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On the importance of np-pairs in the isovector pairing model

Feng Pan[1,2], Chong Qi[3], Lianrong Dai[4,1], Grigor Sargsyan[2], Kristina D. Launey[2] and Jerry P. Draayer[2]

[1] *Department of Physics, Liaoning Normal University - Dalian 116029, China*
[2] *Department of Physics and Astronomy, Louisiana State University - Baton Rouge, LA 70803-4001, USA*
[3] *Department of Physics, KTH Royal Institute of Technology - 10691 Stockholm, Sweden*
[4] *Department of Physics, School of Science, Huzhou University - Huzhou, Zhejiang 313000, China*

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Abstract – It is shown that the isovector np-pair number operator can be realized in the $O(5)$ quasi-spin $SU_\Lambda(2) \otimes SU_I(2)$ basis. The computation of the isovector np-pair number is demonstrated for even-even and odd-odd *ds*-shell nuclei described by the charge-independent mean field plus isovector pairing model restricted within the $O(5)$ seniority-zero subspace, thereby binding energies and low-lying $J = 0^+$ excited states of these *ds*-shell nuclei are fit, along with estimates for the isovector neutron-proton pairing contributions. For reasonable neutron-proton pairing strengths the isovector np-pairing energy contribution to the total binding energy in odd-odd $N = Z$ nuclei is systematically larger than that in the even-even nuclei. In sum, the results suggest that the isovector np-pairing mode is favored in odd-odd $N = Z$ nuclei; and additionally, a decrease in the double binding-energy difference for odd-odd nuclei is primarily due to the symmetry and Wigner energy contributions to the binding energy.

In atomic nuclei it is evident from both theoretical and experimental studies that in addition to neutron-neutron (nn) and proton-proton (pp) pairing, neutron-proton (np) pairing is also very important, especially in $N{\sim}Z$ nuclei [1–8]. Though isoscalar $T = 0$ np-pairing in nuclei becomes important in the high-energy regime [9,10], isovector $T = 1$ np-pairing often dominates in the low-energy regime [8,11], where the shell model mean field plus isovector pairing seems to provide a simple and clear picture of the np-pairing correlations [7,12–14]. Exact solution of the charge-independent mean field plus isovector pairing has been reported in [15,16]. However, since the basis used in [15,16] is equivalent to the tensor product of relevant $O(5)$ irreducible representations (irreps) adapted to the $O_T(3) \otimes O_{\mathcal{N}}(2)$ subgroup which is related to the isospin and the number of valence nucleons, it is not possible to define a simple operator that counts the number of isovector np-pairs, so the corresponding two-body np-pairing interaction related to the isovector np-pairing energy contribution is often used to account for the number of isovector np-pairs instead [17].

Besides shell model calculations based on various algorithms with effective interactions [18], which have been proven to be very successful in describing *ds*-shell nuclei, model calculations based on group theoretic or algebraic labelling scheme within a multi-particle subspace of the same shell model configuration is equally feasible [12,13,19], which is equivalent to choosing a complete set of orthonormal eigenstates of a set of commuting operators, or the subgroup invariants [20] constructed from the single-particle creation and annihilation operators of the shell model mean field. Moreover, though the agreement of the standard shell model results [18] with experiment suggests that the isovector and isoscalar pairing interactions are realistic, the actual interaction strengths are subject to considerable uncertainty due to the fact that the competition of the isovector and isoscalar pairing, deformation, and other correlations leads to a very complex picture [8]. Therefore, in order to address the aforementioned issue more directly, in this work, we only consider isovector pairing corresponding to the $J = 0$ and $T = 1$ two-body interactions adopted in the standard shell model calculations [18] within a spherical mean field in the charge-independent form with a Hamiltonian that can be written as [7,12–16]

$$\hat{H}_0 = \sum_{i=1}^{p} \epsilon_i \, \hat{n}_i - G \sum_{\rho} A_\rho^+ A_\rho, \tag{1}$$

where p is the number of j-orbits of the mean field considered, ϵ_i is the single-particle energy in the i-th orbit, $\hat{n}_i = \sum_{m\,m_t} a^\dagger_{j_i m\,t m_t} a_{j_i m\,t m_t}$ is the valence nucleon number operator in the i-th j-orbit, $A^+_\rho = \sum_{i=1}^p A^+_\rho(i)$ and $A_\rho = \sum_{i=1}^p A_\rho(i)$ $[\rho = -1, 0, 1]$ are collective $J = 0$ and $T = 1$ nn-, np-, and pp-pairing operators with $A^+_\rho(i) = \frac{1}{2}\sqrt{2j_i+1}(a^\dagger_{j_i t} \times a^\dagger_{j_i t})^{01}_{0\rho}$, $A_\rho(i) = (A^+_\rho(i))^\dagger$, in which $(a^\dagger_{j_i t} \times a^\dagger_{j_i t})^{01}_{0\rho}$ stands for the angular momentum and isospin coupling with $j_i \otimes j_i \downarrow 0$ and $t \otimes t \downarrow 1$, and $G > 0$ is the overall isovector pairing strength. Here $a^\dagger_{j_i m\,t m_t}$ ($a_{j_i m\,t m_t}$) is the creation (annihilation) operator for a valence nucleon in the i-th j-orbit with isospin $t = 1/2$ and angular momentum projection m and isospin projection m_t. In order to demonstrate the evaluation of the number of isovector np-pairs, some low-lying $J^\pi = 0^+$ level energies of even-even and odd-odd $A = 18$–28 nuclei up to the middle of the ds-shell outside the ^{16}O core are fit by the charge-independent mean field plus isovector pairing. To fit binding energies and some low-lying $\mathrm{J}^\pi = 0^+$ level energies of these nuclei, in addition to $\hat{H}_0$ shown in (1), the Coulomb energy and the symmetry energy (with the isospin-dependent part of the Wigner energy contribution) are considered with the same model Hamiltonian as that used in [21]:

$$\begin{aligned}\hat{H} \;=\; & -\mathrm{BE}(^{16}\mathrm{O}) + \overline{\epsilon}(\hat{n})\,\hat{n} + \hat{H}_0 + E_{\mathrm{c}}(A,Z) \\ & -E_{\mathrm{c}}(16{,}8) + \alpha_{\mathrm{sym}}(A)\mathbf{T}\cdot\mathbf{T},\end{aligned} \tag{2}$$

where $\mathrm{BE}(^{16}\mathrm{O}) = 127.619\,\mathrm{MeV}$ is the binding energy of the ^{16}O core taken as the experimental value, $\overline{\epsilon}(n)$ is the average binding energy per valence nucleon in the ds-shell with $j_1 = 1/2$, $j_2 = 3/2$, and $j_3 = 5/2$ orbits, of which the function of $\hat{n}$ is determined from a best fit to binding energies of all ds-shell nuclei considered,

$$E_{\mathrm{c}}(A,Z) = 0.699\frac{Z(Z-1)}{A^{1/3}}\left(1 - \frac{0.76}{(Z(Z-1))^{1/3}}\right)\ (\mathrm{MeV}) \tag{3}$$

is the Coulomb energy [22], and

$$\alpha_{\mathrm{sym}}(A) = \frac{1}{A}\left(134.4 - \frac{203.6}{A^{1/3}}\right)\ (\mathrm{MeV}) + \delta\alpha(A) \tag{4}$$

is the parameter of the symmetry energy, of which the first term is taken to be the empirical global symmetry energy parameter provided in [22], while $\delta\alpha(A)$ is adjusted according to the experimental binding energy of the nucleus with a given mass number A needed to account for the local deviation from the first term when the Hamiltonian (2) is used. The experimentally deduced single-particle energies above the ^{16}O core with $\epsilon_1 = \epsilon_{1s_{1/2}} = -3.27\,\mathrm{MeV}$, $\epsilon_2 = \epsilon_{0p_{3/2}} = 0.94\,\mathrm{MeV}$, $\epsilon_3 = \epsilon_{0d_{5/2}} = -4.14\,\mathrm{MeV}$ [21] are taken. In order to get a better fitting quality for the low-lying $J^\pi = 0^+$ level energies, the overall isovector pairing strength is taken as $G = 1\,\mathrm{MeV}$ for all the nuclei fitted, which is very close to the value used in [23] with $G = 20/A\,\mathrm{MeV}$. Comparing to the standard shell model calculations in the ds-shell [18], the empirical Coulomb energy, the symmetry energy, and the average binding energy terms are introduced to replace the other two-body interaction terms of the standard shell model calculations [18], except for $J = 1$ and $T = 0$ isoscalar pairing related to the $J = 1$ and $T = 0$ matrix elements of the two-body interaction in the standard shell model calculation [18], which is not included in (2). Therefore, (2) can be considered as an approximation to the standard shell model Hamiltonian focusing on the isovector pairing only, from which the number of the isovector np-pairs in low-lying $J^\pi = 0^+$ states can be estimated reasonably under the condition that the $J = 1$ and $T = 0$ isoscalar pairing can be neglected.

