UNIQUENESS AND STABILITY OF ENTROPY SHOCKS TO THE ISENTROPIC EULER SYSTEM IN A CLASS OF INVISCID LIMITS FROM A LARGE FAMILY OF NAVIER-STOKES SYSTEMS

MOON-JIN KANG AND ALEXIS F. VASSEUR

ABSTRACT. We prove the uniqueness and stability of entropy shocks to the isentropic Euler systems among all vanishing viscosity limits of solutions to associated Navier-Stokes systems. To take into account the vanishing viscosity limit, we show a contraction property for any large perturbations of viscous shocks to the Navier-Stokes system. The contraction estimate does not depend on the strength of the viscosity. This provides a good control on the inviscid limit process. We prove that, for any initial value, there exist a vanishing viscosity limit to solutions of the Navier-Stokes system. The convergence holds in a weak topology. However, this limit satisfies some stability estimates measured by the relative entropy with respect to an entropy shock. In particular, our result provides the uniqueness of entropy shocks to the shallow water equation in a class of inviscid limits of solutions to the viscous shallow water equations.

CONTENTS

1. Introduction

We consider the vanishing viscosity limit ($\nu \to 0$) of the one-dimensional barotropic Navier-Stokes system in the Lagrangian coordinates:

$$(1.1) \qquad \begin{cases} v_t^\nu - u_x^\nu = 0, \\ u_t^\nu + p(v^\nu)_x = \nu \left(\frac{\mu(v^\nu)}{v^\nu} u_x^\nu \right)_x, \end{cases}$$

where v denotes the specific volume, u is the fluid velocity, and $p(v)$ is the pressure law. We consider the case of a polytropic perfect gas where the pressure is given by

$$(1.2) \qquad p(v) = v^{-\gamma}, \quad \gamma > 1,$$

with γ the adiabatic constant. Here, μ denotes the viscosity coefficient given by

$$(1.3) \qquad \mu(v) = bv^{-\alpha}.$$

Notice that if $\alpha > 0$, $\mu(v)$ degenerates near the vacuum, i.e., near $v = +\infty$. Very often, the viscosity coefficient is assumed to be constant, i.e., $\alpha = 0$. However, in the physical context the viscosity of a gas depends on the temperature (see Chapman and Cowling [6]). In the barotropic case, the temperature depends directly on the density ($\rho = 1/v$). The viscosity is expected to degenerate near the vacuum as a power of the density, which is translated into $\mu(v) = bv^{-\alpha}$ in terms of v with $\alpha > 0$.

At least formally, as $\nu \to 0$, the limit system of (1.1) is given by the isentropic Euler system:

$$(1.4) \qquad \begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0. \end{cases}$$

The idea of approximating inviscid gases by viscous gases with vanishing viscosity is due to the seminal paper by Stokes [46]. The vanishing viscosity limit has been used later to construct entropy solutions to the isentropic Euler system, see DiPerna [20, 21], Hoff-Liu [30], Goodman-Xin [25], Yu [52]. Recently, Chen and Perepelitsa [8] proved the convergence of solutions to the Navier-Stokes system with $\alpha = 0$ towards an entropy solution of the isentropic Euler system with finite energy initial data. Some results exist for the inviscid limit of the Navier-Stokes-Fourier system in very special cases, see for instance Feireisl [23] and [50], and the references cited therein.

In this article, we prove the existence of vanishing viscosity limits of solutions to (1.1) in some weak sense, and obtain a stability estimate of those limits. We then present the class of inviscid limits, in which the entropy shocks to (1.4) are unique.

Our result provides an answer, in the case of a shock, to the conjecture: The compressible Euler system admits a unique entropy solution in the class of vanishing viscosity solutions to the associated compressible Navier-Stokes system. As a breakthrough result related to this conjecture, Bianchini-Bressan [2] constructed a globally-in-time unique entropy solution to a strictly hyperbolic $n \times n$ system with small BV initial datum, which is obtained from vanishing "artificial" viscosity limit of the associate parabolic system. However, to the best

of our knowledge, there is no result on uniqueness of discontinuous entropy solutions in the class of vanishing physical viscosity solutions to the Navier-Stokes systems.

Previous uniqueness results for special discontinuous solutions (as solutions to the Riemann problem) required suitable regularity like locally BV or strong trace properties (see Chen-Frid-Li [7] and Vasseur et al. [35, 39, 49]). Unfortunately, the global-in-time propagation of those regularities is unknown in general (except for the system with $\gamma = 3$ see [47]). In the multi-D case, De Lellis-Székelyhidi[18] and Chiodaroli et al. [9, 10, 11, 12] showed non-uniqueness of entropy solutions. They showed that entropy solutions to the isentropic Euler systems in more than one space dimension are not unique, by constructing infinitely many entropy solutions based on the convex integration method [17, 18].

It is well known that the system (1.1) admits viscous shock waves connecting two end states (v_-, u_-) and (v_+, u_+), provided the two end states satisfy the Rankine-Hugoniot condition and the Lax entropy condition (see Matsumura and Wang [40]):

$$
(1.5) \qquad \exists\, \sigma \quad \text{s.t.} \quad \left\{ \begin{array}{l} -\sigma(v_+ - v_-) - (u_+ - u_-) = 0, \\ -\sigma(u_+ - u_-) + p(v_+) - p(v_-) = 0, \end{array} \right.
$$
$$
\text{and either } v_- > v_+ \text{ and } u_- > u_+ \text{ or } v_- < v_+ \text{ and } u_- > u_+ \text{ holds.}
$$

In other words, for given constant states (v_-, u_-) and (v_+, u_+) satisfying (1.5), there exists a viscous shock wave $(\tilde{v}^\nu, \tilde{u}^\nu)(x - \sigma t)$ as a solution of

$$
(1.6) \qquad \left\{ \begin{array}{l} -\sigma(\tilde{v}^\nu)' - (\tilde{u}^\nu)' = 0, \\ -\sigma(\tilde{u}^\nu)' + p(\tilde{v}^\nu)' = \nu\left(\frac{\mu(\tilde{v}^\nu)}{\tilde{v}^\nu}(\tilde{u}^\nu)' \right)' \\ \lim_{\xi \to \pm\infty}(\tilde{v}^\nu, \tilde{u}^\nu)(\xi) = (v_\pm, u_\pm). \end{array} \right.
$$

Here, if $v_- > v_+$, $(\tilde{v}, \tilde{u})(x - \sigma t)$ is a 1-shock wave with velocity $\sigma = -\sqrt{-\frac{p(v_+) - p(v_-)}{v_+ - v_-}}$, whereas if $v_- < v_+$, that is a 2-shock wave with $\sigma = \sqrt{-\frac{p(v_+) - p(v_-)}{v_+ - v_-}}$.

Let $(\bar{v}, \bar{u})$ be an associated entropy (inviscid) shock wave connecting the two end states (v_-, u_-) and (v_+, u_+) satisfying (1.5) as follows:

$$
(1.7) \qquad (\bar{v}, \bar{u})(x - \sigma t) = \left\{ \begin{array}{ll} (v_-, u_-) & \text{if } x - \sigma t < 0, \\ (v_+, u_+) & \text{if } x - \sigma t > 0. \end{array} \right.
$$

As mentioned above, our goal is to show the uniqueness of the entropy shock to (1.4) in a suitable class, based on a generalization of our recent result [33] on the contraction property of viscous shocks to (1.1). More precisely, we prove the contraction of any large perturbations of viscous shocks to (1.1) in the case of $0 < \alpha \le \gamma \le \alpha + 1$ and $\gamma > 1$, which improves the special case $\gamma = \alpha$ in [33]. The contraction holds up to a shift, and is measured by a weighted relative entropy. Notice that since the relative entropy is locally quadratic, the contraction measured by the relative entropy can be regarded as L^2-type contraction. To prove the contraction, we employ the new approach introduced by the authors [33], which basically uses the relative entropy method. The relative entropy method has been extensively used in studying the contraction (or stability) of viscous (or inviscid) shock waves (see [13, 14, 31, 32, 33, 35, 36, 37, 38, 42, 43, 44, 49, 50]).

1.1. **Main results.** To handle the stability and uniqueness of the entropy shocks, we use the relative entropy associated to the entropy of (1.4) as follows: For any functions v_1, u_1, v_2, u_2,

$$(1.8) \qquad \eta((v_1, u_1)|(v_2, u_2)) := \frac{|u_1 - u_2|^2}{2} + Q(v_1|v_2),$$

where $Q(v_1|v_2)$ is the relative functional associated with the strictly convex function

$$Q(v) := \frac{v^{-\gamma+1}}{\gamma - 1}, \quad v > 0,$$

that is,

$$Q(v_1|v_2) := Q(v_1) - Q(v_2) - Q'(v_2)(v_1 - v_2).$$

However, the first components v_1 that we will consider are limit of Navier-Stokes equations, for which we obtain only uniform bounds in L^1. Therefore, the limit can be a measure in t, x. This is actually physical, and is related to the possible appearance of cavitation. For this reason, we need to extend the definition of relative entropy to measures defined on $\mathbb{R}^+ \times \mathbb{R}$. We will restrict the definition in the case where we compare a measure dv with a simple function $\bar{v}$ only taking two values v_- and v_+. Let v_a denote the Radon-Nikodym derivative of dv with respect to the Lebesgue measure and dv_s its singular part, i.e., $dv = v_a \, dt \, dx + dv_s$. The relative entropy is then itself a measure defined as

$$(1.9) \qquad dQ(v|\bar{v})(t, x) := Q(v_a|\bar{v}) \, dtdx + |Q'(\overline{V}(t, x))|dv_s(t, x),$$

where we need to define $\overline{V}$ everywhere. Denote $\Omega_M = \{t, x : \bar{v}(t, x) = \max(v_-, v_+)\}$, we set

$$\overline{V}(t, x) = \begin{cases} \max(v_-, v_+) & \text{for } (t, x) \in \overline{\Omega_M} \text{ (closure of } \Omega_M), \\ \min(v_-, v_+) & \text{for } (t, x) \in (\overline{\Omega_M})^c. \end{cases}$$

Note that $|Q'(\max(v_-, v_+))| \le |Q'(\min(v_-, v_+))|$. Also, note that if $v \in L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}) + \mathcal{M}(\mathbb{R}))$, then $dQ(v|\bar{v})$ is defined in $L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}) + \mathcal{M}(\mathbb{R}))$, where $\mathcal{M}$ denotes the space of nonnegative Radon measures.

For the global-in-time existence of solutions to (1.1), we introduce the function space:

$$\mathcal{X}_T := \{(v, u) \in \mathbb{R}^+ \times \mathbb{R} \mid v - \underline{v}, \; u - \underline{u} \in L^\infty(0, T; H^1(\mathbb{R})),$$
$$u - \underline{u} \in L^2(0, T; H^2(\mathbb{R})), \; v^{-1} \in L^\infty((0, T) \times \mathbb{R})\},$$

where $\underline{v}$ and $\underline{u}$ are smooth monotone functions such that

$$(1.10) \qquad \underline{v}(x) = v_\pm \quad \text{and} \quad \underline{u}(x) = u_\pm \quad \text{for } \pm x \ge 1.$$

The first theorem is on stability and uniqueness of the entropy shocks to (1.4):

Theorem 1.1. *Let $\gamma > 1$ and $\alpha, b > 0$ be any constants satisfying $\alpha \le \gamma \le \alpha + 1$. For each $\nu > 0$, consider the system (1.1)-(1.3). For a given constant state $(v_-, u_-) \in \mathbb{R}^+ \times \mathbb{R}$, there exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and any $(v_+, u_+) \in \mathbb{R}^+ \times \mathbb{R}$ satisfying (1.5) with $|p(v_-) - p(v_+)| = \varepsilon$, the following holds.*
Let $(\tilde{v}^\nu, \tilde{u}^\nu)$ be a viscous shock connecting the two end states (v_-, u_-) and (v_+, u_+) as a solution of (1.6).
Then for a given initial datum (v^0, u^0) of (1.4) satisfying

$$(1.11) \qquad \mathcal{E}_0 := \int_{-\infty}^{\infty} \eta\big((v^0, u^0)|(\bar{v}, \bar{u})\big) dx < \infty,$$

the following is true.

(i) (Well-prepared initial data) There exists a sequence of smooth functions $\{(v_0^\nu, u_0^\nu)\}_{\nu>0}$ such that

$$\lim_{\nu \to 0} v_0^\nu = v^0, \quad \lim_{\nu \to 0} u_0^\nu = u^0 \quad a.e., \quad v_0^\nu > 0,$$

(1.12)
$$\lim_{\nu \to 0} \int_{\mathbb{R}} \left(\frac{1}{2} \left(u_0^\nu + \nu \left(p(v_0^\nu)^{\frac{\alpha}{\gamma}} \right)_x - \tilde{u}^\nu - \nu \left(p(\tilde{v}^\nu)^{\frac{\alpha}{\gamma}} \right)_x \right)^2 + Q(v_0^\nu | \tilde{v}^\nu) \right) dx = \mathcal{E}_0.$$

(ii) For a given $T > 0$, let $\{(v^\nu, u^\nu)\}_{\nu>0}$ be a sequence of solutions in $\mathcal{X}_T$ to (1.1) with the initial datum (v_0^ν, u_0^ν) as above. Then there exist limits v_∞ and u_∞ such that as $\nu \to 0$ (up to a subsequence),

(1.13)
$$v^\nu \rightharpoonup v_\infty, \quad u^\nu \rightharpoonup u_\infty \quad \text{in } \mathcal{M}_{\mathrm{loc}}((0,T) \times \mathbb{R}) \text{ (space of locally bounded Radon measures),}$$

where v_∞ lies in $L^\infty(0,T,L^\infty(\mathbb{R}) + \mathcal{M}(\mathbb{R}))$.
In addition, there exist shift $X_\infty \in BV((0,T))$ and constant $C > 0$ such that $dQ(v_\infty | \bar{v}) \in L^\infty(0,T; \mathcal{M}(\mathbb{R}))$, and for almost every $t \in (0,T)$,

(1.14)
$$\int_{\mathbb{R}} \frac{|u_\infty(t,x) - \bar{u}(x - X_\infty(t))|^2}{2} dx + \left(\int_{x \in \mathbb{R}} dQ(v_\infty | \bar{v}(x - X_\infty(\cdot))) \right)(t) \le C\mathcal{E}_0.$$

Moreover, the shift X_∞ satisfies

(1.15)
$$|X_\infty(t) - \sigma t| \le \frac{C}{|v_- - v_+|} \left(\mathcal{E}_0 + (1+t)\sqrt{\mathcal{E}_0} \right).$$

Therefore, entropy shocks (1.7) (with small amplitude) of the isentropic Euler system (1.4) are stable and unique in the class of weak inviscid limits of solutions to the Navier-Stokes system (1.1).

Remark 1.1. *1. By (1.14), the limits v_∞, u_∞ satisfy $v_\infty \in \bar{v} + L^\infty(0,T; L^\infty(\mathbb{R}) + \mathcal{M}(\mathbb{R}))$ and $u_\infty \in \bar{u} + L^\infty(0,T; L^2(\mathbb{R}))$, where $\mathcal{M}(\mathbb{R})$ is the set of bounded Radon measures on $\mathbb{R}$. The control in measure of v_∞ is due to the fact that $Q(v|\bar{v}) \ge c_2|v - \bar{v}|$ for $v \ge 3v_-$ (see (3.8) in Lemma 3.1). Especially, v_∞ may have some measure concentration at the limit. This corresponds physically to cavitation (the creation of bubbles in the fluid) and appearance of vacuum. It is interesting to see that this does not affect the contraction property (and the uniqueness of the shock at the limit).*
2. Theorem 1.1 provides the stability and uniqueness of weak Euler shocks in the wide class of weak inviscid limits of solutions to the Navier-Stokes system.
Indeed, for the uniqueness, if $\mathcal{E}_0 = 0$, then (1.14) and (1.15) imply that for a.e. $t \in (0,T)$,

$$\int_{\mathbb{R}} \frac{|u_\infty(t,x) - \bar{u}(x - \sigma t)|^2}{2} dx + \int_{\mathbb{R}} Q(v_a(t,x) | \bar{v}(x - \sigma t)) dx = 0,$$

where $dv_\infty = v_a \, dt \, dx + dv_s$, and the singular part v_s vanishes. Therefore, we have

$$u_\infty(t,x) = \bar{u}(x - \sigma t), \qquad v_\infty(t,x) = \bar{v}(x - \sigma t), \quad a.e. \ (t,x) \in [0,T] \times \mathbb{R}.$$

3. In fact, the smallness of amplitude of shocks is not needed in the proof of Theorem 1.1. The constraint is due to Theorem 1.2.
4. It is worth emphasizing from the assumption on α and γ that Theorem 1.1 also holds in the case of the shallow water equations (i.e., $\gamma = 2$) in a class of inviscid limits of solutions to the viscous shallow water equations (i.e., $\gamma = 2$, $\alpha = 1$). We refer to Gerbeau-Perthame

[24] *for a derivation of the viscous shallow water equations from the incompressible Navier-Stokes equations with free boundary.*

Remark 1.2. *For the global-in-time existence (and uniqueness) of any large solutions to* (1.1) *in* $\mathcal{X}_T$, *we refer to* [41] *in the case of* $\alpha < 1/2$ *and* $\gamma > 1$. *More precisely, they proved that* $\rho = 1/v$ *and* u *satisfy*

$$\rho - \underline{\rho}, u - \underline{u} \in L^\infty(0, T; H^1(\mathbb{R})), \ u - \underline{u} \in L^2(0, T; H^2(\mathbb{R})), \ \rho^{-1} \in L^\infty((0, T) \times \mathbb{R}).$$

This implies that there exists a solution in $\mathcal{X}_T$ *to the system* (1.1) *with* $\alpha < 1/2$ *and* $\gamma > 1$, *since the system* (1.1) *is equivalent to the one in the Eulerian coordinates for such strong solutions. The result of* [41] *was extended by Haspot* [28] *to the case of* $\alpha \in (1/2, 1]$. *Recently, Constantin-Drivas-Nguyen-Pasqualotto* [15, Theorem 1.6] *extended it to the case of* $\alpha \geq 0$ *and* $\gamma \in [\alpha, \alpha + 1]$ *with* $\gamma > 1$, *but they handled it on the periodic domain. Recently, the authors* [34] *extends the result* [15, Theorem 1.6] *to the case where smooth solutions connect possibly two different limits at the infinity on the whole space, which implies our solution space* $\mathcal{X}_T$.

The starting point of the proof of Theorem 1.1 is to derive a uniform-in-ν estimate for any large perturbations of viscous shocks to (1.1). It is equivalent to obtain the contraction property of any LARGE perturbations of viscous shocks to (1.1) with a fixed $\nu = 1$:

$$(1.16) \qquad \begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \left(\frac{\mu(v)}{v} u_x \right)_x. \end{cases}$$

As in [33], we first introduce the following relative functional $E(\cdot|\cdot)$ to measure the contraction:

for any functions v_1, u_1, v_2, u_2,

$$(1.17) \qquad E((v_1, u_1)|(v_2, u_2)) := \frac{1}{2} \left(u_1 + \left(p(v_1)^{\frac{\alpha}{\gamma}} \right)_x - u_2 - \left(p(v_2)^{\frac{\alpha}{\gamma}} \right)_x \right)^2 + Q(v_1|v_2),$$

where the constants γ, α are in (1.2) and (1.3). The functional E is associated to the BD entropy (see Bresch-Desjardins [3, 4, 5]). Since $Q(v_1|v_2)$ is positive definite, (1.17) is also positive definite, that is, for any functions (v_1, u_1) and (v_2, u_2) we have $E((v_1, u_1)|(v_2, u_2)) \geq 0$, and

$$E((v_1, u_1)|(v_2, u_2)) = 0 \text{ a.e.} \quad \Leftrightarrow \quad (v_1, u_1) = (v_2, u_2) \text{ a.e.}$$

The following result provides a contraction property measured by the relative functional (1.17).

Theorem 1.2. *For any* $\gamma > 1$ *and* $\alpha, b > 0$ *satisfying* $\alpha \leq \gamma \leq \alpha + 1$, *consider the system* (1.16) *with* (1.2)-(1.3). *For a given constant state* $(v_-, u_-) \in \mathbb{R}^+ \times \mathbb{R}$, *there exist constants* $\varepsilon_0, \delta_0 > 0$ *such that the following is true.*
For any $\varepsilon < \varepsilon_0$, $\delta_0^{-1}\varepsilon < \lambda < \delta_0$, *and any* $(v_+, u_+) \in \mathbb{R}^+ \times \mathbb{R}$ *satisfying* (1.5) *with* $|p(v_-) - p(v_+)| = \varepsilon$, *there exists a smooth monotone function* $a : \mathbb{R} \to \mathbb{R}^+$ *with* $\lim_{x \to \pm\infty} a(x) = 1 + a_\pm$ *for some constants* a_-, a_+ *with* $|a_+ - a_-| = \lambda$ *such that the following holds.*
Let $\tilde{U} := (\tilde{v}, \tilde{u})$ *be the viscous shock connecting* (v_-, u_-) *and* (v_+, u_+) *as a solution of* (1.6) *with* $\nu = 1$. *For a given* $T > 0$, *let* $U := (v, u)$ *be a solution in* $\mathcal{X}_T$ *to* (1.16) *with a initial datum* $U_0 := (v_0, u_0)$ *satisfying* $\int_{-\infty}^{\infty} E(U_0|\tilde{U})dx < \infty$. *Then there exists a shift*

$X \in W^{1,1}((0,T))$ *such that*

$$(1.18) \quad \begin{aligned}
\int_{-\infty}^{\infty} & a(x)E\big(U(t,x+X(t))|\tilde{U}(x)\big)\,dx \\
& + \delta_0 \frac{\varepsilon}{\lambda} \int_0^T \int_{-\infty}^{\infty} |\sigma a'(x)| Q\left(v(t,x+X(t))|\tilde{v}(x)\right)\,dx dt \\
& + \delta_0 \int_0^T \int_{-\infty}^{\infty} a(x) v^{\gamma-\alpha}(t,x+X(t)) \big|\partial_x\big(p(v(t,x+X(t)))-p(\tilde{v}(x))\big)\big|^2 dx dt \\
& \leq \int_{-\infty}^{\infty} a(x) E\big(U_0(x)|\tilde{U}(x)\big)\,dx,
\end{aligned}$$

and

$$(1.19) \quad \begin{aligned}
& |\dot{X}(t)| \leq \frac{1}{\varepsilon^2}(1+f(t)), \\
& \text{for some positive function } f \text{ satisfying} \quad \|f\|_{L^1(0,T)} \leq \frac{2\lambda}{\delta_0 \varepsilon} \int_{-\infty}^{\infty} E(U_0|\tilde{U})\,dx.
\end{aligned}$$

Remark 1.3. *1. Theorem 1.2 provides a contraction property for viscous shocks with suitably small amplitude parametrized by $\varepsilon = |p(v_-) - p(v_+)|$. This smallness together with (1.5) implies $|v_- - v_+| = \mathcal{O}(\varepsilon)$ and $|u_- - u_+| = \mathcal{O}(\varepsilon)$. For such a fixed small shock, the contraction holds for any large solutions to (1.16), without any smallness condition imposed on U_0. This implies that the contraction still holds for any large solutions to (1.1), which provides a weak compactness to prove Theorem 1.1 as the inviscid limit problem ($\nu \to 0$). 2. In (1.18), the dissipation terms will be used to show the convergence of $\{u^\nu\}_{\nu>0}$ in (1.13).*

Remark 1.4. *The contraction property is non-homogenous in x, as measured by the function $x \to a(x)$. This is consistant with the hyperbolic case (with $\nu = 0$). In the hyperbolic case, it was shown in [42] that a homogenous contraction cannot hold for the full Euler system. However, the contraction property is true if we consider a non-homogenous pseudo-distance [49] providing the so-called a-contraction [35]. Our main result shows that the non-homogeneity of the pseudo-distance can be chosen of a similar size as the strength of the shock (as measured by the quantity λ).*

The rest of the paper is as follows. We explain main ideas of proofs of the mains results in Section 2. In Section 3, we provides a transformation of the system (1.16), and an equivalent version of Theorem 1.2, and useful inequalities. Section 4 is dedicated to the proof of Theorem 3.1. Finally, Section 5 is dedicated to the proof of the main Theorem.

2. Ideas of the proof.

We describe in this section the methodology and main ideas of our results.

Uniform estimates with respect to the viscosity. The main results of this paper boil down to the proof of stability of the viscous shocks to the Navier-Stokes equations UNIFORMLY with respect to the strength of the viscosity. This can be obtained by considering only the case of the viscosity $\nu = 1$, replacing the notion of stability by the notion of contraction, valid even for large perturbations (Theorem 1.2).

Indeed, if (v^ν, u^ν) is a solution of (1.1), then

$$v(t,x) = v^\nu(\nu t, \nu x), \qquad u(t,x) = u^\nu(\nu t, \nu x)$$

is a solution to (1.16), i.e., the Navier-Stokes equations with $\nu = 1$. Note that, even if the initial perturbation $(v_0^\nu - \tilde{v}^\nu, u_0^\nu - \tilde{u}^\nu)$ is small, let say of order $\mathcal{E}$, then the perturbation $(v_0 - \tilde{v}, u_0 - \tilde{u})$ is big (of order $\mathcal{E}/\nu$):

$$\int_\mathbb{R} E(U_0(\xi)|\tilde{U}(\xi))\,d\xi = \frac{1}{\nu}\int_\mathbb{R} E_\nu(U_0^\nu(\xi)|\tilde{U}^\nu(\xi))\,d\xi = \frac{\mathcal{E}}{\nu},$$

where $E(\cdot|\cdot)$ is defined in (1.17), and the rescaled ν-dependent functional $E_\nu(\cdot|\cdot)$ is defined in (5.4).

However, a contraction independent of the size of the perturbation in the case $\nu = 1$, as in Theorem 1.2, provides, after rescaling, a similar contraction for any values ν:

$$\int_\mathbb{R} a(\xi/\nu)E_\nu(U^\nu(t,\xi + \nu X(t/\nu))|\tilde{U}^\nu(\xi))\,d\xi \leq \int_\mathbb{R} a(\xi/\nu)E_\nu(U_0^\nu|\tilde{U}^\nu)\,d\xi.$$

This gives a uniform stability result with respect to ν provided that the weight function a is uniformly bounded from below and from above, that we have a control on the shift which is independent of the transformation

$$X(t) \longrightarrow \nu X\left(\frac{t}{\nu}\right),$$

and that we have a uniform bound of $\int_\mathbb{R} E_\nu(U_0^\nu|\tilde{U}_0)\,d\xi$. The first two conditions are verified by Theorem 1.2 thanks to (1.19), and considering $\lambda < 1$, and the last one is verified thanks to the well-prepared initial data (1.12).

The contraction when $\nu = 1$: Theorem 3.1. This result is a generalization of the result in [33] where only the case $\alpha = \gamma$ was considered. The extension introduces severe technical difficulties. A key to the extension is the local minimization explained below.

Step one: Considering a new velocity variable. We need to control the growth of the perturbation due to the hyperbolic terms (flux functionals). Thanks to the relative entropy method, the linear fluxes are easier to handle (the relative functional of linear quantity vanishes). Therefore, the main hyperbolic quantities to control are the pressure terms depending only on the specific volume v. At the core of the method, we are using a generalized Poincaré inequality Proposition A.1, first proved in [33]. The Navier-Stokes system can be seen as a degenerate parabolic system. But the diffusion is in the other variable, the velocity variable u. Bresch and Desjardins (see [5, 3, 4]) showed that compressible Navier-Stokes systems have a natural perturbed velocity quantity associated to the viscosity:

$$h^\nu = u^\nu + \nu\left(p(v^\nu)^{\frac{\alpha}{\gamma}}\right)_x.$$

Remarkably, the system in the variables (v^ν, h^ν) exhibits a diffusion in the v variable (the Smoluchowski equation), rather than in the velocity variable. For this reason, we are working with the natural relative entropy of this system, which corresponds to the usual relative entropy of the associated p-system in the $U_h^\nu = (v^\nu, h^\nu)$ variable:

$$\eta(U_h^\nu|\tilde{U}_h^\nu) = E_\nu(U^\nu|\tilde{U}^\nu).$$

For the rest of the proof of this theorem, we consider only $\nu = 1$ and work only in the new set of variable (v, h). To simplify the notation, we denote $U = (v, h)$ from now on.

Step 2: Evolution of the relative entropy. Computing the evolution of the relative entropy in Lemma 4.2, we get

$$\frac{d}{dt} \int_{-\infty}^{\infty} a(\xi)\eta\big(U(t,\xi+X(t))|\tilde{U}_\varepsilon(\xi)\big) d\xi$$
$$= \dot{X}(t)Y(U(t,\cdot+X(t))) + \mathcal{J}^{bad}(U(t,\cdot+X(t))) - \mathcal{J}^{good}(U(t,\cdot+X(t))).$$

The functional $\mathcal{J}^{good}(U)$ is non-negative (good term) and can be split into three terms (see (4.10), (4.18)):

$$\mathcal{J}^{good}(U) = \mathcal{J}_1^{good}(U) + \mathcal{G}_2(U) + \mathcal{D}(U),$$

where only $\mathcal{J}_1^{good}(U)$ depends on h (and actually does not depend on v). The term $\mathcal{D}(U)$ corresponds to the diffusive term (which depends on v only, thanks to the transformation of the system).

Step 3: Construction of the shift. The shift $X(t)$ produces the term $\dot{X}(t)Y(U)$. The key idea of the technique is to take advantage of this term when $Y(U(t,\cdot))$ is not two small, by compensating all the other terms via the choice of the velocity of the shift (see (4.20)). Specifically, we algebraically ensure that the contraction holds as long as $|Y(U(t))| \geq \varepsilon^2$. The rest of the analysis is to ensure that when $|Y(U(t))| \leq \varepsilon^2$, the contraction still holds.

The condition $|Y(U(t))| \leq \varepsilon^2$ ensures a smallness condition that we want to fully exploit. This is where the non-homogeneity of the semi-norm is crucial. In the case where the function a is constant, $Y(U)$ is a linear functional in U. The smallness of $Y(U)$ gives only that a certain weighted mean value of U is almost null. However, when a is decreasing, $Y(U)$ becomes convex. The smallness $Y(U(t)) \leq \varepsilon^2$ implies, for this fixed time t (See Lemma 4.4 with (4.14) and (4.1)):

$$(2.1) \qquad \int_{\mathbb{R}} \varepsilon e^{-C\varepsilon|\xi|} Q(v(t,\xi+X(t))|\tilde{v}_\varepsilon(\xi)) \, d\xi \leq C \left(\frac{\varepsilon}{\lambda}\right)^2.$$

This gives a control in L^2 for moderate values of v, and in L^1 for big values of v, in the layer region ($|\xi - X(t)| \lesssim 1/\varepsilon$).

The problem now looks, at first glance, as a typical problem of stability with a smallness condition. There are, however, three major difficulties: The bad term $\mathcal{J}^{bad}(U)$ has some terms depending on the variable h for which we do not have diffusion, we have some smallness in v, only for a very weak norm, and only localized in the layer region. More importantly, the smallness is measured with respect to the smallness of the shock. It basically says that, considering only the moderate values of v: the perturbation is not bigger than ε/λ (which is still very big with respect to the size of the shock ε). Actually, as we will see later, it is not possible to consider only the linearized problem: Third order terms appear in the expansion using the smallness condition (the energy method involving the linearization would have only second order term in ε).

In the argument, for the values of t such that $|Y(U(t))| \leq \varepsilon^2$, we construct the shift as a solution to the ODE: $\dot{X}(t) = -Y(U(t,\cdot+X(t)))/\varepsilon^4$. From this point, we forget that $U = U(t,\xi)$ is a solution to (3.3) and $X(t)$ is the shift. That is, we leave out $X(t)$ and the t-variable of U. Then we show that for any function U satisfying $|Y(U)| \leq \varepsilon^2$, we have

$$(2.2) \qquad -\frac{1}{\varepsilon^4}Y^2(U) + \mathcal{J}^{bad}(U) - \mathcal{J}^{good}(U) \leq 0.$$

This is the main Proposition 4.1 (actually, the proposition is slightly stronger to ensure the control of the shift). This implies clearly the contraction. From now on, we are focusing on the proof of this statement.

