Spectral singularities with directional sensitivity

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We propose a class of spectral singularities that are sensitive to the direction of excitation and arise in nonlinear systems with broken parity symmetry. These spectral singularities are sensitive to the direction of the incident beam and result in diverging transmission and reflection for the left (right) incident, while the transmission and reflection of the right (left) side of the system remain finite. For the pedagogical reason, first we review the scattering formalism in nonlinear systems using an abstract δ-function model. Then, using a parity symmetry broken nonlinear system consisting of two δ functions, one linear and the other nonlinear, we prove the existence of our proposed spectral singularities. Finally, we use an experimentally feasible realistic model based on coupled disk resonators to demonstrate the spectral singularity with directional sensitivity (SSDS). Our proposed SSDS might have applications in the design of nonlinear sensors and might provide a solution for the hole-burning problem in pumped laser ring resonators.

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I. INTRODUCTION

While the study of complex functions has a long history, in recent years it has attracted more attention. This is due to their mathematically peculiar features and their applications in optical systems. One of the unique features of complex potentials, for instance, is the existence of different types of singularities. Exceptional points (EPs) are among such singularities with topological characteristics. EPs arise when the Hamiltonian of the corresponding system becomes defective and the eigenvalues and their associated eigenstates coalesce. While direct physical identification of the exceptional points is difficult, their strong influence on the dynamics can be observed [1–4]. Exceptional point singularities enable stop lights [5], real-entire flat bands [6], unidirectional invisibility [7], topological energy transfer [8], enhanced sensitivity [9,10], and robust zero mode at will [11], to name a few.

Another type of singularity is spectral singularity related to the completeness of the continuous spectrum and that can satisfy outgoing boundary conditions [12]. In other words, spectral singularities do not correspond to square-integrable eigenfunctions. Within the scattering matrix formalism, such singularities identify the lasing threshold of cavities with gain [13,14], where the cavity gain corresponds to a negative imaginary part of the refractive index. The notion of such spectral singularities can be extended to the semi-infinite lattices [15], nonlinear potentials [16], and nonreciprocal cavities in the presence of magnetic elements [17]. In the latest one, the presence of a gyrotropic element together with the broken inversion symmetry in a 1D heterostructure results in asymmetric stationary inflection points where the group velocity of the wave vector in one direction becomes zero, while in the opposite direction group velocity finds a finite and nonzero value. The asymmetric inflection points are the first modes that reach the lasing threshold and thus result in a robust unidirectional lasing. Unidirectional lasing modes have been of interest in recent years due to their potential applications. Another approach to obtain directional emission is based on the strong asymmetric backscattering in the vicinity of an exceptional point [18]. Nonlinear coupling between the clockwise and counterclockwise propagating waves in an ultrahigh-Q whispering-gallery microresonator can produce a chiral emission [19]. Topological insulator lattices can generate directional lasing at the edge of the lattice in the presence of a gain mechanism [20]. Another type of spectral singularities with unidirectional response can be obtained from the interplay of parity-time symmetry and Fano resonances [21]. In such spectral singularities, without breaking the reciprocity, one is able to obtain a simultaneous unidirectional lasing and unidirectional reflectionless mode. For such a mode, one side reflection tends to infinity, the other side reflection becomes zero, and the transmission coefficient remains finite. These singularities emerge from the resonance trapping and delay time associated with the reflected signal residing in the gain or loss part of the parity-time symmetric cavity [21].

In this paper, we introduce a class of spectral singularities with sensitivity to the direction of excitation. Such spectral singularities do not generate directional lasing; however, their source of emission comes from a specific direction. In one dimension, such spectral singularities pick up fluctuation coming from one direction. Thus, lasing emission is activated by the fluctuation from one side. Our proposed spectral singularities appear in one-dimensional (1D) nonlinear systems with broken parity and can be expanded to multichannel nonlinear systems with broken parity symmetry in each channel. While we are interested in mathematically proving the existence of such spectral singularities,

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one might be able to find their application in directional sensing and suppression of spatial hole burning in ring lasers [22].

