UNIQUENESS OF A PLANAR CONTACT DISCONTINUITY FOR 3D COMPRESSIBLE EULER SYSTEM IN A CLASS OF ZERO DISSIPATION LIMITS FROM NAVIER-STOKES-FOURIER SYSTEM

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ABSTRACT. We prove the stability of a planar contact discontinuity without shear, a family of special discontinuous solutions for the three-dimensional full Euler system, in the class of vanishing dissipation limits of the corresponding Navier-Stokes-Fourier system. We also show that solutions of the Navier-Stokes-Fourier system converge to the planar contact discontinuity when the initial datum converges to the contact discontinuity itself. This implies the uniqueness of the planar contact discontinuity in the class that we are considering. Our results give an answer to the open question, whether the planar contact discontinuity is unique for the multi-D compressible Euler system. Our proof is based on the relative entropy method, together with the theory of a-contraction up to a shift and our new observations on the planar contact discontinuity.

CONTENTS

1. Introduction

We consider the Navier-Stokes-Fourier system in three dimensions with periodic boundary: for any $x = (x_1, x_2, x_3) \in \mathbb{T}^3$, $t \geq 0$,

$$(1.1) \quad \begin{cases} \partial_t \rho + \operatorname{div}_x(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x p = \nu \operatorname{div}_x \mathbb{S}, \\ \partial_t \left(\rho \left(\frac{|u|^2}{2} + e \right) \right) + \operatorname{div}_x \left(\left(\rho \left(\frac{|u|^2}{2} + e \right) + p \right) u \right) = \operatorname{div}_x(\kappa \nabla_x \theta) + \nu \operatorname{div}_x(\mathbb{S}u), \end{cases}$$

where the functions $\rho = \rho(t,x), u(t,x) = (u_1, u_2, u_3)^\top(t,x), e = e(t,x), \theta = \theta(t,x)$ and $p = p(t,x)$ represent respectively the fluid density, velocity, specific internal energy, absolute temperature and the pressure.

The aim of this paper is to investigate the uniqueness, and stability of special discontinuous solutions, known as contact discontinuities without dissipation, of the associated Euler equation (with $\nu = 0$ and $\kappa = 0$). The study is based on a careful study of the zero dissipation limit of the Navier-Stokes-Fourier system, for vanishing viscosities ($\nu \to 0$) and heat conductivities ($\kappa \to 0$) . We make the following assumptions on the physical system (1.1).

Assume that the viscous stress tensor $\mathbb{S}$, with the coeffient $\nu > 0$, is given by

$$\mathbb{S} = \mu(\theta)(\nabla u + (\nabla u)^\top) + \lambda(\theta)(\operatorname{div} u) Id_{3 \times 3},$$

where $(\nabla u)^\top$ denotes the transpose of the matrix ∇u, and $Id_{3 \times 3}$ represents the 3×3 identity matrix. We assume that $\mu(\theta)$ and $\lambda(\theta)$ depend linearly on θ, that is:

$$(1.2) \quad \mu(\theta) = \mu_1 \theta, \quad \lambda(\theta) = \lambda_1 \theta,$$

where μ_1 and λ_1 are both constants satisfying the physical constraints $\mu_1 > 0$ and $2\mu_1 + 3\lambda_1 > 0$. By the Fourier laws, the heat flux is given by $-\kappa \nabla_x \theta$ in $(1.1)_3$ with $\kappa > 0$ denoting the heat-conductivity coefficient. Here we assume that both the parameters ν and κ are the positive vanishing coefficients.

The pressure p is a function of ρ and θ of the form:

$$(1.3) \quad p(\rho, \theta) = R\rho\theta + p_e(\rho), \quad p_e(\rho) = a\rho^\gamma.$$

The first part of the pressure coincides with the ideal gas laws, while the second part p_e is an elastic pressure (sometimes called cold pressure) proportional to the isentropic pressure of the ideal gas, with the adiabatic constant $\gamma > 1$ and both R and a are positive constants.

We assume that the specific internal energy e has the following form:

$$(1.4) \quad e(\rho, \theta) = \frac{a\rho^{\gamma-1}}{\gamma - 1} + Q_1(\theta), \quad Q_1(\theta) = \begin{cases} \frac{R}{\gamma-1}\theta & \text{if } \theta \geq \theta_* \\ c_* \theta^2 & \text{if } 0 < \theta < \theta_*, \end{cases}$$

where θ_* is some small positive constant, and the positive constant c_* is chosen such that Q_1 is Lipschitz in θ. Note that $Q_1'(\theta) > 0$ for all $\theta > 0$. The first term of (1.4) is the energy associated to the elastic pressure of (1.3). For $\theta > \theta_*$, (1.4) is consistent with the ideal gas law, and together with (1.3), ensures that

$$(1.5) \quad p = (\gamma - 1)\rho e, \quad \text{for } \theta \geq \theta_*.$$

The definition of Q_1 for extremely cold temperature $\theta < \theta_*$ imposes the validity of the third law of thermodynamic, which states that the entropy cannot blow up when the temperature approaches the absolute zero. The validity of the third law of thermodynamic is important in our study. Note that it requires to deviate from the ideal gas dynamics at least for very cold temperatures (see below (1.8)).

We can now compute the entropy functional of the system, following the second law of thermodynamics:

$$\theta ds = pd(\frac{1}{\rho}) + de. \tag{1.6}$$

The entropy takes the form

$$s = -R\ln\rho + s_1(\theta), \tag{1.7}$$

where $s_1(\theta)$ satisfy

$$s_1(\theta) = \int_0^\theta \frac{Q_1'(z)}{z}dz = \begin{cases} \frac{R}{\gamma-1}\ln\theta & \text{if } \theta \geq \theta_* \\ 2c_*\theta & \text{if } 0 < \theta < \theta_*. \end{cases} \tag{1.8}$$

Note that the above state equation for the entropy s satisfy the third law of thermodynamic, that is, the entropy approaches a constant value as temperature approaches absolute zero.

If $\nu = 0$ and $\kappa = 0$, then the corresponding 3D compressible Euler system reads as

$$\begin{cases} \partial_t\rho + \text{div}_x(\rho u) = 0, \\ \partial_t(\rho u) + \text{div}_x(\rho u \otimes u) + \nabla_x p = 0, \\ \partial_t\left(\rho\left(\frac{|u|^2}{2} + e\right)\right) + \text{div}_x\left(\left(\rho\left(\frac{|u|^2}{2} + e\right) + p\right)u\right) = 0. \end{cases} \tag{1.9}$$

In this paper, we aim to show that a planar contact discontinuity without shear for 3D Euler system (1.9) is stable and unique in the class of vanishing dissipation limit ($\kappa \to 0+$ and $\nu \to 0+$) of solutions to 3D compressible Navier-Stokes-Fourier system (1.1).

1.1. **Main result.** First we describe a planar contact discontinuity solution to the 3D compressible Euler equations (1.9) for moderate temperature, that is, $\theta_\pm > \theta_*$. If we consider the following Riemann initial data

$$(\rho, u, \theta)(x, t = 0) = \begin{cases} (\rho_-, u_-, \theta_-) & x_1 < 0 \\ (\rho_+, u_+, \theta_+) & x_1 > 0, \end{cases} \tag{1.10}$$

where $\rho_\pm > 0, \theta_\pm > \theta_*, u_\pm = (u_{1\pm}, 0, 0)^t$ are prescribed constants without shear for the 3D Euler system (1.9) with $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, then the Riemann problem (1.9)-(1.10) admits a planar contact discontinuity solution (as a self-similar solution in x_1)

$$(\rho, u, \theta)(x, t) = \begin{cases} (\rho_-, u_-, \theta_-) & x_1 < u_{1*}t, \\ (\rho_+, u_+, \theta_+) & x_1 > u_{1*}t, \end{cases} \tag{1.11}$$

provided that

$$u_{1-} = u_{1+} := u_{1*}, \quad p_- = p_+, \tag{1.12}$$

where $p_\pm := R\rho_\pm\theta_\pm + p_e(\rho_\pm)$.

By Galilean invariance, we can assume that $u_{1*} = 0$ without loss of generality. Then the planar contact discontinuity solution is a stationary solution.

Therefore, we consider a planar, stationary and shifted contact discontinuity solution $(\bar\rho, \bar u, \bar\theta)$ to the 3D compressible Euler equations (1.9) on $\mathbb{T}^3$ defined as follows:

$$(\bar\rho, \bar u, \bar\theta)(x) := \begin{cases} (\rho_-, 0, \theta_-), & 0 < x_1 < \frac{1}{2}, \quad x = (x_1, x_2, x_3) \in \mathbb{T}^3 \\ (\rho_+, 0, \theta_+), & \frac{1}{2} < x_1 < 1, \end{cases} \tag{1.13}$$

with

$$\bar p := p_- = p_+ \quad \text{on } \mathbb{T}^3. \tag{1.14}$$

Since $\theta_\pm > \theta_*$, it follows from (1.3) and (1.5) that the contact discontinuity solution $(\bar{\rho}, \bar{u}, \bar{\theta})$ satisfies the ideal gas equality:

$$p_\pm = (\gamma - 1)\rho_\pm e_\pm.$$

Thus, the end states $\mathcal{E}_\pm$ for the total energy satisfy

$$\mathcal{E}_- = \rho_- e_- = \frac{p_-}{\gamma - 1} = \frac{p_+}{\gamma - 1} = \rho_+ e_+ =: \mathcal{E}_+.$$

Therefore, we set

(1.15) $$\bar{\mathcal{E}} := \mathcal{E}_- = \mathcal{E}_+ \quad \text{on } \mathbb{T}^3.$$

The main theorem is on the uniqueness and stability of the planar contact discontinuity solution (1.13) to the 3D compressible Euler equations (1.9) in the class of zero dissipation limits (as $\kappa \to 0+$ and $\nu \to 0+$) of admissible weak solutions to the 3D compressible Navier-Stokes-Fourier equations (1.1) in $\mathbb{T}^3$.

For statement of the main theorem, we first consider a convex function

(1.16) $$\mathcal{F}(y) := \begin{cases} 0 & \text{if } \rho_* := \min\{\rho_-, \rho_+\} \leq y \leq \max\{\rho_-, \rho_+\} =: \rho^*, \\ (y - \rho_*)^2 & \text{if } y \leq \rho_*, \\ (y - \rho^*)^2 & \text{if } y \geq \rho^*, \end{cases}$$

Theorem 1.1. *Consider the Euler system (1.9) with (1.3)-(1.4). Assume $\gamma > 2$. Let $\bar{U} := (\bar{\rho}, \bar{\rho}\bar{u}, \bar{\mathcal{E}})$ be a stationary contact discontinuity (1.13) with (1.14)-(1.15). Let U_0 be an initial datum such that*

$$\int_{\mathbb{T}^3} (-\rho s)(U_0 | \bar{U}) dx < \infty,$$

with the relative entropy $(-\rho s)(U_0 | \bar{U})$ defined in (1.20) and

(1.17) $$(-\rho s)(U_0 | \bar{U}) \in L^\infty(\Omega), \quad \text{for some neighborhood } \Omega \text{ of the plane } x_1 = \frac{1}{2}.$$

For any given $T > 0$, let $(\rho^{\kappa,\nu}, m^{\kappa,\nu}, \mathcal{E}^{\kappa,\nu})$ be an admissible weak solution of (1.1) on $[0, T]$ in the sense of Definition 2.1 in Section 2, where $m^{\kappa,\nu} := \rho^{\kappa,\nu} u^{\kappa,\nu}$ and $\mathcal{E}^{\kappa,\nu} := \rho^{\kappa,\nu} \left(\frac{|u^{\kappa,\nu}|^2}{2} + e^{\kappa,\nu} \right)$.

Then, there exists a limit $(\rho, m, \mathcal{E})$ such that, up to a subsequence, as $\kappa \to 0$ and $\nu \to 0$,

(1.18)
$$\rho^{\kappa,\nu} \rightharpoonup \rho \quad \text{weakly} - * \quad \text{in} \quad L^\infty(0, T; L^2(\mathbb{T}^3)),$$
$$m^{\kappa,\nu} \rightharpoonup m \quad \text{weakly} \quad \text{in} \quad L^2(0, T; L^{\frac{4}{3}}(\mathbb{T}^3)),$$
$$\mathcal{E}^{\kappa,\nu} \rightharpoonup \mathcal{E} \quad \text{weakly} - * \quad \text{in} \quad L^\infty(0, T; \mathcal{M}(\mathbb{T}^3)),$$

where $\mathcal{M}$ denotes the space of Radon measures, and

(1.19)
$$\sup_{[0,T]} \int_{\mathbb{T}^3} \left(\mathcal{F}(\rho) + \frac{|m|^2}{2\rho} \right) dx + \|\mathcal{E} - \bar{\mathcal{E}}\|_{L^\infty(0,T;\mathcal{M}(\mathbb{T}^3))}$$
$$\leq C \sqrt{\int_{\mathbb{T}^3} (-\rho s)(U_0 | \bar{U}) dx} + C \int_{\mathbb{T}^3} (-\rho s)(U_0 | \bar{U}) dx.$$

Furthermore, let $(U_0^n)_{n \in \mathbb{N}}$ be a sequence of initial data such that

$$\int_{\mathbb{T}^3} (-\rho s)(U_0^n | \bar{U}) dx \to 0 \quad \text{as } n \to \infty$$

Then, any inviscid limit $(\rho^n, m^n, \mathcal{E}^n)$ satisfying (1.18)-(1.19) and corresponding to U_0^n satisfies that, up to a subsequence, as $n \to \infty$,

$$\rho^n \rightharpoonup \bar{\rho} \quad weakly - * \quad in \quad L^\infty(0, T; L^2(\mathbb{T}^3)),$$

$$m^n \to 0(= \bar{\rho}\bar{u}) \quad in \quad L^\infty(0, T; L^{\frac{4}{3}}(\mathbb{T}^3)),$$

$$\mathcal{E}^n \to \bar{\mathcal{E}} \quad in \quad L^\infty(0, T; \mathcal{M}(\mathbb{T}^3)).$$

Therefore, the contact discontinuity is stable and unique in the class of vanishing dissipation limits of solutions to (1.1).

