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Unique Continuation Properties of Over-Determined Static Boussinesq Problems with Application to Uniform Stabilization of Dynamic Boussinesq Systems

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Abstract
We consider several direct and adjoint Boussinesq static problems under different types of over-determined conditions. We then conclude, in each case, that the solution pair corresponding to {fluid velocity, scalar temperature} must vanish identically on the whole domain, so that the pressure is then constant (Unique Continuation Property). In going from the direct to the adjoint problem, the coupling operators between the fluid and the thermal equations switch places. As a result, the adjoint Boussinesq system has a more favorable structure than the direct Boussinesq system and hence yields UCP results under weaker requirements; typically, a reduction by one or even two units on the number of components of the fluid vector being involved in the assumptions. To illustrate: in the key direct Boussinesq problem, over-determination consists of the additional vanishing of the solution pair in a common arbitrarily small subset of the interior. In contrast, in the corresponding adjoint Boussinesq problem, only the first $(d-1)$ components of the d-dimensional fluid velocity vector need to be assumed as vanishing on the interior subset. These UCPs for the adjoint problem are critical ingredients in the solution of corresponding uniform stabilization problems of (direct) dynamic Boussinesq systems by suitable finite dimensional feedback controls. They allow one to verify a corresponding Kalman algebraic condition for controllability.

Keywords Static Boussinesq system · Unique continuation properties

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1 Introductions, Statement of Main Results.

Let Ω be an open, connected, smooth, bounded domain in $\mathbb{R}^d$, with emphasis on $d = 2, 3$, with boundary $\Gamma = \partial\Omega$. We next define an equilibrium solution $\{y_e, \theta_e, p_e\}$, depending on $x \in \Omega$, of the steady-state Boussinesq system in Ω. Let ν_0 be the kinematic viscosity coefficient, κ the thermal diffusivity. The term e_d denotes the vector $\{0, \dots, 0, 1\}$. Moreover, $\gamma = g/\bar{\theta}$, where g is the acceleration due to gravity and $\bar{\theta}$ is the reference temperature. Our starting point is

Theorem 1 *Consider the following steady-state Boussinesq system in Ω*

$$\begin{cases} -\nu_0 \Delta y_e + (y_e \cdot \nabla) y_e - \gamma(\theta_e - \bar{\theta}) e_d + \nabla p_e = f(x) & \text{in } \Omega \quad (1.1a) \\ -\kappa \Delta \theta_e + y_e \cdot \nabla \theta_e = g(x) & \text{in } \Omega \quad (1.1b) \\ \operatorname{div} y_e = 0 & \text{in } \Omega \quad (1.1c) \\ y_e = 0, \quad \theta_e = 0 & \text{on } \Gamma. \quad (1.1d) \end{cases}$$

Let $1 < q < \infty$. For any $f, g \in (L^q(\Omega))^d \times L^q(\Omega)$, there exists a solution (not necessarily unique) $\{y_e, \theta_e, p_e\} \in (W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega))^d \times (W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)) \times (W^{1,q}(\Omega)/\mathbb{R})$.

See [2], [3, Theorem 3.2, p. 41; and Theorem 4.4, p. 43], [4, Theorem 4.4, p. 283; and Theorem 5.10, p. 292] for $q \neq 2$. In the Hilbert space setting, see [13,21,27,43,51]. Notice that (1.1c) and (1.1d) for y_e imply that the vector $\nabla y_e \cdot \nu = \frac{\partial y_e}{\partial \nu}$ is tangential to Γ [6, Lemma 3.3.1, p 35]; moreover, $y_e \cdot \nabla \theta_e = \operatorname{div}(\theta_e y_e)$.

Orientation In this paper, we consider several direct and adjoint Boussinesq static problems under different types of over-determined conditions. We then conclude, in each case, that the solution pair corresponding to {fluid velocity, scalar temperature} must vanish identically on the whole domain, so that the pressure is then constant (Unique Continuation Property). Such investigation, while of interest in itself within the class of over-determined elliptic problems, is here prompted and dictated by an altogether different source: the uniform stabilization problem of dynamic Boussinesq systems with finitely many localized feedback controllers; more specifically, by paper [32]. In it, the feedback stabilizing controllers are localized on an arbitrarily small open interior sub-domain $\omega \subset \Omega$. In point of fact, for such uniform stabilization objective of the dynamic Boussinesq equation, the UCPs that count refer to the adjoint static Boussinesq system, not the original (direct) one. This is in line with established literature on the subject. An account of the literature is given in Sect. 1.3, see also Remark 2. The role of the adjoint problem is illustrated in Sect. 3. For the present coupled PDE system of a fluid equation and a thermal equation, it happens that in going from the direct to the adjoint problem, the coupling operators between the fluid and the thermal equations switch places. As a result, the adjoint Boussinesq system has a more favorable structure than the direct Boussinesq system and hence yields UCP results under weaker requirements; typically, a reduction by one or even two units in the number of components of the fluid vector being involved in the assumptions. To illustrate: in the key direct Boussinesq problem (Theorem 2 with interior sub-domain $\omega \subset \Omega$ as in

Fig. 1), over-determination consists of the additional vanishing of the solution pair in a common arbitrarily small subset ω of the interior. In contrast, in the corresponding adjoint Boussinesq result (Theorem 5, for a similar interior sub-domain ω such as in Fig. 1), only the first $(d-1)$ components of the d-dimensional fluid velocity vector need to be assumed as vanishing on the interior subset. All these results – both for the direct and the adjoint static Boussinesq system – do not require Boundary Conditions such as in (1.3e). In addition, for interior sub-domains ω whose boundary shares at least a portion of the boundary Γ of Ω and which satisfy an additional geometrical condition extracted in Definition 1 for $d = 2, 3$ or Definition 2 for $d = 4$, only $(d-2)$ components of the d-dimensional fluid velocity vector need to be assumed as vanishing on ω; in particular, no fluid component for $d = 2$. See Theorem 6, for subsets ω such as in Figss. 5 and 6 for $d = 2$; and Fig. 10 for $d = 3$. However, for these results involving only $(d-2)$ components of the fluid vector, a boundary condition such as (1.21b) for $d = 2$, or (1.29), (1.30) for $d = 3$ or (A.3) for $d = 4$ is needed. One can thus say that in seeking UCP results for the adjoint Boussinesq static system, the thermal equation 'helps' the fluid equation. This is due to the structure of the vector $e_d = \{0, \dots, 0, 1\}$. As said, these UCPs for the adjoint Boussinesq static systems are critical ingredients in the solution of uniform stabilization problems near an equilibrium solution pair of corresponding dynamic (direct) Boussinesq systems by suitable finite dimensional feedback controls. In fact, the Boussinesq system being parabolic, the by now standard approach, introduced in [44] and pursued at first in [34–36,45] for progressively more challenging classical parabolic problems, applies. It consists in splitting the function space of the solution space into two parts: a finite dimensional unstable component, and an infinite dimensional stable component. It is in the analysis of the finite dimensional unstable component of the dynamics that a UCP for an adjoint problem is critically invoked: to assert controllability, hence, stabilizability with an arbitrarily large rate [52, Thm 2.9, p 44] of such unstable dynamics. More precisely, it is verification of the algebraic Kalman (or Hautus) rank condition that requires a UCP for a suitable adjoint Boussinesq static problem. Application of our adjoint UCPs to stabilization problems of the (direct) dynamic Boussinesq system are discussed in Sect. 3.

1.1 The (Original) Static Boussinesq Problem

Let ω be an arbitrary open, connected, smooth subset of Ω, thus of positive measure. Let $\phi(x) = \{\phi_1(x), \dots, \phi_d(x)\}$ be a d-vector, $h(x)$ and $p(x)$ be two scalar functions, all depending on the d-variable $x \in \Omega$. They are the time-independent counterpart of the fluid velocity vector, the scalar temperature and the scalar pressure in the time-dependent Boussinesq system. See Sect. 3.

With y_e obtained from Theorem 1, we define the first order Oseen perturbation by

$$L_e(\phi) \equiv (y_e \cdot \nabla)\phi + (\phi \cdot \nabla) y_e \qquad \text{in } \Omega. \tag{1.2}$$

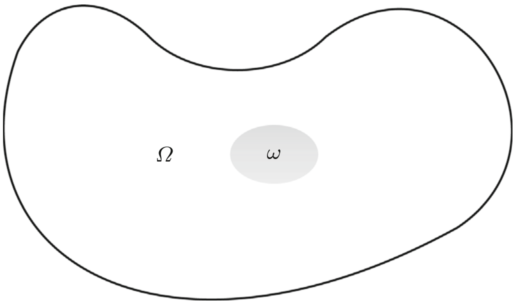


Fig. 1 The pair $\{\omega, \Omega\}$ in Theorem 2 and in Theorem 5

Theorem 2 (UCP, First version of direct problem) *Let* $\{\phi, h, p\} \in (W^{2,q}(\Omega))^d \times W^{2,q}(\Omega) \times W^{1,q}(\Omega)$, $q > d$, *solve the following static Boussinesq problem*

$$\begin{cases} -\nu_0 \Delta\phi + L_e(\phi) + \nabla p - \gamma h e_d = \lambda\phi & \text{in } \Omega & (1.3a) \\ \operatorname{div} \phi = 0 & \text{in } \Omega & (1.3b) \\ -\kappa\Delta h + y_e \cdot \nabla h + \phi \cdot \nabla\theta_e = \lambda h & \text{in } \Omega & (1.3c) \\ \phi \equiv 0, \quad h \equiv 0 & \text{in } \omega & (1.3d) \end{cases}$$

with over-determination on ω *in* (1.3d). *Then*

$$\phi \equiv 0, \quad h \equiv 0, \quad p \equiv const \quad \text{in } \Omega. \tag{1.4}$$

We explicitly point out that the B.C.s

$$\phi\big|_\Gamma \equiv 0, \quad h\big|_\Gamma \equiv 0 \text{ on } \Gamma \tag{1.3e}$$

are not needed in Theorem 2. If (1.3e) were to be added to (1.3a)–(1.3d), we would obtain an over-determined eigenproblem. The proof of Theorem 2 is given in Sect. 2.

Theorem 2 admits a generalization to the Riemannian setting.

Theorem 1.1R *The same result holds true in the case* Ω *is an open bounded set in a complete,* d*-dimensional Riemannian manifold of class* C^3*, with* C^3*-metric* $g : (M, g)$.

In particular, this holds true in case problem (1.3a)–(1.3c) *is still defined in a Euclidean setting, but the differential operators are now of smooth (say,* C^3*)-variable coefficients in space. Here,* $(M, g) = (\mathbb{R}^d, g)$*, where* $g = \sum_{i,j=1}^d g_{ij} dx_i dx_j$ *where* $\{g_{ij}(x)\} = \{a_{ij}(x)\}^{-1}$ *is a positive symmetric matrix defined in terms of the coefficients* $a_{ij}(x) = a_{ji}(x)$ *of the second-order uniformly elliptic partial differential operator* $\mathcal{A} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ *in place of* Δ.

Proof of Theorem 1.1R The proof in the Riemannian setting is essentially the same mutatis mutandis. The Riemannian version of the critical Theorem 10 is now available from [49, Cor. 4.2, Eq. (4.12), p. 368], [50, Cor. 4.2, Eq. (4.22), p. 345]. □

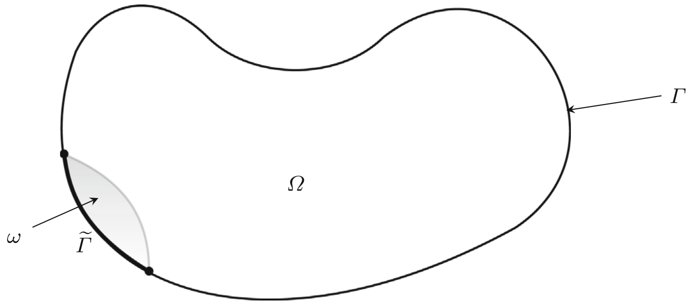


Fig. 2 The pair $\{\omega, \widetilde{\Gamma}\}$ in Theorem 3

We next provide two additional UCP results for the (direct) static Boussinesq system. Their proof is short, as it will be reduced to the validity of Theorem 2 and, in the case of Theorem 4 below, also to the validity of the UCP of the Oseen problem, recalled in Appendix B.

Theorem 3 (UCP, Second version of direct problem) *Let now ω be an arbitrary, open, connected, smooth subset of Ω, thus of positive measure, which moreover is an internal localized collar of the (arbitrarily small) subportion $\widetilde{\Gamma}$ of the boundary $\Gamma = \partial\Omega$, touching the boundary at $\widetilde{\Gamma}$. See* Fig. 2. *Let $\{\phi, h, p\} \in (W^{2,q}(\Omega))^d \times W^{2,q}(\Omega) \times W^{1,q}(\Omega)$, $q > d$, solve the problem*

$$\begin{cases} -\nu_0\Delta\phi + L_e(\phi) + \nabla p - \gamma h e_d = \lambda\phi & \text{in } \Omega \quad (1.5a)\\ \operatorname{div}\phi = 0 & \text{in } \Omega \quad (1.5b)\\ -\kappa\Delta h + y_e\cdot\nabla h + \phi\cdot\nabla\theta_e = \lambda h & \text{in } \Omega \quad (1.5c)\\ h\big|_{\widetilde{\Gamma}} \equiv 0 & (1.5d)\\ \left.\dfrac{\partial h}{\partial\nu}\right|_{\widetilde{\Gamma}} \equiv 0, \quad \phi \equiv 0 \text{ in } \omega & (1.5e) \end{cases}$$

with over-determination in (1.5e). *Then*

$$\phi \equiv 0, \quad h \equiv 0, \quad p \equiv const \quad in\ \Omega. \tag{1.6}$$

Proof of Theorem 3 **Step 1** The assumption $\phi \equiv 0$ in ω from (1.5e), used in Eq. (1.5c) yields

$$\begin{cases} -\kappa\Delta h + y_e\cdot\nabla h = \lambda h & \text{in } \omega \quad (1.7a)\\ h\big|_{\widetilde{\Gamma}} \equiv 0, \quad \left.\dfrac{\partial h}{\partial\nu}\right|_{\widetilde{\Gamma}} \equiv 0 & \widetilde{\Gamma} \subset \partial\omega \quad (1.7b) \end{cases}$$

after recalling also the conditions on h from (1.5d) and (1.5e). It is then a standard result that (1.7) implies

$$h \equiv 0 \qquad \text{in } \omega. \tag{1.8}$$

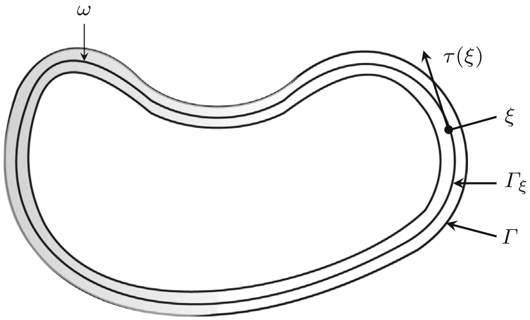


Fig. 3 The pair $\{\omega, \Gamma\}$ in Theorem 4. Internal Collar ω of Fully Boundary Γ

In fact, one first extends h in ω by zero across $\widetilde{\Gamma}$ outside Ω into a set ω_{ext}, next shows that the extended function defined by h = solution of (1.7a)–(1.7b) in ω, and $h \equiv 0$ in ω_{ext}, satisfies $h \in W^{2,q}(\omega \cup \omega_{\text{ext}})$ [29, p 75]. Then, the classical Aronszajin-Cordes uniqueness theorem [26, Vol III, p 3], or Carleman's theorem [12], or [9, p 162; p 263], or [41, pp 59–61] for $y_e \in W^{2,q}(\omega) \hookrightarrow C(\overline{\omega})$ for $2q > d$ [1, p 97], [28, p 79], implies $h \equiv 0$ in ω, as desired in (1.8).

Step 2 Conclusion $h \equiv 0$ in ω from (1.8) and the assumption $\phi \equiv 0$ in ω from (1.5e) are precisely the over-determined conditions in (1.3d). Application of Theorem 2 then yields

$$\phi \equiv 0, \quad h \equiv 0, \quad p \equiv \text{const} \quad \text{in } \Omega, \tag{1.9}$$

and Theorem 3 is proved. $\square$

The setting of the next result is somewhat different as it requires the subtle UCP recalled in Appendix B, which was originally provided in [37, Lemma 2, p 138] and was invoked critically in [38, Thm 6.2] as well as in [31, Lemma 4.3; Problem #2 , Appendix D] in connection with boundary tangential stabilization of the N-S system. To describe it, for purposes of illustration, let ω be at first an arbitrary collar (layer) of the boundary Γ in the interior of Ω, $\omega \subset \Omega$ (Fig. 3). For each point $\xi \in \omega$, we consider the (sufficiently smooth) curve ($d = 2$) or surface ($d = 3$) Γ_ξ, which is the parallel translation of the boundary Γ, passing through $\xi \in \omega$ and lying in ω. Let $\tau(\xi)$ be a unit tangent vector to the oriented curve Γ_ξ at ξ, if $d = 2$; and let $\tau(\xi) = [\tau_1(\xi), \tau_2(\xi)]$ be an orthonormal system of oriented tangent vectors lying on the tangent plane to the surface Γ_ξ at ξ, if $d = 3$, and obtained as isothermal parametrization via a 1-1 conformal mapping of a suitable open set in $\mathbb{R}^2$ with canonical basis $e_1 = \{1, 0\}, e_2 = \{0, 1\}$. See [37, Appendix] for details and references. We shall in particular allow and study the case where ω is a localized collar based on an arbitrarily small, connected portion $\widetilde{\Gamma}$ of the boundary Γ (Fig. 4).

Theorem 4 (UCP, Third version of direct problem) *Let ω be an internal localized collar supported by the subportion $\widetilde{\Gamma}$ of the boundary Γ as in* Fig. 4 *(in particular an internal localized collar of the full boundary Γ, as in* Fig. 3*). Let $\{\phi, h, p\} \in$*

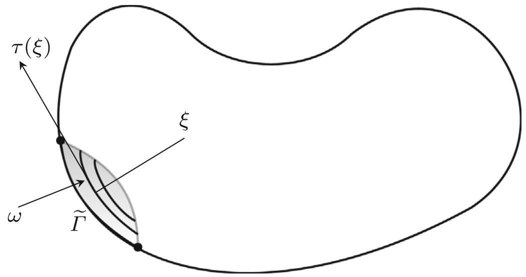


Fig. 4 The pair $\{\omega, \widetilde{\Gamma}\}$ in Theorem 4. Internal Localized Collar ω of Subportion $\widetilde{\Gamma}$ of Boundary Γ

$(W^{2,q}(\Omega))^d \times W^{2,q}(\Omega) \times W^{1,q}(\Omega)$, $q > d$, *solve the problem*

$$\begin{cases} -\nu_0 \Delta\phi + L_e(\phi) + \nabla p - \gamma h e_d = \lambda\phi & \textit{in } \Omega \quad (1.10a)\\ \operatorname{div} \phi = 0 & \textit{in } \Omega \quad (1.10b)\\ -\kappa \Delta h + y_e \cdot \nabla h + \phi \cdot \nabla\theta_e = \lambda h & \textit{in } \Omega \quad (1.10c)\\ \phi\big|_{\widetilde{\Gamma}} \equiv 0, \quad \left.\dfrac{\partial\phi}{\partial\nu}\right|_{\widetilde{\Gamma}} \equiv 0 & \quad (1.10d)\\ h \equiv 0, \quad \phi\cdot\tau \equiv 0 & \textit{in } \omega \quad (1.10e) \end{cases}$$

recalling (1.2)*, where τ is the tangential vector described above, for the pair $\{\omega, \widetilde{\Gamma}\}$ illustrated in* Fig. 4*; or the pair $\{\omega, \widetilde{\Gamma} = \Gamma\}$ illustrated in* Fig. 3*. Then*

$$\phi \equiv 0, \quad h \equiv 0, \quad p \equiv const \quad in\ \Omega. \tag{1.11}$$

Proof of Theorem 4 **Step 1** The assumption $h \equiv 0$ in ω from (1.10e), used in Eq. (1.10a) yields the following over-determined problem

$$\begin{cases} -\nu_0 \Delta\phi + L_e(\phi) + \nabla p = \lambda\phi & \text{in } \omega \quad (1.12a)\\ \operatorname{div} \phi = 0 & \text{in } \omega \quad (1.12b)\\ \phi\big|_{\widetilde{\Gamma}} \equiv 0, \quad \left.\dfrac{\partial\phi}{\partial\nu}\right|_{\widetilde{\Gamma}} \equiv 0, \quad \phi\cdot\tau \equiv 0 \text{ in } \omega & \quad (1.12c) \end{cases}$$

after recalling also (1.10d)–(1.10e), see Fig. 4.