Step 4: Maximization in h for v fixed. We need to get rid of the dependence on the h variable from the bad parts $\mathcal{J}^{bad}(U)$. The idea in [33] (for $\gamma = \alpha$) is to maximize the bad term with respect to h for v fixed:

$$\mathcal{B}(v) = \sup_h \left(\mathcal{J}^{bad}(v, h) - \mathcal{J}_1^{good}(h) \right).$$

We then had an inequality depending only on v and $\partial_x v$ (through $\mathcal{D}(U)$) for which we can apply a generalized Poincaré inequality. This does not work anymore when $\gamma \neq \alpha$. This is because $\mathcal{B}(v)$ involves powers of $p(v)$ which cannot be controlled by the good terms due to big values of $p(v)$. The new idea is to maximize in h ONLY for the fixed values of v such that $p(v) - p(\tilde{v}_\varepsilon) \leq \delta_3$ for a constant δ_3 to be determined (and depending on the Poincaré inequality). This leads to the decomposition (4.17), (4.18). The bad terms involving values $p(v) - p(\tilde{v}_\varepsilon) > \delta_3$ can now be controlled using additional information from the unconditional estimate $|Y(U)| \leq \varepsilon^2$ (see (4.37), (4.70)).

Step 5: Expansion in ε. Although we have no control on the supremum of $|p(v) - p(\tilde{v}_\varepsilon)|$, we can control independently the contribution of the values $|p(v) - p(\tilde{v}_\varepsilon)| \geq \delta_3$ in Proposition 4.3 (for the same δ_3 related to the maximization process above. The coefficient δ_3 can be chosen very small, but independent of ε and of ε/λ). The last step is to perform an expansion in the size of the shock ε every small, uniformly in v (but for a fixed small value of δ). As in [33], the expansion has to be done up to the third order. It leads to the exact same generic expression (A.5). The generalized nonlinear Poincaré inequality, Proposition A.1 concludes the proof.

The inviscid limit: Theorem 1.1.

We have now a stability result uniform with respect to the viscosity. It is natural to expect a stability result on the corresponding inviscid limit. The result, however, is not immediate. Several difficulties have to be overcome. First, due to the BD representation as above, the stability result for ν fixed is on the quantities:

$$U_h^\nu = (v^\nu, h^\nu), \quad h^\nu = u^\nu + \nu \left(p(v^\nu)^{\frac{\alpha}{\gamma}} \right)_x.$$

This is the reason we need a compatibility condition on the family of initial values U_0^ν. This also leads to a very weak convergence (in measure in (t, x) only). The next difficulty is that for small values of v, the relative entropy control only the L^1 norm of $Q(v) = 1/v^{\gamma-1}$. Therefore the pressure $p(v) = 1/v^\gamma$ cannot be controlled at all. Therefore, we do not control the time derivative of u in any distributional sense in x. Moreover, we have to study carefully the effect of small and big values of v together with big values of $|u|$ through truncations (see (5.9) and (5.13)). This is particularly important to pass to the limit on the shift in the contraction inequality (note that the shift converges only in $L_{\text{loc}}^p(\mathbb{R}^+)$, for $1 \leq p < \infty$). Finally, it has to be shown that the shift converges to σt when the perturbation converges to 0. This can be obtained, thanks to the convergence of v in $C^0(\mathbb{R}^+, W_{\text{loc}}^{-s,1}(\mathbb{R}))$. It is interesting to note that the continuity (in time) of v is enough. We do not obtain any such control on u (nor h).

3. Preliminaries

3.1. Transformation of the system (1.16).

We here provides an equivalent version of Theorem 1.2 as in [33]. First of all, since the strength of the coefficient b in $\mu(v)$ does not affect our analysis, as in [33], we set $b = \gamma$ (for simplification) and introduce a new effective velocity

$$h := u + \left(p(v)^{\frac{\alpha}{\gamma}}\right)_x.$$

The system (1.16) with $\mu(v) = \gamma v^{-\alpha}$ is then transformed into

$$(3.1) \qquad \begin{cases} v_t - h_x = -\left(v^\beta p(v)_x\right)_x \\ h_t + p(v)_x = 0, \end{cases}$$

where $\beta := \gamma - \alpha$. Notice that the above system has a parabolic regularization on the specific volume, contrary to the regularization on the velocity for the original system (1.16). This is better for our analysis, since the hyperbolic part of the system is linear in u (or h) but nonlinear in v (via the pressure). This effective velocity was first introduced by Shelukhin [45] for $\alpha = 0$, and in the general case (in Eulerian coordinates) by Bresch-Desjardins [3, 4, 5], and Haspot [27, 26, 29]. It was also used in [51].

As mentioned in Theorem 1.2, we consider shock waves with suitably small amplitude ε. For that, let $(\tilde{v}_\varepsilon, \tilde{u}_\varepsilon)(x - \sigma_\varepsilon t)$ denote a shock wave with amplitude $|p(v_-) - p(v_+)| = \varepsilon$ as a solution of (1.6) with $\mu(v) = \gamma v^{-\gamma}$. Then, setting $\tilde{h}_\varepsilon := \tilde{u}_\varepsilon + \left(p(\tilde{v}_\varepsilon)^{\frac{\alpha}{\gamma}}\right)_x$, the shock wave $(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)(x - \sigma_\varepsilon t)$ satisfies

$$(3.2) \qquad \begin{cases} -\sigma_\varepsilon \tilde{v}_\varepsilon' - \tilde{h}_\varepsilon' = -\left(\tilde{v}_\varepsilon^\beta p(\tilde{v}_\varepsilon)'\right)' \\ -\sigma_\varepsilon \tilde{h}_\varepsilon' + p(\tilde{v}_\varepsilon)' = 0 \\ \lim_{\xi \to \pm\infty}(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)(\xi) = (v_\pm, u_\pm). \end{cases}$$

For simplification of our analysis, we rewrite (3.1) into the following system, based on the change of variable $(t, x) \mapsto (t, \xi = x - \sigma_\varepsilon t)$:

$$(3.3) \qquad \begin{cases} v_t - \sigma_\varepsilon v_\xi - h_\xi = -\left(v^\beta p(v)_\xi\right)_\xi \\ h_t - \sigma_\varepsilon h_\xi + p(v)_\xi = 0 \\ v|_{t=0} = v_0, \quad h|_{t=0} = u_0. \end{cases}$$

For the global-in-time existence of solutions to (3.3), we consider the function space:

$$(3.4) \qquad \begin{aligned} \mathcal{H}_T := \{(v, h) \in \mathbb{R}^+ \times \mathbb{R} \mid v - \underline{v} &\in C(0, T; H^1(\mathbb{R})), \\ h - \underline{u} &\in C(0, T; L^2(\mathbb{R})), \ v^{-1} \in L^\infty((0, T) \times \mathbb{R})\}, \end{aligned}$$

where $\underline{v}$ and $\underline{u}$ are as in (1.10).

Theorem 1.2 is a direct consequence of the following theorem on the contraction of shocks to the system (3.3).

Theorem 3.1. *Assume $\gamma > 1$ and $\alpha > 0$ satisfying $\alpha \leq \gamma \leq \alpha + 1$. For a given constant state $(v_-, u_-) \in \mathbb{R}^+ \times \mathbb{R}$, there exist constants $\varepsilon_0, \delta_0 > 0$ such that the following holds. For any $\varepsilon < \varepsilon_0$, $\delta_0^{-1}\varepsilon < \lambda < \delta_0$, and any $(v_+, u_+) \in \mathbb{R}^+ \times \mathbb{R}$ satisfying (1.5) with $|p(v_-) - p(v_+)| = \varepsilon$, there exists a smooth monotone function $a : \mathbb{R} \to \mathbb{R}^+$ with $\lim_{x \to \pm\infty} a(x) = 1 + a_\pm$ for some constants a_-, a_+ with $|a_- - a_+| = \lambda$ such that the following holds.*

Let $\tilde{U}_\varepsilon := (\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)$ be a viscous shock connecting (v_-, u_-) and (v_+, u_+) as a solution of (3.2). For a given $T > 0$, let $U := (v, h)$ be a solution in $\mathcal{H}_T$ to (3.3) with a initial datum $U_0 := (v_0, u_0)$ satisfying $\int_{-\infty}^{\infty} \eta(U_0|\tilde{U}_\varepsilon)dx < \infty$, there exists a shift function $X \in W^{1,1}((0,T))$ such that

$$
\begin{aligned}
(3.5) \quad & \int_{\mathbb{R}} a(\xi)\eta\big(U(t,\xi+X(t))|\tilde{U}_\varepsilon(\xi)\big)d\xi \\
& + \delta_0 \frac{\varepsilon}{\lambda} \int_0^T \int_{-\infty}^{\infty} \sigma a'(\xi) Q\left(v(t,\xi+X(t))|\tilde{v}_\varepsilon(\xi)\right) d\xi dt \\
& + \delta_0 \int_0^T \int_{-\infty}^{\infty} a(\xi) v^{\gamma-\alpha}(t,\xi+X(t)) \Big| \partial_x\big(p(v(t,\xi+X(t))) - p(\tilde{v}_\varepsilon(\xi)))\big) \Big|^2 d\xi dt \\
& \leq \int_{\mathbb{R}} a(\xi)\eta\big(U_0|\tilde{U}_\varepsilon\big)d\xi,
\end{aligned}
$$

and

$$
(3.6) \quad |\dot{X}(t)| \leq \frac{1}{\varepsilon^2}\Big(f(t) + C\int_{-\infty}^{\infty} \eta(U_0|\tilde{U}_\varepsilon)d\xi + 1\Big) \quad \text{for a.e. } t \in [0,T],
$$

for some positive function f satisfying $\quad \|f\|_{L^1(0,T)} \leq \frac{2\lambda}{\delta_0\varepsilon}\int_{-\infty}^{\infty} \eta(U_0|\tilde{U}_\varepsilon)d\xi.$

Remark 3.1. *1. In [33], the authors proved Theorem 1.2 in the case of $\alpha = \gamma$. Therefore, it suffices to prove the remaining cases where $0 < \alpha < \gamma \leq \alpha+1$. That is, $\beta = \gamma-\alpha \in (0,1]$. 2. Notice that it is enough to prove Theorem 3.1 for 1-shocks. Indeed, the result for 2-shocks is obtained by the change of variables $x \to -x$, $u \to -u$, $\sigma_\varepsilon \to -\sigma_\varepsilon$.*
Therefore, from now on, we only consider a 1-shock $(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)$, i.e., $v_- > v_+$, $u_- > u_+$, and

$$
(3.7) \quad \sigma_\varepsilon = -\sqrt{-\frac{p(v_+) - p(v_-)}{v_+ - v_-}}.
$$

Remark 3.2. *As mentioned in Remark 1.2, we consider the solution $(v,u) \in \mathcal{X}_T$ to (1.16). Then, (3.3) admits the solution (v,h) in $\mathcal{H}_T$. Indeed, since $v_t = u_x \in L^2(0,T;H^1(\mathbb{R}))$ by $(1.16)_1$, we have $v - \underline{v} \in C(0,T;H^1(\mathbb{R}))$. To show $h - \underline{u} \in C(0,T;L^2(\mathbb{R}))$, we first find that for $(v,u) \in \mathcal{X}_T$,*

$$
h - \underline{u} = u - \underline{u} + \frac{\alpha}{\gamma}p(v)^{\frac{\alpha}{\gamma}-1}v_x \in L^\infty(0,T;L^2(\mathbb{R})).
$$

Moreover, together with the fact that $v \in L^\infty((0,T) \times \mathbb{R})$ by Sobolev embedding, we find that

$$
\begin{aligned}
u_t &= -p'(v)v_x + \frac{d}{dv}\Big(\frac{\mu(v)}{v}\Big)v_x u_x + \frac{\mu(v)}{v}u_{xx} \in L^2(0,T;L^2(\mathbb{R})), \\
\Big(p(v)^{\frac{\alpha}{\gamma}-1}v_x\Big)_t &= (\frac{\alpha}{\gamma}-1)p(v)^{\frac{\alpha}{\gamma}-2}v_t v_x + p(v)^{\frac{\alpha}{\gamma}-1}v_{xt} \in L^2(0,T;L^2(\mathbb{R})),
\end{aligned}
$$

which implies $h_t \in L^2(0,T;L^2(\mathbb{R}))$, and therefore $h - \underline{u} \in C(0,T;L^2(\mathbb{R}))$.

3.2. **Global and local estimates on the relative quantities.** We here present useful inequalities on Q and p that are crucially used for the proofs of main results. First, the following lemma provides some global inequalities on the relative function $Q(\cdot|\cdot)$ corresponding to the convex function $Q(v) = \frac{v^{-\gamma+1}}{\gamma-1}$, $v > 0$, $\gamma > 1$.

Lemma 3.1. *For given constants $\gamma > 1$, and $v_- > 0$, there exist constants $c_1, c_2 > 0$ such that the following inequalities hold.*
1) For any $w \in (0, v_-)$,

(3.8)
$$Q(v|w) \geq c_1 |v - w|^2, \quad \text{for all } 0 < v \leq 3v_-,$$
$$Q(v|w) \geq c_2 |v - w|, \quad \text{for all } v \geq 3v_-.$$

2) Moreover if $0 < w \leq u \leq v$ or $0 < v \leq u \leq w$ then

(3.9)
$$Q(v|w) \geq Q(u|w),$$

and for any $\delta_* > 0$ there exists a constant $C > 0$ such that if, in addition, $v_- > w > v_- - \delta_*/2$ and $|w - u| > \delta_*$, we have

(3.10)
$$Q(v|w) - Q(u|w) \geq C|u - v|.$$

3) For any $w \in (v_-/4, v_-)$,

(3.11)
$$|p(v) - p(w)| \leq c_5 |v - w|, \quad \text{for all } v \geq v_-/2,$$

Proof. We refer to [33, Lemma 2.4, 2.5]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Next, we use (3.8) and (3.9) above to prove the following lemma, which is used for the proof of Theorem 1.1.

Lemma 3.2. *For given constants $\gamma, M > 1$, there exist constants $C > 0$ and $k_0 > 1$ such that the following inequalities hold.*
1) For any $k \geq 3M$ and $w \in (M^{-1}, M)$,

(3.12)
$$\max\{(k^{-1} - v)_+, (v - k)_+\} \leq CQ(v|w), \quad \text{for any } v > 0.$$

2) For any $w_1, w_2 \in (M^{-1}, M)$,

(3.13)
$$Q(v|w_1) \leq CQ(v|w_2), \quad \text{for any } v \in (0, k_0^{-1}) \cup (k_0, \infty).$$

Proof. • *proof of* (3.12) : i) If $k^{-1} \leq v \leq k$, then $\max\{(k^{-1} - v)_+, (v - k)_+\} = 0 \leq CQ(v|w)$.
ii) If $0 < v < k^{-1}$, we have

(3.14)
$$\max\{(k^{-1} - v)_+, (v - k)_+\} = k^{-1} - v < k^{-1}.$$

Since $v < k^{-1} \leq M^{-1}/3$, we use (3.8) and (3.9) to have

$$Q(v|w) \geq Q(M^{-1}/3|w) \geq c_1 |M^{-1}/3 - w|^2.$$

Moreover, since $w > M^{-1}$, we have

$$Q(v|w) \geq c_1 4M^{-2}/9 \geq (c_1 4M^{-1}/3)k^{-1},$$

which together with (3.14) implies the desired inequality.
iii) If $v > k$, we have

$$\max\{(k^{-1} - v)_+, (v - k)_+\} = v - k \leq v - 3M.$$

Likewise, using (3.8), we have

$$Q(v|w) \geq c_2 |v - w|.$$

Since $v > 3M > w$, we have

$$Q(v|w) \geq c_2(v - 3M),$$

which completes the desired inequality.

• *proof of* (3.13) : We set $C := 2 \max\left\{\frac{Q'(M)}{Q'(M^{-1})}, 1\right\}$. Since $Q' < 0$, there exists $k_1 > 1$ such that for all $v > k_1$,

$$-Q'(M)v \geq (1-C)Q(v) + \big(Q'(M)M - Q(M)\big) - C\big(Q'(M^{-1})M^{-1} - Q(M^{-1})\big).$$

Moreover, since Q' is increasing and $\frac{d}{dv}\big(Q'(v)v - Q(v)\big) > 0$, we have

$$\begin{aligned}
\big(-CQ'(w_2) + Q'(w_1)\big)v &\geq \big(-CQ'(M^{-1}) + Q'(M)\big)v \geq -Q'(M)v \\
&\geq (1-C)Q(v) + \big(Q'(M)M - Q(M)\big) - C\big(Q'(M^{-1})M^{-1} - Q(M^{-1})\big) \\
&\geq (1-C)Q(v) + \big(Q'(w_1)w_1 - Q(w_1)\big) - C\big(Q'(w_2)w_2 - Q(w_2)\big).
\end{aligned}$$

which together with the definition of $Q(\cdot|\cdot)$ yields that

$$Q(v|w_1) \leq CQ(v|w_2), \quad \text{for all } v > k_1.$$

On the other hand, since $Q(v) \to +\infty$ as $v \to 0+$, there exists $k_0 > k_1$ such that for all $v < k_0^{-1}$,

$$Q(v) \geq \big(CQ'(M^{-1}) - Q'(M)\big)v + \big(Q'(M)M - Q(M)\big) - C\big(Q'(M^{-1})M^{-1} - Q(M^{-1})\big).$$

Then we have

$$\begin{aligned}
(C-1)Q(v) &\geq Q(v) \\
&\geq \big(CQ'(M^{-1}) - Q'(M)\big)v + \big(Q'(M)M - Q(M)\big) - C\big(Q'(M^{-1})M^{-1} - Q(M^{-1})\big) \\
&\geq \big(CQ'(w_2) - Q'(w_1)\big)v + \big(Q'(w_1)w_1 - Q(w_1)\big) - C\big(Q'(w_2)w_2 - Q(w_2)\big),
\end{aligned}$$

which yields that $Q(v|w_1) \leq CQ(v|w_2)$ for all $v < k_0^{-1}$. $\qquad\square$

We present now some local estimates on $p(v|w)$ and $Q(v|w)$.

Lemma 3.3. *For given constants $\gamma > 1$ and $v_- > 0$ there exist positive constants C and δ_* such that for any $0 < \delta < \delta_*$, the following is true.*
1) For any $(v, w) \in \mathbb{R}_+^2$ satisfying $|p(v) - p(w)| < \delta$ and $|p(w) - p(v_-)| < \delta$,

$$(3.15) \qquad p(v|w) \leq \left(\frac{\gamma+1}{2\gamma}\frac{1}{p(w)} + C\delta\right)|p(v) - p(w)|^2.$$

2) For any $(v, w) \in \mathbb{R}_+^2$ such that $|p(w) - p(v_-)| \leq \delta$, and satisfying either $Q(v|w) < \delta$ or $|p(v) - p(w)| < \delta$,

$$(3.16) \qquad |p(v) - p(w)|^2 \leq CQ(v|w).$$

Proof. We refer to [33, Lemma 2.6]. $\qquad\square$

4. Proof of Theorem 3.1

Throughout this section, C denotes a positive constant which may change from line to line, but which stays independent on ε (the shock strength) and λ (the total variation of the function a). We will consider two smallness conditions, one on ε, and the other on ε/λ. In the argument, ε will be far smaller than ε/λ .

4.1. **Properties of small shock waves.** In this subsection, we present useful properties of the 1-shock waves $(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)$ with small amplitude ε. In the sequel, without loss of generality, we consider the 1-shock wave $(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)$ satisfying $\tilde{v}_\varepsilon(0) = \frac{v_- + v_+}{2}$. Notice that the estimates in the following lemma also hold for $\tilde{h}_\varepsilon$ since we have $\tilde{h}'_\varepsilon = \frac{p'(\tilde{v}_\varepsilon)}{\sigma_\varepsilon}\tilde{v}'_\varepsilon$ and $C^{-1} \le \frac{p'(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \le C$. But, since the below estimates for $\tilde{v}_\varepsilon$ are enough in our analysis, we give the estimates only for $\tilde{v}_\varepsilon$.

Lemma 4.1. *We fix $v_- > 0$ and $h_- \in \mathbb{R}$. Then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the following is true. Let $\tilde{v}_\varepsilon$ be the 1-shock wave with amplitude $|p(v_-) - p(v_+)| = \varepsilon$ and such that $\tilde{v}_\varepsilon(0) = \frac{v_- + v_+}{2}$. Then, there exist constants $C, C_1, C_2 > 0$ such that*

$$(4.1) \qquad -C^{-1}\varepsilon^2 e^{-C_1 \varepsilon |\xi|} \le \tilde{v}'_\varepsilon(\xi) \le -C\varepsilon^2 e^{-C_2 \varepsilon |\xi|}, \quad \forall \xi \in \mathbb{R}.$$

Therefore, as a consequence, we have

$$(4.2) \qquad \inf_{\left[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right]} |v'_\varepsilon| \ge C\varepsilon^2.$$

Proof. Using $v_-/2 < \tilde{v}_\varepsilon < v_-$, the proof follows the same arguments as in [33, Lemma 2.1]. Therefore, we omit its details. $\qquad\square$

4.2. **Relative entropy method.** Our analysis is based on the relative entropy. The method is purely nonlinear, and allows to handle rough and large perturbations. The relative entropy method was first introduced by Dafermos [16] and Diperna [19] to prove the L^2 stability and uniqueness of Lipschitz solutions to the hyperbolic conservation laws endowed with a convex entropy.

To use the relative entropy method, we rewrite (3.3) into the following general system of viscous conservation laws:

$$(4.3) \qquad \partial_t U + \partial_\xi A(U) = \begin{pmatrix} -\partial_\xi \left(v^\beta \partial_\xi p(v) \right) \\ 0 \end{pmatrix},$$

where

$$U := \begin{pmatrix} v \\ h \end{pmatrix}, \quad A(U) := \begin{pmatrix} -\sigma_\varepsilon v - h \\ -\sigma_\varepsilon h + p(v) \end{pmatrix}.$$

The system (4.3) has a convex entropy $\eta(U) := \frac{h^2}{2} + Q(v)$, where $Q(v) = \frac{v^{-\gamma+1}}{\gamma-1}$, i.e., $Q'(v) = -p(v)$.
Using the derivative of the entropy as

$$(4.4) \qquad \nabla\eta(U) = \begin{pmatrix} -p(v) \\ h \end{pmatrix},$$

the above system (4.3) can be rewritten as

$$(4.5) \qquad \partial_t U + \partial_\xi A(U) = \partial_\xi \left(M(U)\partial_\xi \nabla\eta(U) \right),$$

where $M(U) = \begin{pmatrix} v^\beta & 0 \\ 0 & 0 \end{pmatrix}$, and (3.2) can be rewritten as

$$(4.6) \qquad \partial_\xi A(\tilde{U}_\varepsilon) = \partial_\xi \left(M(\tilde{U}_\varepsilon)\partial_\xi \nabla\eta(\tilde{U}_\varepsilon) \right).$$

Consider the relative entropy function defined by

$$\eta(U|V) = \eta(U) - \eta(V) - \nabla\eta(V)(U - V),$$

and the relative flux defined by

$$A(U|V) = A(U) - A(V) - \nabla A(V)(U - V).$$

Let $G(\cdot; \cdot)$ be the flux of the relative entropy defined by

$$G(U; V) = G(U) - G(V) - \nabla \eta(V)(A(U) - A(V)),$$

where G is the entropy flux of η, i.e., $\partial_i G(U) = \sum_{k=1}^2 \partial_k \eta(U) \partial_i A_k(U)$, $\quad 1 \le i \le 2$.
Then, for our system (4.3), we have

(4.7)
$$\eta(U|\tilde{U}_\varepsilon) = \frac{|h - \tilde{h}_\varepsilon|^2}{2} + Q(v|\tilde{v}_\varepsilon),$$

$$A(U|\tilde{U}_\varepsilon) = \begin{pmatrix} 0 \\ p(v|\tilde{v}_\varepsilon) \end{pmatrix},$$

$$G(U; \tilde{U}_\varepsilon) = (p(v) - p(\tilde{v}_\varepsilon))(h - \tilde{h}_\varepsilon) - \sigma_\varepsilon \eta(U|\tilde{U}_\varepsilon),$$

where the relative pressure is defined as

(4.8)
$$p(v|w) = p(v) - p(w) - p'(w)(v - w).$$

We consider a weighted relative entropy between the solution U of (4.5) and the viscous
shock $\tilde{U}_\varepsilon := \begin{pmatrix} \tilde{v}_\varepsilon \\ \tilde{h}_\varepsilon \end{pmatrix}$ in (3.2) up to a shift $X(t)$:

$$a(\xi)\eta\big(U(t, \xi + X(t))|\tilde{U}_\varepsilon(\xi)\big).$$

where a is a smooth weight function.

In Lemma 4.2, we will derive a quadratic structure on $\frac{d}{dt} \int_\mathbb{R} a(\xi)\eta\big(U(t, \xi + X(t))|\tilde{U}_\varepsilon(\xi)\big)d\xi$.
For that, we introduce a simple notation: for any function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and the shift
$X(t)$,

$$f^{\pm X}(t, \xi) := f(t, \xi \pm X(t)).$$

We also introduce the function space:

$$\mathcal{H} := \{(v, h) \in \mathbb{R}^+ \times \mathbb{R} \mid v^{-1}, v \in L^\infty(\mathbb{R}), \ h - \tilde{h}_\varepsilon \in L^2(\mathbb{R}), \ \partial_\xi\big(p(v) - p(\tilde{v}_\varepsilon)\big) \in L^2(\mathbb{R})\},$$

on which the functionals $Y, \mathcal{J}^{bad}, \mathcal{J}^{good}$ in (4.10) are well-defined for all $t \in (0, T)$.

Remark 4.1. *As mentioned in Remark 3.2, we consider the solution $(v, h) \in \mathcal{H}_T$ to (3.3).
Then, using the fact that $v_\xi \in C(0, T; L^2(\mathbb{R}))$, $\tilde{v}'_\varepsilon \in L^2(\mathbb{R})$, and $v^{-1}, v \in C(0, T; L^\infty(\mathbb{R}))$,
we find*

$$\partial_\xi\big(p(v) - p(\tilde{v}_\varepsilon)\big) \in C(0, T; L^2(\mathbb{R})),$$

which implies $(v, h)(t) \in \mathcal{H}$ for all $t \in [0, T]$.

Lemma 4.2. *Let $a : \mathbb{R} \to \mathbb{R}^+$ be any positive smooth bounded function whose derivative
is bounded and integrable. Let $\tilde{U}_\varepsilon := \begin{pmatrix} \tilde{v}_\varepsilon \\ \tilde{h}_\varepsilon \end{pmatrix}$ be the viscous shock in (3.2). For any solution
$U \in \mathcal{H}_T$ to (4.5), and any absolutely continuous shift $X : [0, T] \to \mathbb{R}$, the following holds.*

(4.9)
$$\frac{d}{dt} \int_\mathbb{R} a(\xi)\eta(U^X(t, \xi)|\tilde{U}_\varepsilon(\xi))d\xi = \dot{X}(t)Y(U^X) + \mathcal{J}^{bad}(U^X) - \mathcal{J}^{good}(U^X),$$

where

(4.10)

$$Y(U) := -\int_{\mathbb{R}} a'\eta(U|\tilde{U}_\varepsilon)d\xi + \int_{\mathbb{R}} a\partial_\xi \nabla\eta(\tilde{U}_\varepsilon)(U - \tilde{U}_\varepsilon)d\xi,$$

$$\mathcal{J}^{bad}(U) := \int_{\mathbb{R}} a'\big(p(v) - p(\tilde{v}_\varepsilon)\big)(h - \tilde{h}_\varepsilon)d\xi + \sigma_\varepsilon \int_{\mathbb{R}} a\partial_\xi \tilde{v}_\varepsilon p(v|\tilde{v}_\varepsilon)d\xi$$

$$- \int_{\mathbb{R}} a'v^\beta \big(p(v) - p(\tilde{v}_\varepsilon)\big)\partial_\xi\big(p(v) - p(\tilde{v}_\varepsilon)\big)d\xi - \int_{\mathbb{R}} a'\big(p(v) - p(\tilde{v}_\varepsilon)\big)(v^\beta - \tilde{v}_\varepsilon^\beta)\partial_\xi p(\tilde{v}_\varepsilon)d\xi$$

$$- \int_{\mathbb{R}} a\partial_\xi\big(p(v) - p(\tilde{v}_\varepsilon)\big)(v^\beta - \tilde{v}_\varepsilon^\beta)\partial_\xi p(\tilde{v}_\varepsilon)d\xi,$$

$$\mathcal{J}^{good}(U) := \frac{\sigma_\varepsilon}{2}\int_{\mathbb{R}} a'\left|h - \tilde{h}_\varepsilon\right|^2 d\xi + \sigma_\varepsilon \int_{\mathbb{R}} a'Q(v|\tilde{v}_\varepsilon)d\xi + \int_{\mathbb{R}} av^\beta \left|\partial_\xi\big(p(v) - p(\tilde{v}_\varepsilon)\big)\right|^2 d\xi.$$

Remark 4.2. *In what follows, we will define the weight function a such that $\sigma_\varepsilon a' > 0$. Therefore, $-\mathcal{J}^{good}$ consists of three good terms, while $\mathcal{J}^{bad}$ consists of bad terms.*

Proof. To derive the desired structure, we use here a change of variable $\xi \mapsto \xi - X(t)$ as

(4.11)
$$\int_{\mathbb{R}} a(\xi)\eta(U^X(t,\xi)|\tilde{U}_\varepsilon(\xi))d\xi = \int_{\mathbb{R}} a^{-X}(\xi)\eta(U(t,\xi)|\tilde{U}_\varepsilon^{-X}(\xi))d\xi.$$

Then, using the same computation in [33, Lemma 2.3] (see also [48, Lemma 4]), we have

$$\frac{d}{dt}\int_{\mathbb{R}} a^{-X}(\xi)\eta(U(t,\xi)|\tilde{U}_\varepsilon^{-X}(\xi))d\xi$$

$$= -\dot{X}\int_{\mathbb{R}} a'^{-X}\eta(U|\tilde{U}_\varepsilon^{-X})d\xi + \int_{\mathbb{R}} a^{-X}\left[\Big(\nabla\eta(U) - \nabla\eta(\tilde{U}_\varepsilon^{-X})\Big)\Big(-\partial_\xi A(U) + \partial_\xi\Big(M(U)\partial_\xi\nabla\eta(U)\Big)\Big)\right.$$

$$\left. - \nabla^2\eta(\tilde{U}_\varepsilon^{-X})(U - \tilde{U}_\varepsilon^{-X})\Big(-\dot{X}\partial_\xi\tilde{U}_\varepsilon^{-X} - \partial_\xi A(\tilde{U}_\varepsilon^{-X}) + \partial_\xi\Big(M(\tilde{U}_\varepsilon^{-X})\partial_\xi\nabla\eta(\tilde{U}_\varepsilon^{-X})\Big)\Big)\right]d\xi$$

$$= \dot{X}\Big(-\int_{\mathbb{R}} a'^{-X}\eta(U|\tilde{U}_\varepsilon^{-X})d\xi + \int_{\mathbb{R}} a^{-X}\partial_\xi\nabla\eta(\tilde{U}_\varepsilon^{-X})(U - \tilde{U}_\varepsilon^{-X})\Big) + I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 := -\int_{\mathbb{R}} a^{-X}\partial_\xi G(U;\tilde{U}_\varepsilon^{-X})d\xi,$$

$$I_2 := -\int_{\mathbb{R}} a^{-X}\partial_\xi\nabla\eta(\tilde{U}_\varepsilon^{-X})A(U|\tilde{U}_\varepsilon^{-X})d\xi,$$

$$I_3 := \int_{\mathbb{R}} a^{-X}\Big(\nabla\eta(U) - \nabla\eta(\tilde{U}_\varepsilon^{-X})\Big)\partial_\xi\Big(M(U)\partial_\xi\big(\nabla\eta(U) - \nabla\eta(\tilde{U}_\varepsilon^{-X})\big)\Big)d\xi,$$

$$I_4 := \int_{\mathbb{R}} a^{-X}\Big(\nabla\eta(U) - \nabla\eta(\tilde{U}_\varepsilon^{-X})\Big)\partial_\xi\Big(\big(M(U) - M(\tilde{U}_\varepsilon^{-X})\big)\partial_\xi\nabla\eta(\tilde{U}_\varepsilon^{-X})\Big)d\xi$$

$$I_5 := \int_{\mathbb{R}} a^{-X}(\nabla\eta)(U|\tilde{U}_\varepsilon^{-X})\partial_\xi\Big(M(\tilde{U}_\varepsilon^{-X})\partial_\xi\nabla\eta(\tilde{U}_\varepsilon^{-X})\Big)d\xi.$$

Using (4.7) and (4.4), we have

$$I_1 = \int_{\mathbb{R}} a'^{-X} G(U; \tilde{U}_\varepsilon^{-X}) d\xi = \int_{\mathbb{R}} a'^{-X} \Big(\big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big)\big(h - \tilde{h}_\varepsilon^{-X}\big) - \sigma_\varepsilon \eta(U|\tilde{U}_\varepsilon^{-X}) \Big) d\xi$$

$$= \int_{\mathbb{R}} a'^{-X} \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big)\big(h - \tilde{h}_\varepsilon^{-X}\big) d\xi - \frac{\sigma_\varepsilon}{2} \int_{\mathbb{R}} a'^{-X} \left| h - \tilde{h}_\varepsilon \right|^2 d\xi - \sigma_\varepsilon \int_{\mathbb{R}} a'^{-X} Q(v|\tilde{v}_\varepsilon) d\xi,$$

$$I_2 = - \int_{\mathbb{R}} a^{-X} \partial_\xi \tilde{h}_\varepsilon^{-X} p(v|\tilde{v}_\varepsilon^{-X}) d\xi.$$

