Calc. Var. (2018) 57:138
https://doi.org/10.1007/s00526-018-1405-z

Calculus of Variations

Sharp frequency bounds for eigenfunctions of the Ornstein–Uhlenbeck operator

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Received: 23 September 2017 / Accepted: 28 July 2018 / Published online: 22 August 2018

Abstract
We prove sharp bounds for the growth rate of eigenfunctions of the Ornstein–Uhlenbeck operator and its natural generalizations. The bounds are sharp even up to lower order terms and have important applications to geometric flows.

Mathematics Subject Classification 53C44

1 Introduction

The Ornstein–Uhlenbeck operator (or drift Laplacian), $\mathcal{L}$ on $\mathbf{R}^n$ is the second order operator $\mathcal{L}u = \Delta u - \langle \nabla f, \nabla u \rangle$, where $f = \frac{|x|^2}{4}$. It is self-adjoint with respect to the Gaussian L^2 inner product whose norm is $\|u\|_{L^2}^2 = \int u^2 \, \mathrm{e}^{-f}$. The drift Laplacian is important in geometric flows (see, e.g., [7] and [8]) and the results here are ingredients in the proof of the René Thom gradient conjecture for the arrival time function; see [5,6].

We study here the rate of growth of drift eigenfunctions u with $\mathcal{L}\, u = -\lambda u$. It is easy to see that if $\mathcal{L}u = 0$ and $\|u\|_{L^2} < \infty$, then u must be constant. More generally, if $\mathcal{L}u = -\lambda u$ and $\|u\|_{L^2} < \infty$, then λ is a half-integer and u is a polynomial of degree 2λ. When $n = 1$, these polynomials are the Hermite polynomials and the equation $\mathcal{L}u = -\lambda u$ is known as Hermite's equation. Hermite's equation has a dichotomy where either a solution is polynomial, or it grows faster than any exponential.

Our results apply to drift Schrödinger equations with more general weight functions f and a zero-th order term V. Namely, consider solutions u of

$$\mathcal{L}_f \, u \equiv \Delta \, u - \langle \nabla f, \nabla u \rangle = -V \, u \, , \tag{1.1}$$

Communicated by L. Ambrosio.

The authors were partially supported by NSF Grants DMS 1404540, DMS 1206827 and DMS 1707270.

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for some function V and some C^1 function $f(x) = f(|x|)$, where f only depends on the distance to the origin. For the Ornstein–Uhlenbeck operator, $f(r) = \frac{r^2}{4}$ and, thus, $f'(r) = \frac{r}{2}$.

We will define a *frequency function* $U : \mathbf{R}^+ \to \mathbf{R}$ that measures the rate of polynomial growth of a function u at each scale. We delay the precise definition to (2.3), but emphasize that $U \equiv d$ if $u(x) = |x|^d$, while $U \to \infty$ if u grows exponentially. We show:

Theorem 1.1 *Suppose that $f'(r) \geq \frac{r}{2}$. Given $\epsilon > 0$ and $\delta > 0$, there exist $r_1 > 0$ such that if $U(\bar{r}_1) \geq \delta + 2\sup\{0, V\}$ for some $\bar{r}_1 \geq r_1$, then for all $r \geq R(\bar{r}_1, \epsilon, \delta, n, \sup V)$*

$$U(r) > \frac{r^2}{2} - n - 2\sup V - \epsilon. \tag{1.2}$$

We will construct examples that show that the lower bound for U is sharp in all dimensions; not only is the quadratic coefficient $\frac{1}{2}$ sharp, but also the constant $-n$ cannot be improved.

The theorem is also sharp in the dependence on the sup V. Namely, if $V = \frac{k}{2}$ is a positive half-integer, then the polynomial solutions mentioned above have U asymptotic to k. Thus, the threshold $\delta + 2\sup V$ is sharp. Furthermore, we will see that (1.2) is also sharp in sup V.

Theorem 1.1 shows that there is a sharp dichotomy for the growth: either U is bounded and u grows at most polynomially, or u grows at least like $r^{-n-2\sup V}\, e^{\frac{r^2}{4}}$.

For eigenfunctions of the Ornstein–Uhlenbeck operator, where $f(r) = \frac{r^2}{4}$, we also get a lower bound for the derivative of the frequency:

Theorem 1.2 *If $f(x) = \frac{|x|^2}{4}$ and $\mathcal{L}u + \lambda u = 0$, then either $\limsup_{r\to\infty} U(r) \leq 2|\lambda|$ or there exists R so that for all $r \geq R$*

$$U' \geq \frac{r}{2}\left(1 + \frac{r^2}{2n+4+4U(r)-r^2} - \frac{2n+8\lambda}{2n+4U(r)-r^2}\right) + \frac{O(r^{1-n})}{2n+4U(r)-r^2}, \tag{1.3}$$

where $O(r^{1-n})$ is a term that is bounded by a constant times r^{1-n}.

If we set $W = U - \frac{r^2}{4} + \frac{n}{2}$, then (1.3) becomes $W' \geq \frac{r}{8}\left(\frac{r^2}{W+1} - \frac{2n+8\lambda}{W}\right)$ up to lower order terms. Integrating leads to the bound $U \geq \frac{1}{2}r^2 - n - 1 - 2\lambda$, which is slightly worse than (1.2). However, this inequality gives a (positive) derivative bound for all values of U.

Our arguments are quite flexible and generalize. For instance:

Theorem 1.3 *Suppose that $f'(r) \geq \frac{r}{2}$. Let M be an open manifold with nonnegative Ricci curvature, Euclidean volume growth and Green's function G. Fix $x_0 \in M$ and let b be given by $b^{2-n} = G(x_0, \cdot)$. Given $\epsilon > 0$ and $\delta > 0$, there exist $r_1 > 0$ such that if $\mathcal{L}_{f(b)}u = 0$ and $U(\bar{r}) \geq \delta$ for some $\bar{r} \geq r_1$, then for all $r \geq R(\bar{r}, M, \epsilon, \delta)$*

$$U(r) > \frac{r^2}{2} - n - \epsilon. \tag{1.4}$$

In this theorem, $\mathcal{L}_{f(b)}u = \Delta - \langle \nabla u, \nabla f(b)\rangle$ and I, D, and U are defined in terms of b; see (4.1), (4.2) and (4.3).

2 The sharp lower bound for U

In this section, $f : \mathbf{R}^n \to \mathbf{R}$ is a function that only depends on the distance to the origin. With slight abuse of notation we write $f(x) = f(|x|)$ and denote $\partial_r f$ by f'.

Define quantities $I(r)$, $D(r)$, and the frequency $U(r)$ by

$$I(r) = r^{1-n} \int_{\partial B_r} u^2 \, , \tag{2.1}$$

$$D(r) = r^{2-n} \int_{\partial B_r} u u_r = r^{2-n} \, \mathrm{e}^{f(r)} \int_{B_r} \left(|\nabla u|^2 - V \, u^2 \right) \mathrm{e}^{-f} \, , \tag{2.2}$$

$$U(r) = \frac{D}{I} . \tag{2.3}$$

The frequency U is the logarithmic derivative of $\frac{1}{2} \log I$, i.e., $(\log I)' = \frac{2U}{r}$, and thus measures the polynomial rate of growth of $\sqrt{I}$. This frequency was recently used by Bernstein, [2], to study the asymptotic structure of ends of shrinkers for mean curvature flow. It is analogous to a similar quantity for harmonic functions known as Almgren's frequency function, [1], cf. [3,9–12].