Step 2 To the over-determined Oseen problem (1.12), we then apply [37, Lemma 2, p 138] recalled also in Appendix B. We obtain

$$\phi \equiv 0, \qquad p \equiv \text{const} \quad \text{in } \omega. \tag{1.13}$$

Step 3 Thus, we again, by (1.13) and (1.10e), are reduced to the conditions $\phi \equiv 0$, $h \equiv 0$ in ω of Theorem 2, as applied to the Boussinesq static problem. Theorem 2 then applies and yields

$$\phi \equiv 0, \quad h \equiv 0, \quad p \equiv \text{const} \quad \text{in } \Omega, \tag{1.14}$$

and Theorem 4 is proved. □

1.2 The Adjoint Static Boussinesq Problem

As noted in the Orientation, it turns out that in the study of uniform stabilization of the Boussinesq dynamic problem, what is needed is a UCP of the **adjoint** problem (1.18) below, not of the original problem (1.3). This is further discussed in Sect. 3. It is well-known that testing the Kalman algebraic controllability of the unstable finite dimensional projection of a parabolic dynamics involves the **adjoint** problem/operator. Thus, with reference to the Oseen perturbation $L_e(\cdot)$ in (1.2), we now introduce its adjoint

$$L_e^*(\phi) = (y_e \cdot \nabla)\phi + \nabla^{\perp} y_e \cdot \phi. \tag{1.15}$$

It will be justified in Sect. 3 that in taking the adjoint of the fluid equation (1.3a) and the heat equation (1.3c), the coupling operators

$$\text{[from the NS equation]} \quad \mathcal{C}_\gamma h = -\gamma P_q(he_d), \quad \mathcal{C}_\gamma \in \mathcal{L}(L^q(\Omega), L^q_\sigma(\Omega)), \tag{1.16}$$

$$\text{[from the heat equation]} \quad \mathcal{C}_{\theta_e} z = z \cdot \nabla\theta_e, \quad \mathcal{C}_{\theta_e} \in \mathcal{L}(L^q_\sigma(\Omega), L^q(\Omega)). \tag{1.17}$$

switch places, so that the adjoint of $\mathcal{C}_\gamma$ acts now on the thermal equation, while the adjoint of $\mathcal{C}_{\theta_e}$ acts now on the fluid equation. This has the beneficial impact that the resulting adjoint UCP holds true with only the first $(d-1)$ components, $\{\phi_1, \phi_2, \dots, \phi_{d-1}\}$ of the d-dimensional ϕ-variable, assumed as vanishing in the internal subset ω (Fig. 1), Theorem 5 below; or even $(d-2)$ components such as for instance $\{\phi_1, \phi_2, \dots, \phi_{d-2}\}$ assumed as vanishing on ω, in the more specific setting of Theorem 6 (Figs. 5 and 6). This was stated in [32, Sect. 1].

It is computed in [32, Theorem 1.4] that the **adjoint** of problem (1.3a)–(1.3c) is by (1.15)

$$\begin{cases} -\nu_0 \Delta\phi + L_e^*(\phi) + \nabla p + h\nabla\theta_e = \lambda\phi & \text{in } \Omega, \quad (1.18a)\\ -\kappa\Delta h + y_e \cdot \nabla h - \gamma\phi \cdot e_d = \lambda h & \text{in } \Omega, \quad (1.18b)\\ \operatorname{div} \phi = 0 & \text{in } \Omega. \quad (1.18c) \end{cases}$$

The next result is the UCP that is needed in the uniform stabilization of the Boussinesq system in [30]. See Sect. 3.

Theorem 5 (UCP, First version, adjoint problem) *Let ω be an arbitrary, open, connected, smooth subset of Ω, thus of positive measure, as in Theorem 2 (Fig. 1). Let $\{\phi, h, p\} \in (W^{2,q}(\Omega))^d \times W^{2,q}(\Omega) \times W^{1,q}(\Omega)$, $q > d$ solve problem (1.18) along with the over-determination condition*

$$h \equiv 0, \quad \{\phi_1, \phi_2, \dots, \phi_{d-1}\} \equiv 0 \quad \textit{in } \omega. \tag{1.18d}$$

Then,

$$\phi \equiv 0, \quad h \equiv 0, \quad p \equiv const \quad in\ \Omega. \tag{1.19}$$

Proof We recall that e_d is the d-vector $e_d = \{0, \ldots, 0, 1\}$. Then $h \equiv 0$ in ω, as assumed in (1.18d), implies $\phi_d \equiv 0$ in ω, by (1.18b). Combining this with (1.18d), we then obtain

$$h \equiv 0 \quad \text{in } \omega, \qquad \phi \equiv 0 \quad \text{in } \omega. \tag{1.20}$$

Then the same proof of Theorem 2 applies and yields the conclusion (1.19). This is so since the differential term $L_e^*(\phi)$ in (1.15) is first order as is L_e in (1.2), while the term $\gamma\phi \cdot e_d$ in (1.18b) is of zero order as the term $\phi \cdot \nabla\theta_e$ in (1.10c). Thus the same estimates of the proof of Theorem 2 apply. □

Theorem 1.1R *A Riemannian version of Theorem 5, thus a counterpart of Theorem 1.1R, holds true.*

Remark 1 So far, all UCP results expressed by Theorem 2 through Theorem 4 (original Boussinesq system) as well as Theorem 5 (adjoint Boussinesq system) make **no use** of the B.C. (1.3e): $\phi\big|_\Gamma \equiv 0, h\big|_\Gamma \equiv 0$. The situation is somewhat different, however, in the next UCP result, still for the adjoint Boussinesq problem. More precisely, there are two differences. First, a broad geometrical difference regarding the allowed class of interior subsets ω. Namely, to begin with, the small interior subset ω will have to touch the boundary Γ at a non-empty set $\widetilde{\Gamma} = \partial\omega \cap \Gamma \neq \emptyset$, see Figs. 5 and 6 and, further, satisfy some appropriate geometrical conditions. The class of allowed subsets ω is singled out in Definition 1 below, with emphasis on the physical dimensions $d = 2, 3$. It can be readily extended to any d at the price of increased complications. See Appendix 1 for $d = 4$. For such class, we may further reduce by 2 unites the number of components of the d-dimensional fluid vector $\phi = \{\phi_1, \ldots, \phi_d\}$ required to vanish on ω in the over-determination condition and still claim a corresponding UCP of the adjoint Boussinesq static system. Second, there is an analytical difference. In fact, now in addition to the over-determination condition $h \equiv 0$ on ω of the thermal component, the proof does require some version of a boundary condition for the fluid vector ϕ, however *somewhat weaker* than $\phi\big|_{\widetilde{\Gamma}} \equiv 0$ and a-fortiori surely weaker than (1.3e) [$h\big|_{\widetilde{\Gamma}} = 0$ is inherited from $h \equiv 0$ in ω]. In fact, only a subset of components of $\phi = \{\phi_1, \ldots, \phi_d\}$ will have to vanish on a suitable portion Γ_0 of $\widetilde{\Gamma} = \partial\omega \cap \Gamma$, see Figs. 5 and 6. More explicitly, the class of allowed subsets ω is singled out in Definition 1, with emphasis on the physical dimensions $d = 2, 3$ and in Definition 2 for $d = 4$. With respect to such class of subsets ω, the requirements of over-determination on ω of the fluid vector ϕ are as follows:

(i) if $d = 2$, then no over-determination assumption on the vanishing of the fluid vector $\phi = \{\phi_1, \phi_2\}$ on the full subset ω is required: only that $\phi_1\big|_{\Gamma_0} = 0$, where Γ_0 is a suitable subset of $\widetilde{\Gamma}$, $\Gamma_0 \subset \widetilde{\Gamma} = \partial\omega \cap \Gamma$, singled out in Definition 1 (i), (ii$_2$). See Figs. 5 and 6. In which case, $h \equiv 0$ on ω is the only over-determined condition (on ω) needed to claim the corresponding UCP for the required class of subset ω; and this implies $h\big|_{\widetilde{\Gamma}} \equiv 0$.

(ii) if $d = 3$, the over-determination condition on the fluid vector $\phi = \{\phi_1, \phi_2, \phi_3\}$ reduces to:

(ii_1) either $\phi_1 \equiv 0$ on ω, along with the B.C. $\phi_2\big|_{\Gamma_0} = 0$;
(ii_2) or else $\phi_2 \equiv 0$ on ω, along with the B.C. $\phi_1\big|_{\Gamma_0} = 0$

with $\Gamma_0 \subset \widetilde{\Gamma}$ as in Definition 1, (i), (ii_3). Each condition is being accompanied by a corresponding (different) geometric condition on ω and $\Gamma_0 \subset \widetilde{\Gamma}$ as in Definition 1. Of course, the over-determination $h \equiv 0$ on ω remains.

For clarity we shall introduce the next definition by cases

Definition 1 Let ω be an open, connected subset of Ω, thus of positive measure, satisfying the preliminary condition

(i) the intersection between the boundary $\partial\omega$ of ω and the boundary Γ of Ω is non-empty: $\widetilde{\Gamma} = \partial\omega \cap \Gamma \neq \emptyset$.
Moreover,
(ii_2) let $d = 2$. If P is an arbitrary point of ω, then the line ℓ_P passing through P and parallel to the x_1-axis meets the intersection $\widetilde{\Gamma}$. Let $\mathcal{T}_Q$ be the totality (collection) of all points Q where the line ℓ_P meets the portion $\widetilde{\Gamma} = \partial\omega \cap \Gamma \neq \emptyset$ in (i), as P runs over ω. Let next Γ_0 be a connected component, or the union of connected components of $\mathcal{T}_Q$, such that any such line ℓ_P, $P \in \omega$, hits Γ_0 at just one point. See Fig. 5 (Γ_0 consists of one connected component) and Fig. 6 (Γ_0 consists of two connected components) for positive illustrations; and Figs. 7 and 8 for negative illustrations. The illustration in Fig. 7 cannot be made positive by taking a smaller ω, unlike the illustration in Fig. 8.
(ii_3) Let $d = 3$. There are two cases:

(ii_3) **Case 1.** If P is an arbitrary point of ω, then the plane π_P passing through P and parallel to the coordinate (x_1, x_3)-plane meets the intersection $\widetilde{\Gamma} = \partial\omega \cap \Gamma \neq \emptyset$ at a curve $\mathcal{C}_P$. See Fig. 11.
(ii_3) **Case 2.** If P is an arbitrary point of ω, then the plane π_P passing through P and parallel to the coordinate (x_2, x_3)-plane meets the intersection $\widetilde{\Gamma} = \partial\omega \cap \Gamma \neq \emptyset$ at a curve $\mathcal{C}_P$.

In each case, let $\mathcal{T}_{\mathcal{C}_P}$ be the totality (collection) of all curves $\mathcal{C}_P$ where the plane π_P meets the portion $\widetilde{\Gamma} = \partial\omega \cap \Gamma \neq \emptyset$ in (i), as P runs over ω. Let Γ_0 be a connected component, or the union of connected components of $\mathcal{T}_{\mathcal{C}_P}$, such that any such plane π_P, $P \in \omega$, hits Γ_0 at just one curve.

Theorem 6 *(UCP, Second version, adjoint problem, $d = 2$) Let $d = 2$. Let $\{\omega, \Omega\}$ be a pair satisfying Definition 1, (i); Case (ii_2). Thus, for any point $P \in \omega$, there is at least one point $Q \in \widetilde{\Gamma} \equiv \partial\omega \cap \Gamma$ of intersection between a line ℓ_P parallel to the x_1-axis and passing through the point P and the set $\widetilde{\Gamma}$. Let*

$$\begin{aligned}\Gamma_0 = \ & \textit{connected component(s) of the set } \big\{Q : Q \in \widetilde{\Gamma} \cap \ell_P, \ell_P \textit{ the line parallel to} \\ & \textit{the } x_1 \textit{-axis and passing through the point } P \in \omega\big\}, \textit{ as } P \textit{ runs over } \omega \\ & \textit{such that any } \ell_P \textit{ hits } \Gamma_0 \textit{ at just one point, as in Def 1 (i), } (ii^2). \end{aligned} \tag{1.21a}$$

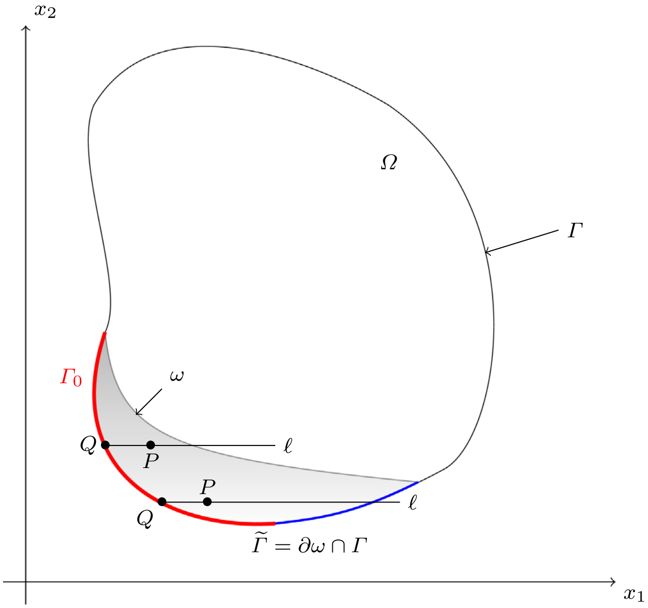


Fig. 5 Pair $\{\omega, \Omega\}$ where Def. 1, $d = 2$ applies. Covered by Theorem 6. Γ_0 consists of one connected component of intersection points Q

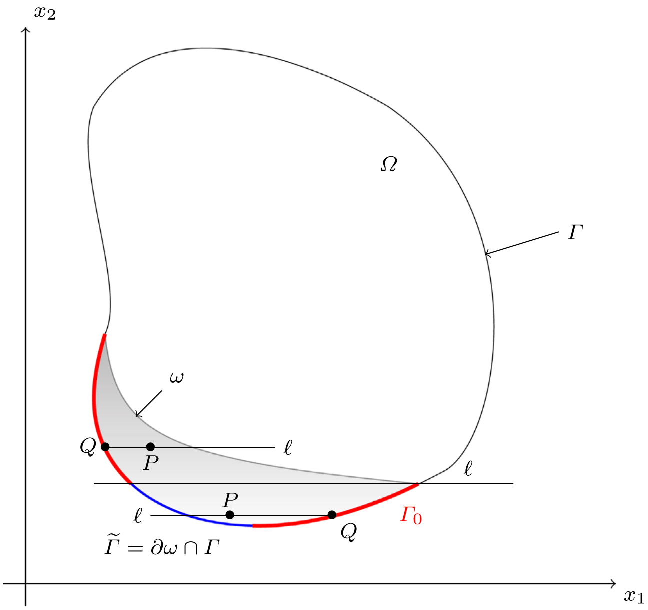


Fig. 6 For same pair $\{\omega, \Omega\}$ as in Fig. 5, a different choice of $\Gamma_0 \subset \widetilde{\Gamma} = \partial\omega \cap \Gamma$. Case covered by Theorem 6, $d = 2$. Γ_0 consists of two disjoint connected components of intersection points Q

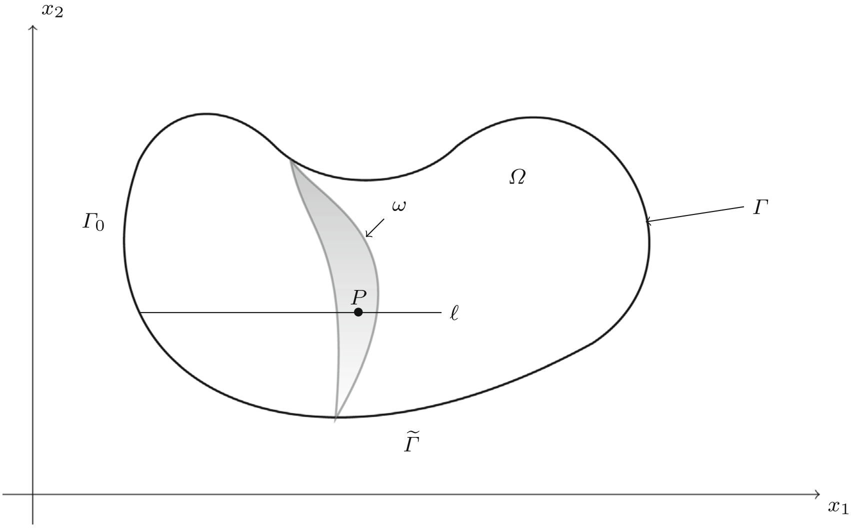


Fig. 7 Pair $\{\omega, \Omega\}$ where Def. 1 fails, $d = 2$. Not covered by Theorem 6, but covered by Theorem 5

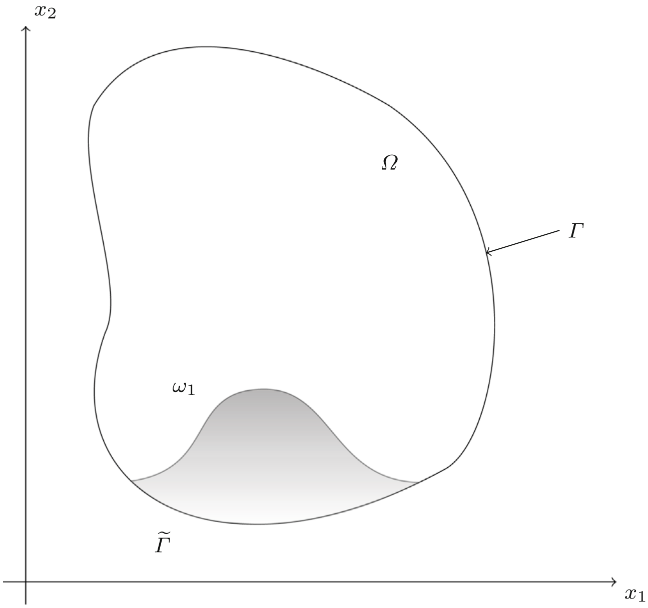


Fig. 8 Pair $\{\omega_1, \Omega\}$ where Def. 1 fails, $d = 2$. Not covered by Theorem 6, but covered by Theorem 5

Let $\{\phi, h, p\} \in (W^{2,q}(\Omega))^d \times W^{2,q}(\Omega) \times W^{1,q}(\Omega)$, $q > d$, *satisfy the adjoint Boussinesq problem* (1.18). *With* $\phi = \{\phi_1, \phi_2\}$, *let*

$$\phi_1\big|_{\Gamma_0} \equiv 0. \tag{1.21b}$$

Moreover, assume the over-determined condition

$$h \equiv 0 \quad in\ \omega. \tag{1.22}$$

Then,

$$\phi \equiv 0, \qquad h \equiv 0, \qquad p \equiv const \quad in\ \Omega. \tag{1.23}$$

