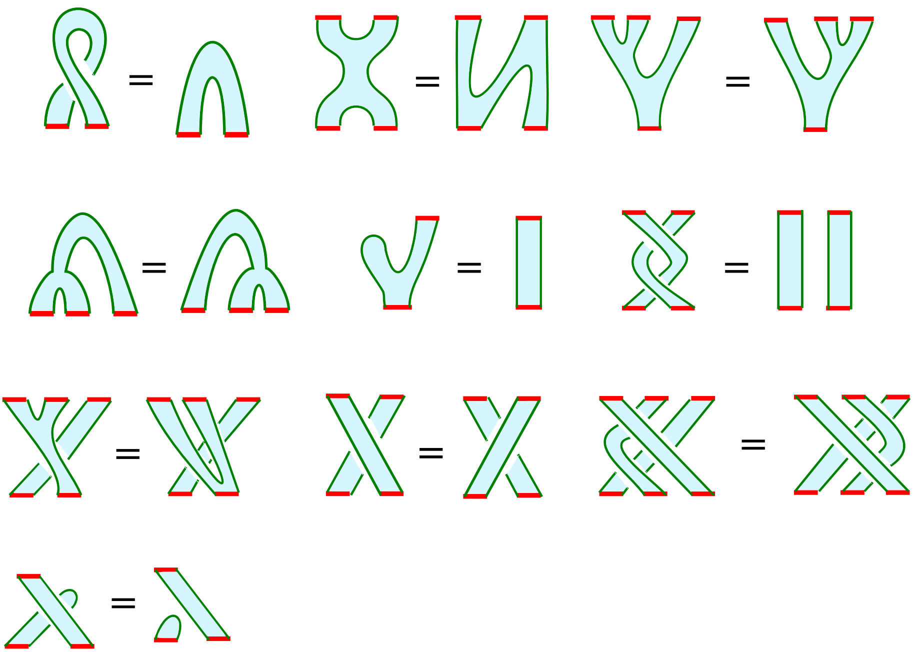

Fig. 5. Some relations in TFS

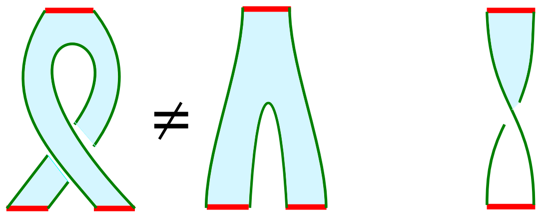

Fig. 6. Left: object 1 is not commutative Frobenius. That the two diagrams on the left are not diffeomorphic rel horizontal boundary can be seen easily by examining the matchings on the six corner points in each diagram provided by side boundaries. The two matchings of the six points are different, a sufficient condition for the two cobordisms not to be diffeomorphic rel boundary. Right: a diagram that's not a morphism in TFS

it is the empty surface. Diffeomorphism equivalence classes of floating tf-surfaces are in a bijection with endomorphisms of the object 0 of TFS. Such a surface is a disjoint union of its connected components, and a component is uniquely determined by its pair $(\ell+1, g)$, $\ell, g \in \mathbb{Z}_+$, the number of boundary components and the genus, respectively. Any such pair is realized by some surface, since pairs $(0+1, 0)$, $(1+1, 0)$, and $(0+1, 1)$ are realized by a disk, an annulus, a flat punctured torus, see Fig. 7, and taking band-connected sum of surfaces with invariants (ℓ_1+1, g_1), (ℓ_2+1, g_2) yields a surface with the invariant $(\ell_1+\ell_2+1, g_1+g_2)$. Choose a closed connected tf-surface $S_{\ell+1,g}$, one for each value of these parameters.

Connected morphisms from 0 to 0 in TFS are in a bijection with $S_{\ell+1,g}$ as above. Endomorphisms of 0 in TFS is a free commutative monoid on generators $S_{\ell+1,g}$, over all $\ell, g \in \mathbb{Z}_+$,

$$\mathrm{End}_{\mathrm{TFS}}(0) \cong \langle S_{\ell+1,g} \rangle_{\ell,g\geq 0}.$$

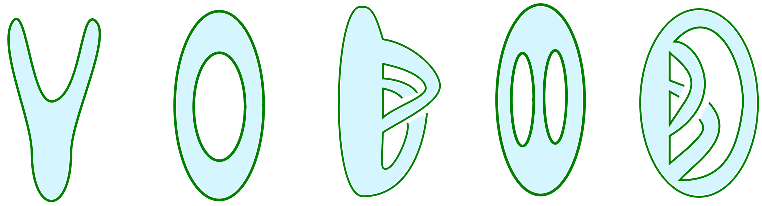

Fig. 7. Examples of closed connected tf-surfaces $S_{\ell+1,g}$ for small values of ℓ and g. We explicitly write $\ell+1$ to remember that a surface always have at least one boundary component

An element $a \in \mathrm{End}_{\mathrm{TFS}}(0)$ has a unique presentation as a finite product of $S_{\ell+1,g}$'s with positive integer multiplicities,

$$a = \prod_{i=1}^{k} S^{r_i}_{\ell_i+1,g_i}, \ r_i \in \{1, 2, \dots\}.$$

Consider a tf-surface S describing a morphism from n to m in TFS. It may have some *floating* connected components, that is, those that are disjoint from the horizontal boundary of S. Each of these components is homeomorphic to $S_{\ell+1,g}$ as above for a unique ℓ, g. Components of S that have non-empty horizontal boundary are called *viewable* or *visible* components. Any component of S is either *floating* or *viewable*. We call S *viewable* if it has no floating components. The empty cobordism is viewable.

The commutative monoid $\mathrm{End}_{\mathrm{TFS}}(0)$ acts on the set $\mathrm{Hom}_{\mathrm{TFS}}(n, m)$ by taking a cobordism to its disjoint union with a floating cobordism. Any morphism $S \in \mathrm{Hom}_{\mathrm{TFS}}(n, m)$ has a unique presentation $S = S_0 \cdot S_1$ where $S_0 \in \mathrm{End}_{\mathrm{TFS}}(0)$, S_1 is a viewable cobordism in $\mathrm{Hom}_{\mathrm{TFS}}(n, m)$ and dot $\cdot$ denotes the monoid action. In particular, $\mathrm{Hom}_{\mathrm{TFS}}(n, m)$ is a free $\mathrm{End}_{\mathrm{TFS}}(0)$-set with a "basis" of viewable cobordisms.

Let us specialize to viewable cobordisms S. All connected components of S are viewable and determine a set-theoretic partition of $n + m$ horizontal boundary intervals of S. Let us label these boundary intervals from left to right by $1, 2, \dots, n$ for the bottom intervals and $1', \dots, m'$ for the top intervals.

Each viewable component contains a non-empty subset of this set of intervals and together viewable components give a decomposition λ of this set into disjoint sets. We denote by D^m_n the set of partitions of these $n + m$ intervals, so that $\lambda \in D^m_n$. To further understand the structure of morphisms, we restrict to the case of connected S, thus a surface with one viewable connected component. All horizontal intervals are in S.

The surface S and its horizontal boundary segments inherit orientation from $\mathbb{R} \times [0, 1]$ and from induced orientations of the top and bottom boundary of $\mathbb{R} \times [0, 1]$, see Fig. 8.

We use the convention of reversing the orientation on the source (bottom) part of the boundary of a cobordism, see Fig. 8. Consequently, bottom intervals $I_1, \dots, I_n$ in ∂S are oppositely oriented from the rest of the boundary, while top intervals $I_{1'}, \dots, I_{m'}$ are oriented compatibly with the side boundary orientations, inherited from that of S and in turn inherited from the orientation of $\mathbb{R} \times [0, 1]$. In Fig. 8 right we shrank "tentacles" of S into the "core" of S to make it easier to see compatible and reverse orientations of the horizontal boundary segments of S.

We can now classify isomorphism classes of connected tf-cobordisms S from n to m. Such a cobordism has $\ell + 1$ boundary circles and genus g. On $\ell + 1$ boundary circles

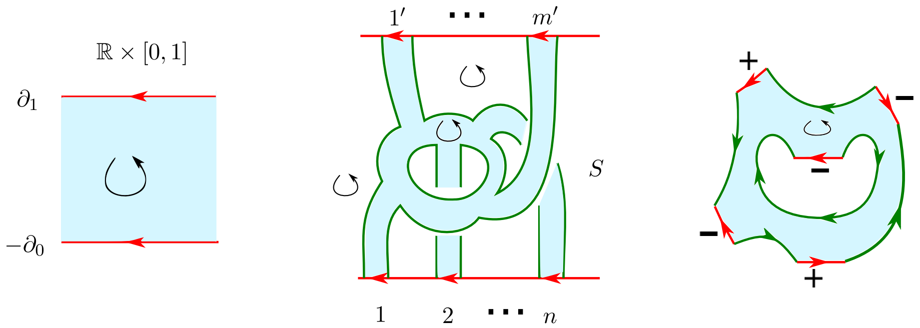

Fig. 8. Orientation convention for $\mathbb{R} \times [0, 1]$, its top and bottom boundary, surface S and its horizontal and side boundary

choose $n+m$ non-overlapping intervals and label them $1, \ldots, n, 1', \ldots, m'$. Choose an orientation of the interval $1'$ or, if $m = 0$, orientation of interval 1.

The orientation of the interval $1'$ induces an orientation of that boundary component of S and hence of S itself. One then gets induced orientations for all boundary components of S. Horizontal parts of ∂S for the intervals $2', \ldots, m'$ are then oriented compatibly with the boundary, while those corresponding to the intervals $1, \ldots, n$ in the opposite way from that for the boundary.

Horizontal intervals on the $\ell + 1$ boundary components determine a partition of

$$\mathbb{N}_n^m := \{1, \ldots, n, 1', \ldots, m'\}$$

into $\ell + 1$ disjoint subsets, possibly with some subsets empty. Orientations of boundary components induce a cyclic order on elements of each subset, where one goes along a component in the direction of its orientation and records horizontal intervals that one encounters. We call an instance of this data a locally cyclic partition of $\mathbb{N}_n^m$ together with a choice of genus $g \geq 0$. Denote the set of locally cyclic partitions of $\mathbb{N}_n^m$ by $D_{n,cyc}^m$ and by $D_{n,cyc}^m(\ell)$ if the number of components is fixed to be $\ell + 1$. This time, empty components are allowed. They correspond to components of ∂S disjoint from the boundary $\mathbb{R} \times \{0, 1\}$ of the strip. We have

$$D_{n,cyc}^m = \bigsqcup_{\ell \geq 0} d_{n,cyc}^m(\ell).$$

For the example in Fig. 8 we have $n = 3$, $m = 2$, the set of horizontal intervals is $\{1, 2, 3, 1', 2'\}$, there are two components ($\ell = 1$), and the cyclic orders are $(1', 1, 2', 3)$ and (2).

Vice versa, suppose given (ℓ, g) as above, a locally cyclic partition $\lambda \in D_{n,cyc}^m(\ell)$ of $\mathbb{N}_n^m$ into $\ell + 1$ subsets, possibly with some subsets empty, with a cyclic order on each subset. To such data we can assign a connected thin flat surface $S(\lambda, g)$ of genus g with the horizontal boundary these $n+m$ intervals, $\ell+1$ boundary components, and horizontal intervals placed according to the cyclic order for the subset along each component.

For another example, for $n = 4$, $m = 3$, the partition $\{(1', 4, 2', 2), (3', 1, 3), ()\}$, which includes one copy of the empty set, with cyclic orders as indicated and genus $g = 2$ the resulting tf-surface is shown in Fig. 9 right.

This bijection between connected morphisms from n to m and elements of the set $D_{n,cyc}^m \times \mathbb{Z}_+$ leads to a classification of morphisms in TFS. An arbitrary morphism

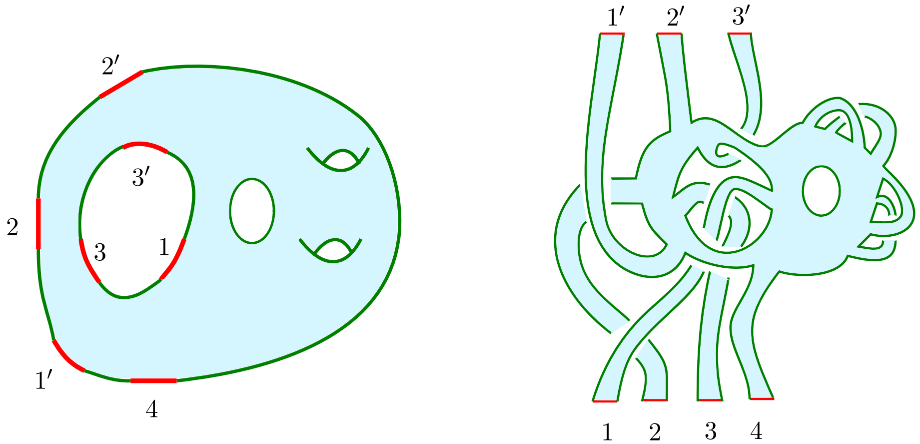

Fig. 9. Left: converting the partition and genus data into a surface with boundary and labelled edges on the boundary. One boundary component (inner right) does not carry labelled edges, since the partition contains one copy of the empty set. Genus two is indicated by schematically showing two handles. Right: stretching out labelled edges into corresponding horizontal intervals to produce a morphism in TFS

$S \in \mathrm{Hom}_{\mathrm{TFS}}(n, m)$ is the union of the viewable subcobordism of S and the floating subcobordism. The latter are classified by elements of $\mathrm{Hom}_{\mathrm{TFS}}(0, 0)$ and admit a very explicit description, via pairs (ℓ, g) of the number of circles minus one and the genus of each connected component. The viewable subcobordism S' of S determines a partition of $\mathbb{N}_n^m$ by with the set of horizontal intervals for each component of S' being a part of that partition. Each part of this partition is non-empty.