This article is structured as follows. In Sec. II, we will review the scattering formalism for an abstract model, namely, a nonlinear δ function, and calculate the spectral singularities associated with it. In Sec. III, we will construct a system composed of a linear and a nonlinear δ function with broken parity symmetry. We construct the scattering matrix and calculate the spectral singularities that appear in the system. Finally, in Sec. IV, we discuss the appearance of spectral singularity with directional sensitivity (SSDS) in a coupled disk resonator system composed of two coupled resonators, one linear and the other nonlinear, that are coupled to a 1D transmission line. We will draw our conclusion in Sec. V.

II. SCATTERING FORMALISM FOR A NONLINEAR POTENTIAL AND ITS SPECTRAL SINGULARITIES

In this section, we will briefly review the basic method for treating the scattering properties of a nonlinear δ-function potential and constructing the scattering matrix in a 1D nonlinear problem. Furthermore, we connect the scattering matrix S to the spectral singularities. Finally, in this section, we calculate the spectral singularities associated with our setup. We show that the spectral singularities in a single nonlinear δ-function potential are reciprocal. In other words, at the spectral singularity, all the reflection and transmission coefficients tend to infinity with the same slope. This occurs because the system preserves the parity symmetry even when it has a nonlinear component.

Let us consider a system in which its permittivity is given by $\epsilon(x) = n_0 + [n + \chi|E(x)|^2]\delta(x)$, where $\delta(x)$ is the Dirac delta function and E is the electric field. In this permittivity, n_0 is the background permittivity (we assume it is equal to one), $n = n_r + in_i$, and $\chi = \chi_r + i\chi_i$, where $n_{r,i} \in \Re$ and $\chi_{i,r} \in \Re$ are the linear and nonlinear complex perturbations to the background permittivity. Such permittivity can be realized with very thin layers of coated materials. In this arrangement, a time-harmonic electric field of frequency ω obeys the 1D Helmholtz equation,

$$\frac{d^2}{dx^2}E(x) + \frac{\omega^2}{c^2}\epsilon(x)E(x) = 0. \quad (1)$$

In Eq. (1), c is the speed of light in the vacuum. On the left and right side of the δ-function potential, Eq. (1) admits the solution $E^-(x) = E_f^- \exp(ikx) + E_b^- \exp(-ikx)$ for $x \leqslant 0$ and $E^+(x) = E_f^+ \exp(ikx) + E_b^+ \exp(-ikx)$ for $x \geqslant 0$, where the wave vector $k = \frac{\sqrt{n_0}\omega}{c}$.

Although the problem at hand is nonlinear, one can still use the S-matrix formalism to treat the scattering properties of it. More precisely, the amplitudes of the ingoing and outgoing propagating waves outside the scattering domain, namely, (E_f^-, E_b^+) and (E_b^-, E_f^+), respectively, are related through a nonlinear 2×2 scattering matrix S,

$$\begin{pmatrix} E_b^- \\ E_f^+ \end{pmatrix} = S\begin{pmatrix} E_f^- \\ E_b^+ \end{pmatrix}. \quad (2)$$

In the above formulation, the elements of the scattering matrix are related to the transmission and reflection amplitudes for the left and right incidents, namely,

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} r_l & t_r \\ t_l & r_r \end{pmatrix}. \quad (3)$$

In Eq. (3), $r_{l(r)}$ and $t_{l(r)}$ are the reflection and transmission amplitude for the left l (right r) incident waves, respectively. In nonreciprocal structures, the elements of the S matrix in Eq. (3) might be different from each other. In linear systems and in the absence of the magnetic field or spatiotemporal modulation, reciprocity states that $S_{12} = S_{21}$ [23].

In the reciprocal systems and at the real frequencies, spectral singularities have been identified as the poles of the scattering matrix which are equivalent to the blowup of the transmission and reflection amplitudes. This defines the lasing points in which for no in-going field there is an outgoing field solution in Eq. (2). Generally, for an open scattering system with no embedded gain medium, the poles of the scattering matrix occur at complex frequencies with a negative imaginary part. By introducing gain in the system, the absolute value of the imaginary part of such frequencies decreases until reaching the lasing threshold where one of the poles reaches the real axes. The real part of the frequency of this pole describes the frequency of the first lasing mode. In some systems, such as ring lasers, several modes can reach the lasing threshold at the same time, the so-called multimode lasing, which is an undesired phenomenon as it distributes gain power between several modes and reduces the lasing power at the desired mode.