Proof As in the proof of Theorem 5, the condition $h \equiv 0$ on ω in (1.22) implies by (1.18b)

$$\phi_2 \equiv 0 \text{ in } \omega,\ d = 2. \tag{1.24}$$

In the case $d = 2$, the condition $\phi_2 \equiv 0$ in ω in (1.24) then implies by (1.18c)

$$\text{div } \phi = \phi_{1x_1} + \phi_{2x_2} \equiv \phi_{1x_1} \equiv 0 \text{ in } \omega;\ \text{ hence } \phi_1(x_1, x_2) \equiv c(x_2) \text{ in } \omega, \tag{1.25}$$

where $c(x_2)$ is a function constant w.r.t. x_1 and depending only on x_2 on ω. Next, let $P = \{x_1(P), x_2(P)\}$ be an arbitrary point of ω. Consider the line ℓ passing through the point P and parallel to the x_1-axis (Figs. 5 and 6). On such a line ℓ, the value $\phi_1(x_1, x_2(P)) = c(x_2(P))$ is constant w.r.t. x_1, as long as ℓ intersects ω. In particular, this constant value is equal to the value of ϕ_1 on the point $Q \in \Gamma_0 \subset \widetilde{\Gamma} = \partial\omega \cap \Gamma$, guaranteed to exist by Def. 1 (ii$_2$). Thus

$$\phi_1(x_1, x_2(P)) = \phi_1\big|_Q = 0; \qquad (x_1, x_2(P)) \in \omega \tag{1.26}$$

by assumption (1.21b). But P is arbitrary in ω. Thus (1.26) yields

$$\phi_1 \equiv 0 \quad \text{in } \omega. \tag{1.27}$$

Then (1.27) along with $h \equiv 0$ in ω by (1.22), allows us to fall into Theorem 5 (for $d = 2$) and obtain conclusion (1.23). Theorem 6 is proved. □

Theorem 7 (UCP, Second version, adjoint problem, $d = 3$) *Let $d = 3$. Let $\{\omega, \Omega\}$ be a pair satisfying Definition 1, (i) and either (ii$_3$) Case 1, or else (ii$_3$) Case 2, respectively. Thus, if P is an arbitrary point of ω, then the plane π_P passing through P and either parallel to the (x_1, x_3)-coordinate plane (Case 1), or else parallel to the (x_2, x_3)-coordinate plane (Case 2), meets the intersection $\widetilde{\Gamma} \equiv \partial\omega \cap \Gamma$ at a curve $\mathcal{C}_P$. Let*

$$\begin{aligned} \Gamma_0 = \Big\{ & \textit{connected component(s) of the union of such curves } \mathcal{C}_P \\ & \textit{as } P \textit{ runs over } \omega,\ \textit{such that any such plane } \pi_P,\ P \in \omega,\ \textit{hits } \Gamma_0 \textit{ at just} \\ & \textit{one curve, as in Definition 1 (i), (ii}^3\textit{) Case 1, or (ii}^3\textit{) Case 2} \Big\}. \end{aligned} \tag{1.28}$$

Let $\{\phi, h, p\} \in (W^{2,q}(\Omega))^d \times W^{2,q}(\Omega) \times W^{1,q}(\Omega)$, $q > d$, satisfy the adjoint Boussinesq problem (1.18). With $\phi = \{\phi_1, \phi_2, \phi_3\}$, assume

$$\phi_2\big|_{\Gamma_0} = 0 \quad under\ (ii_3),\ Case\ 1 \tag{1.29}$$

$$\phi_1\big|_{\Gamma_0} = 0 \qquad \textit{under } (ii_3), \ \textit{Case 2} \tag{1.30}$$

as well as the over-determined condition

$$h \equiv 0 \ \ \textit{in } \omega \quad \textit{and} \quad \begin{cases} \phi_1 \equiv 0 \textit{ in } \omega, \textit{under } (ii_3) \textit{ Case 1}, & (1.31) \\ \phi_2 \equiv 0 \textit{ in } \omega, \textit{under } (ii_3) \textit{ Case 2}. & (1.32) \end{cases}$$

Then, in either case

$$\phi \equiv 0, \qquad h \equiv 0, \qquad p \equiv \textit{const} \quad \textit{in } \Omega. \tag{1.33}$$

Proof As in the proof of Theorem 5, the conditon $h \equiv 0$ in ω in (1.31), (1.32) implies via (1.18b)

$$\phi_3 \equiv 0 \text{ in } \omega, d = 3. \tag{1.34}$$

Case $d = 3$, *under* (ii_3), *Case 1*: so now by (1.34) and (1.31),

$$h \equiv 0 \quad \text{in } \omega, \qquad \phi_1 \equiv 0 \quad \text{in } \omega, \qquad \phi_3 \equiv 0 \quad \text{in } \omega. \tag{1.35}$$

The divergence condition (1.18c) then implies via (1.35)

$$\text{div } \phi = \phi_{1x_1} + \phi_{2x_2} + \phi_{3x_3} \equiv \phi_{2x_2} \equiv 0 \quad \text{in } \omega; \text{ hence } \phi_2(x_1, x_2, x_3) = c(x_1, x_3) \quad \text{in } \omega, \tag{1.36}$$

where $c(x_1, x_3)$ denotes a function constant w.r.t. x_2 and depending only on x_1 and x_3 on ω. Let $P = \{x_1(P), x_2(P), x_3(P)\}$ be an arbitrary point of ω. Consider the plane π_P passing through the point P and parallel to the $\{x_1, x_3\}$-coordinate plane. As the point $\{x_1, x_2(P), x_3\}$ of ω runs over the plane π_P, the value $\phi_2(x_1, x_2(P), x_3) = c(x_1, x_3)$ is independent of $x_2(P)$, as long as such plane π_P intersects ω. By Def. 1 (ii$_3$), Case 1, such plane π_P meets the intersection $\widetilde{\Gamma} = \partial\omega \cap \Gamma$ at some curve $\mathcal{C} = \mathcal{C}_P \subset \Gamma_0$, Γ_0 in (1.28). Thus

$$\phi_2(x_1, x_2(P), x_3) = \phi_2\big|_{\mathcal{C}} = 0, \quad (x_1, x_2(P), x_3) \in \omega \tag{1.37}$$

by recalling assumption (1.28) and (1.29). But P is an arbitrary point of ω. Thus (1.37) yields

$$\phi_2 \equiv 0 \quad \text{in } \omega. \tag{1.38}$$

Then (1.38) along with $\phi_1 \equiv 0$ on ω and $h \equiv 0$ on ω by (1.35) allows us to fall into Theorem 5 ($d = 3$) and yield conclusion (1.33).

Case $d = 3$, *under* (ii_3), *Case 2*. The proof is the same mutatis mutandis, starting now from

$$h \equiv 0 \quad \text{in } \omega, \qquad \phi_2 \equiv 0 \quad \text{in } \omega, \qquad \phi_3 \equiv 0 \quad \text{in } \omega \tag{1.39}$$

by (1.32) and (1.34). □

Theorem 8 *[UCP, Third version, adjoint problem]*

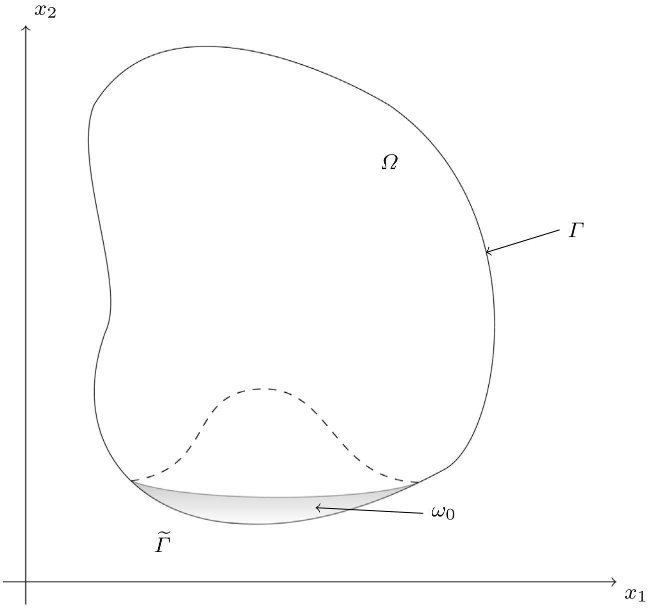


Fig. 9 Rescuing Fig. 8: Pair $\{\omega_0, \Omega\}$, $\omega_0 \subset \omega_1$, covered by Theorem 6, $d = 2$

(a) *Let $d = 2$. Let $\{\omega, \Omega\}$ be a pair satisfying Definition 1, (i); Case (ii$_2$) as in Theorem 6. Let $\{\phi, h, p\} \in (W^{2,q}(\Omega))^d \times W^{2,q}(\Omega) \times W^{1,q}(\Omega)$, $q > d$, satisfy the adjoint Boussinesq problem* (1.18). *With $\phi = \{\phi_1, \phi_2\}$ assume*

$$\phi_1\big|_{\Gamma_0} \equiv 0, \tag{1.40}$$

Γ_0 defined in (1.21a). *Assume, in addition, the over-determined conditions*

$$h\big|_{\Gamma_1} \equiv 0, \quad \frac{\partial h}{\partial \nu}\bigg|_{\Gamma_1} \equiv 0, \quad \Gamma_1 = \text{ arbitrarily small subportion of } \widetilde{\Gamma} = \partial\omega \cap \Gamma \tag{1.41}$$

$$\phi_2 \equiv 0 \text{ in } \omega. \tag{1.42}$$

Then,

$$\phi \equiv 0, \quad h \equiv 0, \quad p \equiv const \quad \text{in } \Omega. \tag{1.43}$$

(b) *Let $d = 3$. Let $\{\omega, \Omega\}$ be a pair satisfying Definition 1, (i) and either (ii$_3$) Case 1, or else (ii$_3$) Case 2, respectively as in Theorem 7. Let $\{\phi, h, p\} \in (W^{2,q}(\Omega))^d \times W^{2,q}(\Omega) \times W^{1,q}(\Omega)$, $q > d$, satisfy the adjoint Boussinesq problem* (1.18). *With $\phi = \{\phi_1, \phi_2, \phi_3\}$, assume*

$$\phi_2\big|_{\Gamma_0} = 0 \quad \text{under (ii}_3\text{), Case 1,} \tag{1.44}$$

$$\phi_1\big|_{\Gamma_0} = 0 \quad \text{under (ii}_3\text{), Case 2,} \tag{1.45}$$

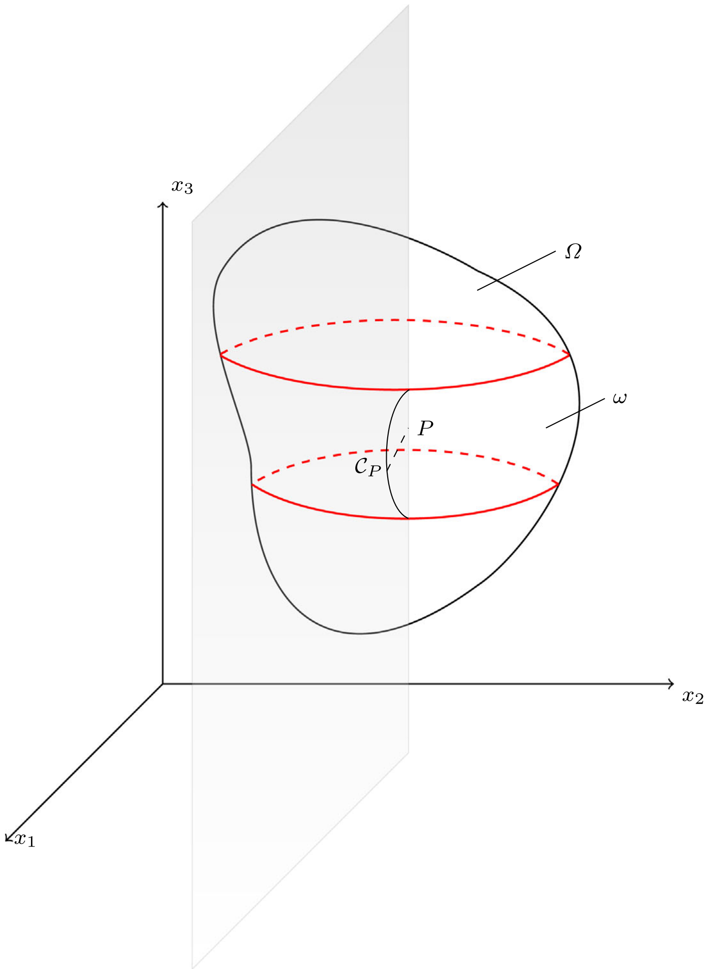


Fig. 10 Pair $\{\omega, \Omega\}$ where Def. 1 (i), (ii$_3$) Case 1 applies, $d = 3$. Covered by Theorem 7

where Γ_0 is defined in (1.28). *In addition, assume the over-determined conditon*

$$\phi_3 \equiv 0 \textit{ in } \omega \textit{ and } \begin{cases} \phi_1 \equiv 0 \textit{ in } \omega, \textit{ under (ii}_3\textit{), Case 1}, & (1.46) \\ \phi_2 \equiv 0 \textit{ in } \omega, \textit{ under (ii}_3\textit{), Case 2}, & (1.47) \end{cases}$$

as well as

$$h\big|_{\Gamma_1} \equiv 0, \qquad \frac{\partial h}{\partial \nu}\bigg|_{\Gamma_1} \equiv 0, \quad \Gamma_1 = \textit{ arbitrarily small subportion of } \widetilde{\Gamma} = \partial\omega \cap \Gamma \tag{1.48}$$

Then,

$$\phi \equiv 0, \quad h \equiv 0, \quad p \equiv const \quad \textit{in } \Omega. \tag{1.49}$$

Proof (a) $d = 2$. Assumption $\phi_2 \equiv 0$ in ω in (1.42) along with $e_d = \{0, 1\}$ used in the thermal equation (1.18b) yields the problem

$$\begin{cases} -\kappa\Delta h + y_e \cdot \nabla h = \lambda h \quad \text{in } \omega & (1.50a) \\ h\big|_{\Gamma_1} \equiv 0, \quad \left.\dfrac{\partial h}{\partial \nu}\right|_{\Gamma_1} \equiv 0 \quad \Gamma_1 = \text{ arbitrarily small subportion of } \widetilde{\Gamma} = \partial\omega \cap \Gamma & (1.50b) \end{cases}$$

recalling also (1.41). As noted in the proof of Theorem 3, problem (1.50) then implies

$$h \equiv 0 \quad \text{in } \omega. \tag{1.51}$$

Then (1.51) along with (1.40) allows us to fall into Theorem 6 (for $d = 2$) and yield conclusion (1.43).

(b) $d = 3$. Now it is assumption $\phi_3 \equiv 0$ in ω in (1.46) along with $e_d = \{0, 0, 1\}$ that yields problem (1.50) recalling now (1.48). Then again we obtain

$$h \equiv 0 \quad \text{in } \omega. \tag{1.52}$$

Thus, (1.52) along with assumptions (1.46) (Case 1) or (1.47) (Case 2), as well as (1.44) (Case 1) or (1.45) (Case 2) allows us to fall into Theorem 7 ($d = 3$) and yield conclusion (1.49).

□

Theorem 9 *(UCP, Fourth version, adjoint problem) Let $d = 3$. Let $\{\omega, \Omega\}$ be a pair satisfying Definition 1, (i) as in Figs. 2 or in 4. Let $\{\phi, h, p\} \in (W^{2,q}(\Omega))^d \times W^{2,q}(\Omega) \times W^{1,q}(\Omega)$, $q > d$, satisfy the adjoint Boussinesq problem (1.18) along with the B.C.*

$$\phi\big|_{\widetilde{\Gamma}} \equiv 0, \quad \left.\frac{\partial \phi}{\partial \nu}\right|_{\widetilde{\Gamma}} \equiv 0, \quad \widetilde{\Gamma} = \textit{(arbitrarily small) } \partial\omega \cap \Gamma \tag{1.53}$$

$$h \equiv 0 \textit{ in } \omega, \qquad \phi \cdot \tau \equiv 0 \textit{ in } \omega, \tag{1.54}$$

in the notation of Theorem 4, see Fig. 4. Then,

$$\phi \equiv 0, \qquad h \equiv 0, \qquad p \equiv const \quad in\ \Omega. \tag{1.55}$$

Proof Assumption (1.53) and (1.54) yield the following over-determined Oseen problem by (1.18a)

$$\begin{cases} -\nu_0\Delta\phi + L_e^*(\phi) + \nabla p = \lambda\phi & \text{in } \omega, & (1.56a) \\ \operatorname{div} \phi \equiv 0 & \text{in } \omega, & (1.56b) \\ \phi\big|_{\widetilde{\Gamma}} \equiv 0, \quad \left.\dfrac{\partial \phi}{\partial \nu}\right|_{\widetilde{\Gamma}} \equiv 0, \quad \phi \cdot \tau \equiv 0 & \text{in } \omega. & (1.56c) \end{cases}$$

As in the proof of Theorem 4, the over-determined Oseen problem (1.56) implies

$$\phi = \{\phi_1, \phi_2, \phi_3\} \equiv 0 \text{ in } \omega, \qquad p \equiv \text{const} \quad \text{in } \omega, \tag{1.57}$$

by virtue of [37, Lemma 2, p 138], recalled in Appendix B. Thus, a-fortiori, the assumptions of Theorem 5 are satisfied. Then Theorem 5 implies conclusion (1.55). $\square$

1.3 Literature

The results for the adjoint Boussinesq static problem – namely, Theorem 5 and, respectively, Theorem 6 ($d = 2$) and 7 ($d = 3$) and Theorem A.3 ($d = 4$), with the reduction of one, respectively, two components on the over-determination of the fluid vector $\phi = \{\phi_1, \ldots, \phi_d\}$ as in (1.18d), respectively, as in (1.22) [Case $d = 2$, with $d - 2 = 0$ components of ϕ] and (1.31), (1.32) [Case $d = 3$, with $d - 2 = 1$ components of ϕ] are in line with the open-loop controllability results in [11,14,25] [15].

A proof yielding, say Theorem 2 with $d = 2$ and with limited regularity of the solution was given in [42]. Theorem 2 is in line with an ' observability inequality ' for the corresponding time dependant problem needed in the study of local controllability to the origin or to a trajectory given in [17]. It improves on the prior observability inequality in [25].

Regarding results on UCP concerning only the fluid equation, we make the following (non-exhaustive) comments. References [16,19] provide a UCP for the Stokes problem (rather than the corresponding Oseen problem as in Theorem A.1 of Appendix A) with implications on approximate controllability. The UCP property of Theorem Theorem A.1, Appendix A has been shown via Carleman's estimates in [5] by first transforming Ω in a "bent" half-space with a parabolic boundary, next selecting the Melrose-Sjostrand form for the Laplacian, and finally applying the Carleman estimates in integral form from [26]. A different proof, directly on Ω, and this time with no use of the condition $\boldsymbol{\varphi} \equiv 0$ on Γ, was later given in [48], also via use of (different) Carleman-type estimates for the Laplacian. Results on UCP for Stokes and Oseen operators (with small y_e) are given in [46,47] in the case of boundary over-determination [10].

2 Proof of Theorem 2

Step 0 Without loss of generality we may normalize the constants $\nu_0 = \kappa = \gamma \equiv 1$. Via (1.2), we can then rewrite Equations (1.3a), (1.3c) combined as in (2.1a) below, along with (1.3b) and the overdetermination (1.3d)

$$
\begin{cases}
(-\Delta)\begin{bmatrix}\phi\\ h\end{bmatrix} + (y_e\cdot\nabla)\begin{bmatrix}\phi\\ h\end{bmatrix} + (\phi\cdot\nabla)\begin{bmatrix}y_e\\ \theta_e\end{bmatrix} + \begin{bmatrix}\nabla p\\ 0\end{bmatrix} - \begin{bmatrix}0\\ \vdots\\ 0\\ h\\ 0\end{bmatrix} = \lambda\begin{bmatrix}\phi\\ h\end{bmatrix} \quad \text{in } \Omega & (2.1a)\\
\operatorname{div}\phi \equiv 0 \text{ in } \Omega \quad \text{and} \quad \phi \equiv 0,\ h \equiv 0 \text{ in } \omega & (2.1b)
\end{cases}
$$

The above problem (2.1a)–(2.1b) is not quite the overdetermined Oseen problem in Appendix A in the variable $u = \{\phi, h\}$. We shall apply the Carleman estimate approach and techniques employed in [48] for the Oseen problem, with appropriate modifications.

Case 1 We write initially the proof for the case where ω is at a positive distance from $\partial\Omega$: $\operatorname{dist}(\partial\Omega, \partial\omega) > 0$. (Figs. 8 and 9).