1.2. **Remarks for the main result.** • The existence of global solutions for (1.1) should be constructed following the theory developed by Feireisl [18]. However, note that our assumption especially (1.2), is not compatible with the hypotheses as stated in [18]. For this reason, we leave the construction of these solutions to a future work. Assumption (1.2) is needed to get the uniform bound of $\nu\|\nabla u^{\kappa,\nu}\|^2_{L^2(0,T;L^2(\mathbb{T}^3))}$ (see (2.11)) from the entropy dissipation (2.6), which is crucial for our asymptotic analysis. Also, we mention that the assumption on $\gamma > 2$ is crucial for the uniform bound (2.10) in Remark 2.1, which is useful in Section 3.4.

• It is well known that Lipschitz solutions to the Euler system (1.9) are stable and unique in the class of admissible weak (or entropic) solutions (see Dafermos [14] and DiPerna [17]). However, the situation for discontinuous solutions is far more complicated. Especially, for the case of entropic shocks or shear flows, the uniqueness is usually not true. Recently, results on the non-uniqueness were obtained by the convex integration method introduced by De Lellis and Székelyhidi [15, 16]. Using this method, Chiodaroli, De Lellis and Kreml [8] showed the existence of Riemann initial data generated by an entropy shock for which there exist infinitely many bounded entropy solutions. We also refer to [2, 7, 8, 9, 10, 19, 35, 34] for other related studies on the non-uniqueness of entropic shocks.

These results have been extended by Březina, Kreml and Mácha [3] to the case of planar contact discontinuity, for the case of the 2D isentropic Euler system with the Chaplygin gas pressure law.

In the case of rarefaction waves (that are discontinuous only at $t = 0$), the uniqueness was proved in the class of entropic solutions to the multi-D Euler system (see [5, 20, 21]). The time-asymptotic stability and vanishing viscosity limit of isentropic Navier-Stokes equatios/Navier-Stokes-Fourier equations to the planar rarefaction wave of 2D/3D compressible Euler equations could be found in [43, 42, 41, 40].

• In Theorem 1.1, we consider the simplest discontinuous solution for (1.9). Note that this solution corresponds to the second characteristic field in the 1D setting, and is a fundamental building block in the study of small BV entropy solutions (see for instance [13, 45]). Our result shows its stability in the 3D setting. Note that it is not known if this solution is unique among the class of entropy solutions for the multi-D compressible Euler system.

An important feature of Theorem 1.1 is that it proves the stability in the class of zero dissipation limits of the Navier-Stokes-Fourier system rather than entropy solutions of the Euler system. In addition, our theorem shows the convergence of solutions of (1.1) to the contact discontinuity when the initial value converges to the contact discontinuity itself. Note that we do not need any *a priori* regularity on the dissipation limits, which are automatically obtained by the entropy bound. Especially, the limits do not need to be solutions to the Euler system. In the 3D setting, it is not known whether global solutions

of the Euler system can be constructed for a large class of initial data. This open question makes working in the large class of dissipation limits appealing.

Replacing the notion of weak solutions to Euler by the one of inviscid limit is already interesting and important in the 1D setting. We first mention results on the uniqueness of entropy shocks in the class of entropy solutions satisfying the locally BV regularity (see Chen-Frid-Li [6]) or the strong trace property (see Vasseur et al. [29, 39, 51] and Krupa [37]). However, the global-in-time propagation of those regularities remains open (except for the system with $\gamma = 3$ see [50]). Recently, Kang and Vasseur [32] proved the uniqueness and stability of entropy shocks for the 1D isentropic Euler system in the class of inviscid limits of solutions to the corresponding Navier-Stokes system. This gives an answer, for the case of entropy shock, to the conjecture: The compressible Euler system admits a unique entropy solution in the class of vanishing viscosity solutions to the associated compressible Navier-Stokes system (as the physical viscous system for the Euler). As a comprehensive study related to this conjecture, Bianchini and Bressan [4] obtained a global unique entropy solution to a 1D strictly hyperbolic $n \times n$ system with small BV initial datum, which is obtained from vanishing "artificial" viscosity limit of the associate parabolic system.

Theorem 1.1 answers the conjecture in the case of contact discontinuity for the full Euler system (1.9) with (1.3)-(1.4).

• Our proof is based on the theory of a-contraction up to a shift, first developed in the one dimensional hyperbolic case in [29, 51]. The main idea is the construction of a weight function (the a function) which is both bounded and bounded by below, such that the discontinuous solutions enjoy a contraction property for the corresponding weighted relative entropy, up to a shift. With the exception of the scalar case, considered by Leger in [38], the contraction property is usually not verified without weight (see [46]). In the case of shocks, the method was extended to 1D Navier-Stokes in [32], and to the inviscid limit in [30] (see also [1, 11, 12, 27, 26, 31, 36, 37, 48, 49] for other developments of the theory).

For the 1D study in the hyperbolic case of the contact discontinuity, the correct weighted function is explicit and given by θ_- on the left and θ_+ on the right (see [47]). It was used to study the zero dissipation limit in 1D in this context in [52]. For the other studies on the vanishing dissipation limit to the 1D Riemann solution which may contain shock and rarefaction waves and contact discontinuity, one can refer [24, 23, 25, 22] and references therein. Note that the limit was proved in [22] for the generic 1D Riemann solution.

Our result in Theorem 1.1 can be seen as the extension of the work of [52] in the multi-D setting. It is the first application of the method for systems in multi-D (see [33] for an application to the multi-D scalar case).

1.3. **Ideas of the proof.** Let us denote $U = (\rho, \rho u, \mathcal{E})$ the conservative variables, and $\eta(U) = -\rho s$ the entropy. Since η is convex, the relative entropy

$$(1.20) \qquad\qquad \eta(U|\bar{U}) = \eta(U) - \eta(\bar{U}) - d\eta(\bar{U}) \cdot (U - \bar{U})$$

define a pseudo distance of U to $\bar{U}$ which is locally (for U and $\bar{U}$ bounded) equivalent to $|U - \bar{U}|^2$. Consider now $U(t,x)$ solution to the system (1.1), and $\bar{U}_-, \bar{U}_+$ the left and right states of the given contact discontinuity. The general idea of the theory of a-contraction with shift for the hyperbolic case ($\nu = \kappa = 0$) is to find disjoint shifted domains for all t, $\Omega_-(t)$,

$\Omega_+(t)$ such that $\Omega_-(t) \cup \Omega_+(t) = \mathbb{T}^3$, and coefficients $a_- > 0, a_+ > 0$ such that

$$t \to a_- \int_{\Omega_-(t)} \eta(U(t,x)|\bar{U}_-) \, dx + a_+ \int_{\Omega_+(t)} \eta(U(t,x)|\bar{U}_+) \, dx$$

is decreasing in time. Note that $p_+ = p_- = \bar{p}$, therefore in the Lagrangian variables, the second law of thermodynamics gives that

$$(\mathbf{1}_{\{x_1 < 1/2\}}\theta_- + \mathbf{1}_{\{x_1 > 1/2\}}\theta_+)ds = \bar{p} \, d(1/\rho) + de$$

is an exact form globally on $\mathbb{T}^3$. And so, at least formally, the contraction holds for $a_- = \theta_-$, $a_+ = \theta_+$, $\Omega_-(t)$ the set $\{x < 1/2\}$ transported by the flow $\{u(s,\cdot), 0 < s < t\}$, and $\Omega_+(t)$ the set $\{x \geq 1/2\}$ transported by the same flow (see [47]). Still formally, considering $(\rho(t,x), \rho(t,x)u(t,x), \rho(t,x)[e(t,x) + \frac{|u(t,x)|^2}{2}])$ a solution to the Euler equation (1.9), and ψ the solution to the transport equation:

(1.21) $$\partial_t \psi + u \cdot \nabla \psi = 0, \qquad \psi(0) = \mathbf{1}_{\{x_1 > 1/2\}},$$

the following function

(1.22) $$\int (\theta_-(1 - \psi(t,x))\eta(U(t,x)|\bar{U}_-) + \theta_+\psi(t,x)\eta(U(t,x)|\bar{U}_+)) \, dx$$

would be non-increasing in time.

The main obstruction to the study above is that the velocity u solution to Euler system is not smooth enough to construct solution to (1.21). It also assumes some non-oscillatory behavior (strong traces) at the boundary of $\Omega_\pm(t)$, properties that are not known to exist for solutions to Euler.

The idea is then to consider the Navier-Stokes-Fourier system instead, which provides solutions $u \in L^2(0,T;H^1(\mathbb{T}^3))$. With this regularity, the flow (1.21) can be constructed. Following [12, 52], we consider the extra viscous ν and κ terms as source terms to be controlled. We first pass into the limit as κ goes to zero. For this limit, extra regularizations have to be performed, via convolutions, both on u and ψ. When $\kappa = 0$, we pass to the limit in the regularization terms, using controls on Lions commutators (see Lemma 3.6).

When the last limit ν goes to zero, the regularity on the velocity u is lost, and so $\Omega_-(t)$ and $\Omega_+(t)$ can become mixed to each other. Note that $\mathcal{E}_- = \mathcal{E}_+ = \bar{\mathcal{E}}$, since the contact discontinuity has values in the regime where the gas verifies the ideal gas equality (1.5). Therefore it can be shown in Lemma 2.2 that the weighted relative entropy (1.22) controls uniformly

$$\int_{\mathbb{T}^3} \mathcal{F}(\rho) + \widetilde{\mathcal{G}}(\mathcal{E} - \bar{\mathcal{E}}) + \frac{\rho|u|^2}{2},$$

where $\widetilde{\mathcal{G}}$ is the convex function:

$$\begin{aligned}\widetilde{\mathcal{G}}(y) &= |y|^2, && \text{for } |y| < 1, \\ &= 2|y| - 1 && \text{for } |y| \geq 1.\end{aligned}$$

Note that the function $\widetilde{\mathcal{G}}$ is equivalent to $\min(|y|^2, |y|)$, and since we are in a bounded domain, this provides a control on the L^1-norm of $\mathcal{E} - \bar{\mathcal{E}}$. However the function $\mathcal{F}$ is degenerated, and is equal to zero for $\rho_* \leq \rho \leq \rho^*$. Consequently, (1.22) controls the perturbations in both velocity and energy. However, due to the possible mixing of the area Ω_- and Ω_+, (1.22) controls only the value of the density outside $[\rho_*, \rho^*]$. This degeneracy is similar to the situation of the pressureless Euler system in [28]. Thus, we recover the control of the density in the similar way as [28], using the continuity equation. Indeed, if $\rho u = 0$ at

the limit (which is a quantity controlled by the relative entropy), then it follows from the continuity equation that $\partial_t \rho = 0$, and so $\rho = \bar{\rho}$.

The rest of the paper is as follows. In Section 2, we first present the notion on admissible weak solutions to (1.1). Then we provide the explicit form for the weighted relative entropy, and construct nonlinear functionals uniformly controlled by it. Section 3 is dedicated to the proof of the main theorem.

2. Preliminaries

In this section, we first present the Definition 2.1 as mentioned in Theorem 1.1 on admissible weak solutions of (1.1), together with useful bounds in Remark 2.1. Then we compute the explicit form for the weighted relative entropy, and construct nonlinear functionals controlled by it.

2.1. Admissible weak solutions for the Navier-Stokes-Fourier system (1.1). .

Definition 2.1. *We say that $(\rho, \rho u, \mathcal{E}, \rho s)$ with $\mathcal{E} := \rho \left(\frac{|u|^2}{2} + e \right)$ is an admissible weak solution to the 3D compressible Navier-Stokes-Fourier system (1.1)-(1.4) in $[0, T] \times \mathbb{T}^3$ with an initial data (ρ_0, u_0, θ_0) if the following holds:*
(i) (Mass conservation equation) For any test function $\varphi \in \mathcal{D}([0, T) \times \mathbb{T}^3)$,

$$(2.1) \qquad \int_0^\infty \int_{\mathbb{T}^3} \left(\rho \partial_\tau \varphi + \rho u \cdot \nabla \varphi \right) dx d\tau + \int_{\mathbb{T}^3} \rho_0(x) \varphi(x, 0) dx = 0,$$

and

$$(2.2) \qquad \rho \in C([0, T]; L^\gamma_{\text{weak}}(\mathbb{T}^3)).$$

(ii) (Momentum conservation equation) For any $\vec{\varphi} \in \left[\mathcal{D}([0, T) \times \mathbb{T}^3) \right]^3$,

$$(2.3) \quad \int_0^\infty \int_{\mathbb{T}^3} \left(\rho u \cdot \partial_\tau \vec{\varphi} + \rho u \otimes u : \nabla \vec{\varphi} + p \, \text{div} \vec{\varphi} - \nu \mathbb{S} : \nabla \vec{\varphi} \right) dx d\tau + \int_{\mathbb{T}^3} \rho_0 u_0(x) \cdot \vec{\varphi}(x, 0) dx = 0.$$

(iii) (Energy inequality)

$$(2.4) \qquad \int_{\mathbb{T}^3} \mathcal{E}(x, t) dx \leq \int_{\mathbb{T}^3} \mathcal{E}(x, 0) dx < \infty, \qquad \forall t \in [0, T],$$

where $\mathcal{E}|_{t=0}$ satisfies the compatibility condition:

$$\mathcal{E}|_{t=0} = \rho_0 \left(\frac{|u_0|^2}{2} + \frac{a \rho_0^{\gamma-1}}{\gamma - 1} + Q_1(\theta_0) \right),$$

and

$$(2.5) \qquad \limsup_{t \to 0+} \int_{\mathbb{T}^3} \mathcal{E}(x, t) dx = \int_{\mathbb{T}^3} \mathcal{E}(x, 0) dx.$$