An easy calculation together with that $\operatorname{div}\left(\mathrm{e}^{-f} \, \nabla v\right) = \mathrm{e}^{-f} \, \mathcal{L}_f \, v$ shows

$$\frac{d}{dr} \left(r^{1-n} \int_{\partial B_r} v \right) = r^{1-n} \int_{\partial B_r} \frac{dv}{dr} = r^{1-n} \left(\mathrm{e}^{f(r)} \int_{B_r} \mathcal{L}_f \, v \, \mathrm{e}^{-f} \right) . \tag{2.4}$$

Using (2.4) with $\mathcal{L}_f \, u = 0$ gives that the spherical average of a $\mathcal{L}_f$-harmonic function is constant in r.

Lemma 2.1 *If $\mathcal{L}_f \, u + V \, u = 0$, then*

$$I'(r) = \frac{2D(r)}{r} \, , \tag{2.5}$$

$$(\log I)'(r) = \frac{2U(r)}{r} \, , \tag{2.6}$$

$$D'(r) = \frac{2-n}{r} \, D + f'(r) \, D + r^{2-n} \int_{\partial B_r} \left(|\nabla u|^2 - V \, u^2 \right) . \tag{2.7}$$

Proof Since $\mathcal{L}_f \, u^2 = 2 \, |\nabla u|^2 - 2 \, V \, u^2$, (2.4) gives

$$I'(r) = 2 r^{1-n} \, \mathrm{e}^{f(r)} \int_{B_r} \left(|\nabla u|^2 - V \, u^2 \right) \mathrm{e}^{-f} = \frac{2D(r)}{r} . \tag{2.8}$$

This gives the first two claims. Differentiating (2.2) gives (2.7). □

Define a (non-linear) first order differential operator on positive functions g on $(0, \infty)$ by

$$P_{f,\lambda} \, g = (\log g)' + \frac{n-2}{r} - f' + \frac{g}{r} + \frac{r \, \lambda}{g} . \tag{2.9}$$

We will later use that if $f_2' \geq f_1'$, then $P_{f_1,\lambda} \, g \geq P_{f_2,\lambda} \, g$.

The key will be that U is a sub-solution of P:

Lemma 2.2 *If $\mathcal{L}_f \, u + V \, u = 0$ and $\infty > U(r) > 0$, then*

$$P_{f, \sup V} \, U \geq 0 . \tag{2.10}$$

Proof The Cauchy–Schwarz inequality

$$\frac{D^2}{r} = r^{3-2n} \left(\int_{\partial B_r} u u_r \right)^2 \leq I \, r^{2-n} \int_{\partial B_r} u_r^2 \leq I \, r^{2-n} \int_{\partial B_r} |\nabla u|^2 \tag{2.11}$$

together with (2.7) gives

$$D'(r) \geq \frac{2-n}{r} D + f'(r)\, D + \frac{U}{r} D - r \sup V \frac{D}{U}, \tag{2.12}$$

Since $(\log U)' = \frac{D'}{D} - \frac{2U}{r}$ and $D(r) > 0$, dividing (2.12) by D gives (2.10). □

The next lemma shows a maximum principle for the operator $P_{f,\lambda}$.

Lemma 2.3 *Suppose that $g,\, h : \mathbf{R} \to (0, \infty)$ are C^1 and satisfy for $r \geq r_1$*

$$P_{f,\lambda}\, h \geq 0 > P_{f,\lambda}\, g. \tag{2.13}$$

If $h(R) > g(R)$ for some $R \geq r_1$, then $h(r) > g(r)$ for all $r \geq R$.
Moreover, if $\epsilon > 0 \geq \lambda$, and g satisfies

$$-\frac{\epsilon}{r} \geq P_{f,\lambda}\, g \text{ for } r \geq r_1\,, \tag{2.14}$$

then there exists $R = R(h(r_1), g(r_1), r_1, \epsilon)$ so that $h \geq g$ for $r \geq R$.

Proof We will prove the first claim by contradiction. Suppose not, then there exists $s > R$ such that $h(s) = g(s)$ and $h(t) > g(t)$ for all $s > t \geq R$. This implies that

$$(\log h)'(s) \leq (\log g)'(s) = \frac{g'}{g}. \tag{2.15}$$

On the other hand, by assumption $P_{f,\lambda}\, h \geq 0$ and thus

$$(\log h)'(s) \geq \frac{2-n}{s} + f'(s) - \frac{h(s)}{s} - s\,\frac{\lambda}{h(s)} = \frac{2-n}{s} + f'(s) - \frac{g(s)}{s} - s\,\frac{\lambda}{g(s)}. \tag{2.16}$$

Together these two inequalities gives that $P_{f,\lambda}\, g \geq 0$ which is the desired contradiction.

The second claim will follow from the first once we show that there is some $R \geq r_1$ so that $h > g$ for some r with $R \geq r \geq r_1$. To see this, we suppose that $h \leq g$ for $r_1 \leq r \leq R$ and then get an upper bound on R. On this interval, since $\lambda \leq 0$ we get that

$$(\log h)'(s) - (\log g)'(s) \geq P_{f,\lambda}\, h - P_{f,\lambda}\, g \geq \frac{\epsilon}{s}. \tag{2.17}$$

Integrating this from r_1 to R gives

$$1 \geq \frac{h(R)}{g(R)} \geq \frac{h(r_1)}{g(r_1)} \left(\frac{R}{r_1}\right)^{\epsilon}. \tag{2.18}$$

Thus, we see that $R^{\epsilon} \leq r_1^{\epsilon} \frac{g(r_1)}{h(r_1)}$. □

Lemma 2.4 *Suppose that $f(r) = \frac{r^2}{4}$, $\epsilon > 0$ and let $g(r) = \frac{r^2}{2} - n - \epsilon - 2\lambda$, then there exists $r_1 = r_1(\epsilon, n)$ so that for $r \geq r_1$*

$$-\frac{\epsilon}{2r} \geq P_{f,\lambda}\, g\,, \tag{2.19}$$

Proof Choose r_1 so that for $r \geq r_1$

$$\frac{2(\lambda+1)}{1 - 2(n+\epsilon+2\lambda)/r^2} \leq 2 + 2\lambda + \frac{1}{2}\epsilon. \tag{2.20}$$

For $r \geq r_1$, (2.20) implies that

$$\frac{2r(\lambda+1)}{r^2-2(n+\epsilon+2\lambda)} = \frac{1}{r}\left(\frac{2(\lambda+1)}{1-2(n+\epsilon+2\lambda)/r^2}\right) \leq \frac{2+2\lambda+\frac{1}{2}\epsilon}{r}. \tag{2.21}$$

Using the definitions of f and g, we get

$$-P_{f,\lambda}\,g = \frac{2-n}{r} + f' - \frac{g}{r} - \frac{r\lambda}{g} - \frac{g'}{g} = \frac{2+\epsilon+2\lambda}{r} - \frac{2r(\lambda+1)}{r^2-2(n+\epsilon+2\lambda)} \geq \frac{\epsilon}{2r}. \tag{2.22}$$

□

Combining the two previous results and Lemma 2.2 (to see that $P_{f,\sup V}\,U \geq 0$) gives Theorem 1.1 in the case where $\lambda \leq 0$. The argument for a general λ is similar but a little more involved since we need a replacement for the second half of Lemma 2.3. We will deal with this in the next subsection.