Step 1 Since $u = \{\phi, h\} \equiv 0$ in ω by (1.3d) = (2.1b), then (2.1a) yields $\nabla p \equiv 0$ in ω, hence $p =$ const in ω. We may then take $p \equiv 0$ in ω, as p is only identified up to a constant. Then we have

$$
u\big|_{\partial\omega} = \begin{bmatrix}\phi\\ h\end{bmatrix}\Bigg|_{\partial\omega} \equiv 0; \quad \frac{\partial u}{\partial\nu}\Bigg|_{\partial\omega} = \frac{\partial}{\partial\nu}\begin{bmatrix}\phi\\ h\end{bmatrix}\Bigg|_{\partial\omega} \equiv 0; \quad p\big|_{\partial\omega} \equiv 0; \quad \frac{\partial p}{\partial\nu}\Bigg|_{\partial\omega} \equiv 0. \tag{2.2}
$$

Step 2 *The cut-off function* χ. Let χ be a smooth, non-negative, cut function defined as follows:

$$
\chi \equiv \begin{cases} 1 \quad \text{on } \omega \cup \Omega_1 & (2.3a)\\ 0 \quad \text{on } \Omega_0 & (2.3b)\end{cases} ; \qquad \operatorname{supp}\chi \subset [\Omega_1 \cup \Omega^* \cup \omega],
$$

while monotonically decreasing from 1 to 0 in Ω^*, with $\chi \equiv 0$ also in a small layer within Ω^* bordering Ω_0 (Figs. 8 and 9). Here:

(i) Ω_1 is a smooth sub-domain of Ω surrounding ω, and $\partial\omega$ is the interface between ω and Ω_1 (Fig. 8). Thus, $\partial\omega =$ internal boundary of Ω_1;
(ii) In turn, Ω^* is a smooth sub-domain of Ω surrounding Ω_1 and the external boundary of Ω_1 is the interface between Ω^* and Ω_1. Thus, [external boundary of Ω_1] = [internal boundary of Ω^*].
(iii) In turn, Ω_0 is a smooth sub-domain of Ω: $\Omega_0 \equiv \Omega \backslash \{\omega \cup \Omega_1 \cup \Omega^*\}$.

Step 3 *The* $(\chi\phi)$*-problem*. Multiply the ϕ-equation in (2.1a) (i.e. (1.3a)) by χ and obtain via (1.2)

$$
(-\Delta)(\chi\phi) + L_e(\chi\phi) + \nabla(\chi p) = \lambda(\chi\phi) + F_\chi \quad \text{in } \Omega \tag{2.4}
$$

$$
F_\chi = F_\chi(\phi, p, h) = F_\chi^{1,0}(\phi, p) + \begin{bmatrix}0\\ 0\\ (\chi h)\end{bmatrix} \tag{2.5}
$$

$$F_\chi^{1,0}(\phi, p) = [\chi, \Delta]\phi - [\chi, L_e]\phi + [\nabla, \chi]p \tag{2.6a}$$
$$= \text{first order in } \phi; \text{ zero order in } p \tag{2.6b}$$
$$\text{supp } F_\chi^{1,0} \subset \Omega^* \tag{2.6c}$$

Notice that (2.6c) holds true since $\chi \equiv 1$ on Ω_1, on ω, and on a small layer within Ω^*, so that on the union of these three sets we have that $F_\chi^{1,0} \equiv 0$. We recall that the commutator $[\chi, \Delta]$ is of order $0 + 2 - 1 = 1$; the commutator $[\chi, L_e]$ is of order $0 + 1 - 1 = 0$; the commutator $[\nabla, \chi]$ is of order $1 + 0 - 1 = 0$.

Step 4 *The* (χh)*-problem*. Next, we multiply the h-equation in (2.1a) = (1.3c) by χ and obtain

$$(-\Delta)(\chi h) + (y_e \cdot \nabla)(\chi h) = \lambda(\chi h) + G_\chi(h, \phi) \quad \text{in } \Omega \tag{2.7}$$
$$G_\chi = G_\chi(h, \phi) = G_\chi^1(h) - (\chi\phi) \cdot \nabla\theta_e \tag{2.8}$$

$$G_\chi^1(h) = [\chi, \Delta]h - y_e \cdot [\chi, \nabla]h = \text{first order in } h \tag{2.9a}$$
$$\text{supp } G_\chi^1 \subset \Omega^*. \tag{2.9b}$$

Notice that (2.9b) holds true, since as in the case for (2.6c), we have that $G_\chi^1 \equiv 0$ on $\omega \cup \Omega_1 \cup$ [a small layer withinΩ^*], since $\chi \equiv 1$ on such union.

Step 5 *The* (χu)*-problem*, $\chi u = \{\chi\phi, \chi h\}$. We combine Step 3 and Step 4 and obtain recalling (1.2):

$$(-\Delta)\left(\chi \begin{bmatrix} \phi \\ h \end{bmatrix}\right) + (y_e \cdot \nabla)\left(\chi \begin{bmatrix} \phi \\ h \end{bmatrix}\right) + \begin{bmatrix} \big((\chi\phi) \cdot \nabla\big) y_e \\ 0 \end{bmatrix} + \begin{bmatrix} \nabla(\chi p) \\ 0 \end{bmatrix}$$
$$= \lambda\left(\chi \begin{bmatrix} \phi \\ h \end{bmatrix}\right) + \begin{bmatrix} F_\chi^{1,0}(\phi, p) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ (\chi h) \end{bmatrix} \\ G_\chi^1(h) - (\chi\phi) \cdot \nabla\theta_e \end{bmatrix} \text{ in } \Omega. \tag{2.10}$$

Moreover, let (Fig. 8)

$$D = \partial\omega \cup \{\text{external boundary of } \Omega^*\} = \partial[\Omega_1 \cup \Omega^*]. \tag{2.11}$$

Since $\chi \equiv 0$ on Ω_0 and in a small layer of Ω^* bordering Ω_0 (Figs. 8 and 9), then $(\chi u) = \{(\chi\phi), (\chi h)\}$ and (χp) have zero Cauchy data on the [external boundary of Ω^*] = [interior boundary of Ω_0].

Moreover, since $u \equiv \{\phi, h\} \equiv 0$ in ω and $p \equiv 0$ in ω by Step 1, then $(\chi u) = \{(\chi\phi), (\chi h)\}$ and (χp) have zero Cauchy data on $\partial\omega$ see (2.2). Thus, recalling D in (2.11) and $u \equiv \{\phi, h\}$:

$$(\chi u)\big|_D \equiv 0, \quad \left.\frac{\partial(\chi u)}{\partial\nu}\right|_D \equiv 0, \quad (\chi p)\big|_{\partial\omega} \equiv 0, \quad \left.\frac{\partial(\chi p)}{\partial\nu}\right|_{\partial\omega} \equiv 0, \tag{2.12}$$

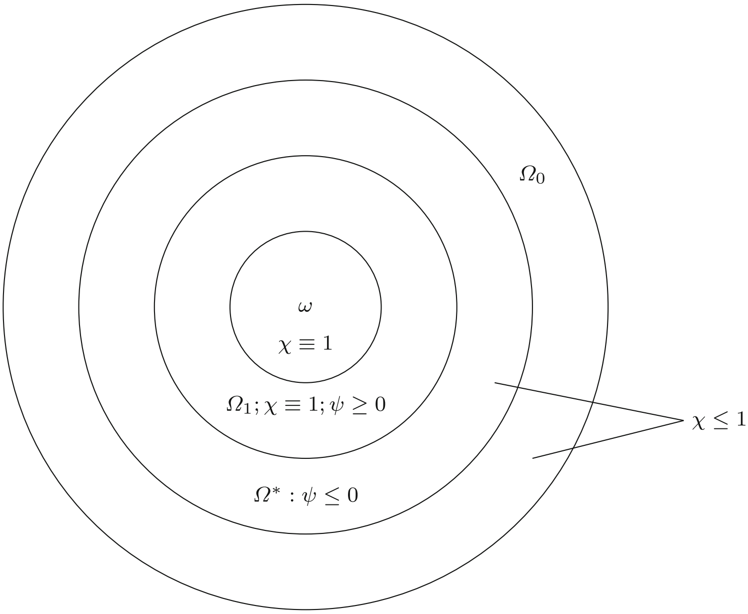


Fig. 11 The cut-off function χ and the strictly convex function ψ. See also Fig. 12

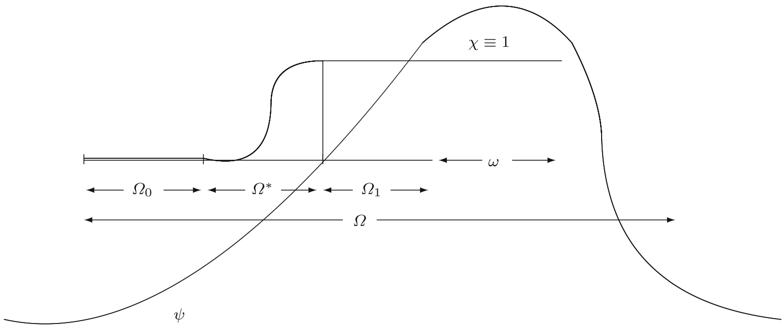


Fig. 12 Choice of ψ in Step 15, (2.36) and (2.37)

where ν denotes here the unit vector outward with respect to $[\Omega^* \cup \Omega_1]$, Fig. 8.

Step 6 *A pointwise Carleman estimate.* We shall invoke the following pointwise Carleman estimate for the Laplacian from [40, Corollary 4.2, Eq. (4.15), p. 73], [39, Corollary 4.3, p. 254].

Theorem 10 *The following pointwise estimate holds true at each point x of a bounded domain G in $\mathbb{R}^d$ for an H^2-function w, where $\epsilon > 0$ and $0 < \delta_0 < 1$ are arbitrary*

$$\delta_0 \left[2\rho\tau - \frac{\epsilon}{2}\right] e^{2\tau\psi(x)}|\nabla w(x)|^2 + [4\rho k^2 \tau^3 (1 - \delta_0) + \mathcal{O}(\tau^2)]e^{2\tau\psi(x)}|w(x)|^2$$
$$\leq \left(1 + \frac{1}{\epsilon}\right) e^{2\tau\psi(x)}|\Delta w(x)|^2 + \text{div } V_w(x), \quad x \in G. \tag{2.13}$$

Here: $\psi(x)$ is any strictly convex function over G, with no critical points in $\overline{G}$, see Fig. 12, to be chosen below in Step 15 where $G = \Omega_1 \cup \Omega^*$; $\rho > 0$ is a constant, defined by $\mathcal{H}_\psi(x) \geq \rho I$, $x \in \overline{G}$, where $\mathcal{H}_\psi$ denotes the (symmetric) Hessian matrix of $\psi(x)$ [40, Eq. (1.1.6), p. 45]; $k > 0$ is a constant, defined by: inf $|\nabla\psi(x)| = k > 0$, where the inf is taken over G [40, Eq. (1.1.7), p. 45]; and τ is a free positive parameter, to be chosen sufficiently large. For what follows, it is not critical to recall what div $V_w(x)$ is, only that, via the divergence theorem, we have

$$\int_G \text{div } V_w(x)dx = \int_{\partial G} V_w(x) \cdot \nu \, d\sigma = 0, \tag{2.14}$$

whenever the Cauchy data *of w vanish on its boundary ∂G: $w|_{\partial G} \equiv 0$; $\nabla w|_{\partial G} \equiv 0$. In* (2.14)*, ν is a unit normal vector outward with respect to G.*

Step 7 *Pointwise Carleman estimates for (χu), $u = \{\phi, h\}$.* Next, we apply estimate (2.13) with $w = (\chi u)$ solution of problem (2.10). For definiteness, we select $\delta_0 = \frac{1}{2}$, $\epsilon = \frac{1}{2}$. We obtain

$$\begin{aligned} &\left[\rho\tau - \frac{1}{8}\right] e^{2\tau\psi(x)} |\nabla(\chi u)(x)|^2 + [2\rho k^2 \tau^3 + \mathcal{O}(\tau^2)] e^{2\tau\psi(x)} |(\chi u)(x)|^2 \\ &\quad \leq 3 e^{2\tau\psi(x)} |\Delta(\chi u)(x)|^2 + \text{div } V_{(\chi u)}(x), \quad x \in G. \end{aligned} \tag{2.15}$$

Next, we integrate (2.15) over the domain $G \equiv [\Omega_1 \cup \Omega^*]$ (Fig. 8), thus obtaining

$$\begin{aligned} &\left[\rho\tau - \frac{1}{8}\right] \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} |\nabla(\chi u)(x)|^2 dx \\ &\quad + [2\rho k^2\tau^3 + \mathcal{O}(\tau^2)] \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} |(\chi u)(x)|^2 dx \\ &\quad \leq 3 \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} |\Delta(\chi u)(x)|^2 dx + \int_{\partial[\Omega_1 \cup \Omega^*]} \cancel{V_{(\chi u)}(x)} \cdot \nu \, dD, \end{aligned} \tag{2.16}$$

where, on the RHS of (2.16), the boundary integral over $D \equiv \partial[\Omega_1 \cup \Omega^*] =$ the boundary of $[\Omega_1 \cup \Omega^*]$, see (2.11) and Fig. 8, vanishes in view of (2.14) with $w = (\chi u)$ having null Cauchy data on D, by virtue of (the LHS of) (2.12).

Step 8 (Bound on the RHS of (2.16)) Here, we estimate the RHS of (2.16). Returning to the (χu)-problem (2.10), we rewrite it over $G = [\Omega_1 \cup \Omega^*]$ as

$$\begin{aligned} \Delta\left(\chi \begin{bmatrix} \phi \\ h \end{bmatrix}\right) &= (y_e \cdot \nabla)\left(\chi \begin{bmatrix} \phi \\ h \end{bmatrix}\right) + \begin{bmatrix} \big((\chi\phi)\cdot\nabla\big) y_e \\ 0 \end{bmatrix} + \begin{bmatrix} \nabla(\chi p) \\ 0 \end{bmatrix} \\ &\quad -\lambda\left(\chi \begin{bmatrix} \phi \\ h \end{bmatrix}\right) - \begin{bmatrix} F_\chi^{1,0}(\phi, p) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ (\chi h) \end{bmatrix} \\ G_\chi^1(h) - (\chi\phi)\cdot\nabla\theta_e \end{bmatrix} \end{aligned} \tag{2.17}$$

and multiply across by $e^{\tau\psi(x)}$ to get

$$
\begin{aligned}
e^{\tau\psi(x)}\Delta\left(\chi\begin{bmatrix}\phi\\ h\end{bmatrix}\right) &= (e^{\tau\psi(x)}y_e\cdot\nabla)\left(\chi\begin{bmatrix}\phi\\ h\end{bmatrix}\right)+\begin{bmatrix}\left(e^{\tau\psi(x)}(\chi\phi)\cdot\nabla\right)y_e\\ 0\end{bmatrix}\\
&\quad+\begin{bmatrix}e^{\tau\psi(x)}\nabla(\chi p)\\ 0\end{bmatrix}-\lambda e^{\tau\psi(x)}\left(\chi\begin{bmatrix}\phi\\ h\end{bmatrix}\right)-e^{\tau\psi(x)}\begin{bmatrix}F_\chi^{1,0}(\phi,p)+\begin{bmatrix}0\\ \vdots\\ 0\\ (\chi h)\end{bmatrix}\\ G_\chi^1(h)-(\chi\phi)\cdot\nabla\theta_e\end{bmatrix}
\end{aligned}
\tag{2.18}
$$

Recalling $y_e\in(W^{2,q}(\Omega))^d$, $\theta_e\in W^{2,q}(\Omega)$ by Theorem 1, as well as the embedding $W^{1,q}(\Omega)\hookrightarrow C(\overline{\Omega})$ for $q>d$, [1, p 97, for Ω having cone property] [28, p. 79, requiring C^1-boundary], we have $|\nabla y_e(x)|+|\nabla\theta_e(x)|\le C_{y_e,\theta_e}$, $x\in\Omega$, for $q>d$, as assumed. In view of this, we return to (2.18) and obtain

$$
\begin{aligned}
e^{2\tau\psi(x)}\left|\Delta\left(\chi\begin{bmatrix}\phi\\ h\end{bmatrix}\right)(x)\right|^2 &\le c_e e^{2\tau\psi(x)}\left\{\left|\nabla\left(\chi\begin{bmatrix}\phi\\ h\end{bmatrix}\right)(x)\right|^2+|(\chi\phi)(x)|^2\right\}\\
&\quad+c_\lambda e^{2\tau\psi(x)}\left|\left(\chi\begin{bmatrix}\phi\\ h\end{bmatrix}\right)(x)\right|^2+e^{2\tau\psi(x)}|\nabla(\chi p)(x)|^2+e^{2\tau\psi(x)}\left|\begin{bmatrix}F_\chi^{1,0}(\phi,p)(x)\\ G_\chi^1(h)(x)\end{bmatrix}\right|^2,\quad x\in G
\end{aligned}
\tag{2.19}
$$

$c_e=$ a constant depending on y_e and θ_e, $c_\lambda=|\lambda|^2+1$. Thus, integrating (2.19) over $G\equiv[\Omega_1\cup\Omega^*]$ as required by (2.16) yields with $u=\begin{bmatrix}\phi\\ h\end{bmatrix}$

$$
\begin{aligned}
\int_{\Omega_1\cup\Omega^*}e^{2\tau\psi(x)}|\Delta(\chi u)(x)|^2dx &\le C_{\lambda,e}\int_{\Omega_1\cup\Omega^*}e^{2\tau\psi(x)}[|\nabla(\chi u)(x)|^2+|(\chi u)(x)|^2]dx\\
&\quad+\int_{\Omega_1\cup\Omega^*}e^{2\tau\psi(x)}|\nabla(\chi p)(x)|^2dx+\int_{\Omega_1\cup\Omega^*}e^{2\tau\psi(x)}\left|\begin{bmatrix}F_\chi^{1,0}(\phi,p)(x)\\ G_\chi^1(h)(x)\end{bmatrix}\right|^2dx,
\end{aligned}
\tag{2.20}
$$

$C_{\lambda,e}=$ a constant depending on λ, y_e and θ_e.

We now recall from (2.6) and (2.9) that $F_\chi^{1,0}(\phi,p)$ is an operator which is first order in ϕ and zero order in p, while $G_\chi^1(h)$ is first order in h; and moreover, that their support is in Ω^* : supp $F_\chi^{1,0}\subset\Omega^*$, supp $G_\chi^1\subset\Omega^*$. Thus, (2.20) becomes explicitly, still with $u=\begin{bmatrix}\phi\\ h\end{bmatrix}$:

$$
\begin{aligned}
\int_{\Omega_1\cup\Omega^*}e^{2\tau\psi(x)}|\Delta(\chi u)(x)|^2dx &\le C_{\lambda,e}\int_{\Omega_1\cup\Omega^*}e^{2\tau\psi(x)}[\nabla(\chi u)(x)|^2+|(\chi u)(x)|^2]dx\\
&\quad+\int_{\Omega_1\cup\Omega^*}e^{2\tau\psi(x)}|\nabla(\chi p)(x)|^2dx\\
&\quad+c_\chi\int_{\Omega^*}e^{2\tau\psi(x)}[|\nabla u(x)|^2+|u(x)|^2+|p(x)|^2]dx,
\end{aligned}
\tag{2.21}
$$

which is the sought-after bound on the last term of the RHS of (2.16). In (2.21), c_χ is a constant depending on χ.

Step 9 *(Final estimate for* (χu)*-problem* (2.10), $u = \begin{bmatrix} \phi \\ h \end{bmatrix}$.) We substitute (2.21) into the RHS of inequality (2.16), and obtain

$$\begin{aligned}
&\left[\rho\tau - \frac{1}{8}\right] \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} |\nabla(\chi u)(x)|^2 dx \\
&\quad + \left[2\rho k^2 \tau^3 + \mathcal{O}(\tau^2)\right] \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} |(\chi u)(x)|^2 dx \\
&\leq C_{\lambda,e} \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} [|\nabla(\chi u)(x)|^2 + |(\chi u)(x)|^2] dx \\
&\quad + 3 \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} |\nabla(\chi p)(x)|^2 dx \\
&\quad + c_\chi \int_{\Omega^*} e^{2\tau\psi(x)} [|\nabla u(x)|^2 + |u(x)|^2 + |p(x)|^2] dx. \qquad (2.22)
\end{aligned}$$

Moving the first integral term on the RHS of inequality (2.22) to the LHS of such inequality then yields for τ sufficiently large:

$$\begin{aligned}
&\left\{\left[\rho\tau - \frac{1}{8}\right] - C_{\lambda,e}\right\} \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} |\nabla(\chi u)(x)|^2 dx \\
&\quad + \left[2\rho k^2 \tau^3 + \mathcal{O}(\tau^2) - C_{\lambda,e}\right] \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} |(\chi u)(x)|^2 dx \\
&\leq 3 \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} |\nabla(\chi p)(x)|^2 dx \\
&\quad + c_\chi \int_{\Omega^*} e^{2\tau\psi(x)} [|\nabla u(x)|^2 + |u(x)|^2 + |p(x)|^2] dx. \qquad (2.23)
\end{aligned}$$

Inequality (2.23) is our final estimate for the (χu)-problem in (2.10), (2.6), (2.9).