Recently, new spectral singularities have been introduced which result in a unidirectional lasing mode. Specifically, in the presence of reciprocity, at these spectral singularities only one reflection, namely, one of the diagonal terms of the S matrix in Eq. (3), tends to infinity and the other elements of the S matrix remain finite [21]. In the nonreciprocal system and in the presence of a magnetic field, it has been shown that such spectral singularities result in the infinite transmission and reflection in one side of the structure. In this case, only one row of the S matrix in Eq. (3) tends to infinity, while the other row remains finite, which is equivalent to having either left transmission coefficient $T_l = |t_l|^2 \to \infty$ and right reflection coefficient $R_r = |r_r|^2 \to \infty$ or right transmission coefficient $T_r = |t_r|^2 \to \infty$ and left reflection coefficient $R_l = |r_l|^2 \to \infty$ [17].

The spectral singularities that we are looking for here have different characteristics from the aforementioned ones or the conventional spectral singularities[24,25]. Specifically, while at these spectral singularities only specific elements of the scattering matrix in Eq. (3) tend to infinity, they do not cause any directional lasing. At these singularities, only one reflection coefficient $R_{l(r)}$ and one transmission coefficient $T_{l(r)}$ tend to infinity, while the other reflection coefficient $R_{r(l)}$ and transmission coefficient $T_{r(l)}$ remain finite. In other words, one column of the scattering matrix S in Eq. (3) tends to infinity, while the other column remains finite. Thus, one can claim that these spectral singularities are sensitive to the direction of incident fluctuations. In Sec. III, we show that a parity broken nonlinear scattering system enables us to realize such spectral singularities with directional sensitivity (SSDS).

Let us go back to the parity symmetric single nonlinear δ-function problem. The transmission and reflection amplitudes for left and right incident waves can be obtained from the boundary conditions

$$E_b^+ = 0 \quad \text{and} \quad E_f^- = 0, \tag{4}$$

respectively, and are defined as

$$\begin{aligned} t_l &\equiv \frac{E_f^+}{E_f^-}, \quad r_l \equiv \frac{E_b^-}{E_f^-}, \\ t_r &\equiv \frac{E_b^-}{E_b^+}, \quad r_r \equiv \frac{E_f^+}{E_b^+}. \end{aligned} \tag{5}$$

For the left incident field, continuity of the field at the δ-function potential, namely,

$$E^-(x)|_{x=0} = E^+(x)|_{x=0}, \tag{6}$$

and the discontinuity of it at $x = 0$, namely,

$$\left.\frac{dE^+}{dx}\right|_{x=0} - \left.\frac{dE^-}{dx}\right|_{x=0} = -k^2[n + \chi|E^+(0)|^2]E^+(0), \tag{7}$$

provide us with two conditions to find E_f^- and E_b^- as a function of E_f^+ (here, we normalize it to one) for the left incident beam. Notice that inserting the first boundary condition of Eq. (4) into Eq. (7) reduces it to

$$\left.\frac{dE^+}{dx}\right|_{x=0} - \left.\frac{dE^-}{dx}\right|_{x=0} = -k^2(n + \chi)E^+(0), \tag{8}$$

which means that in this specific case, nonlinearity acts similar to a linear potential and the field intensity does not play any role even if the potential at hand is nonlinear. This is a peculiar feature of the δ function and does not hold with nonlinear slab potentials. Solving Eqs. (6) and (8) simultaneously results in the transmission, $S_{21} \equiv t_l$, and reflection, $S_{11} \equiv r_l$, amplitudes for the left incident wave,

$$t_l = \frac{2i}{k(n+\chi) + 2i}, \quad r_l = -\frac{k(n+\chi)}{k(n+\chi) + 2i}. \tag{9}$$

For the right incident field, Eq. (6) remains the same, while Eq. (8) modifies to

$$\left.\frac{dE^+}{dx}\right|_{x=0} - \left.\frac{dE^-}{dx}\right|_{x=0} = -k^2(n + \chi)E^-(0), \tag{10}$$

where we have used the second boundary condition given in Eq. (4). Normalizing the E_b^- to one and following similar steps as the left incident wave results in the transmission $S_{12} \equiv t_r$ and reflection $S_{22} \equiv r_r$ amplitudes for the right incident wave, and by comparing them with their corresponding left incident, ones shows that

$$t_r = t_l, \quad r_r = r_l. \tag{11}$$

Equation (11) is a known result where a nonlinear medium with parity symmetry does not lead to any asymmetric transport [26]. Using Eqs. (3), (9) and (11), we can construct the scattering matrix S.