By integration by parts, we have

$$I_3 = \int_{\mathbb{R}} a^{-X} \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big) \partial_\xi \Big(v^\beta \partial_\xi \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big) \Big) d\xi$$

$$= - \int_{\mathbb{R}} a^{-X} v^\beta |\partial_\xi \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big)|^2 d\xi - \int_{\mathbb{R}} a'^{-X} v^\beta \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big) \partial_\xi \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big) d\xi,$$

$$I_4 = \int_{\mathbb{R}} a^{-X} \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big) \partial_\xi \Big((v^\beta - \tilde{v}_\varepsilon^\beta) \partial_\xi p(\tilde{v}_\varepsilon^{-X}) \Big) d\xi$$

$$= - \int_{\mathbb{R}} a'^{-X} \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big)(v^\beta - \tilde{v}_\varepsilon^\beta) \partial_\xi p(\tilde{v}_\varepsilon^{-X}) d\xi$$

$$- \int_{\mathbb{R}} a^{-X} \partial_\xi \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big)(v^\beta - \tilde{v}_\varepsilon^\beta) \partial_\xi p(\tilde{v}_\varepsilon^{-X}) d\xi.$$

Since it follows from (4.6) and (4.4) that

$$I_5 = \int_{\mathbb{R}} a^{-X} (\nabla \eta)(U|\tilde{U}_\varepsilon^{-X}) \partial_\xi A(\tilde{U}_\varepsilon^{-X}) d\xi = \int_{\mathbb{R}} a^{-X} p(v|\tilde{v}_\varepsilon^{-X}) \Big(\partial_\xi \tilde{h}_\varepsilon^{-X} + \sigma_\varepsilon \partial_\xi \tilde{v}_\varepsilon^{-X} \Big) d\xi,$$

we have some cancellation

$$I_2 + I_5 = \sigma_\varepsilon \int_{\mathbb{R}} a^{-X} \partial_\xi \tilde{v}_\varepsilon^{-X} p(v|\tilde{v}_\varepsilon^{-X}) d\xi.$$

Therefore, we have

$$\frac{d}{dt} \int_{\mathbb{R}} a^{-X} \eta(U|\tilde{U}_\varepsilon^{-X}) d\xi$$

$$= \dot{X} \Big(- \int_{\mathbb{R}} a'^{-X} \eta(U|\tilde{U}_\varepsilon^{-X}) d\xi + \int_{\mathbb{R}} a^{-X} \partial_\xi \nabla \eta(\tilde{U}_\varepsilon^{-X})(U - \tilde{U}_\varepsilon^{-X}) d\xi \Big)$$

$$+ \int_{\mathbb{R}} a'^{-X} \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big)\big(h - \tilde{h}_\varepsilon^{-X}\big) d\xi - \frac{\sigma_\varepsilon}{2} \int_{\mathbb{R}} a'^{-X} \left| h - \tilde{h}_\varepsilon \right|^2 d\xi - \sigma_\varepsilon \int_{\mathbb{R}} a'^{-X} Q(v|\tilde{v}_\varepsilon) d\xi$$

$$+ \sigma_\varepsilon \int_{\mathbb{R}} a^{-X} \partial_\xi \tilde{v}_\varepsilon^{-X} p(v|\tilde{v}_\varepsilon^{-X}) d\xi - \int_{\mathbb{R}} a'^{-X} v^\beta \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big) \partial_\xi \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big) d\xi$$

$$- \int_{\mathbb{R}} a'^{-X} \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big) \Big(v^\beta - (\tilde{v}_\varepsilon^\beta)^{-X} \Big) \partial_\xi p(\tilde{v}_\varepsilon^{-X}) d\xi$$

$$- \int_{\mathbb{R}} a^{-X} \partial_\xi \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big) \Big(v^\beta - (\tilde{v}_\varepsilon^\beta)^{-X} \Big) \partial_\xi p(\tilde{v}_\varepsilon^{-X}) d\xi - \int_{\mathbb{R}} a^{-X} v^\beta |\partial_\xi \big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big)|^2 d\xi.$$

Again, we use a change of variable $\xi \mapsto \xi + X(t)$ to have

$$
\frac{d}{dt}\int_{\mathbb{R}} a\eta(U^X|\tilde{U}_\varepsilon)d\xi
$$

$$
= \dot{X}\Big(-\int_{\mathbb{R}} a'\eta(U^X|\tilde{U}_\varepsilon)d\xi + \int_{\mathbb{R}} a\partial_\xi\nabla\eta(\tilde{U}_\varepsilon)(U^X - \tilde{U}_\varepsilon)d\xi\Big)
$$

$$
+ \int_{\mathbb{R}} a'\big(p(v^X) - p(\tilde{v}_\varepsilon)\big)(h^X - \tilde{h}_\varepsilon)d\xi - \frac{\sigma_\varepsilon}{2}\int_{\mathbb{R}} a'\left|h^X - \tilde{h}_\varepsilon\right|^2 d\xi - \sigma_\varepsilon\int_{\mathbb{R}} a'Q(v^X|\tilde{v}_\varepsilon)d\xi
$$

$$
+ \sigma_\varepsilon\int_{\mathbb{R}} a\partial_\xi\tilde{v}_\varepsilon p(v^X|\tilde{v}_\varepsilon)d\xi - \int_{\mathbb{R}} a'(v^\beta)^X\big(p(v^X) - p(\tilde{v}_\varepsilon)\big)\partial_\xi\big(p(v^X) - p(\tilde{v}_\varepsilon)\big)d\xi
$$

$$
- \int_{\mathbb{R}} a'\big(p(v^X) - p(\tilde{v}_\varepsilon)\big)\left((v^\beta)^X - \tilde{v}_\varepsilon^\beta\right)\partial_\xi p(\tilde{v}_\varepsilon)d\xi
$$

$$
- \int_{\mathbb{R}} a\partial_\xi\big(p(v^X) - p(\tilde{v}_\varepsilon)\big)\left((v^\beta)^X - \tilde{v}_\varepsilon^\beta\right)\partial_\xi p(\tilde{v}_\varepsilon)d\xi - \int_{\mathbb{R}} a(v^\beta)^X|\partial_\xi(p(v^X) - p(\tilde{v}_\varepsilon))|^2 d\xi,
$$

which provides the desired representation. $\qquad\square$

4.3. Construction of the weight function. We define the weight function a by

$$
(4.12) \qquad\qquad a(\xi) = 1 - \lambda\frac{p(\tilde{v}_\varepsilon(\xi)) - p(v_-)}{[p]},
$$

where $[p] := p(v_+) - p(v_-)$. We briefly present some useful properties on the weight a. First of all, the weight function a is positive and decreasing, and satisfies $1 - \lambda \leq a \leq 1$. Since $[p] = \varepsilon$, $p'(v_-/2) \leq p'(\tilde{v}_\varepsilon) \leq p'(v_-)$ and

$$
(4.13) \qquad\qquad a' = -\lambda\frac{\partial_\xi p(\tilde{v}_\varepsilon)}{[p]},
$$

we have

$$
(4.14) \qquad\qquad |a'| \sim \frac{\lambda}{\varepsilon}|\tilde{v}_\varepsilon'|.
$$

4.4. Maximization in terms of $h - \tilde{h}_\varepsilon$. In order to estimate the right-hand side of (4.15), we will use Proposition 4.2, i.e., a sharp estimate with respect to $v - \tilde{v}_\varepsilon$ when $v - \tilde{v}_\varepsilon \ll 1$, for which we need to rewrite $\mathcal{J}^{bad}$ on the right-hand side of (4.9) only in terms of v near $\tilde{v}_\varepsilon$, by separating $h - \tilde{h}_\varepsilon$ from the first term of $\mathcal{J}^{bad}$. Therefore, we will rewrite $\mathcal{J}^{bad}$ into the maximized representation in terms of $h - \tilde{h}_\varepsilon$ in the following lemma. However, we will keep all terms of $\mathcal{J}^{bad}$ in a region $\{p(v) - p(\tilde{v}_\varepsilon) > \delta\}$ for small values of v, since we use the estimate (4.70) to control the first term of $\mathcal{J}^{bad}$ in that region.

Lemma 4.3. *Let* $a : \mathbb{R} \to \mathbb{R}^+$ *be as in* (4.12), *and* $\tilde{U}_\varepsilon = \left(\begin{smallmatrix}\tilde{v}_\varepsilon \\ \tilde{h}_\varepsilon\end{smallmatrix}\right)$ *be the viscous shock in* (3.2). *Let* δ *be any positive constant. Then, for any* $U \in \mathcal{H}$,

$$
(4.15) \qquad\qquad \mathcal{J}^{bad}(U) - \mathcal{J}^{good}(U) = \mathcal{B}_\delta(U) - \mathcal{G}_\delta(U),
$$

where

(4.16)
$$\mathcal{B}_\delta(U) := \sigma_\varepsilon \int_\mathbb{R} a\partial_\xi \tilde{v}_\varepsilon p(v|\tilde{v}_\varepsilon) d\xi + \frac{1}{2\sigma_\varepsilon} \int_\mathbb{R} a'|p(v) - p(\tilde{v}_\varepsilon)|^2 \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta\}} d\xi$$

$$+ \int_\mathbb{R} a'\big(p(v) - p(\tilde{v}_\varepsilon)\big)(h - \tilde{h}_\varepsilon)\mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) > \delta\}} d\xi$$

$$- \int_\mathbb{R} a'v^\beta\big(p(v) - p(\tilde{v}_\varepsilon)\big)\partial_\xi\big(p(v) - p(\tilde{v}_\varepsilon)\big) d\xi - \int_\mathbb{R} a'\big(p(v) - p(\tilde{v}_\varepsilon)\big)(v^\beta - \tilde{v}_\varepsilon^\beta)\partial_\xi p(\tilde{v}_\varepsilon) d\xi$$

$$- \int_\mathbb{R} a\partial_\xi\big(p(v) - p(\tilde{v}_\varepsilon)\big)(v^\beta - \tilde{v}_\varepsilon^\beta)\partial_\xi p(\tilde{v}_\varepsilon) d\xi,$$

$$\mathcal{G}_\delta(U) := \frac{\sigma_\varepsilon}{2} \int_\mathbb{R} a' \left| h - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \right|^2 \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta\}} d\xi + \frac{\sigma_\varepsilon}{2} \int_\mathbb{R} a' \left| h - \tilde{h}_\varepsilon \right|^2 \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) > \delta\}} d\xi$$

$$+ \sigma_\varepsilon \int_\mathbb{R} a'Q(v|\tilde{v}_\varepsilon) d\xi + \int_\mathbb{R} av^\beta |\partial_\xi\big(p(v) - p(\tilde{v}_\varepsilon)\big)|^2 d\xi.$$

Remark 4.3. *Since $\sigma_\varepsilon a' > 0$ and $a > 0$, $-\mathcal{G}_\delta$ consists of four good terms.*

Proof. For any fixed $\delta > 0$, we first rewrite $\mathcal{J}^{bad}$ and $-\mathcal{J}^{good}$ into

$$\mathcal{J}^{bad}(U) := \underbrace{\int_\mathbb{R} a'\big(p(v) - p(\tilde{v}_\varepsilon)\big)(h - \tilde{h}_\varepsilon)\mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta\}} d\xi}_{=:J_1}$$

$$+ \int_\mathbb{R} a'\big(p(v) - p(\tilde{v}_\varepsilon)\big)(h - \tilde{h}_\varepsilon)\mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) > \delta\}} d\xi + \sigma_\varepsilon \int_\mathbb{R} a\partial_\xi \tilde{v}_\varepsilon p(v|\tilde{v}_\varepsilon) d\xi$$

$$- \int_\mathbb{R} a'v^\beta\big(p(v) - p(\tilde{v}_\varepsilon)\big)\partial_\xi\big(p(v) - p(\tilde{v}_\varepsilon)\big) d\xi - \int_\mathbb{R} a'\big(p(v) - p(\tilde{v}_\varepsilon)\big)(v^\beta - \tilde{v}_\varepsilon^\beta)\partial_\xi p(\tilde{v}_\varepsilon) d\xi$$

$$- \int_\mathbb{R} a\partial_\xi\big(p(v) - p(\tilde{v}_\varepsilon)\big)(v^\beta - \tilde{v}_\varepsilon^\beta)\partial_\xi p(\tilde{v}_\varepsilon) d\xi,$$

and

$$- \mathcal{J}^{good}(U) := \underbrace{-\frac{\sigma_\varepsilon}{2} \int_\mathbb{R} a' \left| h - \tilde{h}_\varepsilon \right|^2 \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta\}} d\xi}_{=:J_2} - \frac{\sigma_\varepsilon}{2} \int_\mathbb{R} a' \left| h - \tilde{h}_\varepsilon \right|^2 d\xi \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) > \delta\}} d\xi$$

$$- \sigma_\varepsilon \int_\mathbb{R} a'Q(v|\tilde{v}_\varepsilon) d\xi - \int_\mathbb{R} av^\beta \left| \partial_\xi\big(p(v) - p(\tilde{v}_\varepsilon)\big) \right|^2 d\xi.$$

Applying the quadratic identity $\alpha x^2 + \beta x = \alpha(x + \frac{\beta}{2\alpha})^2 - \frac{\beta^2}{4\alpha}$ with $x := h - \tilde{h}_\varepsilon$ to the integrands of $J_1 + J_2$, we find

$$-\frac{\sigma_\varepsilon}{2} \left| h - \tilde{h}_\varepsilon \right|^2 + \big(p(v) - p(\tilde{v}_\varepsilon)\big)(h - \tilde{h}_\varepsilon) = -\frac{\sigma_\varepsilon}{2} \left| h - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \right|^2 + \frac{1}{2\sigma_\varepsilon}|p(v) - p(\tilde{v}_\varepsilon)|^2.$$

Therefore, we have the desired representation (4.15)-(4.16). $\square$

4.5. **Main proposition.** The main proposition for the proof of Theorem 3.1 is the following.

Proposition 4.1. *There exist $\varepsilon_0, \delta_0, \delta_3 \in (0, 1/2)$ such that for any $\varepsilon < \varepsilon_0$ and $\delta_0^{-1}\varepsilon < \lambda < \delta_0$, the following is true.*
For any $U \in \mathcal{H} \cap \{U \mid |Y(U)| \le \varepsilon^2\}$,

(4.17)
$$\mathcal{R}(U) := -\frac{1}{\varepsilon^4}Y^2(U) + \mathcal{B}_{\delta_3}(U) + \delta_0\frac{\varepsilon}{\lambda}|\mathcal{B}_{\delta_3}(U)|$$
$$- \mathcal{G}_1^-(U) - \mathcal{G}_1^+(U) - \left(1 - \delta_0\frac{\varepsilon}{\lambda}\right)\mathcal{G}_2(U) - (1 - \delta_0)\mathcal{D}(U) \le 0,$$

where Y and $\mathcal{B}_{\delta_3}$ are as in (4.10) and (4.16), and $\mathcal{G}_1^-, \mathcal{G}_1^+, \mathcal{G}_2, \mathcal{D}$ denote the four terms of $\mathcal{G}_{\delta_3}$ as follows:

(4.18)
$$\mathcal{G}_1^-(U) := \frac{\sigma_\varepsilon}{2}\int_{\Omega^c} a'|h - \tilde{h}_\varepsilon|^2\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}}d\xi,$$
$$\mathcal{G}_1^+(U) := \frac{\sigma_\varepsilon}{2}\int_{\Omega} a'\left(h - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon}\right)^2\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)\le\delta_3\}}d\xi,$$
$$\mathcal{G}_2(U) := \sigma_\varepsilon\int_{\mathbb{R}} a'Q(v|\tilde{v}_\varepsilon)d\xi,$$
$$\mathcal{D}(U) := \int_{\mathbb{R}} av^\beta|\partial_\xi(p(v) - p(\tilde{v}_\varepsilon))|^2 d\xi.$$

4.6. **Proof of Theorem 3.1 from Proposition 4.1.** We will first show how Proposition 4.1 implies Theorem 3.1.

For any fixed $\varepsilon > 0$, we consider a continuous function Φ_ε defined by

(4.19)
$$\Phi_\varepsilon(y) = \begin{cases} \frac{1}{\varepsilon^2}, & \text{if } y \le -\varepsilon^2, \\ -\frac{1}{\varepsilon^4}y, & \text{if } |y| \le \varepsilon^2, \\ -\frac{1}{\varepsilon^2}, & \text{if } y \ge \varepsilon^2. \end{cases}$$

Let $\varepsilon_0, \delta_0, \delta_3$ be the constants in Proposition 4.1. Then, let ε, λ be any constants such that $0 < \varepsilon < \varepsilon_0$ and $\delta_0^{-1}\varepsilon < \lambda < \delta_0 < 1/2$.

We define a shift function $X(t)$ as a solution of the nonlinear ODE:

(4.20)
$$\begin{cases} \dot{X}(t) = \Phi_\varepsilon(Y(U^X))\left(2|\mathcal{J}^{bad}(U^X)| + 1\right), \\ X(0) = 0, \end{cases}$$

where Y and $\mathcal{J}^{bad}$ are as in (4.10).
Then, for the solution $U \in \mathcal{H}_T$, there exists a unique absolutely continuous shift X on $[0, T]$. Indeed, since $\tilde{v}'_\varepsilon, \tilde{h}'_\varepsilon, a'$ are bounded, smooth and integrable, using $U \in \mathcal{H}_T$ together with the change of variables $\xi \mapsto \xi - X(t)$ as in (4.11), we find that there exists $a, b \in L^2(0, T)$ such that
$$\sup_{x\in\mathbb{R}}|F(t, x)| \le a(t) \quad \text{and} \quad \sup_{x\in\mathbb{R}}|\partial_x F(t, x)| \le b(t), \quad \forall t \in [0, T],$$
where $F(t, X)$ denotes the right-hand side of the ODE (4.20). For more details on the existence and uniqueness theory of the ODE, we refer to [13, Lemma A.1].

Based on (4.9) and (4.20), to get the contraction estimate (3.5), it is enough to prove that for almost every time $t > 0$,

(4.21)
$$\Phi_\varepsilon(Y(U^X))\left(2|\mathcal{J}^{bad}(U^X)| + 1\right)Y(U^X) + \mathcal{J}^{bad}(U^X) - \mathcal{J}^{good}(U^X) \le 0.$$

We define

$$\mathcal{F}(U) := \Phi_\varepsilon(Y(U))\Big(2|\mathcal{J}^{bad}(U)| + 1\Big)Y(U) + \mathcal{J}^{bad}(U) - \mathcal{J}^{good}(U), \quad \forall U \in \mathcal{H}.$$

From (4.19), we have

$$(4.22) \qquad\qquad \Phi_\varepsilon(Y)\Big(2|\mathcal{J}^{bad}| + 1\Big)Y \leq \begin{cases} -2|\mathcal{B}_{\delta_3}|, & \text{if } |Y| \geq \varepsilon^2, \\ -\frac{1}{\varepsilon^4}Y^2, & \text{if } |Y| \leq \varepsilon^2. \end{cases}$$

Hence, for all $U \in \mathcal{H}$ satisfying $|Y(U)| \geq \varepsilon^2$, we have

$$\mathcal{F}(U) \leq -|\mathcal{J}^{bad}(U)| - \mathcal{J}^{good}(U) \leq 0.$$

Using (4.15), (4.22) and Proposition 4.1, we find that for all $U \in \mathcal{H}$ satisfying $|Y(U)| \leq \varepsilon^2$,

$$\mathcal{F}(U) \leq -\delta_0\frac{\varepsilon}{\lambda}|\mathcal{B}_{\delta_3}(U)| - \delta_0\frac{\varepsilon}{\lambda}\mathcal{G}_2(U) - \delta_0\mathcal{D}(U) \leq 0.$$

Since $\delta_0 < 1/2$, these two estimates show that for every $U \in \mathcal{H}$ we have

$$\mathcal{F}(U) \leq -\delta_0\frac{\varepsilon}{\lambda}|\mathcal{B}_{\delta_3}(U)| - \delta_0\frac{\varepsilon}{\lambda}\mathcal{G}_2(U) - \delta_0\mathcal{D}(U).$$

Thus, using the above estimates with $U = U^X$, together with (4.9), (4.21) and the definition of $\mathcal{I}^{good}$, we find that for a.e. $t > 0$,

(4.23)
$$\frac{d}{dt}\int_\mathbb{R} a\eta(U^X|\tilde{U}_\varepsilon) + \delta_0\frac{\varepsilon}{\lambda}\mathcal{G}_2(U^X) + \delta_0\mathcal{D}(U^X)d\xi = \mathcal{F}(U^X) + \delta_0\frac{\varepsilon}{\lambda}\mathcal{G}_2(U^X) + \delta_0\mathcal{D}(U^X)$$

$$\leq -|\mathcal{J}^{bad}(U^X)|\mathbf{1}_{\{|Y(U^X)|\geq\varepsilon^2\}} - \delta_0\frac{\varepsilon}{\lambda}|\mathcal{B}_{\delta_3}(U^X)|\mathbf{1}_{\{|Y(U^X)|\leq\varepsilon^2\}} \leq 0.$$

Therefore we have

$$(4.24) \qquad \int_\mathbb{R} a\eta(U^X|\tilde{U}_\varepsilon)d\xi + \delta_0\frac{\varepsilon}{\lambda}\mathcal{G}_2(U^X) + \delta_0\mathcal{D}(U^X) \leq \int_\mathbb{R} a\eta(U_0|\tilde{U}_\varepsilon)d\xi < \infty,$$

which completes (3.5).

To estimate $|\dot{X}|$, we first observe that (4.19) and (4.20) yield

$$(4.25) \qquad\qquad |\dot{X}| \leq \frac{1}{\varepsilon^2}(2|\mathcal{J}^{bad}(U^X)| + 1), \quad \text{for a.e. } t \in (0, T),$$

Notice that it follows from (4.23) and $1/2 \leq a \leq 1$ by $\delta_0 < 1/2$ that

$$(4.26) \quad \int_0^T \left(|\mathcal{J}^{bad}(U^X)|\mathbf{1}_{\{|Y(U^X)|\geq\varepsilon^2\}} + \delta_0\frac{\varepsilon}{\lambda}|\mathcal{B}_{\delta_3}(U^X)|\mathbf{1}_{\{|Y(U^X)|\leq\varepsilon^2\}}\right)dt \leq 2\int_\mathbb{R} \eta(U_0|\tilde{U}_\varepsilon)d\xi.$$

To estimate $|\mathcal{J}^{bad}(U^X)|$ globally in time, using (4.15) and the definitions of $\mathcal{I}^{good}$ and $\mathcal{G}_{\delta_3}$, we find that

$$
\begin{aligned}
&|\mathcal{J}^{bad}(U^X)| \\
&= |\mathcal{J}^{bad}(U^X)|\mathbf{1}_{\{|Y(U^X)|\geq\varepsilon^2\}} + |\mathcal{J}^{bad}(U^X)|\mathbf{1}_{\{|Y(U^X)|\leq\varepsilon^2\}} \\
&= |\mathcal{J}^{bad}(U^X)|\mathbf{1}_{\{|Y(U^X)|\geq\varepsilon^2\}} + |\mathcal{J}^{good}(U^X) + \mathcal{B}_{\delta_3}(U^X) - \mathcal{G}_{\delta_3}(U^X)|\mathbf{1}_{\{|Y(U^X)|\leq\varepsilon^2\}} \\
&\leq |\mathcal{J}^{bad}(U^X)|\mathbf{1}_{\{|Y(U^X)|\geq\varepsilon^2\}} + |\mathcal{B}_{\delta_3}(U^X)|\mathbf{1}_{\{|Y(U^X)|\leq\varepsilon^2\}} \\
&\quad + \frac{|\sigma|}{2}\int_{\mathbb{R}}|a'|\left|\left(h^X - \tilde{h}_\varepsilon\right)^2 - \left(h^X - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon}\right)^2\right|\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)\leq\delta_3\}}d\xi \\
&\leq |\mathcal{J}^{bad}(U^X)|\mathbf{1}_{\{|Y(U^X)|\geq\varepsilon^2\}} + |\mathcal{B}_{\delta_3}(U^X)|\mathbf{1}_{\{|Y(U^X)|\leq\varepsilon^2\}} \\
&\quad + C\int_{\mathbb{R}}|a'|\left(\left(h^X - \tilde{h}_\varepsilon\right)^2 + \left(p(v^X) - p(\tilde{v}_\varepsilon)\right)^2\right)\mathbf{1}_{\{p(v^X)-p(\tilde{v}_\varepsilon)\leq\delta_3\}}d\xi.
\end{aligned}
$$

Since for any v satisfying $p(v) - p(\tilde{v}_\varepsilon) \leq \delta_3$, there exists a positive constant c_* such that $v > c_*^{-1}$ and $|p(v) - p(\tilde{v}_\varepsilon)| \leq c_*$, we use (3.11) and (3.8) to have

$$
\begin{aligned}
&\int_{\mathbb{R}}\left(p(v) - p(\tilde{v}_\varepsilon)\right)^2\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)\leq\delta_3\}}d\xi \\
&\leq c_*\int_{v>c_*^{-1}}|p(v) - p(\tilde{v}_\varepsilon)|\mathbf{1}_{\{v\geq 3v_-\}}d\xi + \int_{v>c_*^{-1}}|p(v) - p(\tilde{v}_\varepsilon)|^2\mathbf{1}_{\{v\leq 3v_-\}}d\xi \\
&\leq C\int_{v>c_*^{-1}}\left(|v - \tilde{v}_\varepsilon|\mathbf{1}_{\{v\geq 3v_-\}} + |v - \tilde{v}_\varepsilon|^2\mathbf{1}_{\{v\leq 3v_-\}}\right)d\xi \leq C\int_{\mathbb{R}}Q(v|\tilde{v}_\varepsilon)d\xi.
\end{aligned}
$$

Therefore, using $a' \leq C\delta_0$ and $\delta_0 \leq \frac{1}{2} \leq a$, we have

$$
|\mathcal{J}^{bad}(U^X)| \leq |\mathcal{J}^{bad}(U^X)|\mathbf{1}_{\{|Y(U^X)|\geq\varepsilon^2\}} + |\mathcal{B}_{\delta_3}(U^X)|\mathbf{1}_{\{|Y(U^X)|\leq\varepsilon^2\}} + C\int_{\mathbb{R}}a\eta(U^X|\tilde{U}_\varepsilon)d\xi,
$$

which together with (4.24) and (4.25) implies

$$
|\dot{X}| \leq \frac{1}{\varepsilon^2}\left[\left(|\mathcal{J}^{bad}(U^X)|\mathbf{1}_{\{|Y(U^X)|\geq\varepsilon^2\}} + |\mathcal{B}_{\delta_3}(U^X)|\mathbf{1}_{\{|Y(U^X)|\leq\varepsilon^2\}}\right) + C\int_{\mathbb{R}}\eta(U_0|\tilde{U}_\varepsilon)d\xi + 1\right],
$$

and (4.26) implies

$$
\int_0^T\left(|\mathcal{J}^{bad}(U^X)|\mathbf{1}_{\{|Y(U^X)|\geq\varepsilon^2\}} + |\mathcal{B}_{\delta_3}(U^X)|\mathbf{1}_{\{|Y(U^X)|\leq\varepsilon^2\}}\right)dt \leq \frac{2\lambda}{\delta_0\varepsilon}\int_{\mathbb{R}}\eta(U_0|\tilde{U}_\varepsilon)d\xi.
$$

Hence we complete (3.6).

The rest of this section is dedicated to the proof of Proposition 4.1.

4.7. **Expansion in the size of the shock.** We define the following functionals:

$$Y_g(v) := -\frac{1}{2\sigma_\varepsilon^2} \int_{\mathbb{R}} a' |p(v) - p(\tilde{v}_\varepsilon)|^2 d\xi - \int_{\mathbb{R}} a' Q(v|\tilde{v}_\varepsilon) d\xi - \int_{\mathbb{R}} a \partial_\xi p(\tilde{v}_\varepsilon)(v - \tilde{v}_\varepsilon) d\xi$$

$$+ \frac{1}{\sigma_\varepsilon} \int_{\mathbb{R}} a \partial_\xi \tilde{h}_\varepsilon \big(p(v) - p(\tilde{v}_\varepsilon) \big) d\xi,$$

$$\mathcal{I}_1(v) := \sigma_\varepsilon \int_{\mathbb{R}} a \partial_\xi \tilde{v}_\varepsilon p(v|\tilde{v}_\varepsilon) d\xi,$$

(4.27)

$$\mathcal{I}_2(v) := \frac{1}{2\sigma_\varepsilon} \int_{\mathbb{R}} a' |p(v) - p(\tilde{v}_\varepsilon)|^2 d\xi,$$

$$\mathcal{G}_2(v) := \sigma_\varepsilon \int_{\mathbb{R}} a' Q(v|\tilde{v}_\varepsilon) d\xi,$$

$$\mathcal{D}(v) := \int_{\mathbb{R}} a \, v^\beta |\partial_\xi(p(v) - p(\tilde{v}_\varepsilon))|^2 d\xi.$$

Note that all these quantities depend only on v (not on h).

Proposition 4.2. *For any constant $C_2 > 0$, there exist $\varepsilon_0, \delta_3 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, and any $\lambda, \delta \in (0, \delta_3)$ such that $\varepsilon \le \lambda$, the following is true. For any function $v : \mathbb{R} \to \mathbb{R}^+$ such that $\mathcal{D}(v) + \mathcal{G}_2(v)$ is finite, if*

(4.28)
$$|Y_g(v)| \le C_2 \frac{\varepsilon^2}{\lambda}, \qquad \|p(v) - p(\tilde{v}_\varepsilon)\|_{L^\infty(\mathbb{R})} \le \delta_3,$$

then

(4.29)
$$\mathcal{R}_{\varepsilon,\delta}(v) := -\frac{1}{\varepsilon\delta} |Y_g(v)|^2 + \mathcal{I}_1(v) + \delta |\mathcal{I}_1(v)|$$
$$+ \mathcal{I}_2(v) + \delta\left(\frac{\varepsilon}{\lambda}\right) |\mathcal{I}_2(v)| - \left(1 - \delta\left(\frac{\varepsilon}{\lambda}\right)\right) \mathcal{G}_2(v) - (1 - \delta)\mathcal{D}(v) \le 0,$$

where note that $\mathcal{I}_1, \mathcal{I}_2 \ge 0$.

Proof. The proof is almost the same as that of [33, Proposition 3.4], because $Y_g, \mathcal{I}_1, \mathcal{I}_2, \mathcal{G}_2$ defined in (4.27) are the exactly same functionals as in [33, Proposition 3.4], and the diffusion $\mathcal{D}$ is slightly different but has the same expansion. Note that this proposition corresponds to an expansion in $p(v)$ near $p(\tilde{v}_\varepsilon)$ (up to δ_3) for $p(\tilde{v}_\varepsilon)$ close to $p(v_-)$ (up to ε). It is therefore natural that the expansion is similar to the case of $\alpha = \gamma$ as in [33, Proposition 3.4], since the viscosity is almost constant near $p(v_-)$. For completeness, the main part of the proof is given in Appendix A. $\qquad \square$

4.8. **Truncation of the big values of $|p(v) - p(\tilde{v}_\varepsilon)|$.** In order to use Proposition 4.2 in the proof of Proposition 4.1, we need to show that the values for $p(v)$ such that $|p(v) - p(\tilde{v}_\varepsilon)| \ge \delta_3$ have a small effect. However, the value of δ_3 is itself conditioned to the constant C_2 in the proposition. Therefore, we need first to find a uniform bound on Y_g which is not yet conditioned on the level of truncation k.

We consider a truncation on $|p(v) - p(\tilde{v}_\varepsilon)|$ with a constant $k > 0$. Later we will consider the case $k = \delta_3$ as in Proposition 4.2. But for now, we consider the general case k to estimate the constant C_2. For that, let ψ_k be a continuous function defined by

(4.30)
$$\psi_k(y) = \inf\left(k, \sup(-k, y)\right).$$

We then define the function $\bar{v}_k$ uniquely (since the function p is one to one) as

(4.31) $$p(\bar{v}_k) - p(\tilde{v}_\varepsilon) = \psi_k(p(v) - p(\tilde{v}_\varepsilon)).$$

We have the following lemma (see [33, Lemma 3.2]).