Step 10 *The* (χp)*-problem.* We need to estimate the first integral term on the RHS of inequality (2.23). This will be accomplished in (2.31) below. To this end, we need to obtain preliminarily the PDE-problem satisfied by (χp) on $G \equiv \Omega_1 \cup \Omega^*$. This task will be accomplished in this step. Accordingly, we return to the ϕ-Eq. (2.1a) = (1.3a), take here the operation of "div" across, use div $\phi \equiv 0$ from (2.1b) = (1.3b), and obtain, recalling $L_e(\phi)$ in (1.2)

$$\Delta p = -\text{div } L_e(\phi) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \dfrac{\partial h}{\partial x_d} \end{bmatrix} \text{ in } \Omega, \qquad (2.24a)$$

where, actually [46, Eq. (5.21)], [47, Eq. (3.24)],

$$\text{div } L_e(\phi) = 2\{(\partial_x y_e \cdot \nabla)\phi\} = 2\{\partial_x \phi \cdot \nabla) y_e\} \text{ is a first-order differential operator in } \phi. \tag{2.24b}$$

The proof of (2.24b) uses div $\phi \equiv 0$ and div $y_e \equiv 0$ in Ω from (2.1b) = (1.3b) and (1.1c). Next, multiply (2.24a) by χ. We obtain

$$\begin{cases} \Delta(\chi p) = -\text{div } L_e(\chi\phi) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \dfrac{\partial}{\partial x_d}(\chi h) \end{bmatrix} + T_\chi^{0,0,1}(\phi, h, p) \qquad \text{in } \Omega; & (2.25\text{a}) \\ \left.\dfrac{\partial(\chi p)}{\partial \nu}\right|_D = 0, \qquad (\chi p)|_D = 0, \qquad D = \partial[\Omega_1 \cup \Omega^*]. & (2.25\text{b}) \end{cases}$$

$$T_\chi^{0,0,1}(\phi, h, p) \equiv [\Delta, \chi]p + [\text{div } L_e, \chi]\phi + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \left[\chi, \dfrac{\partial}{\partial x_d}\right] h \end{bmatrix} \tag{2.25c}$$

$$= \text{zero order in } \phi \text{ by (2.24b); zero order in } h; \text{ first order in } p; \text{ supp } T_\chi^{0,0,1} \subset \Omega^*,$$

while the B.C.s (2.25b) on the boundary D defined by (2.11) follow for 2 reasons: (i) the RHS of (2.12) on (χp) on $\partial\omega$; actually, the RHS of (2.2) since $\chi \equiv 1$ on ω; (ii) $\chi \equiv 0$ up to the external boundary of Ω^* and a small layer of Ω^* bordering Ω_0, so that $(\chi p) = 0$, $\frac{\partial(\chi p)}{\partial \nu} = 0$, on such external boundary of Ω^*. Thus, (2.25b) is justified. In (2.25b), the reason for supp $T_\chi^{0,0,1} \subset \Omega^*$ is the same as in (2.6c) and (2.9b).

Next, we apply the pointwise Carleman estimate (2.13) to problem (2.25a)–(2.25b), that is for $w = (\chi p)$. We obtain with $G = \Omega_1 \cup \Omega^*$:

$$\begin{aligned} &\delta_0 \left[2\rho\tau - \frac{\epsilon}{2}\right] e^{2\tau\psi(x)} |\nabla(\chi p)(x)|^2 + \left[4\rho k^2 \tau^3 (1-\delta_0) + \mathcal{O}(\tau^2)\right] e^{2\tau\psi(x)} |(\chi p)(x)|^2 \\ &\quad \le \left(1 + \frac{1}{\epsilon}\right) e^{2\tau\psi(x)} |\Delta(\chi p)(x)|^2 + \text{div } V_{(\chi p)}(x), \quad x \in G. \end{aligned} \tag{2.26}$$

Again, it is not critical to recall what div $V_{(\chi p)}(x)$ is, only the vanishing relationship (2.14) (for $w = (\chi p)$) on an appropriate bounded domain G. Indeed, we shall take again $G = \Omega_1 \cup \Omega^*$, and integrate inequality (2.26) over with $G \equiv \Omega_1 \cup \Omega^*$ (after selecting again $\delta_0 = \frac{1}{2}$, $\epsilon = \frac{1}{2}$), and obtain

$$\left[\rho\tau - \frac{1}{8}\right] \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} |\nabla(\chi p)(x)|^2 dx$$

$$
\begin{aligned}
&+ \left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right] \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} |(\chi p)(x)|^2 dx \\
&\leq 3 \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} |\Delta(\chi p)(x)|^2 dx + \int_{\partial[\Omega_1\cup\Omega^*]} \cancel{V_{(\chi p)}(x) \cdot} \nu \, dD, \quad (2.27)
\end{aligned}
$$

where, on the RHS of (2.27), the boundary integral over $D \equiv \partial[\Omega_1 \cup \Omega^*] = [\partial\omega \cup$ external boundary of $\Omega^*]$, see (2.11) and Fig. 8, again vanishes in view of (2.25b) for $w = (\chi p)$. Thus, the vanishing of the last integral term of (2.27) is justified.

Step 11 Here we now estimate the last integral term on the RHS of (2.27).

We multiply Eq.(2.25a) by $e^{\tau\psi(x)}$, thus obtaining

$$
e^{\tau\psi(x)}\Delta(\chi p) = -e^{\tau\psi(x)}\text{div } L_e(\chi\phi) + e^{\tau\psi(x)} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \dfrac{\partial}{\partial x_d}(\chi h) \end{bmatrix} + e^{\tau\psi(x)} T_\chi^{0,0,1}(\phi, h, p) \tag{2.28}
$$

$$
\begin{aligned}
e^{2\tau\psi(x)} |\Delta(\chi p)(x)|^2 \leq c e^{2\tau\psi(x)} \Bigg\{ &|\text{div } L_e(\chi\phi)(x)|^2 \\
&+ \left|\frac{\partial}{\partial x_d}(\chi h)(x)\right|^2 + |T_\chi^{0,0,1}(\phi, h, p)(x)|^2 \Bigg\}, \quad x \in G. \quad (2.29)
\end{aligned}
$$

We now integrate (2.29) over $G \equiv [\Omega_1 \cup \Omega^*]$. In doing so, we recall from (2.24b) that [div L_e] is a first-order operator, and accordingly, from (2.25c), that $T_\chi^{0,0,1}(\phi, h, p)$ is an operator which is zero order in ϕ and h; and first order in p; and that $T_\chi^{0,0,1}(\phi, h, p)$ has support in Ω^*. We thus obtain from (2.29)

$$
\begin{aligned}
&\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} |\Delta(\chi p)(x)|^2 dx \\
&\leq C_{y_e} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left[|\nabla(\chi\phi)(x)|^2 + |(\chi\phi)(x)|^2 + \left|\frac{\partial}{\partial x_d}(\chi h)(x)\right|^2 \right] dx \\
&+ C_\chi \int_{\Omega^*} e^{2\tau\psi(x)} \left[|\nabla p(x)|^2 + |p(x)|^2 + |\phi(x)|^2 + |h(x)|^2 \right] dx \qquad (2.30)
\end{aligned}
$$

with constant C_χ depending on χ.

Step 12 (Final estimate of the (χp)-problem.) We now substitute (2.30) into the RHS of (2.27), divide across by $[\rho\tau - \frac{1}{8}] > 0$ for τ large and obtain

$$
\begin{aligned}
&\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} |\nabla(\chi p)(x)|^2 dx + \frac{[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)]}{\left[\rho\tau - \frac{1}{8}\right]} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} |(\chi p)(x)|^2 dx \\
&\leq \frac{C_{y_e}}{(\rho\tau - \frac{1}{8})} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left[|\nabla(\chi\phi)(x)|^2 + |(\chi\phi)(x)|^2 + \left|\frac{\partial}{\partial x_d}(\chi h)(x)\right|^2 \right] dx
\end{aligned}
$$

$$+\frac{C_\chi}{\left(\rho\tau-\frac{1}{8}\right)}\int_{\Omega^*} e^{2\tau\psi(x)}\left[|\nabla p(x)|^2+|p(x)|^2+|\phi(x)|^2+|h(x)|^2\right]dx. \quad (2.31)$$

Inequality (2.31) is our final estimate on the (χp)-problem (2.25a).

Step 13 (Combining the (χu)-estimate (2.23) with the (χp)-estimate (2.31)) We return to estimate (2.23) and add to each side the term

$$\frac{[2\rho k^2\tau^3+\mathcal{O}(\tau^2)]}{[\rho\tau-\frac{1}{8}]}\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}|(\chi p)(x)|^2dx$$

to get

$$\begin{aligned}
&\left\{\left[\rho\tau-\frac{1}{8}\right]-C_{\lambda,e}\right\}\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}|\nabla(\chi u)(x)|^2dx\\
&\quad+\left[2\rho k^2\tau^3+\mathcal{O}(\tau^2)-C_{\lambda,e}\right]\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}|(\chi u)(x)|^2dx\\
&\quad+\frac{\left[2\rho k^2\tau^3+\mathcal{O}(\tau^2)\right]}{\left[\rho\tau-\frac{1}{8}\right]}\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}|(\chi p)(x)|^2dx\\
&\leq 3\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}|\nabla(\chi p)(x)|^2dx\\
&\quad+\frac{\left[2\rho k^2\tau^3+\mathcal{O}(\tau^2)\right]}{\left[\rho\tau-\frac{1}{8}\right]}\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}|(\chi p)(x)|^2dx\\
&\quad+c_\chi\int_{\Omega^*} e^{2\tau\psi(x)}[|\nabla u(x)|^2+|u(x)|^2+|p(x)|^2]dx. \qquad (2.32)
\end{aligned}$$

Next, we substitute inequality (2.31) for the first two integral terms on the RHS of (2.32), and obtain

$$\begin{aligned}
&\left\{\left[\rho\tau-\frac{1}{8}\right]-C_{\lambda,e}\right\}\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}|\nabla(\chi u)(x)|^2dx\\
&\quad+\left\{\left[2\rho k^2\tau^3+\mathcal{O}(\tau^2)\right]-C_{\lambda,e}\right\}\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}|(\chi u)(x)|^2dx\\
&\quad+\frac{\left[2\rho k^2\tau^3+\mathcal{O}(\tau^2)\right]}{\left[\rho\tau-\frac{1}{8}\right]}\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}|(\chi p)(x)|^2dx\\
&\leq\frac{C_{y_e}}{\left(\rho\tau-\frac{1}{8}\right)}\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}\left[|\nabla(\chi\phi)(x)|^2+|(\chi\phi)(x)|^2+\left|\frac{\partial}{\partial x_d}(\chi h)(x)\right|^2\right]dx\\
&\quad+\frac{C_\chi}{\left(\rho\tau-\frac{1}{8}\right)}\int_{\Omega^*} e^{2\tau\psi(x)}\left[|\nabla p(x)|^2+|p(x)|^2+|\phi(x)|^2+|h(x)|^2\right]dx\\
&\quad+c_\chi\int_{\Omega^*} e^{2\tau\psi(x)}\left[|\nabla u(x)|^2+|u(x)|^2+|p(x)|^2\right]dx. \qquad (2.33)
\end{aligned}$$

Recalling that $u = \begin{bmatrix}\phi \\ h\end{bmatrix}$, we re-write (2.33) explicitly as

$$\begin{aligned}
&\left\{\left[\rho\tau - \frac{1}{8}\right] - C_{\lambda,e}\right\} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left[|\nabla(\chi\phi)(x)|^2 + |\nabla(\chi h)(x)|^2\right] dx \\
&\quad + \left\{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right] - C_{\lambda,e}\right\} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left[|(\chi\phi)(x)|^2 + (\chi h)(x)|^2\right] dx \\
&\quad + \frac{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right]}{\left[\rho\tau - \frac{1}{8}\right]} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} |(\chi p)(x)|^2 dx \\
&\quad \leq \frac{C_{y_e}}{\left(\rho\tau - \frac{1}{8}\right)} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left[|\nabla(\chi\phi)(x)|^2 + |(\chi\phi)(x)|^2 + \left|\frac{\partial}{\partial x_d}(\chi h)(x)\right|^2\right] dx \\
&\quad + \frac{C_\chi}{\left(\rho\tau - \frac{1}{8}\right)} \int_{\Omega^*} e^{2\tau\psi(x)} \left[|\nabla p(x)|^2 + |p(x)|^2 + |\phi(x)|^2 + |h(x)|^2\right] dx \\
&\quad + c_\chi \int_{\Omega^*} e^{2\tau\psi(x)} \left[|\nabla\phi(x)|^2 + |\nabla h(x)|^2 + |\phi(x)|^2 + |h(x)|^2 + |p(x)|^2\right] dx.
\end{aligned} \tag{2.34}$$

Step 14 (Final estimate of problem (2.1a)–(2.1b).) Finally, we combine the integral terms with the same integrand on the LHS of (2.34) and obtain the final sought-after estimate which we formalize as a lemma.

Lemma 1 *The following inequality holds true for all τ sufficiently large:*

$$\begin{aligned}
&\left\{\left[\rho\tau - \frac{1}{8}\right] - C_{\lambda,e} - \frac{C_{y_e}}{\left(\rho\tau - \frac{1}{8}\right)}\right\} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left[|\nabla(\chi\phi)(x)|^2 + |\nabla(\chi h)(x)|^2\right] dx \\
&\quad + \left\{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right] - C_{\lambda,e} - \frac{C_{y_e}}{\left(\rho\tau - \frac{1}{8}\right)}\right\} \\
&\quad \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left[|(\chi\phi)(x)|^2 + (\chi h)(x)|^2\right] dx \\
&\quad + \frac{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right]}{\left[\rho\tau - \frac{1}{8}\right]} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} |(\chi p)(x)|^2 dx \\
&\quad \leq \frac{C_\chi}{\left(\rho\tau - \frac{1}{8}\right)} \int_{\Omega^*} e^{2\tau\psi(x)} \left[|\nabla p(x)|^2 + |p(x)|^2 + |\phi(x)|^2 + |h(x)|^2\right] dx \\
&\quad + c_\chi \int_{\Omega^*} e^{2\tau\psi(x)} \left[|\nabla\phi(x)|^2 + |\nabla h(x)|^2 + |\phi(x)|^2 + |h(x)|^2 + |p(x)|^2\right] dx.
\end{aligned} \tag{2.35}$$

We note explicitly 2 critical features of estimate (2.35): the integral terms on its LHS are over $[\Omega_1 \cup \Omega^*]$; while the integral terms on its RHS are over Ω^*.

As already noted, (2.35) is the ultimate estimate regarding the original problem (2.1a)–(2.1b).

Step 15 (The choice of weight function $\psi(x)$.) We now choose the strictly convex function $\psi(x)$ as follows (Figs. 11 and 12, as well as Fig. 9):

$$\psi(x) \geq 0 \text{ on } \Omega_1 \text{ where } \chi \equiv 1 \text{ by (2.3a), so that } e^{2\tau\psi(x)} \geq 1 \text{ on } \Omega_1; \tag{2.36}$$

$$\psi(x) \leq 0 \text{ on } \Omega_0 \cup \Omega^*; \text{ where } \chi < 1, \text{ so that } e^{2\tau\psi(x)} \leq 1 \text{ on } \Omega^*, \tag{2.37}$$

in such a way that $\psi(x)$ has no critical point in $\Omega \setminus \omega$, as required by Theorem 10 (ψ no critical points on $G = \Omega_1 \cup \Omega^*$): that is, the critical point(s) of ψ will fall on ω, outside the region $G = \Omega_1 \cup \Omega^*$ where we have integrated.

Having chosen $\psi(x)$ as in (2.36), (2.37) with no critical points in $\Omega \setminus \omega$—i.e., no critical points on $G = \Omega_1 \cup \Omega^*$—we return to the basic estimate (2.35), with τ sufficiently large (Fig. 12). On the LHS of (2.35), we retain only integration over Ω_1, where $\psi \geq 0$, hence $e^{2\tau\psi} \geq 1$ and $\chi \equiv 1$ by (2.3a), so that $(\chi u) \equiv u$ on Ω_1, that is, $(\chi\phi) \equiv \phi$ on Ω_1 and $(\chi h) \equiv h$ on Ω_1. On the RHS of (2.35) we have $\psi \leq 0$ on Ω^*, hence $e^{2\tau\psi} \leq 1$ on Ω^*. We thus obtain from (2.35) for τ sufficiently large

$$\begin{aligned}
&\left\{\left[\rho\tau - \frac{1}{8}\right] - C_{\lambda,e} - \frac{C_{y_e}}{(\rho\tau - \frac{1}{8})}\right\} \int_{\Omega_1} \left[|\nabla\phi(x)|^2 + |\nabla h(x)|^2\right] dx \\
&\quad + \left\{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right] - C_{\lambda,e} - \frac{6C_{y_e}}{(\rho\tau - \frac{1}{8})}\right\} \int_{\Omega_1} \left[|\phi(x)|^2 + |h(x)|^2\right] dx \\
&\quad + \frac{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right]}{\left[\rho\tau - \frac{1}{8}\right]} \int_{\Omega_1} |p(x)|^2 dx \\
&\leq \frac{C_\chi}{(\rho\tau - \frac{1}{8})} \int_{\Omega^*} \left[|\nabla p(x)|^2 + |p(x)|^2 + |\phi(x)|^2 + |h(x)|^2\right] dx \\
&\quad + c_\chi \int_{\Omega^*} \left[|\nabla\phi(x)|^2 + |\nabla h(x)|^2 + |\phi(x)|^2 + |h(x)|^2 + |p(x)|^2\right] dx.
\end{aligned} \tag{2.38}$$

For τ sufficiently large, inequality (2.38) is of the type

$$\begin{aligned}
&\left(\tau - \text{const} - \frac{1}{\tau}\right) \int_{\Omega_1} \left[|\nabla\phi(x)|^2 + |\nabla h(x)|^2\right] dx \\
&\quad + \left(\tau^3 - \text{const} - \frac{1}{\tau}\right) \int_{\Omega_1} \left[|\phi(x)|^2 + |h(x)|^2\right] dx + (\tau^2) \int_{\Omega_1} |p(x)|^2 dx \\
&\leq \frac{c}{\tau} \int_{\Omega^*} \left[|\nabla p(x)|^2 + |p(x)|^2 + |\phi(x)|^2 + |h(x)|^2\right] dx \\
&\quad + \text{const} \int_{\Omega^*} \left[|\nabla\phi(x)|^2 + |\nabla h(x)|^2 + |\phi(x)|^2 + |h(x)|^2 + |p(x)|^2\right] dx
\end{aligned} \tag{2.39a}$$

or setting as usual $u = \{\phi, h\}$, we re-write (2.39a) equivalently as

$$
\begin{aligned}
&\left(\tau - \text{const} - \frac{1}{\tau}\right) \int_{\Omega_1} |\nabla u(x)|^2 dx + \left(\tau^3 - \text{const} - \frac{1}{\tau}\right) \\
&\int_{\Omega_1} |u(x)|^2 dx + (\tau^2) \int_{\Omega_1} |p(x)|^2 dx \\
&\leq \frac{c}{\tau} \int_{\Omega^*} \left[|\nabla p(x)|^2 + |p(x)|^2 + |u(x)|^2 \right] dx \\
&\quad + \text{const} \int_{\Omega^*} \left[|\nabla u(x)|^2 + |u(x)|^2 + |p(x)|^2 \right] dx \qquad (2.39b) \\
&\leq \frac{c}{\tau} C_1(p, u; \Omega^*) + \text{const}\ C_2(p, u; \Omega^*). \qquad (2.39c)
\end{aligned}
$$

In going from (2.39b) to (2.39c), we have emphasized in the notation that we are working with a fixed solution $\{u, p\}$ of problem (2.1a)–(2.1b), so that the integrals on the RHS of (2.39b)) are fixed numbers $C_1(p, u; \Omega^*)$ and $C_2(p, u; \Omega^*)$, depending on such fixed solution $\{u, p\}$ as well as Ω^*, $u = \{\phi, h\}$. Inequality (2.39) is more than we need. On its LHS, we may drop the ∇u-term over Ω_1; and alternatively either keep only the u-term over Ω_1, and divide the remaining inequality across by $(\tau^3 - \text{const} - \frac{1}{\tau})$ for τ large; or else keep only the p-term over Ω_1 and divide the corresponding inequality across by τ^2. We obtain, respectively,

$$
\int_{\Omega_1} |u(x)|^2 dx \leq \left(\frac{C}{\tau^3}\ \frac{1}{\tau}\right) C_1(p, u; \Omega^*) + \frac{\text{const}}{\tau^3} C_2(p, u; \Omega^*) \to 0 \text{ as } \tau \to +\infty; \tag{2.40}
$$

$$
\int_{\Omega_1} |p(x)|^2 dx \leq \left(\frac{C}{\tau^2}\ \frac{1}{\tau}\right) C_1(p, u; \Omega^*) + \frac{\text{const}}{\tau^2} C_2(p, u; \Omega^*) \to 0 \text{ as } \tau \to +\infty. \tag{2.41}
$$

We thus obtain

$$
u(x) \equiv \{\phi(x), h(x)\} \equiv 0 \text{ in } \Omega_1; \qquad p(x) \equiv 0 \text{ in } \Omega_1. \tag{2.42}
$$

and recalling (1.3d) and Step 1

$$
u(x) \equiv \{\phi(x), h(x)\} \equiv 0, \quad p(x) \equiv 0 \text{ in } \omega \cup \Omega_1. \tag{2.43}
$$

The implication: Step 1 $\Longrightarrow$ (2.43) is ilustrated by Fig. 13.