As mentioned earlier, the poles of this scattering matrix at real frequencies identify the spectral singularities or the lasing modes. From Eqs. (9) and (11), we observe that the spectral singularities occur for pure imaginary values of n and χ and are given by $k = -\frac{2}{n_i+\chi_i}$. Furthermore, with our choice of coordinate to have the outgoing fields, wave vector k must be a positive variable and thus $n_i + \chi_i < 0$, meaning that although the δ function can contain partial loss either in its linear $n_i > 0$ or nonlinear part $\chi_i > 0$, it must provide a net gain coming from its nonlinear $\chi_i < 0$ or linear $n_i < 0$ part. This conclusion proves our previous discussion where we mentioned that one needs to incorporate a sufficiently strong gain medium to reach the lasing threshold.

III. SPECTRAL SINGULARITIES WITH DIRECTIONAL SENSITIVITY

Armed with the method of calculating the spectral singularities given in Sec. II, in this section we calculate the spectral singularity associated with a parity symmetry broken nonlinear system. To demonstrate the existence of the spectral singularities with directional sensitivity, we consider a system in which its permittivity is given by $\epsilon(x) = n_0 + n_1\delta(x+a) + [n_2 + \chi|E(x)|^2]\delta(x-a)$. In this permittivity, $n_1 = n_{1r} + in_{1i}$, $n_2 = n_{2r} + in_{2i}$, and $\chi = \chi_r + i\chi_i$, where $n_{1r}, n_{2r}, n_{1i}, n_{2i}$, and $\chi_{i,r} \in \Re$ are the linear and nonlinear complex perturbations to the background permittivity. The first δ function, which is given by the perturbed refractive index n_1, is placed at $x = -a$, and the second one, with amplitude $n_2 + \chi|E(x)|^2$, is placed at $x = a$.

Similar to the previous scattering problem, a time-harmonic electric field of frequency ω obeys the 1D Helmholtz equation which is given in Eq. (1) with the following general solution:

$$\begin{aligned} E^-(x) &= E_f^- \exp(ikx) + E_b^- \exp(-ikx), & x \leqslant -a, \\ E^m(x) &= E_f^m \exp(ikx) + E_b^m \exp(-ikx), & -a \leqslant x \leqslant a, \\ E^+(x) &= E_f^+ \exp(ikx) + E_b^+ \exp(-ikx), & x \geqslant a, \end{aligned}$$

with the wave vector $k = \frac{\sqrt{n_0}\omega}{c} > 0$.

Following similar steps as the one discussed in Sec. II, first we calculate the scattering properties of the medium for the left incident field. In this case, we assume $E_f^+ = 1$, $E_b^+ = 0$. Using the continuity of the field and discontinuity of its derivative at $x = a$ and $x = -a$, one can show that $E_f^m = 1 - \frac{1}{2}ik(n_2 + \chi)$, $E_b^m = \frac{1}{2}ike^{2iak}(n_2 + \chi)$ and the transmission and reflection amplitudes for a left incident beam are given by

$$\begin{aligned} t_l &= \frac{4}{k^2 n_1 \zeta\eta - 2ik(n_1 + \eta) + 4}, \\ r_l &= \frac{e^{-2iak}[kn_1(k\eta + 2i) - ke^{4iak}(kn_1 - 2i)\eta]}{k^2 n_1 \zeta\eta - 2ik(n_1 + \eta) + 4}. \end{aligned} \tag{12}$$

In the above equations, the new parameters are defined as $\zeta \equiv \exp(4iak) - 1$ and $\eta = n_2 + \chi$.