(iii) (Entropy dissipation) For any non-negative test function $\varphi \in \mathcal{D}([0, T) \times \mathbb{T}^3)$,

$$(2.6) \qquad \int_0^\infty \int_{\mathbb{T}^3} \left[\rho s \varphi_\tau + \rho s u \cdot \nabla \varphi - \kappa \frac{\nabla \theta}{\theta} \cdot \nabla \varphi + \kappa \frac{|\nabla \theta|^2}{\theta^2} \varphi + \frac{\nu \mathbb{S} : \nabla_x u}{\theta} \varphi \right] dx d\tau$$
$$+ \int_{\mathbb{T}^3} (\rho s)(x, 0) \varphi(x, 0) dx \leq 0,$$

where $(\rho s)|_{t=0}$ satisfies the compatibility condition:

$$(2.7) \qquad (\rho s)|_{t=0} = -R\rho_0 \ln \rho_0 + \rho_0 s_1(\theta_0),$$

and

$$(2.8) \qquad \limsup_{t\to 0+} \int_{\mathbb{T}^3} (\rho s)(x,t)\eta(x)dx = \int_{\mathbb{T}^3} (\rho s)(x,0)\eta(x)dx, \quad \forall \eta \in \mathcal{D}(\mathbb{T}^3).$$

Remark 2.1. *If $\gamma > 2$, then by (2.4), (2.6) and (1.7), there exists a constant C independent of κ, ν (but depending on the initial data) such that*

$$(2.9) \qquad \|\rho\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \leq C,$$

$$(2.10) \qquad \|\rho s\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \leq C,$$

and

$$(2.11) \qquad \nu\|\nabla u\|_{L^2(0,T;L^2(\mathbb{T}^3))}^2 + \kappa \left\|\frac{\nabla \theta}{\theta}\right\|_{L^2(0,T;L^2(\mathbb{T}^3))}^2 \leq C.$$

Indeed, since $\|\rho\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^3))} \leq C$ by (2.4), we have (2.9). The uniform bound (2.11) is obtained by the entropy dissipation of (2.6) with (1.2). Also, note that (1.7), (1.8) and (2.4) imply that

$$\int_{\mathbb{T}^3} |\rho s|^2 dx \leq C\Big(\int_{\mathbb{T}^3} \rho^2 |\ln \rho|^2 \mathbf{1}_{\rho>1} dx + \int_{\mathbb{T}^3} \rho^2 dx + \int_{\mathbb{T}^3} \rho^2 |\ln \theta|^2 \mathbf{1}_{\theta>\theta_*} dx\Big)$$

$$\leq C\Big(\int_{\mathbb{T}^3} \rho^\gamma dx + \int_{\mathbb{T}^3} \rho^2 dx + \int_{\mathbb{T}^3} \rho^2 \theta^{\frac{\gamma-2}{\gamma-1}} \mathbf{1}_{\theta>\theta_*} dx\Big)$$

$$\leq C\Big(\int_{\mathbb{T}^3} \rho^\gamma dx + \int_{\mathbb{T}^3} \rho^2 dx + \int_{\mathbb{T}^3} (\rho^\gamma + \rho\theta)\mathbf{1}_{\theta>\theta_*} dx\Big) \leq C,$$

which gives (2.10).
Note that, to obtain (2.10), we use the setting (1.4) and (1.8) satisfying the third law of thermodynamics. This is the only place we use it.

2.2. **Weighted relative entropy.** We define a function $\mathcal{S} : \mathbb{R}_+ \to \mathbb{R}$ by

$$(2.12) \qquad \mathcal{S}(Q_1(\theta)) := s_1(\theta), \quad \theta > 0.$$

Then the function $\mathcal{S}(Q_1)$ is strictly concave in Q_1. Indeed, since $\mathcal{S}'(Q_1(\theta))Q_1'(\theta) = s_1'(\theta)$, and $s_1'(\theta) = Q_1'(\theta)/\theta$ by (1.8), we have

$$(2.13) \qquad \mathcal{S}'(Q_1(\theta)) = 1/\theta,$$

and thus $\mathcal{S}''(Q_1(\theta))Q_1'(\theta) = -1/\theta^2 < 0$, which yields $S''(Q_1) < 0$ by $Q_1' > 0$.

Let $U := (\rho, m, \mathcal{E})$ be a solution of (1.1) where $m = \rho u$ and $\mathcal{E} = \rho\left(\frac{|u|^2}{2} + e\right)$, and $\bar{U} := (\bar{\rho}, \bar{m}, \bar{\mathcal{E}})$ be the contact discontinuity (1.13) where $\bar{m} = 0$ and $\bar{\mathcal{E}} = \bar{\rho}\bar{e}$. As in [47, 52], we will consider the relative entropy functional weighted by the temperature $\bar{\theta}$ connecting two different constants θ_- and θ_+:

$$(2.14) \qquad \int_{\mathbb{T}^3} \bar{\theta}(-\rho s)(U|\bar{U})dx,$$

where $(-\rho s)(U|\bar{U})$ represents the relative entropy associated with the entropy $-\rho s$, defined as follows: for $\eta(U) := -\rho s$,

$$(2.15) \qquad (\eta)(U|\bar{U}) := \eta(U|\bar{U}) = \eta(U) - \eta(\bar{U}) - d\eta(U) \cdot (U - \bar{U}).$$

In general, the notation (2.15) will be used for a given function η throughout the paper. Note that if $U \mapsto \eta(U)$ is strictly convex, then $\eta(U|\bar{U})$ is positively definite. Thanks to the following lemma, we see that the functional (2.14) can be written as the sum of four sub-functionals that are all positively definite.

Lemma 2.1. Let $U := (\rho, m, \mathcal{E})$ be a solution of (1.1) with $m = \rho u$ and $\mathcal{E} = \rho\left(\frac{|u|^2}{2} + e\right)$. Let $\bar{U} := (\bar{\rho}, \bar{m} = 0, \bar{\mathcal{E}})$ be the planar contact discontinuity (1.13). Then,

$$(2.16) \qquad \bar{\theta}(-\rho s)(U|\bar{U}) = \bar{\theta}R(\rho \ln \rho)(\rho|\bar{\rho}) + \bar{\theta}\rho(-\mathcal{S})(Q_1(\theta)|Q_1(\bar{\theta})) + \frac{p_e(\rho|\bar{\rho})}{\gamma - 1} + \frac{|m|^2}{2\rho}.$$

Here, since all of $\rho \mapsto \rho \ln \rho$, $\rho \mapsto p_e(\rho)$, $Q_1 \mapsto (-\mathcal{S})(Q_1)$ and $(\rho, m) \mapsto \frac{|m|^2}{2\rho}$ are strictly convex, all the terms on the right-hand side of (2.16) are positively definite.

Proof. First of all, by the definitions (1.7) and (2.12), we have

$$(2.17)\ (-\rho s)(U|\bar{U}) = R\rho \ln \rho - \rho\mathcal{S}(Q_1(\theta)) - \left(R\bar{\rho}\ln\bar{\rho} - \bar{\rho}\mathcal{S}(Q_1(\bar{\theta}))\right) - d(-\rho s)(\bar{U}) \cdot (U - \bar{U}).$$

To compute the last term above, we first note that
$$(2.18)$$
$$d(-\rho s)(\bar{U}) = \partial_\rho(-\rho s)(\bar{U})d\rho + \partial_m(-\rho s)(\bar{U})dm + \partial_\mathcal{E}(-\rho s)(\bar{U})d\mathcal{E} = -s(\bar{U})d\rho + \bar{\rho}d(-s)(\bar{U}).$$

To compute $d(-s)(\bar{U})$ above, we use the thermodynamic relation (1.6) with $e = \frac{\mathcal{E}}{\rho} - \frac{|u|^2}{2}$, that is,

$$\theta ds = pd\left(\frac{1}{\rho}\right) + d\left(\frac{\mathcal{E}}{\rho} - \frac{|u|^2}{2}\right) = -\frac{p}{\rho^2}d\rho - \frac{\mathcal{E}}{\rho^2}d\rho + \frac{1}{\rho}d\mathcal{E} - udu.$$

This together with $\bar{u} = 0$ implies

$$\bar{\theta}d(-s)(\bar{U}) = \frac{\bar{p} + \bar{\mathcal{E}}}{\bar{\rho}^2}d\rho - \frac{1}{\bar{\rho}}d\mathcal{E},$$

which together with (2.18) yields

$$\bar{\theta}d(-\rho s)(\bar{U}) = \left(\bar{\theta}(-s)(\bar{U}) + \frac{\bar{p} + \bar{\mathcal{E}}}{\bar{\rho}}\right)d\rho - d\mathcal{E},$$

Thus,

$$(2.19) \qquad \bar{\theta}d(-\rho s)(\bar{U}) \cdot (U - \bar{U}) = \left(\bar{\theta}(-s)(\bar{U}) + \frac{\bar{p} + \bar{\mathcal{E}}}{\bar{\rho}}\right)(\rho - \bar{\rho}) - (\mathcal{E} - \bar{\mathcal{E}}).$$

Therefore, we plug this into (2.17) with (1.7) to obtain

$$\bar{\theta}(-\rho s)(U|\bar{U}) = \bar{\theta}R\rho \ln \rho - \bar{\theta}\rho\mathcal{S}(Q_1(\theta)) - \bar{\theta}R\bar{\rho}\ln\bar{\rho} + \bar{\theta}\bar{\rho}\mathcal{S}(Q_1(\bar{\theta}))$$

$$+ \bar{\theta}\big(-R\ln\bar{\rho} + \mathcal{S}(Q_1(\bar{\theta}))\big)(\rho - \bar{\rho}) \underbrace{- \frac{\bar{p} + \bar{\mathcal{E}}}{\bar{\rho}}(\rho - \bar{\rho}) + (\mathcal{E} - \bar{\mathcal{E}})}_{=:J}.$$

For J, we use $\bar{\mathcal{E}} = \bar{\rho}\bar{e}$ and (1.5) with $\bar{\theta} \geq \theta_*$, to find that

$$\frac{\bar{p} + \bar{\mathcal{E}}}{\bar{\rho}} = \gamma\bar{e},$$

which yields

$$J = -\gamma\bar{e}(\rho - \bar{\rho}) + \rho e - \bar{\rho}\bar{e} + \frac{|m|^2}{2\rho}.$$

Since it follows from (1.3) and (1.4) that

$$\rho e = \frac{p_e(\rho)}{\gamma - 1} + \rho Q_1(\theta),$$

and especially,

$$\gamma\bar{e} = \frac{p'_e(\bar{\rho})}{\gamma - 1} + \gamma Q_1(\bar{\theta}),$$

we use (1.4) and (1.3) to have

$$J = -\frac{p'_e(\bar{\rho})}{\gamma - 1}(\rho - \bar{\rho}) - \gamma Q_1(\bar{\theta})(\rho - \bar{\rho}) + \frac{p_e(\rho)}{\gamma - 1} + \rho Q_1(\theta) - \frac{p_e(\bar{\rho})}{\gamma - 1} - \bar{\rho}Q_1(\bar{\theta}) + \frac{|m|^2}{2\rho}$$

$$= \frac{1}{\gamma - 1}p_e(\rho|\bar{\rho}) - \gamma Q_1(\bar{\theta})(\rho - \bar{\rho}) + \rho Q_1(\theta) - \bar{\rho}Q_1(\bar{\theta}) + \frac{|m|^2}{2\rho}.$$

Moreover, using the fact that $Q_1(\bar{\theta}) = \frac{R}{\gamma-1}\bar{\theta}$ by (1.4), we have

$$J = \frac{1}{\gamma - 1}p_e(\rho|\bar{\rho}) - R\bar{\theta}(\rho - \bar{\rho}) + \rho\big(Q_1(\theta) - Q_1(\bar{\theta})\big) + \frac{|m|^2}{2\rho}.$$

Hence, we have

$$\bar{\theta}(-\rho s)(U|\bar{U}) = \bar{\theta}R\Big(\rho\ln\rho - \bar{\rho}\ln\bar{\rho} - (\ln\bar{\rho} + 1)(\rho - \bar{\rho})\Big)$$

$$- \bar{\theta}\rho\Big(\mathcal{S}(Q_1(\theta)) - \mathcal{S}(Q_1(\bar{\theta})) - \frac{1}{\bar{\theta}}(Q_1(\theta) - Q_1(\bar{\theta}))\Big) + \frac{1}{\gamma - 1}p_e(\rho|\bar{\rho}) + \frac{|m|^2}{2\rho}.$$

Since $\mathcal{S}'(Q_1(\bar{\theta})) = 1/\bar{\theta}$ by (2.13), and $\frac{d(\rho\ln\rho)}{d\rho} = \ln\rho + 1$, we have the desired representation (2.16). $\qquad\square$

2.3. Nonlinear functionals controlled by the weighted relative entropy. We recall the convex function (1.16) and define a nonnegative function $\mathcal{G}$ as follows:

$$(2.20) \qquad \mathcal{G}(y) := \min\{|y|, y^2\}.$$

Lemma 2.2. *Let* $\bar{U} := (\bar{\rho}, \bar{m} = 0, \bar{\mathcal{E}})$ *be the contact discontinuity* (1.13)*, and* $U_\pm := (\rho_\pm, \rho_\pm u_\pm = 0, \mathcal{E}_\pm = \bar{\mathcal{E}})$*. Then, there exists a constant* $C > 0$ *such that for any solution* $U := (\rho, m, \mathcal{E})$ *of* (1.1)*,*

$$(2.21) \qquad \mathcal{F}(\rho) + \mathcal{G}(\mathcal{E} - \bar{\mathcal{E}}) + \frac{|m|^2}{2\rho} \leq C\min\Big\{\theta_-(-\rho s)(U|U_-), \theta_+(-\rho s)(U|U_+)\Big\}.$$

Proof. First, since $\gamma > 2$, there exists a constant $C > 0$ such that $|\rho - \rho_\pm|^2 \leq Cp_e(\rho|\rho_\pm)$ for all $\rho \geq 0$, which together with (1.16) implies

$$\mathcal{F}(\rho) \leq C\min\{p_e(\rho|\rho_+), p_e(\rho|\rho_-)\}, \qquad \forall\rho \geq 0.$$