Finally, we can now push the external boundary of Ω_1 as close as we please to the boundary $\partial\Omega$ of Ω, and thus we finally obtain

$$
u(x) \equiv \{\phi(x), h(x)\} \equiv 0 \text{ in } \Omega, \qquad p(x) \equiv 0 \text{ in } \Omega. \tag{2.44}
$$

Indeed, we have $u \equiv \{\phi, h\} \in (W^{2,q}(\Omega))^d \times (W^{1,q}(\Omega))^d$ and $p \in W^{1,q}(\Omega)$. Moreover, $W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $q > d$ [28, p. 78] as assumed, and more generally $W^{m,q}(\Omega) \hookrightarrow C^k(\overline{\Omega})$ for $qm > d, k = m - \frac{d}{q}$ [28, p. 79]. A fortiori, $u \in (C(\overline{\Omega}))^d$, $p \in$

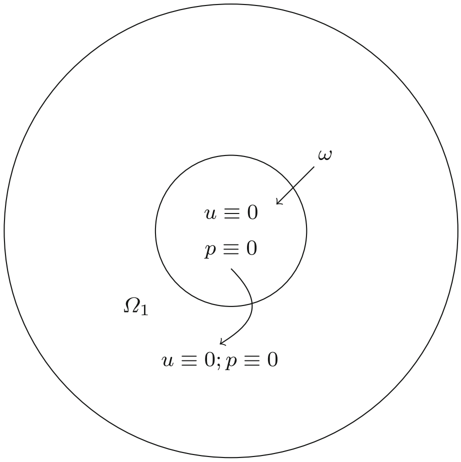
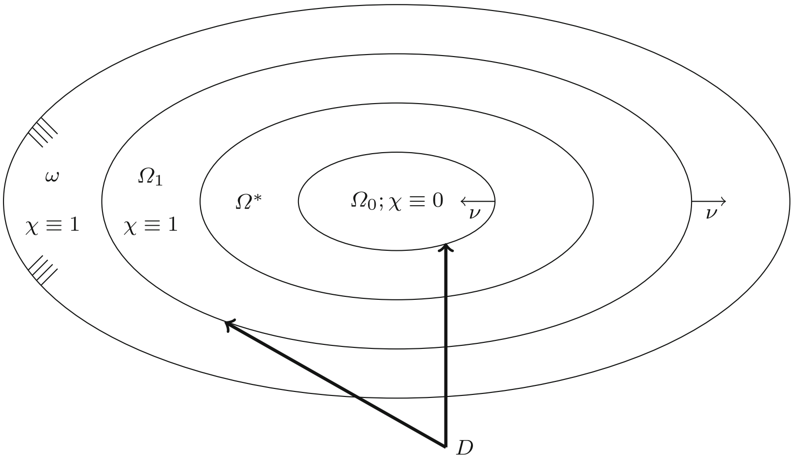


Fig. 13 The implication Step 1 $\rightarrow$ Eq. (2.43)



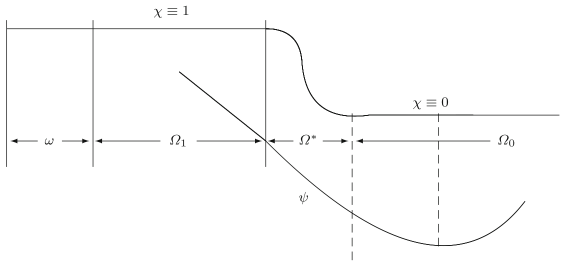
Case 2: Fig. 14 $G = \Omega_1 \cup \Omega^*$; $\partial G = D = \partial\omega \cup$ [internal boundary of Ω^*]. Compare with Case 1, Fig. 11

$C(\overline{\Omega})$, $q > d$, as assumed. Thus, if it should happen that $u(x_1) \neq 0$ at a point $x_1 \in \Omega$ near $\partial\Omega$, hence $u(x) \not\equiv 0$ in a suitable neighborhood N of x_1, then it would suffice to take Ω_1 as to intersect such N to obtain a contradiction.

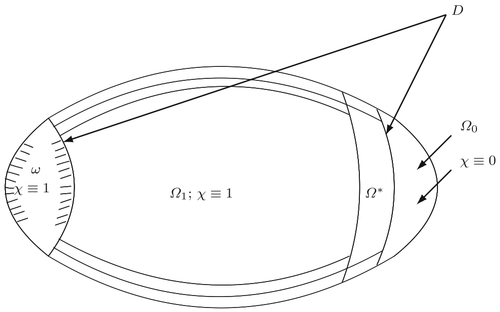
Theorem 1 is proved at least in the Case 1 (Fig. 1).

Case 2. Let ω be a full collar of boundary $\Gamma = \partial\Omega$ (Figs.14 and 15). Then, the above proof of Case 1 can be carried out with sets Ω_1, Ω^*, and Ω_0, as indicated in Fig.14. $\square$

Let now ω be a partial collar of the boundary $\Gamma = \partial\Omega$. Then, the above proof of Case 1 can be carried out with sets Ω_1, Ω^*, and Ω_0, as indicated in Fig.16.



Case 2: Fig. 15 Choice of ψ. Compare with choice of ψ in Case 1, Fig. 12



Case 2: Fig. 16 Selection of sets Ω_1, Ω^*, Ω_0 in Case 2, and corresponding cut-off function χ

3 Applications to Uniform Stabilization of the Dynamic Boussinesq System

It has long been recognized that Unique Continuation Properties of suitable over-determined adjoint eigen-problems play a critical role in the affirmative solution of the (global) uniform stabilization problem of corresponding linear (hence, local uniform stabilization problems of non-linear) unstable parabolic problems. The crux of the matter arises at the finite dimensional analysis. In fact, in implementing the strategy for stabilization of parabolic problems introduced in [44], a key step consists in establishing that the finite dimensional unstable projected system is controllable, by testing the Kalman or Hautus algebraic, full rank characterizing condition. This will then assure [Popov/Wonham, about 1964-5] that such finite dimensional projection can be feedback stabilized with an arbitrarily large decay rate. Reference [52, Theorem 2.9, p 44] calls this property complete stabilizability. A first case testing this approach in the case of *boundary* feedback stabilizing controls for a parabolic problem is [45]. This was soon followed by more challenging papers [34,35], which study the *purely boundary* feedback stabilization problem (by *boundary* controls with *boundary* actuators) of linear, unstable (classical) parabolic equations, by means of a finite dimensional *boundary* feedback, defined in terms of *boundary traces* in the feedback loop. Other cases followed throughout the literature, studied by many authors. We quote in particular: uniform stabilization of unstable Navier-Stokes equations, first by means of interior localized finite dimensional feedback controls [5] (such study was recently extended and improved in [30]), next by means of tangential boundary feedback controllers [6–8,38]. The recent work [31] provides a solution in the affirmative to a recognized open problem in the theory of uniform stabilization of 3-dimensional Navier-Stokes equations in the vicinity of an unstable equilibrium solution, by means of a 'minimal' and 'least' invasive feedback strategy which consists of a control pair $\{v, u\}$ [38]. Here v is a tangential boundary feedback control, acting on an arbitrary small part $\widetilde{\Gamma}$ of the boundary Γ; while u is a localized, interior feedback control, acting tangentially on an arbitrarily small subset ω of the interior supported by $\widetilde{\Gamma}$. The ideal strategy of taking $u = 0$ on ω is not sufficient. A question left open in the literature was: Can such feedback control v of the pair $\{v, u\}$ be asserted to be finite dimensional also in the dimension $d = 3$? [31] gives an affirmative answer to this question, thus establishing an optimal result. To achieve the desired finite dimensionality of the feedback tangential boundary control v, it was necessary to abandon the Hilbert-Sobolev functional setting of past literature and replace it with a critical Besov space setting. These spaces are 'close' to $L^3(\Omega)$ for $d = 3$. They were introduced in [30]. References where solved (and still unsolved) UCPs are studied as critical ingredients of the solutions of corresponding stabilization problems are [37,46,47],[38],[31, Appendix C]. Mathematical techniques used to establish the UCP vary according to the given PDE-problem and corresponding over-determination, as these references document. In this section, we give two applications of the UCPs as pertained to the Boussinesq system. In Sect. 3.1 we consider the case of reference [32] where uniform stabilization of the Boussinesq system is established in the same suitable Besov setting, by means of localized finite dimensional feedback controllers acting on the same (arbitrarily small) *interior* sub-domain ω of the bounded domain Ω where the system evolves. Next in Sect. 3.2 we consider the uniform stabilization case of reference [33] where instead the controller acting on the thermal equation is a *boundary* controller acting on a portion of the boundary $\widetilde{\Gamma}$ that serves as support of the small interior sub-domain ω. Instead, the interior tangential-like controller for the fluid equations is localized precisely on ω (Fig. 2).

3.1 The Problem of Uniform Stabilization of the (Dynamic) Boussinesq System by Finite Dimensional Interior Localized Feedback Controls

This problem is studied in [32]. Its solution requires either the adjoint UCP of Theorem 5, or else the adjoint UCP of Theorem 6 ($d = 2$) or Theorem 7 ($d = 3$), depending on the class of sub-domains ω considered: whether purely interior sub-domains as in Fig. 1; or else sub-domains ω touching the boundary Γ and satisfying the geometrical conditions of Definition 1 ($d = 2$ and $d = 3$) or Definition 2 ($d = 4$) in Appendix 1. See in particular Figs. 5, 6 for $d = 2$. In the first (resp., second) case $(d-1)$-components (resp. $(d-2)$-components) of the d-dimensional fluid vector $\phi = \{\phi_1, \dots, \phi_d\}$ are required to vanish on ω: see (1.18d) of Theorem 5 (resp. (1.22) for $d = 2$ in Theorem 6 and (1.31) or (1.32) of Theorem 7 for $d = 3$), in addition to the condition $h \equiv 0$ in ω. See also Figs. 5, 6 for $d = 2$ and Fig. 10 for $d = 3$. See also Theorem A.3, Equations (C.3)–(C.5) for $d = 4$. While we refer to [32] for the technical and precise mathematical description, we provide here a summary of the results.

Uncontrolled Boussinesq system We complement the notation of Section 1 by letting $Q \equiv (0, T) \times \Omega$ and $\Sigma \equiv (0, T) \times \partial\Omega$ where $T > 0$. As in Sect. 1, let ω be an arbitrary small open smooth sub-domain of the region Ω, $\omega \subset \Omega$, thus of positive measure. Let m denote the characteristic function of ω: $m(\omega) \equiv 1,\ m(\Omega/\omega) \equiv 0$.

Notation In this section, vector-valued functions and corresponding function spaces will be boldfaced. Thus, for instance, for the vector valued (d-valued) velocity field or external force, we shall write say $\mathbf{y}, \mathbf{f} \in \mathbf{L}^q(\Omega)$ rather than $y, f \in (L^q(\Omega))^d$. This is in contrast to Sects. 1 and 2. One justification is that the present section makes reference to the uniform stabilization paper [32], where bold-face notation was used for vector-valued quantities. Thus, the choice of the present section makes it easier to make comparisons with [32].

We start with the Boussinesq system under the action of two localized interior controls $m(x)\mathbf{u}(t, x)$ and $m(x)v(t, x)$ supported on $Q_\omega \equiv (0, \infty) \times \omega$

$$\begin{cases} \mathbf{y}_t - \nu_0 \Delta \mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} - \gamma(\theta - \bar{\theta})\mathbf{e}_d + \nabla\pi = m(x)\mathbf{u}(t, x) + \mathbf{f}(x) & \text{in } Q \quad (3.1a) \\ \theta_t - \kappa\Delta\theta + \mathbf{y}\cdot\nabla\theta = m(x)v(t, x) + g(x) & \text{in } Q \quad (3.1b) \\ \operatorname{div} \mathbf{y} = 0 & \text{in } Q \quad (3.1c) \\ \mathbf{y} = 0,\ \theta = 0 & \text{on } \Sigma \quad (3.1d) \\ \mathbf{y}(0, x) = \mathbf{y}_0, \quad \theta(0, x) = \theta_0 & \text{on } \Omega \quad (3.1e) \end{cases}$$

In the Boussinesq system, $\mathbf{y} = \{y_1, \ldots, y_d\}$ represents the fluid velocity, θ the scalar temperature of the fluid, ν_0 the kinematic viscosity coefficient, κ the thermal diffusivity. The scalar function π is the unknown pressure. The Boussinesq system models heat transfer in a viscous incompressible heat conducting fluid. It consists of the Navier Stokes equation (in the velocity $\mathbf{y}$) coupled with the convection-diffusion equation (for the temperature θ). The external body force $\mathbf{f}(x)$ and the heat source density $g(x)$ may render the overall system *unstable* (in a technical sense, described in [32, Section 3]) and recalled in (3.7b) below. The goal of the paper is to exploit the localised controls, sought to be finite dimensional and in feedback form, in order to stabilize the overall system.

The basic underlying hypothesis is that problem (3.1) with $\mathbf{u} \equiv 0$, $v \equiv 0$ is unstable in the classical parabolic sense [32, Eq. (1.33)]: finitely many unstable eigenvalues of a naturally linearized problem, with analytic semigroup generator.

Technical background First, we introduce the Helmholtz decomposition of $\mathbf{L}^q(\Omega)$

$$\mathbf{L}^q(\Omega) = \mathbf{L}^q_\sigma(\Omega) \oplus \mathbf{G}^q(\Omega). \tag{3.2}$$

where

$$\begin{aligned} \mathbf{L}^q_\sigma(\Omega) &= \overline{\{\mathbf{y} \in \mathbf{C}^\infty_c(\Omega) : \operatorname{div} \mathbf{y} = 0 \text{ in } \Omega\}}^{\|\cdot\|_q} \\ &= \{\mathbf{g} \in \mathbf{L}^q(\Omega) : \operatorname{div} \mathbf{g} = 0;\ \mathbf{g}\cdot\nu = 0 \text{ on } \partial\Omega\}, \\ &\quad \text{for any locally Lipschitz domain } \Omega \subset \mathbb{R}^d, d \geq 2 \\ \mathbf{G}^q(\Omega) &= \{\mathbf{y} \in \mathbf{L}^q(\Omega) : \mathbf{y} = \nabla p,\ p \in W^{1,q}_{loc}(\Omega)\} \text{ where } 1 \leq q < \infty. \end{aligned} \tag{3.3}$$

Both of these are closed subspaces of $\mathbf{L}^q$.

The unique linear, bounded and idempotent (i.e. $P_q^2 = P_q$) projection operator $P_q : \mathbf{L}^q(\Omega) \longrightarrow \mathbf{L}^q_\sigma(\Omega)$ having $\mathbf{L}^q_\sigma(\Omega)$ as its range and $\mathbf{G}^q(\Omega)$ as its null space is called the Helmholtz projection.

When $q \neq 2$, not all domains Ω have a Helmholtz decomposition. However, under various mild assumptions on Ω, the Helmholtz decomposition does exist [20,22,23], [24, Theorem 1.1, p 107, and Theorem 1.2, p 114]. Next, for $1 < q < \infty$ fixed, we introduce the Stokes operator A_q in $\mathbf{L}^q_\sigma(\Omega)$ with Dirichlet boundary conditions

$$A_q \mathbf{z} = -P_q \Delta \mathbf{z}, \quad \mathcal{D}(A_q) = \mathbf{W}^{2,q}(\Omega) \cap \mathbf{W}^{1,q}_0(\Omega) \cap \mathbf{L}^q_\sigma(\Omega). \tag{3.4}$$

Then, the Besov space $\mathbf{B}^{2-\frac{2}{p}}_{q,p}(\Omega)$ is defined as the following special real interpolation space of the $\mathbf{L}^q$ and $\mathbf{W}^{2,q}$ spaces:

$$\mathbf{B}^{2-\frac{2}{p}}_{q,p}(\Omega) = \left(\mathbf{L}^q(\Omega), \mathbf{W}^{2,q}(\Omega)\right)_{1-\frac{1}{p},p}. \tag{3.5}$$

More precisely, $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ is the following subspace

$$\left(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p} = \left\{\mathbf{g} \in \mathbf{B}^{2-\frac{2}{p}}_{q,p}(\Omega) : \ \operatorname{div} \mathbf{g} = 0, \ \mathbf{g} \cdot \nu|_\Gamma = 0\right\} \equiv \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$$
$$\text{if } 0 < 2 - \frac{2}{p} < \frac{1}{q}; \text{ or } 1 < p < \frac{2q}{2q-1}. \tag{3.6}$$

Notice that, in (3.6), the condition $\mathbf{g} \cdot \nu\big|_\Gamma = 0$ is an intrinsic condition of the space $\mathbf{L}^q_\sigma(\Omega)$ in (3.3) of the Helmholtz decomposition, not an extra boundary condition.

This is a key property that justifies the adoption of the space $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ in the uniform stabilization problem via *boundary* localized feedback controls which are finite dimensional also in the case $d = 3$ [31].

Next, let $\mathbb{A}_q$ be the differential (coupled) operator describing the linearized version of the Boussinesq system. See [32, Sect. 1.7]. Let λ_i (respectively $\overline{\lambda}_i$) be one of the M distinct unstable eigenvalues of $\mathbb{A}_q$ (respectively, its suitable adjoint $\mathbb{A}_q^*$) with geometric multiplicity ℓ_i, and let $\boldsymbol{\Phi}_{ij}$ (respectively, $\boldsymbol{\Phi}^*_{ij}$) be the corresponding eigenvectors

$$\mathbb{A}_q \boldsymbol{\Phi}_{ij} = \lambda_i \boldsymbol{\Phi}_{ij} \in \mathcal{D}(\mathbb{A}_q) \qquad \mathbb{A}_q^* \boldsymbol{\Phi}^*_{ij} = \overline{\lambda}_i \boldsymbol{\Phi}^*_{ij} \in \mathcal{D}(\mathbb{A}_q^*). \tag{3.7a}$$
$$\ldots \leq Re\, \lambda_{N+1} < 0 \leq Re\, \lambda_N \leq \ldots \leq Re\, \lambda_1. \tag{3.7b}$$

We now express the eigenvectors Φ^*_{ij} in terms of their coordinates, as $(d+1)$ vectors:

$$\boldsymbol{\Phi}^*_{ij} = \{\boldsymbol{\varphi}^*_{ij}, \psi^*_{ij}\} = \left\{\varphi^{*(1)}_{ij}, \varphi^{*(2)}_{ij}, \ldots, \varphi^{*(d-1)}_{ij}, \varphi^{*(d)}_{ij}, \psi^*_{ij}\right\}, \text{ a } (d+1)\text{-vector}. \tag{3.8}$$

First Case: interior sub-domains ω as in Theorem 5, Fig. 1. Implication of the UCP of Theorem 5 on the Kalman algebraic controllability test.