To calculate the transmission and reflection amplitude of the right incident field, on the other hand, we assume that $E_f^- = 0, E_b^- = 1$. By imposing the boundary conditions at each δ function, namely, the continuity of the field and discontinuity of its derivative at $x = \pm a$, we find that $E_f^m = \frac{1}{2}ikn_1e^{2iak}$, $E_b^m = 1 - \frac{ikn_1}{2}$. Consequently, the transmission and reflection amplitude for a right incident beam are given by

$$t_r = \frac{16e^{4iak}}{k^3\chi|n_1|^2\zeta^2(4i - kn_1\zeta) + 2e^{4iak}[k^2n_1\zeta(2n_2 + 4\chi + ik^2n_1\chi\zeta) + 2k^2\chi n_1^* - 4ik(n_1+n_2) - 4ik\chi + 8] - 4k^2\chi n_1^*},$$

$$r_r = \frac{k^2\chi\zeta^2|n_1|^2(4i - kn_1\zeta) + 4e^{4iak}(k\chi n_1^* - kn_1\eta - 2i\eta) + 4n_1e^{8iak}[kn_2 - 2ik^2n_1\chi\sin^2(2ak) + 2k\chi - 2i] - 4k\chi n_1^*}{k\chi e^{2iak}\zeta n_1^*(kn_1\zeta - 2i)^2 + 2e^{6iak}\{n_1[-ik^2n_1\chi\zeta^2 - 2ke^{4iak}(\eta+\chi) + 2k(\eta+\chi) + 4i] + 4i\eta - 8k^{-1}\}}, \quad (13)$$

where $*$ stands for the complex conjugate.

It is clear that $r_l \neq r_r$ and $t_l \neq t_r$, where the first appears due to the broken parity symmetry and the second occurs due to the coexistence of the nonlinearity and broken parity symmetry [26].

As discussed before, to identify the spectral singularities, we are interested in the zeros of the denominators in Eqs. (12) and (13). Depending on the type of the spectral singularity that we are searching for, we must look for zeros in one or more transmission and reflection amplitudes that occur at the same wave vector for the same parameters, χ, n_1, and n_2. For example, in the conventional spectral singularities, the denominators in Eqs. (12) and (13) must become zero [24]. For unidirectional spectral singularities, we would like to have a divergence only in r_l and possibly in t_r, or in r_r and possibly in t_l [21]. On the other hand, for SSDS, one needs to have only r_l and t_l, or r_r and t_r, tend to infinity. Thus, to have an SSDS, one needs to make sure that the denominators in r_l and t_l are different from r_r and t_r. This allows us to search for the zeros of either of them. As we see, here the broken parity and nonlinearity allow us to have such a possibility. More precisely, the denominators in Eqs. (12) are not the same as the denominators in Eqs. (13), which make it possible to search for the poles of the scattering matrix S such that only one column remains finite while the other column diverges, and thus the system finds itself at the SSDS mode.

In general, finding an analytical solution for an SSDS of the above system is not possible. However, one can find the SSDS points for the left and right incident fields numerically. In an attempt to find the SSDS for the left and right incident field, let us assume that $a = \frac{\pi}{4k^\star}$, where $k^\star$ is the wave vector for which the SSDS mode occurs. This choice does not affect the physics of the problem and it only saves us from doing difficult numerical tasks. For the left incident field, the zeros of the denominators in Eq. (12) occur at

$$k^\star = \pm\frac{4}{\sqrt{6n_1\eta - \eta^2 - n_1^2} \pm i(n_1 + \eta)}. \quad (14)$$

For arbitrary values of n_1, n_2, χ, the wave vectors in Eq. (14) are complex. However, the SSDS wave vectors must be positive and real in order to identify a lasing point. Thus, if there exist n_1, n_2, and χ such that the $k^\star$ in Eq. (14) is real, then we can only accept the positive solution. As an example, let us assume that $n_1 = 2 + i\gamma$, $n_2 = 3 + i\gamma$, and $\chi = 3$. A numerical search shows that when $a = 2.7207$ for $\gamma = -3.4641$, the SSDS wave vector for the left incident field becomes real with value $k^\star = 0.288\,675$. As a result, we expect to have a divergence in the left transmission and left reflection, while the right transmission and right reflection remain finite. In Fig. 1(a), we have plotted the transmissions and reflections for the above δ functions. In agreement with our analytical and numerical predictions, the left transmission and reflection amplitudes diverge at $k^\star = 0.288\,675$, while the t_r and r_r remain finite. The right transmission and reflection show a strong amplification at $k_0 \approx 0.3 > k^\star$, which is different from a diverging behavior. The difference between the wave vectors associated with the right amplification and left SSDS, $\Delta k \equiv |k_0 - k^\star|$, is proportional to χ and tends to zero when the nonlinearity coefficient χ decreases to zero. It is known that at the spectral singularity, the phase must obtain a π shift [21]. Figure 1(b) depicts the phase of the transmission and reflection amplitudes. We see that π jump occurs at the $k^\star$ for the phases of the left transmission ϕ_{t_l} and left reflection ϕ_{r_l}, while the curves associated with the right reflection ϕ_{r_r} and transmission ϕ_{t_r} are smooth curves.