Lemma 4.4. *For a fixed $v_- \geq 0$, $u_- \in \mathbb{R}$, there exists $C_2, k_0, \varepsilon_0, \delta_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$, $\varepsilon/\lambda \leq \delta_0$ with $\lambda < 1/2$, the following is true whenever $|Y(U)| \leq \varepsilon^2$:*

(4.32) $$\int_\mathbb{R} |a'||h - \tilde{h}_\varepsilon|^2 \, d\xi + \int_\mathbb{R} |a'| Q(v|\tilde{v}_\varepsilon) \, d\xi \leq C \frac{\varepsilon^2}{\lambda},$$

(4.33) $$|Y_g(\bar{v}_k)| \leq C_2 \frac{\varepsilon^2}{\lambda}, \qquad \text{for every } k \leq k_0.$$

We now fix the constant δ_3 of Proposition 4.2 associated to the constant C_2 of Lemma 4.4. Without loss of generality, we can assume that $\delta_3 < k_0$ (since Proposition 4.2 is valid for any smaller δ_3). From now on, we set (without confusion)

$$\bar{v} := \bar{v}_{\delta_3}, \qquad \bar{U} := (\bar{v}, h), \qquad \mathcal{B} := \mathcal{B}_{\delta_3}, \qquad \mathcal{G} := \mathcal{G}_{\delta_3}.$$

Note from Lemma 4.4 that

(4.34) $$|Y_g(\bar{v})| \leq C_2 \frac{\varepsilon^2}{\lambda}.$$

We first recall the terms Y in (4.10) as

$$Y = -\int_\mathbb{R} a' \frac{|h - \tilde{h}_\varepsilon|^2}{2} d\xi - \int_\mathbb{R} a' Q(v|\tilde{v}_\varepsilon) d\xi + \int_\mathbb{R} a\left(-\partial_\xi p(\tilde{v}_\varepsilon)(v - \tilde{v}_\varepsilon) + \partial_\xi \tilde{h}_\varepsilon (h - \tilde{h}_\varepsilon)\right) d\xi.$$

In what follows, for simplification, we use the notation:

$$\Omega := \{\xi \mid (p(v) - p(\tilde{v}_\varepsilon))(\xi) \leq \delta_3\}.$$

We split Y into four parts Y_g, Y_b, Y_l and Y_s as follows:

$$Y = Y_g + Y_b + Y_l + Y_s,$$

where

$$Y_g := -\frac{1}{2\sigma_\varepsilon^2} \int_\Omega a' |p(v) - p(\tilde{v}_\varepsilon)|^2 d\xi - \int_\Omega a' Q(v|\tilde{v}_\varepsilon) d\xi - \int_\Omega a \partial_\xi p(\tilde{v}_\varepsilon)(v - \tilde{v}_\varepsilon) d\xi$$
$$+ \frac{1}{\sigma_\varepsilon} \int_\Omega a \partial_\xi \tilde{h}_\varepsilon \big(p(v) - p(\tilde{v}_\varepsilon)\big) d\xi,$$

$$Y_b := -\frac{1}{2} \int_\Omega a' \left(h - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon}\right)^2 d\xi$$
$$- \frac{1}{\sigma_\varepsilon} \int_\Omega a' \big(p(v) - p(\tilde{v}_\varepsilon)\big)\left(h - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon}\right) d\xi,$$

$$Y_l = \int_\Omega a \partial_\xi \tilde{h}_\varepsilon \left(h - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon}\right) d\xi,$$

$$Y_s = -\int_{\Omega^c} a' Q(v|\tilde{v}_\varepsilon) d\xi - \int_{\Omega^c} a \partial_\xi p(\tilde{v}_\varepsilon)(v - \tilde{v}_\varepsilon) d\xi$$
$$- \int_{\Omega^c} a' \frac{|h - \tilde{h}_\varepsilon|^2}{2} d\xi + \int_{\Omega^c} a \partial_\xi \tilde{h}_\varepsilon (h - \tilde{h}_\varepsilon) d\xi.$$

Notice that Y_g consists of the terms related to $v - \tilde{v}_\varepsilon$, while Y_b and Y_l consist of terms related to $h - \tilde{h}_\varepsilon$. While Y_b is quadratic, and Y_l is linear in $h - \tilde{h}_\varepsilon$. Since $\{\xi \mid |p(v) - p(\tilde{v}_\varepsilon)| \leq \delta_3\} \subset \Omega$, $Y_g(\bar{U})$ is the same as $Y_g(v)$ in Proposition 4.2. Therefore we need show that $Y_g(U) - Y_g(\bar{U})$, $Y_b(U)$, $Y_l(U)$ and $Y_s(U)$ are negligible by the good term $\mathcal{G}$.

For the bad terms $\mathcal{B}_{\delta_3}$ in (4.16), we will use the following notations :

$$(4.35) \qquad \mathcal{B}_{\delta_3} = \mathcal{B}_1 + \mathcal{B}_2^- + \mathcal{B}_2^+ + \mathcal{B}_3 + \mathcal{B}_4 + \mathcal{B}_5,$$

where

$$\mathcal{B}_1 = \sigma_\varepsilon \int_{\mathbb{R}} a \partial_\xi \tilde{v}_\varepsilon p(v|\tilde{v}_\varepsilon) d\xi,$$

$$\mathcal{B}_2^- = \int_{\Omega^c} a' \big(p(v) - p(\tilde{v}_\varepsilon)\big)(h - \tilde{h}_\varepsilon) d\xi, \qquad \mathcal{B}_2^+ = \frac{1}{2\sigma_\varepsilon} \int_\Omega a' |p(v) - p(\tilde{v}_\varepsilon)|^2 d\xi,$$

$$\mathcal{B}_3 = -\int_{\mathbb{R}} a' v^\beta \big(p(v) - p(\tilde{v}_\varepsilon)\big) \partial_\xi \big(p(v) - p(\tilde{v}_\varepsilon)\big) d\xi,$$

$$\mathcal{B}_4 = -\int_{\mathbb{R}} a' \big(p(v) - p(\tilde{v}_\varepsilon)\big)(v^\beta - \tilde{v}_\varepsilon^\beta) \partial_\xi p(\tilde{v}_\varepsilon) d\xi,$$

$$\mathcal{B}_5 = -\int_{\mathbb{R}} a \partial_\xi \big(p(v) - p(\tilde{v}_\varepsilon)\big)(v^\beta - \tilde{v}_\varepsilon^\beta) \partial_\xi p(\tilde{v}_\varepsilon) d\xi.$$

We also recall the notations $\mathcal{G}_1^-, \mathcal{G}_1^+, \mathcal{G}_2, \mathcal{D}$ in (4.18) for the good terms.

We now state the following proposition.

Proposition 4.3. *There exist constants* $\varepsilon_0, \delta_0, C, C^* > 0$ *(in particular, C depends on the constant δ_3 in Proposition 4.1) such that for any $\varepsilon < \varepsilon_0$ and $\delta_0^{-1}\varepsilon < \lambda < \delta_0 < 1/2$, the following statements hold true.*

1. For any U such that $|Y(U)| \leq \varepsilon^2$,

$$(4.36) \qquad |\mathcal{B}_1(U) - \mathcal{B}_1(\bar{U})| \leq C \frac{\varepsilon}{\lambda} \left(\mathcal{D}(U) + \big(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\big) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(\bar{U}) \right),$$

$$(4.37) \qquad |\mathcal{B}_2^-(U)| \leq \delta_0 \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U) \right) + \frac{1}{2} \mathcal{G}_1^-(U),$$

$$(4.38) \qquad |\mathcal{B}_2^+(U) - \mathcal{B}_2^+(\bar{U})| \leq \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U),$$

$$(4.39) \qquad |\mathcal{B}_3(U)| + |\mathcal{B}_4(U)| + |\mathcal{B}_5(U)| \leq C\delta_0 \left(\mathcal{D}(U) + \big(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\big) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(\bar{U}) \right),$$

$$(4.40) \qquad |\mathcal{B}_{\delta_3}(U)| \leq C^* \frac{\varepsilon^2}{\lambda} + C\sqrt{\delta_0} \mathcal{D}(U).$$

2. For any U such that $|Y(U)| \leq \varepsilon^2$ and $\mathcal{D}(U) \leq \frac{C^*}{4} \frac{\varepsilon^2}{\lambda}$,

$$(4.41)$$
$$|Y_g(U) - Y_g(\bar{U})|^2 + |Y_b(U)|^2 + |Y_l(U)|^2 + |Y_s(U)|^2$$
$$\leq C \frac{\varepsilon^2}{\lambda} \left(\sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U) + \big(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\big) + \mathcal{G}_1^-(U) + \left(\frac{\lambda}{\varepsilon}\right)^{1/4} \mathcal{G}_1^+(U) + \left(\frac{\varepsilon}{\lambda}\right)^{1/4} \mathcal{G}_2(\bar{U}) \right).$$

To prove this proposition, we will control the bad terms in different ways for each case of small or big values of v, which all correspond to the big values of $|p(v) - p(\tilde{v}_\varepsilon)|$ (as

$|p(v) - p(\tilde{v}_\varepsilon)| \geq \delta_3$). For that, we set

(4.42) $\qquad p(\bar{v}_s) - p(\tilde{v}_\varepsilon) := \psi_{\delta_3}^s \big(p(v) - p(\tilde{v}_\varepsilon)\big), \quad p(\bar{v}_b) - p(\tilde{v}_\varepsilon) := \psi_{\delta_3}^b \big(p(v) - p(\tilde{v}_\varepsilon)\big),$

where $\psi_{\delta_3}^s$ and $\psi_{\delta_3}^b$ are one-sided truncations of ψ_{δ_3} defined in (4.30), i.e.,

$$\psi_{\delta_3}^s(y) = \inf(\delta_3, y), \quad \psi_{\delta_3}^b(y) = \sup(-\delta_3, y).$$

Notice that the function $\bar{v}_s$ (resp. $\bar{v}_b$) represents the truncation of small (resp. big) values of v corresponding to $|p(v) - p(\tilde{v}_\varepsilon)| \geq \delta_3$.

By comparing the definitions of (4.31) and (4.42), we see

(4.43)
$$
\begin{aligned}
(p(\bar{v}_s) - p(\tilde{v}_\varepsilon))\,\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)\geq-\delta_3\}} &= (p(\bar{v}) - p(\tilde{v}_\varepsilon))\,\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)\geq-\delta_3\}}, \\
(p(\bar{v}_b) - p(\tilde{v}_\varepsilon))\,\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)\leq\delta_3\}} &= (p(\bar{v}) - p(\tilde{v}_\varepsilon))\,\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)\leq\delta_3\}}.
\end{aligned}
$$

We also note that

(4.44)
$$
\begin{aligned}
p(v) - p(\bar{v}_s) &= (p(v) - p(\tilde{v}_\varepsilon)) + (p(\tilde{v}_\varepsilon) - p(\bar{v}_s)) \\
&= \left(I - \psi_{\delta^3}^s\right)(p(v) - p(\tilde{v}_\varepsilon)) \\
&= ((p(v) - p(\tilde{v}_\varepsilon)) - \delta_3)_+, \\
p(\bar{v}_b) - p(v) &= (p(\bar{v}_b) - p(\tilde{v}_\varepsilon)) + (p(\tilde{v}_\varepsilon) - p(v)) \\
&= \left(\psi_{\delta^3}^b - I\right)(p(v) - p(\tilde{v}_\varepsilon)) \\
&= (-(p(v) - p(\tilde{v}_\varepsilon)) - \delta_3)_+, \\
|p(v) - p(\bar{v})| &= |(p(v) - p(\tilde{v}_\varepsilon)) + (p(\tilde{v}_\varepsilon) - p(\bar{v}))| \\
&= |(I - \psi_\delta)(p(v) - p(\tilde{v}_\varepsilon))| \\
&= (|p(v) - p(\tilde{v}_\varepsilon)| - \delta_3)_+.
\end{aligned}
$$

Therefore, using (4.31), (4.42) and (4.44), we have

(4.45)
$$
\begin{aligned}
\mathcal{D}(U) &= \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\tilde{v}_\varepsilon))|^2 d\xi \\
&= \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\tilde{v}_\varepsilon))|^2 (\mathbf{1}_{\{|p(v)-p(\tilde{v}_\varepsilon)|\leq\delta_3\}} + \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}} + \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)<-\delta_3\}}) d\xi \\
&= \mathcal{D}(\bar{U}) + \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\bar{v}_s))|^2 d\xi + \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\bar{v}_b))|^2 d\xi \\
&\geq \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\bar{v}_s))|^2 d\xi + \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\bar{v}_b))|^2 d\xi.
\end{aligned}
$$

On the other hand, since $Q(v|\tilde{v}_\varepsilon) \geq Q(\bar{v}|\tilde{v}_\varepsilon)$, we have

(4.46) $\quad |\sigma_\varepsilon| \int_{\mathbb{R}} |a'| Q(v|\tilde{v}_\varepsilon)\, d\xi \geq \mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) = |\sigma_\varepsilon| \int_{\mathbb{R}} |a'|\, (Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon))\, d\xi \geq 0,$

which together with (4.32) yields

(4.47) $$0 \leq \mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) \leq \mathcal{G}_2(U) \leq C\frac{\varepsilon^2}{\lambda}.$$

We first present a series of following lemmas.

Lemma 4.5. *Under the same assumption as Proposition 4.3, we have*

$$(4.48) \qquad \int_{\mathbb{R}} |a'||p(v) - p(\bar{v}_b)|^2 d\xi + \int_{\mathbb{R}} |a'||p(v) - p(\bar{v}_b)| d\xi \leq \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U),$$

$$(4.49) \qquad \int_{\mathbb{R}} |a'| \left| |p(v) - p(\tilde{v}_\varepsilon)|^2 - |p(\bar{v}_b) - p(\tilde{v}_\varepsilon)|^2 \right| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)\leq\delta_3\}} d\xi \leq \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U).$$

$$(4.50) \quad \int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v})|^2 d\xi + \int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v})| d\xi \leq C\lambda^2 \left(\mathcal{D}(U) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(U) \right),$$

$$(4.51) \quad \int_{\mathbb{R}} |a'|^2 \left| v^\beta |p(v) - p(\tilde{v}_\varepsilon)|^2 - \bar{v}^\beta |p(\bar{v}) - p(\tilde{v}_\varepsilon)|^2 \right| d\xi \leq C\lambda^2 \left(\mathcal{D}(U) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(U) \right).$$

Proof. **Proof of** (4.48): We split the proof into two steps.

Step 1: Note first that since $(y - \delta_3/2)_+ \geq \delta_3/2$ whenever $(y - \delta_3)_+ > 0$, we have

$$(4.52) \quad (y - \delta_3)_+ \leq (y - \delta_3/2)_+ \mathbf{1}_{\{y-\delta_3>0\}} \leq (y - \delta_3/2)_+ \left(\frac{(y - \delta_3/2)_+}{\delta_3/2} \right) \leq \frac{2}{\delta_3}(y - \delta_3/2)_+^2.$$

Hence, to show (4.48), it is enough to show it only for the quadratic part, with $\bar{v}_b$ defined with $\delta_3/2$ instead of δ_3. We will keep the notation $\bar{v}_b$ in Step 2 below.

Step 2: First, using (4.45), we find that for any $\xi \in \mathbb{R}$,

$$|p(v) - p(\bar{v}_b)(\xi)| \leq \int_{\xi_0}^{\xi} \left| \partial_\xi \left(p(v) - p(\bar{v}_b) \right) \right| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)<-\delta_3\}} d\xi$$

$$(4.53) \qquad\qquad\qquad \leq C \int_{\xi_0}^{\xi} v^{\beta/2} \left| \partial_\xi \left(p(v) - p(\bar{v}_b) \right) \right| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)<-\delta_3\}} d\xi$$

$$\leq C \sqrt{|\xi| + \frac{1}{\varepsilon}} \sqrt{\mathcal{D}(U)}.$$

For any ξ such that $|(p(v) - p(\bar{v}))(\xi)| > 0$, we have from (4.44) that $|(p(v) - p(\tilde{v}_\varepsilon))(\xi)| > \delta_3$. Thus using (3.11) and (3.8), we have $Q(v(\xi)|\tilde{v}_\varepsilon(\xi)) \geq \alpha$, for some constant $\alpha > 0$ depending only on δ_3. Thus,

$$(4.54) \qquad\qquad\qquad\qquad \mathbf{1}_{\{|p(v)-p(\bar{v})|>0\}} \leq \frac{Q(v|\tilde{v}_\varepsilon)}{\alpha}.$$

Since $\mathbf{1}_{\{|p(v)-p(\bar{v}_b)|>0\}} \leq \mathbf{1}_{\{|p(v)-p(\bar{v})|>0\}}$, using (4.53), (4.54) and (4.32), we estimate

$$\int_{\mathbb{R}} |a'||p(v)-p(\bar{v}_b)|^2 d\xi$$

$$\leq \int_{|\xi|\leq \frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}} |a'||p(v)-p(\bar{v}_b)|^2 d\xi + \int_{|\xi|\geq \frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}} |a'||p(v)-p(\bar{v}_b)|^2 d\xi$$

$$(4.55) \qquad \leq \left(\sup_{\left[-\sqrt{\frac{\lambda}{\varepsilon^3}},\sqrt{\frac{\lambda}{\varepsilon^3}}\right]} |p(v)-p(\bar{v}_s)| \right) \int_{|\xi|\leq \frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}} |a'|\mathbf{1}_{\{|p(v)-p(\bar{v})|>0\}} d\xi$$

$$+ C\mathcal{D}(U) \int_{|\xi|\geq \frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}} |a'| \left(|\xi| + \frac{1}{\varepsilon} \right) d\xi$$

$$\leq C\mathcal{D}(U) \left(\sqrt{\frac{\lambda}{\varepsilon^3}} \int_{\mathbb{R}} |a'|\frac{Q(v|\tilde{v}_\varepsilon)}{\alpha} d\xi + 2\int_{|\xi|\geq \frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}} |a'||\xi| d\xi \right).$$

Therefore we have

$$\int_{\mathbb{R}} |a'||p(v)-p(\bar{v}_b)|^2 d\xi \leq C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U).$$

Indeed, using (4.32) and (4.1) (recalling $|a'| = (\lambda/\varepsilon)|\tilde{v}'_\varepsilon|$), we have

$$\int_{\mathbb{R}} |a'||p(v)-p(\bar{v})|^2 d\xi \leq C\mathcal{D}(U) \left(\sqrt{\frac{\varepsilon}{\lambda}} + \lambda\varepsilon \int_{|\xi|\geq \frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}} e^{-c\varepsilon|\xi|}|\xi| d\xi \right),$$

and for the last term, we take δ_0 small enough such that for any $\varepsilon/\lambda \leq \delta_0$,

$$\lambda\varepsilon \int_{|\xi|\geq \frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}} e^{-c\varepsilon|\xi|}|\xi| d\xi = \frac{\lambda}{\varepsilon} \int_{|\xi|\geq \sqrt{\frac{\lambda}{\varepsilon}}} e^{-c|\xi|}|\xi| d\xi \leq \frac{\lambda}{\varepsilon} \int_{|\xi|\geq \sqrt{\frac{\lambda}{\varepsilon}}} e^{-\frac{c}{2}|\xi|} d\xi = \frac{2\lambda}{c\varepsilon} e^{-\frac{c}{2}\sqrt{\frac{\lambda}{\varepsilon}}} \leq \sqrt{\frac{\varepsilon}{\lambda}}.$$

As mentioned in Step 1, recall that $\bar{v}_b = \bar{v}_{b_{\delta_3/2}}$ in the above estimate. Then using (4.44), we have

$$\int_{\mathbb{R}} |a'|^2 v^\beta |p(v)-p(\bar{v}_{b_{\delta_3}})|^2 d\xi = \int_{\mathbb{R}} |a'|^2 v^\beta (-(p(v)-p(\tilde{v}_\varepsilon)) - \delta_3)_+^2 d\xi$$

$$(4.56) \qquad \leq \int_{\mathbb{R}} |a'|^2 v^\beta (-(p(v)-p(\tilde{v}_\varepsilon)) - \delta_3/2)_+^2 d\xi$$

$$= \int_{\mathbb{R}} |a'|^2 v^\beta |p(v)-p(\bar{v}_{b_{\delta_3/2}})|^2 d\xi$$

$$\leq C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U).$$

For the linear part, using (4.44) and (4.52) with $y := -(p(v)-p(\tilde{v}_\varepsilon))$, we have

$$\int_{\mathbb{R}} |a'|^2 v^\beta |p(v)-p(\bar{v}_{b_{\delta_3}})| d\xi \leq \frac{2}{\delta_3} \int_{\mathbb{R}} |a'|^2 v^\beta |p(v)-p(\bar{v}_{b_{\delta_3/2}})|^2 d\xi$$

$$(4.57) \qquad \leq C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U).$$

Hence, we obtain (4.48).

Proof of (4.49)**:** Since it follows from (4.43) that

$$|p(\bar{v}_b) - p(\tilde{v}_\varepsilon)|\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)\le\delta_3\}} = |p(\bar{v}) - p(\tilde{v}_\varepsilon)|\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)\le\delta_3\}} \le \delta_3,$$

using (4.48), we have

(4.58)
$$\int_{\mathbb{R}} |a'|\Big||p(v) - p(\tilde{v}_\varepsilon)|^2 - |p(\bar{v}_b) - p(\tilde{v}_\varepsilon)|^2\Big|\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)\le\delta_3\}}\,d\xi$$

$$= \int_{\mathbb{R}} |a'||p(v) - p(\bar{v}_b)||p(v) + p(\bar{v}_b) - 2p(\tilde{v}_\varepsilon)|\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)\le\delta_3\}}\,d\xi$$

$$\le \int_{\mathbb{R}} |a'||p(v) - p(\bar{v}_b)|\left(|p(v) - p(\bar{v}_b)| + 2|p(\bar{v}_b) - p(\tilde{v}_\varepsilon)|\right)\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)\le\delta_3\}}\,d\xi$$

$$\le \int_{\mathbb{R}} |a'|\left(|p(v) - p(\bar{v}_b)|^2 + 2\delta_3|p(v) - p(\bar{v}_b)|\right)d\xi \le C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U).$$

Proof of (4.50)**:** Thanks to (4.52), it is enough to show the quadratic part, with $\bar{v}$ defined with $\delta_3/2$ instead of δ_3. For this case, we will keep the notations $\bar{v}_s$ and $\bar{v}_b$ below without confusion.

We first decompose the quadratic part into two parts:

$$\int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v})|^2\,d\xi = \underbrace{\int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v}_b)|^2\,d\xi}_{=:Q_b} + \underbrace{\int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v}_s)|^2\,d\xi}_{=:Q_s}.$$

First, using the condition $\beta = \gamma - \alpha \le 1$, we have

$$|Q_b| = \int_{\mathbb{R}} |a'|^2 \frac{v^\beta|p(v) - p(\bar{v}_b)|^2}{|v - \bar{v}|}|v - \bar{v}|\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)<-\delta_3/2\}}\,d\xi$$

$$\le C\int_{\mathbb{R}} |a'|^2 |v - \bar{v}|\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)<-\delta_3/2\}}\,d\xi.$$

To control the right hand side, we use (3.10) as follows: If $|v - \bar{v}| > 0$, using (3.11), we find

$$|\bar{v} - \tilde{v}_\varepsilon| \ge \min(c_5^{-1}\delta_3/2, v_-/2 - \varepsilon_0).$$

Taking δ_* in 2) of Lemma 3.1 such that $\varepsilon_0 \le \delta_*/2$ and $\min(c_5^{-1}\delta_3, v_-/2 - \varepsilon_0) \ge \delta_*$, we use (3.10) with $w = \tilde{v}_\varepsilon$, $u = \bar{v}$ and $v = v$ to find that there exists a constant $C > 0$ such that

(4.59)
$$C|v - \bar{v}| \le Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon).$$

Therefore, using $|a'| \le \varepsilon\lambda$, we find

(4.60)
$$|Q_b| \le C\int_{\mathbb{R}} |a'|^2\big(Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)\big)\,d\xi \le C\varepsilon\lambda\mathcal{G}_2(U).$$

On the other hand, to control Q_s, we will first derive a point-wise estimate (4.63) as below:

Using $|a'| = (\lambda/\varepsilon)|\tilde{v}_\varepsilon'|$, together with (4.2) and (4.32), we get

$$2\varepsilon\int_{-1/\varepsilon}^{1/\varepsilon} Q(v|\tilde{v}_\varepsilon)\,d\xi \quad\le\quad \frac{2\varepsilon}{\inf_{[-1/\varepsilon,1/\varepsilon]}|a'|}\int_{\mathbb{R}} |a'|Q(v|\tilde{v}_\varepsilon)\,d\xi$$

$$\le\quad C\frac{\varepsilon}{\lambda\varepsilon}\frac{\varepsilon^2}{\lambda} = C\left(\frac{\varepsilon}{\lambda}\right)^2.$$

Therefore, there exists $\xi_0 \in [-1/\varepsilon, 1/\varepsilon]$ such that $Q(v(\xi_0), \tilde{v}_\varepsilon(\xi_0)) \leq C(\varepsilon/\lambda)^2$. For δ_0 small enough, and using (3.16), we have

$$|(p(v) - p(\tilde{v}_\varepsilon))(\xi_0)| \leq C\frac{\varepsilon}{\lambda}.$$

Thus, if δ_0 is small enough such that $C\varepsilon/\lambda \leq \delta_3/2$, then we have from the definition of $\bar{v}_s$ that

$$(p(v) - p(\bar{v}_s))(\xi_0) = 0.$$

Therefore, for any $\xi \in \mathbb{R}$,

$$|v^{\beta/2}(p(v) - p(\bar{v}_s))(\xi)| = \left| \int_{\xi_0}^\xi \partial_\xi \left(v^{\beta/2}(p(v) - p(\bar{v}_s)) \right) d\zeta \right|.$$

To control the right-hand side by the good terms, we observe that since $v^{\beta/2} = p(v)^{-(\gamma-\alpha)/2\gamma}$, we have

$$\partial_\xi \left(v^{\beta/2}(p(v) - p(\bar{v}_s)) \right) = \partial_\xi \left(p(v)^{-(\gamma-\alpha)/2\gamma}((p(v) - p(\bar{v}_s))) \right)$$

$$= p(v)^{-(\gamma-\alpha)/2\gamma} \partial_\xi((p(v) - p(\bar{v}_s)))$$

$$- \frac{\gamma-\alpha}{2\gamma} p(v)^{-(\gamma-\alpha)/2\gamma} \frac{p(v) - p(\bar{v}_s)}{p(v)} \partial_\xi \left[((p(v) - p(\tilde{v}_\varepsilon)) + p(\tilde{v}_\varepsilon)) \right]$$

$$= v^{\beta/2} \partial_\xi((p(v) - p(\bar{v}_s))) - \frac{\gamma-\alpha}{2\gamma} v^{\beta/2} \underbrace{\frac{p(v) - p(\bar{v}_s)}{p(v)} \partial_\xi((p(v) - p(\tilde{v}_\varepsilon)))}_{=:K}$$

$$- \frac{\gamma-\alpha}{2\gamma} v^{\beta/2} \frac{p(v) - p(\bar{v}_s)}{p(v)} \partial_\xi p(\tilde{v}_\varepsilon).$$

In particular, note that (by the definition of $\bar{v}_s$) the part K above can be rewritten by

$$K = \frac{p(v) - p(\bar{v}_s)}{p(v)} \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3/2\}} \partial_\xi((p(v) - p(\tilde{v}_\varepsilon))) = \frac{p(v) - p(\bar{v}_s)}{p(v)} \partial_\xi((p(v) - p(\bar{v}_s))).$$

Then, using $|\partial_\xi p(\tilde{v}_\varepsilon)| \leq C|\tilde{v}_\varepsilon'|$ and

$$\frac{p(v) - p(\bar{v}_s)}{p(v)} \leq C,$$

we have

$$|\partial_\xi \left(v^{\beta/2}(p(v) - p(\bar{v}_s)) \right)| \leq C v^{\beta/2}(|\partial_\xi((p(v) - p(\bar{v}_s)))| + |\tilde{v}_\varepsilon'|).$$

Therefore, using (4.45), we have that for any $\xi \in \mathbb{R}$,

$$|v^{\beta/2}(p(v) - p(\bar{v}_s))(\xi)| = \left| \int_{\xi_0}^\xi \partial_\xi \left(v^{\beta/2}(p(v) - p(\bar{v}_s)) \right) d\xi \right|$$

$$\leq \int_{\xi_0}^\xi \left| \partial_\xi \left(v^{\beta/2}(p(v) - p(\bar{v}_s)) \right) \right| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3/2\}} d\xi$$

$$\leq \int_{\xi_0}^\xi v^{\beta/2}(|\partial_\xi((p(v) - p(\bar{v}_s)))| + |\tilde{v}_\varepsilon'|) \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3/2\}} d\xi$$

$$\leq C\sqrt{|\xi| + \frac{1}{\varepsilon}} \left(\sqrt{\mathcal{D}(U)} + \sqrt{\int_\mathbb{R} a|\tilde{v}_\varepsilon'|^2 v^\beta \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3/2\}} d\xi} \right).$$

Using the condition $\beta = \gamma - \alpha > 0$, we have

(4.61)
$$\int_{\mathbb{R}} a|\tilde{v}_\varepsilon'|^2 v^\beta \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3/2\}} d\xi = \int_{\mathbb{R}} a|\tilde{v}_\varepsilon'|^2 \frac{v^\beta}{|v-\tilde{v}_\varepsilon|^2}|v - \tilde{v}_\varepsilon|^2 \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3/2\}} d\xi$$
$$\leq C \int_{\mathbb{R}} a|\tilde{v}_\varepsilon'|^2 |v - \tilde{v}_\varepsilon|^2 \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3/2\}} d\xi.$$

In addition, using (3.8) and $|\tilde{v}_\varepsilon'| \leq C\frac{\varepsilon}{\lambda}|a'|$, we have

(4.62)
$$\int_{\mathbb{R}} a|\tilde{v}_\varepsilon'|^2 |v - \tilde{v}_\varepsilon|^2 \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3/2\}} d\xi \leq C\frac{\varepsilon^2}{\lambda^2} \int_{\mathbb{R}} |a'| Q(v|\tilde{v}_\varepsilon) d\xi.$$

Therefore we obtain that

(4.63)
$$\forall \xi \in \mathbb{R}, \quad |v^{\beta/2}(p(v) - p(\bar{v}_s))(\xi)| \leq C\sqrt{|\xi| + \frac{1}{\varepsilon}} \left(\sqrt{\mathcal{D}(U)} + \frac{\varepsilon}{\lambda}\sqrt{\mathcal{G}_2(U)} \right).$$

Now, using (4.14) with (4.1), we have

$$Q_s \leq C\left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U) \right) \lambda^2 \varepsilon \int_{\mathbb{R}} e^{-C|\varepsilon\xi|}(|\varepsilon\xi| + 1) d\xi$$
$$\leq C\lambda^2 \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U) \right).$$

Therefore, this and (4.60) complete the estimate:

$$\int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v})|^2 \, d\xi \leq C\lambda^2 \left(\mathcal{D}(U) + \frac{\varepsilon}{\lambda}\mathcal{G}_2(U) \right).$$

Hence using the similar estimates as in (4.56) and (4.57) (i.e., using (4.44) and (4.52) with $y := |p(v) - p(\tilde{v}_\varepsilon)|$), we obtain (4.50).
Proof of (4.51): We first separate it into two parts:

$$\int_{\mathbb{R}} |a'|^2 \left| v^\beta |p(v) - p(\tilde{v}_\varepsilon)|^2 - \bar{v}^\beta |p(\bar{v}) - p(\tilde{v}_\varepsilon)|^2 \right| d\xi$$
$$\leq \underbrace{\int_{\mathbb{R}} |a'|^2 v^\beta \left| |p(v) - p(\tilde{v}_\varepsilon)|^2 - |p(\bar{v}) - p(\tilde{v}_\varepsilon)|^2 \right| d\xi}_{=:I_1} + \underbrace{\int_{\mathbb{R}} |a'|^2 \left| v^\beta - \bar{v}^\beta \right| |p(\bar{v}) - p(\tilde{v}_\varepsilon)|^2 \, d\xi}_{=:I_2}.$$

Using the same arguments as in (4.58), it follows from (4.50) that

$$I_1 \leq \int_{\mathbb{R}} |a'|^2 v^\beta \left(|p(v) - p(\bar{v})|^2 + 2\delta_3 |p(v) - p(\bar{v})| \right) d\xi$$
$$\leq C\lambda^2 \left(\mathcal{D}(U) + \frac{\varepsilon}{\lambda}\mathcal{G}_2(U) \right).$$

For I_2, we first separate I_2 into two parts:

$$I_2 = \underbrace{\int_{\mathbb{R}} |a'|^2 \left| v^\beta - \bar{v}_b^\beta \right| |p(\bar{v}_b) - p(\tilde{v}_\varepsilon)|^2 \, d\xi}_{=:I_2^b} + \underbrace{\int_{\mathbb{R}} |a'|^2 \left| v^\beta - \bar{v}_s^\beta \right| |p(\bar{v}_s) - p(\tilde{v}_\varepsilon)|^2 \, d\xi}_{=:I_2^s}.$$