Thus, it follows from (2.16) that

$$\mathcal{F}(\rho) + \frac{|m|^2}{2\rho} \leq C \min \left\{ \theta_-(-\rho s)(U|U_-), \theta_+(-\rho s)(U|U_+) \right\}.$$

Therefore, it remains to prove that

(2.22) $$\mathcal{G}(\mathcal{E} - \bar{\mathcal{E}}) \leq C \min \left\{ \theta_-(-\rho s)(U|U_-), \theta_+(-\rho s)(U|U_+) \right\}.$$

Indeed, using the following Lemma 2.3 together with the convex open set $\Omega = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ and the strict convex function $\eta = -\rho s$, we find that for each $\bar{\mathcal{E}} = \mathcal{E}_\pm$,

$$\mathcal{G}(\mathcal{E} - \bar{\mathcal{E}}) = \min \left\{ |\mathcal{E} - \bar{\mathcal{E}}|, |\mathcal{E} - \bar{\mathcal{E}}|^2 \right\} \leq \min \left\{ |U - U_\pm|, |U - U_\pm|^2 \right\}$$
$$\leq C(-\rho s)(U|U_\pm) \leq C\theta_\pm(-\rho s)(U|U_\pm),$$

which together with $\bar{\mathcal{E}} = \mathcal{E}_- = \mathcal{E}_+$ implies (2.22). $\qquad\qquad\square$

Lemma 2.3. *Let Ω is a convex open subset of $\mathbb{R}^n$. Let $\eta : \Omega \to \mathbb{R}$ be a strictly convex function. The relative function $\eta(\cdot|\cdot)$ (defined as in (2.15)) satisfies the following: For any $z_0 \in \Omega$, there exists a constant $C > 0$ such that*

$$\eta(z|z_0) \geq C \min \left\{ |z - z_0|, |z - z_0|^2 \right\}, \quad \forall z \in \Omega.$$

Proof. Since Ω is open, for any fixed $z_0 \in \Omega$ there exists a constant $r > 0$ such that

$$\{ z \mid |z - z_0| = r \} \subset \Omega.$$

For any vector $v \in \mathbb{R}^n$ such that $|v| = r$, we define a non-negative function $f_v : \mathbb{R} \to \mathbb{R}_+$ by

$$f_v(t) := \eta(z_0 + tv \mid z_0).$$

Then,

$$f_v'(t) = \Big(d\eta(z_0 + tv) - d\eta(z_0) \Big) \cdot v.$$

Since η is strictly convex on the convex set Ω, $f_v''(t) = v(d^2\eta)(z_0 + tv)v^t > 0$ on the interval $I = \{ t \mid z_0 + tv \in \Omega \}$, which implies that f_v is strictly convex on I. Since $f_v'(0) = 0$, we have

$$\Big(d\eta(z_0 + v) - d\eta(z_0) \Big) \cdot v = f_v'(1) > 0.$$

Moreover, since $v \mapsto \Big(d\eta(z_0 + v) - d\eta(z_0) \Big) \cdot v$ is continuous on the compact set $|v| = r$, there exists a constant $C(r)$ such that

$$f_v'(1) \geq C(r) \quad \forall v \text{ with } |v| = r.$$

Thus, for all v with $|v| = r$,

$$f_v'(t) \geq C(r) \quad \forall\, t \geq 1,$$

which implies that for all v with $|v| = r$,

$$\forall t \geq 2, \qquad f_v(t) = f_v(1) + \int_1^t f_v' \geq \int_1^t f_v' \geq C(r)(t - 1) \geq \frac{C(r)}{2}t.$$

Now, for any $z \in \Omega$ with $|z - z_0| \geq 2r$, by putting $v = \frac{z - z_0}{|z - z_0|}r$ and $t = \frac{|z - z_0|}{r}$ above, we have

$$\eta(z|z_0) = f_v(t) \geq \frac{C(r)}{2r}|z - z_0|.$$

On the other hand, the strict convexity of η and the definition of the relative function imply that there exists a constant $C_r > 0$ such that for any $z \in \Omega$ with $|z - z_0| \leq 2r$,

$$\eta(z|z_0) \geq C_r |z - z_0|^2.$$

Those two estimates imply the desired estimate. $\qquad\square$

3. Proof of Theorem 1.1

As explained in Section 1.3, we will construct the ψ function transported by the velocity field, with regularizations.

3.1. **Construction of shift functions.** For the velocity $u^{\kappa,\nu}$ being the weak solution to (1.1), we first consider a family of spatial-mollifications of u as

$$\bar{u}_\delta^{\kappa,\nu}(x,t) := \int_{\mathbb{R}^3} u^{\kappa,\nu}(x-y,t)\eta_\delta(y)dy,$$

where η_δ denotes the mollifier defined by

$$(3.1) \qquad\qquad \eta_\delta(x) := \frac{1}{\delta^3}\eta\Big(\frac{|x|}{\delta}\Big) \quad \text{for any } \delta > 0,$$

for a non-negative smooth function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} \eta = 1$.
Note that since u is periodic in x, so is $\bar{u}_\delta^{\kappa,\nu}$. Indeed, for each $i = 1, 2, 3$,

$$\bar{u}_\delta^{\kappa,\nu}(x+e_i,t) = \int_{\mathbb{R}^3} \bar{u}^{\kappa,\nu}(x+e_i-y,t)\eta_\delta(y)dy = \int_{\mathbb{R}^3} \bar{u}^{\kappa,\nu}(x-y,t)\eta_\delta(y)dy = \bar{u}_\delta^{\kappa,\nu}(x,t).$$

For the smooth velocity field $\bar{u}_\delta^{\kappa,\nu}$, we define $\psi_\delta^{\kappa,\nu}$ as the unique solution of the following transport equation:

$$(3.2) \qquad \begin{cases} \partial_t\psi + \bar{u}_\delta^{\kappa,\nu} \cdot \nabla\psi = 0, \\ \psi(x,0) =: \psi_0(x) = \begin{cases} 0, & \text{if } 0 < x_1 < \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} < x_1 < 1, \end{cases} \quad x = (x_1, x_2, x_3) \in \mathbb{T}^3. \end{cases}$$

For any fixed $(x,t) \in \mathbb{R}^3 \times [0,T]$, we define a characteristic curve $X(\tau;x,t)$ generated by $u_\delta^{\kappa,\nu}$, passing through x at $\tau = t$ as follows:

$$(3.3) \qquad \begin{cases} \dfrac{d}{d\tau}X(\tau;x,t) = \bar{u}_\delta^{\kappa,\nu}(X(\tau;x,t),\tau), \\ X(\tau = t;x,t) = x. \end{cases}$$

Note that since $\bar{u}_\delta^{\kappa,\nu}$ is smooth in x and $\|\bar{u}_\delta^{\kappa,\nu}\|_{C^1(\mathbb{T}^3)} \in L^2(0,T)$, the above ODE has a unique absolutely continuous solution $X(\tau;x,t)$ on $\tau \in [0,T]$.
Then, since it follows from (3.2) and (3.3) that

$$(3.4) \qquad\qquad \psi_\delta^{\kappa,\nu}(x,t) = \psi_0(X(0;x,t)),$$

it holds that

$$(3.5) \qquad\qquad 0 \leq \psi_\delta^{\kappa,\nu}(x,t) \leq 1, \quad \forall(x,t) \in \mathbb{T}^3 \times [0,T].$$

Remark 3.1. *Note that the solution $\psi_\delta^{\kappa,\nu}(x,t)$ is periodic in x. Indeed, since $\bar{u}_\delta^{\kappa,\nu}$ is periodic in $x \in \mathbb{T}^3$, it follows from (3.3) that for each $i = 1, 2, 3$,*

$$\frac{d}{d\tau}\big(X(\tau;x+e_i,t)-e_i\big) = \frac{d}{d\tau}X(\tau;x+e_i,t) = \bar{u}_\delta^{\kappa,\nu}(X(\tau;x+e_i,t),\tau) = \bar{u}_\delta^{\kappa,\nu}\big(X(\tau;x+e_i,t)-e_i,\tau\big),$$

where e_i $(i = 1, 2, 3)$ is the standard unit vector. Thus, we find that for each $i = 1, 2, 3$,

$$\begin{cases} \dfrac{d}{d\tau}\big(X(\tau; x + e_i, t) - e_i\big) = \bar{u}_\delta^{\kappa,\nu}\big(X(\tau; x + e_i, t) - e_i, \tau\big), \\ \big(X(\tau; x + e_i, t) - e_i\big)|_{\tau=t} = x. \end{cases}$$

Therefore, by the uniqueness of solutions to (3.3), for each $i = 1, 2, 3$,

$$X(\tau; x, t) = X(\tau; x + e_i, t) - e_i, \quad \forall \tau \in [0, T].$$

This together with the periodicity of ψ_0 implies

$$\psi_\delta^{\kappa,\nu}(x + e_i, t) = \psi_0(X(0; x + e_i, t)) = \psi_0(X(0; x + e_i, t) - e_i) = \psi_0(X(0; x, t)) = \psi_\delta^{\kappa,\nu}(x, t).$$

Hence, $\psi_\delta^{\kappa,\nu}(x, t)$ is periodic in $x \in \mathbb{T}^3$.

Since the solution $\psi_\delta^{\kappa,\nu}$ to the equation (3.2) is still discontinuous, we consider a family of mollifications of $\psi_\delta^{\kappa,\nu}$ as

$$(3.6) \qquad \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}(x, t) := \int_{\mathbb{R}^3} \psi_\delta^{\kappa,\nu}(x - y, t)\eta_\varepsilon(y)dy,$$

where η_ε denotes the mollifier defined by in (3.1) with the parameter δ replaced by another parameter ε.

Note that by (3.5),

$$(3.7) \qquad 0 \le \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}(x, t) \le 1, \quad \forall(x, t), \ \forall \varepsilon > 0.$$

3.2. Weighted entropy inequality.

Lemma 3.1. *Let $U := (\rho, m, \mathcal{E})$ be a solution of (1.1) in the sense of Definition 2.1, and $\Psi := \theta_-\big(1 - \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\big) + \theta_+\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}$. Then,*

$$\begin{aligned} (3.8) \quad & \int_{\mathbb{T}^3}(-\rho s)(U(x, t))\Psi(x, t)dx - \int_{\mathbb{T}^3}(-\rho s)(U_0)\Psi(x, 0)dx \\ & + \min\{\theta_+, \theta_-\}\int_0^t \int_{\mathbb{T}^3}\Big(\kappa\frac{|\nabla\theta|^2}{\theta^2} + \frac{\mathbb{S} : \nabla u}{\theta}\Big)(x, \tau)dxd\tau \\ & + \kappa(\theta_+ - \theta_-)\int_0^t \int_{\mathbb{T}^3}\frac{\nabla\theta}{\theta} \cdot \nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}dxd\tau \\ & + (\theta_+ - \theta_-)\int_0^t \int_{\mathbb{T}^3}(\rho s)(U(x, \tau))\Big(\partial_\tau\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} + u^{\kappa,\nu} \cdot \nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\Big)(x, \tau)dxd\tau \le 0 \end{aligned}$$

Proof. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-negative smooth function such that $\phi(s) = \phi(-s)$, $\int_\mathbb{R}\phi = 1$ and $\text{supp }\phi = [-1, 1]$, and let

$$\phi_\zeta(s) := \frac{1}{\zeta}\phi\big(\frac{s - \zeta}{\zeta}\big) \quad \text{for any } \zeta > 0.$$

Then for a given $t \in (0, T)$, and any $\zeta < t/2$, we define a non-negative smooth function

$$(3.9) \qquad \varphi_{t,\zeta}(s) := \int_0^s \Big(\phi_\zeta(z) - \phi_\zeta(z - t)\Big)dz.$$

For test functions of the entropy inequality (2.6), we consider a family of non-negative functions

$$\vec{\varphi}(x, \tau) := \Psi(x, \tau)\varphi_{t,\zeta}(\tau),$$

where $\Psi := \theta_- \left(1 - \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\right) + \theta_+ \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}$.