For convenience, we choose the proton and neutron quasi-spin $SU_\Lambda(2) \otimes SU_I(2)$ group as the subgroup of $O(5)$ with generators of the proton quasi-spin group $SU_\Lambda(2)\{\varsigma^{(i)}_+ = A^\dagger_1(i),\ \varsigma^{(i)}_- = A_1(i),\ \varsigma^{(i)}_0 = \hat{n}_{\pi,i}/2 - \Omega_i/2\}$ and those of the neutron quasi-spin group $SU_I(2)\{\tau^{(i)}_+ = A^\dagger_{-1}(i),\ \tau^{(i)}_- = A_{-1}(i),\ \tau^{(i)}_0 = \hat{n}_{\nu,i}/2 - \Omega_i/2\}$. Details of the relation between the group chain $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ and $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ are provided in [24] and shown in the Supplementary Material `Supplementarymaterial.pdf` (SM), which provides details of the $O(5)$ basis and the reduced matrix elements needed in the calculation. For any orbit, the irreducible representation (irrep) of $O(5)$ is denoted by $(\Omega - v/2, t)$, where the $O(5)$ seniority number of valence nucleons v and the reduced isospin t indicate that there are v nucleons coupled to the isospin t, which are free from the angular momentum $J = 0$ and $T = 1$ pairs. The orthonormal basis vectors of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2) \supset U_\Lambda(1) \otimes U_I(1)$ are labeled as [12,13,24]

$$\left|\begin{array}{c} (\Omega - v/2,\ t) \\ \Lambda = \dfrac{1}{2}(\Omega - v_\pi)\ I = \dfrac{1}{2}(\Omega - v_\nu) \\ \mu \qquad \nu \end{array}\right\rangle, \tag{5}$$

where μ and ν are eigenvalues of ς_0 and τ_0, respectively, Λ and I are the proton and neutron quasi-spin quantum numbers, in which v_π and v_ν are the proton and neutron seniority numbers, respectively, with [24]

$$\begin{aligned} v_\pi &= (v/2 - t + p) + q, \\ v_\nu &= (v/2 + t - p) + q, \end{aligned} \tag{6}$$

where $p = 0, 1, \ldots, 2t$ and $q = 0, 1, \ldots, \Omega - v/2 - t$, indicating that there are v_π and v_ν valence protons and neutrons not coupled to $J = 0$ pp- and nn-pairs, respectively. However, there may be a portion of the valence protons and neutrons that are coupled into $J = 0$ np-pairs, which is indicated by the number q. As is clearly indicated in (6), besides the number of valence protons or neutrons in the np-pairs q, there are $v/2 - t + p$ valence protons and $v/2 + t - p$ valence neutrons with total v valence nucleons that are coupled to the reduced isospin t

with $J \neq 0$. Thus, q is the number of isovector np-pairs in a given orbit, which shows that the isovector np-pair number operator is well defined in the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ labelling scheme.

The Hamiltonian (2) is diagonalized within the $O(5)$ seniority-zero subspace spanned by the tensor product basis $\otimes_{i=1}^{p}(\Omega_i, 0)$ in the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2) \supset U_\Lambda(1) \otimes U_I(1)$ labelling scheme with $p = 3$. Eigenstates of (2) within the $O(5)$ seniority-zero $J^\pi = 0^+$ subspace are denoted as

$$|\xi; n, M_T\rangle = \sum_{n_i M_T(i)} \sum_{q_i=0}^{\Omega_i} C^{\xi;\ q_1,\ldots,q_p}_{n_1 m_T(1),\ldots,n_p m_T(p)} \times \left| \begin{array}{lll} (\Omega_1, 0); & \ldots; & (\Omega_p, 0) \\ \frac{1}{2}(\Omega_1 - q_1); & \ldots; & \frac{1}{2}(\Omega_p - q_p) \\ n_1, m_T(1); & \ldots; & n_p, m_T(p) \end{array} \right\rangle, \quad (7)$$

where the eigenstate $|\xi; n, M_T\rangle$ with total number of valence nucleons $n = \sum_{i=1}^{p} n_i$ and total isospin projection $M_T = \sum_{i=1}^{p} m_T(i)$ is expended in terms of the p copies of the $O(5)$ tensor product basis $\otimes_{i=1}^{p}(\Omega_i, 0)$ in the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2) \supset U_\Lambda(1) \otimes U_I(1)$ labelling scheme with

$$\begin{aligned} I_i &= \Lambda_i = \frac{1}{2}(\Omega_i - q_i), \\ \mu_i &= \frac{1}{4}(n_i + 2m_T(i) - 2\Omega_i), \\ \nu_i &= \frac{1}{4}(n_i - 2m_T(i) - 2\Omega_i), \end{aligned} \quad (8)$$

for $i = 1, 2, 3$, $C^{\xi;q_1,\ldots,q_p}_{n_1 m_T(1),\ldots,n_p m_T(p)}$ is the corresponding expansion coefficient, and ξ labels the ξ-th eigenstate with the same n and M_T. Since $M_T = \sum_{i=1}^{p} m_T(i)$ is a good quantum number, this diagonalization scheme is similar to the M_T-scheme realized in the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2) \supset U_\Lambda(1) \otimes U_I(1)$ basis, which is equivalent to the M_T-scheme with $J^\pi = 0^+$ states adopted in the standard shell model calculation [18]. The details of the $O(5)$ basis and the reduced matrix elements needed in the calculation are shown in the SM. Moreover, it is clearly shown that the eigenstate (7) is a linear combination of the $O(5)$ tensor product states with different proton and neutron seniority numbers $v_{\pi i} = v_{\nu i} = q_i$ $(i = 1, 2, 3)$ for $0 \leq q_i \leq \Omega_i$, where q_i is exactly the number of $T = 1$ np-pairs in the i-th orbit. The main difference from the standard shell model basis is that only $J = 0$ and $T = 1$ pairing operators are used in the construction of (7). Other $O(5)$ seniority-nonzero states constructed by $(J = \text{even}, T = 1)$ and $(J = \text{odd}, T = 0)$ pair operators coupled to $J = 0$ and at least with the total $O(5)$ seniority number $v = 4$ are not considered as an approximation, of which $J^\pi = 0^+$ isovector pairing matrix elements are comparatively small, so that these seniority-nonzero states have less contribution to the low-lying $J^\pi = 0^+$ states as long as the isoscalar pairing interaction is negligible. As shown in our previous works [21,25–27], such an approximation seems adequate as far as binding energies and a few low-lying $J^\pi = 0^+$ level energies are concerned.

The best fit to the binding energies and a few lowest $J^\pi = 0^+$ level energies of these nuclei yields

$$\overline{\epsilon}(\hat{n}) = -2.3325 - 0.2000\hat{n} - 0.0125\hat{n}^2 \text{ (MeV)}, \quad (9)$$

of which the first constant is very close to the value of the average binding energy per valence nucleon with $\epsilon_{\text{avg}} = -2.301$ MeV used in [21], and the contribution from the second term to the binding, which is related to the other two-body interactions in the standard shell model calculation [18] becomes smaller because a relatively larger pairing strength is used in the present calculation, while the third term is related to the three-body interaction as an additional correction. The parameter $\delta\alpha(A)$ (in MeV), which is obtained from the fitting, is given by $\delta\alpha(18) = -0.025$, $\delta\alpha(20) = -0.700$, $\delta\alpha(22) = -0.940$, $\delta\alpha(24) = -0.500$, $\delta\alpha(26) = 1.900$, and $\delta\alpha(28) = -0.005$. Since the binding energies and a few low-lying $J^\pi = 0^+$ level energies of even-even and odd-odd $A = 18$–28 nuclei were fit together, there remain deviations between the fitted and experimental binding energies, with a root mean square deviation $\sigma_{\text{BE}} = 0.32$ MeV, except ^{22}F and ^{22}Al for which $J^\pi = 0^+$ level energies are not available experimentally. Table 1 shows the lowest experimentally known $J^\pi = 0^+$ level energies (in MeV) of these even-even and odd-odd *ds*-shell nuclei. The corresponding shell model results obtained by using the KSHELL code [28] with the USD (W) interaction [29] are also provided for comparison. Though direct computation of the overlaps of the $J = 0^+$ states of this model with those of the shell model is not an easy task, the results provided in table 1 show that the low-lying $J = 0^+$ level energies fitted by the $O(5)$ model restricted within the $O(5)$ seniority-zero subspace are comparable with those of the standard shell model calculation. The root mean square deviation for these excited $J^\pi = 0^+$ level energies is $\sigma_{\text{level}} = 1.29$ MeV in this $O(5)$ model and 0.36 MeV in the standard shell model, while the average deviation of the excited levels, $\phi = \sum_i |E^i_{\text{Th}} - E^i_{\text{Exp}}| / \sum_i E^i_{\text{Exp}}$, where the sum is over all the excited level energies of these nuclei fitted, is $\phi = 16.1\%$ in the $O(5)$ model and 4.0% in the standard shell model calculation, respectively.