With reference to (3.8), we introduce the following corresponding d-vector

$$\widehat{\boldsymbol{\Phi}}^*_{ij} = \left\{\widehat{\boldsymbol{\varphi}^*_{ij}}, \psi^*_{ij}\right\} = \left\{\varphi^{*(1)}_{ij}, \varphi^{*(2)}_{ij}, \dots \varphi^{*(d-1)}_{ij}, \psi^*_{ij}\right\}, \text{ a } d\text{-vector} \tag{3.9}$$

obtained from $\boldsymbol{\Phi}^*_{ij}$ by omitting the d-component $\varphi^{*(d)}_{ij}$ of the vector $\boldsymbol{\Phi}^*_{ij}$. Next, construct the following matrix U_i of size $\ell_i \times K$, $K = \sup\{\ell_i : i = 1, \dots, M\}$

$$U_i = \begin{bmatrix} (\mathbf{u}_1, \widehat{\boldsymbol{\Phi}}^*_{i1})_\omega & \dots & (\mathbf{u}_K, \widehat{\boldsymbol{\Phi}}^*_{i1})_\omega \\ (\mathbf{u}_1, \widehat{\boldsymbol{\Phi}}^*_{i2})_\omega & \dots & (\mathbf{u}_K, \widehat{\boldsymbol{\Phi}}^*_{i2})_\omega \\ \vdots & \ddots & \vdots \\ (\mathbf{u}_1, \widehat{\boldsymbol{\Phi}}^*_{i\ell_i})_\omega & \dots & (\mathbf{u}_K, \widehat{\boldsymbol{\Phi}}^*_{i\ell_i})_\omega \end{bmatrix} : \ell_i \times K. \tag{3.10}$$

Here with

$$\mathbf{u}_k = [\mathbf{u}^1_k, u^2_k] = [(u^1_k)^{(1)}, (u^1_k)^{(2)} \dots (u^1_k)^{(d-1)}, u^2_k] \in \widehat{\mathbf{L}}^q_\sigma(\Omega) \times L^q(\Omega) \tag{3.11a}$$

$$\widehat{\mathbf{L}}^q_\sigma(\Omega) \equiv \text{ the space obtained from } \mathbf{L}^q_\sigma(\Omega) \text{ after omitting the } d\text{-coordinate,} \tag{3.11b}$$

we have defined the duality pairing over ω as

$$(\mathbf{u}_k, \widehat{\boldsymbol{\Phi}}^*_{ij})_\omega = \left(\begin{bmatrix} \mathbf{u}^1_k \\ u^2_k \end{bmatrix}, \begin{bmatrix} \widehat{\boldsymbol{\varphi}}^*_{ij} \\ \psi^*_{ij} \end{bmatrix}\right)_\omega = \int_\omega [\mathbf{u}^1_k \cdot \widehat{\boldsymbol{\varphi}}^*_{ij} + u^2_k \psi^*_{ij}] d\omega$$

$$= (\mathbf{u}^1_k, \widehat{\boldsymbol{\varphi}}^*_{i1})_{\widehat{\mathbf{L}}^q(\omega), \widehat{\mathbf{L}}^{q'}(\omega)} + (u^2_k, \psi^*_{ij})_{L^q(\omega), L^{q'}(\omega)} \tag{3.12}$$

$$= \int_\omega \begin{bmatrix} (u^1_k)^{(1)} \\ (u^1_k)^{(2)} \\ \vdots \\ (u^1_k)^{(d-1)} \\ u^2_k \end{bmatrix} \cdot \begin{bmatrix} \varphi^{*(1)}_{ij} \\ \varphi^{*(2)}_{ij} \\ \vdots \\ \varphi^{*(d-1)}_{ij} \\ \psi^*_{ij} \end{bmatrix} d\omega \tag{3.13}$$

The controllability Kalman/Hautus algebraic condition of the finite-dimensional projection of the linearized dynamics is given by [32, Eq. (4.10)]

$$\text{rank } U_i = \text{full} = \ell_i, \quad i = 1, \dots, M. \tag{3.14}$$

M= number of distinct unstable eigenvalues in (3.7b). Thus, given the distinct (unstable) eigenvalues $\overline{\lambda}_i$ of $\mathbb{A}^*_q$, $i = 1, 2, \dots M$, we need to show that the corresponding vectors $\widehat{\boldsymbol{\Phi}}^*_{i1}, ..\widehat{\boldsymbol{\Phi}}^*_{il_i}$ (defined in (3.9))

$$\begin{tikzcd}[column sep=small] & & \overline{\lambda}_i \arrow[dll, dash] \arrow[dl, dash] \arrow[dr, dash] & \\ \widehat{\boldsymbol{\Phi}}^*_{i1}, & \widehat{\boldsymbol{\Phi}}^*_{i2}, & \cdots & \widehat{\boldsymbol{\Phi}}^*_{i\ell_i} \end{tikzcd} \tag{3.15}$$

are linearly independent in $\widehat{\mathbf{L}}^{q'}(\omega)$, where ℓ_i = geometric multiplicity of $\overline{\lambda}_i$. It is at this point that the UCP of Theorem 5 is critically invoked.

Lemma 2 *The UCP of Theorem 5 for the class of interior subset ω there considered, Fig. 1, implies that for each $\overline{\lambda}_i$ in (3.7a) (the fact that $\overline{\lambda}_i$ is unstable is immaterial), the corresponding vector $\widehat{\Phi}^*_{i1}, \cdots, \widehat{\Phi}^*_{i\ell_i}$ defined in (3.9) are linearly independent in $\widehat{\mathbf{L}}^{q'}(\omega)$. Hence, the full rank condition (3.14) holds true for infinitely many choices of the vector $\{\mathbf{u}^1_k, u^2_k\}$.*

The proof of this result is given in [32, Appendix B]. A similar proof in the Second Case to follow is given in Lemma 3 below. Such Lemma 2 plays a critical role in establishing the following uniform stabilization result for the dynamic Boussinesq system [32].

Local well-posedness and uniform (exponential) stabilization of the original non-linear $\{\mathbf{y}, \theta\}$-problem (3.1) in a neighborhood of an unstable equilibrium solution $\{\mathbf{y}_e, \theta_e\}$, by means of a finite dimensional explicit, spectral based feedback control pair $\{\mathbf{u}, v\}$ localized on ω.

Theorem 11 *Let $1 < p < {}^6/_5, q > 3, d = 3$; and $1 < p < {}^4/_3, q > 2, d = 2$ so that (3.6) holds true. Consider the original Boussinesq problem (3.1). Let $\{\mathbf{y}_e, \theta_e\}$ be a given equilibrium solution pair as guaranteed by Theorem 1 for the steady state problem (1.1). Assume the instability condition (3.7b). For a constant $\rho > 0$, let the initial condition $\{\mathbf{y}_0, \theta_0\}$ in (3.1e) be in $\mathbf{V}^{q,p}(\Omega) \equiv \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times L^q(\Omega)$ and satisfy*

$$\mathcal{V}_\rho \equiv \left\{ \{\mathbf{y}_0, \theta_0\} \in \mathbf{V}^{q,p}(\Omega) : \|\mathbf{y}_0 - \mathbf{y}_e\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} + \|\theta_0 - \theta_e\|_{L^q(\Omega)} \leq \rho \right\}, \quad \rho > 0. \tag{3.16}$$

If $\rho > 0$ is sufficiently small, then

(i) for each $\{\mathbf{y}_0, \theta_0\} \in \mathcal{V}_\rho$, there exists an interior finite dimensional feedback control pair

$$\begin{bmatrix} \mathbf{u} \\ v \end{bmatrix} = \begin{bmatrix} F^1\left(\begin{bmatrix} \mathbf{y} - \mathbf{y}_e \\ \theta - \theta_e \end{bmatrix}\right) \\ F^2\left(\begin{bmatrix} \mathbf{y} - \mathbf{y}_e \\ \theta - \theta_e \end{bmatrix}\right) \end{bmatrix} = F\left(\begin{bmatrix} \mathbf{y} - \mathbf{y}_e \\ \theta - \theta_e \end{bmatrix}\right) = \sum_{k=1}^{K} \left(P_N \begin{bmatrix} \mathbf{y} - \mathbf{y}_e \\ \theta - \theta_e \end{bmatrix}, \mathbf{p}_k \right)_\omega \mathbf{u}_k \tag{3.17}$$

such that the closed loop problem corresponding to (3.1)

$$
\begin{cases}
\mathbf{y}_t - \nu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} - \gamma(\theta-\bar{\theta})\mathbf{e}_d + \nabla\pi = m\left(F^1\left(\begin{bmatrix}\mathbf{y}-\mathbf{y}_e\\ \theta-\theta_e\end{bmatrix}\right)\right) + \mathbf{f}(x) & \text{in } Q \quad (3.18a)\\
\theta_t - \kappa\Delta\theta + \mathbf{y}\cdot\nabla\theta = m\left(F^2\left(\begin{bmatrix}\mathbf{y}-\mathbf{y}_e\\ \theta-\theta_e\end{bmatrix}\right)\right) + g(x) & \text{in } Q \quad (3.18b)\\
\operatorname{div}\mathbf{y} = 0 & \text{in } Q \quad (3.18c)\\
\mathbf{y} = 0, \quad \theta = 0 & \text{on } \Sigma \quad (3.18d)\\
\mathbf{y}|_{t=0} = \mathbf{y}_0, \quad \theta|_{t=0} = \theta_0 & \text{in } \Omega \quad (3.18e)
\end{cases}
$$

has a unique solution $\{\mathbf{y}, \theta\} \in C\big([0,\infty); \mathbf{V}^{q,p}(\Omega) \equiv \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times L^q(\Omega)\big)$. *Here we have:*

(i$_1$) K *= largest geometric multiplicity of distinct unstable eigenvalues;*
(i$_2$) $\mathbf{u}_K = \{\mathbf{u}^1_K, u^2_K\}$, *where the vector* $\mathbf{u}^1_K$ *acting on the fluid d-dimensional component* $\mathbf{y}$ *is of reduced dimension* $(d-1)$, *rather than d, in line with the structure of the* $\widehat{\Phi}^*_{ij}$*-vector in* (3.9), *that is,* $\mathbf{u}^1_K = \{u^{(1)}_K, u^{(2)}_K, \ldots, u^{(d-1)}_K\}$, *as in* (3.11a).

(ii) Moreover, perhaps for $\rho > 0$ *even smaller, such solution exponentially stabilizes the equilibrium solution* $\{\mathbf{y}_e, \theta_e\}$ *in the space* $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times L^q(\Omega) \equiv \mathbf{V}^{q,p}(\Omega)$*: there exist constants* $\widetilde{\gamma} > 0$ *and* $M_{\widetilde{\gamma}} \geq 1$ *such that said solution satisfies*

$$
\|\mathbf{y}(t)-\mathbf{y}_e\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} + \|\theta(t)-\theta_e\|_{L^q(\Omega)} \leq M_{\widetilde{\gamma}} e^{-\widetilde{\gamma}t}\left(\|\mathbf{y}_0-\mathbf{y}_e\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} + \|\theta_0-\theta_e\|_{L^q(\Omega)}\right), \tag{3.19}
$$

$t \geq 0$, $\{\mathbf{y}_0, \theta_0\} \in \boldsymbol{\mathcal{V}}_\rho$.

Second Case (anticipated in [32, end of Remark 1.4]): classes of sub-domains ω touching the boundary Γ, singled out in Definition 1, $d = 2, 3$; Figs. 5, 6, 10; or Definition 2, $d = 4$. Implication of the UCP of Theorem 6 ($d = 2$) and Theorem 7 ($d = 3$) on the Kalman algebraic controllability test.

Let $d = 2$. Consider the class of subsets ω satisfying Definition 1, (i), (ii$_2$). For each unstable eigenvalue $\overline{\lambda}_i$, see (3.7b), out of the 3-dimensional eigenvector $\Phi^*_{ij} = \{\varphi^{*(1)}_{ij}, \varphi^{*(2)}_{ij}, \psi^*_{ij}\}$ in (3.8), $j = 1, \ldots, \ell_i$ = geometric multiplicity, we extract only the last component ψ^*_{ij} corresponding to the thermal unknown, while we disregard the first two components corresponding to the fluid vector. We then ask whether the vectors

$$
\{\psi^*_{i1}, \psi^*_{i2}, \ldots, \psi^*_{i\ell_i}\} \text{ are linearly independent on } L^{q'}(\omega). \tag{3.20}
$$

For scalar element $u_1, u_2, \dots, u_K$, $K = \sup\{\ell_i, i = 1, \dots, M\}$, we consider the matrix U_i defined by

$$U_i = \begin{bmatrix} (u_1, \psi_{i1}^*)_\omega & \dots & (u_K, \psi_{i1}^*)_\omega \\ (u_1, \psi_{i2}^*)_\omega & \dots & (u_K, \psi_{i2}^*)_\omega \\ \vdots & \ddots & \vdots \\ (u_1, \psi_{i\ell_i}^*)_\omega & \dots & (u_K, \psi_{i\ell_i}^*)_\omega \end{bmatrix}. \tag{3.21}$$

Let now $d = 3$. For each unstable eigenvalue $\overline{\lambda}_i$, out of the 4-dimensional eigenvector $\Phi_{ij}^* = \{\varphi_{ij}^{*(1)}, \varphi_{ij}^{*(2)}, \varphi_{ij}^{*(3)}, \psi_{ij}^*\}$ in (3.8), we select the following 2-dimensional vectors, both with 2 reduced components of the fluid vector:

$$\begin{aligned} &\text{either the vector } \widetilde{\Phi}_{ij}^*(1) = \{\varphi_{ij}^{*(1)}, \psi_{ij}^*\}, \text{ for the class of sub-domains } \omega \text{ satisfying} \\ &\qquad\qquad \text{Definition 1, (i), (ii}_3\text{), Case 1, Eq. (1.31)}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} &\text{or else the vector } \widetilde{\Phi}_{ij}^*(2) = \{\varphi_{ij}^{*(2)}, \psi_{ij}^*\}, \text{ for the class of sub-domains } \omega \text{ satisfying} \\ &\qquad\qquad \text{Definition 1, (i), (ii}_3\text{), Case 2, Eq. (1.32)}. \end{aligned} \tag{3.23}$$

In each case, we then ask whether the vectors

$$\left\{\widetilde{\Phi}_{i1}^*(1), \widetilde{\Phi}_{i2}^*(1), \dots, \widetilde{\Phi}_{i\ell_i}^*(1)\right\} \text{ are linearly independent in } L^{q'}(\omega) \times L^{q'}(\omega), \tag{3.24}$$

or respectively the vectors

$$\left\{\widetilde{\Phi}_{i1}^*(2), \widetilde{\Phi}_{i2}^*(2), \dots, \widetilde{\Phi}_{i\ell_i}^*(2)\right\} \text{ are linearly independent in } L^{q'}(\omega) \times L^{q'}(\omega). \tag{3.25}$$

Accordingly, for 2-dimensional vector $\mathbf{u}_1, \dots, \mathbf{u}_K$, we consider the matrix $U_i(k)$ defined by

$$U_i(k) = \begin{bmatrix} (\mathbf{u}_1, \widetilde{\Phi}_{i1}^*(k))_\omega & \dots & (\mathbf{u}_K, \widetilde{\Phi}_{i1}^*(k))_\omega \\ (\mathbf{u}_1, \widetilde{\Phi}_{i2}^*(k))_\omega & \dots & (\mathbf{u}_K, \widetilde{\Phi}_{i2}^*(k))_\omega \\ \vdots & \ddots & \vdots \\ (\mathbf{u}_1, \widetilde{\Phi}_{i\ell_i}^*(k))_\omega & \dots & (\mathbf{u}_K, \widetilde{\Phi}_{i\ell_i}^*(k))_\omega \end{bmatrix}, \qquad k = 1 \text{ or } k = 2. \tag{3.26}$$

In either case, $k = 1$ or $k = 2$, the controllability Kalman algebraic conditon for the linearized dynamics [32] is given by

$$\operatorname{rank} U_i(k) = \ell_i, \text{ respectively}, \quad i = 1, \dots, M. \tag{3.27}$$

It is at this point that the UCP of Theorem 6 ($d = 2$) or Theorem 7 ($d = 3$) is critically invoked.

Lemma 3 *The UCP of Theorem 6 ($d = 2$) or Theorem 7 ($d = 3$) for the class of boundary touching subsets ω there considered for each case $d = 2$ or $d = 3$*

according to Definition 1 implies that for each $\overline{\lambda}_i$ in (3.7a) (the fact that $\overline{\lambda}_i$ is unstable is immaterial), the corresponding vectors $\{\psi^*_{i1}, \cdots, \psi^*_{i\ell_i}\}$ defined in (3.8), or $\{\widetilde{\Phi}^*_{i1}(k), \cdots, \widetilde{\Phi}^*_{i\ell_i}(k)\}$, $k = 1$ or $k = 2$ defined in (3.22), (3.23) are linearly independent in $\widehat{\mathbf{L}}^{q'}(\omega)$; thereby supporting statement (3.20) for $d = 2$, as well as statements (3.24) and (3.25) for $d = 3$. Hence, the full rank condition rank $U_i = \ell_i$, see (3.21) for $d = 2$, and rank $U_i(k) = \ell_i$, see (3.27) for $d = 3$ for the two subclasses of subsets ω in (3.22) or (3.23) hold true for infinitely many choices of the vector $\mathbf{u}_k = \{\mathbf{u}^1_k, u^2_k\}$, with fluid component $\mathbf{u}^1_k$ of reduced dimension $d - 2 = 1$ (rather than of dimension $d = 3$), and the thermal component u^2_k also of dimension 1.