Next, for each part of the partition, remove the connected components of S' for all other parts, downsizing to just one component S''. Relabel the horizontal intervals for S'' into $1, 2, \ldots, n''$ and $1, 2, \ldots, m''$. Then such components S'' are classified by data as above: a locally cyclic partition of $\mathbb{N}_{n''}^{m''}$ (possibly with empty subsets included) and a choice of genus $g \geq 0$.

Putting the steps of this algorithm together gives a classification of morphisms from n to m in TFS.

2.3. Endomorphisms of 1 *and homs between* 0 *and* 1 *in* TFS. The category TFS is rigid symmetric monoidal, with the unit object 0 and the generating self-dual object 1, with all objects being tensor powers of the generating object, $n = 1^{\otimes n}$.

In the rest of this section, since we only consider the category TFS, we may write $\mathrm{Hom}(n, m)$ instead of $\mathrm{Hom}_{\mathrm{TFS}}(n, m)$, $\mathrm{End}(n)$ instead of $\mathrm{End}_{\mathrm{TFS}}(n)$, etc.

Connected endomorphisms of 1: Endomorphisms $\mathrm{End}(1) = \mathrm{End}_{\mathrm{TFS}}(1)$ of the object 1 in the category TFS constitute a monoid. Consider the submonoid $\mathrm{End}^c(1)$ of $\mathrm{End}(1)$ that consists of connected endomorphisms of 1. Define endomorphisms $b_1, b_2, b_3 \in \mathrm{End}^c(1)$ via diagrams in Fig. 10.

Note that b_2 has equivalent presentations, as shown in Fig. 11. The last diagram is not a tf-surface, but describes a diffeomorphism class of one (rel boundary). The tf-cobordism b_2 has genus one and one boundary component, with two horizontal segments labelled 1 and $1'$ on it, which uniquely determines it as an element of $\mathrm{End}(1)$.

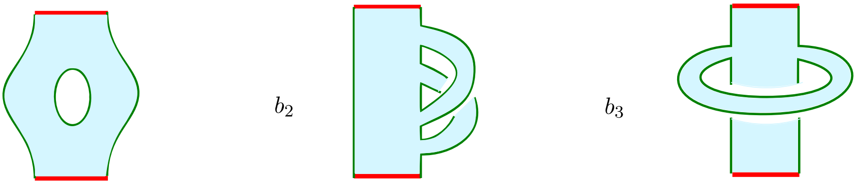

Fig. 10. Endomorphisms b_1, b_2, b_3

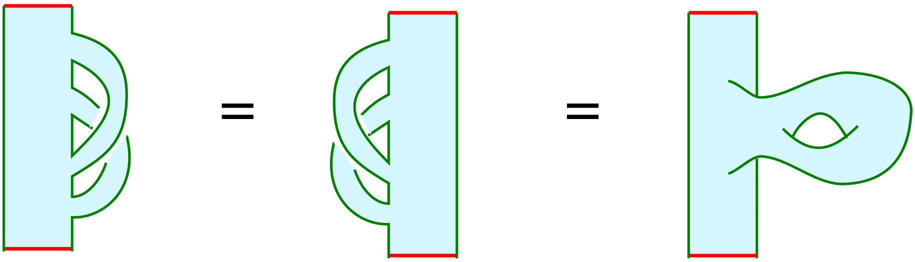

Fig. 11. Presentations of b_2. The diagram on the right is not a thin flat presentation but shows a cobordism that can be deformed to a diagram in TFS

We refer to b_1 as the "hole" cobordism, b_2 as the "handle" cobordism, b_3 as the "cross" cobordism.

Proposition 2.1. *The endomorphisms* $b_1, b_2, b_3 \in \mathrm{End}_{\mathrm{TFS}}(1)$ *pairwise commute:*

$$b_1b_2 = b_2b_1, \ b_1b_3 = b_3b_1, \ b_2b_3 = b_3b_2.$$

Proof. Note that the product with b_1 just adds a hole with no horizontal segments on it to a connected cobordism. Product with b_2 adds a handle to a connected cobordism. □

Proposition 2.2. • $\mathrm{End}^c(1)$ *is a commutative monoid generated by commuting elements* b_1, b_2, b_3 *with an additional defining relation*

$$b_3^2 = b_1b_3.$$

• $\mathrm{End}^c(1)$ *consists of the following distinct elements:*

$$b_1^n b_2^m, \ b_1^n b_2^m b_3, \quad n, m \geq 0.$$

Proof. A cobordism $S \in \mathrm{End}^c(1)$ is a connected surface with $\ell + 1$ boundary circles, genus g, and two horizontal intervals on it. If the intervals are on the same connected component of the boundary, $S = b_1^{\ell} b_2^{g}$. If the intervals lie on distinct boundary components then $\ell \geq 1$ and $S = b_1^{\ell-1} b_2^{g} b_3$. □

Spaces $\mathrm{Hom}(0, 1)$ *and* $\mathrm{Hom}(1, 0)$*:* An element $y \in \mathrm{Hom}(0, 1)$ is a tf-cobordism with one horizontal interval, at the top. It is a product y_1y_0 of one viewable component $y_1 \in \mathrm{Hom}(0, 1)$ and a closed cobordism $y_0 \in \mathrm{Hom}(0, 0)$. Assume that y is viewable, thus connected, since it has a unique horizontal segment. Then y is determined by the number $\ell + 1$ of its boundary components and the genus g and can be written as

$$y = b_1^{\ell} b_2^{g} \iota,$$

where ι is the morphism $0 \longrightarrow 1$ shown in Fig. 3 on far left. Note that $b_3\iota = b_1\iota$.

Proposition 2.3. *A morphism* $y \in Hom_{\mathrm{TFS}}(0,1)$ *has a unique presentation* $y = b_1^{\ell} b_2^{g} \iota \cdot y_0$*, where* $y_0 \in \mathrm{End}(0)$ *is a floating cobordism.*

Reflecting cobordisms about the horizontal line, we obtain a classification of elements in $\mathrm{Hom}_{\mathrm{TFS}}(1,0)$.

Proposition 2.4. *A morphism* $y \in Hom_{TFS}(1,0)$ *has a unique presentation* $y = y_0 \cdot \epsilon b_1^{\ell} b_2^{g}$*, where* $y_0 \in \mathrm{End}(0)$ *is a floating cobordism.*

Endomorphism monoid $\mathrm{End}(1)$. Recall that we continue with a minor abuse of notation, where we denote by 1 the generating object of TFS, also use it as the label for the bottom left horizontal interval of a cobordism in $\mathrm{Hom}(n,m)$, and use it convenionally as the label for the first natural number.

An element y of $\mathrm{End}_{\mathrm{TFS}}(1)$ may be one of the two types:

1. Horizontal intervals 1 and $1'$ belong to the same connected component of y.
2. Intervals 1 and $1'$ belong to different connected components of y.

Denote by U_i the set of elements of type $i \in \{1,2\}$, so that

$$\mathrm{End}(1) = U_1 \sqcup U_2 \,. \tag{3}$$

The set U_2 is closed under left and right multiplication by elements of $\mathrm{End}(1)$, thus constitutes a 2-sided ideal in this monoid. The set U_1 is a unital submonoid in $\mathrm{End}(1)$. These maps

$$U_1 \longrightarrow \mathrm{End}(1) \longleftarrow U_2$$

upgrade decomposition (3). The monoid U_1 is commutative and naturally decomposes

$$U_1 \cong \mathrm{End}^c(1) \times \mathrm{End}(0)$$

into the direct product, both terms of which we have already described. The direct product corresponds to splitting an element of U_1 into the viewable connected component and a floating cobordism.

Likewise, an element y of U_2 splits into a floating cobordism y_0 and a viewable one y_1. A viewable element y_1 of U_2 consists of two connected components, one bounding horizontal inteval 1, the other bounding $1'$. Such an element can be written as

$$y_1 = b_1^{\ell_1} b_2^{g_1} \iota \,\cdot\, \epsilon b_1^{\ell_2} b_2^{g_2},$$

with a general $y \in U_2$ given by

$$y = b_1^{\ell_1} b_2^{g_1} \iota \cdot y_0 \cdot \epsilon b_1^{\ell_2} b_2^{g_2}.$$

Multiplication of two viewable elements as above produces an additional connected component, see Fig. 12, where by the (ℓ, g) coupon we denote the endomorphism $b_1^{\ell} b_2^{g}$ of 1.

Remark. Unlike the monoids $\mathrm{End}(0)$, $\mathrm{End}^c(1)$, and their direct product U_1, monoid $\mathrm{End}(1)$ and its subsemigroup U_2 are not commutative.

3. Linearizations of the Category TFS

In this section we work over a field $\mathbf{k}$, but the construction and some results may be generalized to an arbitrary commutative ring R (or a commutative ring with additional conditions, such as being noetherian). A definitive starting reference for recognizable series with coefficients in commutative rings is Hazewinkel [11].

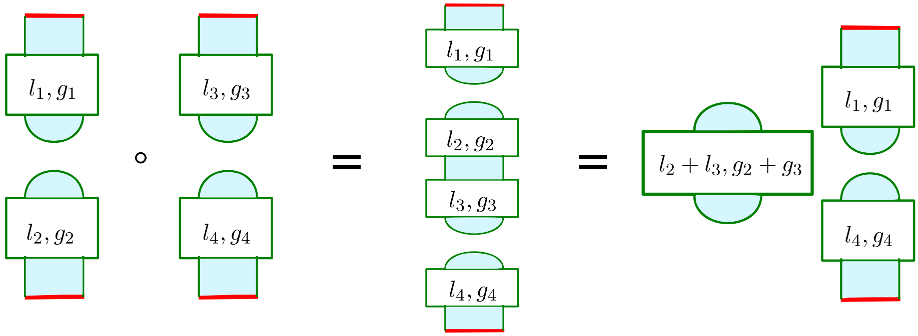

Fig. 12. Product of two viewable elements of U_2 produces a floating component $\epsilon b_1^{\ell_2+\ell_3} b_2^{g_2+g_3} \iota = S_{\ell_2+\ell_3+1, g_2+g_3}$, in addition to the components bounding intervals 1 and 1′

3.1. Categories **k**TFS *and* VTFS_α *for recognizable* α. *Category* **k**TFS. Starting with TFS we can pass to its preadditive closure **k**TFS. Objects of **k**TFS are the same as those of TFS, that is, non-negative integers $n \in \mathbb{Z}_+$. A morphism in **k**TFS from n to m is a finite **k**-linear combination of morphisms from n to m in TFS. In particular, $\mathrm{Hom}_{\mathbf{k}\mathrm{TFS}}(n, m)$ is a **k**-vector space with a basis $\mathrm{Hom}_{\mathrm{TFS}}(n, m)$. Composition of morphisms is defined in the obvious way.

Category **k**TFS is a **k**-linear preadditive category. It is also a rigid symmetric monoidal category.

Power series α. The ring $\mathrm{Hom}_{\mathbf{k}\mathrm{TFS}}(0, 0)$ of endomorphisms of the unit object 0 of **k**TFS is naturally isomorphic to the monoid algebra of $\mathrm{Hom}_{\mathrm{TFS}}(0, 0)$. The latter is a free commutative monoid on generators $S_{\ell+1,g}$, over all $\ell, g \in \mathbb{Z}_+$, so that

$$\mathrm{Hom}_{\mathbf{k}\mathrm{TFS}}(0, 0) \cong \mathbf{k}[S_{\ell+1,g}]_{\ell,g\in\mathbb{Z}_+}$$

is the polynomial algebra on countably many generators, parametrized by pairs (ℓ, g) of non-negative integers. Homomorphisms of **k**-algebras

$$\mathrm{Hom}_{\mathbf{k}\mathrm{TFS}}(0, 0) \longrightarrow \mathbf{k}$$

are in a bijection with doubly-infinite sequences

$$\alpha = (\alpha_{\ell,g})_{\ell,g\in\mathbb{Z}_+}, \quad \alpha_{\ell,g} \in \mathbf{k}.$$

The bijection associates to a sequence α the homomorphism, also denoted α,

$$\mathrm{Hom}_{\mathrm{TFS}}(0, 0) \cong \mathbf{k}[S_{\ell+1,g}]_{\ell,g\in\mathbb{Z}_+} \xrightarrow{\alpha} \mathbf{k}, \quad \alpha(S_{\ell+1,g}) = \alpha_{\ell,g}.$$

Sequences α are also in a bijection with *multiplicative* **k**-valued evaluations of floating cobordisms in TFS. These evaluations are maps from the set of floating cobordisms (endomorphisms of object 0) in TFS to **k** that take disjoint union of cobordisms to the product of evaluations,

$$\alpha(S \sqcup S') = \alpha(S) \cdot \alpha(S').$$

Thus, α is a map of sets

$$\alpha : \mathbb{Z}_+ \times \mathbb{Z}_+ \longrightarrow \mathbf{k}$$

that we can think of a $\mathbb{Z}_+ \times \mathbb{Z}_+$-matrix with coefficients in $\mathbf{k}$

$$\alpha = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2} & \alpha_{0,3} & \cdots \\ \alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \cdots \\ \alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \cdots \\ \alpha_{3,0} & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We encode α into power series in two variables T_1, T_2:

$$Z_\alpha(T_1, T_2) = \sum_{k,g\geq 0} \alpha_{k,g} T_1^k T_2^g, \quad \alpha = (\alpha_{k,g})_{k,g\in\mathbb{Z}_+}, \quad \alpha_{k,g} \in \mathbf{k}. \tag{4}$$

A doubly-infinite sequence α can also be thought of as a linear functional on the space of polynomials in two variables:

$$\alpha \in \mathbf{k}[T_1, T_2]^* := \mathrm{Hom}_{\mathbf{k}}(\mathbf{k}[T_1, T_2], \mathbf{k}).$$

We assume that α is not identically zero (the theory is trivial otherwise). Then $\ker(\alpha) \subset \mathbf{k}[T_1, T_2]$ is a codimension one subspace.