In addition, when the ground state of the nucleus is not a $J^\pi = 0^+$ state, which cannot be determined from present calculation for $J = 0^+$ states only, the eigenenergy of (2) is given by

$$E(\xi, T, J = 0) = -\text{BE}(Z, N) + E_{\text{ex}}(\xi, T, J = 0), \quad (10)$$

where $E_{\text{ex}}(\xi, T, J = 0) > 0$ is the excitation energy of the ξ-th excited state with isospin T and $J = 0$. The theoretical value of BE(Z, N) is adjusted to reproduce a reasonable value of the excitation energy $E_{\text{ex}}(\xi, T, J = 0)$. Due to the Coulomb energy contribution and the freedom in adjusting the binding energy with a reasonable value of the excitation energy in this case, there is about a few

Table 1: Binding energy (BE) (in MeV) and a few lowest $J^\pi = 0^+$ level energies (in MeV) of the 22 even-even and odd-odd *ds*-shell nuclei fit by (2) (Th), where T_ξ denotes the ξ-th excited state with the same T, the experimental data (Exp) are taken from [30] and [31] for the evaluated isobaric analogue states, "–" indicates the corresponding energy level was not observed in experiments, and the standard shell model results (SM) are obtained by using the KSHELL code [28] with the USD (W) interaction [29], and the parameters of (2) are the same as those used in fitting the binding energies.

^{18}O	Exp	Th	SM	^{18}F	Exp	Th	SM	^{18}Ne	Exp	Th	SM
BE	139.81	140.00			137.37	137.31			132.14	132.04	
$T_\xi = 1_g$	0	0	0	$T_\xi = 1_1$	1.04	1.04	1.19	$T_\xi = 1_g$	0	0	0
$T_\xi = 1_2$	3.63	5.71	4.32	$T_\xi = 1_2$	4.75	6.75	5.51	$T_\xi = 1_2$	3.58	5.71	4.32
^{20}O	Exp	Th	SM	^{20}F	Exp	Th	SM	^{20}Ne	Exp	Th	SM
BE	151.37	151.20			154.40	154.40			160.65	160.41	
$T_\xi = 2_g$	0	0	0	$T_\xi = 1_1$	3.53	1.23	3.49	$T_\xi = 0_g$	0	0	0
$T_\xi = 2_2$	4.46	5.07	5.04	$T_\xi = 2_1$	6.52	6.80	6.52	$T_\xi = 0_2$	6.73	5.90	6.76
								$T_\xi = 1_1$	13.64	11.33	13.64
								$T_\xi = 2_1$	16.73	16.90	16.66
^{20}Na	Exp	Th	SM	^{20}Mg	Exp	Th	SM	^{22}O	Exp	Th	SM
BE	145.97	145.97			134.56	133.84			162.04	161.45	
$T_\xi = 1_1$	3.09	1.48	3.49	$T_\xi = 2_g$	0	0	0	$T_\xi = 3_g$	0	0	0
$T_\xi = 2_1$	6.53	7.05	6.52	$T_\xi = 2_2$	–	5.07	5.04	$T_\xi = 3_2$	4.91	4.35	4.62
^{22}Ne	Exp	Th	SM	^{22}Na	Exp	Th	SM	^{22}Mg	Exp	Th	SM
BE	177.77	178.23			174.15	174.14			168.58	168.86	
$T_\xi = 1_g$	0	0	0	$T_\xi = 1_1$	0.66	0.36	0.66	$T_\xi = 1_g$	0	0	0
$T_\xi = 1_2$	6.24	5.03	6.34	$T_\xi = 1_2$	–	5.39	7.01	$T_\xi = 1_2$	5.95	5.03	6.34
^{22}Si	Exp	Th	SM	^{24}Ne	Exp	Th	SM	^{24}Na	Exp	Th	SM
BE	133.28	133.33			191.84	191.60			193.52	193.52	
$T_\xi = 3_g$	0	0	0	$T_\xi = 0_g$	0	0	0	$T_\xi = 1_1$	3.68	0.37	3.33
				$T_\xi = 2_2$	4.77	4.30	4.66	$T_\xi = 2_1$	5.97	6.24	5.88
^{24}Mg	Exp	Th	SM	^{24}Al	Exp	Th	SM	^{24}Si	Exp.	Th.	SM
BE	198.26	198.85			183.59	183.59			172.01	171.52	
$T_\xi = 0_g$	0	0	0	$T_\xi = 1_1$	–	0.48	3.33	$T_\xi = 2_g$	0	0	0
$T_\xi = 0_2$	6.43	5.16	7.56	$T_\xi = 2_1$	5.96	6.35	5.88				
$T_\xi = 1_1$	13.04	10.49	12.87								
$T_\xi = 2_1$	15.44	16.35	15.43								

^{26}Mg	Exp	Th	SM	^{26}Al	Exp	Th	SM	^{26}Si	Exp	Th	SM	^{28}Si	Exp	Th	SM
BE	216.68	216.78			211.89	211.89			206.04	206.09			247.74	247.67	
$T_\xi = 1_g$	0	0	0	$T_\xi = 1_1$	0.23	0.23	0.08	$T_\xi = 1_g$	0	0	0	$T_\xi = 0_g$	0	0	0
$T_\xi = 1_2$	3.59	4.24	3.68	$T_\xi = 1_2$	3.75	4.47	3.76	$T_\xi = 1_2$	3.36	4.24	3.68	$T_\xi = 0_2$	4.98	4.25	5.01
$T_\xi = 1_3$	4.97	5.13	5.20	$T_\xi = 1_3$	5.20	5.36	5.29	$T_\xi = 1_3$	4.83	5.13	5.20	$T_\xi = 1_1$	10.27	10.27	10.29

hundreds of keV energy difference in these excitation energies of mirror nuclei with $J \neq 0$ ground state as shown in table 1.

Panel (a) of fig. 1 shows the double binding-energy difference defined as [32]

$$\delta V_{\rm pn} = \frac{1}{4}({\rm BE}(Z,N) - {\rm BE}(Z-2,N) - {\rm BE}(Z,N-2) + {\rm BE}(Z-2,N-2)). \quad (11)$$

These results show that the $\delta V_{\rm pn}$ are comparatively smaller for odd-odd $N = Z$ nuclei with $A = 22$ and 26. Since the one- and two-body interaction dominating average binding energy term (9) and the Coulomb energy term of (2) only contribute an overall Z- and N-independent constant to $\delta V_{\rm pn}$, the symmetry energy seems to be the main source that alters the pairing gap staggering pattern. This is consistent with the claim made in [5] that

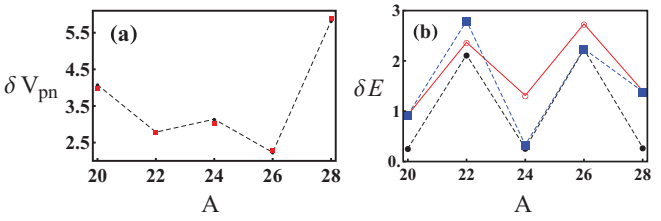

Fig. 1: (a) The double binding-energy difference $\delta V_{\rm pn}$ (in MeV) defined in (11) for even-even and odd-odd $N = Z$ ds-shell nuclei, where the (red) solid squares are the experimental data, and the (black) dots connected with the dashed lines are the results of the present model. (b) The double pairing energy difference (in MeV) defined in (12), where the (red) open circles connected with the solid lines are calculated from (12) with the isovector np-pairing energy, the (black) solid dots connected with the dashed lines are calculated from (12) with the total pairing energy, and the (blue) solid squares connected with the dashed lines are $\widetilde{\delta V_{\rm pn}}$ values calculated from (11) with $\widetilde{\rm BE}(Z,N)$.

the double binding-energy difference (11) actually provides evidence for the importance of the Wigner energy contribution to the binding energy, and additionaly demonstrates that the isovector np-pairing energy contribution is relatively large in the even-even $N = Z$ nuclei [21]. Alternatively, instead of ${\rm BE}(Z,N)$, we also calculated the double pairing energy difference defined as

$$\delta E = \frac{1}{4}(E^{(1)}(Z,N) - E^{(1)}(Z-2,N) - E^{(1)}(Z,N-2) + E^{(1)}(Z-2,N-2)), \quad (12)$$

where

$$E^{(1)}(Z,N) = \langle \xi = 1, n, TM_T | \hat{H}_{\rm P} | \xi = 1, n, TM_T \rangle \quad (13)$$

is the lowest $J^\pi = 0^+$ state expectation value of either $\hat{H}_P = G\sum_\rho A^+_\rho A_\rho$ or $\hat{H}_P = G\, A^+_0 A_0$, of which the former is the total pairing energy contribution, while the latter is the isovector np-pairing energy contribution to the binding energy. By substituting ${\rm BE}(Z,N)$ used in (11) with

$$\widetilde{\rm BE}(Z,N) = {\rm BE}(Z,N) + \frac{1}{A}\left(134.4 - \frac{203.6}{A^{1/3}}\right) T(T+1)\ ({\rm MeV}), \quad (14)$$

which removes the symmetry energy contribution to the binding energy, the resultant $\widetilde{\delta V_{\rm pn}}$ obtained from (11) should be close to the double pairing energy difference (12). And indeed, as shown in panel (b) of fig. 1, the value of $\widetilde{\delta V_{\rm pn}}$ is very close to the δE values calculated with the total pairing energy and the np-pairing energy contribution to the binding energy. Most noticeably, in comparison to $\delta V_{\rm pn}$, the staggering pattern is clearly evident, and, to the contrary, the actual np-pairing energy in the odd-odd $N = Z$ nuclei turns out to be comparatively strong, which shows that the Wigner energy contribution plays the important role in the lowest $J^\pi = 0^+$ states of odd-odd $N = Z$ nuclei. Table 2 shows the actual nn-, pp-, and np-pairing energy at the ground state or the lowest $J^\pi = 0^+$ excited state of (2) for these nuclei defined by

$$\begin{aligned} E^{(1)}_{\rm np} &= G\langle \xi = 1, n, TM_T | A^+_0 A_0 | \xi = 1, n, TM_T \rangle, \\ E^{(1)}_{\rm nn} &= G\langle \xi = 1, n, TM_T | A^+_{-1} A_{-1} | \xi = 1, n, TM_T \rangle, \\ E^{(1)}_{\rm pp} &= G\langle \xi = 1, n, TM_T | A^+_1 A_1 | \xi = 1, n, TM_T \rangle, \end{aligned} \quad (15)$$

and the percentage of the isovector np-pairing energy with respect to the total pairing energy

$$\eta_{\rm np} = E^{(1)}_{\rm np}/(E^{(1)}_{\rm np} + E^{(1)}_{\rm nn} + E^{(1)}_{\rm pp}). \quad (16)$$

