Sharp Interior and Boundary Regularity of the SMGTJ-Equation with Dirichlet or Neumann Boundary Control

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Abstract We consider the third order (in time) linear equation known as SMGTJ-equation, as defined on a multidimensional bounded domain and subject to either Dirichlet or Neumann boundary control. We then establish corresponding sharp interior and boundary regularity results.

Keywords SMGTJ - Equation · Boundary Regularity

1 Introduction: The SMGTJ Equation

Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary $\Gamma = \partial\Omega$, as specified below. In this paper, we consider the problem of sharp interior and boundary **regularity** of the linear version of a third order (in time) PDE **with non-homogeneous term on the boundary**. Both Dirichlet and Neumann boundary terms will be considered. The equation, which should be called SMGTJ [for G. G. Stokes (1851), F. K. Moore and W. E. Gibson (1960), P. A. Thompson (1972) and P. M. Jordan (2004)], see [12, 13, 33, 37, 39], arises in a variety of physical contexts such as: effects of the radiation of heat on the propagation of sound; propagation of disturbances in a gas subject to relaxation effects; behavior of viscoelastic materials; propagation of acoustic waves, etc. In particular, if in classical models in nonlinear acoustics (Kuznetsov equation, Westervelt equation, Kokhlov–Zobolotskaya–Kuznetsov equation), one replaces the Fourier Law for the heat flux with the Maxwell-Cattaneo Law (to avoid the paradox of infinite speed of propagation), one obtains a third order in time PDE, whose linear part is the one considered in the present paper; that is [12, 13]

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = 0 \quad \text{in } (0, T] \times \Omega, \tag{1.1}$$

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J. Banasiak et al. (eds.), *Semigroups of Operators – Theory and Applications*,
Springer Proceedings in Mathematics & Statistics 325,
https://doi.org/10.1007/978-3-030-46079-2_22

where $\tau > 0$, $b > 0$, $c^2 > 0$ are fixed constants, whose physical meaning is not relevant here. See [34].

We are taking Ω in $\mathbb{R}^3$, as this is the physically significant setting. However, the mathematical analysis works on any $\mathbb{R}^d$, $d = 1, 2, \ldots$ in the Dirichlet case and in $\mathbb{R}^d$, $d = 2, 3, \ldots$ in the Neumann case.

Part A: Dirichlet Case

2 Linear Third Order SMGTJ-Equation with Non-homogeneous Dirichlet Boundary Term

Henceforth we shall take $\tau = 1$ in (1.1) w.l.o.g.

If the linear third order equation (1.1) is written in terms of the pressure, then Dirichlet non-homogeneous boundary terms are appropriate [14]. We then consider the following mixed problem in the unknown $y(t, x)$:

$$\begin{cases} y_{ttt} + \alpha y_{tt} - c^2\Delta y - b\Delta y_t = 0 & \text{in } Q = (0, T] \times \Omega \quad (2.1a) \\ y\big|_{t=0} = y_0; \quad y_t\big|_{t=0} = y_1; \quad y_{tt}\big|_{t=0} = y_2 & \text{in } \Omega \quad (2.1b) \\ y\big|_{\Sigma} = g & \text{in } \Sigma = (0, T] \times \Gamma \quad (2.1c) \end{cases}$$

2.1 *Case* $g \equiv 0$.

A rather comprehensive study of this case was carried out in [34] in the constant coefficient case via semigroup/functional analytic techniques, and in [15, 16] in the variable coefficient case via energy methods. Here we shall only report a subset of these results which are relevant to the present paper. Define the positive self adjoint operator on $H = L^2(\Omega)$:

$$Af = -\Delta f, \qquad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \tag{2.2}$$

so that problem (2.1a), (2.1b) and (2.1c) (with $g = 0$) can be re-written abstractly as

$$u_{ttt} + \alpha u_{tt} + c^2 Au + bAu_t = 0 \quad \text{on } H = L^2(\Omega), \tag{2.3}$$

along with I.C. u_0, u_1, u_2. We re-write it as a first order problem as

$$\frac{d}{dt}\begin{bmatrix} u \\ u_t \\ u_{tt} \end{bmatrix} = G\begin{bmatrix} u \\ u_t \\ u_{tt} \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -c^2 A & -bA & -\alpha I \end{bmatrix}. \tag{2.4}$$

Introduce the following spaces:

$$U_0 = H \times H \times H \tag{2.5a}$$

$$U_1 \equiv \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{2}}) \times H; \qquad U_2 \equiv \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \tag{2.5b}$$

$$U_3 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times H; \qquad U_4 \equiv \mathcal{D}(A^{\frac{3}{2}}) \times \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \tag{2.5c}$$

Theorem 2.1 ([34, Sect. 2]) *The operator G in (2.4) generates a s.c. group e^{Gt} on each of the spaces U_1, U_2, U_3, U_4 with appropriate domains so that*

$$\begin{bmatrix} u(t) \\ u_t(t) \\ u_{tt}(t) \end{bmatrix} = e^{Gt} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} \in C([0,T]; U_i), \quad i = 1, 2, 3, 4 \tag{2.6}$$

for $[u_0, u_1, u_2] \in U_i$, $i = 1, 2, 3, 4$. Below we shall emphasize the case U_3, whereby then

$$G : U_3 \supset \mathcal{D}(G) = \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \longrightarrow U_3 \tag{2.4 bis}$$

The group generation property points out that the third order equation (2.3) has a 'hyperbolic' character. In fact, as in [34], rewrite (2.3) as

$$(u_t + \alpha u)_{tt} + bA\left(\frac{c^2}{b}u + u_t\right) = 0. \tag{2.7}$$

This suggests introducing a new variable, as in [34]

$$\text{either } z = \frac{c^2}{b}u + u_t, \qquad \text{or else } \xi = u_t + \alpha u \tag{2.8}$$

(i) Thus,

$$\text{If } \alpha = \frac{c^2}{b}, \text{ then (2.7)} \Longrightarrow z_{tt} + bAz = 0 \ (z = \xi), \tag{2.9}$$

the pure abstract wave equation.

(ii) Otherwise,

$$z = \frac{c^2}{b}u + u_t = (\alpha u + u_t) - \gamma u, \qquad \gamma = \alpha - \frac{c^2}{b} \tag{2.10}$$

$$(u_t + \alpha u)_{tt} = z_{tt} + \gamma u_{tt} = z_{tt} + \gamma\left(z - \frac{c^2}{b}u\right)_t. \tag{2.11}$$

Substituting (2.8), (2.10), (2.11) in (2.7) leads to the following hyperbolic-dominated system

$$\begin{cases} z_{tt} = -bAz - \gamma z_t + \gamma \dfrac{c^2}{b} z - \gamma \left(\dfrac{c^2}{b}\right)^2 u & \text{(2.12a)} \\ u_t = -\dfrac{c^2}{b} u + z & \text{(2.12b)} \end{cases}$$

(model #2 in [34, Sect. 2]) coupling the hyperbolic z-equation with the scalar ODE in u.

2.2 *Case* $y_0 = 0$, $y_1 = 0$, $y_2 = 0$, $g \neq 0$.

In this case, we seek to obtain sharp regularity of the map

$$g \longrightarrow \left\{ y, y_t, y_{tt}, \frac{\partial y}{\partial \nu}\bigg|_{\Sigma} \right\} \tag{2.13}$$

from the Dirichlet boundary datum g to the interior solution $\{y, y_t, y_{tt}\}$ and the Neumann boundary trace $\dfrac{\partial y}{\partial \nu}\bigg|_{\Sigma}$ of problem (2.1a), (2.1b) and (2.1c). Our main result in the present Part A (Dirichlet) is the following

Theorem 2.2 *With reference to problem (2.1a)–(2.1c) with zero I.C., we have the following interior regularity results:*

$$g \in L^2(0, T; L^2(\Gamma)) \cap C([0, \epsilon); L^2(\Gamma)), \quad g(0) = 0, \quad \epsilon > 0 \textit{ small}$$

$$\Longrightarrow \begin{cases} y \in C([0, T]; L^2(\Omega)) & \text{(2.14a)} \\ y_t \in C([0, T]; [\mathcal{D}(A^{\frac{1}{2}})]' = H^{-1}(\Omega)), & \text{(2.14b)} \\ y_{tt} \in L^2(0, T; [\mathcal{D}(A)]') & \text{(2.14c)} \\ \dfrac{\partial y}{\partial \nu}\bigg|_{\Sigma} \in H^{-1}(\Sigma)) & \text{(2.14d)} \end{cases}$$

For (2.14a) and (2.14b), see (3.20) *and* (3.21)*, respectively. For (2.14c), see* (3.31a). *Finally for (2.14d) see (4.1) of Theorem 4.1. Moreover,*

$$\begin{cases} g \in C([0, T]; L^2(\Gamma)) \\ g(0) = 0 \end{cases} \Longrightarrow y_{tt} \in C([0, T]; [\mathcal{D}(A)]'), \tag{2.14e}$$

see (3.31b)*, all the maps being continuous.*

Remark 2.1 The results of Theorem 2.2 should be compared with the following results for general second order hyperbolic equations, even with variable coefficients, which we however report only for the canonical wave equation.

Theorem 2.3 ([23, 25, 26]) *Consider the following mixed problem, where* Ω *is a bounded domain in* $\mathbb{R}^d$, $d \geq 1$, *with sufficiently smooth boundary* Γ*:*

$$\begin{cases} w_{tt} = \Delta w & \text{in } Q = (0, T] \times \Omega & (2.15a) \\ w\big|_{t=0} = 0; \quad w_t\big|_{t=0} = 0 & \text{in } \Omega & (2.15b) \\ w\big|_{\Sigma} = g & \text{in } \Sigma = (0, T] \times \Gamma & (2.15c) \end{cases}$$

Then

$$\begin{cases} g \to \left\{ w, w_t, w_{tt}, \dfrac{\partial w}{\partial \nu}\Big|_{\Sigma} \right\} \text{ continuously} \\ L^2(0, T; L^2(\Gamma)) \to C\left([0, T]; L^2(\Omega) \times H^{-1}(\Omega) \times H^{-2}(\Omega)\right) \times H^{-1}(\Sigma), \end{cases} \tag{2.16}$$

where $H^{-1}(\Sigma)$ = *dual of* $\left\{h \in H_0^1(\Sigma)\right\}$ *i.e. with* $h(\cdot, 0) = 0$ *and* $h(\cdot, T) = 0$ *on* Γ *(but actually,* $h(\cdot, T) = 0$ *is not needed).*

Indeed, our proof of Theorem 2.2 in Sect. 3 (interior regularity) and Sect. 4 (boundary regularity) will critically be based on Theorem 2.3. This result is reported also in [31, Chap. 10, Sect. 5]. The proof of Theorem 2.3 is by PDE-techniques, either directly [25, 26], or much more conveniently, by duality [23].

In fact, consider the following problem, dual of problem (2.15a)–(2.15c)

$$\begin{cases} \phi_{tt} = \Delta \phi + f & \text{in } Q & (2.17a) \\ \phi\big|_{t=T} = \phi_0; \quad \phi_t\big|_{t=T} = \phi_1 & \text{in } \Omega & (2.17b) \\ \phi\big|_{\Sigma} = 0 & \text{in } \Sigma & (2.17c) \end{cases}$$

Theorem 2.4 ([26], [23, Lemma 2.1, p. 154]) *The following (sharp, hidden) trace regularity holds true*

$$\int_0^T \int_\Gamma \left(\frac{\partial \phi}{\partial \nu} \right)^2 d\Sigma = \mathcal{O}_T \left(\left\| \{\phi_0, \phi_1\} \right\|^2_{H_0^1(\Omega) \times L^2(\Omega)} + \| f \|^2_{L^1(0,T;L^2(\Omega))} \right). \tag{2.18}$$

Since [23] it has been ascertained that a most convenient roadmap is to first show (by PDE-techniques) Theorem 2.4 and then obtain Theorem 2.3 on $\{w, w_t, w_{tt}\}$ by duality.

Problem (2.12a), (2.12b) can be likewise re-written in an abstract form as

$$\frac{d}{dt} \begin{bmatrix} z \\ z_t \\ u \end{bmatrix} = \mathbb{A} \begin{bmatrix} z \\ z_t \\ u \end{bmatrix}, \quad \mathbb{A} = \begin{bmatrix} 0 & I & 0 \\ -bA + \gamma \frac{c^2}{b} I & -\gamma I & -\gamma \left(\frac{c^2}{b} \right) I \\ I & 0 & -\frac{c^2}{b} I \end{bmatrix}. \tag{2.19}$$

where $\mathbb{A}$ likewise generates a s.c. group $e^{\mathbb{A}t}$ on each of the following spaces [34, Sect. 2]

$$H_0 = \mathcal{D}(A^{\frac{1}{2}}) \times H \times H \tag{2.20a}$$

$$H_1 \equiv \mathcal{D}(A^{\frac{1}{2}}) \times H \times \mathcal{D}(A^{\frac{1}{2}}); \qquad H_2 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A) \tag{2.20b}$$

$$H_3 \equiv \mathcal{D}(A^{\frac{1}{2}}) \times H \times \mathcal{D}(A); \qquad H_4 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{3}{2}}) \tag{2.20c}$$

The H_i spaces for $\mathbb{A}$ are the perfect courterpart of the spaces U_i for G, $i = 0, \dots, 4$. One has $e^{Gt} = Me^{\mathbb{A}t}M^{-1}$ where the operator M and its inverse M^{-1} are given explicitly in [34].

The constant $\gamma = \alpha - \frac{c^2}{b}$ plays a critical role in the stability of the s.c. group e^{Gt} on U_i, equivalently of the s.c. group $e^{\mathbb{A}t}$ on H_i. Indeed, e^{Gt} is uniformly stable on each U_i (with a sharp explicit decay rate) if and only if $\gamma > 0$. The case $\gamma = 0$, see (2.9) corresponds to the point spectrum $\sigma_p(\mathbb{A})$ of $\mathbb{A}$ being on the imaginary axis, while the point $-\frac{c^2}{b}$ is in its continuous spectrum [34]. Paper [4] claims that if $\gamma < 0$, and at least in the 1-D case, the boundary homogeneous Eq. (1.1) admits a chotic and topologically mixing semigroup on Banach spaces of Herzog's type. General criteria for hypercyclic and chaotic semigroups were given in [7] and further extended in [3] with applications in [2].