Using the assumption $\beta \leq 1$, we have

$$I_2^b \leq \delta_3^2 \int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}_b^\beta|}{|v - \bar{v}|} |v - \bar{v}| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)<-\delta_3\}} \, d\xi$$

$$\leq C \int_{\mathbb{R}} |a'|^2 |v - \bar{v}| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)<-\delta_3\}} \, d\xi$$

$$\leq C \int_{\mathbb{R}} |a'|^2 \left(Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon) \right) d\xi \leq C\varepsilon\lambda\mathcal{G}_2(U).$$

Likewise, we use $\beta > 0$ to have

$$I_2^s \leq \delta_3^2 \int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}_s^\beta|}{|v - \bar{v}|} |v - \bar{v}| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}} \, d\xi$$

$$\leq C \int_{\mathbb{R}} |a'|^2 |v - \bar{v}| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}} \, d\xi \leq C\varepsilon\lambda\mathcal{G}_2(U).$$

$\square$

Lemma 4.6. *Under the same assumption as Proposition 4.3, we have*

$$(4.64) \qquad \int_{\mathbb{R}} |a'| \, |p(v|\tilde{v}_\varepsilon) - p(\bar{v}|\tilde{v}_\varepsilon)| \, d\xi$$

$$\leq C\sqrt{\frac{\varepsilon}{\lambda}} \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(\bar{U}) \right) + C \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) \right),$$

$$(4.65) \qquad \int_{\mathbb{R}} |a'| \, |Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)| \, d\xi + \int_{\mathbb{R}} |a'||v - \bar{v}| \, d\xi \leq C \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) \right).$$

Proof. Following the proof of [33, Lemma 3.3] together with (4.47), we have

$$\int_{\mathbb{R}} |a'| \, |Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)| \, d\xi + \int_{\mathbb{R}} |a'||v - \bar{v}| \, d\xi$$

$$\leq C \int_{\mathbb{R}} |a'| \left(Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon) \right) d\xi \leq C \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) \right).$$

Following the proof of [33, Lemma 3.3], we have

$$\int_{\mathbb{R}} |a'| \, |p(v|\tilde{v}_\varepsilon) - p(\bar{v}|\tilde{v}_\varepsilon)| \, d\xi$$

$$\leq \underbrace{\int_{\mathbb{R}} |a'| \, |p(v) - p(\bar{v})| \, d\xi}_{=:I_1} + C \underbrace{\int_{\mathbb{R}} |a'| \left(Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon) \right) d\xi}_{=:I_2}.$$

First, using (4.47), we have

$$I_2 \leq C \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) \right).$$

We separate I_1 into three parts:

$$I_1 = \underbrace{\int_{\mathbb{R}} |a'| \, |p(v) - p(\bar{v}_b)| \, d\xi}_{=:I_{11}} + \underbrace{\int_{\mathbb{R}} |a'| \, |p(v) - p(\bar{v}_s)| \, \mathbf{1}_{\{v<v_-/2\}} \, d\xi}_{=:I_{12}} + \underbrace{\int_{\mathbb{R}} |a'| \, |p(v) - p(\bar{v}_s)| \, \mathbf{1}_{\{v\geq v_-/2\}} \, d\xi}_{=:I_{13}}.$$

By (4.48), we have

$$I_{11} \leq C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U).$$

For I_{12}, we first observe that

$$v^\beta |p(v) - p(\bar{v}_s)|^2 = p(v)^{-\frac{\gamma-\alpha}{\gamma}} |p(v) - p(\bar{v}_s)|^2 \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}}$$

(4.66)
$$= \left(\frac{|p(v)-p(\bar{v}_s)|}{p(v)}\right)^{\frac{\gamma-\alpha}{\gamma}} \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}} |p(v) - p(\bar{v}_s)|^{\frac{\gamma+\alpha}{\gamma}}$$

$$\geq C\delta_3^{\frac{\gamma-\alpha}{\gamma}} |p(v) - p(\bar{v}_s)|^{\frac{\gamma+\alpha}{\gamma}}.$$

Since (by the smallness of ε_0 and δ_3)

$$|p(v) - p(\bar{v}_s)|\, \mathbf{1}_{\{v<v_-/2\}} \geq |p(v) - p(\tilde{v}_\varepsilon)|\, \mathbf{1}_{\{v<v_-/2\}} - |p(\bar{v}_s) - p(\tilde{v}_\varepsilon)|\, \mathbf{1}_{\{v<v_-/2\}}$$

(4.67)
$$\geq |p(v) - p(\tilde{v}_\varepsilon)|\, \mathbf{1}_{\{v<v_-/2\}} - \delta_3$$

$$\geq |p(v_-/2) - p(3v_-/4)| - \delta_3$$

$$\geq \frac{1}{2}|p(v_-/2) - p(3v_-/4)|,$$

using (4.66), we have

$$|p(v) - p(\bar{v}_s)|\, \mathbf{1}_{\{v<v_-/2\}} \leq C|p(v) - p(\bar{v}_s)|^{\frac{\gamma+\alpha}{\gamma}} \leq Cv^\beta |p(v) - p(\bar{v}_s)|^2.$$

Then it follows from (4.63) that for all $\xi \in \mathbb{R}$,

(4.68) $$|p(v) - p(\bar{v}_s)|\, \mathbf{1}_{\{v<v_-/2\}}(\xi) \leq C\left(|\xi| + \frac{1}{\varepsilon}\right)\left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U)\right).$$

Therefore, using (4.68) together with the same estimate as in (4.55), we have

(4.69)
$$I_{12} \leq \int_{|\xi|\leq\frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}} |a'||p(v) - p(\bar{v}_s)|\mathbf{1}_{\{v<v_-/2\}}\, d\xi + \int_{|\xi|\geq\frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}} |a'||p(v) - p(\bar{v}_s)|\mathbf{1}_{\{v<v_-/2\}}\, d\xi$$

$$\leq C\left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U)\right)\left(\sqrt{\frac{\lambda}{\varepsilon^3}}\int_{\mathbb{R}} |a'|Q(v|\tilde{v}_\varepsilon)\, d\xi + 2\int_{|\xi|\geq\frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}} |a'||\xi|\, d\xi\right)$$

$$\leq C\left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U)\right)\left(\sqrt{\frac{\varepsilon}{\lambda}} + \lambda\varepsilon\int_{|\xi|\geq\frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}} e^{-c\varepsilon|\xi|}|\xi|\, d\xi\right)$$

$$\leq C\sqrt{\frac{\varepsilon}{\lambda}}\left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U)\right).$$

For I_{13}, since $|p(v) - p(\bar{v}_s)|\, \mathbf{1}_{\{v\geq v_-/2\}} \leq p(v_-/2)$, we have

$$I_{13} = \int_{\mathbb{R}} |a'|\frac{|p(v) - p(\bar{v}_s)|}{|v - \bar{v}|}|v - \bar{v}|\mathbf{1}_{\{v\geq v_-/2\}\cap\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}}\, d\xi$$

$$\leq C\int_{\mathbb{R}} |a'||v - \bar{v}|\mathbf{1}_{\{v\geq v_-/2\}\cap\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}}\, d\xi \leq C\left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right).$$

Hence, we have

$$I_1 \leq C\sqrt{\frac{\varepsilon}{\lambda}}\left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U)\right) + C\left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right),$$

which gives (4.64). $\qquad\qquad\square$

Lemma 4.7. *Under the same assumption as Proposition 4.3, we have*

$$(4.70) \qquad \int_{\Omega^c} |a'||p(v) - p(\tilde{v}_\varepsilon)||h - \tilde{h}_\varepsilon| d\xi \leq \delta_0 \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U) \right) + \frac{1}{2}\mathcal{G}_1^-(U),$$

$$(4.71) \qquad \int_{\Omega^c} |a'| \left(Q(\bar{v}|\tilde{v}_\varepsilon) + |\bar{v} - \tilde{v}_\varepsilon| \right) d\xi \leq C\sqrt{\frac{\varepsilon}{\lambda}} \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U) \right).$$

Proof. **Proof of** (4.70)**:** We first separate it into two parts:

$$\int_{\Omega^c} |a'||p(v) - p(\tilde{v}_\varepsilon)||h - \tilde{h}_\varepsilon| \, d\xi$$

$$\leq \underbrace{\int_{\Omega^c} |a'||p(v) - p(\bar{v})||h - \tilde{h}_\varepsilon| \, d\xi}_{=:J_1} + \underbrace{\int_{\Omega^c} |a'||p(\bar{v}) - p(\tilde{v}_\varepsilon)||h - \tilde{h}_\varepsilon| \, d\xi}_{=:J_2}$$

We use the definition of $\bar{v}_s$ and Hölder's inequality to have

$$J_1 = \int_{\Omega^c} |a'||p(v) - p(\bar{v}_s)||h - \tilde{h}_\varepsilon| \, d\xi$$

$$\leq \left(\int_{\Omega^c} |a'||p(v) - p(\bar{v}_s)|^2 d\xi \right)^{1/2} \left(\int_{\Omega^c} |a'||h - \tilde{h}_\varepsilon|^2 d\xi \right)^{1/2}.$$

To estimate $\int_{\mathbb{R}} |a'||p(v) - p(\bar{v}_s)|^2 d\xi$, using (4.66) and (4.63), we find that for any $\xi \in \mathbb{R}$,

$$(4.72) \qquad |(p(v) - p(\bar{v}_s))(\xi)|^2 \leq C \left(|\xi| + \frac{1}{\varepsilon} \right)^{\frac{2\gamma}{\gamma+\alpha}} \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U) \right)^{\frac{2\gamma}{\gamma+\alpha}}.$$

Following the similar arguments as in (4.55), and using (4.72) with $q := \frac{2\gamma}{\gamma+\alpha}$ (note $1 < q < 2$ by $0 < \alpha < \gamma$), we obtain

$$\int_{\mathbb{R}} |a'||p(v) - p(\bar{v}_s)|^2 d\xi$$

$$\leq \int_{|\xi| \leq \frac{1}{\varepsilon}\left(\frac{\lambda}{\varepsilon}\right)^{1/q}} |a'||p(v) - p(\bar{v}_s)|^2 \, d\xi + \int_{|\xi| \geq \frac{1}{\varepsilon}\left(\frac{\lambda}{\varepsilon}\right)^{1/q}} |a'||p(v) - p(\bar{v}_s)|^2 \, d\xi$$

$$\leq C \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U) \right)^q \left(\frac{1}{\varepsilon^q} \int_{\mathbb{R}} |a'|Q(v|\tilde{v}_\varepsilon) \, d\xi + 2 \int_{|\xi| \geq \left(\frac{\lambda}{\varepsilon}\right)^{1/q}} |a'||\xi|^q \, d\xi \right)$$

$$\leq C \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U) \right)^q \left(\frac{1}{\varepsilon^q} \frac{\varepsilon^2}{\lambda} + \frac{\lambda}{\varepsilon^q} \int_{|\xi| \geq \left(\frac{\lambda}{\varepsilon}\right)^{1/q}} |\xi|^q e^{-c|\xi|} \, d\xi \right)$$

$$\leq C \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U) \right)^q \frac{\varepsilon^{2-q}}{\lambda}.$$

Therefore,

$$J_1 \leq C\sqrt{\frac{\varepsilon^{2-q}}{\lambda}} \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U) \right)^{q/2} \left(\int_{\Omega^c} |a'||h - \tilde{h}_\varepsilon|^2 d\xi \right)^{1/2}.$$

Using the Young's inequality (recall $1 < q < 2$), we have

$$J_1 \leq \delta_0 \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U) \right) + \frac{C}{\delta_0} \left(\frac{\varepsilon^{2-q}}{\lambda} \right)^{\frac{1}{2-q}} \left(\int_{\Omega^c} |a'||h - \tilde{h}_\varepsilon|^2 d\xi \right)^{\frac{1}{2-q}}.$$

Since (4.32) yields

$$\left(\frac{\varepsilon^{2-q}}{\lambda}\right)^{\frac{1}{2-q}} \left(\int_{\Omega^c} |a'||h - \tilde{h}_\varepsilon|^2 d\xi\right)^{\frac{1}{2-q}} \leq C \left(\frac{\varepsilon^{2-q}}{\lambda}\right)^{\frac{1}{2-q}} \left(\frac{\varepsilon^2}{\lambda}\right)^{\frac{q-1}{2-q}} \int_{\Omega^c} |a'||h - \tilde{h}_\varepsilon|^2 d\xi$$

$$= C \left(\frac{\varepsilon}{\lambda}\right)^{\frac{2}{2-q}} \int_{\Omega^c} |a'||h - \tilde{h}_\varepsilon|^2 d\xi$$

$$\leq C \left(\frac{\varepsilon}{\lambda}\right)^2 \int_{\Omega^c} |a'||h - \tilde{h}_\varepsilon|^2 d\xi,$$

we have

$$J_1 \leq \delta_0 \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U)\right) + \frac{C}{\delta_0} \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_1^-(U).$$

For J_2, we use $|p(\bar{v}) - p(\tilde{v}_\varepsilon)| \leq \delta_3$ and Young's inequality to have

$$J_2 \leq \delta_3 \int_{\Omega^c} |a'||h - \tilde{h}_\varepsilon|\, d\xi \leq \frac{1}{2}\mathcal{G}_1^-(U) + C \underbrace{\int_{\mathbb{R}} |a'| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}}\, d\xi}_{=:J_{21}}.$$

To control J_{21}, we observe that since $(y - \delta_3/2)_+ \geq \delta_3/2$ whenever $(y - \delta_3)_+ > 0$, we have

$$(4.73) \qquad |p(v) - p(\bar{v}_{\delta_3/2})| = (|p(v) - p(\tilde{v}_\varepsilon)| - \delta_3/2)_+ \geq \frac{\delta_3}{2}\mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}}.$$

Then, using (4.73) and (4.72) (with $q := \frac{2\gamma}{\gamma+\alpha}$) and following the same estimates as in (4.55), we have

$$J_{21} \leq C \int_{\mathbb{R}} |a'||p(v) - p(\bar{v}_{\delta_3/2})|^{2/q} \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}}\, d\xi$$

$$\leq C\sqrt{\frac{\varepsilon}{\lambda}} \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U)\right).$$

Therefore,

$$J_2 \leq \frac{1}{2}\mathcal{G}_1^-(U) + C\sqrt{\frac{\varepsilon}{\lambda}} \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U)\right).$$

Hence we obtain (4.70).

Proof of (4.71): The proof follows from the above estimate for J_{12} as follows:

$$\int_{\Omega^c} |a'| \left(Q(\bar{v}|\tilde{v}_\varepsilon) + |\bar{v} - \tilde{v}_\varepsilon|\right) d\xi \leq C \int_{\mathbb{R}} |a'| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}}\, d\xi \leq C\sqrt{\frac{\varepsilon}{\lambda}} \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U)\right).$$

$\square$

Lemma 4.8. *Under the same assumption as Proposition 4.3, we have*

$$(4.74) \qquad \int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}^\beta|^2}{v^\beta}\, d\xi \leq C\lambda \left(\mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(\bar{U})\right),$$

$$(4.75) \qquad \int_{\mathbb{R}} |a'|^2 \left|\frac{|v^\beta - \tilde{v}_\varepsilon^\beta|^2}{v^\beta} - \frac{|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2}{\bar{v}^\beta}\right| d\xi$$

$$\leq C\lambda \left(\mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(\bar{U})\right).$$

Proof. **Proof of** (4.74)**:** We first have

$$\int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}^\beta|^2}{v^\beta} \, d\xi = \underbrace{\int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}_b^\beta|^2}{v^\beta} \, d\xi}_{=:I_b} + \underbrace{\int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}_s^\beta|^2}{v^\beta} \, d\xi}_{=:I_s} \, .$$

Since $0 < \beta \le 1$, we have

$$I_b \le C \int_{\mathbb{R}} |a'|^2 |v^\beta - \bar{v}_b^\beta| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)<-\delta_3\}} \, d\xi = C \int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}_b^\beta|}{|v - \bar{v}|} |v - \bar{v}| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)<-\delta_3\}} \, d\xi$$

$$\le C \int_{\mathbb{R}} |a'|^2 |v - \bar{v}| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)<-\delta_3\}} \, d\xi \le C \int_{\mathbb{R}} |a'|^2 \left(Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon) \right) \, d\xi$$

$$\le C \varepsilon \lambda \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) \right).$$

For I_s, we separate it into two cases of $\alpha \ge 1$ and $\alpha < 1$.
Case of $\alpha \ge 1$: Since $\beta \le \gamma - 1$ by $\alpha \ge 1$, we observe

$$v^{-\beta} \le v^{-\gamma+1} = Q(v) \quad \text{as} \quad v \to 0,$$

we have

$$I_s = \int_{\mathbb{R}} |a'|^2 \frac{v^{-\beta} |v^\beta - \bar{v}_s^\beta|^2}{Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)} \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}} \left(Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon) \right) \, d\xi$$

$$\le C \int_{\mathbb{R}} |a'|^2 \left(Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon) \right) \, d\xi \le C \varepsilon \lambda \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) \right).$$

Case of $0 < \alpha < 1$: Since $\frac{1-\alpha}{\gamma} < 1$ and

$$v^\beta p(v)^{\frac{1-\alpha}{\gamma}} Q(v) = v^\beta (v^{-\gamma})^{\frac{1-\alpha}{\gamma}} v^{-\gamma+1} = v^0 = 1,$$

using (4.59), we have

$$I_s = \int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}_s^\beta|^2 \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon)>\delta_3\}}}{v^\beta |p(v) - p(\bar{v}_s)|^{\frac{1-\alpha}{\gamma}} \left(Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon) \right)} |p(v) - p(\bar{v}_s)|^{\frac{1-\alpha}{\gamma}} \left(Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon) \right) \, d\xi$$

$$\le C \int_{\mathbb{R}} |a'|^2 |p(v) - p(\bar{v}_s)|^{\frac{1-\alpha}{\gamma}} \left(Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon) \right) \, d\xi.$$

Then, using the fact from (4.66) and (4.63) that

$$|p(v) - p(\bar{v}_s)|^{\frac{\gamma+\alpha}{\gamma}} \le C \left(|\xi| + \frac{1}{\varepsilon} \right) \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right),$$

we have

$$I_s \le C \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right)^{\frac{1-\alpha}{\gamma+\alpha}} \int_{\mathbb{R}} |a'|^2 \left(|\xi| + \frac{1}{\varepsilon} \right)^{\frac{1-\alpha}{\gamma+\alpha}} \left(Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon) \right) d\xi.$$

Notice that since $0 < \frac{1-\alpha}{\gamma+\alpha} < 1$ by $0 < \alpha < 1$, we have

$$|a'| \left(|\xi| + \frac{1}{\varepsilon} \right)^{\frac{1-\alpha}{\gamma+\alpha}} \le C \varepsilon \lambda e^{-c\varepsilon|\xi|} \left(|\xi| + \frac{1}{\varepsilon} \right)^{\frac{1-\alpha}{\gamma+\alpha}}$$

$$\le C \varepsilon^{1-\frac{1-\alpha}{\gamma+\alpha}} \lambda e^{-c\varepsilon|\xi|} \left(|\varepsilon\xi| + 1 \right)^{\frac{1-\alpha}{\gamma+\alpha}} \le C\lambda.$$

Thus, we have

$$I_s \leq C\lambda\left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U)\right)^{\frac{1-\alpha}{\gamma+\alpha}} \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right).$$

Now, using the Young's inequality with $\frac{1}{p} := \frac{1-\alpha}{\gamma+\alpha}$ and $\frac{1}{p} + \frac{1}{p'} = 1$, and then (4.32), we have

$$I_s \leq C\lambda\left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U)\right) + C\lambda\left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right)^{p'}$$

$$\leq C\lambda\left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U)\right) + C\lambda\left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right).$$

Hence we complete the proof.

Proof of (4.75)**:** Since $C^{-1} \leq \bar{v}^\beta \leq C$ and $|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta| \leq C$, we have

$$\int_{\mathbb{R}} |a'|^2 \left|\frac{|v^\beta - \tilde{v}_\varepsilon^\beta|^2}{v^\beta} - \frac{|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2}{\bar{v}^\beta}\right| d\xi$$

$$\leq \int_{\mathbb{R}} |a'|^2 \left(\frac{1}{v^\beta}\left||v^\beta - \tilde{v}_\varepsilon^\beta|^2 - |\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2\right| + \frac{|v^\beta - \bar{v}^\beta|}{v^\beta\bar{v}^\beta}|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2\right) d\xi$$

$$\leq C\int_{\mathbb{R}} |a'|^2 \left(\frac{1}{v^\beta}|v^\beta - \bar{v}^\beta|\left(|v^\beta - \bar{v}^\beta| + 2|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|\right) + \frac{|v^\beta - \bar{v}^\beta|}{v^\beta}\right) d\xi$$

$$\leq C\int_{\mathbb{R}} |a'|^2 \frac{1}{v^\beta}|v^\beta - \bar{v}^\beta|^2 d\xi + C\underbrace{\int_{\mathbb{R}} |a'|^2 \frac{1}{v^\beta}|v^\beta - \bar{v}^\beta| d\xi}_{=:J}.$$

By (4.74), it remains to estimate the term J. For that, we separate it into two parts:

$$J = \underbrace{\int_{\mathbb{R}} |a'|^2 \frac{1}{v^\beta}|v^\beta - \bar{v}^\beta|\mathbf{1}_{\{v<v_-/2\}\cup\{v>2v_-\}} d\xi}_{=:J_1} + \underbrace{\int_{\mathbb{R}} |a'|^2 \frac{1}{v^\beta}|v^\beta - \bar{v}^\beta|\mathbf{1}_{\{v_-/2\leq v\leq 2v_-\}} d\xi}_{=:J_2}.$$

Using the same argument as (4.67) together with the definition of $\bar{v}$, we have

$$|v^\beta - \bar{v}^\beta|\mathbf{1}_{\{v<v_-/2\}\cup\{v>2v_-\}} > C > 0,$$

which yields

$$J_1 \leq C\int_{\mathbb{R}} |a'|^2 \frac{1}{v^\beta}|v^\beta - \bar{v}^\beta|^2 d\xi.$$

Since

$$|v - \bar{v}|\mathbf{1}_{\{v_-/2\leq v\leq 2v_-\}} \leq |p'(v_-/2)||p(v) - p(\bar{v})|,$$

we have

$$J_2 = \int_{\mathbb{R}} |a'|^2 \frac{1}{v^\beta} \frac{|v^\beta - \bar{v}^\beta|}{|v - \bar{v}|}|v - \bar{v}|\mathbf{1}_{\{v_-/2\leq v\leq 2v_-\}} d\xi$$

$$\leq C\int_{\mathbb{R}} |a'|^2|v - \bar{v}|\mathbf{1}_{\{v_-/2\leq v\leq 2v_-\}} d\xi$$

$$\leq C\int_{\mathbb{R}} |a'|^2 v^\beta|p(v) - p(\bar{v})|\mathbf{1}_{\{v_-/2\leq v\leq 2v_-\}} d\xi.$$

Therefore, (4.50) and (4.74) give the desired result.   □

4.8.1. *Proof of Proposition 4.3.* **Proof of** (4.36): It follows from (4.64) together with $|\tilde{v}'_\varepsilon| \leq C\frac{\varepsilon}{\lambda}|a'|$ that

$$|\mathcal{B}_1(U) - \mathcal{B}_1(\bar{U})| \leq C\frac{\varepsilon}{\lambda}\left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(\bar{U}) + \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right)\right).$$

Proof of (4.37): By (4.70), we have

$$|\mathcal{B}_2^-(U)| \leq \delta_0\left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(U)\right) + \frac{1}{2}\mathcal{G}_1^-(U).$$

Proof of (4.38): We use (4.43) and (4.49) to have

$$|\mathcal{B}_2^+(U) - \mathcal{B}_2^+(\bar{U})| = \left|\int_{\mathbb{R}} a'\left(|p(v) - p(\tilde{v}_\varepsilon)|^2 - |p(\bar{v}_b) - p(\tilde{v}_\varepsilon)|\right)\mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta_3\}}d\xi\right| \leq \sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U).$$

Proof of (4.39): Using Young's inequality together with $\frac{1}{2} \leq a \leq 1$, we first find

$$|\mathcal{B}_3(U)| \leq \delta_0\mathcal{D}(U) + \underbrace{\frac{C}{\delta_0}\int_{\mathbb{R}}|a'|^2 v^\beta |p(v) - p(\tilde{v}_\varepsilon)|^2\, d\xi}_{=:\mathcal{B}_6},$$

$$|\mathcal{B}_4(U)| \leq \underbrace{\int_{\mathbb{R}}|a'||\partial_\xi p(\tilde{v}_\varepsilon)|v^\beta|p(v) - p(\tilde{v}_\varepsilon)|^2\, d\xi}_{=:\mathcal{B}_7} + \underbrace{\int_{\mathbb{R}}|a'||\partial_\xi p(\tilde{v}_\varepsilon)|\frac{|v^\beta - \tilde{v}_\varepsilon^\beta|^2}{v^\beta}\, d\xi}_{=:\mathcal{B}_8},$$

$$|\mathcal{B}_5(U)| \leq \delta_0\mathcal{D}(U) + \underbrace{\frac{C}{\delta_0}\int_{\mathbb{R}}|\partial_\xi p(\tilde{v}_\varepsilon)|^2\frac{|v^\beta - \tilde{v}_\varepsilon^\beta|^2}{v^\beta}\, d\xi}_{=:\mathcal{B}_9}.$$

Using (4.51) and (4.75) together with $|\partial_\xi p(\tilde{v}_\varepsilon)| \leq C\frac{\varepsilon}{\lambda}|a'|$ and $\delta_0^{-1}\varepsilon < \lambda < \delta_0$, we have

$$|\mathcal{B}_6(U) - \mathcal{B}_6(\bar{U})| + |\mathcal{B}_7(U) - \mathcal{B}_7(\bar{U})| \leq C\delta_0\left(\mathcal{D}(U) + \frac{\varepsilon}{\lambda}\mathcal{G}_2(U)\right),$$

$$|\mathcal{B}_8(U) - \mathcal{B}_8(\bar{U})| + |\mathcal{B}_9(U) - \mathcal{B}_9(\bar{U})| \leq C\delta_0\left(\mathcal{D}(U) + \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(\bar{U})\right).$$

Therefore, we have

$$\sum_{i=3}^{5}|\mathcal{B}_i(U)| \leq \sum_{i=6}^{9}|\mathcal{B}_i(\bar{U})| + C\delta_0\mathcal{D}(U) + C\delta_0\left(\left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right) + \frac{\varepsilon}{\lambda}\mathcal{G}_2(\bar{U})\right).$$

Since $|a'| \leq C\varepsilon\lambda$, we have

$$\sum_{i=6}^{9}|\mathcal{B}_i(\bar{U})| \leq C\frac{\varepsilon\lambda}{\delta_0}\int_{\mathbb{R}}|a'|\bar{v}^\beta|p(\bar{v}) - p(\tilde{v}_\varepsilon)|^2\, d\xi + C\varepsilon^2\int_{\mathbb{R}}|a'|\frac{|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2}{\bar{v}^\beta}\, d\xi.$$

Using $C^{-1} \leq \bar{v} \leq C$ and (3.16) together with $|p(\bar{v}) - p(\tilde{v}_\varepsilon)| \leq \delta_3$, we have

$$\int_{\mathbb{R}}|a'|\bar{v}^\beta|p(\bar{v}) - p(\tilde{v}_\varepsilon)|^2\, d\xi \leq C\int_{\mathbb{R}}|a'|Q(\bar{v}|\tilde{v}_\varepsilon)\, d\xi \leq C\mathcal{G}_2(\bar{U}).$$

Moreover, since

$$\int_{\mathbb{R}} |a'| \frac{|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2}{\bar{v}^\beta} \, d\xi \le C \int_{\mathbb{R}} |a'| \frac{|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2}{Q(\bar{v}|\tilde{v}_\varepsilon)} Q(\bar{v}|\tilde{v}_\varepsilon) \, d\xi$$

$$\le C \int_{\mathbb{R}} |a'| Q(\bar{v}|\tilde{v}_\varepsilon) \, d\xi \le C\mathcal{G}_2(\bar{U}),$$

we have

$$\sum_{i=6}^{9} |\mathcal{B}_i(\bar{U})| \le C\delta_0 \frac{\varepsilon}{\lambda} \mathcal{G}_2(\bar{U}).$$

Hence we have the desired estimate (4.39).

Proof of (4.40)**:** First, using (3.15), (3.16) and (4.32), we have

$$|\mathcal{B}_1(\bar{U})| + |\mathcal{B}_2^+(\bar{U})| \le C \int_{\mathbb{R}} |a'| Q(\bar{v}|\tilde{v}_\varepsilon) \, d\xi \le C \int_{\mathbb{R}} |a'| Q(v|\tilde{v}_\varepsilon) \, d\xi \le C\frac{\varepsilon^2}{\lambda}.$$

Then, it follows from (4.35), (4.36)-(4.39) and (4.47) that

$$|\mathcal{B}_{\delta_3}(U)| \le C\frac{\varepsilon^2}{\lambda} + C\sqrt{\delta_0} \left(\mathcal{D}(U) + \mathcal{G}_2(U) \right) \le C^* \frac{\varepsilon^2}{\lambda} + C\sqrt{\delta_0} \mathcal{D}(U).$$

Proof of (4.41)**:** We split the proof in three steps.