Then, it follows from (2.6) and $\varphi_{t,\zeta}(0) = 0$ that

$$\int_0^\infty \int_{\mathbb{T}^3} (-\rho s)(U(x,\tau))\Psi(x,\tau)\left(-\varphi_{t,\zeta}'(\tau)\right)dxd\tau + J_\zeta^1 + J_\zeta^2 + J_\zeta^3 \leq 0,$$

where

$$J_\zeta^1 := \int_0^\infty \int_{\mathbb{T}^3} \left(\kappa \frac{|\nabla\theta|^2}{\theta^2} + \nu \frac{\mathbb{S} : \nabla_x u}{\theta}\right)(x,\tau)\Psi(x,\tau)\varphi_{t,\zeta}(\tau)dxd\tau,$$

$$J_\zeta^2 := -\int_0^\infty \int_{\mathbb{T}^3} \kappa \frac{\nabla\theta}{\theta} \cdot \nabla\Psi(x,\tau)\varphi_{t,\zeta}(\tau)dxd\tau,$$

$$J_\zeta^3 := -(\theta_+ - \theta_-)\int_0^\infty \int_{\mathbb{T}^3} (-\rho s)(U(x,\tau))\left(\partial_\tau \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} + u \cdot \nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\right)(x,\tau)\varphi_{t,\zeta}(\tau)dxd\tau.$$

First, since $\Psi \in C([0,T]; C^\infty(\mathbb{T}^3))$ and $\varphi_{t,\zeta}'(\tau) = \phi_\zeta(\tau) - \phi_\zeta(\tau - t)$, (2.7) and (2.8) implies that as $\zeta \to 0$,

$$\int_0^\infty \int_{\mathbb{T}^3} (-\rho s)(U(x,\tau))\Psi(x,\tau)\left(-\varphi_{t,\zeta}'(\tau)\right)dxd\tau \to \int_{\mathbb{T}^3} (-\rho s)(U(x,t))\Psi(x,t)dx - \int_{\mathbb{T}^3} (-\rho s)(U_0)\Psi(x,0)dx.$$

Likewise, as $\zeta \to 0$,

$$J_\zeta^1 \to \int_0^t \int_{\mathbb{T}^3} \left(\kappa \frac{|\nabla\theta|^2}{\theta^2} + \nu \frac{\mathbb{S} : \nabla u}{\theta}\right)(x,\tau)\Psi(x,\tau)dxd\tau =: J_1,$$

$$J_\zeta^2 \to -\int_0^t \int_{\mathbb{T}^3} \kappa \frac{\nabla\theta}{\theta} \cdot \nabla\Psi(x,\tau)dxd\tau =: J_2,$$

$$J_\zeta^3 \to -(\theta_+ - \theta_-)\int_0^t \int_{\mathbb{T}^3} (-\rho s)(U(x,\tau))\left(\partial_\tau \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} + u \cdot \nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\right)(x,\tau)dxd\tau.$$

In particular, using the assumptions (1.2) together with (3.7), we can show

$$J_1 = \int_0^t \int_{\mathbb{T}^3} \left(\kappa \frac{|\nabla\theta|^2}{\theta^2} + \nu|\nabla u|^2\right)(x,\tau)\Psi(x,\tau)dxd\tau$$

$$\geq \min\{\theta_+, \theta_-\} \int_0^t \int_{\mathbb{T}^3} \left(\kappa \frac{|\nabla\theta|^2}{\theta^2} + \nu|\nabla u|^2\right)(x,\tau)dxd\tau.$$

Moreover, since

$$J_2 = \kappa(\theta_+ - \theta_-)\int_0^t \int_{\mathbb{T}^3} \frac{\nabla\theta}{\theta} \cdot \nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} dxd\tau,$$

we have the desired estimate. □

3.3. Weighted relative entropy inequality.

Lemma 3.2. *Let $\bar{U} := (\bar{\rho}, \bar{m}, \bar{\mathcal{E}})$ be the planar contact discontinuity (1.13). Then, there exist constants $C > 0$, C_1, C_2 such that for a solution $U := (\rho, m, \mathcal{E})$ of (1.1) in the sense*

of Definition 2.1, we have

$$\int_{\mathbb{T}^3} \left[\left(1 - \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\right)\theta_-(-\rho s)(U|U_-) + \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\theta_+(-\rho s)(U|U_+)\right](x,t)dx$$

$$+ \min\{\theta_+, \theta_-\} \int_0^t \int_{\mathbb{T}^3} \left(\kappa\frac{|\nabla\theta|^2}{\theta^2} + \nu|\nabla u|^2\right)(x,\tau)dxd\tau$$

(3.10)

$$\leq C\varepsilon + C\int_{\mathbb{T}^3} (-\rho s)(U_0|\bar{U})dx + \kappa C\int_0^t \int_{\mathbb{T}^3} \frac{|\nabla\theta|}{\theta}|\nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}|dxd\tau$$

$$+ \int_0^t \int_{\mathbb{T}^3} \left(C_1\rho + C_2(-\rho s)(U)\right)\left(\partial_\tau\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} + u^{\kappa,\nu}\cdot\nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\right)(x,\tau)dxd\tau.$$

Proof. First, using the definition of the relative functional, we have

$$L(t) := \int_{\mathbb{T}^3} \left[\left(1 - \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\right)\theta_-(-\rho s)(U|U_-) + \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\theta_+(-\rho s)(U|U_+)\right](x,t)dx$$

$$= \int_{\mathbb{T}^3} \Psi(x,t)(-\rho s)(U(x,t))dx$$

$$\underbrace{- \int_{\mathbb{T}^3} \left(1 - \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\right)\left(\theta_-(-\rho s)(U_-) + \theta_- d(-\rho s)(U_-)\cdot(U - U_-)\right)dx}_{=:J_1}$$

$$\underbrace{- \int_{\mathbb{T}^3} \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\left(\theta_+(-\rho s)(U_+) + \theta_+ d(-\rho s)(U_+)\cdot(U - U_+)\right)dx}_{=:J_2}.$$

Then, using (2.19),

$$J_1 = \int_{\mathbb{T}^3} \left(1 - \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\right)\left[\theta_-\rho_-(-s)(U_-) + \left(\theta_-(-s)(U_-) + \frac{p_- + \mathcal{E}_-}{\rho_-}\right)(\rho - \rho_-) - (\mathcal{E} - \mathcal{E}_-)\right]dx$$

$$= \int_{\mathbb{T}^3} \left(1 - \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\right)\left[\left(\theta_-(-s)(U_-) + \frac{p_- + \mathcal{E}_-}{\rho_-}\right)\rho - p_- - \mathcal{E}\right]dx.$$

Likewise,

$$J_2 = \int_{\mathbb{T}^3} \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\left[\left(\theta_+(-s)(U_+) + \frac{p_+ + \mathcal{E}_+}{\rho_+}\right)\rho - p_+ - \mathcal{E}\right]dx.$$

Set

$$\beta_- := \theta_-(-s)(U_-) + \frac{p_- + \mathcal{E}_-}{\rho_-}, \qquad \beta_+ := \theta_+(-s)(U_+) + \frac{p_+ + \mathcal{E}_+}{\rho_+}.$$

Thus, using $\bar{p} = p_- = p_+$, we have

(3.11)

$$L(t) = \int_{\mathbb{T}^3} \Psi(x,t)(-\rho s)(U(x,t))dx - \int_{\mathbb{T}^3} \left[\left(\left(1 - \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\right)\beta_- + \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\beta_+\right)\rho - \bar{p} - \mathcal{E}\right](x,t)dx$$

$$=: M(t), \qquad \forall t \geq 0.$$

Since $\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} \in C([0,T]; C^\infty(\mathbb{T}^3))$, the initial conditions (2.2), (2.5), (2.7) and (2.8) imply

$$M(0) = \lim_{t\to 0+} M(t)$$

$$= \int_{\mathbb{T}^3} \Psi(x,0)(-\rho s)(U_0)dx - \int_{\mathbb{T}^3} \left[\left(\left(1 - \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}(x,0)\right)\beta_- + \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}(x,0)\beta_+\right)\rho_0 - \bar{p} - \mathcal{E}_0\right]dx,$$

which together with (3.11) yields

$$M(0) = \int_{\mathbb{T}^3} \Psi(x,0)(-\rho s)(U_0)dx - \int_{\mathbb{T}^3} \left[\left((1-\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu})\beta_- + \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\beta_+\right)\rho_0 - \bar{p} - \mathcal{E}_0\right]dx$$

$$= \int_{\mathbb{T}^3} \left[(1-\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}(x,0))\theta_-(-\rho s)(U_0|U_-) + \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}(x,0)\theta_+(-\rho s)(U_0|U_+)\right]dx$$

$$=: L_0.$$

Thus, we have

$$L(t) - L_0 = M(t) - M(0)$$

$$= \int_{\mathbb{T}^3} \left[\Psi(x,t)(-\rho s)(U(x,t)) - \Psi(0,t)(-\rho s)(U_0)\right]dx$$

$$- \int_{\mathbb{T}^3} \left[\rho\left((1-\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu})\beta_- + \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\beta_+\right)(x,t) - \rho_0\left((1-\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu})\beta_- + \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\beta_+\right)(x,0)\right]dx$$

$$+ \int_{\mathbb{T}^3} (\mathcal{E} - \mathcal{E}_0)dx =: R_1(t) + R_2(t) + R_3(t).$$

First, by (2.4), $R_3(t) \leq 0$.

To handle $R_2(t)$, we consider the following test functions for (2.1):

$$\varphi(x,\tau) := \left((1-\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu})\beta_- + \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\beta_+\right)(x,\tau)\varphi_{t,\zeta}(\tau),$$

where $\varphi_{t,\zeta}$ are as in (3.9). Plugging the above test functions into (2.1), and taking $\zeta \to 0$ together with the same argument as before, we have

$$R_2(t) = (\beta_+ - \beta_-)\int_0^t \int_{\mathbb{T}^3} \rho(x,\tau)\left(\partial_\tau \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} + u \cdot \nabla \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\right)(x,\tau)dxd\tau$$

Therefore, using (3.8), we have

$$L(t) \leq L_0 - \min\{\theta_+, \theta_-\}\int_0^t \int_{\mathbb{T}^3} \left(\kappa\frac{|\nabla\theta|^2}{\theta^2} + \nu|\nabla u|^2\right)(x,\tau)dxd\tau$$

$$+ \kappa|\theta_+ - \theta_-|\int_0^t \int_{\mathbb{T}^3} \frac{|\nabla\theta|}{\theta}|\nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}|dxd\tau$$

$$+ \int_0^t \int_{\mathbb{T}^3} \left(C_1\rho + C_2(-\rho s)(U)\right)\left(\partial_\tau \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} + u \cdot \nabla \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\right)(x,\tau)dxd\tau,$$

where $C_1 := \beta_+ - \beta_-$, $C_2 := \theta_+ - \theta_-$.

Since $\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}|_{t=0} = \psi_0 * \eta_\varepsilon$ by (3.4), it follows from (1.17) that

$$L_0 = \int_{\mathbb{T}^3} \left[(1-(\psi_0 * \eta_\varepsilon))\theta_-(-\rho s)(U_0|U_-) + (\psi_0 * \eta_\varepsilon)\theta_+(-\rho s)(U_0|U_+)\right]dx$$

$$\leq C \int_{\mathbb{T}^3} (-\rho s)(U_0|\bar{U})dx + C \int_{1/2-\varepsilon}^{1/2+\varepsilon} (-\rho s)(U_0|\bar{U})dx$$

$$\leq C \int_{\mathbb{T}^3} (-\rho s)(U_0|\bar{U})dx + C\varepsilon.$$

$\square$

Note first that since the velocity u depends on κ, ν, $\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}$ depends on $\kappa, \delta, \varepsilon, \nu$ by the definition (3.2). Thus, the left-hand side of the estimate (3.10) depends on $\kappa, \delta, \varepsilon, \nu$. Therefore, before performing the limit process for (3.10), we may first use Lemma 2.2 to ensure that the left-hand side of (3.10) controls a new nonlinear functional independent of those parameters $\kappa, \delta, \varepsilon, \nu$. Indeed, using Lemmas 2.2 and 3.2 together with (3.7), we have the following Proposition.

Proposition 3.1. *Under the same hypotheses as in Lemma 2.2, there exist constants $C > 0$, C_1, C_2 such that for a solution $U^{\kappa,\nu} := (\rho, m, \mathcal{E})$ of (1.1),*

(3.12)
$$
\int_{\mathbb{T}^3} \left(\mathcal{F}(\rho^{\kappa,\nu}) + \mathcal{G}(\mathcal{E}^{\kappa,\nu} - \bar{\mathcal{E}}) + \frac{|m^{\kappa,\nu}|^2}{2\rho^{\kappa,\nu}} \right) dx + \int_0^t \int_{\mathbb{T}^3} \left(\kappa \frac{|\nabla\theta^{\kappa,\nu}|^2}{(\theta^{\kappa,\nu})^2} + \nu|\nabla u^{\kappa,\nu}|^2 \right)(x,\tau)dxd\tau
$$
$$
\leq C\varepsilon + C \int_{\mathbb{T}^3} (-\rho s)(U_0|\bar{U})dx + \kappa C \int_0^t \int_{\mathbb{T}^3} \frac{|\nabla\theta^{\kappa,\nu}|}{\theta^{\kappa,\nu}} |\nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}| dxd\tau
$$
$$
+ \int_0^t \int_{\mathbb{T}^3} \left(C_1 \rho^{\kappa,\nu} + C_2(-\rho s)(U^{\kappa,\nu}) \right) \left(\partial_\tau \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} + u^{\kappa,\nu} \cdot \nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} \right)(x,\tau)dxd\tau.
$$

The remaining part of the proof is dedicated to the asymptotic analysis on passing to the limits for the parameters $\kappa, \delta, \varepsilon$ and ν in order.

3.4. **Passing to the limit as $\kappa \to 0$.** First of all, since Young's inequality yields

$$
\kappa C \int_0^t \int_{\mathbb{T}^3} \frac{|\nabla\theta^{\kappa,\nu}|}{\theta^{\kappa,\nu}} |\nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}| dxd\tau \leq \frac{\kappa}{2} \int_0^t \int_{\mathbb{T}^3} \frac{|\nabla\theta^{\kappa,\nu}|^2}{(\theta^{\kappa,\nu})^2} dxd\tau + \kappa C \int_0^t \int_{\mathbb{T}^3} |\nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}|^2 dxd\tau,
$$

it follows from (3.12) that

(3.13)
$$
\int_{\mathbb{T}^3} \left(\mathcal{F}(\rho^{\kappa,\nu}) + \mathcal{G}(\mathcal{E}^{\kappa,\nu} - \bar{\mathcal{E}}) + \frac{|m^{\kappa,\nu}|^2}{2\rho^{\kappa,\nu}} \right) dx
$$
$$
\leq C\varepsilon + C \int_{\mathbb{T}^3} (-\rho s)(U_0|\bar{U})dx + J_1 + J_2.
$$

where

$$
J_1 := \kappa C \int_0^t \int_{\mathbb{T}^3} |\nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}|^2 dxd\tau,
$$
$$
J_2 := \int_0^t \int_{\mathbb{T}^3} \left(C_1 \rho^{\kappa,\nu} + C_2(-\rho s)(U^{\kappa,\nu}) \right) \left(\partial_\tau \bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} + u^{\kappa,\nu} \cdot \nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} \right) dxd\tau.
$$

For J_1, using $\nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} = \psi_\delta^{\kappa,\nu} * \nabla\eta_\varepsilon$ by (3.6), and $0 \leq \psi_\delta^{\kappa,\nu} \leq 1$ by (3.5), we have

$$
\int_0^t \int_{\mathbb{T}^3} |\nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}|^2 dxd\tau \leq C(\varepsilon) \quad \text{independent of } \kappa,
$$

from which,

$$
J_1 \to 0 \quad \text{as } \kappa \to 0.
$$

For J_2, we will use the following uniform bound :

(3.14) $$\|u^{\kappa,\nu}\|_{L^2(0,T;H^1(\mathbb{T}^3))} \leq C \quad \text{(independent of } \kappa, \delta, \varepsilon).$$

This follows from (2.11) and

$$
\|u^{\kappa,\nu}\|_{L^2(\mathbb{T}^3)}^2 \leq C \left[\|\nabla u^{\kappa,\nu}\|_{L^2(\mathbb{T}^3)} + \|\rho^{\kappa,\nu}\|_{L^1(\mathbb{T}^3)} \|\sqrt{\rho^{\kappa,\nu}} u^{\kappa,\nu}\|_{L^2(\mathbb{T}^3)}^2 \right] \leq C,
$$

which is obtained by Lemma 3.3 together with (2.11), (2.4), (2.9).