2.3 *Proof of Theorem 2.2: Preliminary Analysis*

We introduce, as usual, the Dirichlet map

$$Dg = \varphi \iff \left\{\Delta\varphi = 0 \text{ in } \Omega, \quad \varphi|_{\Gamma} = g\right\}. \tag{2.21a}$$

$$D : L^2(\Gamma) \to H^{\frac{1}{2}}(\Omega) \subset H^{\frac{1}{2}-2\epsilon}(\Omega) = \mathcal{D}(A^{\frac{1}{4}-\epsilon}), \text{ or } A^{\frac{1}{4}-\epsilon}D \in \mathcal{L}\left(L^2(\Gamma); L^2(\Omega)\right) \tag{2.21b}$$

by elliptic theory [24, 25, 41], with $\epsilon > 0$ arbitrary. One cannot take $\epsilon = 0$, see [31, Remark 3.1.4, p. 186]. At first we shall take $g \in H^1(0, T; L^2(\Gamma))$, so that $g_t \in L^2(0, T; L^2(\Gamma)) = L^2(\Sigma)$. We next return to Eq. (2.1a) and re-write it, as usual [25, 41], [31, Appendix 3B, pp. 420–424], via (2.21a) as

$$y_{ttt} + \alpha y_{tt} - c^2\Delta(y - Dg) - b\Delta(y - Dg)_t = 0 \quad \text{in } Q \tag{2.22}$$

or abstractly, via (2.2), as

$$y_{ttt} + \alpha y_{tt} + c^2 A(y - Dg) + bA(y - Dg)_t = 0 \quad \text{in } H. \tag{2.23}$$

Extending, as usual [25, 31], the original operator A in (2.2): $L^2(\Omega) \supset \mathcal{D}(A) \to L^2(\Omega)$ to $A_e : L^2(\Omega) \to [\mathcal{D}(A^*)]' = [\mathcal{D}(A)]'$; duality $[\ \ ]'$ w.r.t. $H = L^2(\Omega)$ by isomorphism, and retaining the symbol A also for A_e for such extension, we re-write Eq. (2.23) as

$$(y_t + \alpha y)_{tt} + bA\left(\frac{c^2}{b}y + y_t\right) = c^2ADg + bADg_t \in [\mathcal{D}(A)]'. \tag{2.24}$$

Setting as in (2.10)

$$z = \frac{c^2}{b}y + y_t = (\alpha y + y_t) - \gamma y, \qquad \gamma = \alpha - \frac{c^2}{b} \tag{2.25}$$

and proceeding as in going from (2.10) to (2.12a), (2.12b), we re-write problem (2.1a), (2.1b) and (2.1c) as the following hyperbolic-dominated system

$$\begin{cases} z_{tt} = -bAz - \gamma z_t + \gamma\dfrac{c^2}{b}z - \gamma\left(\dfrac{c^2}{b}\right)^2 y + c^2ADg + bADg_t \in [\mathcal{D}(A)]' & (2.26a) \\ y_t = -\dfrac{c^2}{b}y + z & (2.26b) \end{cases}$$

along with the I.C. (we are taking $y_0 = 0$, $y_1 = 0$, $y_2 = 0$)

$$z_0 = \frac{c^2}{b}y_0 + y_1 = 0, \qquad z_1 = \frac{c^2}{b}y_1 + y_2 = 0. \tag{2.26c}$$

3 First Proof of Theorem 2.2 (Interior Regularity): Direct Method

Step 1 The coupling $\gamma\left(\frac{c^2}{b}\right)^2 y = \gamma\left(\frac{c^2}{b}\right)^2 \int_0^t e^{-\frac{c^2}{b}(t-\tau)}z(\tau)d\tau$ between the hyperbolic z-dynamics in (2.26a) and the ODE y-equation in (2.26b) is a mild (lower order) integral term. Thus, essentially w.l.o.g., we may take at first

$$\gamma = 0, \quad \text{i.e. } \alpha = \frac{c^2}{b}, \tag{3.1}$$

see (2.9), to simplify the computations. This will not affect the sought-after regularity of the map in (2.9). The terms z_t, z that by taking $\gamma = 0$ disappear are benign terms for the argument that follows. Thus, we obtain the simplified problem

$$\begin{cases} z_{tt} = -bAz + c^2ADg + bADg_t \in [\mathcal{D}(A)]' & \text{(3.2a)} \\ y_t = -\dfrac{c^2}{b}y + z & \text{(3.2b)} \end{cases}$$

along with zero I.C., where now under the (essentially w.l.o.g.) assumption (3.1), the z-problem is uncoupled; that is, explicitly, in PDE-form

$$\begin{cases} \begin{cases} z_{tt} = b\Delta z & \text{in } Q = (0, T] \times \Omega & \text{(3.3a)} \\ z\big|_{t=0} = 0; \quad z_t\big|_{t=0} = 0 & \text{in } \Omega & \text{(3.3b)} \\ z\big|_{\Sigma} = \dfrac{c^2}{b}g + g_t & \text{in } \Sigma = (0, T] \times \Gamma & \text{(3.3c)} \\ & & \text{(3.3d)} \end{cases} \\ \begin{cases} y_t = -\dfrac{c^2}{b}y + z & \text{(3.4a)} \\ y\big|_{t=0} = 0 & \text{(3.4b)} \end{cases} \end{cases}$$

Orientation Thus, under the (essentially benign) assumption (3.1), the crux of our proof consists in applying to the z-wave equation, either as a mixed problem as in (3.3a)–(3.3c), or else in the abstract form (3.2a), the optimal regularity results (at present for the solution $\{z, z_t\}$ in the interior) from Theorem 2.3 of Remark 2.1 and then use these results for z to obtain corresponding results for y, via (3.4a), (3.4b). In carrying out this strategy, the challenge we face is that we seek to reduce the assumption of regularity of the Dirichlet boundary term $g \in H^1(0, T; L^2(\Gamma))$ to a sort of 'minimal' level, as the term g_t is not present in the original problem (2.1a), (2.1b) and (2.1c), but is sneaked in at the level of the technical step in (2.23). Of course $g \in H^1(0, T; L^2(\Gamma))$ allows one to invoke the results of Theorem 2.3 at once and thus obtain a preliminary (conservative) result: the map

$$g \in H^1(0, T; L^2(\Gamma)) \quad \rightarrow \quad \{z, z_t\} \in C([0, T]; L^2(\Omega) \times H^{-1}(\Omega)) \tag{3.5a}$$

continuously. From here, it then readily follows by (3.4a), (3.4b); as $y_0 = 0$, $z_0 = 0$:

$$y(t) = \int_0^t e^{-\frac{c^2}{b}(t-\tau)} z(\tau) d\tau \in C([0, T]; L^2(\Omega)) \tag{3.5b}$$

$$y_t(t) = \int_0^t e^{-\frac{c^2}{b}(t-\tau)} z_t(\tau) d\tau \in C([0, T]; H^{-1}(\Omega)) \tag{3.5c}$$

continuously. We note for future reference that for the positive self-adjoint operator A in (2.2) we have

$$\mathcal{D}(A^{\frac{1}{2}}) \equiv H_0^1(\Omega), \qquad \text{hence } [\mathcal{D}(A^{\frac{1}{2}})]' \equiv H^{-1}(\Omega) \tag{3.6}$$

(norm-equivalence) [23, Eq. (3.1), p. 17]. Our goal is precisely to refine the regularity results in (3.5a)–(3.5c), i.e. by eventually dropping the H^1-regularity of g in time, via a limit approximation argument.

Step 2 To this end, we shall use critically representation formulas of solutions of second order hyperbolic equations with (presently) Dirichlet non-homogeneous terms, such as the z-problem (3.3a)–(3.3c), by use of cosine/sine operators. Such formulas—(3.8a), (3.9a) below—for **boundary non-homogeneous** second order PDEs were first introduced in [41] in 1977 and used extensively since, e.g. in [25–28], [23, Sect. 3], etc. The author most gratefully acknowledges to have learnt cosine operator theory (originally, for **boundary homogeneous** problems) from Sova [36], Kisynski [17–22], Da Prato–Giusti [6]. This theory for boundary non-homogeneous problems was later collected in Fattorini [10, 11]. For convenience and easy reference, we shall recall critical results as needed in our present development in Appendix A. The (negative self-adjoint) operator $-A : L^2(\Omega) \supset \mathcal{D}(A) \to L^2(\Omega)$ generates a s.c. (self-adjoint) cosine operator $\mathcal{C}(t)$ with corresponding sine operator $\mathcal{S}(t)x = \int_0^t \mathcal{C}(\tau)x d\tau$. As reported in Appendix A after [25, 26, 31, 41], the representation formulae for the solution of the Dirichlet-boundary problem (3.3a)–(3.3c), or its abstract version (3.2a) with henceforth

$$b = 1, \qquad c = 1, \qquad \text{w.l.o.g.} \tag{3.7}$$

are given by ($\mathcal{S}(0) = 0$) [23, p. 172]

$$z(t) = A \int_0^t \mathcal{S}(t-\tau) D g(\tau) d\tau + A \int_0^t \mathcal{S}(t-\tau) D g_t(\tau) d\tau \tag{3.8a}$$

$$= z^{(1)}(t) + z^{(2)}(t) \tag{3.8b}$$

$$z_t(t) = A\int_0^t \mathcal{C}(t-\tau)Dg(\tau)d\tau + A\int_0^t \mathcal{C}(t-\tau)Dg_t(\tau)d\tau \tag{3.9a}$$

$$= z_t^{(1)}(t) + z_t^{(2)}(t). \tag{3.9b}$$

Step 3 We now invoke the optimal regularity theory of the wave (or, generally, 2nd order hyperbolic equations), this time with Dirichlet boundary term $g \in L^2(0,T;L^2(\Gamma))$ and obtain, continuously (Theorem 2.3):

$$g \in L^2(0,T;L^2(\Gamma)) \Longrightarrow \begin{cases} z^{(1)}(t) = A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)d\tau \in C([0,T];L^2(\Omega)) & (3.10a) \\ z_t^{(1)}(t) = A\int_0^t \mathcal{C}(t-\tau)Dg(\tau)d\tau \in C([0,T];H^{-1}(\Omega)) & (3.10b) \end{cases}$$

Step 4 Next, let with $\epsilon > 0$ arbitrarily small

$$g \in L^2(0,T;L^2(\Gamma)) \cap C([0,\epsilon);L^2(\Gamma)), \quad g(0) = 0. \tag{3.11}$$

Integrating by parts ($\mathcal{S}(0) = 0$) we obtain from (3.8a), (3.8b)

$$z^{(2)}(t) = A\int_0^t \mathcal{S}(t-\tau)Dg_t(\tau)d\tau = \Big[A\mathcal{S}(t-\tau)Dg(\tau)\Big]_{\tau=0}^{\tau=t} - A\int_0^t \mathcal{C}(t-\tau)Dg(\tau)d\tau \tag{3.12}$$

$$= \cancel{A\mathcal{S}(0)Dg(t)} - \cancel{A\mathcal{S}(t)Dg(0)} - A\int_0^t \mathcal{C}(t-\tau)Dg(\tau)d\tau$$

or, as in (3.10b):

$$\begin{cases} z^{(2)}(t) = -A\int_0^t \mathcal{C}(t-\tau)Dg(\tau)d\tau = -z_t^{(1)}(t) \in C([0,T];H^{-1}(\Omega)) \\ \text{for } g \text{ as in (3.11), continuously.} \end{cases} \tag{3.13}$$

Thus, by (3.10a) and (3.13) used in (3.8b), we obtain

$$\begin{cases} z(t) = z^{(1)}(t) + z^{(2)}(t) \in C([0,T];H^{-1}(\Omega)) \\ \text{for } g \text{ as in (3.11), continuously.} \end{cases} \tag{3.14}$$

Step 5 Next, returning to the y-equation in (2.26b) [with (3.7) w.l.o.g.] and z given by (3.14) we obtain as in (3.5b), (3.5c)

$$y(t) = \int_0^t e^{-(t-\tau)} z(\tau) d\tau = \int_0^t e^{-(t-\tau)} z^{(1)}(\tau) d\tau + \int_0^t e^{-(t-\tau)} z^{(2)}(\tau) d\tau \quad (3.15a)$$

$$= y^{(1)}(t) + y^{(2)}(t). \quad (3.15b)$$

Returning to (3.10a) for $z^{(1)}$, we obtain conservatively

$$\begin{cases} y^{(1)}(t) = \int_0^t e^{-(t-\tau)} z^{(1)}(\tau) d\tau \in C([0,T]; L^2(\Omega)) \\ \text{for } g \in L^2(0,T; L^2(\Gamma)), \text{ continuously.} \end{cases} \quad (3.16)$$

Next, with $z^{(2)} = -z_t^{(1)}(t)$ as given by (3.13), we compute

$$y^{(2)}(t) = \int_0^t e^{-(t-\tau)} z^{(2)}(\tau) d\tau = -\int_0^t e^{-(t-\tau)} z_t^{(1)}(\tau) d\tau \quad (3.17)$$

$$= -\left[e^{-(t-\tau)} z^{(1)}(\tau) \right]_{\tau=0}^{\tau=t} + \int_0^t e^{-(t-\tau)} z^{(1)}(\tau) d\tau \quad (3.18)$$

$$\begin{cases} y^{(2)}(t) = -z^{(1)}(t) + \cancel{e^{-t} z^{(1)}(0)} + \int_0^t e^{-(t-\tau)} z^{(1)}(\tau) d\tau \in C([0,T]; L^2(\Omega)) \\ \text{for } g \text{ as in (3.11), continuously.} \end{cases} \quad (3.19)$$

as the expression of $z^{(2)} = -z_t^{(1)}(t)$ in (3.13) has such a constraint. As to the regularity noted in (3.19), we invoke (3.10a) for the term $z^{(1)}(t)$, while the same regularity holds true for the second convolution term, this time conservatively.

We conclude by (3.16) on $y^{(1)}$ and (3.19) on $y^{(2)}$ that

$$\begin{cases} y(t) = y^{(1)}(t) + y^{(2)}(t) = -z^{(1)}(t) + 2\int_0^t e^{-(t-\tau)} z^{(1)}(\tau) d\tau \in C([0,T]; L^2(\Omega)) \\ \text{for } g \text{ as in (3.11), continuously.} \end{cases} \quad (3.20)$$

Then (3.20) shows the first result in (2.14a) of Theorem 2.2.

Step 6 Next, with (2.25) = (3.4a), invoke (3.14) for z and (3.20) for y and obtain via (3.6):

$$\begin{cases} y_t = -\frac{c^2}{b} y + z \in C\left([0,T]; H^{-1}(\Omega) = [\mathcal{D}(A^{\frac{1}{2}})]' \right) \\ \text{for } g \text{ as in (3.11), continuously.} \end{cases} \quad (3.21)$$

Then (3.21) shows the second result in (2.14b) of Theorem 2.2.

Step 7 We finally need to establish the regularity of y_{tt}. This will be obtained via (3.4a) from

$$y_{tt} = -\frac{c^2}{b} y_t + z_t. \tag{3.22}$$

Thus, we need to establish the regularity of z_t, as that of y_t is given by (3.21). But $z_t = z_t^{(1)} + z_t^{(2)}$ by (3.9b), where the regularity of $z_t^{(1)} = -z^{(2)}(t)$ by (3.13) was already established in (3.10b).

Step 8 We seek the regularity of $z_t^{(2)}$ from its representation formula in (3.9a), (3.9b). We compute from (3.9b), with g as in (3.11)

$$z_t^{(2)}(t) = A \int_0^t \mathcal{C}(t-\tau) D g_t(\tau) d\tau \quad \text{(by parts, recalling (A.2))} \tag{3.23}$$

$$= \Big[A\mathcal{C}(t-\tau) D g(\tau) \Big]_{\tau=0}^{\tau=t} + A \int_0^t A\mathcal{S}(t-\tau) D g(\tau) d\tau \tag{3.24}$$

$$= ADg(t) - \xcancel{A\mathcal{C}(t)Dg(0)} + AA \int_0^t \mathcal{S}(t-\tau) D g(\tau) d\tau \tag{3.25}$$

$$= ADg(t) + Az^{(1)}(t) \tag{3.26}$$

recalling $\mathcal{C}(0) = I$ and (3.10a). Thus by (3.10a)

$$g \in L^2(0,T; L^2(\Gamma)) \Longrightarrow Az^{(1)}(t) = AA \int_0^t \mathcal{S}(t-\tau) D g(\tau) d\tau \in C([0,T]; [\mathcal{D}(A)]') \tag{3.27}$$

continuously. Moreover, by (2.21b) with any $\epsilon_1 > 0$

$$g \in L^2(0,T; L^2(\Gamma)) \Longrightarrow ADg = A^{\frac{3}{4}+\epsilon_1}(A^{\frac{1}{4}-\epsilon_1} D) g \in L^2(0,T; [\mathcal{D}(A^{\frac{3}{4}+\epsilon_1})]') \subset L^2(0,T; [\mathcal{D}(A)]') \tag{3.28a}$$

$$g \in C([0,T]; L^2(\Gamma)) \Longrightarrow ADg = A^{\frac{3}{4}+\epsilon_1}(A^{\frac{1}{4}-\epsilon_1} D) g \in C([0,T]; [\mathcal{D}(A^{\frac{3}{4}+\epsilon_1})]') \subset C([0,T]; [\mathcal{D}(A)]') \tag{3.28b}$$

continuously. By (3.26), (3.28a) and (3.27)

$$\begin{cases} g \in L^2(0,T; L^2(\Gamma)) \cap C([0,\epsilon); L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow z_t^{(2)}(t) = ADg + Az^{(1)} \in L^2(0,T; [\mathcal{D}(A)]') \tag{3.29a}$$

continuously, as well as by (3.26), (3.28b) and (3.27)

$$\begin{cases} g \in C([0,T]; L^2(\Gamma)) \Longrightarrow z_t^{(2)}(t) = ADg + Az^{(1)} \in C([0,T]; [\mathcal{D}(A)]') \\ g(0) = 0, \end{cases} \tag{3.29b}$$

continuously.

Step 9 In turn, combining (3.10b) for $z_t^{(1)}$ with (3.29a) for $z_t^{(2)}$ in (3.9b), we conclude

$$\begin{cases} g \in L^2(0,T; L^2(\Gamma)) \cap C([0,\epsilon); L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow z_t = z_t^{(1)} + z_t^{(2)} \in L^2(0,T; [\mathcal{D}(A)]') \tag{3.30a}$$

continuously, as well as by (3.29b) on $z_t^{(2)}$ this time

$$\begin{cases} g \in C([0,T]; L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow z_t = z_t^{(1)} + z_t^{(2)} \in C([0,T]; [\mathcal{D}(A)]') \tag{3.30b}$$

continuously. Finally, from (3.22), recalling the regularity of y_t in (2.14b) and the regularity of z_t in (3.30a), we obtain

$$\begin{cases} g \in L^2(0,T; L^2(\Gamma)) \cap C([0,\epsilon); L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow y_{tt} = -\frac{c^2}{b} y_t + z_t \in L^2(0,T; [\mathcal{D}(A)]') \tag{3.31a}$$

continuously, as well as by (2.14b) and (3.30b)

$$\begin{cases} g \in C([0,T]; L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow y_{tt} = -\frac{c^2}{b} y_t + z_t \in C([0,T]; [\mathcal{D}(A)]') \tag{3.31b}$$

continuously. Then (3.31b) shows the result in (2.14c) and (2.14e) of Theorem 2.2.