Step 1: First of all, we use the notations Y_1^s, Y_2^s, Y_3^s and Y_4^s for the terms of Y_s as follows:

$$Y_s = \underbrace{-\int_{\Omega^c} a' Q(v|\tilde{v}_\varepsilon) d\xi}_{=:Y_1^s} \underbrace{-\int_{\Omega^c} a \partial_\xi p(\tilde{v}_\varepsilon)(v - \tilde{v}_\varepsilon) d\xi}_{=:Y_2^s} \underbrace{-\int_{\Omega^c} a' \frac{|h - \tilde{h}_\varepsilon|^2}{2} d\xi}_{=:Y_3^s} + \underbrace{\int_{\Omega^c} a \partial_\xi \tilde{h}_\varepsilon (h - \tilde{h}_\varepsilon) d\xi}_{=:Y_4^s}.$$

We use (4.48),(4.49), (4.65) together with (4.43) to have

$$|Y_g(U) - Y_g(\bar{U})| + |Y_1^s(U) - Y_1^s(\bar{U})| + |Y_2^s(U) - Y_2^s(\bar{U})|$$

(4.76)
$$\le C \int_{\mathbb{R}} |a'| \left(\left| |p(v) - p(\tilde{v}_\varepsilon)|^2 - |p(\bar{v}_b) - p(\tilde{v}_\varepsilon)|^2 \right| + \left| Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon) \right| \right.$$

$$\left. + |v - \bar{v}| + |p(v) - p(\bar{v}_b)| \right) d\xi \le C\sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U) + C \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) \right).$$

On the other hand, (4.71) yields
(4.77)

$$|Y_1^s(\bar{U})| + |Y_2^s(\bar{U})| \le \int_{\Omega^c} |a'| \left(Q(\bar{v}|\tilde{v}_\varepsilon) + |\bar{v} - \tilde{v}_\varepsilon| \right) d\xi \le C\sqrt{\frac{\varepsilon}{\lambda}} \left(\mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right).$$

Next, by the definitions of $\mathcal{G}_1^\pm$ in (4.18), we have

$$|Y_3^s(U)| + |Y_b(U)| \le C\mathcal{G}_1^-(U) + C\mathcal{G}_1^+(U) + C \int_\Omega |a'| |p(v) - p(\tilde{v}_\varepsilon)|^2 \, d\xi$$

$$\le C(\mathcal{G}_1^-(U) + \mathcal{G}_1^+(U) + |\mathcal{B}_{\delta_3}(U)|),$$

Moreover, since

$$\mathcal{G}_1^+(U) \le C \int_\Omega |a'| \left(|h - \tilde{h}_\varepsilon|^2 + |p(v) - p(\tilde{v}_\varepsilon)|^2 \right) d\xi \le C \int_\Omega |a'| |h - \tilde{h}_\varepsilon|^2 \, d\xi + C|\mathcal{B}_{\delta_3}(U)|,$$

using (4.40), we have

$$|Y_3^s(U)| + |Y_b(U)| \leq C \int_{\mathbb{R}} |a'||h - \tilde{h}_\varepsilon|^2 \, d\xi + C^* \frac{\varepsilon^2}{\lambda} + C\sqrt{\delta_0}\mathcal{D}(U).$$

Therefore, using (4.32), (4.47), and the assumption $\mathcal{D}(U) \leq C^*\varepsilon^2/\lambda$, it follows from (4.76), (4.77) and the above estimate that

$$|Y_g(U) - Y_g(\bar{U})| + |Y_1^s(U)| + |Y_2^s(U)| + |Y_3^s(U)| + |Y_b(U)| \leq C\frac{\varepsilon^2}{\lambda}.$$

Step 2: First of all, using Young's inequality and (3.16), (4.38), we estimate

$$\left| \int_\Omega a' \big(p(v) - p(\tilde{v}_\varepsilon) \big) \Big(h - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \Big) d\xi \right|$$

$$\leq \Big(\frac{\lambda}{\varepsilon} \Big)^{1/4} \mathcal{G}_1^+(U) + C \Big(\frac{\varepsilon}{\lambda} \Big)^{1/4} \int_\Omega |a'||p(v) - p(\tilde{v}_\varepsilon)|^2 \, d\xi$$

$$\leq \Big(\frac{\lambda}{\varepsilon} \Big)^{1/4} \mathcal{G}_1^+(U) + C \Big(\frac{\varepsilon}{\lambda} \Big)^{1/4} \big(\mathcal{B}_2^+(\bar{U}) + (\mathcal{B}_2^+(U) - \mathcal{B}_2^+(\bar{U})) \big)$$

$$\leq \Big(\frac{\lambda}{\varepsilon} \Big)^{1/4} \mathcal{G}_1^+(U) + C \Big(\frac{\varepsilon}{\lambda} \Big)^{1/4} \Big(\mathcal{G}_2(\bar{U}) + \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U) \Big).$$

Therefore, this estimate, (4.76) and (4.77) yield

$$|Y_g(U) - Y_g(\bar{U})| + |Y_1^s(U)| + |Y_2^s(U)| + |Y_3^s(U)| + |Y_b(U)|$$

$$\leq C\sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U) + C \big(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) \big) + C\mathcal{G}_1^-(U) + 2 \Big(\frac{\lambda}{\varepsilon} \Big)^{1/4} \mathcal{G}_1^+(U) + C \Big(\frac{\varepsilon}{\lambda} \Big)^{1/4} \mathcal{G}_2(\bar{U}).$$

Step 3: For the remaining terms, using Hölder's inequality together with $|\tilde{h}_\varepsilon'| \leq C\frac{\varepsilon}{\lambda}|a'|$, we estimate

$$|Y_4^s(U)|^2 \leq C \Big(\frac{\varepsilon}{\lambda} \Big)^2 \Big(\int_{\mathbb{R}} |a'| \, d\xi \Big) \int_{\Omega^c} |a'||h - \tilde{h}_\varepsilon|^2 \, d\xi \leq C\frac{\varepsilon^2}{\lambda} \mathcal{G}_1^-(U),$$

$$|Y_l(U)|^2 \leq C \Big(\frac{\varepsilon}{\lambda} \Big)^2 \Big(\int_{\mathbb{R}} |a'| \, d\xi \Big) \int_\Omega |a'| \Big(h - \tilde{h}_\varepsilon - \frac{p(\bar{v}) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \Big)^2 d\xi \leq C\frac{\varepsilon^2}{\lambda} \mathcal{G}_1^+(U).$$

Therefore, this together with Step1 and Step2 yield

$$|Y_g(U) - Y_g(\bar{U})|^2 + |Y_b(U)|^2 + |Y_l(U)|^2 + |Y_s(U)|^2$$

$$\leq \big(|Y_g(U) - Y_g(\bar{U})| + |Y_1^s(U)| + |Y_2^s(U)| + |Y_3^s(U)| + |Y_b(U)| \big)^2 + |Y_4^s(U)|^2 + |Y_l(U)|^2$$

$$\leq C\frac{\varepsilon^2}{\lambda} \Big(\sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U) + \big(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) \big) + \mathcal{G}_1^-(U) + \Big(\frac{\lambda}{\varepsilon} \Big)^{1/4} \mathcal{G}_1^+(U) + \Big(\frac{\varepsilon}{\lambda} \Big)^{1/4} \mathcal{G}_2(\bar{U}) \Big).$$

4.9. **Proof of Proposition 4.1.** We now prove the main Proposition 4.1. We split the proof into two steps, depending on the strength of the dissipation term $\mathcal{D}(U)$.

Step 1: We first consider the case of $\mathcal{D}(U) \geq 4C^* \frac{\varepsilon^2}{\lambda}$, where the constant C^* is defined as in Proposition 4.3. Then using (4.40) and taking δ_0 small enough, we have

$$
\begin{aligned}
\mathcal{R}(U) &\leq -\frac{|Y(U)|^2}{\varepsilon^4} + \left(1 + \delta_0 \frac{\varepsilon}{\lambda}\right)|\mathcal{B}_{\delta_3}(U)| - \mathcal{G}_1^-(U) - \mathcal{G}_1^+(U) - \left(1 - \delta_0 \frac{\varepsilon}{\lambda}\right)\mathcal{G}_2(U) - (1 - \delta_0)\mathcal{D}(U) \\
&\leq 2|\mathcal{B}_{\delta_3}(U)| - (1 - \delta_0)\mathcal{D}(U) \\
&\leq 2C^* \frac{\varepsilon^2}{\lambda} - \left(1 - \delta_0 - 2C\sqrt{\delta_0}\right)\mathcal{D}(U) \\
&\leq 2C^* \frac{\varepsilon^2}{\lambda} - \frac{1}{2}\mathcal{D}(U) \leq 0,
\end{aligned}
$$

which gives the desired result.

Step 2: We now assume the other alternative, i.e., $\mathcal{D}(U) \leq 4C^* \frac{\varepsilon^2}{\lambda}$.
We will use Proposition 4.2 to get the desired result. First of all, we have (4.34), and for the small constant δ_3 of Proposition 4.2 associated to the constant C_2 of (4.34), we have $|p(\bar{v}) - p(\tilde{v}_\varepsilon)| \leq \delta_3$.
Using

$$
Y_g(\bar{U}) = Y(U) - (Y_g(U) - Y_g(\bar{U})) - Y_b(U) - Y_l(U) - Y_s(U),
$$

we have

$$
|Y_g(\bar{U})|^2 \leq 4\left(|Y(U)|^2 + |Y_g(U) - Y_g(\bar{U})|^2 + |Y_b(U)|^2 + |Y_l(U)|^2 + |Y_s(U)|^2\right),
$$

which can be written as

$$
-4|Y(U)|^2 \leq -|Y_g(\bar{U})|^2 + 4|Y_g(U) - Y_g(\bar{U})|^2 + 4|Y_b(U)|^2 + 4|Y_l(U)|^2 + 4|Y_s(U)|^2.
$$

Now, let us take δ_0 small enough such that $\delta_0 \leq \delta_3^9$. (In fact, since we see from the proofs of Lemma 4.5-4.8 that the constants C in Proposition 4.3 depend on δ_3 as algebraically negative power of it, we take δ_0 smaller enough if needed.)
Then we find that for any $\varepsilon < \varepsilon_0 (\leq \delta_3)$ and $\varepsilon/\lambda < \delta_0 (\leq \delta_3^9)$,

$$
\begin{aligned}
\mathcal{R}(U) &\leq -\frac{4|Y(U)|^2}{\varepsilon\delta_3} + \mathcal{B}_{\delta_3}(U) + \delta_0 \frac{\varepsilon}{\lambda}|\mathcal{B}_{\delta_3}(U)| \\
&\quad - \mathcal{G}_1^-(U) - \mathcal{G}_1^+(U) - \left(1 - \delta_0 \frac{\varepsilon}{\lambda}\right)\mathcal{G}_2(U) - (1 - \delta_0)\mathcal{D}(U) \\
&\leq -\frac{|Y_g(\bar{U})|^2}{\varepsilon\delta_3} + \left(\mathcal{B}_1(\bar{U}) + \mathcal{B}_2^+(\bar{U})\right) + \delta_0 \frac{\varepsilon}{\lambda}\left(|\mathcal{B}_1(\bar{U})| + |\mathcal{B}_2^+(\bar{U})|\right) \\
&\quad - \left(1 - \delta_3 \frac{\varepsilon}{\lambda}\right)\mathcal{G}_2(\bar{U}) - (1 - \delta_3)\mathcal{D}(\bar{U}) \\
&\quad + \underbrace{\frac{4}{\varepsilon\delta_3}\left(|Y_g(U) - Y_g(\bar{U})|^2 + |Y_b(U)|^2 + |Y_l(U)|^2 + |Y_s(U)|^2\right)}_{=:J_1} \\
&\quad + \underbrace{\left(1 + \delta_0 \frac{\varepsilon}{\lambda}\right)\left(|\mathcal{B}_1(U) - \mathcal{B}_1(\bar{U})| + |\mathcal{B}_2^+(U) - \mathcal{B}_2^+(\bar{U})| + |\mathcal{B}_2^-(U)| + \sum_{i=3}^{5}|\mathcal{B}_i(U)|\right)}_{=:J_2} \\
&\quad - \mathcal{G}_1^-(U) - \mathcal{G}_1^+(U) - \frac{1}{2}\left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right) - \frac{\delta_3}{2}\frac{\varepsilon}{\lambda}\mathcal{G}_2(\bar{U}) - (\delta_3 - \delta_0)\mathcal{D}(U),
\end{aligned}
$$

where we used $\mathcal{D}(\bar{U}) \le \mathcal{D}(U)$ by (4.45). We claim that J_1, J_2 are controlled by the last line above. Indeed, it follows from (4.36)-(4.39) and (4.41) that for any $\varepsilon/\lambda < \delta_0 (\le \delta_3^9)$,

$$J_1 \le \frac{C}{\delta_3} \frac{\varepsilon}{\lambda} \left(\sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U) + \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right) + \mathcal{G}_1^-(U) + \left(\frac{\lambda}{\varepsilon}\right)^{1/4} \mathcal{G}_1^+(U) + \left(\frac{\varepsilon}{\lambda}\right)^{1/4} \mathcal{G}_2(\bar{U}) \right)$$

$$\le \frac{C}{\delta_3} \left(\frac{\varepsilon}{\lambda}\right)^{1/4} \left(\mathcal{D}(U) + \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right) + \mathcal{G}_1^-(U) + \mathcal{G}_1^+(U) + \frac{\varepsilon}{\lambda}\mathcal{G}_2(\bar{U}) \right)$$

$$\le \frac{1}{4}\delta_3 \left(\mathcal{D}(U) + \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right) + \mathcal{G}_1^-(U) + \mathcal{G}_1^+(U) + \frac{\varepsilon}{\lambda}\mathcal{G}_2(\bar{U}) \right),$$

$$J_2 \le C\sqrt{\delta_0} \left(\mathcal{D}(U) + \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right) + \frac{\varepsilon}{\lambda}\mathcal{G}_2(\bar{U}) \right)$$

$$\le \frac{1}{4}\delta_3 \left(\mathcal{D}(U) + \left(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})\right) + \frac{\varepsilon}{\lambda}\mathcal{G}_2(\bar{U}) \right).$$

Therefore, we have

$$\mathcal{R}(U) \le -\frac{|Y_g(\bar{U})|^2}{\varepsilon \delta_3} + \left(\mathcal{B}_1(\bar{U}) + \mathcal{B}_2^+(\bar{U})\right) + \delta_3 \frac{\varepsilon}{\lambda} \left(|\mathcal{B}_1(\bar{U})| + |\mathcal{B}_2^+(\bar{U})|\right)$$

$$- \left(1 - \delta_3 \frac{\varepsilon}{\lambda}\right)\mathcal{G}_2(\bar{U}) - (1 - \delta_3)\mathcal{D}(\bar{U}).$$

Since the above quantities $Y_g(\bar{U}), \mathcal{B}_1, \mathcal{B}_2^+(\bar{U}), \mathcal{G}_2(\bar{U})$ and $\mathcal{D}(\bar{U})$ depends only on $\bar{v}$ through $\bar{U}$, and $\mathcal{B}_1(\bar{U}) = \mathcal{I}_1(\bar{v})$ and $\mathcal{B}_2^+(\bar{U}) = \mathcal{I}_2^+(\bar{v})$, it follows from Proposition 4.2 that $\mathcal{R}(U) \le 0$. Hence we complete the proof of Proposition 4.1.

5. Proof of Theorem 1.1

5.1. **Proof of** (1.12) **: Well-prepared initial data.** For a given datum (v^0, u^0) satisfying (1.11), let $\{(v_0^r, u_0^r)\}_{r>0}$ be a sequence of truncations defined by

$$v_0^r = \begin{cases} v^0 \mathbf{1}_{\{r \le v^0 \le r^{-1}\}} & \text{if } -r^{-1} \le x \le r^{-1}, \\ v_- & \text{if } x \le -r^{-1}, \\ v_+ & \text{if } x \ge r^{-1}, \end{cases}$$

and

$$u_0^r = \begin{cases} u^0 \mathbf{1}_{\{-r^{-1} \le u^0 \le r^{-1}\}} & \text{if } -r^{-1} \le x \le r^{-1}, \\ u_- & \text{if } x \le -r^{-1}, \\ u_+ & \text{if } x \ge r^{-1}. \end{cases}$$

Then, we consider a mollification of the above sequence: using $\phi_\nu(x) := \frac{1}{\sqrt{\nu}}\phi_1\left(\frac{x}{\sqrt{\nu}}\right)$ where ϕ_1 is a smooth mollifier with $\text{supp}\phi_1 = [-1, 1]$, consider a double sequence $\{(v_0^{r,\nu}, u_0^{r,\nu})\}_{r,\nu>0}$ defined by

$$v_0^{r,\nu} = v_0^r * \phi_\nu, \quad u_0^{r,\nu} = u_0^r * \phi_\nu.$$

First, we will show

$$(5.1) \qquad \lim_{r \to 0} \lim_{\nu \to 0} \int_{\mathbb{R}} Q\left(v_0^{r,\nu} | \tilde{v}\left(\frac{x}{\nu}\right)\right) dx = \int_{-\infty}^0 Q(v^0 | v_-) dx + \int_0^\infty Q(v^0 | v_+) dx.$$

For a fixed r, since $v_0^{r,\nu} \to v_0^r$ a.e., and $\tilde{v}\left(\frac{x}{\nu}\right) \to v_-$ a.e. $x < 0$ as $\nu \to 0$, using

$$Q\left(v_0^{r,\nu}|\tilde{v}\left(\frac{x}{\nu}\right)\right) - Q(v_0^r|v_-) = \left(Q(v_0^{r,\nu}) - Q(v_0^r)\right) + \left(Q(v_-) - Q\left(\tilde{v}\left(\frac{x}{\nu}\right)\right)\right)$$
$$- \left(Q'\left(\tilde{v}\left(\frac{x}{\nu}\right)\right) - Q'(v_-)\right)\left(v_0^{r,\nu} - \tilde{v}\left(\frac{x}{\nu}\right)\right) - Q'(v_-)\left(\left(v_0^{r,\nu} - v_0^r\right) - \left(\tilde{v}\left(\frac{x}{\nu}\right) - v_-\right)\right),$$

we have

$$Q\left(v_0^{r,\nu}|\tilde{v}\left(\frac{x}{\nu}\right)\right) \to Q(v_0^r|v_-) \quad \text{a.e. } x < 0, \quad \text{as } \nu \to 0.$$

Likewise,

$$Q\left(v_0^{r,\nu}|\tilde{v}\left(\frac{x}{\nu}\right)\right) \to Q(v_0^r|v_+) \quad \text{a.e. } x > 0, \quad \text{as } \nu \to 0.$$

Moreover, since

$$|v_0^r - \bar{v}| \leq \max(r^{-1}, v_\pm)\mathbf{1}_{\{|x| \leq r^{-1}\}},$$

we have

$$Q\left(v_0^{r,\nu}|\tilde{v}\left(\frac{x}{\nu}\right)\right) \leq C_r\left|v_0^{r,\nu} - \tilde{v}\left(\frac{x}{\nu}\right)\right|^2 \leq C_r\left(\left|v_0^{r,\nu} - \bar{v}\right|^2 + \left|\tilde{v}\left(\frac{x}{\nu}\right) - \bar{v}\right|^2\right)$$
$$\leq \max(r^{-2}, v_\pm^2)\mathbf{1}_{\{|x| \leq r^{-1}+1\}} + \left|\tilde{v} - \bar{v}\right|^2, \quad \nu < 1.$$

Since $\mathbf{1}_{\{|x| \leq r^{-1}+1\}} + \left|\tilde{v} - \bar{v}\right|^2 \in L^1(\mathbb{R})$, the dominated convergence theorem implies

$$\lim_{\nu \to 0} \int_{\mathbb{R}} Q\left(v_0^{r,\nu}|\tilde{v}\left(\frac{x}{\nu}\right)\right) dx = \int_{-\infty}^0 Q(v_0^r|v_-) dx + \int_0^\infty Q(v_0^r|v_+) dx.$$

Furthermore, since $Q(v_0^r|\bar{v}) \leq Q(v^0|\bar{v}) \in L^1(\mathbb{R})$ and $v_0^r \to v^0$ a.e. as $r \to 0$, we have

$$\lim_{r \to 0} \left(\int_{-\infty}^0 Q(v_0^r|v_-) dx + \int_0^\infty Q(v_0^r|v_+) dx\right) = \int_{-\infty}^0 Q(v^0|v_-) dx + \int_0^\infty Q(v^0|v_+) dx,$$

which completes (5.1).

Hence by the diagonal extraction of (5.1), there exists a sequence (still denoted by v_0^ν) such that

$$\lim_{\nu \to 0} \int_{\mathbb{R}} Q\left(v_0^\nu|\tilde{v}\left(\frac{x}{\nu}\right)\right) dx = \int_{-\infty}^0 Q(v^0|v_-) dx + \int_0^\infty Q(v^0|v_+) dx.$$

In particular, we have from the above construction that v_0^ν converges to v^0 in $L^1_{loc}(\mathbb{R})$, and especially :

(5.2) $v_0^\nu \to v^0 \quad \text{in } W_{loc}^{-s,1}(\mathbb{R}), \ s > 0.$

where this convergence will be used in the proof of (5.37).

Using the same argument as above, we show

(5.3)
$$\lim_{r \to 0} \lim_{\nu \to 0} \int_{\mathbb{R}} \frac{1}{2}\left(u_0^{r,\nu} + \nu\left(p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}}\right)_x - \tilde{u}^\nu(x) - \nu\left(p\left(\tilde{v}^\nu(x)\right)^{\frac{\alpha}{\gamma}}\right)_x\right)^2 dx = \int_{-\infty}^\infty \frac{|u^0 - \bar{u}|^2}{2} dx,$$

where $\tilde{u}^\nu(x) = \tilde{u}(x/\nu)$ and $\tilde{v}^\nu(x) = \tilde{v}(x/\nu)$.

Indeed, since $|v_0^r| \le r^{-1}$ for any small $r > 0$, we have

$$\left| \nu \left(p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}} \right)_x \right| = \nu \frac{\alpha}{\gamma} p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}-1} \left| p'(v_0^{r,\nu}) \right| \left| v_0^r * (\phi_\nu(x))_x \right|$$

$$\le \frac{\alpha}{\gamma} p(r^{-1})^{\frac{\alpha}{\gamma}-1} |p'(r)| r^{-1} \int_{\mathbb{R}} \left| \phi_1'(\frac{y}{\sqrt{\nu}}) \right| dy$$

$$\le C(r)\sqrt{\nu},$$

which means

$$\left\| \nu \left(p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}} \right)_x \right\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as } \nu \to 0.$$

Moreover, since $\tilde{v}'\left(\frac{x}{\nu}\right) \to 0$ a.e. as $\nu \to 0$, we have

$$\left(u_0^{r,\nu} + \nu \left(p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}} \right)_x - \tilde{u}\left(\frac{x}{\nu}\right) - \nu \left(p\left(\tilde{v}\left(\frac{x}{\nu}\right)\right)^{\frac{\alpha}{\gamma}}\right)_x \right)^2 \to \begin{cases} |u_0^r - u_-|^2 & \text{for a.e. } x < 0, \\ |u_0^r - u_+|^2 & \text{for a.e. } x > 0. \end{cases}$$

Furthermore, since

$$\left(u_0^{r,\nu} + \nu \left(p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}} \right)_x - \tilde{u}\left(\frac{x}{\nu}\right) - \nu \left(p\left(\tilde{v}\left(\frac{x}{\nu}\right)\right)^{\frac{\alpha}{\gamma}}\right)_x \right)^2$$

$$\le 2\left(|u_0^{r,\nu} - \bar{u}|^2 + \left| \tilde{u}\left(\frac{x}{\nu}\right) - \bar{u}\right|^2 + \left| \nu \left(p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}} \right)_x \right|^2 + \left| \nu \left(p\left(\tilde{v}\left(\frac{x}{\nu}\right)\right)^{\frac{\alpha}{\gamma}}\right)_x \right|^2 \right)$$

$$\le C(r)\left(\mathbf{1}_{\{|x| \le r^{-1}+1\}} + |\tilde{u} - \bar{u}|^2 + \left| \phi_1'\left(\frac{x}{\sqrt{\nu}}\right) \right|^2 + \left| \tilde{v}'\left(\frac{x}{\nu}\right) \right|^2 \right)$$

$$\le C(r)\left(\mathbf{1}_{\{|x| \le r^{-1}+1\}} + |\tilde{u} - \bar{u}|^2 + \left| \phi_1'(x) \right|^2 + \left| \tilde{v}'(x) \right|^2 \right) =: g(x),$$

and $g \in L^1(\mathbb{R})$, the dominated convergence theorem implies

$$\lim_{\nu \to 0} \int_{\mathbb{R}} \frac{1}{2} \left(u_0^{r,\nu} + \nu \left(p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}} \right)_x - \tilde{u}\left(\frac{x}{\nu}\right) - \nu \left(p\left(\tilde{v}\left(\frac{x}{\nu}\right)\right)^{\frac{\alpha}{\gamma}}\right)_x \right)^2 dx$$

$$= \int_{-\infty}^0 \frac{|u_0^r - u_-|^2}{2} dx + \int_0^\infty \frac{|u_0^r - u_+|^2}{2} dx.$$

Furthermore, since $|u_0^r - \bar{u}|^2 \le |u^0 - \bar{u}|^2 \in L^1(\mathbb{R})$ and $u_0^r \to u^0$ a.e. as $r \to 0$, we have

$$\lim_{r \to 0} \left(\int_{-\infty}^0 \frac{|u_0^r - u_-|^2}{2} dx + \int_0^\infty \frac{|u_0^r - u_+|^2}{2} dx \right) = \int_{-\infty}^\infty \frac{|u^0 - \bar{u}|^2}{2} dx,$$

which completes (5.3). Hence, using the diagonal extraction as before, there exists a sequence (still denoted by u_0^ν) such that

$$\lim_{\nu \to 0} \int_{\mathbb{R}} \frac{1}{2} \left(u_0^r + \nu \left(p(v_0^r)^{\frac{\alpha}{\gamma}} \right)_x - \tilde{u}^\nu(x) - \nu \left(p\left(\tilde{v}^\nu(x)\right)^{\frac{\alpha}{\gamma}}\right)_x \right)^2 dx = \int_{-\infty}^\infty \frac{|u^0 - \bar{u}|^2}{2} dx.$$

5.2. **Proof for the main part of Theorem 1.1.** We here present a proof for the second part (ii) of Theorem 1.1.

5.2.1. **Uniform estimates in ν.** Let $\{(v^\nu, u^\nu)\}_{\nu>0}$ be a sequence of solutions on $(0, T)$ to (1.1) with the initial datum (v_0^ν, u_0^ν). Our starting point is to apply Theorem 1.2 to the below functions:

$$v(t, x) = v^\nu(\nu t, \nu x), \quad \tilde{v}(x) := \tilde{v}^\nu(\nu x), \quad u(t, x) = u^\nu(\nu t, \nu x), \quad \tilde{u}(x) := \tilde{u}^\nu(\nu x).$$

That is, using (1.18) in Theorem 1.2 together with (4.14), we have

$$\int_{-\infty}^{\infty} E\big((v, u)(t, x) | (\tilde{v}, \tilde{u})(x - X(t))\big) dx$$
$$+ \int_0^{T/\nu} \int_{-\infty}^{\infty} |\tilde{v}'(x)| Q\left(v(t, x) | \tilde{v}(x - X(t))\right) dx dt$$
$$+ \int_0^{T/\nu} \int_{-\infty}^{\infty} v^{\gamma-\alpha}(t, x) \big|\partial_x\big(p(v(t, x)) - p(\tilde{v}(x - X(t)))\big)\big|^2 dx dt$$
$$\le C \int_{-\infty}^{\infty} E\big((v_0^\nu, u_0^\nu)(\nu x) | (\tilde{v}, \tilde{u})(x)\big) dx.$$

Then by the change of variables $t \mapsto t/\nu, x \mapsto x/\nu$, we have

$$\int_{-\infty}^{\infty} E_\nu\big((v^\nu, u^\nu)(t, x) | (\tilde{v}^\nu, \tilde{u}^\nu)(x - X_\nu(t))\big) dx$$
$$+ \int_0^T \int_{-\infty}^{\infty} |(\tilde{v}^\nu)'(x)| Q\left(v^\nu(t, x) | \tilde{v}^\nu(x - X_\nu(t))\right) dx dt$$
$$+ \nu \int_0^T \int_{-\infty}^{\infty} (v^\nu)^{\gamma-\alpha}(t, x) \big|\partial_x\big(p(v^\nu(t, x)) - p(\tilde{v}^\nu(x - X_\nu(t)))\big)\big|^2 dx dt$$
$$\le C \int_{-\infty}^{\infty} E_\nu\big((v_0^\nu, u_0^\nu)(x) | (\tilde{v}^\nu, \tilde{u}^\nu)(x)\big) dx,$$

where $X_\nu(t) := \nu X(t/\nu)$, and

$$(5.4) \qquad E_\nu((v_1, u_1) | (v_2, u_2)) := \frac{1}{2}\left(u_1 + \nu\left(p(v_1)^{\frac{\alpha}{\gamma}}\right)_x - u_2 - \nu\left(p(v_2)^{\frac{\alpha}{\gamma}}\right)_x\right)^2 + Q(v_1 | v_2).$$

For simplification, we introduce the variables:

$$(5.5) \qquad\qquad h^\nu := u^\nu + \nu\left(p(v^\nu)^{\frac{\alpha}{\gamma}}\right)_x, \quad \tilde{h}^\nu := \tilde{u}^\nu + \nu\left(p(\tilde{v}^\nu)^{\frac{\alpha}{\gamma}}\right)_x,$$

Then, recalling (1.8), the above estimate implies

$$\int_{-\infty}^{\infty} \eta\big((v^\nu, h^\nu)(t, x) | (\tilde{v}^\nu, \tilde{h}^\nu)(x - X_\nu(t))\big) dx$$
$$+ \int_0^T \int_{-\infty}^{\infty} |(\tilde{v}^\nu)'| Q\left(v^\nu(t, x) | \tilde{v}^\nu(x - X_\nu(t))\right) dx dt$$
$$+ \nu \int_0^T \int_{-\infty}^{\infty} (v^\nu)^{\gamma-\alpha} \big|\partial_x\big(p(v^\nu(t, x)) - p(\tilde{v}^\nu(x - X_\nu(t)))\big)\big|^2 dx dt$$
$$\le C \int_{-\infty}^{\infty} E_\nu\big((v_0^\nu, u_0^\nu) | (\tilde{v}^\nu, \tilde{u}^\nu)\big) dx.$$

Therefore, using (1.12), we find that

for any $\delta \in (0,1)$, there exists ν_* such that for all $\nu < \nu_*$,

(5.6)
$$\int_{-\infty}^{\infty} \eta\big((v^\nu, h^\nu)(t,x)|(\tilde{v}^\nu, \tilde{h}^\nu)(x - X_\nu(t))\big) dx$$
$$+ \int_0^T \int_{-\infty}^{\infty} |(\tilde{v}^\nu)'| \, Q\left(v^\nu(t,x)|\tilde{v}^\nu(x - X_\nu(t))\right) dx dt$$
$$+ \nu \int_0^T \int_{-\infty}^{\infty} (v^\nu)^{\gamma-\alpha} \big|\partial_x \big(p(v^\nu(t,x)) - p(\tilde{v}^\nu(x - X_\nu(t)))\big)\big|^2 dx dt$$
$$\leq C\mathcal{E}_0 + \delta,$$

where

$$\mathcal{E}_0 := \int_{-\infty}^{\infty} \eta\big((v^0, u^0)|(\bar{v}, \bar{u})\big) dx.$$

5.2.2. **Proof of** (1.13). We first prove the weak convergence (1.13).

• **Convergence of** $\{v^\nu\}_{\nu>0}$: For the given two end states $v_\pm$, we first fix a constant $M > 1$ such that
$$\big(\min\{v_-, v_+\}, \max\{v_-, v_+\}\big) \subset (M^{-1}, M).$$
Then we fix the constant $k_0 > 1$ in Lemma 3.2, and set

(5.7)
$$K := \max\{3M, k_0\}.$$

For the constant $K > 1$, let $\overline{\psi}$ be a continuous function defined by

(5.8)
$$\overline{\psi}(x) = \begin{cases} x, & \text{if } K^{-1} \leq x \leq K, \\ K^{-1}, & \text{if } x < K^{-1}, \\ K, & \text{if } x > K. \end{cases}$$

Then we set

(5.9)
$$\underline{v}^\nu := \overline{\psi}(v^\nu), \quad v_e^\nu := v^\nu - \underline{v}^\nu.$$

Note that the truncation $\underline{v}^\nu$ will be used in the proof of (1.14) below. Since
$$|v_e^\nu| \leq \max\left\{(K^{-1} - v^\nu)_+, (v^\nu - K)_+\right\},$$
and $M^{-1} < \tilde{v}^\nu(x - X_\nu) < M$, we use (3.12) in Lemma 3.2 to have
$$|v_e^\nu| \leq CQ\big(v^\nu|\tilde{v}^\nu(x - X_\nu)\big).$$
Then it follows from (5.6) that for all $\nu < \nu_*$,
$$\int_{-\infty}^{\infty} |v_e^\nu| dx \leq C\big(\mathcal{E}_0 + 1\big).$$

Therefore, $\{v_e^\nu\}_{\nu>0}$ is bounded in $L^\infty(0, T; L^1(\mathbb{R}))$. Moreover, since the definition (5.9) implies that $\{\underline{v}^\nu\}_{\nu>0}$ is bounded in $L^\infty((0,T) \times \mathbb{R}) \subset L^\infty(0, T; L^1_{loc}(\mathbb{R}))$, we obtain that

(5.10)
$$\{v^\nu\}_{\nu>0} \text{ is bounded in } L^\infty(0, T; L^1_{loc}(\mathbb{R})).$$

Therefore, there exists v_∞ such that

(5.11)
$$v^\nu \rightharpoonup v_\infty \quad \text{in } \mathcal{M}_{loc}((0,T) \times \mathbb{R}),$$

and
$$v_\infty \in L^\infty(0, T; L^\infty(\mathbb{R}) + \mathcal{M}(\mathbb{R})).$$

- **Convergence of $\{u^\nu\}_{\nu > 0}$:**
  We split the proof into two steps.