Lemma 3.3. [18, Lemma 3.2] *Let $v \in W^{1,2}(\Omega)$ for a bounded domain $\Omega \subset \mathbb{R}^N$, and ρ be a non-negative function such that*

$$0 < M \le \int_\Omega \rho \, dx, \quad \int_\Omega \rho^\gamma dx \le E_0,$$

where M, E_0 and $\gamma > 1$ are some constants. Then there exists a constant $C = C(M, E_0)$ such that

$$\|v\|_{L^2(\Omega)}^2 \le C \left(\|\nabla_x v\|_{L^2(\Omega)}^2 + \left(\int_\Omega \rho |v| dx \right) \right).$$

By (3.14), there exists $u^\nu \in L^2(0, T; H^1(\mathbb{T}^3))$ such that

(3.15) $$u^{\kappa, \nu} \rightharpoonup u^\nu \quad \text{weakly in } L^2(0, T; H^1(\mathbb{T}^3)) \quad \text{as } \kappa \to 0.$$

Note that it follows from the uniform bound (3.5), there exists ψ_δ^ν such that

(3.16) $$0 \le \psi_\delta^\nu \le 1$$

and

(3.17) $$\psi_\delta^{\kappa, \nu} \rightharpoonup \psi_\delta^\nu \quad \text{weakly-* in } L^\infty((0, T) \times \mathbb{T}^3) \quad \text{as } \kappa \to 0.$$

To get the desired limit from J_2, we need to rewrite $\partial_\tau \bar{\psi}_{\delta, \varepsilon}^{\kappa, \nu} + u^{\kappa, \nu} \cdot \nabla \bar{\psi}_{\delta, \varepsilon}^{\kappa, \nu}$ as follows: First, take the mollification on the equation (3.2) with the mollifier η_ε, to have

(3.18) $$\partial_t \bar{\psi}_{\delta, \varepsilon}^{\kappa, \nu} + (\overline{\bar{u}_\delta^{\kappa, \nu} \cdot \nabla \psi_\delta^{\kappa, \nu}})_\varepsilon = 0,$$

from which, we have

$$\partial_t \bar{\psi}_{\delta, \varepsilon}^{\kappa, \nu} + u^{\kappa, \nu} \cdot \nabla \bar{\psi}_{\delta, \varepsilon}^{\kappa, \nu} = R_{\delta, \varepsilon}^{\kappa, \nu},$$

where

$$R_{\delta, \varepsilon}^{\kappa, \nu} := u^{\kappa, \nu} \cdot \nabla \bar{\psi}_{\delta, \varepsilon}^{\kappa, \nu} - (\overline{\bar{u}_\delta^{\kappa, \nu} \cdot \nabla \psi_\delta^{\kappa, \nu}})_\varepsilon.$$

equivalently,

$$R_{\delta, \varepsilon}^{\kappa, \nu} := u^{\kappa, \nu} \cdot \nabla \bar{\psi}_{\delta, \varepsilon}^{\kappa, \nu} - \left(\overline{\text{div}(\bar{u}_\delta^{\kappa, \nu} \psi_\delta^{\kappa, \nu})} \right)_\varepsilon + \left(\overline{(\text{div} \bar{u}_\delta^{\kappa, \nu}) \psi_\delta^{\kappa, \nu}} \right)_\varepsilon.$$

By using the above representation, we rewrite J_2 as

$$J_2 := \int_0^t \int_{\mathbb{T}^3} \left(C_1 \rho^{\kappa, \nu} + C_2(-\rho s)(U^{\kappa, \nu}) \right) R_{\delta, \varepsilon}^{\kappa, \nu} dx d\tau,$$

We will first show that there exists $G^\nu \in L^\infty(0, T; L^2(\mathbb{T}^3))$ such that

$$C_1 \rho^{\kappa, \nu} + C_2(-\rho s)(U^{\kappa, \nu}) \to G^\nu \quad \text{strongly in } L^2(0, T; H^{-1}(\mathbb{T}^3)) \quad \text{as } \kappa \to 0.$$

For that, we may use the Aubin-Lions lemma (see [18, Lemma 6.3]):

Lemma 3.4. *Let $\{v_n\}_{n=1}^\infty$ be a sequence of functions such that v_n are uniformly bounded in $L^2(0, T; L^q(\Omega)) \cap L^\infty(0, T; L^1(\Omega))$ with $q > \frac{2N}{N+1}$. Furthermore, assume that*

$$\partial_t v_n \ge g_n, \quad \text{in } \mathcal{D}'((0, T) \times \Omega)$$

where g_n are uniformly bounded in $L^1(0, T; W^{-m,r}(\Omega))$ with $m \ge 1$ and $r > 1$. Then $\{v_n\}_{n=1}^\infty$ contains a subsequence such that

$$v_n \to v \quad \text{in } L^2(0, T; H^{-1}(\Omega)) \quad \text{strongly.}$$

In order to apply Lemma 3.4 to the entropy inequality (2.6):

$$(\rho^{\kappa,\nu}s^{\kappa,\nu})_t + \mathrm{div}(\rho^{\kappa,\nu}u^{\kappa,\nu}s^{\kappa,\nu}) - \kappa\,\mathrm{div}\Big(\frac{\nabla\theta^{\kappa,\nu}}{\theta^{\kappa,\nu}}\Big) + \kappa\frac{|\nabla\theta^{\kappa,\nu}|^2}{(\theta^{\kappa,\nu})^2} + \nu\frac{\mathbb{S}:\nabla u^{\kappa,\nu}}{\theta^{\kappa,\nu}} \geq 0 \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathbb{T}^3),$$

we set

$$(\rho^{\kappa,\nu}s^{\kappa,\nu})_t \geq -\mathrm{div}(\rho^{\kappa,\nu}u^{\kappa,\nu}s^{\kappa,\nu}) + \kappa\,\mathrm{div}\Big(\frac{\nabla\theta^{\kappa,\nu}}{\theta^{\kappa,\nu}}\Big) - \kappa\frac{|\nabla\theta^{\kappa,\nu}|^2}{(\theta^{\kappa,\nu})^2} - \nu\frac{\mathbb{S}:\nabla u^{\kappa,\nu}}{\theta^{\kappa,\nu}} := \sum_{i=1}^{4} g_i^{\kappa}.$$

Since it follows from (2.10) and (3.14) that $\rho^{\kappa,\nu}s^{\kappa,\nu}$ and $u^{\kappa,\nu}$ are respectively uniformly bounded in $L^2(0,T;L^2(\mathbb{T}^3))$ and $L^2(0,T;L^6(\mathbb{T}^3))$, g_1^{κ} is uniformly bounded in $L^2(0,T;W^{-1,\frac{3}{2}}(\mathbb{T}^3))$. Since $\sqrt{\kappa}\frac{\nabla\theta^{\kappa,\nu}}{\theta^{\kappa,\nu}}$ is uniformly bounded in $L^2(0,T;L^2(\mathbb{T}^3))$ by (2.11), g_2^{κ} is uniformly bounded in $L^2(0,T;W^{-1,2}(\mathbb{T}^3))$. Moreover, $g_3^{\kappa} + g_4^{\kappa}$ is uniformly bounded in $L^1(0,T;L^1(\mathbb{T}^3)) \hookrightarrow L^1(0,T;W^{-1,\frac{3}{2}}(\mathbb{T}^3))$. Therefore, by the Aubin-Lions lemma, $\{\rho^{\kappa,\nu}s^{\kappa,\nu}\}_{\kappa>0}$ is pre-compact in $L^2(0,T;W^{-1,2}(\mathbb{T}^3))$. Thus, there exists G_*^{ν} such that

$$(-\rho s)(U^{\kappa,\nu}) \to G_*^{\nu} \quad \text{strongly in} \quad L^2(0,T;H^{-1}(\mathbb{T}^3)) \quad \text{as } \kappa \to 0,$$

in addition, by (2.10),

$$G_*^{\nu} \in L^{\infty}(0,T;L^2(\mathbb{T}^3)).$$

Likewise, applying the Aubin-Lions lemma to the continuity equation:

$$\partial_t\rho^{\kappa,\nu} + \mathrm{div}(\rho^{\kappa,\nu}u^{\kappa,\nu}) = 0, \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathbb{T}^3),$$

and using (2.9) and (3.14), we obtain that there exists $\rho^{\nu} \in L^{\infty}(0,T;L^2(\mathbb{T}^3))$ such that

$$(3.19) \qquad \rho^{\kappa,\nu} \to \rho^{\nu} \quad \text{strongly in} \quad L^2(0,T;H^{-1}(\mathbb{T}^3)) \quad \text{as } \kappa \to 0.$$

Thus, putting $G^{\nu} := C_1\rho^{\nu} + C_2 G_*^{\nu}$, we have

$$(3.20) \qquad \|G^{\nu}\|_{L^{\infty}(0,T;L^2(\mathbb{T}^3))} \leq C, \qquad \text{(independent of } \varepsilon, \delta)$$

and

$$(3.21) \qquad C_1\rho^{\kappa,\nu} + C_2(-\rho s)(U^{\kappa,\nu}) \to G^{\nu} \quad \text{strongly in} \quad L^2(0,T;H^{-1}(\mathbb{T}^3)) \quad \text{as } \kappa \to 0.$$

Next, we will show that

$$(3.22) \qquad R_{\delta,\varepsilon}^{\kappa,\nu} \rightharpoonup R_{\delta,\varepsilon}^{\nu} \quad \text{weakly in} \quad L^2(0,T;H^1(\mathbb{T}^3)) \quad \text{as } \kappa \to 0,$$

where

$$(3.23) \qquad R_{\delta,\varepsilon}^{\nu} := u^{\nu} \cdot \nabla\bar{\psi}_{\delta,\varepsilon}^{\nu} - \big(\overline{\mathrm{div}(\bar{u}_{\delta}^{\nu}\psi_{\delta}^{\nu})}\big)_{\varepsilon} + \big(\overline{(\mathrm{div}\bar{u}_{\delta}^{\nu})\psi_{\delta}^{\nu}}\big)_{\varepsilon}.$$

For that, we may derive a strong convergence of $\psi_{\delta}^{\kappa,\nu}$. Since the equation (3.2) can be rewritten as

$$\partial_t\psi_{\delta}^{\kappa,\nu} = -\mathrm{div}(\bar{u}_{\delta}^{\kappa,\nu}\psi_{\delta}^{\kappa,\nu}) + (\mathrm{div}\bar{u}_{\delta}^{\kappa,\nu})\psi_{\delta}^{\kappa,\nu},$$

the uniform bounds (3.5) and (3.14) imply that $\{\partial_t\psi_{\delta}^{\kappa,\nu}\}_{\kappa>0}$ is bounded in $L^2(0,T;H^{-1}(\mathbb{T}^3))$. Then, this together with the bound (3.5) and the Aubin-Lions Lemma implies

$$(3.24) \qquad \psi_{\delta}^{\kappa,\nu} \to \psi_{\delta}^{\nu} \quad \text{in} \quad L^2(0,T;H^{-1}(\mathbb{T}^3)) \quad \text{as } \kappa \to 0.$$

Moreover, note that (3.14) with the mollification implies

$$(3.25) \qquad \|\bar{u}_{\delta}^{\kappa,\nu}\|_{L^2(0,T;H^2(\mathbb{T}^3))} \leq C \quad \text{(independent of } \kappa)\,.$$

Then, using the following Lemma 3.5 together with (3.14), (3.5),(3.24) and (3.25), we have

$$\left(\overline{\mathrm{div}(\bar{u}_\delta^{\kappa,\nu}\psi_\delta^{\kappa,\nu})}\right)_\varepsilon \rightharpoonup \left(\overline{\mathrm{div}(\bar{u}_\delta^\nu\psi_\delta^\nu)}\right)_\varepsilon \quad \text{weakly} \quad \text{in} \quad L^2(0,T;H^1(\mathbb{T}^3)),$$

$$\left(\overline{((\mathrm{div}\bar{u}_\delta^{\kappa,\nu})\psi_\delta^{\kappa,\nu})}\right)_\varepsilon \rightharpoonup \left(\overline{((\mathrm{div}\bar{u}_\delta^\nu)\psi_\delta^\nu)}\right)_\varepsilon \quad \text{weakly} \quad \text{in} \quad L^2(0,T;H^1(\mathbb{T}^3)).$$

On the other hand, since (3.5) and the mollification yield

$$\|\nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\|_{L^\infty(0,T;L^\infty(\mathbb{T}^3))} \leq C \quad \text{(independent of } \kappa) ,$$

which together with (3.14) implies

(3.26) $$\|u^{\kappa,\nu} \cdot \nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu}\|_{L^2(0,T;H^1(\mathbb{T}^3))} \leq C \quad \text{(independent of } \kappa) .$$

Moreover, since (3.24) with the mollification implies

$$\nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} \to \nabla\bar{\psi}_{\delta,\varepsilon}^\nu \quad \text{in} \quad L^2(0,T;H^1(\mathbb{T}^3)) \quad \text{as } \kappa \to 0,$$

this together with (3.15) and (3.26) implies

$$u^{\kappa,\nu} \cdot \nabla\bar{\psi}_{\delta,\varepsilon}^{\kappa,\nu} \rightharpoonup u^\nu \cdot \nabla\bar{\psi}_{\delta,\varepsilon}^\nu \quad \text{weakly} \quad \text{in} \quad L^2(0,T;H^1(\mathbb{T}^3)) \quad \text{as } \kappa \to 0.$$

Therefore, we obtain the desired convergence (3.22).