Category VTFS_α. Given α, we can form the quotient VTFS_α of category $\mathbf{k}\mathrm{TFS}$ by adding the relation that a floating surface $S_{\ell+1,g}$ of genus g with $\ell + 1$ boundary components evaluates to $\alpha_{\ell,g} \in \mathbf{k}$. Objects of VTFS_α are still non-negative integers n. Morphisms from n to m are finite $\mathbf{k}$-linear combinations of *viewable* cobordisms from n to m. Composition of cobordisms from n to m and from m to k is a cobordism from n to k which may have floating components. These components are removed simultaneously with multiplying the viewable cobordism that remains by the product of $\alpha_{\ell,g}$'s, for every component $S_{\ell+1,g}$.

The space of homs from n to m in this category has a basis of viewable cobordisms from n to m. Letter V in the notation VTFS_α stands for *viewable*.

Recognizable series. Borrowing terminology from control theory [7,10], we say that a linear functional or series α is *recognizable* if $\ker(\alpha)$ contains an ideal $I \in \mathbf{k}[T_1, T_2]$ of finite codimension.

Proposition 3.1. *α is recognizable iff the power series Z_α has the form*

$$Z_\alpha(T_1, T_2) = \frac{P(T_1, T_2)}{Q_1(T_1)Q_2(T_2)}, \tag{5}$$

where $Q_1(T_1)$, $Q_2(T_2)$ are one-variable polynomials and $P(T_1, T_2)$ is a two-variable polynomial, all with coefficients in the field $\mathbf{k}$.

We assume that $Q_1(0) \neq 0,\ Q_2(0) \neq 0$, otherwise at least one of these polynomials is not coprime with $P(T_1, T_2)$ and either T_1 or T_2 cancels out from the numerator and denominator. With the denominator not zero at $T_1, T_2 = 0$ the power series expansion makes sense.

Proof. See [7] for a proof. This result is also mentioned in [10, Remark 2]. To prove it, assume that α is recognizable. We start with the case when $\mathbf{k}$ is algebraically closed. A finite codimension ideal $I \subset \mathbf{k}[T_1, T_2]$ necessarily contains a sum,

$$I_1 \otimes \mathbf{k}[T_2] + \mathbf{k}[T_1] \otimes I_2 \subset I \subset \mathbf{k}[T_1, T_2] \tag{6}$$

for some finite codimension ideals $I_1 \subset \mathbf{k}[T_1]$ and $I_2 \subset \mathbf{k}[T_2]$. To see this, note that the finite affine scheme $\mathrm{Spec}(\mathbf{k}[T_1, T_2]/I)$ is supported over finitely many points of the affine plane $\mathbb{A}^2$. Projecting these points onto the coordinate lines and counting them with multiplicities produces two one-variable polynomials $U_1(T), U_2(T)$ such that I contains the ideal $(U_1(T_1)) + (U_2(T_2))$ of $\mathbf{k}[T_1, T_2]$. We can now take principal ideals $I_i = (U_i(T_i)), i = 1, 2$ to get the inclusion on the LHS of (6). This also gives a quotient map

$$\mathbf{k}[T_1]/(U_1(T_1)) \otimes \mathbf{k}[T_2]/(U_2(T_2)) \longrightarrow \mathbf{k}[T_1, T_2]/I$$

lifting to the identity map on $\mathbf{k}[T_1, T_2]$. Existence of such finite codimension ideals I_1, I_2 over an arbitrary field $\mathbf{k}$ follows as well.

Hence, recognizable series α has the property that $\alpha(U_1(T_1)T_1^k T_2^m) = 0$ for any $k, m \geq 0$. We can assume that $U_1(T)$ is a polynomial of some degree r with the lowest degree term $u_s T^s$ for $s \leq r$ and write

$$U_1(T) = u_r T^r + u_{r-1}T^{r-1} + \cdots + u_{s+1}T^{s+1} + u_s T^s,\ 0 \leq s \leq r,\ u_r, u_s \neq 0,\ u_j \in \mathbf{k}.$$

Then, for any $k, m \geq 0$

$$u_r \alpha_{r+k,m} + u_{r-1}\alpha_{r-1+k,m} + \cdots + u_{s+1}\alpha_{s+k+1,m} + u_s \alpha_{s+k,m} = 0. \tag{7}$$

We obtain a similar relation on the coefficients with U_2 and T_2 in place of U_1 and T_1 and varying the second index. Let us write

$$U_2(T) = v_{r'}T^{r'} + v_{r'-1}T^{r'-1} + \cdots + v_{s'+1}T^{s'+1} + v_{s'}T^{s'},\ 0 \leq s' \leq r',\ v_{r'}, v_{s'} \neq 0,\ v_j \in \mathbf{k}.$$

Then, for any $k, m \geq 0$

$$v_{r'}\alpha_{r'+k,m} + v_{r'-1}\alpha_{r'-1+k,m} + \cdots + v_{s'+1}\alpha_{s'+k+1,m} + v_s \alpha_{s'+k,m} = 0. \tag{8}$$

Consequently, α is eventually recurrent in both T_1 and T_2 directions and its values are determined by $\alpha_{i,j}$ with $0 \leq i < r, 0 \leq j < r'$.

Consider polynomials

$$\widehat{Q}_1(T) = T^r U_1(T^{-1}) = u_s T^r + u_{s+1}T^{r-1} + u_{s+2}T^{r-2} + \cdots + u_r T^{r-s},$$
$$\widehat{Q}_2(T) = T^{r'} U_2(T^{-1}) = v_{s'} T^{r'} + v_{s'+1}T^{r'-1} + v_{s'+2}T^{r'-2} + \cdots + v_{r'} T^{r'-s'}.$$

Form the product

$$\widehat{P}(T_1, T_2) := Z_\alpha(T_1, T_2)\widehat{Q}_1(T_1)\widehat{Q}_2(T_2) = \sum_{i,j \geq 0} w_{i,j} T_1^i T_2^j$$

and examine coefficients of its power series expansion. Formulas (7), (8) show that $w_{i,j} = 0$ if $i \geq r$ of $j \geq r'$. Therefore, $\widehat{P}(T_1, T_2)$ is a polynomial with T_1, T_2 degrees bounded by $r - 1, r' - 1$, respectively. We can then form the quotient

$$\frac{\widehat{P}(T_1, T_2)}{\widehat{Q}_1(T_1)\widehat{Q}_2(T_2)}$$

The numerator and denominator may share common factors, including $T_1^{r-s} T_2^{r'-s'}$. After canceling those out, we arrive at the presentation (5) for $Z_\alpha(T_1, T_2)$.

We leave the proof of the opposite implication of the proposition to the reader or refer to [7].

Note that the proof works for any finite number of variables $T_1, \ldots, T_c$, not only for two. □

The condition that α is *recognizable* can also be expressed via its Hankel matrix H_α. The latter matrix has rows and columns enumerated by pairs $(m,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, equivalently by the monomial basis elements $T_1^m T_2^k$. The $((m_1,k_1),(m_2,k_2))$-entry of H_α is $\alpha_{m_1+m_2,k_1+k_2}$. The following result is proved in [7].

Proposition 3.2. *The series α is recognizable iff the Hankel matrix H_α has finite rank.*

Note that H_α has finite rank iff there exists M such that any $M \times M$ minor of H_α has determinant zero. The rank is $M-1$ if in addition there is an $(M-1)\times(M-1)$ minor with a non-zero determinant.

3.2. Skein category STFS_α. *Recognizable series and commutative Frobenius algebras.* Assume that α is recognizable. Among all finite-codimension ideals $I \subset \ker(\alpha)$ there is a unique largest ideal I_α, given by the sum over all such I. Equivalently, it can be described as follows. There is a homomorphism of $\mathbf{k}[T_1,T_2]$-modules

$$h : \mathbf{k}[T_1,T_2] \longrightarrow \mathbf{k}[T_1,T_2]^* \tag{9}$$

given by sending 1 to α and $z \in \mathbf{k}[T_1,T_2]$ to $z\alpha \in \mathbf{k}[T_1,T_2]^*$ with $(z\alpha)(f) = \alpha(zf)$. The ideal I_α is the kernel of h.

Notice that α descends to a nondegenerate bilinear form on the quotient algebra

$$A_\alpha := \mathbf{k}[T_1,T_2]/I_\alpha. \tag{10}$$

In particular, A_α is a commutative Frobenius algebra on two generators T_1, T_2 with a nondegenerate trace form α.

Vice versa, assume given a commutative Frobenius $\mathbf{k}$-algebra B with the nondegenerate trace form $\beta : B \longrightarrow \mathbf{k}$ and a pair of generators g_1, g_2. To such data we can associate a surjective homomorphism

$$\psi : \mathbf{k}[T_1,T_2] \longrightarrow B,\ \psi(T_i) = g_i,\ i = 1,2,$$

the trace map $\alpha = \beta \circ \psi$ on $\mathbf{k}[T_1,T_2]$ given by composing ψ with β, and recognizable series

$$\alpha_\beta = \sum_{\ell,g\geq 0} \beta(g_1^\ell g_2^g) T_1^\ell T_2^g.$$

Thus, recognizable power series on $\mathbf{k}[T_1,T_2]$ are classified by isomorphism classes of data (B, g_1, g_2, β): a commutative Frobenius algebra B generated by $g_1, g_2 \in B$ and a non-degenerate trace β.

Category STFS_α. We can now define the category STFS_α (where first S stands for "skein") to be a quotient of VTFS_α by the skein relations in the ideal I_α. The category STFS_α has the same objects as all the other cobordism categories we've considered so far, that is, nonnegative integers n. Morphisms from n to m are $\mathbf{k}$-linear combinations of viewable cobordisms modulo the relations in I_α. Precisely, let

$$p(T_1,T_2) = \sum_{i,j} p_{i,j} T_1^i T_2^j \in I_\alpha \tag{11}$$

be a polynomial in the ideal I_α. Given a viewable cobordism x choose a component c of x and denote by $x_c(i,j)$ the cobordism given by inserting i holes and adding j handles

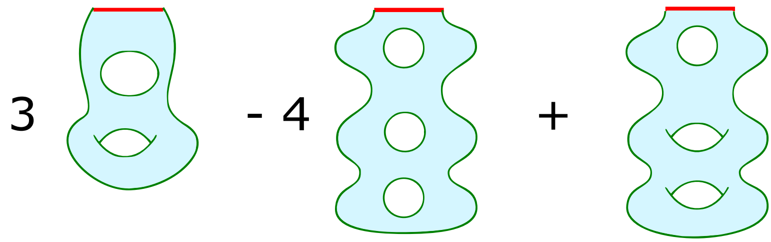

Fig. 13. $b(v)$, for $v = 3T_1T_2 - 4T_1^3 + T_1T_2^2$; handles are shown schematically

to x at the component c. We now mod out the hom space $\mathrm{Hom}_{\mathrm{VTFS}_\alpha}(n, m)$, which is a $\mathbf{k}$-vector space with a basis of all viewable cobordisms from n to m, by the relations

$$\sum_{i,j} p_{i,j}\, x_c(i, j) = 0,$$

one for each component c of x, over all viewable cobordisms x.

It is easy to see that these "skein" relations are compatible with α-evaluation of floating cobordisms. Namely, if instead of a viewable cobordism x we consider a floating cobordism y and choose a component c of y to add holes and handles, resulting in cobordisms $y_c(i, j)$, then

$$\sum_{i,j} p_{i,j}\alpha(y_c(i, j)) = 0.$$

This compatibility condition, immediate from our definition of I_α as the kernel of the module map (9), ensures non-triviality of this quotient and its compatibility with the composition of morphisms.

Viewing VTFS_α as a tensor category, it is enough to write down corresponding relations on homs from 0 to 1 and then mod out by them in the tensor category (by gluing each term in the resulting linear combination of products of holes and handles on a disk to any component along a segment on its side boundary). Choose a generating set $v_1, \ldots, v_r$ of I_α viewed as $\mathbf{k}[T_1, T_2]$-module. Specializing to a single basis element v_j, assume that it is given by the polynomial p on the right hand side of (11). Form the element

$$b(v_j) := \sum_{i,j} p_{i,j} b_1^i b_2^j\, \iota \in \mathrm{Hom}(0, 1).$$

The skein category STFS_α can be defined as the quotient of VTFS_α by the tensor ideal generated by elements $b(v_1), \ldots, b(v_r)$. Figure 13 shows an example of an element $b(v)$.