2.1 The case $\lambda > 0$

The next lemma will replace the second half of Lemma 2.3 when $\lambda > 0$.

Lemma 2.5 *Suppose that $\lambda > 0$ and for $r \geq r_1$ we have that $g, h > 0$, $f' \geq \frac{r}{2}$, $P_{f,\lambda}\,h \geq 0$, $-\frac{\epsilon}{r} \geq P_{f,\lambda}\,g$, and $r\,g' \geq g$. If $\delta > 0$ and $r_2 \geq r_1$ satisfies $\frac{2-n}{r_2} + \frac{r_2}{2} - r_2\,\frac{\lambda}{\delta+2\lambda} > \sqrt{\lambda}$ and $h(r_2) > 2\lambda+\delta$, then there exists $R = R(r_2, g(r_2), \delta, \lambda, \epsilon)$ such that $h(r) \geq g(r)$ for $r \geq R$.*

Proof By the first part of Lemma 2.3, once h is above g it stays above. Thus, we must show that there exists R so that $h(r) \geq g(r)$ for some $r \in (r_2, R)$. We do this in steps: h gets above $\sqrt{\lambda}\,r$, then stays above $\sqrt{\lambda}\,r$ if it is below g, and finally overtakes g.

First, if $2\lambda + \delta \leq h(r) < \sqrt{\lambda}\,r$ for $r \geq r_2$, then $P_{f,\lambda}\,h \geq 0$ implies that

$$(\log h)'(r) \geq \frac{2-n}{r} + \frac{r}{2} - \sqrt{\lambda} - r\,\frac{\lambda}{\delta+2\lambda} > 0. \tag{2.23}$$

Thus, h stays above $2\lambda + \delta$. Moreover, the derivative of the middle term in (2.23) is at least

$$\frac{1}{2} - \frac{\lambda}{\delta+2\lambda} = \frac{\delta}{2\delta+4\lambda} > 0. \tag{2.24}$$

It follows that $\log h$ is not only increasing, but starts to grow at least linearly and, thus, h will overtake $\sqrt{\lambda}\,r$ in a bounded interval.

Suppose $r \geq r_2$ satisfies $\sqrt{\lambda}\,r \leq h(r) < g(r)$. Since $P_{f,\lambda}\,h \geq 0$ and $-\frac{\epsilon}{r} \geq P_{f,\lambda}\,g$, we have

$$\left(\log\frac{h}{g}\right)' = P_{f,\lambda}\,h - P_{f,\lambda}\,g + \frac{g-h}{r} - \lambda\,r\,\frac{g-h}{g\,h} \geq \frac{\epsilon}{r} + (g-h)\left(\frac{1}{r} - \frac{\lambda\,r}{g\,h}\right). \tag{2.25}$$

Therefore, using that $\sqrt{\lambda}\,r \leq h(r) < g(r)$, we have

$$\left(\log\frac{h}{g}\right)'(r) \geq \frac{\epsilon}{r}, \tag{2.26}$$

and hence, using also that $rg' \geq g$ (this is the only place where this is used), we have

$$(\log h)'(r) \geq \frac{\epsilon}{r} + (\log g)'(r) \geq \frac{1+\epsilon}{r}. \tag{2.27}$$

It follows that

$$(h - \sqrt{\lambda}\, r)' \geq \frac{(1+\epsilon)\, h(r)}{r} - \sqrt{\lambda} \geq (1+\epsilon)\sqrt{\lambda} - \sqrt{\lambda} = \epsilon\sqrt{\lambda} > 0. \tag{2.28}$$

We conclude from (2.28) that h stays above $\sqrt{\lambda}\, r$ if h is below g. Finally, (2.26) implies that $\frac{h}{g}$ grows at least polynomially in this region and, thus, h must overtake g in a bounded interval. This completes the proof. □

We can now get rid of the assumption that $r\, g' \geq g$ in Lemma 2.5 to get:

Theorem 2.1 *Suppose that* $\lambda > 0$ *and for* $r \geq r_1$ *we have that* $g, h > 0$, $f' \geq \frac{r}{2}$, $P_{f,\lambda}\, h \geq 0$, $-\frac{\epsilon}{r} \geq P_{f,\lambda}\, g$, *then there exists* $r_2 > 0$ *so that if* $h(s) > 2\lambda + \delta$ *for some* $s \geq r_2$, *then there exists* R *so that* $h(r) \geq g(r)$ *for* $r \geq R$.

In particular, $h(r) \geq \frac{r^2}{2} - n - 2\lambda - \epsilon$ *for* $r \geq R$.

Proof We show the second claim first and then use it to show the first claim. To do that note that if $g_0 = \frac{r^2}{2} - n - 2\lambda - \epsilon$, then $r\, g_0' = r^2 \geq g_0$. Moreover, Lemma 2.4 gives $-\frac{\epsilon}{2r} \geq P_{f,\lambda}\, g_0$. It follows from Lemma 2.5 that for some $R > 0$ and all $r > R$ we have that $h(r) \geq \frac{r^2}{2} - n - 2\lambda - \epsilon$.

To show the first claim, note that in the proof of Lemma 2.5 the only place where the assumption $r\, g' \geq g$ was used was to show that there exists some R so that once $r \geq R$ and $h(r) \geq \sqrt{\lambda}\, r$ the function h would stay above the function $\sqrt{\lambda}\, r$. However, this follows from $h(r) \geq \frac{r^2}{2} - n - 2\lambda - \epsilon$ for r large enough. □

Proof of Theorem 1.1 We have already proven the case $\lambda \leq 0$. The case $\lambda > 0$ follows from Lemma 2.2 and Theorem 2.1. □

2.2 Sharpness of Theorem 1.1

The next theorem uses standard solutions of Hermite's equation to show that Theorem 1.1 is sharp even up to the lower order term.

Theorem 2.2 *For every* n *and* $k \in \mathbf{Z}$, *there is a function* v *on* $\mathbf{R}^n$ *with* $\mathcal{L}v = -\frac{k}{2}\, v$ *whose frequency* U *goes to infinity but for every* $\epsilon > 0$ *has a sequence* r_i *going to infinity with*

$$U(r_i) \leq \frac{1}{2}\, r_i^2 - n - k + \epsilon. \tag{2.29}$$

The second order ODE $\mathcal{L}\, u = 0$ on $\mathbf{R}$, where $f(r) = \frac{r^2}{4}$, has a two-parameter family of solutions. The first solution is a constant. The second, $u_0(x)$, can be normalized to have $u_0(0) = 0$ and $u_0'(0) = 1$. The next lemma shows that $I(r) \approx \frac{1}{r}\, e^{\frac{r^2}{4}}$.