Unfortunately, there is no close form for the right SSDS wave vector. However, one can still numerically locate those wave vectors that diverge the amplitudes given in Eq. (13). For example, when the two δ functions are apart by a distance equal to $a = 4.645\,03$ and $n_1 = 2 - 2.957\,12i$, $n_2 = 3 - 2.957\,12i$, and $\chi = 3$, the SSDS wave vector for the

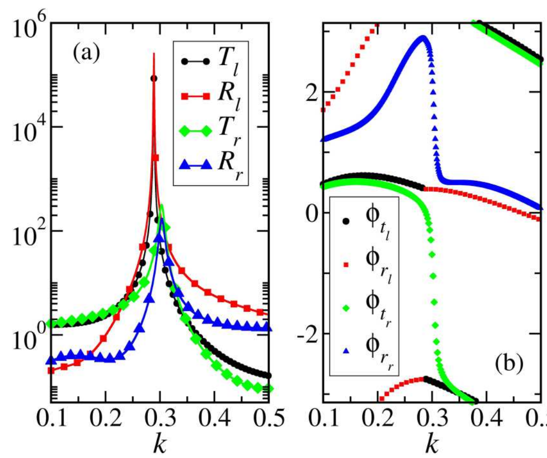

FIG. 1. Left spectral singularity with directional singularity. (a) Transmission and reflection curves vs wave vector k associated with a system composed of two δ functions, one linear with permittivity $n_1 = 2 - 3.4641i$ placed at $x = -2.7207$ (in units of k^{-1}) and the other with $n_2 = 3 - 3.4641i$ and $\chi = 3$ placed at $x = 2.7207$ (in units of k^{-1}). Left transmission and left reflection are diverging at $k \approx 0.288\,675$, while transmission and reflection for the right incident beam remain finite. (b) The phase associated with the t_i, r_i, $i = l, r$. At the SSDS, a π shift occurs for the left reflection (red ■) and transmission (black •), while the right reflection and transmission curves are smooth.

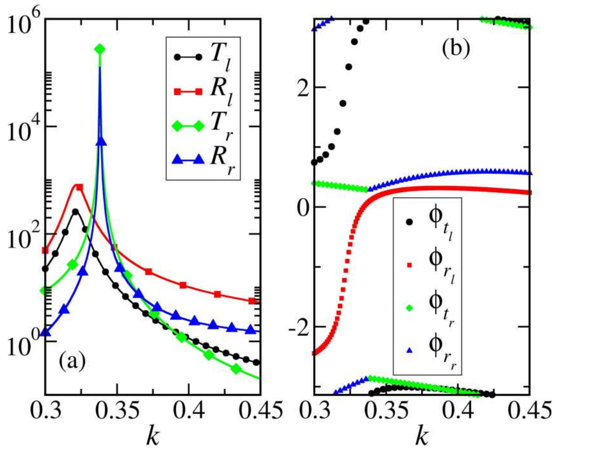

FIG. 2. Right spectral singularity with directional singularity. (a),(b) Similar to the one presented in Fig. 1, with δ functions placed at $x \approx \pm 2.32$ with $n_1 = 2 - 2.95712i$, $n_2 = 3 - 2.95712i$, and $\chi = 3$. Here the SSDS occurs for right transmission and reflection.