It can be seen from the results that $E^{(1)}_{\rm nn}$ in the $N = Z + 2$ nuclei is the same as $E^{(1)}_{\rm pp}$ in the $Z = N + 2$ mirror nuclei, while $E^{(1)}_{\rm nn} = E^{(1)}_{\rm pp}$ in the $N = Z$ nuclei due to the charge-independent approximation adopted. However, $E^{(1)}_{\rm np} = E^{(1)}_{\rm nn} = E^{(1)}_{\rm pp}$ in even-even $N = Z$ nuclei, while $E^{(1)}_{\rm np} > E^{(1)}_{\rm nn} = E^{(1)}_{\rm pp}$ in odd-odd $N = Z$ nuclei, shows that the isovector np-pairing energy contribution to the binding energy is the largest in odd-odd $N = Z$ nuclei in comparison to that in the adjacent $N \neq Z$ and even-even $N = Z$ nuclei.

Finally, the average number of the isovector np-pairs in the lowest $J^\pi = 0^+$ state can be defined as

$$k_{\rm np} = \langle \xi = 1, n, TM_T | \hat{q} | \xi = 1, n, TM_T \rangle, \quad (17)$$

with $\hat{q} = \sum_{i=1}^{p}(\Omega_i - 2\hat{\Lambda}_i)$. Hence, the average number of nn-pairs and that of pp-pairs are given by

$$k_{\rm nn} = (n_\nu - k_{\rm np})/2,\ k_{\rm pp} = (n_\pi - k_{\rm np})/2. \quad (18)$$

These values for the each nucleus at the lowest $J^\pi = 0^+$ state are shown in table 2. Since the number of isovector np-pairs is not a conserved quantity in general, its fluctuation in the lowest $J^\pi = 0^+$ state of these nuclei defined as

$$\Delta k_{\rm np} = (\langle \xi = 1, n, TM_T | (\hat{q} - k_{\rm np})^2 | \xi = 1, n, TM_T \rangle)^{1/2} \quad (19)$$

is also provided. It can be observed from table 2 that the $k_{\rm np}$ value is a definite integer for nuclei with less than or equal to one valence neutron or proton, for which the $k_{\rm np}$ value is also easily countable, while (17) must be used for evaluating $k_{\rm np}$ for nuclei with more valence neutrons and protons. It is obvious that the $k_{\rm np}$ value is indeed relatively large in the odd-odd $N = Z$ nuclei, which is consistent to the larger isovector np-pairing energy contribution to the binding energy shown in table 2, while the average number of the isovector np-pairs $k_{\rm np}$ in the even-even nuclei is considerably small with very large fluctuation. The $\Delta k_{\rm np}$ value in these even-even $N = Z$ nuclei is almost two times of the corresponding average value. Though the isovector np-pair occupation number defined as

$$\zeta_{\rm np} = k_{\rm np}/k, \quad (20)$$

Table 2: The isovector np-, nn-, and pp-pairing energy contribution (in MeV) to the binding energy and the percentage of the isovector np-pairing energy with respect to the total pairing energy $\eta_{\rm np}$ in the 22 even-even and odd-odd *ds*-shell nuclei described within the seniority-zero subspace of the model, where the average number of the isovector np-pairs $k_{\rm np}$ and its fluctuation $\Delta k_{\rm np}$, and the isovector np-pair occupation number $\zeta_{\rm np}$ in the $J^\pi = 0^+$ ground state or the lowest $J^\pi = 0^+$ state of these nuclei are also shown.

Nucleus	n	Isospin	$E^{(1)}_{\rm np}$	$E^{(1)}_{\rm nn}$	$E^{(1)}_{\rm pp}$	$k_{\rm np}$	$\Delta k_{\rm np}$	$k_{\rm nn}$	$k_{\rm pp}$	$\eta_{\rm np}$	$\zeta_{\rm np}$
$^{18}_{8}O_{10}$	2	$T=1$	0	5.036	0	0	0	1	0	0%	0%
$^{18}_{9}F_{9}$	2	$T=1$	5.036	0	0	1	0	0	0	100%	100%
$^{18}_{10}Ne_{8}$	2	$T=1$	0	0	5.036	0	0	0	1	0%	0%
$^{20}_{8}O_{12}$	4	$T=2$	0	7.945	0	0	0	2	0	0%	0%
$^{20}_{9}F_{11}$	4	$T=1$	2.568	2.568	0	1	0	1	0	50%	50%
$^{20}_{10}Ne_{10}$	4	$T=0$	3.707	3.707	3.707	0.497	0.865	0.7515	0.7515	33.33%	24.85%
$^{20}_{11}Na_{9}$	4	$T=1$	2.568	0	2.568	1	0	0	1	50%	50%
$^{20}_{12}Mg_{8}$	4	$T=2$	0	0	7.945	0	0	0	2	0%	0%
$^{22}_{8}O_{14}$	6	$T=3$	0	8.666	0	0	0	3	0	0%	0%
$^{22}_{10}Ne_{12}$	6	$T=1$	2.226	7.356	4.444	0.205	0.606	1.8975	0.8975	15.87%	6.83%
$^{22}_{11}Na_{11}$	6	$T=1$	9.573	2.226	2.226	1.756	0.940	0.622	0.622	68.25%	58.53%
$^{22}_{12}Mg_{10}$	6	$T=1$	2.226	4.444	7.356	0.205	0.606	0.8975	1.8975	15.87%	6.83%
$^{22}_{14}Si_{8}$	6	$T=3$	0	0	8.666	0	0	0	3	0%	0%
$^{24}_{10}Ne_{14}$	8	$T=2$	1.600	8.393	4.756	0.083	0.159	2.9585	0.9585	10.85%	2.08%
$^{24}_{11}Na_{13}$	8	$T=1$	5.061	4.167	2.681	1.404	0.645	1.798	0.798	42.49%	35.10%
$^{24}_{12}Mg_{12}$	8	$T=0$	6.000	6.000	6.000	0.711	1.342	1.6445	1.6445	33.33%	17.78%
$^{24}_{13}Al_{11}$	8	$T=1$	5.061	2.681	4.167	1.404	0.645	0.798	1.798	42.49%	35.10%
$^{24}_{14}Si_{10}$	8	$T=2$	1.600	4.756	8.393	0.083	0.159	0.9585	2.9585	10.85%	2.08%
$^{26}_{12}Mg_{14}$	10	$T=1$	3.615	7.917	7.186	0.234	0.449	2.883	1.883	19.31%	4.68%
$^{26}_{13}Al_{13}$	10	$T=1$	11.489	3.615	3.615	1.792	1.179	1.604	1.604	61.38%	35.84%
$^{26}_{14}Si_{12}$	10	$T=1$	3.615	7.186	7.917	0.234	0.449	1.883	2.883	19.31%	4.68%
$^{28}_{14}Si_{14}$	12	$T=0$	6.847	6.847	6.847	0.585	1.066	2.7075	2.7075	33.33%	9.75%

where $k = n/2$, in the even-even $N = Z$ nuclei is small, the isovector np-pairing energy is still comparable to the nn- or pp-pairing energy. Table 2 shows that the isovector np-pairing energy per pair $E^{(1)}_{\rm np}/k_{\rm np}$ is 2.31 and 4.63 times of $E^{(1)}_{\rm nn}/k_{\rm nn} = E^{(1)}_{\rm pp}/k_{\rm pp}$ in ^{24}Mg and ^{28}Si, respectively. In short, the number of isovector np-pairs is larger in odd-odd $N = Z$ nuclei as described by the charge-independent mean field plus isovector pairing model.