Remark. For a recognizable series α there are unique minimal degree monic polynomials $q_{\alpha,1}, q_{\alpha,2}$,

$$q_{\alpha,1}(x) = x^t + a_{t-1}x^{t-1} + \cdots + a_0, \quad q_{\alpha_2}(x) = x^{t'} + a'_{t'-1}x^{t'-1} + \cdots + a'_0,$$

such that

$$q_{\alpha,1}(T_1) \in I_\alpha, \quad q_{\alpha,2}(T_2) \in I_\alpha.$$

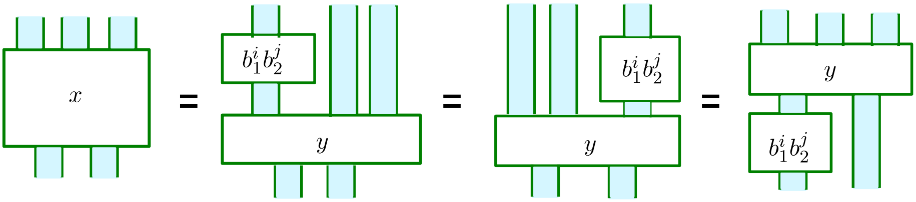

Fig. 14. Factorization of a connected cobordism x into a coupon and a minimal cobordism is shown schematically. Since y is connected, $b_1^i b_2^j$ coupon can be moved to any leg of y

Among skein relations associated to elements of I_α in STFS_α there is a polynomial relation that utilizes only adding holes to a component of the cobordism. This relation is given by the polynomial $q_{\alpha,1}(T_1)$:

$$b_1^t + a_{t-1} b_1^{t-1} + \cdots + a_0 = 0,$$

describing an equality in the ring of endomorphisms of object 1 of STFS_α, where b_1 is the *hole* cobordism, see Fig. 11. Equivalently, it can be rewritten as a relation in Hom(0, 1):

$$(b_1^t + a_{t-1} b_1^{t-1} + \cdots + a_0)\iota = 0,$$

Likewise, there is a skein relation on cobordisms that differ only by genus of a given component. The relation is given by the polynomial $q_{\alpha,2}(T_2)$:

$$b_2^{t'} + a'_{t'-1} b_2^{t'-1} + \cdots + a'_0 = 0,$$

where b_2 is the *handle* morphism, see Fig. 11.

Minimal viewable cobordisms, B_α-companions, and bases of hom spaces of STFS_α.
Consider a connected viewable cobordism x. We say that x is *minimal* if it has genus zero and no *holes*, that is, each boundary component of x contains at least one horizontal segment. Equivalently x is minimal if it cannot be factored into $x' b_1 x''$ or $x' b_2 x''$ for some morphisms x', x''. Note that if such a factorization exists, then there exists one with x'' the identity cobordism and one with x' the identity cobordism. Any viewable connected cobordism x from n to m with $m > 0$ can be written as $(b_1^i b_2^j \otimes \mathrm{id}_{m-1})y$ for some minimal y and, if $n > 0$, as $y(b_1^i b_2^j \otimes \mathrm{id}_{n-1})$ for the same y, see Fig. 14. If one of n or m is zero, only one of these two presentations exist.

Equivalently, a connected viewable cobordism x is minimal if it is *handless* and has no holes.

A viewable cobordism y is called *minimal* if each connected component of y is minimal. A viewable cobordism x factors into a product of a minimal cobordism and "coupons" carrying powers of b_1, b_2, one for each connected component of x. That is, for each connected component c of x count holes and handles on it and then remove them to get a minimal connected component c'. The original component can be recovered by inserting holes and handles back anywhere along c'. For instance, they may be inserted at one of its top or bottom legs by multiplying c' by the corresponding powers of b_1 and b_2 there.

To any viewable x we can associate its minimal counterpart y by removing holes and handles from each connected component of x. Given y, we can recover x by multiplying by appropriate powers of b_1 and b_2 at horizontal intervals for different components of y.

Denote by $\mathcal{M}(n, m)$ the set of minimal viewable cobordisms from n to m.

Proposition 3.3. *$\mathcal{M}(n, m)$ is a finite set.*

Proof. From our classification of morphisms in TFS it is clear that minimal cobordisms from n to m are in a bijection with partitions λ of the set $\mathbb{N}_n^m$ of $n+m$ horizontal intervals, together with a choice of a partition μ_i of each part λ_i of λ and a cyclic order on each part of μ_i. □

Recall finite codimension ideal I_α (the *syntactic* ideal) associated with recognizable series α. Let

$$d_\alpha = \dim(\mathbf{k}[T_1, T_2]/I_\alpha).$$

Choose a set of pairs

$$P_\alpha = \{(i_t, j_t)\}_{t=1}^{d_\alpha}, \quad i_t, j_t \in \mathbb{Z}_+$$

such that monomials $T_1^{i_t} T_2^{j_t}$ constitute a basis of the algebra $\mathbf{k}[T_1, T_2]/I_\alpha$. Denote this basis by B_α. It is well-known [18] that a basis can always be choosen so that the exponents (i_t, j_t) of the monomials, when placed into corresponding points of the square lattice, constitute a partition of d_α, but we do not need this result here.

Choose a minimal cobordism y and assign an element $v_c \in B_\alpha$ to each connected component c of y. This assignment gives rise to a cobordism x obtained from y by inserting cobordisms $b(v_c)$ at all components c of y. For $v_c = T_1^i T_2^j$ we add i holes and j handles to the component c or, equivalently, multiply it at one of its horizontal boundary intervals by $b_1^i b_2^j$.

In this way to $y \in \mathcal{M}(n, m)$ there are assigned d_α^r cobordisms x, where r is the number of components of y. These x are called B_α-companions of y. Denote the set of such x by $B_\alpha(y)$.

Proposition 3.4. *Elements of sets $B_\alpha(y)$, over all $y \in \mathcal{M}(n, m)$, constitute a basis of $Hom_{\mathrm{STFS}_\alpha}(n, m)$.*

In other words, to get a basis of homs from n to m in the skein category STFS_α we take all minimal cobordisms y from n to m and insert a basis element from B_α into each component of y.

Proof. The proposition follows immediately from our construction of STFS_α. One needs to check consistency, that our rules do not force additional relations when composing cobordisms. This is straightforward. □

Corollary 3.5. *Hom spaces in the category* STFS_α *are finite dimensional.*

Remark. In a seeming discrepancy, object 1 of the category TFS is a symmetric Frobenius object but not a commutative Frobenius object, see Fig. 6 left, since the multiplication map $1 \otimes 1 \longrightarrow 1$ does not commute with the permutation endomorphism of $1 \otimes 1$. Yet, in the category STFS_α the state space $\mathrm{Hom}(0, 1)$ of the interval is a commutative Frobenius algebra A_α, defined in (10), with the multiplication on $\mathrm{Hom}(0, 1)$ given by the *thin flat pants* cobordism in Fig. 15 left. This is explained by the observation that the thin flat pants multiplication is commutative in the categories we consider, including TFS and VTFS_α and STFS_α. Indeed, viewable morphisms from 0 to 1 in TFS have the form $b_1^n b_2^m \iota$, and the product of two such morphisms does not depend on their order, see Fig. 15 right. Adding floating components (or passing to linear combinations, or taking quotients) does not break commutativity.

Later, in Section 5.1, in a similar situation we also denote $b_1^n b_2^m \iota$ by $\underline{b}_1^n \underline{b}_2^m$.

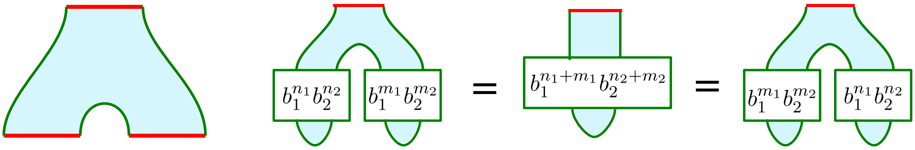

Fig. 15. Left: thin flat pants cobordism from $1 \otimes 1$ to 1. Right: commutativity of multiplication in Hom(0, 1)

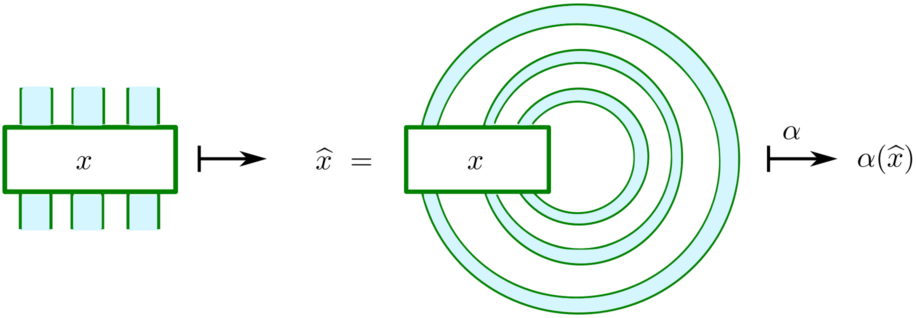

Fig. 16. The trace map: closing endomorphism x of n into $\widehat{x}$ and applying α

3.3. Quotient by negligible morphisms and Karoubi envelopes.. Category TFS_α. Consider the ideal $J_\alpha \subset \mathrm{STFS}_\alpha$ of negligible morphisms, relative to the trace form tr_α associated with α, and form the quotient category

$$\mathrm{TFS}_\alpha := \mathrm{STFS}_\alpha / J_\alpha.$$

The trace form is given on a cobordism x from n to n by closing it via n annuli connecting n top with n bottom circles of the horizontal boundary of x into a floating cobordism $\widehat{x}$ and applying α,

$$\mathsf{tr}_\alpha(x) := \alpha(\widehat{x}).$$

This operation is depicted in Fig. 16.

A morphism $y \in \mathrm{Hom}(n, m)$ is called *negligible* if $\mathsf{tr}_\alpha(zy) = 0$ for any morphism $z \in \mathrm{Hom}(m, n)$. Negligible morphisms constitute a two-sided ideal in the pre-additive category STFS_α.

The quotient category TFS_α has finite-dimensional hom spaces, as does STFS_α (recall that α is recognizable). The trace is nondegenerate on TFS_α and defines perfect bilinear pairings

$$\mathrm{Hom}(n, m) \otimes \mathrm{Hom}(m, n) \longrightarrow \mathbf{k}$$

on its hom spaces. We may call TFS_α the *gligible quotient* of STFS_α, having modded out by the ideal of negligible morphisms.

Let us go back to the category TFS and its linear version **k**TFS. Fix the number n of intervals and consider the vector space V_n with a basis of all viewable tf-surfaces with that boundary, that is viewable cobordisms in TFS from 0 to n. Given α, define a bilinear form on V_n via its values on pairs of basis vectors:

$$(x, y) = \alpha(\overline{y}x) \in \mathbf{k},$$

where $\overline{y}$ is given by reflecting y about a horizontal line to get a cobordism from n to 0, and $\overline{y}x$ is a floating cobordism from 0 to 0 given by composing $\overline{y}$ and x. This bilinear form on V_n is symmetric. Define $A_\alpha(n)$ as the quotient of V_n by the kernel of this bilinear form. Then there is a canonical isomorphism

$$A_\alpha(n) \cong \mathrm{Hom}_{\mathrm{TFS}_\alpha}(0, n)$$

as well as isomorphisms

$$A_\alpha(n+m) \cong \mathrm{Hom}_{\mathrm{TFS}_\alpha}(0, n+m) \cong \mathrm{Hom}_{\mathrm{TFS}_\alpha}(m, n)$$

given by moving m invervals from bottom to top via the duality morphism.

The symmetric group S_n acts by permutation cobordisms on $A_\alpha(n)$. Furthermore, at each circle there is an action of the endomorphism algebra $\mathrm{End}(1) = \mathrm{End}_{\mathrm{TFS}_\alpha}(1)$. Consequently, the cross-product algebra $\mathbf{k}\mathbb{S}_n \ltimes \mathrm{End}(1)^{\otimes k}$ acts on $A_\alpha(n)$.

Multiplication maps

$$A_\alpha(n) \otimes A_\alpha(m) \longrightarrow A_\alpha(n+m)$$

turn the direct sum

$$A_\alpha := \bigoplus_{n\geq 0} A_\alpha(n)$$

into a unital commutative associative graded algebra, with $A_\alpha(0) \cong \mathbf{k}$. All of this data, including the power series $\sum_{n\geq 0} \dim A_\alpha(n) z^n$ encoding dimensions of $A_\alpha(n)$, are invariants of recognizable series α.

In the diagram of five categories and four functors

$$\mathrm{TFS} \longrightarrow \mathbf{k}\mathrm{TFS} \longrightarrow \mathrm{VTFS}_\alpha \longrightarrow \mathrm{STFS}_\alpha \longrightarrow \mathrm{TFS}_\alpha$$

one can get from $\mathbf{k}$TFS to TFS_α in one step, bypassing VTFS_α and STFS_α, by taking the ideal of negligible morphisms in $\mathbf{k}$TFS (for essentially the same trace map, shown in Fig. 16) and modding out by it. It is convenient to introduce those intermediate categories, though. For instance, STFS_α already has finite-dimensional hom spaces and allows to define the analogue of the Deligne category in our case.