4 Proof of Theorem 2.2: Regularity of the Boundary Trace $\left.\frac{\partial y}{\partial \nu}\right|_{\Sigma}$

In this section we shall establish the boundary regularity (2.14d) of Theorem 2.2. It is here repeated for convenience.

Theorem 4.1 *With reference to problem (2.1a)–(2.1c), we have*

$$\begin{cases} g \in L^2(0,T;L^2(\Gamma)) \cap C([0,\epsilon);L^2(\Gamma)) \\ g(0) = 0 \end{cases} \Longrightarrow \left.\frac{\partial y}{\partial \nu}\right|_{\Sigma} \in H^{-1}(\Sigma) \tag{4.1}$$

continuously. Here $H^{-1}(\Sigma) =$ *dual of* $\{h \in H_0^1(\Sigma)\}$ *i.e. with* $h(\cdot,0) = 0$ *and* $h(\cdot,T) = 0$ *on* Γ *(but actually,* $h(\cdot,T) = 0$ *is not needed.)*

Proof **Step 1** We return to the solution representation formula (3.8b) complemented by (3.13), with g as in (3.11), in particular $g(0) = 0$:

$$z(t) = z^{(1)}(t) + z^{(2)}(t) = z^{(1)}(t) - z_t^{(1)}(t) \tag{4.2}$$

$$= A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)d\tau - A\int_0^t \mathcal{C}(t-\tau)Dg_t(\tau)d\tau. \tag{4.3}$$

recalling (3.10a), (3.10b). We next invoke critically the boundary regularity results in [23, Eq. (2.14) of Theorem 2.3, p. 153], recalled in Theorem 2.3, Eq. (2.16) of Remark 2.1 and also (A.5):

$$g \in L^2(0,T;L^2(\Gamma)) \Longrightarrow \left.\frac{\partial z^{(1)}}{\partial \nu}\right|_{\Sigma} = -D^*AA\int_0^t \mathcal{S}(t-\tau)Dg(\tau)d\tau \in H^{-1}(\Sigma). \tag{4.4}$$

Step 2 We now return to (3.20), with g as in (3.11) (and $b = 1$, $c = 1$, w.l.o.g. as in (3.7))

$$y(t) = -z^{(1)}(t) + 2\int_0^t e^{-(t-\tau)}z^{(1)}(\tau)d\tau \tag{4.5}$$

where then, recalling (4.4), we obtain

$$\left.\frac{\partial y}{\partial \nu}\right|_{\Sigma} = -\left.\frac{\partial z^{(1)}}{\partial \nu}\right|_{\Sigma} + 2\int_0^t e^{-(t-\tau)}\frac{\partial z^{(1)}}{\partial \nu}(\tau)d\tau \in H^{-1}(\Sigma) \tag{4.6}$$

for g as in (3.11), continuously. Then (4.6) establishes Theorem 4.1. □

5 A Boundary Trace Result for the *u*-Problem (2.3)

The optimal interior (and boundary) regularity results for second order hyperbolic equations with Dirichlet boundary terms were proven directly in [25, 26] and by duality in [23] via Theorem 2.4. This latter paper then set up the road map to obtain optimal interior (and boundary) regularity results for a variety of other dynamics such as: Schrödinger equations with Dirichlet boundary term, plate-like equations with certain boundary terms, all by duality. Dual results (boundary traces) are of interest in themselves. In this paper, we shall also show an independent dual result, which will be used in Sect. 6 in reproving (essentially) the main Theorem 2.2. The dual result will be given in Sect. 6. In this section, we provide an independent boundary trace result for a problem which is not the dual problem of the original non-homogeneous problem (2.1a)–(2.1c), but which is very closely related to the actual dual problem. This will be presented in Sect. 6. The independent boundary trace results of the present and next sections for the SMGTJ mixed problem will be critically based on a duality result of second order hyperbolic equation, namely Theorem 2.4. In the present section, the object of our interest is problem (2.3), re-written in PDE-form as

$$\begin{cases} u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t = 0 & \text{in } Q = (0, T] \times \Omega \quad (5.1a) \\ u\big|_{t=T} = u_0; \quad u_t\big|_{t=T} = u_1; \quad u_{tt}\big|_{t=T} = u_2 & \text{in } \Omega \quad (5.1b) \\ u\big|_{\Sigma} = 0 & \text{in } \Sigma = (0, T] \times \Gamma \quad (5.1c) \end{cases}$$

$$\text{abstractly} \qquad u_{ttt} + \alpha u_{tt} + c^2 A u + b A u_t = 0 \tag{5.1d}$$

with I.C. at $t = T$ (i.e. backward in time). We shall select the I.C. in the space U_3 defined in (2.5c)

$$\{u_0, u_1, u_2\} \in U_3 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times H. \tag{5.2}$$

Accordingly, in view of Theorem 2.1, we have

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} \in U_3 \to \begin{bmatrix} u(t) \\ u_t(t) \\ u_{tt}(t) \end{bmatrix} = e^{G(T-t)} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} \in C([0, T]; U_3) \tag{5.3}$$

continuously, where e^{Gt} is a s.c. group on U_3 with infinitesimal generator G defined in (2.4). Explicitly,

$$\begin{aligned} [u_0, u_1, u_2] \in U_3 = \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times L^2(\Omega) \Longrightarrow \\ [u, u_t, u_{tt}] \in C([0, T]; \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times L^2(\Omega)) \\ = C([0, T]; [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \times L^2(\Omega)), \end{aligned} \tag{5.4}$$

continuously, to be invoked repeatedly below. The key boundary trace result of the present section is the following Theorem.

Theorem 5.1 *With reference to problem (5.1a)–(5.1b), (5.2), the following trace estimate holds true*

$$\int_0^T \int_\Gamma \left(\frac{\partial u_t}{\partial \nu}\right)^2 d\Sigma = \mathcal{O}_T\left(\left\|\{u_0, u_1, u_2\}\right\|_{U_3}^2\right) \tag{5.5}$$

where $\mathcal{O}$ denotes a constant depending on T, $T > 0$ arbitrary, as well as the equation coefficients.

Remark 5.1 Estimate (5.5) is an illustration of a sharp, so called hidden regularity result, in line with the phenomenon first discovered in [23, 26] for second order hyperbolic equations, and reported (for the wave equation) in Theorem 2.4. The term $u_t \in C([0,T]; H_0^1(\Omega))$ optimally in the interior, and yet $\dfrac{\partial u_t}{\partial \nu} \in L^2(0,T; L^2(\Gamma))$, i.e. it possesses '$\frac{1}{2}$' space regularity better than a formal application of trace theory by reducing the time regularity from C to L^2.

Proof of Theorem 5.1. Step 1 Rewrite Eq. (5.1a) as in (2.7)

$$(u_t + \alpha u)_{tt} - b\Delta\left(\frac{c^2}{b}u + u_t\right) = 0 \quad \text{in } Q \tag{5.6}$$

and introduce now the variable ξ in (2.8) recalling (5.4)

$$\xi = \alpha u + u_t \in C([0,T]; H_0^1(\Omega)), \text{ so that } \frac{c^2}{b}u + u_t = \xi - \gamma u, \quad \gamma = \alpha - \frac{c^2}{b}. \tag{5.7}$$

Rewrite problem (5.1a)–(5.1c) accordingly as

$$\begin{cases} \xi_{tt} - b\Delta\xi + b\gamma\Delta u = 0 & \text{in } Q & (5.8a)\\ \xi\big|_{t=T} = \xi_0 = \alpha u_0 + u_1; \quad \xi_t\big|_{t=T} = \xi_1 = \alpha u_1 + u_2 & \text{in } \Omega & (5.8b)\\ \xi\big|_\Sigma \equiv 0 & \text{in } \Sigma & (5.8c)\end{cases}$$

$$\text{abstractly} \qquad \xi_{tt} + bA\xi - b\gamma Au = 0 \tag{5.8d}$$

with I.C. at $t = T$. For problem (5.8a)–(5.8c), the key sharp/hidden regularity result is

Theorem 5.2 *With reference to problem (5.8a)–(5.8c), we have*

$$\int_0^T \int_\Gamma \left(\frac{\partial \xi}{\partial \nu}\right)^2 d\Sigma = \mathcal{O}_T\left(\left\|\{u_0, u_1, u_2\}\right\|_{U_3}^2\right), \quad U_3 = \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times L^2(\Omega) \tag{5.9}$$

where $\mathcal{O}_T$ denotes a constant depending on T, $T > 0$ arbitrary.

Remark 5.2 Estimate (5.9) is again an illustration of a sharp, 'hidden' regularity result, as $\xi \in C([0,T]; H_0^1(\Omega))$ optimally in the interior, by (5.7), and thus (5.9) does not follow by a formal application of trace theory.

Proof of Theorem 5.2 The proof is critically based on the approach and results of [23] reproduced also in [31, Sect. 10.5.10, p. 958]. Let, as usual, $h(x) = [h_1(x), \dots, h_d(x)] \in C^2(\overline{\Omega})$ be a vector field, such that $h\big|_\Gamma = \nu =$ outward unit normal vector. Let $H(x) = \left[\dfrac{\partial h_i}{\partial x_j}\right], i, j = 1, \dots, d$ be the usual Jacobian matrix. The multiplier method, with multiplier $h \cdot \nabla\xi$, applied to Eq. (5.8a), say with $b = 1$ w.l.o.g., gives the usual identity [31, Eq. (10.5.10.5), p. 959]

$$\begin{aligned}
&\int_\Sigma \frac{\partial \xi}{\partial \nu} h \cdot \nabla\xi d\Sigma + \frac{1}{2}\int_\Sigma \xi_t^2 h\cdot\nu d\Sigma - \frac{1}{2}\int_\Sigma |\nabla\xi|^2 h\cdot\nu d\Sigma \\
&= \int_Q H\nabla\xi\cdot\nabla\xi dQ + \frac{1}{2}\int_Q \xi_t^2 \,\mathrm{div} h\, dQ - \frac{1}{2}\int_Q |\nabla\xi|^2 \,\mathrm{div} h\, dQ \\
&\qquad + b\gamma\int_Q \Delta u h\cdot\nabla\xi dQ + \left[\left(\xi_t,\ h\cdot\nabla\xi\right)_\Omega\right]_0^T,
\end{aligned} \tag{5.10}$$

where from (5.4), (5.7), (5.8c)

$$\begin{cases}
\xi_t \equiv 0 \text{ on } \Sigma; \quad h\cdot\nabla\xi = \dfrac{\partial\xi}{\partial\nu} h\cdot\nu = \dfrac{\partial\xi}{\partial\nu} \text{ on } \Sigma; \quad |\nabla\xi|^2 = \left(\dfrac{\partial\xi}{\partial\nu}\right)^2 \text{ on } \Sigma \\
\xi = \alpha u + u_t \in C([0,T]; H_0^1(\Omega)); \quad |\nabla\xi| \in C([0,T]; L^2(\Omega)) \\
\xi_t \in C([0,T]; L^2(\Omega)); \quad \Delta u \in C([0,T]; L^2(\Omega))
\end{cases} \tag{5.11}$$

continuously on $\{u_0, u_1, u_2\} \in U_3$. Thus, as usual via (5.11), identity (5.10) reduces to

$$\int_0^T\int_\Gamma \left(\frac{\partial\xi}{\partial\nu}\right)^2 d\Sigma = \mathcal{O}_T\left(\left\|\{u_0, u_1, u_2\}\right\|_{U_3}^2\right) \tag{5.12}$$

and Theorem 5.2 is established. In effect, we could get estimate (5.12) at once by applying Theorem 2.4 directly, with Δu as given by (5.11). The above computations based on (5.10) give a flavor of the proof of Theorem 2.4 given in [23, Lemma 2.1, p. 154].

Step 2 From (5.7), we estimate

$$\begin{cases}
\left|\dfrac{\partial u_t}{\partial\nu}\right| - |\alpha|\left|\dfrac{\partial u}{\partial\nu}\right| \le \left|\dfrac{\partial u_t}{\partial\nu} + \alpha\dfrac{\partial u}{\partial\nu}\right| = \left|\dfrac{\partial\xi}{\partial\nu}\right| & (5.13a) \\
\dfrac{1}{2}\left|\dfrac{\partial u_t}{\partial\nu}\right|^2 - \alpha^2\left|\dfrac{\partial u}{\partial\nu}\right|^2 \le \left|\dfrac{\partial\xi}{\partial\nu}\right|^2 & (5.13b)
\end{cases}$$

which used in (5.12) yields

$$\int_0^T\int_\Gamma\left(\frac{\partial u_t}{\partial\nu}\right)^2 d\Sigma \le \mathcal{O}\left(\left\|\{u_0,u_1,u_2\}\right\|_{U_3}^2\right)+2\alpha^2\int_0^T\int_\Gamma\left(\frac{\partial u}{\partial\nu}\right)^2 d\Sigma. \tag{5.14}$$

Step 3 Via $u\in C([0,T];H^2(\Omega))$ from (5.4), we obtain directly via trace theory

$$\int_0^T\int_\Gamma\left(\frac{\partial u}{\partial\nu}\right)^2 d\Sigma \le C_T\left(\left\|\{u_0,u_1,u_2\}\right\|_{U_3}^2\right) \tag{5.15}$$

which inserted in (5.14) yields

$$\int_0^T\int_\Gamma\left(\frac{\partial u_t}{\partial\nu}\right)^2 d\Sigma = \mathcal{O}_T\left(\left\|\{u_0,u_1,u_2\}\right\|_{U_3}^2\right) \tag{5.16}$$

as desired, and Theorem 5.1 is proved.

6 A Dual Result of the Non-homogeneous Dirichlet Problem (2.1a)–(2.1c)

It turns out that the dual problem of the boundary non-homogeneous Dirichlet problem (2.1a)–(2.1c) is actually the problem

$$\begin{cases} v_{ttt}-\alpha v_{tt}+c^2\Delta v-b\Delta v_t=0 & \text{in } Q & (6.1a)\\ v\big|_{t=T}=v_0;\quad v_t\big|_{t=T}=v_1;\quad v_{tt}\big|_{t=T}=v_2 & \text{in } \Omega & (6.1b)\\ v\big|_\Sigma=0 & \text{in } \Sigma & (6.1c)\end{cases}$$

$$\text{abstractly}\qquad v_{ttt}-\alpha v_{tt}-c^2Av+bAv_t=0 \tag{6.2}$$

along with I.C. at $t=T$. This is very closely related to problem (5.1a)–(5.1c), or its abstract version (5.1d), as the present considerations attest. Write (6.1a) as in (2.7) = (5.6).

$$(v_t-\alpha v)_{tt}-b\Delta\left(-\frac{c^2}{b}v+v_t\right)=0 \quad \text{in } Q. \tag{6.3}$$

Set, similarly to (2.8)

$$\eta=v_t-\alpha v \qquad \text{so that } \zeta=-\frac{c^2}{b}v+v_t=\eta+\gamma v,\quad \gamma=\alpha-\frac{c^2}{b} \tag{6.4}$$

and then problem (6.1a)–(6.1c) reduces to

$$\begin{cases} \eta_{tt} - b\Delta\eta - b\gamma\Delta v = 0 & \text{in } Q \quad (6.5a) \\ \eta\big|_{t=T} = \eta_0 = v_1 - \alpha v_0; \quad \eta_t\big|_{t=T} = \eta_1 = v_2 - \alpha v_1 & \text{in } \Omega \quad (6.5b) \\ \eta\big|_\Sigma \equiv 0 & \text{in } \Sigma \quad (6.5c) \end{cases}$$

$$\text{abstractly} \qquad \eta_{tt} + bA\eta + b\gamma A v = 0 \tag{6.5d}$$

with I.C. at $t = T$, to be compared with the ξ-problem (5.8a)–(5.8c) and its abstract version (5.8d). Moreover, in terms of the variable ζ in (6.4), we have, with $\gamma = \alpha - \frac{c^2}{b}$:

$$\zeta = -\frac{c^2}{b}v + v_t = (-\alpha v + v_t) + \gamma v = \eta + \gamma v \tag{6.6}$$

$$(v_t - \alpha v)_{tt} = \zeta_{tt} - \gamma v_{tt} = \zeta_{tt} - \gamma\left(\zeta - \frac{c^2}{b}v\right)_t = \zeta_{tt} - \gamma\zeta_t - \gamma\frac{c^2}{b}v_t \tag{6.7}$$

$$= \zeta_{tt} - \gamma\zeta_t - \gamma\frac{c^2}{b}\zeta - \gamma\left(\frac{c^2}{b}\right)^2 v. \tag{6.8}$$

Hence (6.3) $(v_t - \alpha v)_{tt} + bA\left(-\frac{c^2}{b}v + v_t\right) = 0$ is rewritten by (6.6)–(6.8) as

$$\begin{cases} \zeta_{tt} = -bA\zeta + \gamma\zeta_t + \gamma\frac{c^2}{b}\zeta + \gamma\left(\frac{c^2}{b}\right)^2 v & (6.9a) \\ v_t = \frac{c^2}{b}v + \zeta, & (6.9b) \end{cases}$$

to be compared with the z-problem (2.12a), (2.12b). The ζ-problem (6.9a), (6.9b) and the z-problem in (2.12a), (2.12b) differ only by innocuous changes of signs. Thus, the well-posedness theory developed in [34] and recalled in Theorem 2.1 for the z-problem in (2.12a), (2.12b) applies also to the ζ-problem (6.9a), (6.9b) over a finite time interval. In particular, the implication on the original v-problem (6.1a)–(6.1c)—counterpart of the u-problem in Theorem 2.1 is the following Theorem.