In summary, it is shown that the isvector np-pair number operator can be realized in the neutron and proton quasi-spin $SU_\Lambda(2) \otimes SU_I(2)$ chain, in which the average number of the isovector np-pairs and its fluctuation can be evaluated. The evaluation is demonstrated in the charge-independent mean field plus isovector pairing model, thereby binding energies and low-lying $J^\pi = 0^+$ excited states of even-even and odd-odd $N{\sim}Z$ *ds*-shell nuclei are fit, and in turn, the isovector np-, nn-, and pp-pairing energy contributions to the binding energy in the even-even and odd-odd *ds*-shell nuclei are estimated. The results corroborate earlier findings [25,26] that the isovector np-pairing energy contribution to the lowest $J = 0^+$ level energy in the odd-odd $N = Z$ nuclei is systematically larger than that in the even-even nuclei. It is also observed that the decrease in the double binding-energy difference for the odd-odd nuclei is mainly due to the symmetry energy, which includes the Wigner energy contribution, that alters the pairing staggering patten. Particularly, the average number of the isovector np-pairs in the $J^\pi = 0^+$ ground state or the lowest $J^\pi = 0^+$ excited state of the even-even and odd-odd *ds*-shell nuclei are evaluated, which shows that the average number of the isovector np-pairs $k_{\rm np}$ in the even-even $N = Z$ nuclei is considerably small with very large fluctuation in comparison to that in the odd-odd $N = Z$ nuclei, and leads to the conclusion that the isovector np-pairs increase in number in odd-odd $N = Z$ nuclei. On the other hand, although the isovector np-pair occupation number in the even-even $N = Z$ nuclei is rather small, the isovector np-pairing energy per pair is systematically larger than the nn- or pp-pairing energy per pair. It should be stated that the purpose of this work is mainly to provide the precise definition of the

isovector np-pair number operator. The computation of the isovector np-pair number, however, is demonstrated for the even-even and odd-odd $N\sim Z$ *ds*-shell nuclei described by the isovector pairing model restricted within the $O(5)$ seniority-zero subspace only, where the isoscalar np-pairs are not involved. In order to reveal the actual np-pair contents in these $N\sim Z$ nuclei, other $O(5)$ seniority-nonzero configurations must be considered, for which an alternative $O(8)$ model [33–35] should be more convenient. Nevertheless, as has been shown in our recent work on the $O(8)$ model [36], not only the binding energies and the low-lying $J^\pi = 0^+$ level energies shown in table 1, but also the isovector pairing energy contributions to the binding energies provided in table 2 are the same as those calculated from the $O(8)$ model, where the isoscalar np-pairs are also involved. Therefore, the conclusion of the present work on the isovector pairing energy contribution to the binding energies of these *ds*-shell nuclei is still valid even in the presence of isoscalar np-pairs.

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Supplemental Material to "On the importance of np-pairs in the isovector pairing model"

Feng Pan,[1,2] Chong Qi,[3] Lianrong Dai,[4,1] Grigor Sargsyan,[2] Kristina D. Launey,[2] and Jerry P. Draayer[2]

[1]*Department of Physics, Liaoning Normal University, Dalian 116029, China*
[2]*Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001, USA*
[3]*Department of Physics, KTH Royal Institute of Technology, 10691 Stockholm, Sweden*
[4]*Department of Physics, School of Science, Huzhou University, Huzhou, Zhejiang 313000, China*

1. Fock states in a given j-orbit with the group theoretical classification

In the shell model, let $\{a^\dagger_{j\,m_j,\,t\,m_t},\, a_{j\,m_j,\,t\,m_t}\}$ be a set of the (valence) nucleon creation and annihilation operators in the j-orbit, where m_j is the quantum number of the angular momentum projection, $t = 1/2$ is the quantum number of the isospin, and $m_t = 1/2$ or $-1/2$ is the quantum number of the isospin projection, respectively. It is well known that the total number of many-particle product (Fock) states, Γ, provided by $\{\Xi^\dagger_{\phi_n}|0\rangle = \prod^n_{m_j\,m_t} a^\dagger_{j\,m_j,t\,m_t}|0\rangle\}$, where $|0\rangle$ is the (valence) nucleon vacuum state, and ϕ_n, up to the permutations among n creation operators, stands for the n unequal sub-indices involved, is given by

$$\Gamma = \sum_{n=0}^{4j+2} \frac{(4j+2))!}{n!(4j+2-n)!} = 2^{4j+2}, \tag{1}$$

which is due to the fact that the maximal number of creation operators involved in the nonzero many-particle product states is $4j+2$ restricted by the Pauli excursion. It is obvious [1, 2] that the set of operators $\{Q_{\phi_n,\phi'_{n'}} = \Xi^\dagger_{\phi_n}\Xi_{\phi'_{n'}},\, 1 \le n, n' \le 4j+2\}$ generate the unitary group $U(2^{4j+2})$. The set of the many-particle product (Fock) states $\{\Xi^\dagger_{\phi_1}|0\rangle, \cdots, \Xi^\dagger_{\phi_{4j+2}}|0\rangle\}$ spans a complex linear space for the fundamental irrep $[1, 0, \cdots, 0]$ of $U(2^{4j+2})$. A subset of $\{\Xi^\dagger_{\phi_n}, \Xi_{\phi_n}\}$ with $n = 1, 2$ and $H_{\phi\phi'} = \Xi^\dagger_{\phi_1}\Xi_{\phi'_1}$ generate the $O(8j+5)$ group. Therefore, $U(2^{4j+2}) \supset O(8j+5)$ with the branching rule $[1, 0, \cdots, 0] \downarrow (\frac{1}{2}, \cdots, \frac{1}{2})$, where $(\frac{1}{2}, \cdots, \frac{1}{2})$ with $4j+2$ components to be $\frac{1}{2}$ is a spinor representation of $O(8j+5)$. The largest nontrivial subgroup of $O(8j+5)$ is $O(8j+4)$ generated by $\{\Xi^\dagger_{\phi_2}, \Xi_{\phi_2}, H_{\phi\phi'}\}$ with the branching rule:

$$\begin{array}{c} O(8j+5)\downarrow \quad O(8j+4) \\ (\frac{1}{2}, \cdots, \frac{1}{2}) \downarrow (\frac{1}{2}, \cdots, \frac{1}{2}, \frac{1}{2}) \oplus (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}), \end{array} \tag{2}$$

where the irreducible representation (irrep) $(\frac{1}{2}, \cdots, \frac{1}{2}, \frac{1}{2})$ of $O(8j+4)$ is spanned by $\{\Xi^\dagger_{\phi_n}|0\rangle\}$ with n even, while $(\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$ is spanned by $\{\Xi^\dagger_{\phi_n}|0\rangle\}$ with n odd. There are several important subgroup chains useful to provide various complete basis vectors of the irreps of $O(8j+4)$, among which the following chain

$$O(8j+4) \supset (\mathrm{O}(5) \supset \mathrm{O_T}(3) \otimes \mathrm{O}_{\mathcal{N}}(2)) \otimes (Sp(2j+1) \supset SU_{\mathrm{J}}(2)) \tag{3}$$

is used to label the complete basis vectors with n_j valence nucleons in the j-orbit, where T is the quantum number of the total isospin, J is the quantum number of the total angular momentum, and $\mathcal{N}(j) = n_j/2 - \Omega_j$ with $\Omega_j = j + 1/2$.

The generators of O(5) of this case are (J=0, T=1) pair creation operators $A^+_\mu(j)$, pair annihilation operators $A_\mu(j) = \left(A^+_\mu(j)\right)^\dagger$, with $\mu = +, -, 0$, the number operator of valence nucleons in the j-orbit $\hat{n}_j$, and isospin operators $T_\mu(j)$, with

$$A^+_\mu(j) = \sum_{m_j>0} (-)^{j-m_j} a^\dagger_{jm_j,tm_t} a^\dagger_{j-m_j,tm_t} \quad \text{with} \quad \mu = 1 \text{ or } -1, \tag{4}$$

corresponding to $m_t = 1/2$ or $-1/2$,

$$A^+_0(j) = \sqrt{\frac{1}{2}} \left\{ \sum_{m_j>0} (-)^{j-m_j} a^\dagger_{jm_j,t\,1/2} a^\dagger_{j-m_j,t\,-1/2} + \sum_{m_j>0} (-)^{j-m_j} a^\dagger_{jm_j,t\,-1/2} a^\dagger_{j-m_j,t\,1/2} \right\}, \tag{5}$$

$$\hat{n}_j = \sum_{m_j m_t} a^\dagger_{jm_j,tm_t} a_{jm_j,tm_t}, \quad T_+(j) = \sum_{m_j} a^\dagger_{jm_j,t\,1/2} a_{jm_j,t\,-1/2}, \quad T_-(j) = \sum_{m_j} a^\dagger_{jm_j,t\,-1/2} a_{jm_j,t\,1/2},$$

$$T_0(j) = \frac{1}{2} \sum_{m_j} \left(a^\dagger_{jm_j,t\,1/2} a_{jm_j,t\,1/2} - a^\dagger_{jm_j,t\,-1/2} a_{jm_j,t\,-1/2} \right), \tag{6}$$

of which the commutation relations among the above O(5) generators were explicitly shown in [3]. The generators of $Sp(2j+1)$ are given by

$$\sum_{m_t} \left(a^\dagger_{j,tm_t} \tilde{a}_{j,tm_t} \right)^{(k)}_\rho \tag{7}$$

with $k = 1, 3, \cdots, 2j$, and $\rho = k, k-1, \cdots, -k$ for a given k, where $\tilde{a}_{jm_j,tm_t} = (-)^{j+m_j} a_{j\,-m_j,tm_t}$, and $(A_j B_j)^{(k)}_\rho$ stands for the angular momentum coupling with $j \otimes j \downarrow k$.