Lemma 2.6 *The function* u_0 *is odd,* $u_0'(x) = e^{\frac{x^2}{4}}$, *and for* $x \geq 2$

$$e^{\frac{x^2}{4}} \leq x\, u_0(x) \leq 6\, e^{\frac{x^2}{4}}. \tag{2.30}$$

Moreover, there are functions u_k *for all* $k \in \mathbf{Z}$ *with* $\mathcal{L}u_k = -\frac{k}{2}\, u_k$, *so that* $u_k' = u_{k-1}$ *and, furthermore, there are constants* c_k *so that*

$$|u(x)| \leq c_k\, |x|^{-k-1}\, e^{\frac{x^2}{4}} \text{ for } 1 \leq |x|. \tag{2.31}$$

Proof Since $\mathcal{L}u_0 = u_0'' - \frac{x}{2}\,u_0' = \mathrm{e}^{\frac{x^2}{4}}\left(u_0'\mathrm{e}^{-\frac{x^2}{4}}\right)'$, we see that $\left(u_0'\mathrm{e}^{-\frac{x^2}{4}}\right)$ is constant. Using the normalization $u_0'(0) = 1$, the constant is one. For the lower bound given $r > 2$, we have

$$r\,u_0(r) = r\int_0^r \mathrm{e}^{\frac{x^2}{4}}\,dx \geq \int_0^r x\,\mathrm{e}^{\frac{x^2}{4}}\,dx = 2\,\mathrm{e}^{\frac{x^2}{4}}\big|_0^r = 2\,\mathrm{e}^{\frac{r^2}{4}} - 2 \geq \mathrm{e}^{\frac{r^2}{4}}. \tag{2.32}$$

To get the upper bound, we divide the integral into three parts

$$\begin{aligned} u_0(r) &\leq \int_0^1 \mathrm{e}^{\frac{x^2}{4}}\,dx + \int_1^{r/2} x\,\mathrm{e}^{\frac{x^2}{4}}\,dx + \frac{2}{r}\int_{r/2}^r x\,\mathrm{e}^{\frac{x^2}{4}}\,dx \leq \mathrm{e}^{\frac{1}{4}} + 2\left(\mathrm{e}^{\frac{r^2}{16}} - \mathrm{e}^{\frac{1}{4}}\right) + \frac{4}{r}\mathrm{e}^{\frac{r^2}{4}} \\ &\leq 2\,\mathrm{e}^{\frac{r^2}{16}} + \frac{4}{r}\mathrm{e}^{\frac{r^2}{4}} \leq \frac{6}{r}\mathrm{e}^{\frac{r^2}{4}}, \end{aligned} \tag{2.33}$$

where the last inequality used $r^{-1}\,\mathrm{e}^{\frac{3r^2}{16}}$ is increasing for $r \geq 2$ and $\mathrm{e}^{\frac{r^2}{16}} \leq \frac{1}{r}\mathrm{e}^{\frac{r^2}{4}}$ at $r = 2$.

For $k \geq 0$, we inductively define

$$u_{k+1}(x) = \int_0^x u_k(s)\,ds + d_{k+1}\,, \tag{2.34}$$

where the constant d_{k+1} is chosen to make $\mathcal{L}u_{k+1} = -\frac{k+1}{2}\,u_{k+1}$. To see that we can choose d_{k+1} so that it satisfies the equation, note that

$$\begin{aligned} \left(\mathcal{L}u_{k+1} + \frac{k+1}{2}\,u_{k+1}\right)' &= \left(u_k' - \frac{x}{2}\,u_k + \frac{k+1}{2}\,u_{k+1}\right)' \\ &= u_k'' - \frac{x}{2}\,u_k' - \frac{1}{2}\,u_k + \frac{k+1}{2}\,u_k = 0. \end{aligned} \tag{2.35}$$

Using integration by parts, it is easy to see that u_{k+1} grows one degree slower than u_k and, thus, satisfies (2.31).

We construct the u_k's for k inductively for $k < 0$ by defining $u_k = u_{k+1}'$. Differentiating the equation $\mathcal{L}u_{k+1} = -\frac{k+1}{2}\,u_{k+1}$ gives that $\mathcal{L}u_k = -\frac{k}{2}\,u_k$. The equation for u_{k+2} gives

$$-\frac{k+2}{2}\,u_{k+2} = \mathcal{L}\,u_{k+2} = u_{k+2}'' - \frac{x}{2}\,u_{k+2}' = u_k - \frac{x}{2}\,u_{k+1}. \tag{2.36}$$

Since we already know that u_{k+1} and u_{k+2} satisfy (2.31), it follows that so does u_k. □

Proof of Theorem 2.2 It suffices to construct v_k with $\mathcal{L}v_k = -\frac{k}{2}\,v_k$ where v_k grows at least exponentially and has (for all x sufficiently large)

$$|v_k| \leq C\,\mathrm{e}^{\frac{|x|^2}{2}}\,|x|^{-k-n}. \tag{2.37}$$

This is because the failure of (2.29) for all r_i larger than some fixed R implies $\mathrm{e}^{\frac{|x|^2}{2}}\,|x|^{\epsilon-k-n}$ growth $r \geq R$, contradicting (2.37).

The function u_k from Lemma 2.6 satisfies (2.37) for $n = 1$. For $n > 1$, we set

$$v_k(x_1, \ldots, x_n) = u_k(x_1)u_0(x_2)\ldots u_0(x_n). \tag{2.38}$$

□

3 Lower bound for U'

In this section, we specialize to the Ornstein–Uhlenbeck operator $\mathcal{L}$ where $f(r) = \frac{r^2}{4}$. In this case, $\mathcal{L} f = \frac{n}{2} - f$, $|\nabla f|^2 = f$, and the Hessian of f is diagonal with $f_{ij} = \frac{1}{2}\delta_{ij}$.

The next lemma is a drift version of the classical Rellich identity that is used to prove monotonicity of Almgren's frequency for harmonic functions.

Lemma 3.1 *Suppose that $\mathcal{L} u + V u = 0$ on $\mathbf{R}^n$. Given $r > 0$, we have*

$$2r \int_{\partial B_r} u_r^2 - r \int_{\partial B_r} \left(|\nabla u|^2 - V u^2\right) = (2-n)\, e^{\frac{r^2}{4}} \int_{B_r} |\nabla u|^2 e^{-f} + 2 e^{\frac{r^2}{4}} \int_{B_r} |\nabla u|^2 f e^{-f} + 2 e^{\frac{r^2}{4}} \int_{B_r} V u^2 \left(\frac{n}{2} - f\right) e^{-f} + 2 e^{\frac{r^2}{4}} \int_{B_r} u^2 \langle \nabla V, \nabla f\rangle e^{-f}. \tag{3.1}$$

Proof Using that $f_{ij} = \frac{1}{2}\delta_{ij}$, the divergence of $\langle \nabla f, \nabla u\rangle \nabla u - \frac{1}{2}|\nabla u|^2 \nabla f$ is

$$\left(f_i u_i u_j - \frac{1}{2} u_i^2 f_j\right)_j = f_i u_i u_{jj} + f_i u_{ij} u_j + f_{ij} u_i u_j - \frac{1}{2} u_i^2 f_{jj} - u_{ij} u_i f_j = f_i u_i u_{jj} + \frac{2-n}{4} u_i^2. \tag{3.2}$$