Proof In the case $d = 2$ with reference to (3.20), we seek to establish that

$$\sum_{j=1}^{\ell_i} \alpha_i \psi^*_{ij} = 0 \text{ in } L^{q'}(\omega) \Longrightarrow \alpha_j = 0,\ j = 0, \ldots, \ell_i. \tag{3.28}$$

Instead, in the case $d = 3$, with reference to either (3.24) or else (3.25), for the respective classes of sub-domains ω satisfying Definition 1, (i), and either (ii$_3$) Case 1, Eq. (1.31); or else (ii$_3$) Case 2, Eq. (1.32), respectively, we seek to establish that for k fixed, $k = 1$ or $k = 2$,

$$\sum_{j=1}^{\ell_i} \alpha_i \widetilde{\Phi}^*_{ij}(k) = 0 \text{ in } L^{q'}(\omega) \times L^{q'}(\omega) \Longrightarrow \alpha_j = 0,\ j = 0, \ldots, \ell_i. \tag{3.29}$$

To this end, in both cases $d = 2$ and $d = 3$, we define the vector $\Phi^* = [\boldsymbol{\varphi}^*, \psi^*]$ (we suppress dependence on i) by

$$\Phi^* = \sum_{j=1}^{\ell_i} \alpha_i \Phi^*_{ij} \text{ in } \mathbf{L}^{q'}(\Omega) \times L^{q'}(\Omega) \tag{3.30}$$

that is, Φ^* is a linear combinaton, with the same constants α_j as in (3.28) or (3.29), of the eigenvectors $\Phi^*_{i1}, \ldots, \Phi^*_{i\ell_i}$ of $\mathbb{A}^*_q$ in (3.7a), (3.8). Then $\Phi^* = [\boldsymbol{\varphi}^*, \psi^*]$ is itself an eigenvector of the operator $\mathbb{A}^*_q$ corresponding to the eigenvalue $\overline{\lambda}_i$, as in (3.7a). Thus, we have

$$\mathbb{A}^*_q \Phi^* = \overline{\lambda}_i \Phi^* \text{ in } \mathbf{L}^{q'}(\Omega) \times L^{q'}(\Omega); \qquad \boldsymbol{\varphi}^* \in \mathbf{L}^{q'}(\Omega), \ \ \psi^* \in L^{q'}(\Omega) \tag{3.31}$$

along with

$$\psi^* = \sum_{j=1}^{\ell_i} \alpha_i \psi^*_{ij} \equiv 0 \text{ in } L^{q'}(\omega) \text{ in case } d = 2, \text{ by (3.28)} \tag{3.32}$$

or

$$\widetilde{\Phi}^*(k) = \sum_{j=1}^{\ell_i} \alpha_i \widetilde{\Phi}^*_{ij}(k) \equiv 0 \text{ in } \mathbf{L}^{q'}(\omega) \times L^{q'}(\omega), \text{ in case } d = 3, \text{ by (3.29)}. \quad (3.33)$$

With $\Phi^* = [\boldsymbol{\varphi}^*, \psi^*]$, the PDE-version of $\mathbb{A}_q^* \Phi^* = \lambda \Phi^*$ is

$$-\nu \Delta \boldsymbol{\varphi}^* + L_e^*(\boldsymbol{\varphi}^*) + \psi^* \nabla \theta_e + \nabla \pi = \lambda \boldsymbol{\varphi}^* \quad \text{in } \Omega \quad (3.34a)$$
$$-\kappa \Delta \psi^* + \mathbf{y}_e \cdot \nabla \psi^* - \gamma \boldsymbol{\varphi}^* \cdot \mathbf{e}_d = \lambda \psi^* \quad \text{in } \Omega \quad (3.34b)$$
$$\text{div } \boldsymbol{\varphi}^* = 0 \quad \text{in } \Omega \quad (3.34c)$$
$$\boldsymbol{\varphi}^* = 0, \ \psi^* = 0 \quad \text{on } \Gamma \quad (3.34d)$$

for any d. In addition:

(i) for $d = 2$, we have

$$\psi^* \equiv 0 \text{ in } \omega, \text{ by (3.32)}. \quad (3.35)$$

(ii) Instead for $d = 3$, we have

$$\psi^* \equiv 0 \text{ in } \omega \text{ and } \begin{cases} \varphi_1^* \equiv 0 \text{ in } \omega, \text{ under } k = 1, (\text{ii}_3), \text{ Case 1}, & (3.36) \\ \varphi_2^* \equiv 0 \text{ in } \omega, \text{ under } k = 2, (\text{ii}_3), \text{ Case 2}, & (3.37) \end{cases}$$

by (3.33) with $k = 1$ and $k = 2$ respectively.

Application of the UCP in Theorem 6 ($d = 2$) or Theorem 7 ($d = 3$) implies

$$\Phi^* = [\boldsymbol{\varphi}^*, \psi^*] \equiv 0 \text{ in } \Omega. \quad (3.38)$$

Indeed, in order to reach conclusion (3.38), we note that:

(i) for $d = 2$, there is no need of the B.C. $\varphi^* \equiv 0$, $\psi^* \equiv 0$ on Γ in (3.34d);
(ii) for $d = 3$, there is no need of the B.C. $\psi^*\big|_\Gamma \equiv 0$ in (3.34d) while the B.C. $\boldsymbol{\varphi}^* = \{\varphi_1^*, \varphi_2^*, \varphi_3^*\} \equiv 0$ on Γ is more than enough: compare with (1.29) or (1.30). We would only need $\varphi_2^*\big|_{\Gamma_0} = 0$ under (ii_3), Case 1, or $\varphi_1^*\big|_{\Gamma_0} = 0$ under (ii_3), Case 2, for the boundary set Γ_0 defined in Def. 1.

Going back to (3.30) we use (3.38) and obtain

$$\Phi^* = \sum_{j=1}^{\ell_i} \alpha_j \Phi^*_{ij} \equiv 0 \text{ in } \Omega; \quad \text{hence } \alpha_j = 0, \ j = 1, \dots, \ell_i \quad (3.39)$$

since the eigenvectors $\left\{\Phi^*_{ij}\right\}_{j=1}^{\ell_i}$ are linearly independent in $\mathbf{L}^{q'}(\Omega) \times L^{q'}(\Omega)$. Conclusion (3.28) or (3.29) is established. □

Such Lemma 3 (like Lemma 2) plays a critical role in establishing the following uniform stabilization result for the dynamic Boussinesq system to complement [32].

Theorem 12 *For $d = 2, 3$, consider the class of sub-domains ω touching the boundary Γ, as singled out in Definition 1, Figs. 5 and 6 or 10. Then the perfect counterpart of Theorem 11 holds true, with the additional advantage that the vector $\mathbf{u}_K$ occurring in (3.17) has the following more desirable structure, in line with Lemma 3:*

(i) for $d = 2$, the feedback vector $\mathbf{u}_K$ in (3.17) is now $u_K = u_K^2$ (scalar); that is, it contains only a component coming from the thermal equation. Thus, no fluid component is present in the feedback vector u_K ($d - 2 = 2 - 2 = 0$ fluid components). The thermal equation is sufficient by itself to inject a feedback control capable to produce uniform stabilization of the dynamic Boussinesq system in feedback-form such as in (3.18).

(ii) for $d = 3$, the feedback vector $\mathbf{u}_K$ in (3.17) is given by $\mathbf{u}_K = \{u_K^1, u_K^2\}$, where now the fluid component u_K^1 is scalar ($d - 2 = 3 - 2 = 1$), as acting on the fluid equation.

The rest of Theorem 11 is unchanged, in particular the exponential decay (3.19).

Remark 2 The above results on the reduction by one unit or, respectively, two unites in the number of components of the d-fluid vector $\mathbf{u}_K^1$ in feedback form to achieve the closed loop uniform feedback stabilization results of Theorem 11, respectively Theorem 12, are in line with the open-loop controllability results in [11,14,15,25].

3.2 Uniform Stabilization of the (Dynamic) Boussinesq System by a Localized Interior Feedback Control and a Localized Boundary Feedback Control, Both Finite Dimensional

In the notation of Section 3.1, consider the following Boussinesq problem

$$\begin{cases} \mathbf{y}_t - \nu_0 \Delta \mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{y} - \gamma(\theta - \bar{\theta})\mathbf{e}_d + \nabla \pi = m(x)\mathbf{u}(t, x) + \mathbf{f}(x) & \text{in } Q \quad (3.40\text{a}) \\ \theta_t - \kappa \Delta \theta + \mathbf{y} \cdot \nabla \theta = g(x) & \text{in } Q \quad (3.40\text{b}) \\ \operatorname{div} \mathbf{y} = 0 & \text{in } Q \quad (3.40\text{c}) \\ \mathbf{y} \equiv 0, \quad \theta = v & \text{on } \Sigma \quad (3.40\text{d}) \\ \mathbf{y}(0, x) = \mathbf{y}_0(x), \quad \theta(0, x) = \theta_0(x) & \text{on } \Omega \quad (3.40\text{e}) \end{cases}$$

where $\mathbf{u}$ is the interior control localized in the arbitrarily small, open, connected, smooth subset ω of Ω of positive measure, which moreover is a collar of the subportion $\widetilde{\Gamma}$ of the boundary $\Gamma = \partial\Omega$. See Fig. 2. Moreover, the boundary (Dirichlet) control v for the heat component acts (has compact support) on $\widetilde{\Gamma}$.

As for problem (3.1), we assume that the external body force $\mathbf{f}(x)$ and the heat source density $g(x)$ render the overall system *unstable* (in a technical sense described in [32, Sect. 3]), and recalled also in connection with the Boussinesq system (3.1). This is quantitatively stated in (3.7b) by assuming that there are N eigenvalues (M distinct) of the operator $\mathbb{A}_q$ (equivalently, $\mathbb{A}_q^*$) in (3.7a) that lie on the closed half-plane $\mathbb{C}^+$. While we refer to [33] for the technical and precise mathematical description, it suffices to report here that the successful feedback system (well-posed and stabilizing in a suitable L_q/Besov functional setting by finite dimensional feedback controllers $\mathbf{u}$ and v) is critically based (at the analysis of the finite dimensional projection as in

Sect. 3.1) on the following result. Let, as in (3.7a),

$$\Phi_{ij} = \begin{bmatrix} \boldsymbol{\phi}_{ij} \\ \psi_{ij} \end{bmatrix}, \qquad \Phi^*_{ij} = \begin{bmatrix} \boldsymbol{\phi}^*_{ij} \\ \psi^*_{ij} \end{bmatrix}, \quad j = 1, \ldots, \ell_i \tag{3.41}$$

be the eigenvectors of the operator $\mathbb{A}_q$, respectively $\mathbb{A}^*_q$, corresponding to the unstable eigenvalue λ_i, respectively, $\overline{\lambda}_i$, of geometric multiplicity ℓ_i, stated in (3.7b):

$$\mathbb{A}_q \begin{bmatrix} \boldsymbol{\phi}_{ij} \\ \psi_{ij} \end{bmatrix} = \lambda_i \begin{bmatrix} \boldsymbol{\phi}_{ij} \\ \psi_{ij} \end{bmatrix} \in \mathcal{D}(\mathbb{A}_q), \qquad \mathbb{A}^*_q \begin{bmatrix} \boldsymbol{\phi}^*_{ij} \\ \psi^*_{ij} \end{bmatrix} = \overline{\lambda}_i \begin{bmatrix} \boldsymbol{\phi}^*_{ij} \\ \psi^*_{ij} \end{bmatrix} \in \mathcal{D}(\mathbb{A}^*_q). \tag{3.42}$$

Construct the following matrix

$$U_i = \left[\begin{array}{c|c} \left(f_1, \partial_\nu \psi^*_{i1}\right)_{\widetilde{\Gamma}} \cdots \left(f_{\ell_i}, \partial_\nu \psi^*_{i1}\right)_{\widetilde{\Gamma}} & \left\langle \mathbf{u}_1, \boldsymbol{\phi}^*_{i1}\right\rangle_\omega \cdots \left\langle \mathbf{u}_{\ell_i}, \boldsymbol{\phi}^*_{i1}\right\rangle_\omega \\ \left(f_1, \partial_\nu \psi^*_{i2}\right)_{\widetilde{\Gamma}} \cdots \left(f_{\ell_i}, \partial_\nu \psi^*_{i2}\right)_{\widetilde{\Gamma}} & \left\langle \mathbf{u}_1, \boldsymbol{\phi}^*_{i2}\right\rangle_\omega \cdots \left\langle \mathbf{u}_{\ell_i}, \boldsymbol{\phi}^*_{i2}\right\rangle_\omega \\ \vdots & \vdots \\ \left(f_1, \partial_\nu \psi^*_{i\ell_i}\right)_{\widetilde{\Gamma}} \cdots \left(f_{\ell_i}, \partial_\nu \psi^*_{i\ell_i}\right)_{\widetilde{\Gamma}} & \left\langle \mathbf{u}_1, \boldsymbol{\phi}^*_{i\ell_i}\right\rangle_\omega \cdots \left\langle \mathbf{u}_{\ell_i}, \boldsymbol{\phi}^*_{i\ell_i}\right\rangle_\omega \end{array}\right] \tag{3.43}$$

where $f_1, \ldots, f_{\ell_i}$ are boundary vectors on $\widetilde{\Gamma}$; $\mathbf{u}_1, \ldots, \mathbf{u}_{\ell_i}$ are interior vectors on ω; $(\cdot, \cdot)_{\widetilde{\Gamma}}$ and $(\cdot, \cdot)_\omega$ are suitable duality pairings on $\widetilde{\Gamma}$ and ω. The corresponding controllability Kalman/Hautus algebraic condition of the finite-dimensional projected system is given by

$$\text{rank } U_i = \text{full} = \ell_i, \qquad i = 1, \ldots, M, \tag{3.44}$$

$M = \#$ of distinct unstable eigenvalues. In turn, in the present case, such rank condition is equivalent to the Unique Continuation Property of Theorem 3: (1.5) $\Longrightarrow$ (1.6) of the operator $\mathbb{A}_q$; actually, of its adjoint $\mathbb{A}^*_q$ rather than $\mathbb{A}_q$. Details are given in [33].

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Appendix A Over-Determined Oseen Eigenproblem

We refer to [48] and with the same notation used for problem (1.3), we consider the following Oseen eigenproblem

$$\begin{cases} -\nu_0 \Delta y + L_e(y) + \nabla p = \lambda y & \text{in } \Omega, \quad y = [y_1, \ldots, y_d] \in \mathbb{R}^d & \text{(A.1a)} \\ \text{div } y = 0 & \text{in } \Omega & \text{(A.1b)} \\ y \equiv 0 & \text{in } \omega & \text{(A.1c)} \end{cases}$$

Theorem A.1 *([48]) Let $\lambda \in \mathbb{C}$. Let $y \in (H^2(\Omega))^d$, $p \in H^1(\Omega)$ be any solution of problem (A.1). Then*

$$y \equiv 0 \qquad and \qquad p \equiv const \; in \; \Omega, \tag{A.2}$$

and we can take $p \equiv 0$*, as* p *is identified only up to a constant.*

Appendix B

Theorem A.2 *Let now* $\{\varphi, p\} \in (W^{2,q}(\Omega))^d \times W^{1,q}(\Omega)$ *solve the problem*

$$\begin{cases} (-\nu_o \Delta)\varphi + L_e(\varphi) + \nabla \pi = \lambda \varphi & in \; \Omega \quad \text{(B.1a)} \\ \operatorname{div} \varphi = 0 & in \; \Omega \quad \text{(B.1b)} \\ \varphi|_{\widetilde{\Gamma}} \equiv 0, \quad \left.\dfrac{\partial \varphi}{\partial \nu}\right|_{\widetilde{\Gamma}} \equiv 0, \quad \varphi \cdot \tau \equiv 0 & in \; \omega \quad \text{(B.1c)} \end{cases}$$

where $\widetilde{\Gamma}$ *is an open subset of* $\Gamma = \partial\Omega$ *of positive surface mesaure and* ω *is a local collar of* $\widetilde{\Gamma}$ *(Fig. 4), the tangent vector* τ *being defined in the paragraph preceding Theorem 4. Then, [37, Lemma 2, p 138], [38, Theorem 6.2], [31, Lemma 6.2]*

$$\varphi \equiv 0 \quad and \quad p \equiv const \quad in \; \Omega. \tag{B.2}$$

Appendix C UCP, Adjoint Problem, $d = 4$

In this Appendix 1 we provide the extension to the case $d = 4$ of the UCP of Theorem 6 ($d = 2$) and Theorem 7 ($d = 3$) of the adjoint problem. The class of relevant pairs $\{\omega, \Omega\}$ is singled out in the next definition, an extension of Definition 1.

Definition 2 Let ω be an open, connected subset of Ω, thus of positive measure, satisfying the preliminary conditions:

(i) the intersection between the boundary $\partial\omega$ of ω and the boundary Γ of Ω is non-empty: $\widetilde{\Gamma} = \partial\omega \cap \Gamma \neq \emptyset$.
(ii$_4$) let $d = 4$. There are three cases. If P is an arbitrary point of ω, then the hyperplane π_P passing through P and parallel

Case 1: to the coordinate (x_1, x_2, x_4)-hyperplane;
Case 2: to the coordinate (x_1, x_3, x_4)-hyperplane;
Case 3: to the coordinate (x_2, x_3, x_4)-hyperplane;

meets the intersection $\widetilde{\Gamma}$ at a surface ζ_P.
In each case, let $\mathcal{T}_{\zeta_P}$ be the totality (collection) of all surfaces ζ_P where the hyperplane π_P meets the portion $\widetilde{\Gamma} = \partial\omega \cap \Gamma \neq \emptyset$ in (i), as P runs over ω. Let

$$\begin{aligned} \Gamma_0 = {} & \text{connected component, or union of connected components, of } \mathcal{T}_{\zeta_P}, \\ & \text{such that any such hyperplane } \pi_P,\ P \in \omega,\ \text{hits } \Gamma_0 \text{ at just one surface.} \end{aligned} \tag{C.2}$$

Theorem A.3 *Let $d = 4$. Let $\{\omega, \Omega\}$ be a pair satisfying Definition 2 (i) and either (ii$_4$) Case 1, or (ii$_4$) Case 2, or else (ii$_4$) Case 3. Thus, if P is an arbitrary point of ω, then the hyperplane π_P passing through P and parallel either to the (x_1, x_2, x_4)-coordinate hyperplane (Case 1), or the (x_1, x_3, x_4)-coordinate hyperplane (Case 2), or the (x_2, x_3, x_4)-coordinate hyperplane (Case 3), meets the intersection $\widetilde{\Gamma} = \partial\omega \cap \Gamma$ at a surface ζ_P.*

Let $\{\phi, h, p\} \in (W^{2,q}(\Omega))^d \times W^{2,q}(\Omega) \times W^{1,q}(\Omega)$, $q > d$, satisfy the adjoint Boussinesq problem (1.18). With $\phi = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, recall Γ_0 from (C.2) and assume

$$\phi_3\big|_{\Gamma_0} = 0 \text{ in Case 1}; \qquad \phi_2\big|_{\Gamma_0} = 0 \text{ in Case 2}; \qquad \phi_1\big|_{\Gamma_0} = 0 \text{ in Case 3} \tag{A.3}$$

as well as the over-determined conditions

$$h \equiv 0 \text{ in } \omega \text{ and } \begin{cases} \phi_1 \equiv \phi_2 \equiv 0 \text{ in } \omega, \ \text{in Case 1}, & \text{(C.3)} \\ \phi_1 \equiv \phi_3 \equiv 0 \text{ in } \omega, \ \text{in Case 2}, & \text{(C.4)} \\ \phi_2 \equiv \phi_3 \equiv 0 \text{ in } \omega, \ \text{in Case 3}, & \text{(C.5)} \end{cases}$$

Thus, in either of the three cases, in follows that

$$\phi = \{\phi_1, \phi_2, \phi_3, \phi_4\} \equiv 0, \qquad h \equiv 0, \qquad p \equiv \text{ const}, \quad \text{in } \Omega. \tag{B.6}$$

Proof of Theorem A.3 We prove only Case 1, as the proof of the other two cases is similar, mutatis mutandis. As in the cases of Theorem 6 and 7, the condition $h \equiv 0$ in ω in (C.3)–(C.5) implies via (1.18b)

$$\phi_4 \equiv 0 \text{ in } \omega, \quad d = 4. \tag{A.7}$$

By (A.7), under Case 1 in (C.3), the divergence condition (1.18c) implies

$$\begin{aligned} &\text{div } \phi = \phi_{1x_1} + \phi_{2x_2} + \phi_{3x_3} + \phi_{4x_x} = \phi_{3x_3} \equiv 0 \qquad \text{in } \omega \\ &\text{hence } \phi_3(x_1, x_2, x_3, x_4) = c(x_1, x_2, x_4) \qquad \text{in } \omega, \end{aligned} \tag{A.8}$$

where $c(\cdot)$ denotes a function constant w.r.t. x_3 and depending only on x_1, x_2, x_4. Let $P = \{(x_1(P), x_2(P), x_3(P), x_4(P)\}$ be an arbitrary point of ω. Consider the hyperplane π_P through the point P and parallel to the coordinate (x_1, x_2, x_4)-hyperplane. As the point $\{x_1, x_2, x_3(P), x_4\}$ of ω runs over the hyperplane π_P, the value $\phi_3(x_1, x_2, x_3(P), x_4) = c(x_1, x_2, x_4)$ is independant of $x_3(P)$, as long as such hyperplane intersections ω. By Definition 2 (ii$_4$) Case 1, such hyperplane π_P meets the intersection $\widetilde{\Gamma} = \partial\omega \cap \Gamma$ at some surface ζ_P. Thus,

$$\phi_3(x_1, x_2, x_3(P), x_4) = \phi_3\big|_{\zeta_P} = 0, \quad (x_1, x_2, x_3(P), x_4) \in \omega \tag{A.9}$$

by recalling assumption (A.3) for $\zeta \subset \Gamma_0$, with Γ_0 defined in (C.2). Thus, (A.9) gives $\phi_3(P) = 0$. But P is an arbitrary point of ω. Thus,

$$\phi_3 \equiv 0 \qquad \text{in } \omega. \tag{A.10}$$

Then $\phi \equiv \{\phi_1, \phi_1, \phi_3, \phi_4\} \equiv 0$ in ω by (C.3), (A.10), (A.7). Application of Theorem 5 (with $d = 4$) yields the desired conclusion (B.6).

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