Theorem 6.1 *With reference to problem (6.1a)–(6.1c) we have*

$$\{v_0, v_1, v_2\} \in U_3 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times H \Longrightarrow \{v, v_t, v_{tt}\} \in C([0, T]; U_3) \tag{6.10}$$

as a s.c. group on U_3.

Returning to the v-problem (6.1a)–(6.1c), we obtain the following Theorem.

Theorem 6.2 *With reference to the v-problem (6.1a)–(6.1c), abstractly (6.2), we have*

$$\int_0^T \int_\Gamma \left(\frac{\partial v_t}{\partial \nu}\right)^2 d\Sigma = \mathcal{O}_T\left(\left\|\left\{v_0, v_1, v_2\right\}\right\|^2_{U_3}\right), \quad U_3 = \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times H. \tag{6.11}$$

The proof is exactly the same as that of Theorem 5.1. The first step is showing

Theorem 6.3 *With reference to the η-problem (6.5a)–(6.5d), we have*

$$\int_0^T \int_\Gamma \left(\frac{\partial \eta}{\partial \nu}\right)^2 d\Sigma = \mathcal{O}\left(\left\|\left\{v_0, v_1, v_2\right\}\right\|^2_{U_3}\right). \tag{6.12}$$

The proof of Theorem 6.3 is exactly the same as that of Theorem 5.2 (the difference in sign $+b\gamma$ in (5.8a) versus $-b\gamma$ in (6.5a) is irrelevant, under the common property $\Delta u, \Delta v \in C([0, T]; H)$. Also, apply at once Theorem 2.4.

Next, by duality on the trace result in Theorem 6.2 for the v-problem (6.1a)–(6.1c), we shall re-obtain (in a slightly weaker form) the basic interior regularity result of Theorem 2.2 for $\left\{y, y_t, y_{tt}\right\}$. While the proof of Theorem 2.2 was 'direct', the proof of Theorem 6.4 is 'by duality'.

Theorem 6.4 *With reference to the Dirichlet problem (2.1a)–(2.1c),we have*

$$\begin{cases} g \in C([0, T]; L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow \left\{y, y_t, y_{tt}\right\} \in C([0, T]; H \times [\mathcal{D}(A^{\frac{1}{2}})]' \times [\mathcal{D}(A)]') \tag{6.13}$$

continuously.

Proof **Step 1** We shall first establish the following

Proposition 6.5 *With reference to the Dirichlet problem (2.1a)–(2.1c),we have, for each* $0 < t \le T$*:*

$$\begin{cases} g \in C([0, T]; L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow \left\{y(t), y_t(t), y_{tt}(t)\right\} \in H \times [\mathcal{D}(A^{\frac{1}{2}})]' \times [\mathcal{D}(A)]' \tag{6.14a}$$

$$\Longrightarrow \int_0^t \left[y(\tau), y_t(\tau), y_{tt}(\tau)\right] d\tau \in H \times [\mathcal{D}(A^{\frac{1}{2}})]' \times [\mathcal{D}(A)]' \tag{6.14b}$$

continuously.

Proof of Proposition 6.5. **Step (i)** By Theorem 6.1 and 6.2, we have

$$\{v_0, v_1, v_2\} \in U_3 = \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times H \tag{6.15a}$$

$$\Longrightarrow \begin{cases} \{v, v_t, v_{tt}\} \in C([0,T]; U_3 = \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times H) \\ \text{AND} \\ \dfrac{\partial v_t}{\partial \nu} \in L^2(0,T; L^2(\Gamma)) \end{cases} \tag{6.15b}$$

continuously.

Step (ii) We now invoke the duality identity (B.4) in Appendix B, which we rewrite here for convenience for a generic t, $0 < t \le T$:

$$\begin{aligned} \langle y_{tt}(t) + \alpha y_t(t), v_0\rangle - \langle y_t(t) + \alpha y(t), v_1\rangle + \langle y(t), v_2\rangle \\ -b\,\langle y(t), \Delta v_0\rangle \\ = -\left\langle c^2 g + b g_t, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(0,t;L^2(\Gamma))}. \end{aligned} \tag{6.16}$$

With $g \in L^2(0,t; L^2(\Gamma))$ and under (6.15a), (6.15b) and hence $\dfrac{\partial v}{\partial \nu} \in C([0,t]; H^{\frac{1}{2}}(\Gamma))$, we have regarding the first term on the RHS of (6.16):

$$\left\langle g, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(0,t;L^2(\Gamma))} = \int_0^t \int_\Gamma g \frac{\partial v}{\partial \nu} d\Gamma dt < \infty. \tag{6.17}$$

Step (iii) Under hypothesis (6.14a) for g and under (6.15a), (6.15b) for the adjoint v-problem, we compute the last term on the RHS of (6.16) by parts as follows, for $0 < t \le T$

$$\int_0^t \left(g_t(\tau), \frac{\partial v(\tau)}{\partial \nu}\right)_{L^2(\Gamma)} d\tau = \left[\left(g(\tau), \frac{\partial v(\tau)}{\partial \nu}\right)_{L^2(\Gamma)}\right]_{\tau=0}^{\tau=t} - \int_0^t \left(g(\tau), \frac{\partial v_t(\tau)}{\partial \nu}\right)_{L^2(\Gamma)} d\tau \tag{6.18}$$

$$= \left(g(t), \frac{\partial v(t)}{\partial \nu}\right)_{L^2(\Gamma)} - \cancel{\left(g(0), \frac{\partial v(0)}{\partial \nu}\right)_{L^2(\Gamma)}} - \int_0^t \left(g(\tau), \frac{\partial v_t(\tau)}{\partial \nu}\right)_{L^2(\Gamma)} d\tau \tag{6.19}$$

With $g \in C([0,T]; L^2(\Gamma))$ as in (6.14a) and $\frac{\partial v}{\partial \nu} \in C([0,t]; H^{\frac{1}{2}}(\Gamma))$, we have that the first term on the RHS of (6.19) is well-defined (conservatively). Notice that it can also be re-written via (A.5) as

$$\left(g(t), \frac{\partial v(t)}{\partial \nu}\right)_{L^2(\Gamma)} = \left(g(t), -D^* A v(t)\right)_{L^2(\Gamma)} \\ = -\left(ADg(t), v(t)\right)_{L^2(\Gamma)} \text{ (well-defined)} \tag{6.20}$$

with $v \in C([0,T]; \mathcal{D}(A))$ by (6.10) and $ADg \in C([0,T]; [\mathcal{D}(A)]')$ conservatively by (2.21b) and (6.14a). Then by (6.19), (6.20)

$$\left\langle g_t, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(0,t;L^2(\Gamma))} = -\left(ADg(t), v(t)\right)_{L^2(\Omega)} - \left\langle g, \frac{\partial v_t}{\partial \nu}\right\rangle_{L^2(0,t;L^2(\Gamma))} \tag{6.21}$$

(well-defined, by (6.20) and (6.15b)),

continuously with respect to g as in (6.14a) and $\{v_0, v_1, v_2\}$ as in (6.15a), (6.15b). Thus, the RHS of identity (6.17) is well-defined for g as in (6.14a) and $\{v_0, v_1, v_2\} \in U_3$ as in (6.15a), (6.15b), critically because of Theorem 6.2. Next we turn to the LHS of identity (6.16) and then the argument displayed in (B.15) of Appendix B, yields (6.14a), for any $0 < t \le T$.

Specifically, for each t, $0 < t \le T$ via duality pairing:

$$\left.\begin{aligned} &v_2 \in H \rightarrow y(t) \in H && (6.22)\\ &v_t \in \mathcal{D}(A^{\frac{1}{2}}) \rightarrow y_t(t) + \alpha y(t) \in [\mathcal{D}(A^{\frac{1}{2}})]' && (6.23)\end{aligned}\right\} \Longrightarrow y_t(t) \in \left[\mathcal{D}(A^{\frac{1}{2}})\right]'$$

$$v_0 \in \mathcal{D}(A) \rightarrow y_{tt}(t) + \alpha y_t(t) \in [\mathcal{D}(A)]' \Longrightarrow y_{tt}(t) \in [\mathcal{D}(A)]' \tag{6.24}$$

Thus, (6.23)–(6.24) imply $\{y(t), y_t(t), y_{tt}(t)\} \in H \times [\mathcal{D}(A^{\frac{1}{2}})]' \times [\mathcal{D}(A)]'$. The same argument then gives

$$\int_0^t Y(\tau)d\tau \in H \times [\mathcal{D}(A^{\frac{1}{2}})]' \times [\mathcal{D}(A)]', \text{ for any } 0 < t \le T, \tag{6.25}$$

where we have set $Y(t) = [y(t), y_t(t), y_{tt}(t)]$. Proposition 6.5 is proved.

Step 2 In light of (6.14b)=(6.25), we are in the same situation as in [23, Corollary 3.2, p. 173], whereby then the map

$$t \rightarrow \int_0^t [y(\tau), y_t(\tau), y_{tt}(\tau)]d\tau \tag{6.26}$$

is continuous $[0,T] \rightarrow H \times [\mathcal{D}(A^{\frac{1}{2}})]' \times [\mathcal{D}(A)]'$. Theorem 6.4 is proved. □

Part B: Neumann Case

7 Linear Third Order SMGTJ-Equation with Non-homogeneous Neumann Boundary Term

Likewise we shall take $\tau = 1$ in (1.1) w.l.o.g. If the linear third order equation (1.1) is written in terms of scalar velocity potential (where pressure $= k\dfrac{d}{dt}$ (velocity potential)), then the Neumann non-homogeneous boundary terms are appropriate [14]. In this part B, we shall consider the following mixed problem in the unknown $y(t, x)$:

$$\begin{cases} y_{ttt} + \alpha y_{tt} - c^2\Delta y - b\Delta y_t = 0 & \text{in } Q = (0, T] \times \Omega \quad (7.1a) \\ y\big|_{t=0} = y_0; \quad y_t\big|_{t=0} = y_1; \quad y_{tt}\big|_{t=0} = y_2 & \text{in } \Omega \quad (7.1b) \\ \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma} = g & \text{in } \Sigma = (0, T] \times \Gamma \quad (7.1c) \end{cases}$$

7.1 *Case $g = 0$.*

This case is the perfect counterpart of Sect. 2.1 in the Dirichlet case, under the present setting whereby $H = L^2(\Omega)/\mathbb{R}$ and $-A$ is now the Neumann Laplacian

$$Af = -\Delta f, \quad \mathcal{D}(A) = \left\{ h \in H^2(\Omega) : \frac{\partial h}{\partial \nu} = 0 \text{ on } \Gamma \right\}. \tag{7.2}$$

A is likewise strictly positive self-adjoint on H so that the fractional powers A^θ, $0 < \theta < 1$, are well defined on H. Thus, Equation (2.3) through (2.12a), (2.12b) still hold true now including Theorem 2.1, with $-A$ the Neumann Laplacian in (7.2), rather than the Dirichlet Laplacian as in Sect. 2.1.

7.2 *Case $y_0 = 0$, $y_1 = 0$, $y_2 = 0$, $g \neq 0$*

In this case, we seek to obtain sharp regularity of the map

$$g \to \left\{ y, y_t, y_{tt}; y\big|_{\Sigma} \right\}$$

from the Neumann boundary datum g to the interior solution $\{y, y_t, y_{tt}\}$ and the Dirichlet boundary trace $y\big|_{\Sigma}$. As in Part A, our analysis of the above question will critically fall on the regularity of the wave equation (or a more generally of second order hyperbolic equations) under Neumann boundary control. Unlike the Dirichlet boundary control case invoked critically in Part A, the Neumann boundary control case has two peculiarities: (i) it is dimension dependent (the case $d = 1$ is markedly more regular [31, Sect. 9.8.4, Theorem 9.8.4.1, p. 859]; (ii) it is geometry dependent [29, 30, 38] (reported in [31, p. 739]). Accordingly, set throughout Part B (Neumann)

$$\widehat{\alpha} = \beta = \frac{2}{3} \qquad \text{for a general sufficiently smooth domain} \Omega \text{ in } \mathbb{R}^d,\, d \geq 2 \tag{7.3a}$$

$$\widehat{\alpha} = \beta = \frac{3}{4} \qquad \text{for a parallelepiped in } \mathbb{R}^d,\, d \geq 2 \tag{7.3b}$$

In reference [30] the parameter $\widehat{\alpha}$ refers to interior regularity while β refer to boundary regularity. As we shall invoke a number of results from [30], clarity requires that we keep both of them.

Theorem 7.1 *With reference to problem (7.1a)–(7.1c), we have the following regularity results*

$$g \in L^2(0,T; L^2(\Gamma)) \cap C([0,\epsilon); L^2(\Gamma)), \quad g(0) = 0,\ \epsilon > 0 \textit{ small}$$

$$\Longrightarrow \begin{cases} y \in C([0,T]; H^{\widehat{\alpha}}(\Omega) = \mathcal{D}(A^{\frac{\widehat{\alpha}}{2}})) & (7.4a) \\ y_t \in C([0,T]; H^{\widehat{\alpha}-1}(\Omega) = [\mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})]'), & (7.4b) \\ y_{tt} \in L^2(0,T; H^{\widehat{\alpha}-2}(\Omega) = [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]') & (7.4c) \\ y\big|_{\Sigma} \in H^{2\widehat{\alpha}-1}(\Sigma)). & (7.4d) \end{cases}$$

$$\begin{cases} g \in C([0,T]; L^2(\Gamma)) \\ g(0) = 0 \end{cases} \Longrightarrow y_{tt} \in C([0,T]; H^{\widehat{\alpha}-2}(\Omega) \equiv [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]'). \tag{7.4e}$$

Remark 7.1 The result of Theorem 7.1 should be compared with the following results for general second order hyperbolic equations, even with space-variable coefficients, which we however report only for the canonical wave equation. Thus, let Ω be an open bounded domain in $\mathbb{R}^d$, $d \geq 2$, with smooth boundary Γ. (For $d = 1$, sharper regularity results hold true [31, Sect. 9.8.4, Theorem 9.8.4.1, p. 859])

Theorem 7.2 ([29, 30, 38]) *Consider the mixed problem*

$$\begin{cases} w_{tt} = \Delta w & \text{in } Q = (0, T] \times \Omega & (7.5a) \\ w\big|_{t=0} = 0; \quad w_t\big|_{t=0} = 0 & \text{in } \Omega & (7.5b) \\ \dfrac{\partial w}{\partial \nu}\Big|_{\Sigma} = g & \text{in } \Sigma = (0, T] \times \Gamma. & (7.5c) \end{cases}$$

Recall the constants $\widehat{\alpha}$, β from (7.3a), (7.3b). *With reference to problem (7.1a)–(7.1c), we have the following interior regularity*

$$g \in L^2(0, T; L^2(\Gamma)) \Longrightarrow \begin{cases} w \in C([0, T]; H^{\widehat{\alpha}}(\Omega) = \mathcal{D}(A^{\frac{\widehat{\alpha}}{2}})) & (7.6a) \\ w_t \in C([0, T]; H^{\widehat{\alpha}-1}(\Omega) = [\mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})]') & (7.6b) \end{cases}$$

duality w.r.t. $H = L^2(\Omega)/\mathbb{R}$, as well as the independent boundary (trace) regularity

$$g \in L^2(0, T; L^2(\Gamma)) \Longrightarrow w\big|_{\Sigma} \in H^{2\widehat{\alpha}-1}(\Sigma) \tag{7.7}$$

continuously.