In particular, since $\operatorname{div}_f X \equiv \mathrm{e}^f \operatorname{div}\left(\mathrm{e}^{-f} X\right) = \operatorname{div} X - \langle \nabla f, X\rangle$, we see that

$$\operatorname{div}_f \left(\langle \nabla f, \nabla u\rangle \nabla u - \frac{1}{2}|\nabla u|^2 \nabla f\right) = \langle \nabla f, \nabla u\rangle \mathcal{L} u + \frac{2-n}{4}|\nabla u|^2 + \frac{1}{2}|\nabla u|^2 f,$$

where the equality also used that $|\nabla f|^2 = f$. The divergence theorem gives

$$2r \int_{\partial B_r} u_r^2 - r \int_{\partial B_r} |\nabla u|^2 = (2-n)\, \mathrm{e}^{\frac{r^2}{4}} \int_{B_r} |\nabla u|^2 \mathrm{e}^{-f} + 2\mathrm{e}^{\frac{r^2}{4}} \int_{B_r} |\nabla u|^2 f \mathrm{e}^{-f} + 4 \mathrm{e}^{\frac{r^2}{4}} \int_{B_r} \langle \nabla f, \nabla u\rangle \mathcal{L} u\, \mathrm{e}^{-f}. \tag{3.3}$$

The lemma follows from this and taking div_f of $\frac{1}{2} V u^2 \nabla f$ to get

$$\int_{B_r} \langle \nabla f, \nabla u\rangle\, \mathcal{L} u\, \mathrm{e}^{-f} = -\frac{1}{2} \int_{B_r} V \langle \nabla f, \nabla u^2\rangle \mathrm{e}^{-f} = \frac{1}{2} \int_{B_r} V u^2 \left(\frac{n}{2} - f\right) \mathrm{e}^{-f} - \frac{r}{4} \mathrm{e}^{-\frac{r^2}{4}} \int_{\partial B_r} V u^2 + \frac{1}{2} \int_{B_r} u^2 \langle \nabla V, \nabla f\rangle \mathrm{e}^{-f}. \tag{3.4}$$

□

We specialize next to drift eigenfunctions, i.e., where $V = \lambda$ is constant.

Lemma 3.2 *If $\mathcal{L} u = -\lambda u$ on $\mathbf{R}^n$, then*

$$D'(r) \geq \frac{r}{2} D + 2\frac{U D}{r} - 2 e^{\frac{r^2}{4}} r^{1-n} \int_{B_r} \left(|\nabla u|^2 - \lambda u^2\right) f e^{-f} - 2\lambda e^{\frac{r^2}{4}} r^{1-n} \int_{B_r} u^2 e^{-f}, \tag{3.5}$$

Proof Multiplying Lemma 3.1 by r^{1-n} gives that

$$2r^{2-n}\int_{\partial B_r} u_r^2 - r^{2-n}\int_{\partial B_r}\left(|\nabla u|^2 - \lambda u^2\right) = (2-n)\,\frac{D}{r} + 2\,e^{\frac{r^2}{4}}\,r^{1-n} \times \int_{B_r}\left(|\nabla u|^2 - \lambda u^2\right) f\,e^{-f} + 2\lambda\,e^{\frac{r^2}{4}}\,r^{1-n}\int_{B_r} u^2\,e^{-f}. \tag{3.6}$$

Using this in the formula for D' from Lemma 2.1 gives

$$\begin{aligned} D'(r) &= \frac{2-n}{r}\,D + \frac{r}{2}\,D + r^{2-n}\int_{\partial B_r}\left(|\nabla u|^2 - \lambda u^2\right) \\ &= \frac{r}{2}\,D + 2r^{2-n}\int_{\partial B_r} u_r^2 - 2\,e^{\frac{r^2}{4}}\,r^{1-n}\int_{B_r}\left(|\nabla u|^2 - \lambda u^2\right) f\,e^{-f} \\ &\quad - 2\lambda e^{\frac{r^2}{4}}\,r^{1-n}\int_{B_r} u^2\,e^{-f}. \end{aligned} \tag{3.7}$$

The lemma follows from this since $r^{2-n}\int_{\partial B_r} u_r^2 \geq \frac{UD}{r}$ by (2.11). □

The next corollary shows that U is monotone for drift-harmonic functions.

Corollary 3.1 *If $\mathcal{L}u = 0$ on $\mathbf{R}^n$, then*

$$(\log U)' \geq \frac{2\int_{B_r}|\nabla u|^2\left(\frac{r^2}{4} - f\right)e^{-f}}{r\int_{B_r}|\nabla u|^2\,e^{-f}} \geq 0. \tag{3.8}$$

Proof Dividing by D in (3.5) with $\lambda = 0$, we see that

$$(\log U)' \geq \frac{r}{2} - \frac{2\int_{B_r}|\nabla u|^2\,f\,e^{-f}}{r\int_{B_r}|\nabla u|^2\,e^{-f}} = \frac{2\int_{B_r}|\nabla u|^2\left(\frac{r^2}{4} - f\right)e^{-f}}{r\int_{B_r}|\nabla u|^2\,e^{-f}} \geq 0. \tag{3.9}$$

□

When u is not drift harmonic, then we will need to rewrite the right hand side of equation (3.5). This is done next (we record the result for a general V).

Lemma 3.3 *If $\mathcal{L}\,u + V\,u = 0$ on $\mathbf{R}^n$, then*

$$e^{\frac{r^2}{4}}r^{1-n}\int_{B_r}\left(|\nabla u|^2 - V\,u^2\right) f\,e^{-f} = \frac{r}{4}\,(D - I) + \frac{1}{2}\,e^{\frac{r^2}{4}}r^{1-n}\int_{B_r} u^2\left(\frac{n}{2} - f\right)e^{-f}. \tag{3.10}$$

Proof Observe first that since $\mathcal{L}u = -V\,u$, $\frac{r^{3-n}}{4}\int_{\partial B_r} u u_r = \frac{r}{4}\,D$, and

$$\mathrm{div}_f\,(u\,f\,\nabla u) = \left(|\nabla u|^2 - V\,u^2\right) f + u\,\langle\nabla u, \nabla f\rangle\,, \tag{3.11}$$

we have

$$e^{\frac{r^2}{4}}r^{1-n}\int_{B_r}\left(|\nabla u|^2 - V\,u^2\right) f\,e^{-f} = \frac{r}{4}\,D - e^{\frac{r^2}{4}}r^{1-n}\int_{B_r} u\langle\nabla u, \nabla f\rangle\,e^{-f}. \tag{3.12}$$

Next since $\operatorname{div}_f(u^2 \nabla f) = 2u\langle \nabla u, \nabla f\rangle + u^2\, \mathcal{L} f = 2u\langle \nabla u, \nabla f\rangle + u^2 \left(\frac{n}{2} - f\right)$, we have

$$\mathrm{e}^{\frac{r^2}{4}} r^{1-n} \int_{B_r} u\langle \nabla u, \nabla f\rangle\, \mathrm{e}^{-f} = \frac{r}{4} I - \frac{1}{2} \mathrm{e}^{\frac{r^2}{4}} r^{1-n} \int_{B_r} u^2 \left(\frac{n}{2} - f\right) \mathrm{e}^{-f}. \tag{3.13}$$

Combining these two equations gives the claim. □

As a corollary, we get a lower bound for U'.