For a fixed number of valence particles n_j, the labels of the $O(8j+4)$ irrep are redundant, the complete basis vectors of (3) may be denoted as

$$\left| \begin{matrix} (v_1, v_2) \\ n_j\, \beta\, T\, M_T \end{matrix} ; \alpha\, J\, M_J \right\rangle. \tag{8}$$

In (8), v_1 and v_2, being positive integers or positive half-integers simultaneously, denotes a possible irrep of O(5) with $v_1 \geq v_2 \geq 0$, which are related to the O(5) seniority number of nucleons v and the reduced isospin t with $v_1 = \Omega - v/2$ and $v_2 = t$. v and t indicate that there are v nucleons coupled to the isospin t, which are free from the angular momentum J = 0 and T = 1 pairs. v and t also label the corresponding irrep of $Sp(2j+1)$ simultaneously represented by a two-column Young diagram $\langle 2^v, 1^{2t} \rangle$ with $v+2t$ boxes in the first column and v boxes in the second column. β and α in (8) are multiplicity labels for given T and J needed in the reduction $\mathrm{O(5)} \downarrow \mathrm{O_T(3)} \otimes \mathrm{O_{\mathcal{N}}(2)}$ and $Sp(2j+1) \downarrow SU_{\mathrm{J}}(2)$, respectively.

For the O(5) seniority zero case corresponding to $v = t = 0$ and J=0 discussed in this work, the quantum numbers of $Sp(2j+1)$ and $SU_{\mathrm{J}}(2)$ are thus neglected. In this case, for given number of valence nucleons n_j, the basis vectors (8) can be constructed by using n_j (J=0, T=1) pair creation operators $A^+_\mu(j)$ coupled to isospin T as shown in [4].

Moreover, for a given j-orbit, the generators of O(5) in the canonical $\mathrm{SU}_\Lambda(2) \otimes \mathrm{SU}_I(2)$ basis are denoted as $\{\varsigma_\rho, \tau_\rho, U_{\mu\nu}\}$ with $-1 \leq \rho \leq 1$ and $-\frac{1}{2} \leq \mu, \nu \leq \frac{1}{2}$. where $\{\varsigma_+, \varsigma_-, \varsigma_0\}$ and $\{\tau_+, \tau_-, \tau_0\}$ generate the subgroup $\mathrm{SU}_\Lambda(2)$ and $\mathrm{SU}_I(2)$, respectively, and the double tensor operators $\{U_{\mu\nu}\}$ satisfy the following Hermitian conjugation relation:

$$(U_{\mu\nu})^\dagger = (-)^{\mu+\nu}\, U_{-\mu-\nu}. \tag{9}$$

The commutation relations of these generators are

$$
\begin{aligned}
&[\varsigma_0,\ \varsigma_\pm] = \pm\varsigma_\pm, \quad [\varsigma_+,\ \varsigma_-] = 2\varsigma_0,\\
&[\tau_0,\ \tau_\pm] = \pm\tau_\pm, \quad [\tau_+,\ \tau_-] = 2\tau_0,\\
&[\varsigma_0,\ U_{\mu\nu}] = \mu\, U_{\mu\nu}, \quad [\tau_0,\ U_{\mu\nu}] = \nu\, U_{\mu\nu},\\
&[\varsigma_\pm,\ U_{\mu\nu}] = \sqrt{(\tfrac{1}{2}\mp\mu)(\tfrac{1}{2}\pm\mu+1)}\, U_{\mu\pm1\nu}, \quad [\tau_\pm,\ U_{\mu\nu}] = \sqrt{(\tfrac{1}{2}\mp\nu)(\tfrac{1}{2}\pm\nu+1)}\, U_{\mu\nu\pm1},\\
&[U_{\pm\frac{1}{2}\frac{1}{2}},\ U_{\pm\frac{1}{2}-\frac{1}{2}}] = \pm\varsigma_\pm,\ [U_{\frac{1}{2}\pm\frac{1}{2}},\ U_{-\frac{1}{2}\pm\frac{1}{2}}] = \pm\tau_\pm,\ [U_{\pm\frac{1}{2}\frac{1}{2}},\ U_{\mp\frac{1}{2}-\frac{1}{2}}] = -(\varsigma_0 \pm \tau_0).
\end{aligned}
\tag{10}
$$

The relations of the O(5) generators in the non-canonical $\mathrm{O(5)}\supset\mathrm{O}_T(3)\times\mathrm{O}_{\mathcal{N}}(2)$ basis with those in the canonical $\mathrm{SU}_\Lambda(2)\otimes\ \mathrm{SU}_I(2)$ basis are given by [5, 6]

$$
\begin{aligned}
&A^\dagger_1(j) = \varsigma_+,\ A^\dagger_{-1}(j) = \tau_+,\ A_1(j) = \varsigma_-,\ A_{-1}(j) = \tau_-,\ A^\dagger_0(j) = U_{\frac{1}{2}\frac{1}{2}},\ A_0(j) = -U_{-\frac{1}{2}-\frac{1}{2}},\\
&T_+(j) = -\sqrt{2}U_{\frac{1}{2}-\frac{1}{2}},\ T_-(j) = -\sqrt{2}U_{-\frac{1}{2}\frac{1}{2}},\ T_0(j) = \varsigma_0 - \tau_0,\ \ \hat{\mathcal{N}}(j) = \varsigma_0 + \tau_0.
\end{aligned}
\tag{11}
$$

Since $\mathrm{O(5)}\downarrow\mathrm{O(4)}$ is simply reducible and O(4) is locally isomorphic to $\mathrm{SU}_\Lambda(2)\otimes\mathrm{SU}_I(2)$, instead of $\mathrm{O(5)}\supset\mathrm{O_T(3)}\otimes\mathrm{O}_{\mathcal{N}}(2)$ state classification, the (canonical) branching multiplicity-free orthonormal basis vectors of $\mathrm{O(5)}\supset\mathrm{SU}_\Lambda(2)\otimes\mathrm{SU}_I(2)\supset\mathrm{U}_\Lambda(1)\otimes\mathrm{U}_I(1)$ with

$$
\left|\begin{matrix}(v_1,v_2)\\ \Lambda=\frac{1}{2}(u_1+u_2),\ I=\frac{1}{2}(u_1-u_2)\\ \mu, \qquad\quad \nu\end{matrix}\right\rangle
\tag{12}
$$

are used, where (u_1, u_2) labels possible irrep of O(4) within the given irrep (v_1, v_2) of O(5) restricted by $v_2 \le u_1 \le v_1$ and $-v_2 \le u_2 \le v_2$. The Casimir (invariant) operator of O(5) can be expressed as

$$
\begin{aligned}
&C_2(\mathrm{O(5)}) = 2\,\varsigma\cdot\varsigma + 2\,\tau\cdot\tau + \textstyle\sum_{\mu\nu}(-1)^{\mu+\nu}U_{\mu\nu}U_{-\mu\,-\nu}\\
&= \textstyle\sum_\rho\left(A^\dagger_\rho(j)A_\rho(j) + A_\rho(j)A^\dagger_\rho(j)\right) + \mathbf{T}(j)\cdot\mathbf{T}(j) + \hat{\mathcal{N}}(j)^2,
\end{aligned}
\tag{13}
$$

where $\mathbf{l}\cdot\mathbf{l} = \frac{1}{2}(l_+l_- + l_-l_+) + l_0^2$. Eigenvalues of $C_2(\mathrm{O(5)})$, $\varsigma\cdot\varsigma$, and $\tau\cdot\tau$ under (12) are given by

$$
\begin{aligned}
&\left(\begin{matrix}C_2(\mathrm{O(5)})\\ \varsigma\cdot\varsigma\\ \tau\cdot\tau\end{matrix}\right)\left|\begin{matrix}(v_1,v_2)\\ \Lambda=\frac{1}{2}(u_1+u_2),\ I=\frac{1}{2}(u_1-u_2)\\ \mu, \qquad\quad \nu\end{matrix}\right\rangle =\\
&\left(\begin{matrix}v_1(v_1+3)+v_2(v_2+1)\\ \Lambda(\Lambda+1)\\ I(I+1)\end{matrix}\right)\left|\begin{matrix}(v_1,v_2)\\ \Lambda=\frac{1}{2}(u_1+u_2),\ I=\frac{1}{2}(u_1-u_2)\\ \mu, \qquad\quad \nu\end{matrix}\right\rangle,
\end{aligned}
\tag{14}
$$

where $u_1 = v_1 - q$ and $u_2 = v_2 - p$ with $p = 0, 1, \cdots, 2v_2$ and $q = 0, 1, \cdots, v_1 - v_2$.