The Deligne category DTFS_α *and its gligible quotient* $\underline{\mathrm{DTFS}}_\alpha$. The skein category STFS_α is a rigid symmetric monoidal $\mathbf{k}$-linear category with objects $n \in \mathbb{Z}_+$ and finite-dimensional hom spaces. We form the additive Karoubi closure

$$\mathrm{DTFS}_\alpha := \mathrm{Kar}(\mathrm{STFS}_\alpha^\oplus)$$

by allowing formal finite direct sums of objects in STFS, extending morphisms correspondingly, and then adding idempotents to get a Karoubi-closed category. Category DTFS_α plays the role of the Deligne category in our construction.

In the Deligne category DTFS_α endomorphisms of an object (n, e), where e is an idempotent endomorphism of n, inherit the trace map tr_α into the ground field. Consequently, category DTFS_α also has a two-sided ideal of negligible morphisms $J_{D,\alpha}$. Taking the quotient by this ideal

$$\underline{\mathrm{DTFS}}_\alpha := \mathrm{DTFS}_\alpha / J_{D,\alpha}$$

gives us an idempotent-complete category with non-degenerate symmetric bilinear forms on hom spaces $\mathrm{Hom}(0, (n, e))$, where (n, e) is an object as above, and more generally non-degenerate bilinear pairings on hom spaces

$$\mathrm{Hom}((n, e), (m, e')) \otimes \mathrm{Hom}((m, e'), (n, e)) \longrightarrow \mathbf{k}$$

where e' is an idempotent endomorphism of object m. Due to the symmetry between homs given by the contravariant involution on all categories that we have considered so far (reflection about a horizontal line), the above bilinear pairings can be converted into non-degenerate symmetric bilinear forms on $\mathrm{Hom}((n, e), (m, e'))$ in $\underline{\mathrm{DTFS}}_\alpha$.

Category $\underline{\mathrm{DTFS}}_\alpha$ is also equivalent to the additive Karoubi closure of the category TFS_α, see the commutative square in (12).

3.4. Summary of the categories and functors. Below is a summary for each category that has been considered.

- TFS: the category of thin flat surfaces (tf-surfaces). Its objects are non-negative integers and morphisms are thin flat surfaces.
- **k**TFS: this category has the same objects as TFS; its morphisms are formal finite **k**-linear combinations of morphisms in TFS.
- VTFS_α: in this quotient category of **k**TFS we reduce morphisms to linear combinations of viewable cobordisms. Floating connected components are removed by evaluating them via α.
- STFS_α: to define this category, specialize to recognizable α and add skein relations by modding out by elements of the ideal I_α in $\mathbf{k}[T_1, T_2]$ along each connected component of a surface (T_1 is a hole, T_2 a handle). Hom spaces in this category are finite-dimensional.
- TFS_α: the quotient of STFS_α by the ideal J_α of negligible morphisms. This category is also equivalent (even isomorphic) to the quotients of **k**TFS and VTFS_α by the corresponding ideals of negligible morphisms in them. The trace pairing in TFS_α between $\mathrm{Hom}(n, m)$ and $\mathrm{Hom}(m, n)$ is perfect.
- DTFS_α: it is the analogue of the Deligne category obtained from STFS_α by allowing finite direct sums of objects and then adding idempotents as objects to get a Karoubi-closed category.
- $\underline{\mathrm{DTFS}}_\alpha$: the quotient of DTFS_α by the two-sided ideal of negligible morphisms. This category is equivalent to the additive Karoubi closure of TFS_α and sits in the bottom right corner of the commutative square below.

We arrange these categories and functors, when α is recognizable, into the following diagram:

$$\begin{array}{ccccccccc} \mathrm{TFS} & \longrightarrow & \mathbf{k}\mathrm{TFS} & \longrightarrow & \mathrm{VTFS}_\alpha & \longrightarrow & \mathrm{STFS}_\alpha & \longrightarrow & \mathrm{DTFS}_\alpha \\ & & & & & & \downarrow & & \downarrow \\ & & & & & & \mathrm{TFS}_\alpha & \longrightarrow & \underline{\mathrm{DTFS}}_\alpha \end{array} \tag{12}$$

All seven categories are rigid symmetric monoidal. All but the leftmost category TFS are **k**-linear. Except for the two categories on the far right, the objects of each category are non-negative integers. The four categories on the right all have finite-dimensional hom spaces. The two categories on the far right are additive and Karoubi-closed. The four categories in the middle of the diagram are pre-additive but not additive.

The arrows show functors between these categories considered in the paper. The square is commutative. An analogous diagram of functors and categories can be found in [16] for the category of oriented 2D cobordisms in place of TFS.

It is possible to go directly from **k**TFS to TFS_α by modding out by the ideal of negligible morphisms in the former category. We found it convenient to get to this quotient in several steps, introducing categories VTFS_α and STFS_α along the way.

Remark. For possible future use, it may be convenient to relabel the categories above using shorter strings. For instance, writing $\mathcal{S}$ (for "surfaces") in place of TFS we can rename the categories as follows:

$$\begin{array}{ccccccccc} \mathcal{S} & \longrightarrow & \mathbf{k}\mathcal{S} & \longrightarrow & \mathcal{VS}_\alpha & \longrightarrow & \mathcal{SS}_\alpha & \longrightarrow & \mathcal{DS}_\alpha \\ & & & & & & \downarrow & & \downarrow \\ & & & & & & \mathcal{S}_\alpha & \longrightarrow & \underline{\mathcal{DS}}_\alpha \end{array} \tag{13}$$

For convenience we wrote below short reminders of what these categories are:

$$\begin{array}{ccccccccc} \text{cobordisms} & \longrightarrow & \mathbb{k}\text{-linear} & \longrightarrow & \text{viewable} & \longrightarrow & \text{skein} & \longrightarrow & \text{Deligne (Karoubian)} \\ & & & & & & \downarrow & & \downarrow \\ & & & & & & \text{gligible} & \longrightarrow & \text{gligible and Karoubian} \end{array} \tag{14}$$

If α is not recognizable, we can still define categories VTFS_α, TFS_α and $\underline{\mathrm{DTFS}}_\alpha$ (in the streamlined notation, categories $\mathcal{VS}_\alpha, \mathcal{S}_\alpha$ and $\underline{\mathcal{DS}}_\alpha$), but it is not clear whether these categories may be interesting for some such α.

4. Hilbert Scheme and Recognizable Series

Recognizable series and points on the dual tautological bundle. Recognizable series α gives rise to the ideal I_α in $\mathbf{k}[T_1, T_2]$ of finite codimension $k = d_\alpha$ and the quotient algebra A_α by this ideal, see formula (10) in Section 3.2. This algebra is commutative Frobenius and comes with two generators T_1, T_2 and a non-degenerate trace. The ideal I_α describes a point in the Hilbert scheme of codimension k ideals in $\mathbb{A}^2 = \mathrm{Spec}\, \mathbf{k}[T_1, T_2]$, where

$$k = d_\alpha = \dim A_\alpha.$$

Let us specialize to the ground field $\mathbf{k} = \mathbb{C}$. Denote by Rec_k the set of recognizable series with the *syntactic ideal* I_α of codimension k and refer to $\alpha \in \mathrm{Rec}_k$ as a *recognizable series of codimension* k. Let also

$$\mathrm{Rec} := \bigsqcup_{k \geq 0} \mathrm{Rec}_k, \quad \mathrm{Rec}_{\leq n} := \bigsqcup_{k \leq n} \mathrm{Rec}_k.$$

Consider the Hilbert scheme $H^k = \mathrm{Hilb}^k(\mathbb{C}^2)$ of k points in $\mathbb{C}^2$ or, equivalently, the scheme of codimension k ideals in $\mathbb{C}[T_1, T_2]$, see [20].

Denote by $\mathcal{T}_k$ the tautological bundle over H^k whose fiber over the point associated to an ideal I of codimension k is the space $A_I = \mathbb{C}[T_1, T_2]/I$. Points of the dual bundle $\mathcal{T}_k^\vee$ above a point $I \in H^k$ describe elements of $A_I^* = \mathrm{Hom}_\mathbb{C}(A_I, \mathbb{C})$, that is, linear functionals on A_I. Let

$$\pi \ : \ \mathcal{T}_k^\vee \longrightarrow H^k$$

be the projection of the bundle onto its base. For a point $p \in \mathcal{T}_k^\vee$ the element $\pi(p) \in H^k$ is the projection of p onto the base of the bundle, and we denote by $I_{\pi(p)}$ the corresponding codimension k ideal of $\mathbb{C}[T_1, T_2]$.

The point p also defines a linear functional α_p on

$$A_{\pi(p)} \ := \ \mathbb{C}[T_1, T_2]/I_{\pi(p)}, \quad \alpha_p : A_{\pi(p)} \longrightarrow \mathbb{C},$$

associated to p. This functional lifts to a functional on $\mathbb{C}[T_1, T_2]$, which is recognizable, contains $I_{\pi(p)}$ in its kernel, and has codimension at most k. The latter functional (equivalently, recognizable power series) is also denoted α_p.

This functional has the associated ideal $I_p = I_{\alpha_p} \subset \mathbb{C}[T_1, T_2]$ of finite codimension, the largest ideal in the kernel of functional α_p on $\mathbb{C}[T_1, T_2]$. There is an inclusion of ideals

$$I_{\pi(p)} \subset I_p.$$

For a generic point p on $\mathcal{T}_k^\vee$ this inclusion is the equality $I_{\pi(p)} = I_p$, but for some points p the inclusion is proper.

Another way to describe the ideal I_p is to consider the symmetric bilinear form $(\ ,\ )_p$ on $A_{\pi(p)}$ given by

$$(x, y)_p \ := \ \alpha_p(xy), \quad x, y \in A_{\pi(p)}.$$

The kernel of the form $(\ ,\ )_p$ is an ideal I'_p in $A_{\pi(p)}$ that lifts to the above ideal I_p in $\mathbb{C}[T_1, T_2]$, and there is an isomorphism $I'_p \cong I_p/I_{\pi(p)}$. The inclusion $I_{\pi(p)} \subset I_p$ is proper precisely when I'_p is a nonzero ideal, that is, when the bilinear form $(\ ,\ )_p$ is degenerate.

These ideals are shown in the diagram below, where the two squares on the left are pull-backs. The bottom sequence is short exact, and the top row becomes exact upon replacing $I_{\pi(p)}$ by 0.

$$\begin{array}{ccccccccc}
I_{\pi(p)} & \hookrightarrow & I_p & \hookrightarrow & \mathbb{C}[T_1, T_2] & \twoheadrightarrow & A_p & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \| & & \downarrow \\
0 & \hookrightarrow & I'_p & \longrightarrow & A_{\pi(p)} & \twoheadrightarrow & A_p & \longrightarrow & 0
\end{array}$$

Denote by D_k the subset of $\mathcal{T}_k^\vee$ that consists of points p such that the inclusion $I_{\pi(p)} \subset I_p$ is proper:

$$D_k \ := \ \{p \in \mathcal{T}_k^\vee | I_{\pi(p)} \neq I_p\}.$$

Recognizable power series α_p for $p \in \mathcal{T}_k^\vee$ has codimension k (in our notations, $\alpha_p \in \mathrm{Rec}_k$) precisely when $p \in \mathcal{T}_k^\vee \backslash D_k$.

If $p \in D_k$ so that

$$\mathrm{codim}(I_p) = m < k = \mathrm{codim}(I_{\pi(p)}),$$

then recognizable power series α_p has codimension m strictly less than k and $\alpha_p \in \mathrm{Rec}_m$. For example, if $p \in H^k \subset \mathcal{T}_k^\vee$ is a point in the zero section of $\mathcal{T}_k^\vee$, so that the linear map α_p is identically zero, the ideal $I_p = \mathbb{C}[T_1, T_2]$ has zero codimension and $m = 0$. A mild confusion exists in our notations in this case (and in this case only), for then $p = \pi(p)$.

Going the other way, to a recognizable series α with the associated ideal I_α of codimension $d_\alpha = k$ as above we associate a point p_α of $\mathcal{T}_k^\vee$. It is the point in the fiber of $\mathcal{T}_k^\vee$ over the ideal I_α which describes functional α on $\mathbb{C}[T_1, T_2]$ and the induced functional on the quotient algebra $A_\alpha = A_{I_\alpha}$.

The above discussion implies the proposition below.

Proposition 4.1. *Assigning p_α to $\alpha \in \mathrm{Rec}_k$ and α_p to $p \in \mathcal{T}_k^\vee \backslash D_k$ establishes a bijection*

$$\mathrm{Rec}_k \ \cong \ \mathcal{T}_k^\vee \backslash D_k.$$

In particular, $p_{\alpha_p} = p$ and $\alpha_{p_\alpha} = \alpha$ for p and α as in the proposition, so the two assignments are mutually-inverse bijections. □

Note that the two ideals coincide, $I_{\pi(p)} = I_p$, precisely when α_p is a nondegenerate trace map on $A_{\pi(p)}$. In particular, in this case $A_{\pi(p)}$ is Frobenius. We obtain the following statement.

Proposition 4.2. *Points $p \in \mathcal{T}_k^\vee \backslash D_k$ classify isomorphism classes of data (A, ϵ, t_1, t_2): a commutative Frobenius algebra A over $\mathbb{C}$ of dimension k with a non-degenerate trace ϵ and generators t_1, t_2 of A.*

Not every commutative Frobenius algebra can be generated by two elements, of course.