Corollary 3.2 *If $\mathcal{L} u + \lambda u = 0$ on $\mathbf{R}^n$, then*

$$U'(r) \geq \frac{r}{2} + I^{-1}(r)\, e^{\frac{r^2}{4}}\, r^{1-n} \int_{B_r} u^2 \left(f - \frac{n}{2} - 2\lambda\right) e^{-f}. \tag{3.14}$$

Furthermore, given $\delta > 0$, there exists r_1 so that if $U(\bar{r}) \geq \delta + 2|\lambda|$ for some $\bar{r} \geq r_1$, then there exists R so that for all $r \geq R$

$$U'(r) \geq \frac{r}{2}. \tag{3.15}$$

Proof Combining Lemmas 3.2 and 3.3 gives

$$D'(r) \geq 2\frac{D\,U}{r} + \frac{r}{2} I + \mathrm{e}^{\frac{r^2}{4}}\, r^{1-n} \int_{B_r} u^2 \left(f - \frac{n}{2} - 2\lambda\right) \mathrm{e}^{-f}. \tag{3.16}$$

The first claim follows from this since $U' = \left(\frac{D'}{D} - \frac{2U}{r}\right) U$.

To prove the second claim, we just need to show that there is some $R^2 \geq 2n + 8\lambda$ with

$$\int_{B_R} u^2 \left(f - \frac{n}{2} - 2\lambda\right) \mathrm{e}^{-f} \geq 0. \tag{3.17}$$

This follows immediately since $I(r)\, \mathrm{e}^{-\frac{r^2}{4}}$ grows rapidly by Theorem 1.1. □

Proof (of Theorem 1.2.) We can assume that $\limsup_{r\to\infty} U(r) > 2|\lambda|$. Thus, the second part of Corollary 3.2 applies and $U'(t) \geq \frac{t}{2}$ for $t > r_0$. In particular, $W(t) = U(t) - \frac{t^2}{4}$ satisfies $W' \geq 0$ for $t \geq r_0$. After possibly increasing r_0, we can assume that $r_0^2 > 2n + 8\lambda$ and, moreover, that $W(r_0) > 0$ (using Theorem 1.1).

By Lemma 2.1, for $r > s > r_0$

$$\log \frac{I(s)}{I(r)} = -2\int_s^r \frac{U}{t}\, dt \geq \frac{s^2 - r^2}{4} - 2\, W(r) \int_s^r \frac{1}{t}\, dt = \frac{s^2 - r^2}{4} \log \left(\frac{s}{r}\right)^{2\, W(r)}. \tag{3.18}$$

It follows that for any constant $c \leq r_0^2$

$$\begin{aligned} \mathrm{e}^{\frac{r^2}{4}} \frac{r^{1-n}}{I(r)} \int_{r_0}^r \left(s^2 - c\right) s^{n-1}\, I(s)\, \mathrm{e}^{-\frac{s^2}{4}}\, ds &\geq r^{1-n-2\, W(r)} \\ &\quad \times \int_{r_0}^r \left(s^{n+1+2\, W(r)} - c\, s^{n-1+2\, W(r)}\right) ds \\ &= \frac{r^3}{n+2+2\, W(r)} - \frac{c\, r}{n+2\, W(r)} + \frac{1}{n+2+2\, W(r)} O(r^{1-n-2W(r)}), \end{aligned} \tag{3.19}$$

where $O(r^{1-n-2W(r)})$ is a term that is bounded by a constant (depending on r_0) times $r^{1-n-2W(r)}$. Inserting this in Corollary 3.2 with $c = 2n + 8\lambda$ gives the claim. □

4 Drift harmonic functions on open manifolds

In this section, we will show a natural generalization (Theorem 1.3) of (1.2) to open manifolds with nonnegative Ricci curvature and Euclidean volume growth. In fact, the assumptions on the Ricci curvature and volume growth are only used to show that the function b defined below is proper.

Let again f be a function on $(0, \infty)$ with $f' \geq \frac{r}{2}$. Suppose that M is an open manifold, $b : M \to \mathbf{R}$ is a proper function. For a function $u : M \to \mathbf{R}$, define (cf. [3] and [4])

$$I(r) = r^{1-n} \int_{b=r} u^2 \, |\nabla b| \, , \tag{4.1}$$

$$D(r) = r^{2-n} \, \mathrm{e}^{f(r)} \int_{b \leq r} |\nabla u|^2 \, \mathrm{e}^{-f(b)} \, , \tag{4.2}$$

$$U(r) = \frac{D(r)}{I(r)} . \tag{4.3}$$

We set $\mathcal{L}_f \, u = \Delta \, u - \langle \nabla u, \nabla f(b) \rangle$. It follows that

$$\begin{aligned} I'(r) &= r^{1-n} \int_{b=r} \frac{\nabla b}{|\nabla b|} u^2 + \int_{b=r} u^2 \, \frac{\nabla b}{|\nabla b|^2} \left(r^{1-n} \, |\nabla b| \, d \, \mathrm{Vol} \right) \\ &= r^{1-n} \, \mathrm{e}^{f(r)} \int_{b \leq r} \mathcal{L}_f \, u^2 \, \mathrm{e}^{-f(b)} + \int_{b=r} u^2 \, \frac{\nabla b}{|\nabla b|^2} \left(r^{1-n} \, |\nabla b| \, d \, \mathrm{Vol} \right) , \end{aligned} \tag{4.4}$$

where $d \, \mathrm{Vol}$ is the volume element of the level set of b. The co-area formula gives

$$D'(r) = \frac{2-n}{r} \, D + f'(r) \, D + r^{2-n} \int_{b=r} \frac{|\nabla u|^2}{|\nabla b|} . \tag{4.5}$$

If $\mathcal{L}_f \, u = 0$, then $\mathcal{L}_f \, u^2 = 2 \, |\nabla u|^2$. Therefore

$$D(r) = \frac{1}{2} \, r^{2-n} \, \mathrm{e}^{f(r)} \int_{b \leq r} \mathcal{L}_f \, u^2 \, \mathrm{e}^{-f(b)} = r^{2-n} \int_{b=r} u \left\langle \nabla u, \frac{\nabla b}{|\nabla b|} \right\rangle . \tag{4.6}$$

The Cauchy–Schwarz inequality (cf. (2.11)) gives for $\mathcal{L}_f \, u = 0$

$$\frac{D^2}{r} = r^{3-2n} \left(\int_{b=r} u u_r \right)^2 \leq I \, r^{2-n} \int_{b=r} \frac{|\nabla u|^2}{|\nabla b|} . \tag{4.7}$$

It follows that for $\mathcal{L}_f \, u = 0$

$$D'(r) = \frac{2-n}{r} \, D + f' \, D + r^{2-n} \int_{b=r} \frac{|\nabla u|^2}{|\nabla b|} \geq \frac{2-n}{r} \, D + f' \, D + \frac{U}{r} \, D . \tag{4.8}$$

If $\mathcal{L}_f \, u = 0$ and

$$\frac{\nabla b}{|\nabla b|^2} \left(r^{1-n} \, |\nabla b| \, d \, \mathrm{Vol} \right) = 0 \, , \tag{4.9}$$

then $I' = 2 \, \frac{D}{r}$ and $(\log I)' = \frac{2U}{r}$. Hence, by (4.8)

$$P_{f,0} \, U \geq 0 . \tag{4.10}$$

By [3] if b^{2-n} is harmonic, then (4.9) holds. This is due to the following:

Lemma 4.1 *Let d Vol denote the volume element of the level set of a function v, then*

$$\frac{\nabla v}{|\nabla v|^2}\,(|\nabla v|\, d\, Vol) = \frac{\Delta\, v}{|\nabla v|}\, d\, Vol. \tag{4.11}$$