For a given irrep (v_1, v_2) of O(5), the matrix representations of $\mathrm{O(5)} \supset \mathrm{SU}_\Lambda(2) \otimes \mathrm{SU}_I(2)$ are given by [5, 6]

$$
\left\langle\begin{matrix}\Lambda-\frac{1}{2}\\ I+\frac{1}{2}\end{matrix}\right\|U\left\|\begin{matrix}\Lambda\\ I\end{matrix}\right\rangle = -\left[\frac{(v_1-I+\Lambda+1)(v_2-I+\Lambda)(v_1-\Lambda+I+2)(v_2-\Lambda+I+1)}{2(2\Lambda)(2I+2)}\right]^{\frac{1}{2}},
$$

$$\left\langle \begin{array}{c} \Lambda-\frac{1}{2} \\ I-\frac{1}{2} \end{array} \middle\| U \middle\| \begin{array}{c} \Lambda \\ I \end{array} \right\rangle = \left[\frac{(v_1+I+\Lambda+2)(v_2+I+\Lambda+1)(v_1-\Lambda-I+1)(\Lambda+I-v_2)}{2(2\Lambda)(2I)}\right]^{\frac{1}{2}} \tag{15}$$

with the $\mathrm{SU}_\Lambda(2)\otimes\mathrm{SU}_I(2)$ conjugation relation

$$\left\langle \begin{array}{c} \Lambda \\ I \end{array} \middle\| U \middle\| \begin{array}{c} \Lambda' \\ I' \end{array} \right\rangle = \left[\frac{(2I'+1)(2\Lambda'+1)}{(2I+1)(2\Lambda+1)}\right]^{\frac{1}{2}} (-)^{I'-I+\Lambda'-\Lambda+1} \left\langle \begin{array}{c} \Lambda' \\ I' \end{array} \middle\| U \middle\| \begin{array}{c} \Lambda \\ I \end{array} \right\rangle, \tag{16}$$

where the phase factor shown in [6] has been corrected.

The branching rule of $\mathrm{O}(5)\downarrow\mathrm{SU}_\Lambda(2)\otimes\mathrm{SU}_I(2)$ can be expressed as

$$\begin{array}{ccc} \mathrm{O}(5) & \downarrow & \mathrm{SU}_\Lambda(2) \quad \otimes \quad \mathrm{SU}_I(2) \\ (v_1,v_2) & \downarrow & \bigoplus_{q=0,\ p=0}^{v_1-v_2,\ 2v_2} \left(\Lambda=\frac{1}{2}(v_1+v_2-p-q),\quad I=\frac{1}{2}(v_1-v_2+p-q)\right), \end{array} \tag{17}$$

which can be verified by the sum rule

$$\begin{aligned} \mathrm{Dim}(\mathrm{O}(5),(v_1,v_2)) &= \textstyle\sum_{q=0}^{v_1-v_2}\sum_{p=0}^{2v_2}(v_1+v_2-p-q+1)(v_1-v_2+p-q+1) \\ &= \tfrac{1}{6}(2v_1+3)(v_1-v_2+1)(v_1+v_2+2)(2v_2+1), \end{aligned} \tag{18}$$

where $\mathrm{Dim}(\mathrm{O}(5),(v_1,v_2))$ is the dimension of the $\mathrm{O}(5)$ irrep (v_1,v_2).

2. Matrix elements of the Hamiltonian and its aidgonalization

The Hamiltonian of the charge-independent mean-field plus isovector pairing model used in this work is [6]

$$\hat{H}_0 = \sum_{i=1}^{p}\epsilon_i\,\hat{n}_i - G\sum_{\rho}A_\rho^+A_\rho, \tag{19}$$

where j_i is abbreviated as i, $A_\rho^+=\sum_{i=1}^p A_\rho^+(i)$ and $A_\rho=\sum_{i=1}^p A_\rho(i)$ are collective pairing operators, ϵ_i is the valence nucleon single-particle energy in the i-th orbit, and $G>0$ is the overall pairing interaction strength. The Hamiltonian (19) is digonalized in the subspace of tensor product $\otimes_{i=1}^p\mathrm{O}^{(i)}(5)$ basis when p j-orbits of the shell model are considered, in which each copy of the $\mathrm{O}(5)$ irrep is adapted to the $\mathrm{O}(5)\supset\mathrm{SU}_\Lambda(2)\otimes\mathrm{SU}_I(2)\supset\mathrm{U}_\Lambda(1)\otimes\mathrm{U}_I(1)$ chain. Though the procedure for seniority nonzero cases is the same, in this work, only seniority-zero configuration with total angular momentum J=0 constructed from the tensor product of p copies of the $\mathrm{O}(5)$ irrep $(\Omega_i,0)$ is considered, in which only equal proton and neutron quasispin $I_i=\Lambda_i$ in the i-th orbit is allowed according to (17). Eigenstates of (19) within the seniority-zero $\mathrm{J}^\pi=0^+$ subspace are denoted as

$$|\xi;n,M_T\rangle = \sum_{\Lambda_i n_i M_T(i)} C^{\xi;\ \Lambda_1,\cdots,\Lambda_p}_{n_1 m_T(1),\cdots,n_p m_T(p)} \left| \begin{array}{c} (\Omega_1,0);\ \cdots;\ (\Omega_p,0) \\ \Lambda_1;\ \cdots;\ \Lambda_p \\ n_1,m_T(1);\cdots;n_p,m_T(p) \end{array} \right\rangle, \tag{20}$$

where the eigenstate $|\xi;n,M_T\rangle$ with total number of valence nucleons $n=\sum_{i=1}^p n_i$ and total isospin projection $M_T=\sum_{i=1}^p m_T(i)$ is expended in terms of the p copies of $\mathrm{O}(5)$ tensor product basis $\otimes_{i=1}^p(\Omega_i,0)$ in the $\mathrm{O}(5)\supset\mathrm{SU}_\Lambda(2)\otimes\mathrm{SU}_I(2)\supset\mathrm{U}_\Lambda(1)\otimes\mathrm{U}_I(1)$ labelling scheme with

$$I_i=\Lambda_i,\quad \mu_i=\tfrac{1}{4}(n_i+2m_T(i)-2\Omega_i),\quad \nu_i=\tfrac{1}{4}(n_i-2m_T(i)-2\Omega_i) \tag{21}$$

according to the relations shown in (11), $C^{\xi;\ \Lambda_1,\cdots,\Lambda_p}_{n_1 m_T(1),\cdots,n_p m_T(p)}$ is the corresponding expansion coefficient, and ξ labels the ξ-th eigenstate with the same n and M_T.

Matrix elements of each terms involved in (19) under the O(5) tensor product basis $\otimes_{i=1}^{p}(\Omega_i,0)$ in the $\mathrm{O}(5)\supset\mathrm{SU}_\Lambda(2)\otimes\mathrm{SU}_I(2)\supset\mathrm{U}_\Lambda(1)\otimes\mathrm{U}_I(1)$ labelling scheme can be evaluated according to the results shown in the previous section. Specifically, we have

$$\left\langle \begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1';&\cdots;&\Lambda_p'\\ \mu_1',\nu_1';&\cdots;&\mu_p',\nu_p'\end{matrix}\right|\sum_{i=1}^{p}\epsilon_i\,\hat{n}_i\left|\begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1;&\cdots;&\Lambda_p\\ \mu_1,\nu_1;&\cdots;&\mu_p,\nu_p\end{matrix}\right\rangle=$$
$$\prod_{q=1}^{p}\delta_{\Lambda_q\Lambda_q'}\delta_{\mu_q\mu_q'}\delta_{\nu_q\nu_q'}\sum_{i=1}^{p}2\epsilon_i(\mu_i+\nu_i+\Omega_i),\tag{22}$$

$$\left\langle \begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1';&\cdots;&\Lambda_p'\\ \mu_1',\nu_1';&\cdots;&\mu_p',\nu_p'\end{matrix}\right|A_1^{+}(i)A_1(i)\left|\begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1;&\cdots;&\Lambda_p\\ \mu_1,\nu_1;&\cdots;&\mu_p,\nu_p\end{matrix}\right\rangle=$$
$$\left\langle \begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1';&\cdots;&\Lambda_p'\\ \mu_1',\nu_1';&\cdots;&\mu_p',\nu_p'\end{matrix}\right|\varsigma_{+}^{(i)}\varsigma_{-}^{(i)}\left|\begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1;&\cdots;&\Lambda_p\\ \mu_1,\nu_1;&\cdots;&\mu_p,\nu_p\end{matrix}\right\rangle=$$
$$\prod_{q=1}^{p}\delta_{\Lambda_q\Lambda_q'}\delta_{\mu_q\mu_q'}\delta_{\nu_q\nu_q'}(\Lambda_i-\mu_i+1)(\Lambda_i+\mu_i),\tag{23}$$

$$\left\langle \begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1';&\cdots;&\Lambda_p'\\ \mu_1',\nu_1';&\cdots;&\mu_p',\nu_p'\end{matrix}\right|A_1^{+}(i)A_1(j)\left|\begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1;&\cdots;&\Lambda_p\\ \mu_1,\nu_1;&\cdots;&\mu_p,\nu_p\end{matrix}\right\rangle=$$
$$\left\langle \begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1';&\cdots;&\Lambda_p'\\ \mu_1',\nu_1';&\cdots;&\mu_p',\nu_p'\end{matrix}\right|\varsigma_{+}^{(i)}\varsigma_{-}^{(j)}\left|\begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1;&\cdots;&\Lambda_p\\ \mu_1,\nu_1;&\cdots;&\mu_p,\nu_p\end{matrix}\right\rangle=$$
$$\prod_{q=1}^{p}\delta_{\Lambda_q\Lambda_q'}\prod_{r\neq i\neq j}^{p}\delta_{\mu_r\mu_r'}\delta_{\nu_r\nu_r'}\delta_{\mu_i'\,\mu_i+\frac{1}{2}}\delta_{\mu_j'\,\mu_j-\frac{1}{2}}\sqrt{(\Lambda_i+\mu_i+1)(\Lambda_i-\mu_i)(\Lambda_j-\mu_j+1)(\Lambda_j+\mu_j)}\tag{24}$$