Taking codimension $m \le k$ of I_p into account, one gets the following statement.

Proposition 4.3. *Associating α_p to $p \in \mathcal{T}_k^\vee$ gives a surjective map*

$$\mathcal{T}_k^\vee \longrightarrow \bigsqcup_{m=0}^{k} \mathrm{Rec}_m .$$

Restricting this map to D_k gives a surjective map

$$D_k \longrightarrow \bigsqcup_{m=0}^{k-1} \mathrm{Rec}_m ,$$

while on the complement to D_k this map is the bijection in Proposition 4.1.

Example. The set Rec_0 is a single point corresponding to the zero series α, $\alpha_{i,j} = 0, i, j \in \mathbb{Z}_+$. The ideal for this point is the entire algebra $\mathbb{C}[T_1, T_2]$. Points of Rec_1 correspond to hyperplanes (codimension one subspaces) that are ideals $J = (T_1 - \lambda_1, T_2 - \lambda_2)$ together with a nonzero functional on $\mathbb{C} \cong \mathbb{C}[T_1, T_2/J$, determined by its value λ on 1. Consequently, we can identify $\mathrm{Rec}_1 \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C}^\times$ by sending a point in Rec_1 to the triple $(\lambda_1, \lambda_2, \lambda)$.

Set-theoretic divisor D_k. Quasi-projective variety H^k admits an open cover by affine sets U_λ, over all partitions λ of k, see Theorem 18.4 in [18, Section 18.4], for example. Place partition λ in the corner of the first quadrant of the plane so that it covers nodes (i, j) of the square lattice with $0 \le i < \lambda_{j+1}$. In particular, it covers λ_1 nodes on the x-axis.

Let T_λ be the set of monomials $T_1^i T_2^j$ with $(i, j) \in \lambda$ (in particular, $|T_\lambda| = k$) and T'_λ be the set of complementary monomials, for pairs $(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \backslash \lambda$. Open set $U_\lambda \subset H^k$ consists of ideals I such that monomials in T_λ descend to a basis of $A_I = \mathbb{C}[T_1, T_2]/I$, see [18, Section 18.4] for details.

The vector bundle $\mathcal{T}_k^\vee \longrightarrow H^k$ can be trivialized over U_λ, being naturally isomorphic to the trivial bundle of functions on the set T_λ. A functional p on $\mathbb{C}[T_1, T_2]/I_{\pi(p)}$ is determined by its values on the basis elements $t \in T_\lambda$ of this quotient space.

To describe the points $p \in \mathcal{T}_k^\vee$ with $\pi(p) \in U_\lambda$ consider an arbitrary linear functional $\alpha \in (\mathbb{C}T_\lambda)^*$, given by its values

$$\alpha(T_1^i T_2^j) \in \mathbb{C}, \text{ for } T_1^i T_2^j \in T_\lambda,$$

and an ideal $I \in U_\lambda$. Such pair (α, I) trivializes a pair $(p, \pi(p))$ with $\pi(p) \in U_\lambda$. For a pair $u, v \in T_\lambda$ take the product uv, view it as an element of $A_I = \mathbb{C}[T_1, T_2]/I$, and then write it as a linear combination of elements in T_λ, allowing to apply α to it explicitly.

Consider a matrix M_α where rows and columns are labelled by elements of T_λ and put $\alpha(uv)$ as the entry at the intersection of row u and column v.

Proposition 4.4. *Point p with $\pi(p) \in U_\lambda$ is in the subset D_k iff* $\mathsf{det}(M_\alpha) = 0$.

Proof. Matrix M_α is the *Gram* or *Hankel matrix* of the bilinear form $(x, y) = \alpha(xy)$ on the associative algebra A_I in the basis T_λ. A bilinear form on a finite-dimensional algebra B given by the composition of the multiplication with a fixed linear functional on B is non-degenerate exactly when its Hankel matrix with respect to some (equivalently, any) basis is non-degenerate, that is, has a non-zero determinant. □

Condition that $\mathsf{det}(M_\alpha) = 0$ is locally a codimension one condition (given by a single equation), unless the determinant is identically zero on points $(p, \pi(p))$ with $\pi(p)$ on some irreducible component of the open subset U_λ of the Hilbert scheme. To see that the latter case does not happen, observe that a "generic" point I on the Hilbert scheme H^k corresponds to a semisimple quotient (no nilpotent elements in $\mathbb{C}[T_1, T_2]/I$), with the quotient algebra isomorphic to the product of k fields,

$$\mathbb{C}[T_1, T_2]/I \cong \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}.$$

On this quotient an open subset of linear functionals are non-degenerate, with the associated bilinear forms having trivial kernels. Indeed, a functional α on the algebra $\prod_{i=1}^k \mathbb{C}$ is non-degenerate iff each of its k coefficients is non-zero.

These observations imply the following result.

Proposition 4.5. *D_k is a set-theoretic divisor on the variety $\mathcal{T}_k^\vee$.*

It is straightforward to check that D_k comes from an actual divisor on $\mathcal{T}_k^\vee$. For a finite-dimensional $\mathbb{C}$-vector space V define a one-dimensional vector space

$$\mathsf{det}\, V \ := \ (\Lambda^{\mathrm{top}} V)^\vee = \Lambda^{\mathrm{top}}(V^\vee).$$

The determinant $\mathsf{det}\,\widehat{\alpha}$ of a bilinear form $\widehat{\alpha}\colon V \otimes V \to \mathbb{C}$ is an element $\mathsf{det}\,\widehat{\alpha} \in (\mathsf{det}\, V)^{\otimes 2}$ defined as the determinant of the matrix of $\widehat{\alpha}$. Namely, if $e_1, \ldots, e_k$ is a basis of V and $e^1, \ldots, e^k$ is the dual basis in $V^\vee$, then $e^1 \wedge \cdots \wedge e^k$ is a basis in the one-dimensional space $\mathsf{det}\, V$ and

$$\mathsf{det}\,\widehat{\alpha} := \mathsf{det}||\widehat{\alpha}(e_i, e_j)||\ (e^1 \wedge \cdots \wedge e^k)^{\otimes 2}.$$

A point $p \in \mathcal{T}_k^\vee$ defines a symmetric bilinear form $\widehat{\alpha}_p(x, y) := \alpha_p(xy)$ on the fiber $\mathcal{T}_{\pi(p)} = I_{\pi(p)}$ of the tautological bundle. The determinant of this form is an element of $(\mathsf{det}\, \mathcal{T}_{\pi(p)})^{\otimes 2}$. Hence the pullback line bundle

$$\pi^* \left((\mathsf{det} \mathcal{T})^{\otimes 2} \right) \longrightarrow \mathcal{T}_k^\vee$$

over $\mathcal{T}_k^\vee$ has a canonical section σ_{det} given by $\sigma_{\mathsf{det}}(p) := \mathsf{det}\,\widehat{\alpha}_p$. The set D_k is the divisor of zeroes of this section.

Corollary 4.6. *D_k is the divisor of zeros of the section σ_{det}.*

Each point of $\mathcal{T}_k^\vee \backslash D_k$ gives rise to recognizable series α in two variables and to the corresponding rigid symmetric monoidal categories, as discussed in the Section 3 and summarized in Section 3.4, including category TFS_α, the Deligne category DTFS_α and its gligible quotient $\underline{\mathrm{DTFS}}_\alpha$. It may be interesting to understand these categories for various α's, including finding the analogue of the classification result from [15] on when the category $\underline{\mathrm{DTFS}}_\alpha$ is semisimple.

5. Modifications

5.1. Adding closed surfaces. Category TFS can be enlarged to a category $\mathcal{C}$ with morphisms – oriented 2D cobordisms (surfaces) with corners between oriented 1D manifolds with corners. Extensions of 2D TQFTs to this category have been widely studied [2,17,19,23]. An oriented 1D manifold with corners is diffeomorphic to a disjoint union of finitely-many oriented intervals and circles. We adopt a minimalist approach and choose one manifold for each such diffeomorphism class. Consequently, objects of $\mathcal{C}$ are pairs $\mathbf{n} = (n_1, n_2)$ of non-negative integers, and an object $\mathbf{n}$ is represented by a fixed disjoint union $W(\mathbf{n}) = W(n_1, n_2)$ of n_1 intervals and n_2 circles. Morphisms from $\mathbf{n}$ to $\mathbf{m} = (m_1, m_2)$ are compact oriented 2D cobordisms M, possibly with corners, with both *horizontal* and *side* boundary and corners where these two different boundary types meet:

$$\partial M = \partial_h M \cup \partial_v M, \ \ \partial_h M = W(m_1, m_2) \sqcup (-W(n_1, n_2)).$$

Cobordisms that are diffeomorphic rel boundary define the same morphisms. Category $\mathcal{C}$ contains TFS as a subcategory.

$\mathcal{C}$ is a rigid symmetric monoidal category, with self-dual objects. The unit object $\mathbf{1}$ is the empty one-manifold $W(0, 0)$. Its endomorphism monoid is freely generated by diffeomorphism types of compact connected surfaces with boundary. The latter are classified by surfaces $S_{\ell,g}$ with ℓ boundary components and of genus g, one for each pair (ℓ, g), $\ell, g \in \mathbb{Z}_+$. The difference with endomorphisms of the unit object of TFS is that in $\mathcal{C}$ closed surfaces are allowed, which corresponds to $\ell = 0$ and surfaces $S_{0,g}$, over all $g \in \mathbb{Z}_+$.

Multiplicative evaluations β of endomorphisms of the unit object are again encoded by a power series

$$\widetilde{Z}_\beta(T_1, T_2) = \sum_{k,g\geq 0} \beta_{k,g} T_1^k T_2^g, \ \ \beta = (\beta_{k,g})_{k,g\in\mathbb{Z}_+}, \ \ \alpha_{k,g} \in \mathbf{k}, \tag{15}$$

with the first index shifted by 1 compared to evaluations for TFS. We changed the label from α in evaluations in TFS to β in $\mathcal{C}$ to make it easier to compare evaluations in these two categories. Now the coefficient

$$\beta_{k,g} = \beta(S_{k,g})$$

is the evaluation of connected genus g surface with k boundary components rather than with $k + 1$ components as in the TFS case, see earlier.

To relate these two power series encodings, in formulas (1) and (4) versus (15), start with $Z_\alpha(T_1, T_2)$ as in (4) and also form a one-variable power series

$$Z_\gamma(T_2) = \sum_{k\geq 0} \gamma_k T_2^k, \ \ \gamma_k \in \mathbf{k}.$$

To the pair (Z_α, Z_γ) assign the series

$$\widetilde{Z}_\beta(T_1, T_2) = T_1 Z_\alpha(T_1, T_2) + Z_\gamma(T_2). \tag{16}$$

Adding coefficients of Z_γ to the data provided by Z_α precisely means that we now include evaluations of closed surfaces, via coefficients γ_k (for a closed surface genus k). The scaling factor T_1 in the formula is needed to match the discrepancy in the evaluation conventions in the two categories TFS and $\mathcal{C}$. Formula (16) gives a bijection between series encoded by β and those encoded by (α, γ). Starting from $\widetilde{Z}_\beta$, one recovers Z_α and Z_γ as

$$Z_\gamma(T_2) = \widetilde{Z}_\beta(0, T_2)$$
$$Z_\alpha(T_1, T_2) = (\widetilde{Z}_\beta(T_1, T_2) - \widetilde{Z}_\beta(0, T_2))/T_1.$$

From $\mathcal{C}$ pass to its **k**-linearization $\mathbf{k}\mathcal{C}$ by allowing finite **k**-linear combinations of morphisms in $\mathcal{C}$. Given series β, we can define analogues of categories VTFS_α and TFS_α in (12). Denote these new categories by $\mathcal{VC}_\beta$ and $\mathcal{C}_\beta$:

- In $\mathcal{VC}_\beta$ one evaluates floating components to elements of **k** via β. A connected component is *floating* if it has no horizontal boundary.
- To form category $\mathcal{C}_\beta$ we quotient category $\mathcal{VC}_\beta$ (alternatively, category $\mathbf{k}\mathcal{C}$) by the two-sided ideal of *negligible morphisms*, defined in the same way as for TFS.

We say that evaluation β (or series $\widetilde{Z}_\beta$) is *recognizable* if category $\mathcal{C}_\beta$ have finite-dimensional hom spaces.

Proposition 5.1. *β is recognizable iff the power series $\widetilde{Z}_\beta$ has the form*

$$\widetilde{Z}_\beta(T_1, T_2) = \frac{\widetilde{P}(T_1, T_2)}{\widetilde{Q}_1(T_1)\widetilde{Q}_2(T_2)}, \tag{17}$$

where $\widetilde{Q}_1(T_1)$, $\widetilde{Q}_2(T_2)$ are one-variable polynomials and $\widetilde{P}(T_1, T_2)$ is a two-variable polynomial, all with coefficients in the field **k**.