Next, consider the following problem, dual of problem (7.5a)–(7.5c)

$$\begin{cases} \phi_{tt} = \Delta \phi + f & \text{in } Q & (7.8a) \\ \phi\big|_{t=T} = \phi_0; \quad \phi_t\big|_{t=T} = \phi_1 & \text{in } \Omega & (7.8b) \\ \dfrac{\partial \phi}{\partial \nu}\Big|_{\Sigma} = 0 & \text{in } \Sigma. & (7.8c) \end{cases}$$

Theorem 7.3 ([30, Theorem B(1), p. 118, proved in (2.10) Theorem 2.0, p. 123]) *With reference to problem (7.8a)–(7.8c), let $\phi_0 = \phi_1 = 0$, $f \in L^2(Q)$. Then, with β in* (7.3a), (7.3b)*:*

$$\|\phi\big|_{\Sigma}\|_{H^{\beta}(\Sigma)} = \mathcal{O}_T\left(\|f\|^2_{L^2(Q)}\right). \tag{7.9}$$

Theorem 7.4 ([30, Theorem E with $\theta = 0$, p. 119]) *With reference to problem (7.8a)–(7.8c), let*

$$\{\phi_0, \phi_1\} \in H^{1-\widehat{\alpha}}(\Omega) \times [H^{\widehat{\alpha}}(\Omega)]', \quad f \in [H^{\widehat{\alpha}}(Q)]'. \tag{7.10a}$$

Then we have the following (sharp hidden) trace regularity

$$\|\phi\big|_{\Gamma}\|_{L^2(\Sigma)} = \mathcal{O}\left(\|\{\phi_0, \phi_1\}\|^2_{H^{1-\widehat{\alpha}}(\Omega)\times[H^{\widehat{\alpha}}(\Omega)]'} + \|f\|_{[H^{\widehat{\alpha}}(Q)]'} \cdot\right), \tag{7.10b}$$

Proof **Step 1** The case $\phi_0 = \phi_1 = 0$ is contained in [30, Theorem E with $\theta = 0$, p. 119].

Step 2 Let now $f \equiv 0$. This case is not explicitly contained in [30], but it can be deduced by interpolation from two results in [30].

First we have

$$\{\phi_0, \phi_1\} \in H^1(\Omega) \times L^2(\Omega) \Longrightarrow \phi\big|_\Sigma \in H^\beta(\Sigma) \tag{7.11}$$

continuously. This is [30, Theorem C(1), p. 118]. It is proved as [30, Theorem 7.1, (7.1), (7.2), p. 158]. Next we have

$$\{\phi_0, \phi_1\} \in L^2(\Omega) \times [H^1(\Omega)]' \Longrightarrow \phi\big|_\Gamma \in H^{\widehat{\alpha}-1}(\Sigma) \tag{7.12}$$

continuously. This is [30, Theorem 8.3, (8.7) and (8.8), p. 162].

Finally, by (complex) interpolation between statement (7.11) and statement (7.12) to obtain with $0 < \theta < 1$:

$$\left[H^1(\Omega), L^2(\Omega)\right]_{1-\theta} = H^\theta(\Omega); \quad \text{or } \left[H^1(\Omega), L^2(\Omega)\right]_{\widehat{\alpha}} = H^{1-\widehat{\alpha}}(\Omega); \tag{7.13}$$

for ϕ_0; next

$$\left[L^2(\Omega), [H^1(\Omega)]'\right]_{1-\theta} = [H^{1-\theta}(\Omega)]'; \quad \text{or } \left[L^2(\Omega, [H^1(\Omega)]')\right]_{\widehat{\alpha}} = [H^{\widehat{\alpha}}(\Omega)]'; \tag{7.14}$$

for ϕ_1; finally

$$\left[H^\beta(\Sigma), H^{\widehat{\alpha}-1}(\Sigma)\right]_{1-\theta} = H^{\beta\theta+(1-\theta)(\widehat{\alpha}-1)}(\Sigma) = L^2(\Sigma), \tag{7.15}$$

where $\beta\theta + (1-\theta)(\widehat{\alpha}-1) = 0$ with $\widehat{\alpha} = \beta$ for $\theta = 1 - \widehat{\alpha}$. Thus,

$$\left[H^\beta(\Sigma), H^{\widehat{\alpha}-1}(\Sigma)\right]_{\widehat{\alpha}} = L^2(\Sigma). \tag{7.16}$$

Then Eqs. (7.13)–(7.16) conclude the interpolation argument and the case $f \equiv 0$ is also proved. Theorem 7.4 is established. □

7.3 *Proof of Theorem 7.1: Preliminary Analysis*

We introduce, as usual [26, 31, 41], the Neumann map

$$Nh = \phi \Longleftrightarrow \left\{\Delta\phi = 0 \text{ in } \Omega, \quad \frac{\partial\phi}{\partial\nu}\bigg|_\Gamma = h, \quad \phi \in L^2(\Omega)/\mathbb{R}\right\} \tag{7.17a}$$

$$N : L^2(\Gamma) \to H^{\frac{3}{2}}(\Omega) \quad \subset \quad H^{\frac{3}{2}-2\epsilon}(\Omega) = \mathcal{D}(A^{\frac{3}{4}-\epsilon}), \quad \text{or } A^{\frac{3}{4}-\epsilon}N \in \mathcal{L}(L^2(\Gamma); L^2(\Omega)) \tag{7.17b}$$

for any $\epsilon > 0$. At first we shall take $g \in H^1(0, T; L^2(\Gamma))$, so that $g_t \in L^2(0, T; L^2(\Gamma)) \equiv L^2(\Sigma)$. We next return to Eq. (7.1a) and re-write it, as usual via (7.17a), as

$$y_{ttt} + \alpha y_{tt} - c^2\Delta(y - Ng) - b\Delta(y - Ng)_t = 0 \quad \text{in } Q \tag{7.18}$$

or abstractly, via (7.2), as

$$y_{ttt} + \alpha y_{tt} + c^2 A(y - Ng) + bA(y - Ng)_t = 0 \quad \text{in } H = L^2(\Omega)/\mathbb{R}. \tag{7.19}$$

Extending, as usual, the original operator A in (7.2) $H \supset \mathcal{D}(A) \to H$ to $A_e : H \to [\mathcal{D}(A^*)]' = [\mathcal{D}(A)]'$, duality w.r.t. H by isomorphism, and retaining the symbol A for such an extension, we re-write Eq. (7.19) as

$$(y_t + \alpha y)_{tt} + bA\left(\frac{c^2}{b} y + y_t\right) = c^2 ANg + bANg_t \in [\mathcal{D}(A)]'. \tag{7.20}$$

See [31, Vol. 1, pp. 420–424; Vol. 2, p. 1061]. Setting, as in (2.10)

$$z = \frac{c^2}{b} y + y_t = (\alpha y + y_t) - \gamma y, \qquad \gamma = \alpha - \frac{c^2}{b} \tag{7.21}$$

and proceeding as in going from (2.10) to (2.26a), (2.26b) or (2.23) to (2.26a), (2.26b), we re-write problem (7.1a)–(7.1c) as the following hyperbolic-dominated system

$$\begin{cases} z_{tt} = -bAz - \gamma z_t + \gamma \dfrac{c^2}{b} z - \gamma \left(\dfrac{c^2}{b}\right)^2 y + c^2 ANg + bANg_t \in [\mathcal{D}(A)]' & (7.22a) \\ y_t = -\dfrac{c^2}{b} y + z & (7.22b) \end{cases}$$

along with the I.C. (we are taking $y_0 = 0$, $y_1 = 0$, $y_2 = 0$):

$$z_0 = \frac{c^2}{b} y_0 + y_1 = 0, \qquad z_1 = \frac{c^2}{b} y_1 + y_2 = 0 \tag{7.22c}$$

8 First Proof of Theorem 7.1 (Interior Regularity): Direct Proof

Step 1 (same strategy as in Step 1 of Sect. 3) The coupling between the hyperbolic z-dynamics in (7.22a) and the ODE y-equation in (7.22b) is mild (lower order), as the coupling $\gamma\left(\frac{c^2}{b}\right)^2 y = \gamma\left(\frac{c^2}{b}\right)^2 \int_0^t e^{-\frac{c^2}{b}(t-\tau)} z(\tau)d\tau$, an integral operator. Thus, essentially w.l.o.g., we may take at first

$$\gamma = 0, \quad \text{i.e. } \alpha = \frac{c^2}{b}, \tag{8.1}$$

see (7.21), to simplify the computations. This will not affect the sought-after regularity of the map in (7.4a)–(7.4e). Thus, we obtain the simplified problem

$$\begin{cases} z_{tt} = -bAz + c^2ADg + bADg_t \in [\mathcal{D}(A)]' & (8.2a) \\ y_t = -\frac{c^2}{b}y + z & (8.2b) \end{cases}$$

along with zero I.C., where now under the (essentially w.l.o.g.) assumption (8.1), the z-problem is uncoupled; that is, explicitly, in PDE-form

$$\begin{cases} \begin{cases} z_{tt} = b\Delta z & \text{in } Q = (0,T] \times \Omega & (8.3a) \\ z\big|_{t=0} = 0; \quad z_t\big|_{t=0} = 0 & \text{in } \Omega & (8.3b) \\ z\big|_{\Sigma} = \frac{c^2}{b}g + g_t & \text{in } \Sigma = (0,T] \times \Gamma & (8.3c) \end{cases} \\ \begin{cases} y_t = -\frac{c^2}{b}y + z & (8.4a) \\ y\big|_{t=0} = 0 & (8.4b) \end{cases} \end{cases}$$

Orientation Thus, under the (essentially benign) assumption (8.1), the crux of our proof consists in applying to the wave equation, either as a mixed problem as in (8.3a)–(8.3c), or else in the abstract form (8.2a), the optimal regularity results (at present for the solution $\{z, z_t\}$ in the interior) from [29, 30, 38] reported in Theorem 7.2 for convenience, and then use these results for z to obtain corresponding results for y, via (8.4a), (8.4b). In carrying out this strategy, the challenge is that we seek to reduce the assumption of regularity of the Dirichlet boundary term $g \in H^1(0,T; L^2(\Gamma))$ to a sort of 'minimal' level, as the term g_t is not present in the original problem (7.1a), (7.1b) and (7.1c), but is sneaked in at the level of the technical step in (7.19). Of course $g \in H^1(0,T; L^2(\Gamma))$ allows one to invoke the results of Theorem 7.2 at once and thus obtain a preliminary conservative result: the map

$$g \in H^1(0, T; L^2(\Gamma)) \quad \rightarrow \quad \{z, z_t\} \in C([0, T]; H^{\widehat{\alpha}}(\Omega) \times H^{\widehat{\alpha}-1}(\Omega)) \tag{8.5a}$$

continuously. From here, it then readily follows via (8.4a), (8.4b)

$$y(t) = \int_0^t e^{-\frac{c^2}{b}(t-\tau)} z(\tau) d\tau \in C([0, T]; H^{\widehat{\alpha}}(\Omega)) \tag{8.5b}$$

$$y_t(t) = \int_0^t e^{-\frac{c^2}{b}(t-\tau)} z_t(\tau) d\tau \in C([0, T]; H^{\widehat{\alpha}-1}(\Omega)) \tag{8.5c}$$

continuously, differentiating (8.4a) in t, and using $y_t\big|_{t=0} = 0$. Our goal is precisely to refine the regularity result (8.5a).

Step 2 To this end, we shall use critically two main results: (i) the sharp (interior and boundary) regularity theory of Theorem 7.2 obtained by purely PDE-techniques such as pseudo-differential operators and micro-local analysis, and (ii) representation formulas to express (but not to obtain from) such results. For convenience and easy reference, we shall provide in Appendix A a short account of these results as needed in our present development. The (negative self-adjoint) operator $-A : H \supset \mathcal{D}(A) \rightarrow H$ generates a s.c. (self-adjoint) cosine operator $\mathcal{C}(t)$ with corresponding sine operator $\mathcal{S}(t)x = \int_0^t \mathcal{C}(\tau)x d\tau$. As reported in Appendix A, the representation formulae of problem (8.3a)–(8.3c), or its abstract version (8.2a) with henceforth

$$b = 1, \qquad c = 1, \qquad \text{w.l.o.g.} \tag{8.6}$$

are given by

$$z(t) = A \int_0^t \mathcal{S}(t-\tau) N g(\tau) d\tau + A \int_0^t \mathcal{S}(t-\tau) N g_t(\tau) d\tau \tag{8.7a}$$

$$= z^{(1)}(t) + z^{(2)}(t) \tag{8.7b}$$

$$z_t(t) = A \int_0^t \mathcal{C}(t-\tau) N g(\tau) d\tau + A \int_0^t \mathcal{C}(t-\tau) N g_t(\tau) d\tau \tag{8.8a}$$

$$= z_t^{(1)}(t) + z_t^{(2)}(t) \tag{8.8b}$$

Step 3 We now invoke the sharp regularity theory of Theorem 7.2 of the wave (in fact, generally, of 2nd order hyperbolic equations), this time with Neumann boundary term $g \in L^2(0, T; L^2(\Gamma))$ and obtain, continuously:

$$g \in L^2(0,T;L^2(\Gamma)) \Longrightarrow \begin{cases} z^{(1)}(t) = A\int_0^t \mathcal{S}(t-\tau)Ng(\tau)d\tau \in C([0,T]; H^{\widehat{\alpha}}(\Omega) = \mathcal{D}(A^{\frac{\widehat{\alpha}}{2}})) & (8.9a) \\ z_t^{(1)}(t) = A\int_0^t \mathcal{C}(t-\tau)Ng(\tau)d\tau \in C([0,T]; H^{\widehat{\alpha}-1}(\Omega) = [\mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})]') & (8.9b) \end{cases}$$

Step 4 Next, let with $\epsilon > 0$ arbitrarily small

$$g \in L^2(0, T; L^2(\Gamma)) \cap C([0, \epsilon); L^2(\Gamma)), \quad g(0) = 0. \tag{8.10}$$

Integrating by parts ($\mathcal{S}(0) = 0$) we obtain from (8.7a), (8.7b)

$$z^{(2)}(t) = A\int_0^t \mathcal{S}(t-\tau)Ng_t(\tau)d\tau = \Big[A\mathcal{S}(t-\tau)Ng(\tau)\Big]_{\tau=0}^{\tau=t} - A\int_0^t \mathcal{C}(t-\tau)Ng(\tau)d\tau \tag{8.11}$$

$$= \cancel{A\mathcal{S}(0)Ng(t)} - \cancel{A\mathcal{S}(t)Ng(0)} - A\int_0^t \mathcal{C}(t-\tau)Ng(\tau)d\tau \tag{8.12a}$$

or, as in (8.9b):

$$\begin{cases} z^{(2)}(t) = -A\int_0^t \mathcal{C}(t-\tau)Ng(\tau)d\tau = -z_t^{(1)}(t) \in C([0,T]; H^{\widehat{\alpha}-1}(\Omega) = [\mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})]') \\ \text{for } g \text{ as in (8.10), continuously.} \end{cases} \tag{8.12b}$$

Thus, by (8.9b) and (8.12b) used in (8.7b), we obtain

$$\begin{cases} z(t) = z^{(1)}(t) + z^{(2)}(t) \in C([0,T]; H^{\widehat{\alpha}-1}(\Omega) = [\mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})]') \\ \text{for } g \text{ as in (8.10), continuously.} \end{cases} \tag{8.13}$$

Step 5 Next, returning to the y-equation in (8.2b) [with (8.6) w.l.o.g.] and z given by (8.7b) we obtain

$$y(t) = \int_0^t e^{-(t-\tau)}z(\tau)d\tau = \int_0^t e^{-(t-\tau)}z^{(1)}(\tau)d\tau + \int_0^t e^{-(t-\tau)}z^{(2)}(\tau)d\tau \tag{8.14a}$$

$$= y^{(1)}(t) + y^{(2)}(t) \tag{8.14b}$$

as $y_0 = 0$. Returning to (8.9a), we obtain conservatively

$$\begin{cases} y^{(1)}(t) = \int\limits_0^t e^{-(t-\tau)} z^{(1)}(\tau) d\tau \in C([0,T]; H^{\widehat{\alpha}}(\Omega) = \mathcal{D}(A^{\frac{\widehat{\alpha}}{2}})) \\ \text{for } g \in L^2(0,T; L^2(\Gamma)), \text{ continuously.} \end{cases} \tag{8.15}$$

Next, with $z^{(2)} = -z_t^{(1)}(t)$ as given by (8.12b), we compute via (8.14a), (8.14b)

$$y^{(2)}(t) = \int\limits_0^t e^{-(t-\tau)} z^{(2)}(\tau) d\tau = - \int\limits_0^t e^{-(t-\tau)} z_t^{(1)}(\tau) d\tau \tag{8.16}$$

$$= -\Big[e^{-(t-\tau)} z^{(1)}(\tau) \Big]_{\tau=0}^{\tau=t} + \int\limits_0^t e^{-(t-\tau)} z^{(1)}(\tau) d\tau \tag{8.17}$$

$$\begin{cases} y^{(2)}(t) = -z^{(1)}(t) + \cancel{e^{-t} z^{(1)}(0)} + \int\limits_0^t e^{-(t-\tau)} z^{(1)}(\tau) d\tau \in C([0,T]; H^{\widehat{\alpha}}(\Omega) = \mathcal{D}(A^{\frac{\widehat{\alpha}}{2}})) \\ \text{valid for } g \text{ as in (8.10), continuously.} \end{cases} \tag{8.18}$$

as the expression of $z^{(2)}(t) = -z_t^{(1)}(t)$ in (8.12b) has such a constraint. As to the regularity noted in (8.18), we invoke (8.9a) for the term $z^{(1)}(t)$, while the same regularity holds true for the second convolution term, this time conservatively. We conclude by (8.15) on $y^{(1)}$ and (8.18) on $y^{(2)}$ that

$$\begin{cases} y(t) = y^{(1)}(t) + y^{(2)}(t) = -z^{(1)}(t) + 2\int\limits_0^t e^{-(t-\tau)} z^{(1)}(\tau) d\tau \in C([0,T]; H^{\widehat{\alpha}}(\Omega) = \mathcal{D}(A^{\frac{\widehat{\alpha}}{2}})) \\ \text{for } g \text{ as in (8.10), continuously.} \end{cases} \tag{8.19}$$

Then (8.19) shows the first result in (7.4a) of Theorem 7.1.