for $i\neq j$,

$$\left\langle \begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1';&\cdots;&\Lambda_p'\\ \mu_1',\nu_1';&\cdots;&\mu_p',\nu_p'\end{matrix}\right|A_{-1}^{+}(i)A_{-1}(i)\left|\begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1;&\cdots;&\Lambda_p\\ \mu_1,\nu_1;&\cdots;&\mu_p,\nu_p\end{matrix}\right\rangle=$$
$$\left\langle \begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1';&\cdots;&\Lambda_p'\\ \mu_1',\nu_1';&\cdots;&\mu_p',\nu_p'\end{matrix}\right|\tau_{+}^{(i)}\tau_{-}^{(i)}\left|\begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1;&\cdots;&\Lambda_p\\ \mu_1,\nu_1;&\cdots;&\mu_p,\nu_p\end{matrix}\right\rangle=$$
$$\prod_{q=1}^{p}\delta_{\Lambda_q\Lambda_q'}\delta_{\mu_q\mu_q'}\delta_{\nu_q\nu_q'}(\Lambda_i-\nu_i+1)(\Lambda_i+\nu_i),\tag{25}$$

$$\left\langle \begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1';&\cdots;&\Lambda_p'\\ \mu_1',\nu_1';&\cdots;&\mu_p',\nu_p'\end{matrix}\right|A_{-1}^{+}(i)A_{-1}(j)\left|\begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1;&\cdots;&\Lambda_p\\ \mu_1,\nu_1;&\cdots;&\mu_p,\nu_p\end{matrix}\right\rangle=$$
$$\left\langle \begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1';&\cdots;&\Lambda_p'\\ \mu_1',\nu_1';&\cdots;&\mu_p',\nu_p'\end{matrix}\right|\tau_{+}^{(i)}\tau_{-}^{(j)}\left|\begin{matrix}(\Omega_1,0);&\cdots;&(\Omega_p,0)\\ \Lambda_1;&\cdots;&\Lambda_p\\ \mu_1,\nu_1;&\cdots;&\mu_p,\nu_p\end{matrix}\right\rangle=$$
$$\prod_{q=1}^{p}\delta_{\Lambda_q\Lambda_q'}\prod_{r\neq i\neq j}^{p}\delta_{\mu_r\mu_r'}\delta_{\nu_r\nu_r'}\delta_{\nu_i'\,\nu_i+\frac{1}{2}}\delta_{\nu_j'\,\nu_j-\frac{1}{2}}\sqrt{(\Lambda_i+\nu_i+1)(\Lambda_i-\nu_i)(\Lambda_j-\nu_j+1)(\Lambda_j+\nu_j)}\tag{26}$$

for $i \neq j$,

$$\left\langle \begin{array}{c} (\Omega_1,0);\ \cdots;\ (\Omega_p,0) \\ \Lambda'_1;\ \cdots;\ \Lambda'_p \\ \mu'_1,\nu'_1;\ \cdots;\ \mu'_p,\nu'_p \end{array} \right| A_0^+(i)A_0(i) \left| \begin{array}{c} (\Omega_1,0);\ \cdots;\ (\Omega_p,0) \\ \Lambda_1;\ \cdots;\ \Lambda_p \\ \mu_1,\nu_1;\ \cdots;\ \mu_p,\nu_p \end{array} \right\rangle =$$

$$\left\langle \begin{array}{c} (\Omega_1,0);\ \cdots;\ (\Omega_p,0) \\ \Lambda'_1;\ \cdots;\ \Lambda'_p \\ \mu'_1,\nu'_1;\ \cdots;\ \mu'_p,\nu'_p \end{array} \right| (-1)U^{(i)}_{\frac{1}{2},\frac{1}{2}}U^{(i)}_{-\frac{1}{2},-\frac{1}{2}} \left| \begin{array}{c} (\Omega_1,0);\ \cdots;\ (\Omega_p,0) \\ \Lambda_1;\ \cdots;\ \Lambda_p \\ \mu_1,\nu_1;\ \cdots;\ \mu_p,\nu_p \end{array} \right\rangle =$$

$$\prod_{q\neq i}^{p}\delta_{\Lambda_q\Lambda'_q}\sum_{\Lambda''_i}\langle\Lambda'_i||U||\Lambda''_i\rangle\langle\Lambda''_i||U||\Lambda_i\rangle\prod_{r=1}^{p}\delta_{\mu_r\mu'_r}\delta_{\nu_r\nu'_r}\times$$

$$(-1)\langle\Lambda''_i\mu_i-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}|\Lambda'_i\mu_i\rangle\langle\Lambda''_i\nu_i-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}|\Lambda'_i\nu_i\rangle\langle\Lambda_i\mu_i\tfrac{1}{2}\,-\tfrac{1}{2}|\Lambda''_i\mu_i-\tfrac{1}{2}\rangle\langle\Lambda_i\nu_i\tfrac{1}{2}\,-\tfrac{1}{2}|\Lambda''_i\,\nu_i-\tfrac{1}{2}\rangle, \qquad (27)$$

$$\left\langle \begin{array}{c} (\Omega_1,0);\ \cdots;\ (\Omega_p,0) \\ \Lambda'_1;\ \cdots;\ \Lambda'_p \\ \mu'_1,\nu'_1;\ \cdots;\ \mu'_p,\nu'_p \end{array} \right| A_0^+(i)A_0(j) \left| \begin{array}{c} (\Omega_1,0);\ \cdots;\ (\Omega_p,0) \\ \Lambda_1;\ \cdots;\ \Lambda_p \\ \mu_1,\nu_1;\ \cdots;\ \mu_p,\nu_p \end{array} \right\rangle =$$

$$\left\langle \begin{array}{c} (\Omega_1,0);\ \cdots;\ (\Omega_p,0) \\ \Lambda'_1;\ \cdots;\ \Lambda'_p \\ \mu'_1,\nu'_1;\ \cdots;\ \mu'_p,\nu'_p \end{array} \right| (-1)U^{(i)}_{\frac{1}{2},\frac{1}{2}}U^{(j)}_{-\frac{1}{2},-\frac{1}{2}} \left| \begin{array}{c} (\Omega_1,0);\ \cdots;\ (\Omega_p,0) \\ \Lambda_1;\ \cdots;\ \Lambda_p \\ \mu_1,\nu_1;\ \cdots;\ \mu_p,\nu_p \end{array} \right\rangle = (-1)\langle\Lambda'_i||U||\Lambda_i\rangle\langle\Lambda'_j||U||\Lambda_j\rangle\times$$

$$\prod_{q\neq i\neq j}^{p}\delta_{\Lambda_q\Lambda'_q}\delta_{\mu_q\mu'_q}\delta_{\nu_q\nu'_q}\langle\Lambda_i\mu_i\tfrac{1}{2}\,\tfrac{1}{2}|\Lambda'_i\mu'_i\rangle\langle\Lambda_i\nu_i\tfrac{1}{2}\,\tfrac{1}{2}|\Lambda'_i\nu'_i\rangle\langle\Lambda_j\mu_j\tfrac{1}{2}\,-\tfrac{1}{2}|\Lambda'_j\mu'_j\rangle\langle\Lambda_j\nu_j\tfrac{1}{2}\,-\tfrac{1}{2}|\Lambda'_j\nu'_j\rangle \qquad (28)$$

for $i \neq j$, where $\langle\Lambda_i\mu_i\frac{1}{2}\,\frac{1}{2}|\Lambda'_i\mu'_i\rangle$ and $\langle\Lambda_j\mu_j\frac{1}{2}\,-\frac{1}{2}|\Lambda'_j\mu'_j\rangle$ are the CG coefficients of SU(2), and $\langle\Lambda'_i||U||\Lambda_i\rangle$ is the $SU_\Lambda(2)\otimes SU_I(2)$ reduced matrix element with $I_i=\Lambda_i=(\Omega_i-q_i)/2$ for $q_i=0,1,\cdots,\Omega_i$ shown in (15) and (16), in which q_i is the number of np-pairs in the i-th orbit for this case. Thus, the number of np-pairs in the i-th orbit for given Λ_i can be expressed in terms of the neutron (proton) quasi-spin as

$$q_i = \Omega_i - 2\Lambda_i \qquad (29)$$

for $i=1,2,\cdots,p$.

It is obvious that this diagonalization scheme is equivalent to the M_T-scheme realized in the $\mathrm{O}(5)\supset \mathrm{SU}_\Lambda(2)\otimes\mathrm{SU}_I(2)\supset \mathrm{U}_\Lambda(1)\otimes\mathrm{U}_I(1)$ basis. The results produced from this scheme has been checked against the exact solution of the model up to 3 pairs shown in [3, 7], which shows that the results produced from this scheme are exactly the same as those obtained from the formalism provided in [3, 7].

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