Proof An easy calculation shows that the change in volume element (of the level set) is

$$\operatorname{div}\left(\frac{\nabla v}{|\nabla v|^2}\right) - \langle \nabla_{\frac{\nabla v}{|\nabla v|}}\left(\frac{\nabla v}{|\nabla v|^2}\right), \frac{\nabla v}{|\nabla v|}\rangle = \frac{\Delta\, v}{|\nabla v|^2} - \frac{\nabla v(|\nabla v|)}{|\nabla v|^3}. \tag{4.12}$$

From this the claim follows. □

It follows from (4.10) together with Theorem 2.1 that:

Theorem 4.1 *Suppose that $f' \geq \frac{r}{2}$. Given $\epsilon > 0$ and $\delta > 0$, if $\Delta\, b^{2-n} = 0$, then there exist $r_1 > 0$ so that if $U(\bar{r}) \geq \delta$ for some $\bar{r} \geq r_1$ and $\mathcal{L}_f u = 0$, then for all $r \geq R(\bar{r}, M, \epsilon, \delta)$*

$$U(r) > \frac{1}{2} r^2 - n - \epsilon. \tag{4.13}$$

In particular, it follows from [3] that if M is an open manifold with nonnegative Ricci curvature and Euclidean volume growth and b is given by $b^{2-n} = G$, then b is proper and thus the conclusion of Theorem 4.1 holds giving Theorem 1.3. (Here G is the Green's function.)

5 Approximation of eigenfunctions

Theorem 1.1 implies that if $\mathcal{L}u = -\lambda u$ on $\mathbf{R}^n$, then either u grows at most polynomially or at least as fast as $r^{-p} e^{\frac{r^2}{4}}$ for some power p. In the first case, $\|u\|_{L^2} < \infty$, so u is a polynomial and λ a half-integer. The next theorem gives a local version of this; we will see a more general version of this in the next section.

Theorem 5.1 *Given $k \in \mathbf{Z}$ and R_0, there exist C and R_1 so that if $\mathcal{L}u = -\frac{k}{2} u$ on B_R for some $R \geq R_1$, then there is a polynomial v of degree at most k so that*

$$\sup_{B_{R_0}} |u - v|^2 \leq C\, R^{4n-1+\max\{0,2k+2\}}\, e^{-\frac{R^2}{2}} \int_{B_{R+\frac{1}{R}} \setminus B_{R-\frac{1}{R}}} u^2. \tag{5.1}$$

Proof We will prove this in two steps. Suppose first that $k \leq -1$. Lemma 2.2 gives

$$(\log U)' \geq \frac{2-n}{r} + \frac{r}{2} + \frac{r}{2U} - \frac{U}{r}. \tag{5.2}$$

We will show first that U goes above n on any interval $[r_0, r_0 + 1]$ for $r_0 \geq 2n$. To see this, suppose that $U \leq n$ on such an interval and use (5.2) to get that

$$U' \geq \frac{r}{2} + U\left(\frac{2-n}{r} + \frac{r}{2} - \frac{n}{r}\right) > n. \tag{5.3}$$

This is impossible since $0 \leq U \leq n$, giving the claim. Thus, Theorem 1.1 gives $\bar{R}$ depending on n so that $U(r) > \frac{r^2}{2} - n$ for all $r \geq \bar{R}$. Given $r \geq \bar{R}$, integrating this from r to R gives

$$\log \frac{I(R)}{I(r)} \geq 2\int_r^R \left(\frac{s}{2} - \frac{n}{s}\right) ds = \frac{1}{2}\left(R^2 - r^2\right) - 2n \log \frac{R}{r}. \tag{5.4}$$

Letting $r = \min\{\bar{R}, 2R_0\}$, exponentiating and applying elliptic estimates gives

$$\sup_{B_{R_0}} |u|^2 \leq c\, I(r) \leq C\, R^{2n}\, \mathrm{e}^{-\frac{R^2}{2}}\, I(R). \tag{5.5}$$

The case $k \leq -1$ follows from this since $I(R) \leq c\, R^{2-n} \int_{B_{R+\frac{1}{R}} \setminus B_R} u^2$.

Suppose now that $k \geq 0$ and let w be any $(k+1)$-st partial derivative of u. It follows that $\mathcal{L} w = -\frac{1}{2}\, w$, so (5.5) implies that

$$\sup_{B_{R_0}} |\nabla^{k+1} u|^2 \leq C\, R^{2n}\, \mathrm{e}^{-\frac{R^2}{2}} \int_{\partial B_R} |\nabla^{k+1} u|^2. \tag{5.6}$$

Elliptic estimates on balls of radius R^{-1} centered on ∂B_R give that

$$\sup_{\partial B_R} |\nabla^{k+1} u|^2 \leq C\, R^{2k+2+n} \int_{B_{R+\frac{1}{R}} \setminus B_{R-\frac{1}{R}}} u^2 \tag{5.7}$$

The theorem follows with v given by the degree k Taylor polynomial for u at 0. □

6 Approximate eigenfunctions on cylinders

In this section, we let $M = N \times \mathbf{R}^n$ be a product manifold where N is closed. Let x be coordinates on $\mathbf{R}^n$, define $f = \frac{|x|^2}{4}$ and the drift Laplacian $\mathcal{L} = \Delta - \frac{1}{2} \nabla_x = \Delta_N + \mathcal{L}_{\mathbf{R}^n}$. Given a function u, we define I and D by

$$I(r) = r^{1-n} \int_{|x|=r} u^2, \tag{6.1}$$

$$D(r) = r^{2-n} \int_{|x|=r} u\, u_r = \mathrm{e}^{\frac{r^2}{4}}\, r^{2-n} \int_{|x|<r} \left(|\nabla u|^2 + u \mathcal{L} u\right) \mathrm{e}^{-f}. \tag{6.2}$$

Here u_r denotes the normal derivative of u on the level set $|x| = r$. Since N is compact, f is proper and the integrals exist. It is easy to see that $I' = \frac{2D}{r}$ and $(\log I)' = \frac{2U}{r}$, where the frequency U is given by $U = \frac{D}{I}$.

The next theorem gives a strong approximation for approximate eigenfunctions on M. The theorem is stated for eigenvalue $-\frac{1}{2}$ for simplicity, but can be modified easily for other eigenvalues by arguing as in the previous section. This result is a key ingredient in [5].