Lemma 3.5. *Assume that $\{u_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\}_{n\in\mathbb{N}}$ are sequences such that one of the following holds:*

(i) $u_n \to u$ *in* $L^2(0,T;H^1(\mathbb{T}^3))$ *and* $\psi_n \rightharpoonup \psi$ *weakly* $- *$ *in* $L^\infty((0,T)\times\mathbb{T}^3)$;

(ii) $\psi_n \to \psi$ *in* $L^2(0,T;H^{-1}(\mathbb{T}^3))$ *and*

$$\|\psi_n\|_{L^\infty((0,T)\times\mathbb{T}^3)} + \|u_n\|_{L^2(0,T;H^1(\mathbb{T}^3))} + \|\nabla\mathrm{div}u_n\|_{L^2(0,T;L^2(\mathbb{T}^3))} \leq C \,(\text{independent of } n).$$

Then, for any fixed $\varepsilon > 0$, up to a subsequence,

$$\left(\overline{\mathrm{div}(u_n\psi_n)}\right)_\varepsilon \rightharpoonup \left(\overline{\mathrm{div}(u\psi)}\right)_\varepsilon \quad \text{weakly} \quad \text{in} \quad L^2(0,T;H^1(\mathbb{T}^3)) \quad \text{as } n \to 0,$$

$$\left(\overline{((\mathrm{div}u_n)\psi_n)}\right)_\varepsilon \rightharpoonup \left(\overline{((\mathrm{div}u)\psi)}\right)_\varepsilon \quad \text{weakly} \quad \text{in} \quad L^2(0,T;H^1(\mathbb{T}^3)) \quad \text{as } n \to 0.$$

Proof. If the assumption (i) holds, then

$$u_n\psi_n \rightharpoonup u\psi \quad \text{and} \quad (\mathrm{div}u_n)\psi_n \rightharpoonup (\mathrm{div}u)\psi \quad \text{weakly} \quad \text{in} \quad L^2(0,T;L^2(\mathbb{T}^3)).$$

Thus, thanks to the spatial mollification, we have the desired convergence.
If the assumption (ii) holds, then up to a subsequence,

$$u_n\psi_n \to u\psi \quad \text{and} \quad (\mathrm{div}u_n)\psi_n \to (\mathrm{div}u)\psi \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathbb{T}^3).$$

Moreover, since

$$\|\left(\overline{\mathrm{div}(u_n\psi_n)}\right)_\varepsilon\|_{L^2(0,T;H^1(\mathbb{T}^3))} + \|\left(\overline{((\mathrm{div}u_n)\psi_n)}\right)_\varepsilon\|_{L^2(0,T;H^1(\mathbb{T}^3))} \leq C \quad \text{(independent of } n),$$

we have the desired convergence. $\square$

Therefore, (3.21) and (3.22) implies

$$J_2 \to \int_0^t \int_{\mathbb{T}^3} G^\nu R_{\varepsilon,\delta}^\nu dxd\tau, \quad \forall t \leq T.$$

Hence we have from (3.13) that

(3.27)
$$\liminf_{\kappa \to 0} \int_{\mathbb{T}^3} \left(\mathcal{F}(\rho^{\kappa,\nu}) + \mathcal{G}(\mathcal{E}^{\kappa,\nu} - \bar{\mathcal{E}}) + \frac{|m^{\kappa,\nu}|^2}{2\rho^{\kappa,\nu}} \right) dx$$
$$\leq C\varepsilon + C \int_{\mathbb{T}^3} (-\rho s)(U_0|\bar{U}) dx + \int_0^t \int_{\mathbb{T}^3} G^\nu R^\nu_{\varepsilon,\delta} dx d\tau, \quad \forall t \leq T.$$

Since it follows from (3.15) and (3.19) that

(3.28)
$$m^{\kappa,\nu} = \rho^{\kappa,\nu} u^{\kappa,\nu} \rightharpoonup \rho^\nu u^\nu := m^\nu \quad \text{weakly in} \ \ \mathcal{D}'((0,T) \times \mathbb{T}^3),$$

using the weakly lower semi-continuity of the convex functionals $\rho \mapsto \mathcal{F}(\rho)$ and $(\rho, m) \mapsto \frac{|m|^2}{\rho}$, we have

(3.29)
$$\int_{\mathbb{T}^3} \left(\mathcal{F}(\rho^\nu) + \frac{|m^\nu|^2}{2\rho^\nu} \right) dx \leq \liminf_{\kappa \to 0} \int_{\mathbb{T}^3} \left(\mathcal{F}(\rho^{\kappa,\nu}) + \frac{|m^{\kappa,\nu}|^2}{2\rho^{\kappa,\nu}} \right) dx.$$

On the other hand, since the definition of the functional $\mathcal{G}$ yields that the quanity $f := \mathcal{E}^{\kappa,\nu} - \bar{\mathcal{E}}$ satisfies

(3.30)
$$\int_{\mathbb{T}^3} |f| dx = \int_{|f| \leq 1} |f| dx + \int_{|f| > 1} |f| dx \leq \sqrt{\int_{\mathbb{T}^3} dx} \sqrt{\int_{|f| \leq 1} |f|^2 dx} + \int_{|f| > 1} |f| dx$$
$$\leq \sqrt{\int_{\mathbb{T}^3} \mathcal{G}(f) dx} + \int_{\mathbb{T}^3} \mathcal{G}(f) dx,$$

and it follows from (3.27) that $\int_{\mathbb{T}^3} \mathcal{G}(f) dx$ is uniformly bounded in κ, we find that there exists $\mathcal{E}^\nu \in L^\infty(0,T; \mathcal{M}(\mathbb{T}^3))$ such that

$$\mathcal{E}^{\kappa,\nu} \rightharpoonup \mathcal{E}^\nu \quad \text{weakly} - * \quad \text{in} \ \ L^\infty(0,T; \mathcal{M}(\mathbb{T}^3)).$$

Therefore, this and (3.30) imply

$$\|\mathcal{E}^\nu - \bar{\mathcal{E}}\|_{L^\infty(0,T;\mathcal{M}(\mathbb{T}^3))} \leq \liminf_{\kappa \to 0} \sup_{t \in [0,T]} \left(\sqrt{\int_{\mathbb{T}^3} \mathcal{G}(\mathcal{E}^{\kappa,\nu} - \bar{\mathcal{E}}) dx} + \int_{\mathbb{T}^3} \mathcal{G}(\mathcal{E}^{\kappa,\nu} - \bar{\mathcal{E}}) dx \right).$$

Hence, we obtain from (3.27) and (3.29) that

(3.31)
$$\sup_{t \in [0,T]} \int_{\mathbb{T}^3} \left(\mathcal{F}(\rho^\nu) + \frac{|m^\nu|^2}{2\rho^\nu} \right) dx + \|\mathcal{E}^\nu - \bar{\mathcal{E}}\|_{L^\infty(0,T;\mathcal{M}(\mathbb{T}^3))} \leq C \sup_{t \in [0,T]} \left(\sqrt{\mathcal{A}^\nu_{\varepsilon,\delta}} + \mathcal{A}^\nu_{\varepsilon,\delta} \right),$$

where

$$\mathcal{A}^\nu_{\varepsilon,\delta} := C\varepsilon + \int_{\mathbb{T}^3} (-\rho s)(U_0|\bar{U}) dx + \left| \int_0^t \int_{\mathbb{T}^3} G^\nu R^\nu_{\varepsilon,\delta} dx d\tau \right|.$$

Also, note that the convergences (3.28) and (3.19) imply

(3.32)
$$\partial_t \rho^\nu + \text{div}(m^\nu) = 0 \quad \text{in} \ \ \mathcal{D}'((0,T) \times \mathbb{T}^3).$$

3.5. Passing to the limit as $\delta \to 0$. Since only the term $R^\nu_{\varepsilon,\delta}$ in (3.31) depends on δ, we will pass to the limit on $R^\nu_{\varepsilon,\delta}$ as $\delta \to 0$ as $\delta \to 0$.

We first recall (3.23) as

$$R^\nu_{\delta,\varepsilon} := u^\nu \cdot \nabla \bar\psi^\nu_{\delta,\varepsilon} - \big(\overline{\operatorname{div}(\bar u^\nu_\delta \psi^\nu_\delta)}\big)_\varepsilon + \big(\overline{(\operatorname{div}\bar u^\nu_\delta)\psi^\nu_\delta}\big)_\varepsilon.$$

By (3.16), up to a subsequence, there exists ψ^ν such that $0 \le \psi^\nu \le 1$ and

(3.33) $\qquad\qquad \psi^\nu_\delta \rightharpoonup \psi^\nu \quad$ weakly-* $\quad$ in $\quad L^\infty((0,T) \times \mathbb{T}^3) \quad$ as $\delta \to 0$,

which yields that (by the mollification)

$$\nabla \bar\psi^\nu_{\delta,\varepsilon} \rightharpoonup \nabla \bar\psi^\nu_\varepsilon \quad \text{weakly-*} \quad \text{in} \quad L^\infty(0,T;H^1(\mathbb{T}^3)) \quad \text{as } \delta \to 0.$$

This together with $u^\nu \in L^2(0,T;H^1(\mathbb{T}^3))$ implies

$$u^\nu \cdot \nabla \bar\psi^\nu_{\delta,\varepsilon} \rightharpoonup u^\nu \cdot \nabla \bar\psi^\nu_\varepsilon \quad \text{weakly} \quad \text{in} \quad L^2(0,T;H^1(\mathbb{T}^3)) \quad \text{as } \delta \to 0.$$

Since $u^\nu \in L^2(0,T;H^1(\mathbb{T}^3))$ implies

$$\bar u^\nu_\delta \to u^\nu \quad \text{in} \quad L^2(0,T;H^1(\mathbb{T}^3)) \quad \text{strongly as } \delta \to 0,$$

which together with Lemma 3.5 and (3.33) yields

$$\big(\overline{\operatorname{div}(\bar u^\nu_\delta \psi^\nu_\delta)}\big)_\varepsilon \rightharpoonup \big(\overline{\operatorname{div}\big(u^\nu \psi^\nu\big)}\big)_\varepsilon \quad \text{weakly} \quad \text{in} \quad L^2(0,T;H^1(\mathbb{T}^3)),$$

$$\big(\overline{(\operatorname{div}\bar u^\nu_\delta)\psi^\nu_\delta}\big)_\varepsilon \rightharpoonup \big(\overline{(\operatorname{div}u^\nu)\psi^\nu}\big)_\varepsilon \quad \text{weakly} \quad \text{in} \quad L^2(0,T;H^1(\mathbb{T}^3)).$$

Hence, we have

(3.34) $\qquad\qquad R^\nu_{\varepsilon,\delta} \rightharpoonup R^\nu_\varepsilon \quad \text{weakly} \quad \text{in} \quad L^2(0,T;H^1(\mathbb{T}^3)),$

where

$$R^\nu_\varepsilon := u^\nu \cdot \nabla \bar\psi^\nu_\varepsilon - \big(\overline{\operatorname{div}\big(u^\nu \psi^\nu\big)}\big)_\varepsilon + \big(\overline{(\operatorname{div}u^\nu)\psi^\nu}\big)_\varepsilon.$$

This and (3.20) with (3.31) yield

(3.35) $\quad \displaystyle\sup_{t\in[0,T]} \int_{\mathbb{T}^3} \Big(\mathcal{F}(\rho^\nu) + \frac{|m^\nu|^2}{2\rho^\nu}\Big)dx + \|\mathcal{E}^\nu - \bar{\mathcal{E}}\|_{L^\infty(0,T;\mathcal{M}(\mathbb{T}^3))} \le C \sup_{t\in[0,T]} \Big(\sqrt{\mathcal{A}^\nu_\varepsilon} + \mathcal{A}^\nu_\varepsilon\Big),$

where

$$\mathcal{A}^\nu_\varepsilon := C\varepsilon + \int_{\mathbb{T}^3} (-\rho s)(U_0|\bar U)dx + \left|\int_0^t \int_{\mathbb{T}^3} G^\nu R^\nu_\varepsilon dx d\tau\right|.$$

3.6. Passing to the limit as $\varepsilon \to 0$. We first rewrite R^ν_ε as

$$R^\nu_\varepsilon = \operatorname{div}\big(u^\nu \bar\psi^\nu_\varepsilon\big) - (\operatorname{div}u^\nu)\bar\psi^\nu_\varepsilon - \big(\overline{\operatorname{div}\big(u^\nu \psi^\nu\big)}\big)_\varepsilon + \big(\overline{(\operatorname{div}u^\nu)\psi^\nu}\big)_\varepsilon$$

$$= \underbrace{\Big(\big(\overline{(\operatorname{div}u^\nu)\psi^\nu}\big)_\varepsilon - (\operatorname{div}u^\nu)\bar\psi^\nu_\varepsilon\Big)}_{=:K_1} + \underbrace{\Big(\operatorname{div}\big(u^\nu \bar\psi^\nu_\varepsilon\big) - \big(\overline{\operatorname{div}\big(u^\nu \psi^\nu\big)}\big)_\varepsilon\Big)}_{=:K_2}.$$

First of all, note that

(3.36) $\qquad\qquad \|K_1\|_{L^2((0,T)\times\mathbb{T}^3)} \le C \quad$ (independent of ε).