Step 6 Next, with (8.4a), invoke (8.13) for z and (8.19) for y and obtain via (8.4a)

$$\begin{cases} y_t = -\dfrac{c^2}{b} y + z \in C([0,T]; H^{\widehat{\alpha}-1}(\Omega) = [\mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})]') \\ \text{for } g \text{ as in (8.10), continuously.} \end{cases} \tag{8.20}$$

Then (8.20) shows the second result in (7.4b) of Theorem 7.1.

Step 7 We finally need to establish the regularity of y_{tt}. This will be obtained from

$$y_{tt} = -\frac{c^2}{b} y_t + z_t. \tag{8.21}$$

Thus, we need to establish the regularity of z_t, as the regularity of y_t is given by (8.20). But $z_t = z_t^{(1)} + z_t^{(2)}$, where the regularity of $z_t^{(1)} = -z^{(2)}(t)$ was already established in (8.12b).

Step 8 We seek the regularity of $z_t^{(2)}$ from its representation formula in (8.8a). We compute from (8.8a)

$$z_t^{(2)}(t) = A\int_0^t \mathcal{C}(t-\tau)Ng_t(\tau)d\tau \quad \text{(by parts, recalling (A.2))} \tag{8.22}$$

$$= \Big[A\mathcal{C}(t-\tau)Ng(\tau)\Big]_{\tau=0}^{\tau=t} + A\int_0^t A\mathcal{S}(t-\tau)Ng(\tau)d\tau \tag{8.23}$$

$$= ANg(t) - \cancel{A\mathcal{C}(t)Ng(0)} + AA\int_0^t \mathcal{S}(t-\tau)Ng(\tau)d\tau \tag{8.24}$$

$$= ANg(t) + Az^{(1)}(t) \tag{8.25}$$

recalling $\mathcal{C}(0) = I$ and (8.9a). Thus, again by (8.9a)

$$g \in L^2(0,T;L^2(\Gamma)) \Longrightarrow Az^{(1)}(t) = AA\int_0^t \mathcal{S}(t-\tau)Ng(\tau)d\tau \in C([0,T];[\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]') \tag{8.26}$$

continuously. Moreover, by (7.17b)

$$g \in L^2(0,T;L^2(\Gamma)) \Longrightarrow ANg = A^{\frac{1}{4}+\epsilon_1}(A^{\frac{3}{4}-\epsilon_1}N)g \in L^2(0,T;[\mathcal{D}(A^{\frac{1}{4}+\epsilon_1})]') \subset L^2(0,T;[\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]') \tag{8.27a}$$

as well as

$$g \in C([0,T];L^2(\Gamma)) \Longrightarrow ANg = A^{\frac{1}{4}+\epsilon_1}(A^{\frac{3}{4}-\epsilon_1}N)g \in C([0,T];[\mathcal{D}(A^{\frac{1}{4}+\epsilon_1})]') \subset C([0,T];[\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]') \tag{8.27b}$$

continuously, since $\widehat{\alpha} = \frac{2}{3}$ or $\frac{3}{4}$ by (7.3a), (7.3b), so $1 - \frac{\widehat{\alpha}}{2} = \frac{2}{3}$ or $\frac{5}{8}$ and so $1 - \frac{\widehat{\alpha}}{2} > \frac{1}{4} + \epsilon$. Combining (8.26) and (8.27a) in (8.26), we obtain

$$\begin{cases} g \in L^2(0,T;L^2(\Gamma)) \cap C([0,\epsilon);L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow z_t^{(2)} = ANg + Az^{(1)} \in L^2(0,T;[\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]') \tag{8.28a}$$

as well as, via (8.26) and (8.27b)

$$\begin{cases} g \in C([0,T]; L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow z_t^{(2)} = ANg + Az^{(1)} \in C([0,T]; [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]') \tag{8.28b}$$

continuously.

Step 9 In turn, combining (8.9b) for $z_t^{(1)}$ with (8.28a) for $z_t^{(2)}$ in (8.8b), we conclude

$$\begin{cases} g \in L^2(0,T; L^2(\Gamma)) \cap C([0,\epsilon); L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow z_t = z_t^{(1)} + z_t^{(2)} \in L^2(0,T; [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]') \tag{8.29a}$$

as well as, via (8.28b) for $z_t^{(2)}$

$$\begin{cases} g \in C([0,T]; L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow z_t = z_t^{(1)} + z_t^{(2)} \in C([0,T]; [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]') \tag{8.29b}$$

since $\frac{1-\widehat{\alpha}}{2} < 1 - \frac{\widehat{\alpha}}{2}$.

Step 10 Combining (8.20) on y_t with (8.29a) for z_t, we conclude via (8.21)

$$\begin{cases} g \in L^2(0,T; L^2(\Gamma)) \cap C([0,\epsilon); L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow y_{tt} = -\frac{c^2}{b} y_t + z_t \in L^2(0,T; [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]') \tag{8.30a}$$

as well as, via (8.29b) for z_t

$$\begin{cases} g \in C([0,T]; L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow y_{tt} = -\frac{c^2}{b} y_t + z_t \in C([0,T]; [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]') \tag{8.30b}$$

continuously. Then (8.30a), (8.30b) shows the interior regularity results in (7.4c) and (7.4d). Theorem 7.1 is proved, except for the boundary regularity statement (7.4c), which will be established in Sect. 9.

9 Proof of Theorem 7.1: Regularity of the Boundary Trace $y|_{\Sigma}$

In this section we shall establish the boundary regularity (7.4c) of Theorem 7.1. It is here repeated for convenience.

Theorem 9.1 *With reference to problem (7.1a)–(7.1c), we have*

$$\begin{cases} g \in L^2(0,T; L^2(\Gamma)) \cap C([0,\epsilon); L^2(\Gamma)) \\ g(0) = 0 \end{cases} \Longrightarrow y\big|_{\Sigma} \in H^{2\widehat{\alpha}-1}(\Sigma)) \tag{9.1}$$

continuously.

Proof **Step 1** We return to the solution formula (8.7a) complemented by (8.12b), with g as in (8.10), i.e. as on the LHS of (9.1):

$$z(t) = z^{(1)}(t) + z^{(2)}(t) = z^{(1)}(t) - z_t^{(1)}(t) \tag{9.2}$$

$$= A\int_0^t \mathcal{S}(t-\tau)Ng(\tau)d\tau - A\int_0^t \mathcal{C}(t-\tau)Ng(\tau)d\tau. \tag{9.3}$$

We next invoke critically the boundary regularity results reported in (7.7) of Theorem 7.2 as well as (A.4), (A.6):

$$g \in L^2(0,T;L^2(\Gamma)) \Longrightarrow z^{(1)}\big|_\Sigma = (N^*A)A\int_0^t \mathcal{S}(t-\tau)Ng(\tau)d\tau \in H^{2\widehat{\alpha}-1}(\Sigma). \tag{9.4}$$

Step 2 We return to (8.19), with g as in (8.10)

$$y(t) = -z^{(1)}(t) + 2\int_0^t e^{-(t-\tau)}z^{(1)}(\tau)d\tau \tag{9.5}$$

where then, recalling (9.4), we obtain

$$y\big|_\Sigma = -z^{(1)}\big|_\Sigma + 2\int_0^t e^{-(t-\tau)}z^{(1)}\big|_\Sigma(\tau)d\tau \in H^{2\widehat{\alpha}-1}(\Sigma) \tag{9.6}$$

continuously. Then (9.6) proves Theorem 9.1, hence (7.4c) of Theorem 7.1. □

10 A Boundary Trace Result for the u-Problem

In this section we consider problem (7.1a)–(7.1c) with $g \equiv 0$, rewritten in PDE-form as

$$\begin{cases} u_{ttt} + \alpha u_{tt} - c^2\Delta u - b\Delta u_t = 0 & \text{in } Q = (0,T] \times \Omega \quad (10.1a) \\ u\big|_{t=T} = u_0; \quad u_t\big|_{t=T} = u_1; \quad u_{tt}\big|_{t=T} = u_2 & \text{in } \Omega \quad (10.1b) \\ \left.\dfrac{\partial u}{\partial \nu}\right|_\Sigma = 0 & \text{in } \Sigma = (0,T] \times \Gamma \quad (10.1c) \end{cases}$$

abstractly, with A as in (7.2)

$$u_{ttt} + \alpha u_{tt} + c^2 Au + bAu_t = 0 \tag{10.1d}$$

with I.C. at $t = T$ (i.e. backward in time). This is the counterpart of problem (5.1a)–(5.1d) in the corresponding Dirichlet case. In this section, we shall consider two spaces for the I.C., see (2.5c) for U_3

$$\{u_0, u_1, u_2\} \in U_3 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times H \tag{10.2}$$

$$\{u_0, u_1, u_2\} \in U_5 \equiv \mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}}) \times \mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}}) \times \left[\mathcal{D}(A^{\frac{\widehat{\alpha}}{2}})\right]' \tag{10.3a}$$

$$= \begin{cases} \mathcal{D}(A^{\frac{2}{3}}) \times \mathcal{D}(A^{\frac{1}{6}}) \times \left[\mathcal{D}(A^{\frac{1}{3}})\right]', & \alpha = \dfrac{2}{3} \quad (10.3b) \\ \mathcal{D}(A^{\frac{5}{8}}) \times \mathcal{D}(A^{\frac{1}{8}}) \times \left[\mathcal{D}(A^{\frac{3}{8}})\right]'. & \alpha = \dfrac{3}{4} \quad (10.3c) \end{cases}$$

recalling (7.3a), (7.3b). Notice that the regularity of the components of U_5 is reduced by $\mathcal{D}(A^{\frac{1}{2}})$ from u_0 to u_1 to u_2, in line with the spaces U_3 or U_4 in (2.5c).

Accordingly, in view of Theorem 2.1, we have

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} \in U_i \to \begin{bmatrix} u(t) \\ u_t(t) \\ u_{tt}(t) \end{bmatrix} = e^{G(T-t)} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} \in C([0, T]; U_i) \tag{10.4}$$

continuously, for the solution of (10.1a)–(10.1d), where e^{Gt} is a s.c. group on U_i, with infinitesimal generator G defined in (2.4), with A as in (7.2) in the Neumann case.

Theorem 10.1 *With reference to problem (10.1a)– (10.1d) and to (10.2), the following trace estimates hold true:*

(a)

$$\|u_t\|^2_{H^\beta(\Sigma)} = \mathcal{O}_T\left(\left\|\{u_0, u_1, u_2\}\right\|^2_{U_3}\right), \tag{10.5}$$

where $\beta = \dfrac{2}{3}$ or $\beta = \dfrac{3}{4}$ is defined in (7.3a), (7.3b) and where $\mathcal{O}_T$ denotes a constant depending on $\Sigma = \Gamma \times (0, T]$ and the equation coefficients, but not on U_3.

(b)

$$\int_0^T \int_\Gamma |u_t|^2 d\Gamma dt = \mathcal{O}_T\left(\left\|\{u_0, u_1, u_2\}\right\|^2_{U_5}\right). \tag{10.6}$$

Remark 10.1 The above two trace regularity results do not follow from the interior regularity. They express a 'hidden' regularity property. As to (10.5), the interior regularity of u_t is $u_t \in C([0,T]; \mathcal{D}(A^{\frac{1}{2}}) \equiv H^1(\Omega))$ by (10.4), (10.2), hence $u_t|_\Gamma \in C([0,T]; H^{1/2}(\Gamma))$ by trace theory. Instead (10.5) yields, in particular, $u_t|_\Sigma \in L^2(0,T; H^\beta(\Gamma))$, with $\beta - \frac{1}{2} = \frac{1}{6}$ or $\frac{1}{4}$ stronger in space regularity by (7.3a), (7.3b), for $\beta = \frac{2}{3}$ or $\beta = \frac{3}{4}$. Likewise, as to (10.6), one has the interior regularity $u_t \in C([0,T]; \mathcal{D}(A^{\frac{1}{6}}) \equiv H^{\frac{1}{3}}(\Omega))$ say by (10.3b) for $\widehat{\alpha} = \frac{2}{3}$, which does not yield the trace regularity $u_t|_\Sigma \in L^2(0,T; L^2(\Gamma))$ of (10.6).

Proof of Theorem 10.1: **Step 1** Rewrite (10.1a) as in (2.7)

$$(u_t + \alpha u)_{tt} - b\Delta\left(\frac{c^2}{b}u + u_t\right) = 0 \quad \text{in } Q \tag{10.7}$$

and introduce the new variable ξ in (2.8)

$$\xi = \alpha u + u_t \in \begin{cases} C([0,T]; \mathcal{D}(A^{\frac{1}{2}})) & \text{for } \{u_0, u_1, u_2\} \in U_3 \quad (10.8a) \\ C([0,T]; \mathcal{D}(A^{\frac{1}{6}})) & \text{for } \{u_0, u_1, u_2\} \in U_5, \quad \widehat{\alpha} = \frac{2}{3} \quad (10.8b) \end{cases}$$

in the less regular case $\widehat{\alpha} = \frac{2}{3}$ in (10.3b), as it follows from (10.4) along with (10.2) or (10.3a) respectively,

$$\frac{c^2}{b}u + u_t = \xi - \gamma u, \qquad \gamma = \alpha - \frac{c^2}{b}. \tag{10.8c}$$

Rewrite problem (10.1a)–(10.1c) accordingly as

$$\begin{cases} \xi_{tt} - b\Delta\xi + b\gamma\Delta u = 0 & \text{in } Q \quad (10.9a) \\ \xi\big|_{t=T} = \xi_0 = \alpha u_0 + u_1; \quad \xi_t\big|_{t=T} = \xi_1 = \alpha u_1 + u_2 & \text{in } \Omega \quad (10.9b) \\ \left.\dfrac{\partial\xi}{\partial\nu}\right|_\Sigma \equiv 0 & \text{in } \Sigma \quad (10.9c) \end{cases}$$

and set

$$\xi = \xi^{(1)} + \xi^{(2)} \tag{10.10}$$

$$\begin{cases} \xi^{(1)}_{tt} - b\Delta\xi^{(1)} + b\gamma\Delta u = 0 \\ \xi^{(1)}\Big|_{t=T} = 0, \quad \xi^{(1)}_t\Big|_{t=T} = 0\,; \\ \dfrac{\partial\xi^{(1)}}{\partial\nu}\Big|_{\Sigma} = 0 \end{cases} \qquad \begin{cases} \xi^{(2)}_{tt} - b\Delta\xi^{(2)} = 0 & \text{in } Q \quad (10.11a) \\ \xi^{(2)}\Big|_{t=T} = \xi_0, \quad \xi^{(2)}_t\Big|_{t=T} = \xi_1 & \text{in } Q \quad (10.11b) \\ \dfrac{\partial\xi^{(2)}}{\partial\nu}\Big|_{\Sigma} = 0 & \text{in } Q \quad (10.11c) \end{cases}$$

Step 2

Theorem 10.2 *The following trace estimates hold true (recall β in (7.3a), (7.3b)):*

(a)

$$\left\| \xi\big|_\Sigma \right\|^2_{H^\beta(\Sigma)} = \mathcal{O}_T\left(\left\| \{u_0, u_1, u_2\} \right\|^2_{U_3} \right), \tag{10.12}$$

(b)

$$\begin{cases} \left\| \xi\big|_\Sigma \right\|^2_{L^2(\Sigma)} = \mathcal{O}_T\left(\left\| \{u_0, u_1, u_2\} \right\|^2_{U_5} \right) \\ U_5 \textit{ given by (10.3a)–(10.3c)} \end{cases} \tag{10.13}$$

Proof of Theorem 10.2

(a) With $\{u_0, u_1, u_2\} \in U_3$ in (10.2), we have

$$\Delta u \in C([0, T]; L^2(\Omega)), \tag{10.14}$$

continuously, by (10.4), $i = 3$. We next invoke [30, Theorem B(1), p. 118] (recalled in Theorem 7.3), as applied to the $\xi^{(1)}$-problem in (10.11a)–(10.11c), with RHS as in (10.14) and obtain

$$\xi^{(1)}\big|_\Sigma \in H^\beta(\Sigma), \text{ continuously in } U_3. \tag{10.15}$$

Next, with

$$\xi^{(2)}\Big|_{t=T} = \xi_0 = \alpha u_0 + u_1 \in \mathcal{D}(A^{\frac{1}{2}}) \equiv H^1(\Omega); \quad \xi^{(2)}_t\Big|_{t=T} = \xi_1 = \alpha u_1 + u_2 \in L^2(\Omega), \tag{10.16}$$

we invoke this time [30, Theorem C(1), p 118] (recalled in the (7.11) of Theorem 7.4), as applied to the $\xi^{(2)}$-problem in (10.11a)–(10.11c), with I.C. as in (10.16) and obtain

$$\xi^{(2)}\big|_\Sigma \in H^\beta(\Sigma), \tag{10.17}$$

continuously in $\{\xi_0, \xi_1\} \in H^1(\Omega) \times L^2(\Omega)$, hence continuously in U_3. Combining (10.15) and (10.17) in (10.10), we obtain (10.12) and part (a) of Theorem 10.2 is proved.