It is easy to see that series β is recognizable iff hom spaces

$$\mathrm{Hom}(\mathbf{1}, (1, 0)) \ \text{ and } \ \mathrm{Hom}(\mathbf{1}, (0, 1))$$

in $\mathcal{C}_\beta$ are finite-dimensional. These are the hom spaces from the empty 1-manifold $W(0, 0)$ (representing the unit object **1**) to an interval $W(1, 0)$ and a circle $W(0, 1)$, respectively. Necessity of this condition is obvious. Vice versa, if these homs are finite-dimensional, by analogy with the proof of Proposition 3.1, there are skein relations allowing to reduce some large number of handles (respectively, holes) on a connected component to a linear combination of otherwise identical cobordisms but with fewer handles (respectively holes). The rest of the proof of Proposition 5.1 follows that of Proposition 3.1. □

Corollary 5.2. *Series β is recognizable iff the corresponding series α and γ are both recognizable.*

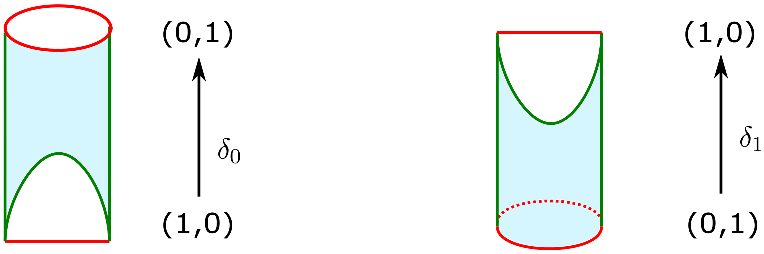

Fig. 17. Maps δ_0 and δ_1

Note that, when α and γ are recognizable, their rational function presentation may have very different denominators,

$$Z_\alpha(T_1, T_2) = \frac{P(T_1, T_2)}{Q_1(T_1)Q_2(T_2)}, \quad Z_\gamma(T_2) = \frac{P_\gamma(T_2)}{Q_\gamma(T_2)},$$

so that

$$\begin{aligned}\widetilde{Z}_\beta(T_1, T_2) &= \frac{T_1 P(T_1, T_2)}{Q_1(T_1)Q_2(T_2)} + \frac{P_\gamma(T_2)}{Q_\gamma(T_2)} \\ &= \frac{T_1 P(T_1, T_2)Q_\gamma(T_2) + Q_1(T_1)Q_2(T_2)P_\gamma(T_2)}{Q_1(T_1)Q_2(T_2)Q_\gamma(T_2)}.\end{aligned}$$

For generic polynomials, there are no cancellations and

$$\widetilde{Q}_1(T_1) = Q_1(T_1), \quad \widetilde{Q}_2(T_2) = Q_2(T_2)Q_\gamma(T_2)$$

are the denominators in the minimal rational presentation (17) for $\widetilde{Z}_\beta$.

For recognizable β, the state spaces

$$A_\beta(1, 0) := \mathrm{Hom}_{\mathcal{C}_\beta}(\mathbf{1}, (1, 0)), \quad A_\beta(0, 1) := \mathrm{Hom}_{\mathcal{C}_\beta}(\mathbf{1}, (0, 1)),$$

of homs from the unit object $\mathbf{1} = (0, 0)$ to the interval and the circle objects, respectively, are both commutative Frobenius algebras. Annuli, viewed as morphisms between $(1, 0)$ and $(0, 1)$, see Fig. 17, give linear maps

$$\delta_0 \ : \ A_\beta(1, 0) \longrightarrow A_\beta(0, 1), \qquad \delta_1 \ : \ A_\beta(0, 1) \longrightarrow A_\beta(1, 0)$$

between the underlying vector spaces.

Consider the *hole* and *handle* endomorphisms b_1, b_2 of the interval and c_1, c_2 of the circle, respectively, in Fig. 18 top.

Multiplications in algebras $A_\beta(1, 0)$ and $A_\beta(0, 1)$ are given by pants and flat pants cobordisms, see Fig. 19, where the cobordisms for the unit and trace morphisms on $A_\beta(1, 0)$ and $A_\beta(0, 1)$ are shown as well.

Take endomorphisms b_1, b_2, c_1, c_2 of the interval and circle and cap them off at the bottom with the unit morphisms ι and ι' for the interval and circle (see Fig. 19) to get elements $\underline{b}_1 = b_1\iota, \underline{b}_2 = b_2\iota$ in $A_\beta(1, 0)$ and elements $\underline{c}_1 = c_1\iota', \underline{c}_2 = c_2\iota'$ in $A_\beta(0, 1)$, shown in Fig. 18.

The analogue of Proposition 2.3 holds in $\mathcal{C}$, and the "interval" Frobenius algebra $A_\beta(1, 0)$ is generated by commuting elements $\underline{b}_1, \underline{b}_2$ (the hole and handle elements).

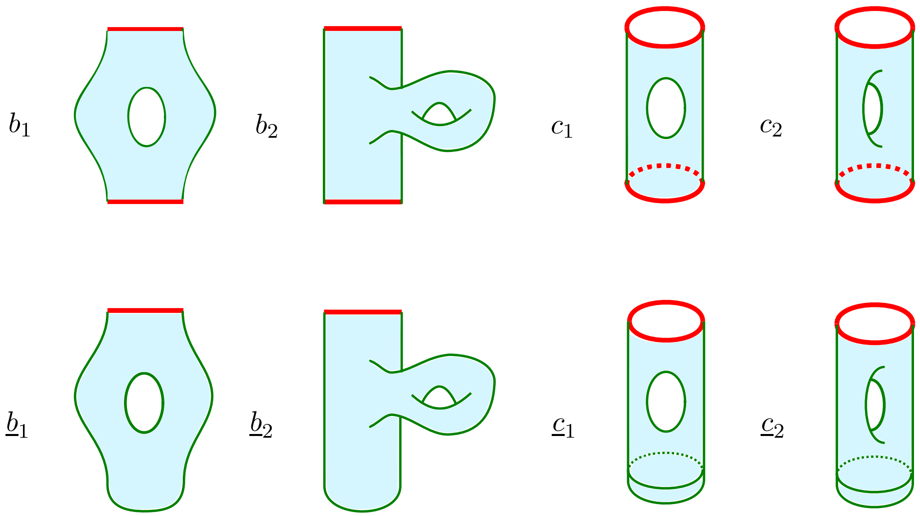

Fig. 18. Endomorphisms b_1, b_2 of the interval, endomorphisms c_1, c_2 of the circle and corrresponding elements $\underline{b}_1, \underline{b}_2$ of $A_\beta(1,0) = \mathrm{Hom}(\mathbf{1}, (1,0))$ and elements $\underline{c}_1, \underline{c}_2 \in A_\beta(0,1) = \mathrm{Hom}(\mathbf{1}, (0,1))$

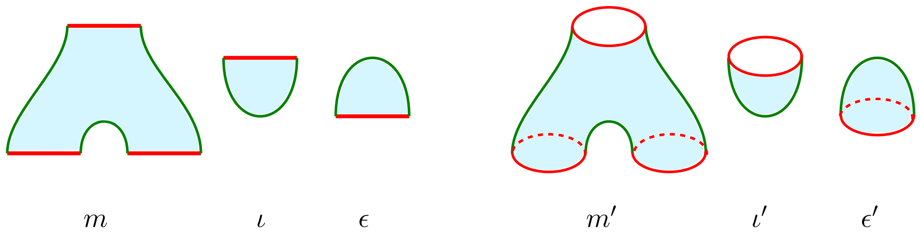

Fig. 19. Flat pants and pants cobordisms, together with the other structure maps ι, ϵ and ι', ϵ' (units ι, ι' and counits ϵ, ϵ') of commutative Frobenius algebras $A_\beta(1,0)$ and $A_\beta(0,1)$

Likewise, the "circle" Frobenius algebra $A_\beta(0,1)$ is generated by commuting hole and handle elements $\underline{c}_1$ and $\underline{c}_2$.

Endomorphisms b_1, b_3 of the interval in the category $\mathcal{C}$ are different (endomorphism b_3 is also shown in Fig. 11), but they induce the same map on $A_\beta(1,0)$, see Fig. 20. There, $x \in A_\beta(1,0)$ can be written as a linear combination of monomials $\underline{b}_1^n \underline{b}_2^m$, with b_3 acting by

$$b_3 \underline{b}_1^n \underline{b}_2^m = \underline{b}_1^{n+1} \underline{b}_2^m = b_1 \underline{b}_1^n \underline{b}_2^m.$$

Trace maps

$$\epsilon \ : \ A_\beta(0,1) \longrightarrow \mathbf{k}, \quad \epsilon' \ : \ A_\beta(1,0) \longrightarrow \mathbf{k}, \tag{18}$$

given by capping off the interval with a disk, respectively the circle with a cap, turn these two commutative algebras into Frobenius algebras (for recognizable β).

Compositions of δ_0 and δ_1 are endomorphisms of the interval and the circle in $\mathcal{C}$ (and in $\mathcal{C}_\beta$) and satisfy

$$\begin{aligned} \delta_1\delta_0 &= b_3, \quad & \delta_0\delta_1 &= c_1, \\ \delta_1 c_1 &= b_1\delta_1, \quad & \delta_1 c_2 &= b_2\delta_1, \\ \delta_0 b_1 &= c_1\delta_0, \quad & \delta_0 b_2 &= c_2\delta_0. \end{aligned}$$

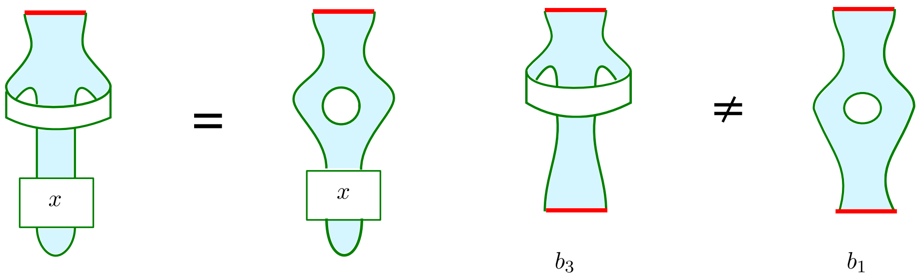

Fig. 20. $b_3 x = b_1 x$ for any $x \in A_\beta(1,0)$. $b_3 \neq b_1$ as $\mathrm{End}((1,0))$ in $\mathcal{C}$ (and in $\mathcal{C}_\beta$, in general)

In particular, maps δ_0, δ_1 intertwine the hole endomorphisms b_1, c_1 of the interval and the circle. They also intertwine the handle endomorphisms b_2, c_2 of the interval and the circle.

Their two compositions produce the hole endomorphisms of the interval and the circle.

The map δ_1 is a surjective unital homomorphism of commutative algebras, while the map δ_0 is an injective homomorphism of cocommutative coalgebras, with comultiplications given by the dual of the multiplications on these Frobenius algebras. In particular, δ_0 respects traces, in the sense that $\epsilon'\delta_0 = \epsilon$.

A recognizable power series β is encoded by a commutative Frobenius algebra (the state space of a circle) $A_\beta(0,1)$ with generators $\underline{c}_1, \underline{c}_2$ and non-degenerate trace map ϵ' such that

$$\beta_{\ell,g} = \epsilon'(\underline{c}_1^{\ell}\underline{c}_2^{g}), \quad \ell, g \in \mathbb{Z}_+. \tag{19}$$

Further unwrapping this data, to a recognizable power series β we can associate

- Two commutative Frobenius algebras $A(1,0) = A_\beta(1,0)$ and $A(0,1) = A_\beta(0,1)$ with generators $\underline{b}_1, \underline{b}_2$ and $\underline{c}_1, \underline{c}_2$, respectively (hole and handle elements).
- Non-degenerate traces ϵ and ϵ' as in (18), subject to (19) and

$$\beta_{\ell+1,g} = \epsilon(\underline{b}_1^{\ell}\underline{b}_2^{g}), \quad \ell, g \in \mathbb{Z}_+.$$

- Linear maps δ_0, δ_1:

$$A_\beta(1,0) \underset{\delta_1}{\overset{\delta_0}{\rightleftarrows}} A_\beta(0,1)$$

 that intertwine the action of handle elements $\underline{b}_2$ and $\underline{c}_2$. The hole elements are given by

$$\underline{b}_1 = \delta_1\delta_0(1), \quad \underline{c}_1 = \delta_0\delta_1(1).$$

- δ_1 is a surjective unital homomorphism of commutative algebras.

The reader may want to constrast the data coming from a recognizable series β as above, with both algebras $A_\beta(0,1)$ and $A_\beta(1,0)$ commutative Frobenius, with that given by a 2-dimensional TQFT with corners [2,17,19,23] where the Frobenius algebra B associated to the interval is not necessarily commutative and the algebra associated to the circle is related to the center of B.

To a recognizable series β there is associated a finite codimension ideal $I_\beta \subset \mathbf{k}[T_1, T_2]$ describing relations on the hole and handle endomorphisms along any component of a surface. Starting with the *viewable* category $\mathcal{VC}_\beta$, described earlier, where

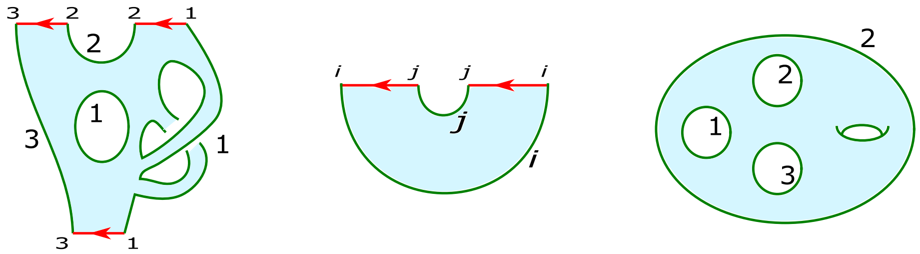

Fig. 21. Left: A morphism in $\mathrm{TFS}^{(r)}$ from the colored interval (3, 1) to the union $(3, 2) \sqcup (2, 1)$ of two colored intervals. Middle: the dual of object (i, j) is the object (j, i). Right: a connected floating component of genus 1 and the sequence (1, 2, 1). It has one boundary circle of colors 1 and 3 each and two circles of color 2

floating components are evaluated via β, we impose relations in I_β on hole and handle endomorphisms along any component. The resulting category, denoted $\mathcal{SC}_\beta$ (the *skein* category) has finite-dimensional hom spaces.