Step 1: We will first show convergence of $\{h^\nu\}_{\nu > 0}$.
For the given two end states $u_\pm$, we first fix a constant $L > 1$ such that
$$\left(\min\{u_-, u_+\}, \max\{u_-, u_+\}\right) \subset \left(-\frac{L}{2}, \frac{L}{2}\right).$$

Then let $\overline{\varphi}$ be a continuous function defined by

(5.12)
$$\overline{\varphi}(x) = \begin{cases} x, & \text{if } |x| \leq L, \\ -L, & \text{if } x < -L, \\ L, & \text{if } x > L. \end{cases}$$

Then we set

(5.13)
$$\underline{h}^\nu := \overline{\varphi}(h^\nu), \quad h_e^\nu := h^\nu - \underline{h}^\nu.$$

Note that the truncation $\underline{h}^\nu$ will be used in the proof of (1.14) below.
Likewise, since
$$|h_e^\nu| \leq \max\left\{(-h^\nu - L)_+, (h^\nu - L)_+\right\},$$

and $-L < \tilde{h}^\nu(x - X_\nu) < L$, we have
$$|h_e^\nu| \leq \max\left\{(-h^\nu - L)_+, (h^\nu - L)_+\right\} \leq |h^\nu - \tilde{h}^\nu(x - X_\nu)|.$$

Then it follows from (5.6) that for all $\nu < \nu_*$,
$$\int_{-\infty}^{\infty} |h_e^\nu|^2 dx \leq C(\mathcal{E}_0 + 1).$$

Therefore, $\{h_e^\nu\}_{\nu > 0}$ is bounded in $L^\infty(0, T; L^2(\mathbb{R}))$. Moreover, since $\{\underline{h}^\nu\}_{\nu > 0}$ is bounded in $L^\infty((0, T) \times \mathbb{R}) \subset L^\infty(0, T; L_{loc}^2(\mathbb{R}))$, we have

(5.14)
$$\{h^\nu\}_{\nu > 0} \text{ is bounded in } L^\infty(0, T; L_{loc}^2(\mathbb{R})).$$

Therefore, there exists $u_\infty \in L^\infty(0, T; L_{loc}^2(\mathbb{R}))$ such that

(5.15)
$$h^\nu \rightharpoonup u_\infty \quad \text{in } L^\infty(0, T; L_{loc}^2(\mathbb{R})).$$

Step 2: We now prove that $u^\nu \rightharpoonup u_\infty$ in $\mathcal{M}_{loc}((0, T) \times \mathbb{R})$.
Since $u^\nu = h^\nu - \nu\left(p(v^\nu)^{\frac{\alpha}{\gamma}}\right)_x$, it is enough to show that

(5.16)
$$\nu\left(p(v^\nu)^{\frac{\alpha}{\gamma}}\right)_x \rightharpoonup 0 \quad \text{in } \mathcal{M}((0, T) \times \mathbb{R}).$$

In fact, since
$$\left(p(v^\nu)^{\frac{\alpha}{\gamma}}\right)_x = \frac{\alpha}{\gamma} p(v^\nu)^{\frac{\alpha - \gamma}{\gamma}} p(v^\nu)_x = \frac{\alpha}{\gamma}(v^\nu)^\beta p(v^\nu)_x, \quad (\text{recall } \beta = \gamma - \alpha),$$

it is enough to show that

(5.17)
$$\nu(v^\nu)^\beta p(v^\nu)_x \rightharpoonup 0 \quad \text{in } \mathcal{M}((0, T) \times \mathbb{R}).$$

For that, we separate $\nu(v^\nu)^\beta p(v^\nu)_x$ into two parts:

$$\nu(v^\nu)^\beta p(v^\nu)_x = \underbrace{\nu(v^\nu)^\beta \Big(p(v^\nu) - p(\tilde{v}^\nu(x - X_\nu(t)))\Big)_x}_{=:J_1} + \underbrace{\nu(v^\nu)^\beta p(\tilde{v}^\nu(x - X_\nu(t)))_x}_{=:J_2}$$

For any $\Psi \in C_c((0,T) \times \mathbb{R})$, using (5.6) and (3.8) together with the condition $0 < \beta \leq 1$, we find that for all $\nu < \nu_*$,

$$\int_{\mathbb{R}^+\times\mathbb{R}} J_1 \Psi dx dt \leq \nu \sqrt{\int_{\mathbb{R}^+\times\mathbb{R}} (v^\nu)^\beta \left|\big(p(v^\nu) - p(\tilde{v}^\nu(x - X_\nu(t)))\big)_x\right|^2 \Psi dx dt} \sqrt{\int_{\mathbb{R}^+\times\mathbb{R}} (v^\nu)^\beta \Psi dx dt}$$

$$\leq C\sqrt{\nu}\sqrt{\mathcal{E}_0 + 1} \sqrt{\int_{\mathbb{R}^+\times\mathbb{R}} (v^\nu)^\beta \big(\mathbf{1}_{\{v^\nu \leq 3v_-\}} + \mathbf{1}_{\{v^\nu \geq 3v_-\}}\big) \Psi dx dt}$$

$$\leq C\sqrt{\nu}\sqrt{\mathcal{E}_0 + 1} \sqrt{1 + \int_{\mathbb{R}^+\times\mathbb{R}} |v^\nu - \tilde{v}^\nu(x - X_\nu(t))|\mathbf{1}_{\{v^\nu \geq 3v_-\}} \Psi dx dt}$$

$$\leq C\sqrt{\nu}\sqrt{\mathcal{E}_0 + 1} \sqrt{1 + \int_{\mathbb{R}^+\times\mathbb{R}} Q\left(v^\nu | \tilde{v}^\nu(x - X_\nu(t))\right) \Psi dx dt}$$

$$\leq C\sqrt{\nu}\sqrt{\mathcal{E}_0 + 1} \sqrt{1 + \int_{\text{supp}(\Psi)} \eta\big((v^\nu, h^\nu)(t,x)|(\tilde{v}^\nu, \tilde{h}^\nu)(x - X_\nu(t))\big) dx dt}$$

$$\leq C\sqrt{\nu}(\mathcal{E}_0 + 1).$$

Likewise, we find that for all $\nu < \nu_*$,

$$\int_{\mathbb{R}^+\times\mathbb{R}} J_2 \Psi dx dt \leq \nu C \int_{\mathbb{R}^+\times\mathbb{R}} p(\tilde{v}^\nu(x - X_\nu(t)))_x \mathbf{1}_{\{v^\nu \leq 3v_-\}} \Psi dx dt$$

$$+ \nu \int_{\mathbb{R}^+\times\mathbb{R}} (v^\nu)^\beta p(\tilde{v}^\nu(x - X_\nu(t)))_x \mathbf{1}_{\{v^\nu \geq 3v_-\}} \Psi dx dt$$

$$\leq \nu C \int_{\mathbb{R}^+\times\mathbb{R}} |(\tilde{v}^\nu)'(x)| \Psi dx dt$$

$$+ C\nu \int_{\mathbb{R}^+\times\mathbb{R}} |(\tilde{v}^\nu)'(x)| Q\left(v^\nu | \tilde{v}^\nu(x - X_\nu(t))\right) \mathbf{1}_{\{v^\nu \geq 3v_-\}} \Psi dx dt$$

$$\leq \nu C \int_{\mathbb{R}} |\tilde{v}'(x)| dx + C\nu(\mathcal{E}_0 + 1).$$

Therefore we have

$$\int_{\mathbb{R}^+\times\mathbb{R}} \nu(v^\nu)^\beta p(v^\nu)_x \Psi dx dt \to 0 \quad \text{as } \nu \to 0,$$

which implies (5.17), and thus,

(5.18) $$u^\nu \rightharpoonup u_\infty \quad \text{in } \mathcal{M}_{\text{loc}}((0,T) \times \mathbb{R}).$$

Hence we complete the proof of (1.13).

5.2.3. **Convergence of $\{X_\nu\}_{\nu>0}$.**

Lemma 5.1. *There exists $X_\infty \in BV(0,T)$ such that*

(5.19) $$X_\nu \to X_\infty \quad \text{in } L^1(0,T), \quad \text{up to subsequence as } \nu \to 0.$$

Proof. First, since $X'_\nu(t) = X'(t/\nu)$, it follows from (1.19) that

$$|X'_\nu(t)| \leq C(1 + f_\nu(t)),$$

where $f_\nu(t) := f\left(\frac{t}{\nu}\right)$. Notice that (1.19) and (1.12) imply that for any $\nu < \nu_*$,

$$\|f_\nu\|_{L^1(0,T)} = \nu\|f\|_{L^1(0,T/\nu)} \leq \nu C \int_{-\infty}^{\infty} \eta\big((v_0^1, h_0^1)|(\tilde{v}, \tilde{h})\big)dx$$

$$\leq C\Big(\int_{-\infty}^{\infty} \eta\big((v^0, u^0)|(\bar{v}, \bar{u})\big)dx + 1\Big).$$

Thus, f_ν is uniformly bounded in $L^1(0,T)$. Therefore, X'_ν is uniformly bounded in $L^1(0,T)$. Moreover, since $X_\nu(0) = 0$ and thus,

$$|X_\nu(t)| \leq Ct + C \int_0^t f_\nu(s)ds,$$

X_ν is also uniformly bounded in $L^1(0,T)$.

Therefore, by the compactness of BV (see for example [1, Theorem 3.23]), we have the desired convergence. $\qquad\square$

5.2.4. **Proof of** (1.14). Consider a mollifier

(5.20) $$\phi_\varepsilon(t) := \frac{1}{\varepsilon}\phi\Big(\frac{t}{\varepsilon}\Big) \quad \text{for any } \varepsilon > 0,$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a nonnegative smooth function such that $\int_{\mathbb{R}} \phi = 1$ and supp $\phi = [-1,1]$. For the truncations $\underline{v}^\nu, \underline{h}^\nu$ defined by (5.9), (5.13) with L, K fixed, we let

$$L_\nu := \int_0^T \phi_\varepsilon(s) \int_{\mathbb{R}} \eta\big((\underline{v}^\nu, \underline{h}^\nu)(s,x)|(\tilde{v}^\nu, \tilde{h}^\nu)(x - X_\nu(s))\big)dxds,$$

Using the definition of the truncations together with (5.6) and $\int_0^T \phi_\varepsilon = 1$, we find that for all $\nu < \nu_*$,

(5.21) $$L_\nu \leq \int_0^T \phi_\varepsilon(s) \int_{\mathbb{R}} \eta\big((v^\nu, h^\nu)(s,x)|(\tilde{v}^\nu, \tilde{h}^\nu)(x - X_\nu(s))\big)dxds \leq C\mathcal{E}_0 + \delta.$$

Then we have the following.

Lemma 5.2. *For the fixed constants L, K, let*

$$R_\nu := \int_0^T \phi_\varepsilon(s) \int_{\mathbb{R}} \eta\big((\underline{v}^\nu, \underline{h}^\nu)(s,x)|(\bar{v}, \bar{u})(x - X_\infty(s))\big)dxds.$$

Then

$$\Big|L_\nu - R_\nu\Big| \to 0 \quad \text{up to a subsequence as } \nu \to 0.$$

Proof. Since $\underline{h}^\nu, \tilde{h}^\nu, \bar{u}$ are bounded, we have

$$\left||\underline{h}^\nu - \tilde{h}^\nu(x - X_\nu)|^2 - |\underline{h}^\nu - \bar{u}(x - X_\infty)|^2\right| \le C\left|\tilde{h}^\nu(x - X_\nu) - \bar{u}(x - X_\infty)\right|.$$

We separate the right-hand side into two parts:

$$|\tilde{h}^\nu(x - X_\nu) - \bar{u}(x - X_\infty)| \le \underbrace{|\tilde{h}^\nu(x - X_\nu) - \bar{u}(x - X_\nu)|}_{=:I_1} + \underbrace{|\bar{u}(x - X_\nu) - \bar{u}(x - X_\infty)|}_{=:I_2}.$$

Since $\tilde{h}^\nu = \tilde{u}^\nu + \nu p(\tilde{v}^\nu)_x$, using $\|\tilde{u}^\nu - \bar{u}\|_{L^1(\mathbb{R})} = \nu\|\tilde{u} - \bar{u}\|_{L^1(\mathbb{R})}$, we have

$$\|I_1\|_{L^1(\mathbb{R})} = \|\tilde{h}^\nu - \bar{u}\|_{L^1(\mathbb{R})} \le \|\tilde{u}^\nu - \bar{u}\|_{L^1(\mathbb{R})} + \nu\|p(\tilde{v}^\nu)_x\|_{L^1(\mathbb{R})} \le C\nu.$$

Moreover, since $\|I_2\|_{L^1(\mathbb{R})} = |u_- - u_+||X_\nu - X_\infty|$, it follows from Lemma 5.1 that

$$\int_0^T \phi_\varepsilon(s) \int_\mathbb{R} \frac{1}{2}\left||\underline{h}^\nu - \tilde{h}^\nu(x - X_\nu(s))|^2 - |\underline{h}^\nu - \bar{u}(x - X_\infty(s))|^2\right|dxds$$

$$\le C\nu + C\int_0^\infty \phi_\varepsilon(s)|X_\nu(s) - X_\infty(s)|ds \to 0.$$

Likewise, since $\underline{v}^\nu$ is bounded, using the definition of $Q(\cdot|\cdot)$, we have

$$\left|Q(\underline{v}^\nu|\tilde{v}^\nu(x - X_\nu)) - Q(\underline{v}^\nu|\bar{v}(x - X_\infty))\right|$$
$$\le |Q(\tilde{v}^\nu(x - X_\nu)) - Q(\bar{v}(x - X_\infty))| + \left|Q'(\tilde{v}^\nu(x - X_\nu))\right|\left|\tilde{v}^\nu(x - X_\nu) - \bar{v}(x - X_\infty)\right|$$
$$\quad + (|\underline{v}^\nu| + |\bar{v}(x - X_\infty)|)\left|Q'(\tilde{v}^\nu(x - X_\nu)) - Q'(\bar{v}(x - X_\infty))\right|$$
$$\le C\left|\tilde{v}^\nu(x - X_\nu) - \bar{v}(x - X_\infty)\right|.$$

Therefore, following the same computations as above, we have the desired result. $\qquad\square$

Recalling (5.8) and (5.12), we now consider

$$\iint_{(0,T)\times\mathbb{R}} \phi_\varepsilon(s)\eta\big((v^\nu, h^\nu)(s, x)|(\bar{v}, \bar{u})(x - X_\infty(s))\big)dxds$$

$$= \underbrace{\iint_{h^\nu \in [-L,L]} \phi_\varepsilon \frac{|\underline{h}^\nu - \bar{u}(x - X_\infty)|^2}{2}dxds}_{=:J_1} + \underbrace{\iint_{h^\nu \notin [-L,L]} \phi_\varepsilon \frac{|h^\nu - \bar{u}(x - X_\infty)|^2}{2}dxds}_{=:J_2}$$

$$+ \underbrace{\iint_{v^\nu \in [K^{-1},K]} \phi_\varepsilon Q\big(\underline{v}^\nu|\bar{v}(x - X_\infty)\big)dxds}_{=:J_3} + \underbrace{\iint_{v^\nu \notin [K^{-1},K]} \phi_\varepsilon Q\big(v^\nu|\bar{v}(x - X_\infty)\big)dxds}_{=:J_4}.$$

Note that (using (5.21))

$$J_1 + J_3 \le R_\nu = (R_\nu - L_\nu) + L_\nu \le (R_\nu - L_\nu) + C\mathcal{E}_0 + \delta.$$

For J_2, we use the fact that since $\bar{u}, \tilde{h}^\nu \in \big(\min\{u_-, u_+\}, \max\{u_-, u_+\}\big) \subset \big(-L/2, L/2\big)$, we find

$$\left|h^\nu - \bar{u}(x - X_\infty)\right| \le 3\left|h^\nu - \tilde{h}^\nu(x - X_\nu)\right| \quad \text{for all } h^\nu \notin [-L, L].$$

Then using (5.6),

$$J_2 \le \frac{9}{2}\iint_{h^\nu \notin [-L,L]} \phi_\varepsilon\left|h^\nu - \tilde{h}^\nu(x - X_\nu)\right|^2 dxds \le C(\mathcal{E}_0 + \delta).$$

Likewise for J_4, since $\bar{v}, \tilde{v}^\nu \in \left(\min\{v_-, v_+\}, \max\{v_-, v_+\}\right) \subset \left(M^{-1}, M\right)$, using (3.13) in Lemma 3.2 with the choice (5.7), we have

$$J_4 \le C \iint_{v^\nu \notin [K^{-1}, K]} \phi_\varepsilon Q\big(v^\nu | \tilde{v}^\nu (x - X_\nu)\big) dx ds \le C(\mathcal{E}_0 + \delta).$$

Therefore, we have

(5.22)
$$\iint_{(0,T)\times\mathbb{R}} \phi_\varepsilon(s) \frac{|h^\nu(t,x) - \bar{u}(x - X_\infty(s))|^2}{2} ds dx$$
$$+ \iint_{(0,T)\times\mathbb{R}} \phi_\varepsilon(s) Q(v^\nu(t,x) | \bar{v}(x - X_\infty(s))) ds dx \le |R_\nu - L_\nu| + C(\mathcal{E}_0 + \delta).$$

Now, it remains to show that the left-hand side of (5.22) is lower semi-continuous with respect to the weak convergences (5.11) and (5.15).
First of all, using the weak lower semi-continuity of the L^2-norm (for example see [22]) together with (5.15), we have

(5.23)
$$\iint_{(0,T)\times\mathbb{R}} \phi_\varepsilon(s) \frac{|u_\infty(t,x) - \bar{u}(x - X_\infty(s))|^2}{2} ds dx$$
$$\le \liminf_{\nu \to 0} \iint_{(0,T)\times\mathbb{R}} \phi_\varepsilon(s) \frac{|h^\nu(t,x) - \bar{u}(x - X_\infty(s))|^2}{2} ds dx.$$

However, since v_∞ is a measure in space as $v_\infty \in L^\infty(0, T; L^\infty(\mathbb{R}) + \mathcal{M}(\mathbb{R}))$, we may use the generalized relative functional (1.9) to handle the measure v_∞.
In the following lemma, we show the weakly lower semi-continuity of the functional

$$dQ(v^\nu | \bar{v}(x - X_\infty))$$

in the left-hand side of (5.22). In fact, Lemma 5.3 deals with more general case where $\{v^\nu\}_{\nu>0}$ is the sequence of measures. Without loss of generality, we only handle the case of $v_- > v_+$, and set

(5.24) $$\Omega_M := \{(t,x) \in (0,T) \times \mathbb{R} \mid x < X_\infty(t)\}.$$

Since $X_\infty \in BV(0,T)$, we have that

(5.25)   Lebesgue measure on $\mathbb{R}^2$ of $\partial\Omega_M$ ($:=$ the boundary of Ω_M) is zero,

and the complement of $\overline{\Omega_M}$ ($:=$ the closure of Ω_M) in $(0,T) \times \mathbb{R}$ is as follows:

(5.26) $$(\overline{\Omega_M})^c = \{(t,x) \in (0,T) \times \mathbb{R} \mid x > X_\infty(t)\}.$$

Note that

(5.27) $$\bar{v}(x - X_\infty(t)) = \begin{cases} v_- & \text{for } (t,x) \in \Omega_M, \\ v_+ & \text{for } (t,x) \in (\overline{\Omega_M})^c. \end{cases}$$

Lemma 5.3. *Assume $v_+ < v_-$. Consider the set (5.24) and the properties (5.25), (5.26). Let $\Phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be any compactly supported nonnegative function. Let $\{v^k\}_{k=1}^\infty$ be a sequence of positive measures in $L^\infty((0,T) \times \mathbb{R}) + \mathcal{M}((0,T) \times \mathbb{R})$ such that for some constant $C_0 > 0$ (independent of k),*

$$\int_{(0,T)\times\mathbb{R}} \Phi(t,x) \, dQ\left(v^k | \bar{v}(x - X_\infty(t))\right)(t,x) \le C_0,$$

where

$$dQ\left(v^k|\bar{v}(x - X_\infty(t))\right)(t,x) := Q\left(v_a^k|\bar{v}(x - X_\infty(t))\right)dtdx + |Q'(\overline{V}(t,x))|dv_s^k(t,x),$$

where $dv^k(t,x) := v_a^k(t,x)dtdx + dv_s^k(t,x)$ *(by Radon-Nikodym's theorem), and*

(5.28) $$\overline{V}(t,x) := \begin{cases} v_- & \text{for } (t,x) \in \overline{\Omega_M}, \\ v_+ & \text{for } (t,x) \in (\overline{\Omega_M})^c. \end{cases}$$

Then, there exists a limit $v_\infty \in L^\infty((0,T) \times \mathbb{R}) + \mathcal{M}((0,T) \times \mathbb{R})$ *such that* $v^k \rightharpoonup v_\infty$ *in* $\mathcal{M}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$*, and*

$$\int_{(0,T)\times\mathbb{R}} \Phi(t,x) \; dQ\left(v_\infty|\bar{v}(x - X_\infty(t))\right)(t,x) \le C_0.$$

Proof. Since v^k are positive measures in $L^\infty((0,T)\times\mathbb{R}) + \mathcal{M}((0,T)\times\mathbb{R})$, Radon-Nikodym's theorem implies that there exist positive measures $v_a^k \in L^\infty(\mathbb{R}) + L^1(\mathbb{R})$ and dv_s^k (singular part of v^k) such that

$$dv^k(t,x) = v_a^k(t,x)dtdx + dv_s^k(t,x).$$

To truncate v_a^k by some big constant, we first use the fact that for any $\varepsilon > 0$, there exists $\xi > 0$ with $\xi > \max(2v_-, 2v_+^{-1})$ such that for all $v > \xi$,

(5.29) $$(|Q'(\bar{\mathbf{v}})| + \varepsilon)v \ge Q(v|\bar{\mathbf{v}}) \ge (|Q'(\bar{\mathbf{v}})| - \varepsilon)v,$$

where $\bar{\mathbf{v}} := \bar{v}(x - X_\infty(t))$. Indeed, this is straightforwardly verified by the definition of the relative functional $Q(\cdot|\cdot)$, and $Q(v) \to 0$ as $v \to \infty$.

For such a constant ξ, we define

$$v_\xi^k := \inf(v_a^k, \xi),$$

and

$$Q_\xi(v) := \begin{cases} Q(v), & \text{if } v \ge \xi^{-1}, \\ Q'(\xi^{-1})(v - \xi^{-1}) + Q(\xi^{-1}), & \text{if } v \le \xi^{-1}. \end{cases}$$

Note that $v \mapsto Q_\xi(v)$ is nonnegative and convex C^1-function on $[0,\infty)$, and $Q_\xi'(\bar{\mathbf{v}}) = Q'(\bar{\mathbf{v}})$ (by $\xi^{-1} < v_+/2 < \bar{\mathbf{v}}$). Then, we consider its relative functional: for any $v_1, v_2 \ge 0$,

$$Q_\xi(v_1|v_2) := Q_\xi(v_1) - Q_\xi(v_2) - Q_\xi'(v_2)(v_1 - v_2).$$

Then, using (5.29), we have

(5.30) $$dQ_\xi(v^k|\bar{\mathbf{v}}) \ge Q_\xi(v_\xi^k|\bar{\mathbf{v}})dtdx + (|Q_\xi'(\overline{V})| - \varepsilon)(dv^k - v_\xi^k dtdx) - 2\varepsilon dv^k,$$

which means that $dQ_\xi(v^k|\bar{\mathbf{v}}) - \left[Q_\xi(v_\xi^k|\bar{\mathbf{v}})dtdx + (|Q_\xi'(\overline{V})| - \varepsilon)(dv^k - v_\xi^k dtdx) - 2\varepsilon dv^k\right]$ is nonnegative measure. Indeed, this is verified as follows: If $v_a^k \le \xi$, then $v_\xi^k = v_a^k$, and so

$$\text{LHS} := Q_\xi(v_a^k|\bar{\mathbf{v}})dtdx + |Q_\xi'(\overline{V})|dv_s^k = Q(v_\xi^k|\bar{\mathbf{v}})dtdx + |Q_\xi'(\overline{V})|dv_s^k \ge \text{RHS},$$

where the last inequality follows from the facts that (by Radon-Nikodym's theorem) the measure $v^k - v_\xi^k$ is positive and $v^k - v_\xi^k = (v_a^k - v_\xi^k) + v_s^k = v_s^k$.
If $v_a^k > \xi$, then $v_\xi^k = \xi$, and using (5.29), LHS $\ge (|Q_\xi'(\bar{\mathbf{v}})| - \varepsilon)v_a^k dtdx + |Q_\xi'(\overline{V})|dv_s^k$. Since $\bar{\mathbf{v}} = \overline{V}$ $dtdx$-a.e. (by (5.27) and (5.28)), we have

$$\text{LHS} \ge (|Q_\xi'(\overline{V})| - \varepsilon)dv^k = (|Q_\xi'(\overline{V})| + \varepsilon)\xi dtdx + (|Q_\xi'(\overline{V})| - \varepsilon)(dv^k - \xi dtdx) - 2\varepsilon\xi dtdx$$

$$= (|Q_\xi'(\bar{\mathbf{v}})| + \varepsilon)\xi dtdx + (|Q_\xi'(\overline{V})| - \varepsilon)(dv^k - v_\xi^k dtdx) - 2\varepsilon\xi dtdx$$

$$\ge Q_\xi(\xi|\bar{\mathbf{v}})dtdx + (|Q_\xi'(\overline{V})| - \varepsilon)(dv^k - v_\xi^k dtdx) - 2\varepsilon\xi dtdx.$$

Thus, using $\xi = v_\xi^k \leq v_a^k \leq v^k$, we have (5.30).

Therefore, we use (5.30) to have

$$C_0 \geq \limsup_{k \to \infty} \int_{(0,T) \times \mathbb{R}} \Phi(t,x) \, dQ\left(v^k | \bar{v}(x - X_\infty(t))\right)(t,x)$$

$$\geq \limsup_{k \to \infty} \int_{(0,T) \times \mathbb{R}} \Phi(t,x) \Big[Q_\xi(v_\xi^k | \bar{v}(x - X_\infty(t))) dt dx$$

$$+ (|Q_\xi'(\overline{V})| - \varepsilon) d(v^k - v_\xi^k) - 2\varepsilon dv^k \Big].$$

We set $\Omega_m := (\overline{\Omega_M})^c$, and define

$$\Omega_m^\delta := \{(t,x) \in \Omega_m \mid d((t,x)|\Omega_m^c) > \delta\}, \quad \forall \delta > 0.$$

Then we define a smooth function ψ_1^δ such that

(5.31)
$$\psi_1^\delta(t,x) := \begin{cases} 1, & \text{on } (\Omega_m^\delta)^c, \\ 0, & \text{on } \Omega_m^{2\delta}. \end{cases}$$

Then, using this together with the facts that (5.27), (5.28) and $|Q_\xi'(v_-)| \leq |Q_\xi'(v_+)|$ by $v_+ < v_-$, we have

$$C_0 \geq \limsup_{k \to \infty} \Big[\int_{\Omega_M} \Phi Q_\xi(v_\xi^k | v_-) dt dx + \int_{\Omega_m} \Phi Q_\xi(v_\xi^k | v_+) dt dx + (|Q_\xi'(v_-)| - \varepsilon) \int \Phi \psi_1^\delta d(v^k - v_\xi^k)$$

$$+ (|Q_\xi'(v_+)| - \varepsilon) \int \Phi(1 - \psi_1^\delta) d(v^k - v_\xi^k) - 2\varepsilon \int \Phi dv^k \Big].$$

Note that since $|v_\xi^k| \leq \xi$ for all k, there exists v_* such that

$$v_\xi^k \rightharpoonup v_* \quad \text{in} \ \ L^\infty.$$

Moreover, since the function $v \mapsto Q_\xi(v|c)$ with any constant c is convex, the weak lower semi-continuity of convex functions (for example, see [22]) implies

$$\liminf_{k \to \infty} \Big[\int_{\Omega_M} \Phi Q_\xi(v_\xi^k | v_-) dt dx + \int_{\Omega_m} \Phi Q_\xi(v_\xi^k | v_+) dt dx \Big]$$

$$\geq \int_{\Omega_M} \Phi Q_\xi(v_* | v_-) dt dx + \int_{\Omega_m} \Phi Q_\xi(v_* | v_+) dt dx.$$

Also, since $v^k \rightharpoonup v_\infty$ in $\mathcal{M}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$, and thus

(5.32) $v^k - v_\xi^k \rightharpoonup v_\infty - v_*$ in $\mathcal{M}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$ (by the uniqueness of the decomposition),

we have
(5.33)

$$C_0 \geq \int_{\Omega_M} \Phi Q_\xi(v_* | v_-) dt dx + \int_{\Omega_m} \Phi Q_\xi(v_* | v_+) dt dx + (|Q_\xi'(v_-)| - \varepsilon) \int \Phi \psi_1^\delta d(v_\infty - v_*)$$

$$+ (|Q'(v_+)| - \varepsilon) \int \Phi(1 - \psi_1^\delta) d(v_\infty - v_*) - 2\varepsilon \int \Phi dv_\infty =: \mathcal{R}.$$

By Radon-Nikodym's theorem, there exist positive measures $v_a \in L^\infty(\mathbb{R}) + L^1(\mathbb{R})$ and dv_s (singular part of v_∞) such that

(5.34)
$$dv_\infty(t,x) = v_a(t,x) dt dx + dv_s(t,x).$$

Note that since the measure $v^k - v_\xi^k$ is positive, it follows from (5.32) and (5.34) that $v_\infty - v_*$, $v_a - v_*$ and dv_s are all nonnegative.

Since $dv_\infty - v_* dtdx = (v_a - v_*)dtdx + dv_s$ (by the uniqueness of the decomposition), we rewrite $\mathcal{R}$ in (5.33) as

$$\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3,$$

where

$$\mathcal{R}_1 := |Q'_\xi(v_-)| \int \Phi \psi_1^\delta dv_s + |Q'_\xi(v_+)| \int \Phi(1 - \psi_1^\delta)dv_s,$$

$$\mathcal{R}_2 := \int_{\Omega_M} \Phi Q_\xi(v_*|v_-)dtdx + |Q'_\xi(v_-)| \int \Phi \psi_1^\delta(v_a - v_*)dtdx$$

$$+ \int_{\Omega_m} \Phi Q_\xi(v_*|v_+)dtdx + |Q'_\xi(v_+)| \int \Phi(1 - \psi_1^\delta)(v_a - v_*)dtdx,$$

$$\mathcal{R}_3 := -3\varepsilon \int \Phi dv_\infty + \varepsilon \int \Phi v_* dtdx.$$

Using $\overline{\Omega_M} \subset (\Omega_m^\delta)^c$ and (5.31), we have

$$\mathcal{R}_1 \geq |Q'_\xi(v_-)| \int_{\overline{\Omega_M}} \Phi dv_s + |Q'_\xi(v_+)| \int_{\Omega_m^{2\delta}} \Phi dv_s.$$

Since Φdv_s is a positive measure, and

(5.35) $$\Omega_m^{2\delta} \nearrow \cup_{\delta>0} \Omega_m^{2\delta} = (\overline{\Omega_M})^c,$$

we have

$$\lim_{\delta \to 0} \int_{\Omega_m^{2\delta}} \Phi dv_s = \int_{(\overline{\Omega_M})^c} \Phi dv_s.$$

Thus,

$$\mathcal{R}_1 \geq \int |Q'_\xi(\overline{V})| \Phi dv_s.$$

For $\mathcal{R}_2$, we use (5.31) to have

$$\mathcal{R}_2 \geq \int_{\Omega_M} \Phi \Big[Q_\xi(v_*|v_-) + |Q'_\xi(v_-)|(v_a - v_*) \Big] dtdx$$

$$+ \int_{\Omega_m^{2\delta}} \Phi \Big[Q_\xi(v_*|v_+) + |Q'_\xi(v_+)|(v_a - v_*) \Big] dtdx.$$

Then, we have

$$\mathcal{R}_2 \geq \int_{\Omega_M} \Phi Q_\xi(v_a|v_-)dtdx + \int_{\Omega_m^{2\delta}} \Phi Q_\xi(v_a|v_+)dtdx,$$

where we used the equality that for any $w_1, w_2 \geq 0$ and any $c > 0$,

$$Q_\xi(w_1 + w_2|c) \leq Q_\xi(w_1|c) + |Q'_\xi(c)|w_2.$$

Indeed, it follows from $Q'_\xi \leq 0$ and the definition of $Q_\xi(\cdot|\cdot)$ that

$$Q_\xi(w_1 + w_2|c) - Q_\xi(w_1|c) - |Q'_\xi(c)|w_2 = Q_\xi(w_1 + w_2) - Q_\xi(w_1) \leq 0.$$

Since (5.35) imiplies

$$\lim_{\delta \to 0} \int_{\Omega_m^{2\delta}} \Phi Q_\xi(v_a|v_+)dtdx = \int_{(\overline{\Omega_M})^c} \Phi Q_\xi(v_a|v_+)dtdx,$$

we use (5.25) to have

$$\mathcal{R}_2 \geq \int \Phi Q_\xi(v_a|\bar{v}(x - X_\infty(t)))dtdx.$$

Therefore, we have

$$\mathcal{R} \geq \int \Phi Q_\xi(v_a|\bar{v}(x - X_\infty(t)))dtdx + \int \Phi|Q'_\xi(\overline{V})|dv_s - 3\varepsilon \int \Phi dv_\infty,$$

that is,

$$\int \Phi Q_\xi(v_a|\bar{v}(x - X_\infty(t)))dtdx + \int \Phi|Q'_\xi(\overline{V})|dv_s \leq \mathcal{R} + 3\varepsilon \int \Phi dv_\infty.$$

Therefore, taking $\xi \to \infty$ and using Fatou's lemma, we have

$$\int \Phi Q(v_a|\bar{v}(x - X_\infty(t)))dtdx + \int \Phi|Q'(\overline{V})|dv_s \leq \mathcal{R} + 3\varepsilon \int \Phi dv_\infty.$$

Then taking $\varepsilon \to 0$, we have

$$\int \Phi Q(v_a|\bar{v}(x - X_\infty(t)))dtdx + \int \Phi|Q'(\overline{V})|dv_s \leq \mathcal{R}$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

To apply Lemma 5.3 to (5.22), we define a smooth function ψ_0^R such that for any $R > 0$,

$$\mathbf{1}_{|x|\leq R} \leq \psi_0^R(x) \leq \mathbf{1}_{|x|\leq 2R}.$$

Then, it follows from (5.22) that

$$(5.36) \quad \begin{aligned} &\iint_{(0,T)\times\mathbb{R}} \phi_\varepsilon(s)\frac{|h^\nu(t,x) - \bar{u}(x - X_\infty(s))|^2}{2}dsdx \\ &+ \iint_{(0,T)\times\mathbb{R}} \phi_\varepsilon(s)\psi_0^R(x)Q(v^\nu(t,x)|\bar{v}(x - X_\infty(s)))dsdx \leq |R_\nu - L_\nu| + C(\mathcal{E}_0 + \delta). \end{aligned}$$

Thus, using Lemma 5.3 together with the weak convergence (5.11), we have

$$\iint_{(0,T)\times\mathbb{R}} \phi_\varepsilon(s)\psi_0^R(x) \, dQ(v_\infty(t,x)|\bar{v}(x - X_\infty(s)))dsdx$$

$$\leq \liminf_{\nu\to 0} \iint_{(0,T)\times\mathbb{R}} \phi_\varepsilon(s)\psi_0^R(x)Q(v^\nu(t,x)|\bar{v}(x - X_\infty(s)))dsdx.$$

Here, the measure v_∞ has the decomposition (5.34), and

$$dQ\left(v_\infty|\bar{v}(x - X_\infty(t))\right)(t,x) = Q\left(v_a|\bar{v}(x - X_\infty(t))\right)dtdx + |Q'(\overline{V}(t,x))|dv_s(t,x),$$

where $\overline{V}(t,x)$ is defined by (5.28) with (5.24).
Then, using $\mathbb{R}^+ \times (-R,R) \nearrow \mathbb{R}^+ \times \mathbb{R}$ as $R \to \infty$, we have

$$\iint_{(0,T)\times\mathbb{R}} \phi_\varepsilon(s) \, dQ(v_\infty(t,x)|\bar{v}(x - X_\infty(s)))dsdx$$

$$\leq \liminf_{\nu\to 0} \iint_{(0,T)\times\mathbb{R}} \phi_\varepsilon(s)\psi_0^R(x)Q(v^\nu(t,x)|\bar{v}(x - X_\infty(s)))dsdx.$$

Therefore, this together with (5.36), (5.23) and Lemma 5.2 yields

$$\iint_{(0,T)\times\mathbb{R}} \phi_\varepsilon(s)\frac{|u_\infty(t,x)-\bar{u}(x-X_\infty(s))|^2}{2}dsdx$$

$$+ \iint_{(0,T)\times\mathbb{R}} \phi_\varepsilon(s)\,dQ(v_\infty(t,x)|\bar{v}(x-X_\infty(s)))dsdx \le C(\mathcal{E}_0+\delta).$$

Taking $\varepsilon \to 0$ (recall (5.20)), we obtain that

$$dQ(v_\infty|\bar{v}(\cdot-X_\infty(\cdot))) \in L^\infty(0,T;\mathcal{M}(\mathbb{R})),$$

and, for a.e. $t \in (0,T)$,

$$\int_\mathbb{R} \frac{|u_\infty(t,x)-\bar{u}(x-X_\infty(t))|^2}{2}dx + \left(\int_{x\in\mathbb{R}} dQ(v_\infty|\bar{v}(x-X_\infty(\cdot)))\right)(t) \le C(\mathcal{E}_0+\delta).$$

Since $\delta > 0$ is arbitrary, we obtain

$$\int_\mathbb{R} \frac{|u_\infty(t,x)-\bar{u}(x-X_\infty(t))|^2}{2}dx + \left(\int_{x\in\mathbb{R}} dQ(v_\infty|\bar{v}(x-X_\infty(\cdot)))\right)(t) \le C\mathcal{E}_0,$$

which gives (1.14).