Since $\psi^\nu \in L^\infty((0,T) \times \mathbb{T}^3) \subset L^q((0,T) \times \mathbb{T}^3)$ for $q < \infty$,

$$\bar\psi^\nu_\varepsilon \to \psi^\nu \quad \text{strongly} \quad \text{in} \quad L^q((0,T) \times \mathbb{T}^3) \quad \text{as } \varepsilon \to 0,$$

which together with $u^\nu \in L^2(0,T;H^1(\mathbb{T}^3))$ implies

$$(\operatorname{div}u^\nu)\bar\psi^\nu_\varepsilon \to (\operatorname{div}u^\nu)\psi^\nu \quad \text{in} \quad L^{r_0}((0,T) \times \mathbb{T}^3) \quad \text{for some } r_0 \in (1,2).$$

Moreover, using $(\mathrm{div}u^\nu)\psi^\nu \in L^2((0,T) \times \mathbb{T}^3)$, we have

$$\overline{\big((\mathrm{div}u^\nu)\psi^\nu\big)}_\varepsilon \to (\mathrm{div}u^\nu)\psi^\nu \quad \text{in} \ \ L^2((0,T) \times \mathbb{T}^3).$$

Thus,

$$K_1 \to 0 \quad \text{in} \ \ L^{r_0}((0,T) \times \mathbb{T}^3),$$

which together with (3.36) yields

$$K_1 \rightharpoonup 0 \quad \text{in} \ \ L^2((0,T) \times \mathbb{T}^3).$$

For K_2, we use the following lemma on the Lions commutator estimate:

Lemma 3.6. [44, Lemma 2.3] *There exists a constant C such that for any $\varepsilon > 0$, any functions $f \in W^{1,\alpha}(\mathbb{T}^3)$ and $g \in L^\beta(\mathbb{T}^3)$ with $0 \le \frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{r} \le 1$ satisfy*

$$\big\|\overline{\big(\mathrm{div}\big(fg\big)\big)}_\varepsilon - \mathrm{div}\big(f\bar{g}_\varepsilon\big)\big\|_{L^r(\mathbb{T}^3)} \le C\|f\|_{W^{1,\alpha}(\mathbb{T}^3)}\|g\|_{L^\beta(\mathbb{T}^3)}.$$

In addition, if $r < \infty$, then

$$\big\|\overline{\big(\mathrm{div}\big(fg\big)\big)}_\varepsilon - \mathrm{div}\big(f\bar{g}_\varepsilon\big)\big\|_{L^r(\mathbb{T}^3)} \to 0 \quad \text{as } \varepsilon \to 0.$$

We may apply Lemma 3.6 to our case: $f = u^\nu, g = \psi^\nu, \alpha = 2, \beta = \infty$ and $r = 2$. Since $\|u^\nu\|_{L^2(0,T;H^1(\mathbb{T}^3))}$ and $\|\psi^\nu\|_{L^\infty((0,T) \times \mathbb{T}^3)}$ are uniformly bounded in ν, Lemma 3.6 implies that
(3.37)
$$\|K_2\|_{L^2(0,T;L^2(\mathbb{T}^3))} \le C\|u^\nu\|_{L^2(0,T;H^1(\mathbb{T}^3))}\|\psi^\nu\|_{L^\infty((0,T) \times \mathbb{T}^3)} \le C \quad \text{(independent of } \nu),$$

and

(3.38) $$\|K_2(\cdot,t)\|_{L^2(\mathbb{T}^3))} \to 0 \quad \text{as } \varepsilon \to 0, \quad \text{for a.e. } t \in (0,T),$$

where we used the fact that $u^\nu(\cdot,t) \in H^1(\mathbb{T}^3)$ and $\psi^\nu(\cdot,t) \in L^2(\mathbb{T}^3)$ for a.e. $t \in (0,T)$, and uniformly in ν.
Thus, it follows from (3.37) and (3.38) that for $q < 2$,

$$\|K_2(\cdot,t)\|_{L^2(\mathbb{T}^3)} \to 0 \quad \text{in} \ \ L^q((0,T)) \quad \text{as } \varepsilon \to 0,$$

which together with (3.37) yields

$$K_2 \rightharpoonup 0 \quad \text{in} \ \ L^2((0,T) \times \mathbb{T}^3) \quad \text{as } \varepsilon \to 0.$$

Therefore,

(3.39) $$R^\nu_\varepsilon \rightharpoonup 0 \quad \text{in} \ \ L^2((0,T) \times \mathbb{T}^3) \quad \text{as } \varepsilon \to 0.$$

which together with (3.20) yields

$$\int_0^t \int_{\mathbb{T}^3} G^\nu R^\nu_\varepsilon \, dx d\tau \to 0 \quad \text{as } \varepsilon \to 0, \quad \forall t \in [0,T].$$

Hence, taking $\varepsilon \to 0$ in (3.35), we have

(3.40)
$$\sup_{[0,T]} \int_{\mathbb{T}^3} \Big(\mathcal{F}(\rho^\nu) + \frac{|m^\nu|^2}{2\rho^\nu}\Big) dx + \|\mathcal{E}^\nu - \bar{\mathcal{E}}\|_{L^\infty(0,T;\mathcal{M}(\mathbb{T}^3))}$$
$$\le C\sqrt{\int_{\mathbb{T}^3}(-\rho s)(U_0|\bar{U})dx} + C\int_{\mathbb{T}^3}(-\rho s)(U_0|\bar{U})dx.$$

3.7. Passing to the limit as $\nu \to 0$. First of all, by (2.9), there exists $\rho \in L^\infty(0, T; L^2(\mathbb{T}^3))$ such that

$$(3.41) \qquad \rho^\nu \rightharpoonup \rho \quad \text{weakly-* in} \quad L^\infty(0, T; L^2(\mathbb{T}^3)).$$

Since the Hölder inequality with $|m^\nu| = \sqrt{\rho^\nu} \frac{|m^\nu|}{\sqrt{\rho^\nu}}$ yields

$$\|m^\nu\|_{L^\infty(0,T;L^{\frac{4}{3}}(\mathbb{T}^3))} \leq \|\sqrt{\rho^\nu}\|_{L^\infty(0,T;L^4(\mathbb{T}^3))} \left\| \frac{m^\nu}{\sqrt{\rho^\nu}} \right\|_{L^\infty(0,T;L^2(\mathbb{T}^3))},$$

the uniform bounds (2.9) and (2.4) imply the uniform bound of m^ν in $L^\infty(0, T; L^{\frac{4}{3}}(\mathbb{T}^3))$. Thus, there exists $m \in L^\infty(0, T; L^{\frac{4}{3}}(\mathbb{T}^3))$ such that

$$(3.42) \qquad m^\nu \rightharpoonup m \quad \text{weakly-* in} \quad L^\infty(0, T; L^{\frac{4}{3}}(\mathbb{T}^3)).$$

Note that it follows from (3.40) that there exists $\mathcal{E} \in L^\infty(0, T; \mathcal{M}(\mathbb{T}^3))$ such that

$$(3.43) \qquad \mathcal{E}^\nu \rightharpoonup \mathcal{E} \quad \text{weakly-* in} \quad L^\infty(0, T; \mathcal{M}(\mathbb{T}^3)).$$

Therefore, (3.43) and the weakly lower semi-continuity of the convex functionals in (3.40) together with (3.41) and (3.42), we have

$$(3.44) \qquad \begin{aligned} &\sup_{[0,T]} \int_{\mathbb{T}^3} \left(\mathcal{F}(\rho) + \frac{|m|^2}{2\rho} \right) dx + \|\mathcal{E} - \bar{\mathcal{E}}\|_{L^\infty(0,T;\mathcal{M}(\mathbb{T}^3))} \\ &\qquad \leq C \sqrt{\int_{\mathbb{T}^3} (-\rho s)(U_0|\bar{U}) dx} + C \int_{\mathbb{T}^3} (-\rho s)(U_0|\bar{U}) dx, \end{aligned}$$

which gives (1.19).

We also obtain from (3.32) that

$$(3.45) \qquad \partial_t \rho + \operatorname{div}(m) = 0 \quad \text{in} \quad \mathcal{D}'((0, T) \times \mathbb{T}^3).$$

3.8. Uniqueness. We here prove the last part of Theorem 1.1, for the uniqueness. Let $(U_0^n)_{n \in \mathbb{N}}$ be a sequence of initial data such that

$$(3.46) \qquad \int_{\mathbb{T}^3} (-\rho s)(U_0^n|\bar{U}) dx \to 0 \quad \text{as } n \to \infty.$$

Then, by (3.44),

$$\liminf_{n \to \infty} \left[\sup_{[0,T]} \int_{\mathbb{T}^3} \left(\mathcal{F}(\rho^n) + \frac{|m^n|^2}{2\rho^n} \right) dx + \|\mathcal{E}^n - \bar{\mathcal{E}}\|_{L^\infty(0,T;\mathcal{M}(\mathbb{T}^3))} \right] = 0.$$

Thus, as $n \to \infty$,

$$\mathcal{E}^n \to \bar{\mathcal{E}} \quad \text{in} \quad L^\infty(0, T; \mathcal{M}(\mathbb{T}^3)),$$

and

$$\frac{|m^n|}{\sqrt{\rho^n}} \to 0 \quad \text{in} \quad L^\infty(0, T; L^2(\mathbb{T}^3)).$$

Since

$$\begin{aligned} \rho^n &= \rho^n \mathbf{1}_{\rho^n \in [\rho_*, \rho^*]} + \rho^n \mathbf{1}_{\rho^n \in (0, \rho_*)} + \rho^n \mathbf{1}_{\rho^n \in (\rho^*, \infty)} \\ &\leq \rho^* + |\rho^n \mathbf{1}_{\rho^n \in (0, \rho_*)} - \rho_*| + \rho_* + |\rho^n \mathbf{1}_{\rho^n \in (\rho^*, \infty)} - \rho^*| + \rho^*, \end{aligned}$$

where ρ_* and ρ^* are defined in (1.16), it follows from (1.16) and $\liminf_{n\to\infty}\sup_{[0,T]}\int_{\mathbb{T}^3}\mathcal{F}(\rho^\varepsilon)dx = 0$ that

$$\|\rho^n\|^2_{L^\infty(0,T;L^2(\mathbb{T}^3))} \le C + C\sup_{[0,T]}\int_{\mathbb{T}^3}\mathcal{F}(\rho^n)dx \le C,$$

which implies that there exists ρ such that

(3.47) $\rho^n \rightharpoonup \rho$ weakly-*   in $L^\infty(0,T;L^2(\mathbb{T}^3))$.

Therefore,

$$\|m^n\|_{L^\infty(0,T;L^{\frac{4}{3}}(\mathbb{T}^3))} \le \|\sqrt{\rho^n}\|_{L^\infty(0,T;L^4(\mathbb{T}^3))}\left\|\frac{m^n}{\sqrt{\rho^n}}\right\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \le C\left\|\frac{m^n}{\sqrt{\rho^n}}\right\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \to 0.$$

Thus, it remains to prove that $\rho = \bar\rho$.

To this end, we will use the equation (3.45):

(3.48) $\partial_t\rho^n + \mathrm{div}_x(m^n) = 0$ in $\mathcal{D}'((0,T)\times\mathbb{T}^3)$.

For test functions for the above equation, we recall the same functions as in (3.9): for any $t \in (0,T)$, and any $\zeta < t/2$, we consider

$$\varphi_{t,\zeta}(s) := \int_0^s \Big(\phi_\zeta(z) - \phi_\zeta(z-t)\Big)dz,$$

where

$$\phi_\zeta(s) := \frac{1}{\zeta}\phi\Big(\frac{s-\zeta}{\zeta}\Big)   \text{ for any } \zeta > 0,$$

and $\phi : \mathbb{R} \to \mathbb{R}$ is a non-negative smooth function such that $\phi(s) = \phi(-s)$, $\int_{\mathbb{R}}\phi = 1$ and $\mathrm{supp}\,\phi = [-1,1]$.

Then we consider the following test functions for the continuity equation:

$$\Phi(x)\varphi_{t,\zeta}(s),$$

where $\Phi \in C^\infty(\mathbb{T}^3)$. That is,

$$\int_0^\infty\int_{\mathbb{T}^3}\Big(\Phi(x)(-\varphi'_{t,\zeta}(s))\rho^n\Big)dxds = \int_0^\infty\int_{\mathbb{T}^3}\Big(\varphi_{t,\zeta}(s)\nabla\Phi\cdot m^n\Big)dxds.$$

Thus,

$$\int_0^\infty\int_{\mathbb{T}^3}\Big(\Phi(x)\phi_\zeta(s-t)\rho^n\Big)dxds = \int_0^\infty\int_{\mathbb{T}^3}\Big(\Phi(x)\phi_\zeta(s)\rho^n\Big)dxds + \int_0^\infty\int_{\mathbb{T}^3}\Big(\varphi_{t,\zeta}(s)\nabla\Phi\cdot m^n\Big)dxds.$$

Note that since $\rho^n \in L^\infty(0,T;L^2(\mathbb{T}^3))$ and $m^n \in L^2(0,T;L^{\frac{4}{3}}(\mathbb{T}^3))$ by (1.18), it follows from (3.48) and Aubin-Lions Lemma that $\rho^n \in C([0,T];L^1(\mathbb{T}^3))$.

Therefore, taking $\zeta \to 0$ above, we find that

$$\int_{\mathbb{T}^3}\Phi(x)\rho^n(x,t)dx = \int_{\mathbb{T}^3}\Phi(x)\rho_0^n(x)dx + \int_0^t\int_{\mathbb{T}^3}\nabla\Phi\cdot m^n dxds.$$

Note that (3.46) yields

(3.49) $\int_{\mathbb{T}^3}|\rho_0^n - \bar\rho|^2dx \le \int_{\mathbb{T}^3}p_e(\rho_0^n|\bar\rho)dx \to 0$ as $n \to \infty$.

Since

$$\left|\int_0^t\int_{\mathbb{T}^3}\nabla\Phi\cdot m^n dxds\right| \le \|\nabla\Phi\|_{L^4(\mathbb{T}^3)}\|m^n\|_{L^\infty(0,T;L^{\frac{4}{3}}(\mathbb{T}^3))} \to 0,$$

taking $n \to \infty$ and using (3.47) and (3.49), we have

$$\int_{\mathbb{T}^3} \Phi(x)\rho(x,t)dx = \int_{\mathbb{T}^3} \Phi(x)\bar{\rho}(x)dx, \quad \forall t \in [0,T].$$

Hence, $\rho = \bar{\rho}$ a.e. on $\mathbb{T}^3 \times [0,T]$.

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