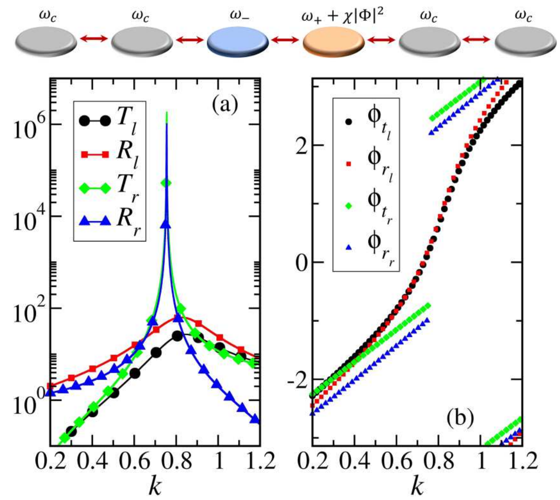

FIG. 3. (a) Transmission and reflection coefficients of the coupled disk resonators depicted in the upper inset. The resonance frequency of the disks at the transmission line is $\omega_c = 0$, the resonance frequency of the linear (blue) disk with gain is $\omega_- = 0.503\,461i$, while the linear part of the resonance frequency of the nonlinear disk is $\omega_+ = 2\omega_-$ and its Kerr coefficient is $\chi = 1$. Right transmission and right reflection are diverging at $k^\star = 0.754\,559$ (in units of couplings), indicating the existence of a right SSDS mode. (b) The corresponding phases. We observe the π phase shift at the spectral singularity (green ♦ and blue ▲).

right incident field becomes $k^\star = 0.338\,167$. This numerical prediction is in agreement with our calculated t_i and r_i, with $i = l, r$, and their corresponding phases in Figs. 2(a) and 2(b).

IV. SPECTRAL SINGULARITY WITH DIRECTIONAL SENSITIVITY IN COUPLED MICRORESONATORS

In the previous section, we used an abstract model to show the existence of SSDS modes. In this section, we identify the SSDS modes in an experimentally feasible system, namely, an array of coupled disk resonators depicted schematically in the upper inset of Fig. 3. The array is composed of two distinct disk resonators, one with a linear resonance frequency and the other with a nonlinear resonance frequency, embedded in an array of coupled disk resonators playing the role of the transmission line. In this system, each disk supports two degenerate modes: a clockwise a^+ and a counterclockwise a^-. Using the coupled mode theory, we express the dynamics of the total field amplitudes, $\Phi = a^+ + a^-$, in each disk [27,28],

$$\begin{aligned} i\dot{\Phi}_n &= -\Phi_{n-1} - \Phi_{n+1} + \omega_c \Phi_n \quad (|n| \geqslant 2),\\ i\dot{\Phi}_n &= -\delta_{n\pm1,2}\Phi_{n\pm1} - \delta_{n,\pm1}\Phi_\pm + \omega_c\Phi_n \quad (n = \pm 1),\\ i\dot{\Phi}_+ &= -\Phi_1 - \Phi_- + (\omega_+ + \chi|\Phi_+|^2)\Phi_+,\\ i\dot{\Phi}_- &= -\Phi_{-1} - \Phi_+ + \omega_-\Phi_-, \end{aligned} \tag{15}$$

where we used Kronecker δ notation

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{16}$$

Furthermore, Φ_n, and $\Phi_\pm$ are the total field amplitudes at the disk n, and nonlinear (linear) cavity with subindex $+(-)$, respectively. The $n \geqslant 1$ refers to disks after the nonlinear disk with resonance ω_+, and $n \leqslant -1$ refers to disks before the disk with resonance ω_- . Note that we have assumed all the couplings are equal (normalized to one) and the nonlinear disk has a Kerr-type nonlinearity. The resonance frequency of the disks in the chain is denoted by ω_c, which we assume is equal to zero without loss of generality. Moreover, the resonance of the nonlinear disk is $(\omega_+ + \chi|\Phi_+|^2)$, while the resonance of the linear disk with field amplitude ϕ_- is given by ω_-. The chain supports the dispersion relation $\omega = -2\cos(k)$, with $-\pi \leqslant k \leqslant \pi$. In the elastic scattering process for which $\Phi = \psi e^{-i\omega t}$, the stationary modal amplitudes of the system have the asymptotic behavior $\psi_n = F_l e^{ik(n+1)} + B_l e^{-ik(n+1)}$ for $n \leqslant -1$ and $\psi_n = F_r e^{ik(n-1)} + B_r e^{-ik(n-1)}$ for $n \geqslant 1$, respectively.