(b) Now with $\{u_0, u_1, u_2\} \in U_5$ in (10.3a), we have $u \in C\left([0,T]; H^{2-\widehat{\alpha}}(\Omega) \equiv \mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})\right)$ by (10.4) and hence continuously

$$\Delta u = Au \in C\left([0,T]; \left[H^{\widehat{\alpha}}(\Omega)\right]'\right), \quad \widehat{\alpha} = \frac{2}{3} \text{ or } \widehat{\alpha} = \frac{3}{4} \text{ as in (7.3a)–(7.3b).} \tag{10.18}$$

We next invoke [30, Theorem E, with $\theta = 0$, p. 119] (recalled in the Theorem 7.4), as applied to the $\xi^{(1)}$-problem in (10.11a)–(10.11c), with RHS as in (10.18) and zero I.C. and obtain

$$\xi^{(1)}\big|_{\Sigma} \in L^2(\Sigma), \text{ continuously in } U_5. \tag{10.19}$$

Next, with

$$\xi^{(2)}\Big|_{t=T} = \xi_0 = \alpha u_0 + u_1 \in H^{1-\widehat{\alpha}}(\Omega) = \mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}}), \quad \left[\mathcal{D}(A^{\frac{1}{6}}) \equiv H^{\frac{1}{3}}(\Omega) \text{ for } \widehat{\alpha} = \frac{2}{3}\right] \tag{10.20a}$$

$$\xi_t^{(2)}\Big|_{t=T} = \xi_1 = \alpha u_1 + u_2 \in \left[H^{\widehat{\alpha}}(\Omega)\right]' \equiv \left[\mathcal{D}(A^{\frac{\widehat{\alpha}}{2}})\right]', \quad \left[\left[\mathcal{D}(A^{\frac{1}{3}})\right]' \equiv \left[H^{\frac{2}{3}}(\Omega)\right]' \text{ for } \widehat{\alpha} = \frac{2}{3}\right] \tag{10.20b}$$

we obtain by Theorem 7.4, Eq. (7.10b) [with $f \equiv 0$]

$$\xi^{(2)}\big|_{\Sigma} \in L^2(\Sigma) \tag{10.21}$$

continuously in $\{\xi_0, \xi_1\} \in H^{1-\widehat{\alpha}}(\Omega) \times \left[H^{\widehat{\alpha}}(\Omega)\right]'$, hence continuously in U_5. Combining (10.19) with (10.21) in (10.10), we obtain (10.13) and part 10.2 of Theorem 10.2 is proved. □

Step 3 (continuing the proof of Theorem 10.1b)

(a) Thus, by Theorem 10.2a, we have for $\{u_0, u_1, u_2\} \in U_3$ and β in (7.3a), (7.3b)

$$\xi\big|_{\Sigma} = \alpha u + u_t\big|_{\Sigma} \in H^{\beta}(\Sigma) \tag{10.22}$$

while $u \in C([0,T]; \mathcal{D}(A)) \subset C([0,T]; H^2(\Omega))$ by (10.4), (10.2) implies

$$u\big|_{\Sigma} \in C([0,T]; H^{\frac{3}{2}}(\Omega)) \subset H^{\beta}(\Sigma) \tag{10.23}$$

continuously in U_3. Then, (10.22) and (10.23) imply $u_t\big|_{\Sigma} \in H^{\beta}(\Sigma)$ and (10.5) of Theorem 10.1a is proved.

(b) Similarly, by Theorem 10.2b, we have for $\{u_0, u_1, u_2\} \in U_5$,

$$\xi\big|_{\Sigma} = \alpha u + u_t\big|_{\Sigma} \in L^2(\Sigma) \tag{10.24}$$

continuously, while $u \in C([0,T]; H^{2-\widehat{\alpha}}(\Omega))$ by (10.4), (10.3a) implies

$$u\big|_{\Sigma} \in C([0,T]; H^{\frac{3}{2}-\widehat{\alpha}}(\Gamma)) \subset L^2(\Sigma) \tag{10.25}$$

continuously in U_5. Then, (10.24) and (10.25) imply $u_t\big|_{\Sigma} \in L^2(\Sigma)$ and (10.6) of Theorem 10.1b is established. □

11 A Dual Result of the Neumann Problem (7.1a)–(7.1c)

The present section is the Neumann counterpart of the Dirichlet Sect. 6. Therefore, it will simply list the counterpart results. The dual problem of the boundary non-homogeneous Neumann problem (7.1a)–(7.1c) is the problem

$$\begin{cases} v_{ttt} - \alpha v_{tt} + c^2 \Delta v - b\Delta v_t = 0 & \text{in } Q & (11.1a) \\ v\big|_{t=T} = v_0; \quad v_t\big|_{t=T} = v_1; \quad v_{tt}\big|_{t=T} = v_2 & \text{in } \Omega & (11.1b) \\ \left.\dfrac{\partial v}{\partial \nu}\right|_{\Sigma} = 0 & \text{in } \Sigma & (11.1c) \end{cases}$$

$$\text{abstractly} \qquad v_{ttt} - \alpha v_{tt} - c^2 A v + bAv_t = 0 \tag{11.2}$$

with $-$A the Neumann Laplician in (7.2). This is very closely related to problem (10.1a)–(10.1c), or its abstract version (10.1d) for reasons similar to those given in Sect. 6. The relevant results are as follows

Theorem 11.1 *With reference to the v-problem (11.1a)–(11.1c), abstractly* (11.2)*, we have*

$$\{v_0, v_1, v_2\} \in U_i \Longrightarrow \{v, v_t, v_{tt}\} \in C([0,T]; U_i), \quad i = 3, 5 \tag{11.3a}$$

$$U_3 = \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times H, \qquad U_5 = \mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}}) \times \mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}}) \times \left[\mathcal{D}(A^{\frac{\widehat{\alpha}}{2}})\right]'. \tag{11.3b}$$

This is the counterpart of Theorem 6.1 in the Dirichlet case, at least for $i = 3$.

Theorem 11.2 *With reference to problem (11.1a)–(11.1c), we have*

(a)

$$\|v_t\|^2_{H^\beta(\Sigma)} = \mathcal{O}_T\left(\left\|\{v_0, v_1, v_2\}\right\|^2_{U_3}\right), \tag{11.4}$$

where β is defined in (7.3a)*,* (7.3b)*.*

(b)

$$\int_0^T \int_\Gamma |v_t|^2 d\Gamma dt = \mathcal{O}_T\left(\left\|\{v_0, v_1, v_2\}\right\|^2_{U_5}\right). \tag{11.5}$$

This is the counterpart of Theorem 10.1. It is a sharp, hidden trace regularity result.

Next, by duality on the trace result (11.5) for the v-problem (11.1a)–(11.1c), we shall re-obtain (in a slightly weaker form) the basic interior regularity result of Theorem 7.1.

Theorem 11.3 *With reference to problem (7.1a)–(7.1c), we have*

$$\begin{cases} g \in C([0,T]; L^2(\Gamma)) \\ g(0) = 0 \end{cases} \Longrightarrow \{y, y_t, y_{tt}\} \in C\left([0,T]; \mathcal{D}(A^{\frac{\widehat{\alpha}}{2}}) \times [\mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})]' \times [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]'\right), \tag{11.6}$$

continuously.

Proof The proof is the conceptual counterpart of Theorem 6.4 in the Dirichlet case, subject to further technicalities proper of the Neumann problem.

Step 1 We shall first establish the following.

Proposition 11.4 *With reference to the Neumann problem (7.1a)–(7.1c), we have, for each* $0 < t \leq T$*:*

$$\begin{cases} g \in C([0,T]; L^2(\Gamma)) \\ g(0) = 0, \end{cases} \Longrightarrow \left\{y(t), y_t(t), y_{tt}(t)\right\} \in \mathcal{D}(A^{\frac{\widehat{\alpha}}{2}}) \times [\mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})]' \times [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]' \tag{11.7a}$$

$$\Longrightarrow \int_0^t \left[y(\tau), y_t(\tau), y_{tt}(\tau)\right] d\tau \in \mathcal{D}(A^{\frac{\widehat{\alpha}}{2}}) \times [\mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})]' \times [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]' \tag{11.7b}$$

continuously.

Proof of Proposition 11.4. It is based by duality on Theorem 11.2b, Eq. (11.5), counterpart of estimate (10.6).
Step (i) By Theorem 11.1 and 11.2b, we have

$$\left\{v_0, v_1, v_2\right\} \in U_5 = \mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}}) \times \mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}}) \times \left[\mathcal{D}(A^{\frac{\widehat{\alpha}}{2}})\right]' \tag{11.8a}$$

$$\Longrightarrow \begin{cases} \left\{v, v_t, v_{tt}\right\} \in C([0,T]; U_5) \\ \text{AND} \\ v_t\big|_\Gamma \in L^2(0,T; L^2(\Gamma)) \end{cases} \tag{11.8b}$$

continuously.
Step (ii) We now invoke the duality identity (B.5) in Appendix B, Neumann case, and obtain for a generic t, $0 < t \leq T$ (notation denotes duality pairing):

$$
\begin{aligned}
\langle y_{tt}(t) + \alpha y_t(t), v_0\rangle - \langle y_t(t) + \alpha y(t), v_1\rangle + \langle y(t), v_2\rangle - b\,\langle y(t), \Delta v_0\rangle \\
= \left\langle c^2 g + b g_t, v\right\rangle_{L^2(0,t;L^2(\Gamma))}.
\end{aligned}
\tag{11.9}
$$

With $g \in L^2(0,t;L^2(\Gamma))$ and under (11.8a), (11.8b) and hence $v \in C([0,T]; H^{2-\widehat{\alpha}}(\Omega)$ and $v\big|_\Sigma \in C([0,T]; H^{\frac{3}{2}-\widehat{\alpha}}(\Gamma)$ [say $v\big|_\Sigma \in C([0,t]; H^{\frac{4}{3}-\frac{1}{2}}(\Gamma)) = H^{\frac{5}{6}}(\Gamma))$ for $\widehat{\alpha} = \frac{2}{3}$], we have regarding the first term on the RHS of (11.9):

$$
\left\langle g, v\big|_\Gamma\right\rangle_{L^2(0,t;L^2(\Gamma))} = \int_0^t \int_\Gamma g v\big|_\Gamma d\Gamma dt < \infty. \tag{11.10}
$$

Step (iii) Under hypothesis (11.7a), (11.7b) and under (11.8a), (11.8b) for the adjoint v-problem, we compute the last term on the RHS of (11.9) by parts as follows, for $0 < t \le T$

$$
\begin{aligned}
\int_0^t \left(g_t(\tau), v\big|_\Gamma(\tau)\right)_{L^2(\Gamma)} d\tau &= \Big[\left(g(\tau), v\big|_\Gamma(\tau)\right)_{L^2(\Gamma)}\Big]_{\tau=0}^{\tau=t} \\
&\quad - \int_0^t \left(g(\tau), v_t\big|_\Gamma(\tau)\right)_{L^2(\Gamma)} d\tau \\
&= \left(g(t), v\big|_\Gamma(t)\right)_{L^2(\Gamma)} - \cancel{\left(g(0), v\big|_\Gamma(0)\right)_{L^2(\Gamma)}} \\
&\quad - \int_0^t \left(g(\tau), v_t\big|_\Gamma(\tau)\right)_{L^2(\Gamma)} d\tau
\end{aligned}
\tag{11.11}
$$

With $g \in C([0,T]; L^2(\Gamma))$ as in (11.7a), (11.7b) and $v\big|_\Sigma \in C([0,t]; H^{\frac{3}{2}-\widehat{\alpha}}(\Gamma))$, we have that the first term on the RHS of (11.11) is well-defined (conservatively). Notice that it can also be re-written as

$$
\begin{aligned}
\left(g(t), v_t\big|_\Gamma\right)_{L^2(\Gamma)} &= \left(g(t), N^* A v(t)\right)_{L^2(\Gamma)} \\
&= (ANg(t), v(t))_{L^2(\Gamma)} \text{ (well-defined)}
\end{aligned}
\tag{11.12}
$$

with $v \in C([0,T]; \mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}}))$ by (11.8a) and $ANg \in C([0,T]; [\mathcal{D}(A^{\frac{1}{4}+\epsilon_1})]') \subset C([0,T]; [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]')$. Here we have invoked (7.17b) for N, (11.7a) for g as well as $\frac{1}{4} + \epsilon_1 < 1 + \frac{\widehat{\alpha}}{2}$. Then by (11.11) and (11.12),

$$
\begin{aligned}
\left\langle g_t, v\big|_\Gamma\right\rangle_{L^2(0,t;L^2(\Gamma))} = (ANg(t), v(t))_{L^2(\Omega)} - \left\langle g, v_t\big|_\Gamma\right\rangle_{L^2(0,t;L^2(\Gamma))} \\
\text{(well-defined, by (11.12) and (11.8b)),}
\end{aligned}
\tag{11.13}
$$

continuously, with respect to g as in (11.7a), (11.7b) and $\{v_0, v_1, v_2\}$ as in (11.8a), (11.8b). Thus, the RHS of identity (11.9) is well-defined for such g in (11.7a), (11.7b) and $\{v_0, v_1, v_2\} \in U_5$, critically because of Theorem 11.2b, Eq. (11.5), used in (11.8b). Next we turn to the LHS of identity (11.9) and then the duality argument below—the counterpart of the argument in B.8 or in (6.22)–(6.24) in the Dirichlet case—yields (11.7a) also in the present Neumann case. Specifically, for each $t, 0 < t \leq T$, via duality pairing in the LHS of identity (11.9), we obtain recalling $\{v_0, v_1, v_2\} \in U_5$ in (10.3a):

$$\left.\begin{array}{l} v_2 \in [\mathcal{D}(A^{\frac{\widehat{\alpha}}{2}})]' \to y(t) \in \mathcal{D}(A^{\frac{\alpha}{2}}) \\ v_1 \in \mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}}) \to y_t(t) + \alpha y(t) \in [\mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})]' \end{array}\right\} \Longrightarrow y_t(t) \in \left[\mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})\right]' \qquad (11.14)\ (11.15)$$

$$v_0 \in \mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}}) \Longrightarrow y_{tt}(t) + \alpha y_t(t) \in [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]' \qquad (11.16)$$

$$\Longrightarrow y_{tt}(t) \in [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]' \qquad (11.17)$$

since $\dfrac{1-\widehat{\alpha}}{2} < 1 - \dfrac{\widehat{\alpha}}{2}$, and hence $\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}}) \subset \mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})$, hence $[\mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})]' \subset [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]'$. Thus, conclusions (11.14), (11.15), (11.17) establish statement (11.7a). A similar argument gives (11.7b) and Proposition 11.4 proved.