From the skein category we can pass to the already defined gligible quotient $\mathcal{C}_\beta$ by taking the quotient of $\mathcal{SC}_\beta$ by the ideal of negligible morphisms. This ideal comes from the trace on $\mathcal{SC}_\beta$ or, equivalently, from the bilinear form given by pairing of cobordisms.

Taking the additive Karoubi closure of $\mathcal{SC}_\beta$ results in the Deligne category $\mathcal{DC}_\beta$.

Taking the quotient of $\mathcal{DC}_\beta$ by the ideal of negligible morphisms produces the category $\underline{\mathcal{DC}}_\beta$. Alternatively, this category is equivalent to the additive Karoubi closure of $\mathcal{C}_\beta$, and the square below is commutative.

The following diagram summarizes these categories and functors (compare with (12), (14), and [16]).

$$\begin{array}{ccccccccc} \mathcal{C} & \longrightarrow & \mathbf{k}\mathcal{C} & \longrightarrow & \mathcal{VC}_\beta & \longrightarrow & \mathcal{SC}_\beta & \longrightarrow & \mathcal{DC}_\beta \\ & & & & & & \downarrow & & \downarrow \\ & & & & & & \mathcal{C}_\beta & \longrightarrow & \underline{\mathcal{DC}}_\beta \end{array} \quad (20)$$

Each of the four categories in the vertices of the commutative square has finite-dimensional hom spaces between its objects.

5.2. Coloring side boundaries of cobordisms. Fix a natural number $r \geq 1$ and consider a modification $\mathrm{TFS}^{(r)}$ of the category TFS where side boundaries of cobordisms are colored by numbers from 1 to r. Let $\mathbb{N}_r = \{1, \ldots, r\}$ be the set of colors. A morphisms in $\mathrm{TFS}^{(r)}$ is a tf-surface x, up to rel boundary diffeomorphisms, such that each side (or vertical) boundary component of x carries a label from $\mathbb{N}_r$. Coloring of x induces a coloring on the set of corners of x, that is, on endpoints of the one-manifold $\partial_h x$ which is the horizontal boundary of x, see Fig. 21.

Consequently, each boundary interval I of x, being oriented, gets an induced ordered sequence $(r_1(I), r_0(I))$ of two colors. We consider a skeletal version of $\mathrm{TFS}^{(r)}$, choosing only one object for each isomorphism class. An object a then is determined by the $r \times r$ matrix $M = M(a)$ with the (i, j)-entry the number of intervals in a colored (i, j).

Thus, objects a are described by $r \times r$ matrices of non-negative integers counting number of colored intervals in a. We can call these objects *r-colored* or *r-labelled* thin one-manifolds or *r-boundary colored* thin one-manifolds. An object can also be described by a list of colored intervals in it.

This skeletal version is still rigid tensor, with the obvious tensor product. The unit object $\mathbf{1} = \emptyset$ corresponds to the matrix of size 0×0.

The notion of a connected component, floating and viewable components of a morphism are defined as for TFS. Commutative monoid $\mathrm{End}(\emptyset)$ of endomorphisms of the empty one-manifold $\emptyset$ is a free abelian monoid generated by diffeomorphism classes of connected floating r-colored tf-surfaces. Such a surface S is classified by its genus $g \geq 0$ and a sequence of r non-negative integers $\mathbf{n} = (n_1, \dots, n_r)$, where n_i is the number of boundary components of color i. Denote such component by $S_{\mathbf{n},g}$. Figure 21 right shows the component $S_{(1,2,1),1}$.

For each color $i \leq r$ there is an embedding of TFS into $\mathrm{TFS}^{(r)}$ by coloring each side boundary of morphisms in TFS by i. Each horizontal interval is then an (i, i)-interval.

For a morphisms between two objects in $\mathrm{TFS}^{(r)}$ to exist, there must exist a suitable matching between the colorings of their endpoints. For instance, there are no morphisms from the empty object $\emptyset$ to (i, j) interval if $i \neq j$, since the i and j endpoints must belong to the same side interval and have the same coloring. There are morphisms from $\emptyset$ to $(i, j) \sqcup (j, i)$ but no morphisms from $\emptyset$ to $(i, j) \sqcup (i, j)$ for $i \neq j$, since matching the two i's via a side interval is not possible with our orientation setup.

As usual, denote by $\mathbf{k}\mathrm{TFS}^{(r)}$ the $\mathbf{k}$-linear version of $\mathrm{TFS}^{(r)}$, with the same objects as $\mathrm{TFS}^{(r)}$ and morphisms – kk-linear combinations of morphisms in $\mathrm{TFS}^{(r)}$.

The construction of evaluation categories and recognizable (or rational) series can be extended from TFS to $\mathrm{TFS}^{(r)}$ in a direct way.

An evaluation α is a multiplicative homomorphism from the monoid $\mathrm{End}(\emptyset)$ of floating colored tf-surfaces to a field $\mathbf{k}$. Such an evaluation is determined by its values on connected floating surfaces $S_{\mathbf{n},g}$. Let

$$Z_\alpha(T_0, \dots, T_r) = \sum_{\mathbf{n},g} \alpha_{\mathbf{n},g}\, T_0^g T^{\mathbf{n}}, \quad \alpha_{\mathbf{n},g} \in \mathbf{k} \tag{21}$$

be a formal power series in $r + 1$ variables, with

$$T^{\mathbf{n}} \;:=\; T_1^{n_1} \dots T_r^{n_r}, \quad \mathbf{n} = (n_1, \dots, n_r), \quad n_i \in \mathbb{Z}_+$$

where $T^{\mathbf{n}}$ is a monomial in $T_1, \dots, T_r$. Thus, T_0 is the *genus* variable and $T_1, \dots, T_r$ are *color* variables. Coefficient $\alpha_{\mathbf{n},g}$ at $T_0^g T_1^{n_1} \dots T_r^{n_r}$ encodes the evaluation of floating connected surface $S_{\mathbf{n},g}$.

Since each component of a tf-surface has non-empty boundary, coefficients at T_0^g, with $\mathbf{n} = \mathbf{0} = (0, \dots, 0)$ do not appear in this formal sum. We set them to zero and extend the sum to these indices by setting

$$\alpha_{\mathbf{0},g} = 0, \quad g \in \mathbb{Z}_+. \tag{22}$$

Thus, our power series has the property that

$$Z_\alpha(T_0, 0, \dots, 0) = 0. \tag{23}$$

We can also view α as a linear map of vector spaces

$$\alpha \;:\; \mathbf{k}[T_0, \dots, T_r] \longrightarrow \mathbf{k}$$

subject to condition (22), that is, $\alpha(T_0^g) = 0$, $g \geq 0$.

To α we assign the category $\mathrm{VTFS}_\alpha^{(r)}$, the quotient of $\mathbf{k}\mathrm{TFS}^{(r)}$ by the relations that a connected floating component diffeomorphic to $S_{\mathbf{n},g}$ evaluates to $\alpha_{\mathbf{n},g}$. This is the category of *viewable* r-colored tf-surfaces with the α-evaluation.

Categor $\mathrm{VTFS}_\alpha^{(r)}$ carries a natural trace form given on an endomorphism x of an object a by closing x into a floating surface $\widehat{x}$ and evaluating this surface via α, see Fig. 16, where now side boundaries are r-colored. If x is not a single cobordism but a linear combination, we use linearity of the trace to define $\mathsf{tr}_\alpha(x) = \alpha(\widehat{x})$.

Denote by J_α the two-sided ideal of negligible morphisms in $\mathrm{VTFS}_\alpha^{(r)}$ for this trace map. Define the *gligible* cobordism category $\mathrm{TFS}_\alpha^{(r)}$ as the quotient of $\mathrm{VTFS}_\alpha^{(r)}$ by the ideal J_α:

$$\mathrm{TFS}_\alpha^{(r)} := \mathrm{VTFS}_\alpha^{(r)}/J_\alpha.$$

We say that evaluation α is *rational* or *recognizable* if category $\mathrm{TFS}_\alpha^{(r)}$ has finite-dimensional hom spaces.

Proposition 5.3. *The following properties are equivalent.*

(1) *α is recognizable.*
(2) *Hom spaces Hom$(\emptyset, (i,i))$ from the empty one-manifold to the (i,i)-interval are finite-dimensional in $\mathrm{TFS}_\alpha^{(r)}$ for all $i = 1, \ldots, r$.*
(3) *Power series Z_α has the form*

$$Z_\alpha(T_0, \ldots, T_r) = \frac{P(T_0, \ldots, T_r)}{Q_0(T_0)Q_1(T_1)\ldots Q_r(T_r)},$$

where P is a polynomial in $r+1$ variables and $Q_0, \ldots, Q_r$ are one-variable polynomials, with $Q_i(0) \neq 0$, $i = 0, \ldots, r$.

Polynomials Q_i can be normalized so that $Q_i(0) = 1$ for all i. Power series Z_α also satisfies equation (23).

Proof. The proof is essentially the same as in $r = 1$ case, when all side components carry the same color and there is no need to mention colors. Proof of Proposition 3.1 carries directly to the case of arbitrary r. □

Take any floating component S and a monomial $T = T_0^g T_1^{n_1} \ldots T_r^{n_r}$. Define $S(T)$ as the surface S with additional g handles and additional n_i holes with boundary colored i, for $i = 1, \ldots, r$.

Given a linear combination $y = \sum \mu_i T_i$ of monomials, define $S(y) = \sum \mu_i S(T_i)$ as the linear combination of corresponding floating surfaces. Evaluation $\alpha(S(y))$ is an element of the ground field $\mathbf{k}$.

Given α, we can then define the syntactic ideal $I_\alpha \subset \mathbf{k}[T_0, \ldots, T_r]$. Namely, $y \in I_\alpha$ if $\alpha(S(y)) = 0$ for any floating S.

Proposition 5.4. *α is recognizable iff the ideal I_α has finite codimension in* $\mathbf{k}[T_0, \ldots, T_r]$.

Thus, for recognizable α, one can define the *skein* category $\mathrm{STFS}_\alpha^{(r)}$ as the quotient of $\mathrm{VTFS}_\alpha^{(r)}$ by the relations that inserting any $y \in I_\alpha$ into a cobordism is zero. Category $\mathrm{STFS}_\alpha^{(r)}$ has finite-dimensional hom spaces. It also has the ideal of negligible morphisms, with the quotient category isomorphic to $\mathrm{TFS}_\alpha^{(r)}$. One can then define the analogue of the Deligne category for $\mathrm{STFS}_\alpha^{(r)}$ by taking its additive Karoubi closure and define the glibigle quotient of the latter. The resulting diagram of categories and functors below mirrors diagrams (12), (20), and the corresponding diagram in [16]. The square is commutative.

$$\begin{array}{ccccccccc} \mathrm{TFS}^{(r)} & \longrightarrow & \mathbf{k}\mathrm{TFS}^{(r)} & \longrightarrow & \mathrm{VTFS}_\alpha^{(r)} & \longrightarrow & \mathrm{STFS}_\alpha^{(r)} & \longrightarrow & \mathrm{DTFS}_\alpha^{(r)} \\ & & & & & & \downarrow & & \downarrow \\ & & & & & & \mathrm{TFS}_\alpha^{(r)} & \longrightarrow & \underline{\mathrm{DTFS}}_\alpha^{(r)} \end{array} \tag{24}$$

Condition (23) on the power series Z_α seems rather unnatural. It can be removed by passing to the larger category, as in Section 5.1, where now closed components are allowed. Objects of the new category that extends $\mathrm{TFS}^{(r)}$ are disjoint unions of oriented intervals (with endpoints colored by elements of $\mathbb{N}_r$) and circles. Morphisms are two-dimensional oriented cobordisms between these collections, with side boundary intervals and side circles colored by elements of $\mathbb{N}_r$. In the definition of evaluation α we can now omit condition (22) or, equivalently, restriction (23) on the power series Z_α.

Definition and basic properties or recognizable series now work as in the $\mathrm{TFS}^{(r)}$ case. In the analogue of Proposition 5.3 for this modification, property (2) is replaced by the condition that the state space of the circle is finite-dimensional (hom space $\mathrm{Hom}(\emptyset, \mathbb{S}^1)$ is finite-dimensional). This is due to the surjection from the state space of the circle to that of the interval (i, i) induced by the map δ_1 in Fig. 18 with the side (vertical) interval colored i. It is straightforward to set up the analogue of the diagram (24) of categories and functors for this case as well, for recognizable α.

Acknowledgement. M.K. was partially supported by the NSF grant DMS-1807425 while working on this paper. Y. Q. was partially supported by the NSF grant DMS-1947532. L.R. was partially supported by the NSF grant DMS-1760578.

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Communicated by S. Gukov