Theorem 6.1 *There exist $\bar{R}$ and C depending on n so that if v is a function on $\{|x| \leq R\}$, where $\bar{R} \leq R$, and*

(1) $\left|-\frac{1}{2}\, v^2 + v\, \mathcal{L} v\right| \leq \psi^2 + \epsilon \left(\frac{v^2}{2} + |\nabla v|^2\right)$, *where ψ is a function and $\epsilon < \frac{1}{2}$,*

then we get for any $\Lambda \in (0, 1/2)$ that

$$\int_{|x|<4n} v^2\, e^{-f} \leq \frac{2}{\Lambda}\, \|\psi\|_{L^2}^2 + C\, I(R)\, R^{2n}\, e^{-\frac{(1-\epsilon-\Lambda)\, R^2}{2(1+\epsilon+\Lambda)^2}}. \tag{6.3}$$

In the proof, we will need a modified version of the frequency. Define $E(r)$ by

$$E(r) = r^{2-n}\,\mathrm{e}^{\frac{r^2}{4}} \int_{|x|<r} \left\{ |\nabla v|^2 + \frac{1}{2}\,v^2 \right\} \mathrm{e}^{-f}$$
$$= D(r) - r^{2-n}\,\mathrm{e}^{\frac{r^2}{4}} \int_{|x|<r} \left(v\mathcal{L}v - \frac{1}{2}\,v^2 \right) \mathrm{e}^{-f}.$$

We define a modified frequency U_E by

$$U_E(r) = \frac{E(r)}{I(r)}. \tag{6.4}$$

Lemma 6.1 *We have*

$$(\log U_E)' \geq \frac{2-n}{r} + \frac{r}{2} + \frac{r}{2\,U_E} + \frac{U}{r}\left(\frac{D}{E} - 2\right). \tag{6.5}$$

Proof Differentiating gives that

$$E'(r) = \frac{2-n}{r}\,E + \frac{r}{2}\,E + \frac{r}{2}\,I + r^{2-n}\int_{|x|=r} |\nabla v|^2 \geq \frac{2-n}{r}\,E + \frac{r}{2}\,E + \frac{r}{2}\,I + \frac{U\,D}{r}, \tag{6.6}$$

where the inequality used the Cauchy–Schwarz inequality, (2.11). The lemma follows from this since $\frac{I'}{I} = \frac{2U}{r}$. □

Proof of Theorem 6.1 We get (6.3) immediately if

$$\int_{|x|<4n} v^2 \mathrm{e}^{-f} < \frac{2}{\Lambda}\,\|\psi\|_{L^2}^2. \tag{6.7}$$

Suppose, therefore, that (6.7) fails. Given any $r \geq 4n$, it follows from (1) that

$$|D - E| \leq \epsilon\,E + r^{2-n}\,\mathrm{e}^{\frac{r^2}{4}}\,\|\psi\|_{L^2}^2 \leq \epsilon\,E + \Lambda\,r^{2-n}\,\mathrm{e}^{\frac{r^2}{4}} \int_{|x|<4n} \frac{v^2}{2}\,\mathrm{e}^{-f} \leq (\epsilon + \Lambda)\,E. \tag{6.8}$$

Therefore, if $4n \leq r$, then: $0 \leq I'(r)$,

$$|U - U_E|\,(r) \leq (\epsilon + \Lambda)\,U_E(r)\,, \tag{6.9}$$
$$(\log U_E)' \geq \frac{2-n}{r} + \frac{r}{2} + \frac{r}{2\,U_E} - (1 + \epsilon + \Lambda)^2\,\frac{U_E}{r}\,, \tag{6.10}$$

where the last inequality also used Lemma 6.1.

We first show that $\max_{[4n,8n]} U_E \geq n$. To see this, suppose instead that $U_E < n$ on $[4n, 8n]$ and use (6.10) to get

$$(\log U_E)' > \frac{2n}{U_E}. \tag{6.11}$$

Multiplying by U_E, we get an interval of length $4n$ where $0 < U_E < n$ but $2n < U_E'$. This is impossible, so we conclude that $\max_{[4n,8n]} U_E \geq n$ as claimed.

We claim that there exists $\bar{R} = \bar{R}(n) \geq 5n$ so that for all $r \geq \bar{R}$ we have

$$U_E(r) > \frac{r^2 - 2n}{2(1 + \epsilon + \Lambda)^2}. \tag{6.12}$$

The key is that if (6.12) fails for some $r \geq 4n$, then (6.10) implies that

$$(\log U_E)' \geq \frac{2}{r} + \frac{r}{2\,U_E} \geq \frac{6}{r}. \tag{6.13}$$

On the other hand, for $r \geq 4n$, we have

$$\left(\log \frac{r^2 - 2n}{2(1+\epsilon+\Lambda)^2}\right)' = \frac{2r}{r^2 - 2n} < \frac{3}{r}. \tag{6.14}$$

Integrating (6.13) and (6.14) and using that $\max_{[4n,8n]} U_E \geq n$, gives an upper bound for the maximal interval where (6.12) fails. The first derivative test, (6.13), and (6.14) imply that once (6.12) holds for some $R \geq 4n$, then it also holds for all $r \geq R$. This gives the claim.

Using (6.9) and (6.12), we get for $r \geq \bar{R}$ that

$$U(r) \geq (1-\epsilon-\Lambda)U_E(r) > \frac{(1-\epsilon-\Lambda)}{(1+\epsilon+\Lambda)^2}\left(\frac{r^2}{2} - n\right) \equiv \kappa\left(\frac{r^2}{2} - n\right), \tag{6.15}$$

where the last equality defines κ. Integrating this from $\bar{R}$ to R gives that

$$\log \frac{I(R)}{I(\bar{R})} \geq 2\int_{\bar{R}}^{R} \frac{U(r)}{r}\,dr \geq \kappa \int_{\bar{R}}^{R}\left(r - \frac{2n}{r}\right) dr = \kappa\left(\frac{R^2 - \bar{R}^2}{2} - 2n\,\log\frac{R}{\bar{R}}\right). \tag{6.16}$$

Since $\bar{R}$ is uniformly bounded, exponentiating gives that

$$\sup_{4n \leq r \leq \bar{R}} I(r) = I(\bar{R}) \leq c_n\, I(R)\, R^{2n\kappa}\, \mathrm{e}^{-\frac{\kappa}{2}R^2}. \tag{6.17}$$

We use the reverse Poincaré to get the integral bound on $|x| < 4n$. Let $\eta \leq 1$ be a cutoff that is one on $\{|x| < 4n\}$, zero for $|x| > 5n$, and has $|\nabla\eta| \leq 1$. Integration by parts gives

$$\int \eta^2\left(|\nabla v|^2 + \frac{v^2}{2}\right)\mathrm{e}^{-f} = -\int\left(2\eta v\langle\nabla v, \nabla\eta\rangle + \eta^2\left(v\mathcal{L}v - \frac{v^2}{2}\right)\right)\mathrm{e}^{-f}. \tag{6.18}$$

Using (2) on the last term (note that $\epsilon < 1/2$) and absorbing the first term gives

$$\begin{aligned}\frac{1}{2}\int \eta^2\left(|\nabla v|^2 + \frac{v^2}{2}\right)\mathrm{e}^{-f} &= \|\psi\|_{L^2}^2 - \int (2\eta v\langle\nabla v, \nabla\eta\rangle)\,\mathrm{e}^{-f} \\ &\leq \|\psi\|_{L^2}^2 + \frac{1}{2}\int \eta^2 v|\nabla v|^2\,\mathrm{e}^{-f} + 2\int |\nabla\eta|^2 v^2\,\mathrm{e}^{-f}. \end{aligned} \tag{6.19}$$

Since $\eta = 1$ for $|x| < 4n$ and $|\nabla\eta| \leq 1$ is only nonzero for $4n < |x| < 5n$, it follows that

$$\int_{\{|x|<4n\}} v^2\,\mathrm{e}^{-f} \leq 4\,\|\psi\|_{L^2}^2 + 8\int_{\{4n<|x|<5n\}} v^2\,\mathrm{e}^{-f} \leq 4\,\|\psi\|_{L^2}^2 + C\,I(5n), \tag{6.20}$$

where we used that $I'(r) \geq 0$ for $r \geq 4n$. Combining (6.17) and (6.20) gives (6.3). □

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