5.2.5. **Proof of $v_\infty(t=0)=v^0$.** In order to prove (1.15), we will first prove the continuity of v_∞ at $t=0$:

(5.37)
$$v_\infty(t=0) = v^0.$$

We first claim that

(5.38)
$$\{u^\nu\}_{\nu>0} \text{ is bounded in } L^2(0,T;L^1_{loc}(\mathbb{R})).$$

For that, recall from (5.5) that $u^\nu = h^\nu - \nu\big(p(v^\nu)^{\frac{\alpha}{\gamma}}\big)_x$. First, we have (5.14). To get a uniform boundedness of $\nu\big(p(v^\nu)^{\frac{\alpha}{\gamma}}\big)_x$, we use the same estimates as in Step 2 for the proof of (5.16). Indeed, since

$$\nu\left|\big(p(v^\nu)^{\frac{\alpha}{\gamma}}\big)_x\right| = \nu\frac{\alpha}{\gamma}(v^\nu)^\beta\left|p(v^\nu)_x\right|$$

$$\le \nu(v^\nu)^\beta\left|\big(p(v^\nu)-p(\tilde{v}^\nu(x-X_\nu(t)))\big)_x\right| + \nu(v^\nu)^\beta\left|p(\tilde{v}^\nu(x-X_\nu(t)))_x\right|$$

$$\le C\left(\sqrt{\nu}(v^\nu)^\beta\left|\big(p(v^\nu)-p(\tilde{v}^\nu(x-X_\nu(t)))\big)_x\right|^2 + \big(1+Q\left(v^\nu|\tilde{v}^\nu(x-X_\nu(t))\right)\big)\right.$$

$$\left.+ \left|(\tilde{v}^\nu)'\right|\big(1+Q\left(v^\nu|\tilde{v}^\nu(x-X_\nu(t))\right)\big)\right),$$

using (5.6), we have

$$\nu\big(p(v^\nu)^{\frac{\alpha}{\gamma}}\big)_x \text{ is uniformly bounded in } L^2(0,T;L^1_{loc}(\mathbb{R})).$$

Therefore, we have (5.38).

Then, (5.38) together with the equation $v^\nu_t - u^\nu_x = 0$ in (1.1) implies

$$v^\nu_t \text{ is uniformly bounded in } L^2(0,T;W^{-1,1}_{loc}(\mathbb{R})).$$

Hence, by Aubin-Lions lemma, this and (5.10) together with (5.11) imply that (up to a subsequence)

$$v^\nu \to v_\infty \text{ in } C([0,T];W^{-s,1}_{loc}(\mathbb{R})), \quad s > 0,$$

which together with (5.2) completes the proof of (5.37).

5.2.6. **Proof of** (1.15). First of all, since $X_\infty \in BV((0,T))$, there exists a positive constant $r = r(T)$ such that $\|X_\infty\|_{L^\infty((0,T))} = r$. Then we consider a nonnegative smooth function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(x) = \psi(-x)$, and $\psi'(x) \le 0$ for all $x \ge 0$, and $|\psi'(x)| \le 2/r$ for all $x \in \mathbb{R}$, and

$$\psi(x) = \begin{cases} 1, & \text{if } |x| \le r, \\ 0, & \text{if } |x| \ge 2r. \end{cases}$$

On the other hand, let $\theta : \mathbb{R} \to \mathbb{R}$ be a nonnegative smooth function such that $\theta(s) = \theta(-s)$, $\int_\mathbb{R} \theta = 1$ and supp $\theta = [-1, 1]$, and let

$$\theta_\delta(s) := \frac{1}{\delta}\theta\left(\frac{s-\delta}{\delta}\right) \quad \text{for any } \delta > 0.$$

Then for a given $t \in (0, T)$, and any $\delta < t/2$, we define a nonnegative smooth function

$$\varphi_{t,\delta}(s) := \int_0^s \left(\theta_\delta(\tau) - \theta_\delta(\tau - t)\right) d\tau.$$

Since $v_t^\nu - u_x^\nu = 0$ by $(1.1)_1$, it follows from (1.13) that the limits v_∞ and u_∞ satisfy

(5.39) $$\int_{[0,T]\times\mathbb{R}} \left(\varphi'_{t,\delta}(s)\psi(x)dv_\infty(s,x) - \varphi_{t,\delta}(s)\psi'(x)u_\infty(s,x)dsdx\right) = 0.$$

Since $\varphi'_{t,\delta}(s) = \theta_\delta(s) - \theta_\delta(s-t)$, we decompose the left-hand side above into three parts as

$$I_1^\delta + I_2^\delta + I_3^\delta = 0,$$

where

$$I_1^\delta := \int_{[0,T]\times\mathbb{R}} \theta_\delta(s)\psi(x)dv_\infty(s,x),$$

$$I_2^\delta := -\int_{[0,T]\times\mathbb{R}} \theta_\delta(s-t)\psi(x)dv_\infty(s,x),$$

$$I_3^\delta := -\int_0^T \int_\mathbb{R} \varphi_{t,\delta}(s)\psi'(x)u_\infty(s,x)dxds.$$

Using (5.37) and the fact that $\int_\mathbb{R} \psi(x)v_\infty(s,dx)$ is continuous in s, we find that as $\delta \to 0$:

$$I_1^\delta \to \int_\mathbb{R} \psi(x)v^0(x)dx, \quad I_2^\delta \to -\int_\mathbb{R} \psi(x)v_\infty(t,dx),$$

and

$$I_3^\delta \to -\int_0^T \int_\mathbb{R} \psi'(x)u_\infty(s,x)dxds.$$

Therefore, it follows from (5.39) that

$$\underbrace{\int_\mathbb{R} \psi(x)\left(v_\infty(t,dx) - v^0(x)dx\right)}_{=:J_1} + \underbrace{\int_0^T \int_\mathbb{R} \psi'(x)u_\infty(s,x)dxds}_{=:J_2} = 0.$$

To show (1.15) from the above equation, we will use the stability estimate (1.14) and the Rankine-Hugoniot condition.

For that, we decompose J_1 into three parts:

$$J_1 = J_{11} + J_{12} + J_{13},$$

where

$$J_{11} = \int_{\mathbb{R}} \psi(x)\big(v_\infty(t, dx) - \bar{v}(x - X_\infty(t))dx\big),$$

$$J_{12} = \int_{\mathbb{R}} \psi(x)\big(\bar{v}(x - X_\infty(t)) - \bar{v}(x)\big)dx,$$

$$J_{13} = \int_{\mathbb{R}} \psi(x)\big(\bar{v}(x) - v^0(x)\big)dx.$$

Likewise, we decompose J_2 into two parts:

$$J_2 = J_{21} + J_{22},$$

where

$$J_{21} = \int_0^t \int_{\mathbb{R}} \psi'(x)\big(u_\infty(s, x) - \bar{u}(x - X_\infty(s)))\big)dxds,$$

$$J_{22} = \int_0^t \int_{\mathbb{R}} \psi'(x)\bar{u}(x - X_\infty(s))dxds.$$

Since $|X_\infty(t)| \le r$ for all $t \in (0, T)$, we have

$$J_{12} + J_{22} = (v_- - v_+)X_\infty(t) + t(u_- - u_+).$$

Then using the Rankine-Hugoniot condition $(1.5)_1$, i.e., $\sigma = -\frac{u_- - u_+}{v_- - v_+}$, we have

$$J_{12} + J_{22} = \big(X_\infty(t) - \sigma t\big)(v_- - v_+).$$

To control J_{11} by the initial perturbation $\mathcal{E}_0 = \int_{-\infty}^\infty \eta\big((v^0, u^0)|(\bar{v}, \bar{u})\big)dx$, we recall the (unique) decomposition of the measure v_∞ by

$$dv_\infty(t, dx) = v_a(t, x)dx + v_s(t, dx).$$

Using (3.8), we have

$$|J_{11}| \le \int_{-2r}^{2r} \big|v_a(t, x) - \bar{v}(x - X_\infty(t))\big|\mathbf{1}_{\{v \le 3v_-\}}dx + \int_{-2r}^{2r} \big|v_a(t, x) - \bar{v}(x - X_\infty(t))\big|\mathbf{1}_{\{v \ge 3v_-\}}dx$$

$$+ \int_{\mathbb{R}} \psi(x)v_s(t, dx)$$

$$\le \frac{1}{\sqrt{c_1}} \int_{-2r}^{2r} \sqrt{Q\big(v_a(t, x)|\bar{v}(x - X_\infty(t))\big)}dx + \frac{1}{c_2} \int_{\mathbb{R}} Q\big(v_a(t, x)|\bar{v}(x - X_\infty(t))\big)dx$$

$$+ \frac{1}{|Q'(v_-)|} \int_{\mathbb{R}} \psi(x)|Q'(\overline{V})|v_s(t, dx),$$

where note that $|Q'(\overline{V})| \ge |Q'(v_-)| > 0$ by (5.28).
Thus, we use the stability estimate (1.14) to have

$$|J_{11}| \le C\sqrt{r}\sqrt{\mathcal{E}_0} + C\mathcal{E}_0.$$

Using the same estimates as above, and (1.12), we have

$$|J_{13}| \le C\sqrt{r}\sqrt{\mathcal{E}_0} + C\mathcal{E}_0.$$

Likewise,

$$|J_{21}| \le \frac{2}{r} \int_0^t \int_{[-2r, -r] \cup [r, 2r]} \big|u_\infty(s, x) - \bar{u}(x - X_\infty(s))\big|dxds \le \frac{C}{\sqrt{r}}t\sqrt{\mathcal{E}_0}.$$

Hence we have

$$(v_- - v_+)|X_\infty(t) - \sigma t| \leq C\Big(\mathcal{E}_0 + (1+t)\sqrt{\mathcal{E}_0}\Big),$$

which completes the proof.

APPENDIX A. PROOF OF PROPOSITION 4.2

We rewrite the functionals $Y_g, \mathcal{I}_1, \mathcal{I}_2, \mathcal{G}_2, \mathcal{D}$ with respect to the following variables

$$w := p(v) - p(\tilde{v}_\varepsilon), \quad y := \frac{p(\tilde{v}_\varepsilon(\xi)) - p(v_-)}{p(v_+) - p(v_-)}.$$

Since $p(\tilde{v}_\varepsilon(\xi))$ is increasing in ξ, we use the change of variable $\xi \in \mathbb{R} \mapsto y \in [0, 1]$. Notice that $a = 1 - \lambda y$ and $|a - 1| \leq \delta_3$ by (4.12), and

$$(A.1) \qquad \frac{dy}{d\xi} = \frac{p(\tilde{v}_\varepsilon)'}{p(v_+) - p(v_-)}, \quad \text{where } |p(v_+) - p(v_-)| = \varepsilon.$$

As in [33, Proposition 3.4], we use the same notations:

$$W := \frac{\lambda}{\varepsilon}w, \qquad \alpha_\gamma := \frac{\gamma\sqrt{-p'(v_-)}p(v_-)}{\gamma + 1} > 0.$$

First of all, note that $Y_g, \mathcal{I}_1, \mathcal{I}_2, \mathcal{G}_2$ are respectively the same functionals as $Y_g, \mathcal{B}_1, \mathcal{B}_2, \mathcal{G}_2$ in [33, Proposition 3.4] except for the term $\frac{1}{2}\int_\mathbb{R} a''|p(v) - p(\tilde{v}_\varepsilon)|^2 d\xi$ in $\mathcal{B}_2$, which is negligible by $\mathcal{I}_2$ because of $|a''| \leq C\varepsilon|a'|$ (see [33, (2.29) and (3.31)]). Thus, it follows from [33, (3.39), (3.34), (3.33), (3.35)] that

$$-2\alpha_\gamma \frac{\lambda^2}{\varepsilon^3} \frac{|Y_g|^2}{\varepsilon\delta_3} \leq -\frac{\alpha_\gamma}{\delta_3\sigma^4}\left|\int_0^1 W^2\,dy + 2\int_0^1 W\,dy\right|^2 + C\delta_3\int_0^1 W^2\,dy,$$

$$2\alpha_\gamma \frac{\lambda^2}{\varepsilon^3}|\mathcal{I}_1| \leq (1 + C(\varepsilon_0 + \delta_3))\int_0^1 W^2\,dy,$$

$$(A.2) \qquad 2\alpha_\gamma \frac{\lambda^2}{\varepsilon^3}|\mathcal{I}_2| \leq \left(\frac{\alpha_\gamma}{\sigma}\left(\frac{\lambda}{\varepsilon}\right) + C(\varepsilon_0 + \delta_3)\right)\int_0^1 W^2\,dy,$$

$$-2\alpha_\gamma \frac{\lambda^2}{\varepsilon^3}\mathcal{G}_2 \leq \left(-\frac{\alpha_\gamma}{\sigma}\left(\frac{\lambda}{\varepsilon}\right) + C\delta_3\right)\int_0^1 W^2\,dy + \frac{2}{3}\int_0^1 W^3\,dy + C\varepsilon_0\int_0^1 |W|^3\,dy.$$

Therefore, it remains to estimate the diffusion $\mathcal{D}$ as follows:
First, by the change of variable, we have

$$\mathcal{D} = \int_0^1 (1 - \lambda y)|\partial_y w|^2 v^\beta\left(\frac{dy}{d\xi}\right)dy.$$

Since it follows from (3.2) that

$$\tilde{v}_\varepsilon^\beta p(\tilde{v}_\varepsilon)' = \sigma_\varepsilon(\tilde{v}_\varepsilon - v_-) + \frac{p(\tilde{v}_\varepsilon) - p(v_-)}{\sigma_\varepsilon},$$

using (A.1), we find

$$(A.3) \qquad \varepsilon\tilde{v}_\varepsilon^\beta \frac{dy}{d\xi} = \frac{1}{\sigma_\varepsilon}\Big(\sigma_\varepsilon^2(\tilde{v}_\varepsilon - v_-) + p(\tilde{v}_\varepsilon) - p(v_-)\Big).$$

Since the right-hand side of (A.3) is the same as the one in the proof of [33, Lemma 3.1], we have

$$\frac{\tilde{v}_\varepsilon^\beta}{y(1-y)}\frac{dy}{d\xi} = \frac{\varepsilon}{\sigma_\varepsilon(v_- - v_+)}\left(\frac{\tilde{v}_\varepsilon - v_-}{p(\tilde{v}_\varepsilon) - p(v_-)} + \frac{\tilde{v}_\varepsilon - v_+}{p(v_+) - p(\tilde{v}_\varepsilon)}\right).$$

Thus, it follows from the proof of [33, Lemma 3.1] that

$$\left|\frac{\tilde{v}_\varepsilon^\beta}{y(1-y)}\frac{dy}{d\xi} - \frac{\varepsilon}{2\alpha_\gamma}\right| \le C\varepsilon^2.$$

Then, using $|(v^\beta/\tilde{v}_\varepsilon^\beta) - 1| \le C\delta_3$, we have

$$\begin{aligned}
\mathcal{D} &\ge (1-\lambda)\int_0^1 |\partial_y w|^2 v^\beta\left(\frac{dy}{d\xi}\right)dy = (1-\lambda)\int_0^1 |\partial_y w|^2 \frac{v^\beta}{\tilde{v}_\varepsilon^\beta}\tilde{v}_\varepsilon^\beta\left(\frac{dy}{d\xi}\right)dy\\
&\ge (1-\lambda)\left(\frac{\varepsilon}{2\alpha_\gamma} - C\varepsilon^2 - C\delta_3\right)\int_0^1 y(1-y)|\partial_y w|^2\, dy\\
&\ge \frac{\varepsilon}{2\alpha_\gamma}(1 - C(\delta_3 + \varepsilon_0))\int_0^1 y(1-y)|\partial_y w|^2 dy.
\end{aligned}$$

After the normalization, we obtain

(A.4) $$-2\alpha_\gamma\frac{\lambda^2}{\varepsilon^3}\mathcal{D} \le -(1 - C(\varepsilon_0 + \delta_3))\int_0^1 y(1-y)|\partial_y W|^2 dy.$$

To finish the proof, we first observe that for any $\delta < \delta_3$,

$$\begin{aligned}
\mathcal{R}_{\varepsilon,\delta}(v) &\le -\frac{1}{\varepsilon\delta_3}|Y_g(v)|^2 + (1+\delta_3)|\mathcal{I}_1(v)|\\
&\quad + \left(1 + \delta_3\left(\frac{\varepsilon}{\lambda}\right)\right)|\mathcal{I}_2(v)| - \left(1 - \delta_3\left(\frac{\varepsilon}{\lambda}\right)\right)\mathcal{G}_2(v) - (1-\delta_3)\mathcal{D}(v).
\end{aligned}$$

Then, (A.2) and (A.4) together with $\varepsilon_0 \le \delta_3$ imply

$$\begin{aligned}
2\alpha_\gamma\left(\frac{\lambda^2}{\varepsilon^3}\right)\mathcal{R}_{\varepsilon,\delta}(v) &\le -\frac{1}{C_\gamma\delta_3}\left(\int_0^1 W^2\, dy + 2\int_0^1 W\, dy\right)^2 + (1 + C_*\delta_3)\int_0^1 W^2\, dy\\
&\quad + \frac{2}{3}\int_0^1 W^3\, dy + C_*\delta_3\int_0^1 |W|^3\, dy - (1 - C_*\delta_3)\int_0^1 y(1-y)|\partial_y W|^2\, dy.
\end{aligned}$$

To finish the proof, we use the nonlinear Poincaré type inequality [33, Proposition 3.3] as follow:

Proposition A.1. [33, Proposition 3.3] *For a given $C_1 > 0$, there exists $\delta_2 > 0$, such that for any $\delta < \delta_2$ the following is true.*
For any $W \in L^2(0,1)$ such that $\sqrt{y(1-y)}\partial_y W \in L^2(0,1)$, if $\int_0^1 |W(y)|^2\, dy \le C_1$, then

(A.5) $$\begin{aligned}
&-\frac{1}{\delta}\left(\int_0^1 W^2\, dy + 2\int_0^1 W\, dy\right)^2 + (1+\delta)\int_0^1 W^2\, dy\\
&+ \frac{2}{3}\int_0^1 W^3\, dy + \delta\int_0^1 |W|^3\, dy - (1-\delta)\int_0^1 y(1-y)|\partial_y W|^2\, dy \le 0.
\end{aligned}$$

First, using the same estimate as in [33, (3.38)], we find the constant $C_1 > 0$ such that

$$\int_0^1 W^2\, dy \le C_1.$$

Then, let us fix the value of the δ_2 of Proposition A.1 corresponding to the constant C_1. We consider $\bar{\delta} = \max(C_\gamma, C_*)\delta_3$, and choose δ_3 small enough, such that $\bar{\delta}$ is smaller than δ_2. Then we have

$$2\alpha_\gamma \left(\frac{\lambda^2}{\varepsilon^3}\right) \mathcal{R}_{\varepsilon,\delta}(v) \leq -\frac{1}{\delta_2}\left(\int_0^1 W^2\, dy + 2\int_0^1 W\, dy\right)^2 + (1+\delta_2)\int_0^1 W^2\, dy$$

$$+ \frac{2}{3}\int_0^1 W^3\, dy + \delta_2 \int_0^1 |W|^3\, dy - (1-\delta_2)\int_0^1 y(1-y)|\partial_y W|^2\, dy.$$

Therefore, using Proposition A.1, we have

$$2\alpha_\gamma \left(\frac{\lambda^2}{\varepsilon^3}\right) \mathcal{R}_{\varepsilon,\delta}(v) \leq 0,$$

which completes the proof.

References

[1] L. Ambrosio, N. Fusco, and D. Pallara. Functions of bounded variation and free discontinuity problems,. *Oxford Mathematical Monographs*, 2000.

[2] S. Bianchini and A. Bressan. Vanishing viscosity solutions to nolinear hyperbolic systems. *Ann. of Math.*, 166:223–342, 2005.

[3] D. Bresch and B. Desjardins. Existence of global weak solutions for 2d viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.*, 238:211–223, 2003.

[4] D. Bresch and B. Desjardins. On the construction of approximate solutions for the 2d viscous shallow water model and for compressible navier-stokes models,. *J. Math. Pures Appl.*, 86(9):362–368, 2006.

[5] D. Bresch, B. Desjardins, and C.K. Lin. On some compressible fluid models: Korteweg, lubrication, and shallow water systems,. *Comm. Partial Differential Equations,*, 28:843–868, 2003.

[6] S. Chapman and T.G. Cowling. The mathematical theory of non-uniform gases. *Cambridge University Press, London*, 3rd ed., 1970.

[7] G.-Q. Chen, H. Frid, and Y. Li. Uniqueness and stability of Riemann solutions with large oscillation in gas dynamics. *Comm. Math. Phys.*, 228(2):201–217, 2002.

[8] G.-Q. Chen and M. Perepelitsa. Vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for compressible fluid flow,. *Comm. Pure Appl. Math.*, 63(11):1469–1504, 2010.

[9] E. Chiodaroli. A counterexample to well-posedness of entropy solutions to the compressible Euler system. *J. Hyperbolic Differ. Equ.*, 11(3):493–519, 2014.

[10] E. Chiodaroli, C. De Lellis, and O. Kreml. Global ill-posedness of the isentropic system of gas dynamics. *Comm. Pure Appl. Math.*, 68(7):1157–1190, 2015.

[11] E. Chiodaroli, E. Feireisl, and O. Kreml. On the weak solutions to the equations of a compressible heat conducting gas. *Ann. Inst. H. Poincaré Anal. Non Linaire*, 32(1):225–243, 2015.

[12] E. Chiodaroli and O. Kreml. On the energy dissipation rate of solutions to the compressible isentropic Euler system. *Arch. Ration. Mech. Anal.*, 214(3):1019–1049, 2014.

[13] K. Choi, M.-J. Kang, Y. Kwon, and A. Vasseur. Contraction for large perturbations of traveling waves in a hyperbolic-parabolic system arising from a chemotaxis model. *https://arxiv.org/abs/1904.12169*.

[14] K. Choi and A. Vasseur. Short-time stability of scalar viscous shocks in the inviscid limit by the relative entropy method. *SIAM J. Math. Anal.*, 47:1405–1418, 2015.

[15] P. Constantin, T. D. Drivas, H. Q. Nguyen, and F. Pasqualotto. Compressible fluids and active potentials,. *Ann. Inst. H. Poincaré Anal. Non Linéaire, To appear*, 2019.

[16] C. M. Dafermos. The second law of thermodynamics and stability. *Arch. Rational Mech. Anal.*, 70(2):167–179, 1979.

[17] C. De Lellis and L. Székelyhidi. The Euler equations as a differential inclusion. *Ann. of Math.*, 170(3):1417–1436, 2009.

[18] C. De Lellis and L. Székelyhidi. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, 195(1):225–260, 2010.

[19] R. J. DiPerna. Uniqueness of solutions to hyperbolic conservation laws. *Indiana Univ. Math. J.*, 28(1):137–188, 1979.

[20] R. J. DiPerna. Convergence of approximate solutions to conservation laws. *Arch. Rational Mech. Anal.*, 82(1):27–70, 1983.

[21] R. J. DiPerna. Convergence of the viscosity method for isentropic gas dynamics. *Comm. in Math. Phys.*, 91:1–30, 1983.

[22] L. C. Evans. Weak convergence methods for nonlinear partial differential equations,. *CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC,,* 74, 1990.

[23] E. Feireisl. Vanishing dissipation limit for the Navier-Stokes-Fourier system. *Commun. Math. Sci.*, 14(6):1535–1551, 2016.

[24] J.-F. Gerbeau and B. Perthame. Derivation of viscous saint-venant system for laminar shallow water; numerical validation,. *Discrete Contin. Dyn. Syst. Ser. B*, 1(1)(2001):89–102, 2018.

[25] J. Goodman and Z. Xin. Viscous limits for piecewise smooth solutions to systems of conservation laws. *Arch. Rational Mech. Anal.*, 121:235–265, 1992.

[26] B. Haspot. New formulation of the compressible navier-stokes equations and parabolicity of the density,. *arXiv:1411.5501.*

[27] B. Haspot. Porous media, fast diffusion equations and the existence of global weak solution for the quasi-solution of compressible Navier-Stokes equations,. *Hyperbolic Problems: Theory, Numerics, Applications-Methods and Applications of Analysis,,* 20(2):141–164, 2013.

[28] B. Haspot. Existence of global strong solution for the compressible Navier-Stokes equations with degenerate viscosity coefficients in 1D,. *posted on arXiv,,* 2014.

[29] B. Haspot. Weak-strong uniqueness for compressible Navier-Stokes system with degenerate viscosity coefficient and vacuum in one dimension. *Commun. Math. Sci.*, 15(3):587–591, 2017.

[30] D. Hoff and T. P. Liu. The inviscid limit for the Navier-Stokes equations of compressible isentropic flow with shock data. *Indiana Univ. Math. J.*, 38:861–915, 1989.

[31] M.-J. Kang. L^2-type contraction for shocks of scalar viscous conservation laws with strictly convex flux. *https://arxiv.org/pdf/1901.02969.pdf.*

[32] M.-J. Kang. Non-contraction of intermediate admissible discontinuities for 3-D planar isentropic magnetohydrodynamics. *Kinet. Relat. Models*, 11(1):107–118, 2018.

[33] M.-J. Kang and A. Vasseur. Contraction property for large perturbations of shocks of the barotropic Navier-Stokes system. *J. Eur. Math. Soc. (JEMS), To appear. https://arxiv.org/pdf/1712.07348.pdf.*

[34] M.-J. Kang and A. Vasseur. Global smooth solutions for 1D barotropic Navier-Stokes equations with a large class of degenerate viscosities. *https://arxiv.org/abs/1907.12938.*

[35] M.-J. Kang and A. Vasseur. Criteria on contractions for entropic discontinuities of systems of conservation laws. *Arch. Ration. Mech. Anal.*, 222(1):343–391, 2016.

[36] M.-J. Kang and A. Vasseur. L^2-contraction for shock waves of scalar viscous conservation laws. *Annales de l'Institut Henri Poincaré (C) : Analyse non linéaire*, 34(1):139156, 2017.

[37] M.-J. Kang, A. Vasseur, and Y. Wang. L^2-contraction for planar shock waves of multi-dimensional scalar viscous conservation laws. *http://arxiv.org/pdf/1609.01825.pdf.*

[38] N. Leger. L^2 stability estimates for shock solutions of scalar conservation laws using the relative entropy method. *Arch. Ration. Mech. Anal.*, 199(3):761–778, 2011.

[39] N. Leger and A. Vasseur. Relative entropy and the stability of shocks and contact discontinuities for systems of conservation laws with non-BV perturbations. *Arch. Ration. Mech. Anal.*, 201(1):271–302, 2011.

[40] A. Matsumura and Y. Wang. Asymptotic stability of viscous shock wave for a one-dimensional isentropic model of viscous gas with density dependent viscosity,. *Methods Appl. Anal.*, 17:279–290, 2010.

[41] A. Mellet and A. Vasseur. Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations. *SIAM J. Math. Anal.*, 39(4):1344–1365, 2007/08.

[42] D. Serre and A. Vasseur. L^2-type contraction for systems of conservation laws. *J. Éc. polytech. Math.*, 1:1–28, 2014.

[43] D. Serre and A. Vasseur. About the relative entropy method for hyperbolic systems of conservation laws. *Contemp. Math. AMS*, 658:237–248, 2016.

[44] D. Serre and A. Vasseur. The relative entropy method for the stability of intermediate shock waves; the rich case. *Discrete Contin. Dyn. Syst.*, 36(8):4569–4577, 2016.

[45] V.V. Shelukhin. On the structure of generalized solutions of the one-dimensional equations of a poly-tropic viscous gas,. *J. Appl. Math. Mech., 48(1984), 665–672; translated from Prikl. Mat. Mekh. 48(1984), no. 6, 912–920.*

[46] G.G. Stokes. On a difficulty in the theory of sound,. *Philos. Mag.*, 33:349–356, 1848.

[47] A. Vasseur. Time regularity for the system of isentropic gas dynamics with $\gamma = 3$. *Comm. Partial Differential Equations*, 24(11-12):1987–1997, 1999.

[48] A. Vasseur. Recent results on hydrodynamic limits. In *Handbook of differential equations: evolutionary equations. Vol. IV*, Handb. Differ. Equ., pages 323–376. Elsevier/North-Holland, Amsterdam, 2008.

[49] A. Vasseur. Relative entropy and contraction for extremal shocks of conservation laws up to a shift. In *Recent advances in partial differential equations and applications*, volume 666 of *Contemp. Math.*, pages 385–404. Amer. Math. Soc., Providence, RI, 2016.

[50] A. Vasseur and Y. Wang. The inviscid limit to a contact discontinuity for the compressible navier-stokes-fourier system using the relative entropy method. *SIAM J. Math. Anal.*, 47(6):4350–4359, 2015.

[51] A. Vasseur and L. Yao. Nonlinear stability of viscous shock wave to one-dimensional compressible isentropic Navier-Stokes equations with density dependent viscous coefficient,. *Commun. Math. Sci.*, 14(8):2215–2228, 2016.

[52] S. H. Yu. Zero-dissipation limit of solutions with shocks for systems of hyperbolic conservation laws. *Arch. Rational Mech. Anal.*, 146:275–370, 1999.

(Moon-Jin Kang)
DEPARTMENT OF MATHEMATIC & RESEARCH INSTITUTE OF NATURAL SCIENCES,
SOOKMYUNG WOMEN'S UNIVERSITY, SEOUL 140-742, KOREA
    E-mail address: moonjinkang@sookmyung.ac.kr

(Alexis F. Vasseur)
DEPARTMENT OF MATHEMATICS,
THE UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TX 78712, USA
    E-mail address: vasseur@math.utexas.edu