Step 2 In light of (11.7b), we apply [23, Corollary 3.2, p. 173] to obtain that the map (as in (6.26))

$$t \to \int_0^t [y(\tau), y_t(\tau), y_{tt}(\tau)] d\tau$$

is continuous $[0, T] \to \mathcal{D}(A^{\frac{\widehat{\alpha}}{2}}) \times [\mathcal{D}(A^{\frac{1-\widehat{\alpha}}{2}})]' \times [\mathcal{D}(A^{1-\frac{\widehat{\alpha}}{2}})]'$. Theorem 11.3 is proved. □

Acknowledgements The results of the present paper were obtained in Spring 2016, when the author was an invited visitor of the NSF-sponsored Institute of Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, on the occasion of the year-long thematic program “Control Theory and its Applications”, September 1, 2015–June 30, 2016. They were presented by the author at the the following conferences venues: (i) Optimal Control for Evolutionary PDEs, Cortona, Italy, June 20–24, 2016; (ii) Semigroups of Operators: Theory and Applications, Kazimierz Dolny, Poland, September 30-October 5, 2018; (iii) Colloquium at Florida International University, November 2018; (iv) XVIII Workshop on Partial Differential Equations, Petropolis, Brazil, September 2019. The author wishes to thank the IMA for its most efficient hospitality and ideal working conditions. The author is most pleased to join the other participants in celebrating the 85th birthday of Professor Jan Kisynski, whom the author was privileged to meet, and receive much math information from, in the Fall of 1974. Many happy returns to Professor Kisynski!
Research partially supported by the National Science Foundation under Grant DMS 1713506.

Appendix A

1. **Cosine Operators**. While we refer to standard work [6, 10, 11, 17, 22, 36, 40] etc for the topic of cosine operator theory on Banach space, we include here only a few results which are used and invoked in the text with reference to a Hilbert space H ($H = L^2(\Omega)$ in Part A; $H = L^2(\Omega)/\mathbb{R}$ in Part B). In line with the text, we let $(-A)$ be the (strictly positive) self-adjoint infinitesimal generator of a strongly continuous (self-adjoint) cosine operator family $\mathcal{C}(t)$ with sine operator $\mathcal{S}(t)x = \int_0^t \mathcal{C}(\tau)x d\tau$, $x \in H$, with $A^{\frac{1}{2}}\mathcal{S}(t)$ strongly continuous:

$$\mathcal{S}(t-\tau) = \mathcal{S}(t)\mathcal{C}(\tau) - \mathcal{C}(t)\mathcal{S}(\tau) \tag{A.1a}$$

$$\mathcal{C}(t-\tau) = \mathcal{C}(t)\mathcal{C}(\tau) - A\mathcal{S}(t)\mathcal{S}(\tau), \quad \tau, t \in \mathbb{R} \tag{A.1b}$$

We have

$$\frac{d^2\mathcal{C}(t)x}{dt^2} = -A\mathcal{C}(t)x, x \in \mathcal{D}(A); \qquad \frac{d\mathcal{C}(t)x}{dt} = -A\mathcal{S}(t)x, x \in \mathcal{D}(A^{\frac{1}{2}}), \tag{A.2}$$

$\mathcal{C}(t)$ is even on H, $\mathcal{C}(0) = I$; $\mathcal{S}(t)$ is odd on H, $\mathcal{S}(0) = 0$. The above formulae (A.2) on H with $H \supset \mathcal{D}(A) \to H$ can be extended to $[\mathcal{D}(A)]'$ with A now the extension $A_e : H \to [\mathcal{D}(A)]'$, which we still denote by A.

2. **Representation formulae of non-homogeneous boundary control for wave (second order hyperbolic) equations** [25–27, 31, 41],[23, Sect. 3]
Dirichlet case We return to the Dirichlet non-homogeneous w-problem in (2.15a)–(2.15c). Let D be the Dirichlet map in (2.21a), (2.21b) and $(-A)$ be the Dirichlet Laplacian in (2.2). Then

$$w(t) = A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)d\tau; \quad w_t(t) = A\int_0^t \mathcal{C}(t-\tau)Dg(\tau)d\tau. \tag{A.3}$$

Neumann case We now return to the Neumann non-homogeneous w-problem in (7.5a)–(7.5c). Let N be the Neumann map in (7.17a), (7.17b) and $(-A)$ be the Neumann Laplacian in (7.2). Then

$$w(t) = A\int_0^t \mathcal{S}(t-\tau)Ng(\tau)d\tau; \quad w_t(t) = A\int_0^t \mathcal{C}(t-\tau)Ng(\tau)d\tau. \tag{A.4}$$

3. **Operator formulae for traces**
Let $(-A)$ be the Dirichlet Laplacian in (2.2) and D the Dirichlet map in (2.21a), (2.21b). Then [41], [29, p. 181]

$$D^*A^*\phi = -\frac{\partial\phi}{\partial\nu}, \quad \phi \in \mathcal{D}(A), \tag{A.5}$$

which can be extended to all $\phi \in H^{\frac{3}{2}+\epsilon}(\Omega) \cap H_0^1(\Omega)$, $\epsilon > 0$.
Let now $(-A)$ be the Neumann Laplacian in (7.2) and N the Neumann map in (7.17a), (7.17b). Then [41], [29, p. 196]

$$N^* A^* \phi = \phi\big|_\Gamma \quad \phi \in \mathcal{D}(A), \tag{A.6}$$

which can be extended to all $\phi \in H^{\frac{3}{2}+\epsilon}(\Omega) \cap H_0^1(\Omega)$, $\epsilon > 0$, with $\frac{\partial \phi}{\partial \nu}\big|_\Gamma = 0$.

Appendix B The Dual Problem of the Boundary Non-homogeneous Problem (2.1a)–(2.1c). A PDE-Approach

In this Appendix we consider the following two problems:

Problem #1 (2.1a)–(2.1c)

$$\begin{cases} y_{ttt} + \alpha y_{tt} - c^2 \Delta y - b \Delta y_t = 0 & \text{in } Q = (0, T] \times \Omega \quad \text{(B.1a)} \\ y\big|_{t=0} = y_0; \quad y_t\big|_{t=0} = y_1; \quad y_{tt}\big|_{t=0} = y_2 & \text{in } \Omega \quad \text{(B.1b)} \\ \text{and either} & \\ \text{Dirichlet-control } y\big|_\Sigma = g & \text{in } \Sigma = (0, T] \times \Gamma \quad \text{(B.1c)} \\ \text{or else} & \\ \text{Neumann-control } \dfrac{\partial y}{\partial \nu}\Big|_\Sigma = g & \text{in } \Sigma. \quad \text{(B.1d)} \end{cases}$$

Problem #2 With $T > 0$ arbitrary,

$$\begin{cases} v_{ttt} - \alpha v_{tt} + c^2 \Delta v - b \Delta v_t = 0 & \text{in } Q \quad \text{(B.2a)} \\ v\big|_{t=T} = v_0; \quad v_t\big|_{t=T} = v_1; \quad v_{tt}\big|_{t=T} = v_2 & \text{in } \Omega \quad \text{(B.2b)} \\ \text{and either} & \\ \text{Dirichlet homogeneous B.C. } v\big|_\Sigma \equiv 0 & \text{in } \Sigma \quad \text{(B.2c)} \\ \text{or else} & \\ \text{Neumann homogeneous B.C. } \dfrac{\partial v}{\partial \nu}\Big|_\Sigma = 0 & \text{in } \Sigma. \quad \text{(B.2d)} \end{cases}$$

The v-problem (B.2a)–(B.2d) is dual to the y-problem (B.1a)–(B.1d) for zero I.C.: $y_0 = y_1 = y_2 = 0$, in the sense specified below

Theorem B.1 *(i) Under the appropriate regularity assumptions on the data:* $\{y_0, y_1, y_2\}$, g, *and* $\{v_0, v_1, v_2\}$*—to be made explicit below—the following identity holds true, where* $\langle\, , \,\rangle_\Omega$ *denotes the duality pairing with respect to* $H = L^2(\Omega)$ *and* $\langle\, , \,\rangle_\Gamma$ *denotes the duality pairing with respect to* $L^2(\Gamma)$*:*

$$\langle y_{tt}(T)+\alpha y_t(T), v_0\rangle_\Omega - \langle y_t(T)+\alpha y(T), v_1\rangle_\Omega + \langle y(T), v_2\rangle_\Omega - b\,\langle y(T), \Delta v_0\rangle_\Omega$$

$$+\langle y_0, -v_{tt}(0)+\alpha v_t(0)+b\Delta v(0)\rangle_\Omega + \langle y_1, v_t(0)-\alpha v(0)\rangle_\Omega - \langle y_2, v(0)\rangle_\Omega$$

$$-\left\langle c^2\frac{\partial y}{\partial \nu}+b\frac{\partial y_t}{\partial \nu}, v\right\rangle_{L^2(\Sigma)} + \left\langle c^2 y + b y_t, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(\Sigma)} = 0. \tag{B.3}$$

(ii) Consider the Dirichlet non-homogeneous condition (B.1c) with zero I.C. $y_0 = y_1 = y_2 = 0$, coupled with the corresponding homogeneous Dirichlet condition (B.2c). Then identity (B.3) specializes to

$$\langle y_{tt}(T)+\alpha y_t(T), v_0\rangle_\Omega - \langle y_t(T)+\alpha y(T), v_1\rangle_\Omega + \langle y(T), v_2\rangle_\Omega - b\,\langle y(T), \Delta v_0\rangle_\Omega$$
$$= -\left\langle c^2 g + b g_t, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(0,T;L^2(\Gamma))}. \tag{B.4}$$

(iii) Consider the Neumann non-homogeneous condition (B.1d) with zero I.C. $y_0 = y_1 = y_2 = 0$, coupled with the corresponding homogeneous Neumann condition (B.2d). Then identity (B.3) *specializes to*

$$\langle y_{tt}(T)+\alpha y_t(T), v_0\rangle_\Omega - \langle y_t(T)+\alpha y(T), v_1\rangle_\Omega + \langle y(T), v_2\rangle_\Omega$$
$$-b\,\langle y(T), \Delta v_0\rangle_\Omega$$
$$= \langle c^2 g + b g_t, v\rangle_{L^2(0,T;L^2(\Gamma))}. \tag{B.5}$$

Proof **Step 1** Multiply (B.1a) by v and integrate by parts. We obtain:

$$① = \int_\Omega \int_0^T y_{ttt} v\, dt d\Omega = \langle y_{tt}(T), v(T)\rangle_\Omega - \langle y_{tt}(0), v(0)\rangle_\Omega - \langle y_t(T), v_t(T)\rangle_\Omega$$
$$+\langle y_t(0), v_t(0)\rangle_\Omega + \langle y(T), v_{tt}(T)\rangle_\Omega - \langle y(0), v_{tt}(0)\rangle_\Omega - \int_\Omega \int_0^T y v_{ttt} dQ \tag{B.6}$$

$$② = \int_\Omega \int_0^T y_{tt} v\, dt d\Omega = \langle y_t(T), v(T)\rangle_\Omega - \langle y_t(0), v(0)\rangle_\Omega - \langle y(T), v_t(T)\rangle_\Omega$$
$$+\langle y(0), v_t(0)\rangle_\Omega + \int_\Omega \int_0^T y v_{tt} dQ \tag{B.7}$$

$$③ = \int_0^T \int_\Omega \Delta y v\, d\Omega dt = \int_0^T \left[\int_\Omega y \Delta v d\Omega + \int_\Gamma \frac{\partial y}{\partial \nu} v d\Gamma - \int_\Gamma y \frac{\partial v}{\partial \nu} d\Gamma\right] dt \tag{B.8}$$

$$④ = \int_0^T \int_\Omega \Delta y_t v \, d\Omega dt = \int_0^T \left[\int_\Omega y_t \Delta v d\Omega + \int_\Gamma \frac{\partial y_t}{\partial \nu} v d\Gamma - \int_\Gamma y_t \frac{\partial v}{\partial \nu} d\Gamma \right] dt \tag{B.9}$$

Step 2 We sum up: $① + \alpha② - c^2③ - b④ = 0$ and obtain

$$\begin{aligned} &\langle y_{tt}(T) + \alpha y_t(T), v(T)\rangle_\Omega - \langle y_t(T) + \alpha y(T), v_t(T)\rangle_\Omega + \langle y(T), v_{tt}(T)\rangle_\Omega - b\,\langle y(T), \Delta v(T)\rangle_\Omega \\ &+ \langle y(0), -v_{tt}(0) + \alpha v_t(0) + b\Delta v(0)\rangle_\Omega + \langle y_1, v_t(0) - \alpha v(0)\rangle_\Omega - \langle y_2, v(0)\rangle_\Omega \\ &- \int_Q y\left[v_{ttt} - \alpha v_{tt} + c^2 \Delta v - b \Delta v_t\right] dQ - \left\langle c^2 \frac{\partial y}{\partial \nu} + b\frac{\partial y_t}{\partial \nu}, \nu \right\rangle_{L^2(\Sigma)} + \left\langle c^2 y + b y_t, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(\Sigma)} = 0. \end{aligned} \tag{B.10}$$

Step 3 The $\int_Q$-term in (B.10) vanishes because of (B.2a). Next, we use the I.C. in (B.2b) for the v-problem at $t = T$, and identity (B.10) reduces to (B.3). Part (i) is proved.

Step 4 In the Dirichlet case, use $y\big|_\Sigma = 0$ in (B.1c) and $v\big|_\Sigma \equiv g$ in (B.2c). Then, identity (B.3) reduces to identity (B.4).

Step 5 In the Neumann case, use $\left.\frac{\partial y}{\partial \nu}\right|_\Sigma = g$ in (B.1d) and $\left.\frac{\partial v}{\partial \nu}\right|_\Sigma \equiv 0$ in (B.2d). Then identity (B.3) reduces to identity (B.5).

The next is a preliminary result.

Corollary B.2 *With reference to the Dirichlet-Problem # 1 in (B.1a)–(B.1d) with I.C. $y_0 = y_1 = y_2 = 0$ and corresponding Dirichlet Problem # 2 in (B.2a)–(B.2d), assume*

$$g \in H^1(0, T_1; L^2(\Gamma)), \quad \{v_0, v_1, v_2\} \in U_3 = \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times H. \tag{B.11}$$

Then, for any $t, 0 < t \le T_1$:

$$y(t), y_t(t), y_{tt}(t) \in H \times [\mathcal{D}(A^{\frac{1}{2}})]' \times [\mathcal{D}(A)]'. \tag{B.12}$$

Proof We have Theorem 6.1 for any $0 < T_1 < \infty$:

$$\{v_0, v_1, v_2\} \in U_3 \Longrightarrow \{v, v_t, v_{tt}\} \in C\left([0, T_1]; U_3 = \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times H\right) \tag{B.13}$$

so that just by trace theory

$$\frac{\partial v}{\partial \nu} \in C\left([0, T]; H^{\frac{1}{2}}(\Gamma)\right) \tag{B.14}$$

Thus, the RHS of identity (B.4) is well defined by (B.11), (B.14), finite on any finite time interval. Here T is an arbitrary point $0 < T \le T_1$. We then focus on the LHS of identity (B.4) to make sure that each term is well defined as a duality pairing w.r.t. $H = L^2(\Omega)$. We obtain

$$\left.\begin{array}{r}\left.\begin{array}{r} v_2 = v_{tt}(T) \in H \Longrightarrow y(T) \in H \\ v_1 = v_t(T) \in \mathcal{D}(A^{\frac{1}{2}}) \Longrightarrow y_t(T) + \alpha y(T) \in [\mathcal{D}(A^{\frac{1}{2}})]' \end{array}\right\} \Longrightarrow y_t(T) \in [\mathcal{D}(A^{\frac{1}{2}})]' \\ v_0 = v(T) \in \mathcal{D}(A) \Longrightarrow y_{tt}(T) + \alpha y_t(T) \in [\mathcal{D}(A)]' \end{array}\right\} \Longrightarrow y_{tt}(T) \in [\mathcal{D}(A)]'. \tag{B.15}$$

This takes care of the first three therms on the LHS of (B.4). Notice then that the fourth term $\langle y(T), \Delta v_0\rangle_\Omega$ is likewise automatically well-posed with $v_0 \in \mathcal{D}(A)$, $\Delta v_0 \in H = L^2(\Omega)$, $y(T) \in H$. □

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