For the left incident field where $B_r = 0$ and $F_r = 1$, it is easy to show that

$$\begin{aligned} t_l &= \frac{e^{3ik} - e^{5iq}}{1 + e^{ik}(\omega_- + \beta) + e^{2ik}(\beta\omega_- - 1)},\\ r_l &= \frac{-e^{3ik}[\omega_- + e^{ik}(\omega_- + e^{ik})\beta]}{1 + e^{ik}(\omega_- + \beta) + e^{2ik}(\beta\omega_- - 1)}, \end{aligned} \tag{17}$$

and the left SSDS mode wave vector is given by

$$k^\star = -i\ln\left(\frac{-2}{\omega_- + \beta \pm \sqrt{(\beta - \omega_-)^2 + 4}}\right), \tag{18}$$

where $\beta = \chi + \omega_+$. Assuming $B_l = 1$ and $F_l = 0$, the right transmission and right reflection can also be calculated as

$$t_r = \frac{e^{3ik} - e^{5ik}}{\chi\omega\_^*(1+e^{ik}\omega\_)^2 + e^{3ik}\chi\omega\_^2 + e^{ik}(\omega\_ + \beta) + e^{2ik}[\omega\_(2\chi+\omega_+) - 1] + 1},$$
$$r_r = \frac{e^{4ik}\{1 - e^{-ik}(1+e^{ik}\omega\_)[\chi|\omega\_|^2 + 2e^{-ik}\chi\mathrm{Re}(\omega\_) + i\sin(k)(2\chi\omega\_ + 1) + \cos(k) + \beta]\}}{e^{ik}(2\chi|\omega\_|^2 + \omega\_ + \beta) + e^{2ik}\{\omega\_[\chi(|\omega\_|^2+2) + \omega_+] - 1\} - i\chi\mathrm{Im}(\omega\_) + e^{3ik}\chi\omega\_^2 + \chi\mathrm{Re}(\omega\_) + 1}. \quad (19)$$

In Eq. (19), $\mathrm{Re}(\omega\_)$ and $\mathrm{Im}(\omega\_)$ are the real and imaginary parts of the $\omega\_$, respectively. The wave vector for a right SSDS has a very long closed form and thus we do not report it here. Similar to the nonlinear δ-function system, one can numerically search for $\omega_\pm$ and χ such that they make the $k^\star$, associated with a left SSDS [given in Eq. (18)] or right SSDS (not shown here), positive and real. For instance, the left SSDS occurs at $k^\star = 0.950\,497$ (normalized to the units of the couplings) when $\chi = 1, \omega\_ = 0.641\,322i$, and $\omega_+ = 2\omega\_$, while the right SSDS occurs at $k^\star = 0.754\,559$ when $\chi = 1, \omega\_ = 0.503\,461i$, and $\omega_+ = 2\omega\_$. Thus, for this system, the singularity mode (lasing threshold) appears first for the right fluctuations as a right SSDS is triggered by lower gain value. After that point, the system becomes nonlinear and our treatment is no longer valid. Thus, $k^\star = 0.950\,497$ is not a physical solution. In Fig. 3, we reported the transmissions and reflections and their corresponding phases for $\chi = 1, \omega\_ = 0.503\,461$, and $\omega_+ = 2\omega\_$.

V. CONCLUSION

In conclusion, we have developed a scattering matrix formalism for nonlinear systems, and then, using an abstract model of two δ functions, one nonlinear and one linear, we have shown that a nonlinear system with broken parity symmetry can have peculiar spectral singularities with sensitivity to the direction of excitation, the so-called SSDS. We discussed the possibility of getting left or right SSDS depending on the distance between the two δ functions. Furthermore, we found the right SSDS in an experimentally feasible system, namely, coupled disk resonators. While we have used a parity symmetry broken system to show the existence of SSDS, such singularities might be found with any broken symmetry that changes $\vec{k} \to -\vec{k}$ and in the presence of any kind of nonlinear process. The nonlinear process might occur due to the existence of a nonlinear material or using external driving, the so-called temporal modulation. While we discussed the SSDS in 1D systems, by using a similar method one can possibly extend the same notion to higher dimensions or systems with several emission channels. In higher dimensions, one should carefully define the directions. Our proposed spectral singularity might have applications in directional sensing and might be considered as a solution for hole-burning problems in laser cavities.

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[28] One can write the same set of equations for the subtraction of the clockwise and counterclockwise modes where all the couplings change their sign and the dispersion relation becomes $\omega = \omega_c + 2\cos(q)$. We carried out the same analysis and obtained the same results.