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Weak and strong semigroups in structural acoustic Kirchhoff-Boussinesq interactions with boundary feedback ☆

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Abstract

We consider a structural-acoustic wall problem in three dimensions, in which the structural wall is modeled by a 2D Kirchhoff-Boussinesq plate and the acoustic medium is subject to boundary damping. For this model we study the existence of a continuous nonlinear semigroup associated with the model in the finite energy space. We show that strong/weak continuity of the semigroups depends on the support of the boundary damping. The complications are related to supercritical nonlinearity exhibited by the plate along with the compromised boundary regularity of the acoustic waves. Compensated compactness methods along with a hidden boundary regularity of hyperbolic traces are exploited in order to establish weak (resp. strong) generation of a nonlinear semigroup subjected to feedback forces placed on the boundary of the acoustic medium.

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1. Introduction

We consider structural acoustic interactions comprising of an acoustic medium modeled by an acoustic wave propagating in a 3-dimensional bounded domain with hard and elastic walls coupled with an elastic structure which is modeled by a plate equation with *rough* nonlinear effects resulting from internal and external forces affecting the structure. The coupling is of hybrid type and the plate oscillations act as a force on the acoustic medium, while the acoustic velocity induces oscillations of the structure. Structural acoustic models have acquired considerable attention in both engineering and mathematical literature, cf. [9,12,10,23,30,27] and references therein. This is due to an array of technological applications which includes noise suppression in acoustic chambers and pressure reduction in the cockpit of a helicopter. The associated modeling and resulting PDE systems became a rich source of mathematical analysis, see e.g. [7,13,27] and references therein. The basic dynamics is an interaction between the acoustics waves hitting the elastic walls which then, through oscillations, provide a feedback transferred back to the acoustic environment. Not surprisingly, the way both dynamics interact on a common interface (elastic wall) is the key element and the main carrier of propagation of effects emitted by each component. Mathematically, this part has been a source of challenges and recent discoveries. While each component of the system may have a well understood dynamical behavior, the interface effects introduce new phenomenological peculiarities which lead to new effects emerging for the overall structure.

As an acoustic medium domain we consider $\Omega \subset \mathbb{R}^3$ an open, bounded domain with boundary $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_0}$, where Γ_1, Γ_0 are relatively open and $\Gamma_1 \cap \Gamma_0 = \emptyset$. The structure wall will be represented by the portion Γ_0 of the boundary, which will be assumed flat. The latter is only assumed for simplicity in order to focus on the nonlinear aspects of the model. Curved walls and shells can also be considered by using intrinsic geometry tools [22]. The dynamics of the acoustic medium Ω are associated with the velocity potential function z, while the oscillating dynamics on the wall Γ_0 will be represented by the vertical displacement w. The structure is subject to an internal restoring nonlinear force $\operatorname{div}\{|\nabla w|^2 \nabla w\}$ and an external semilinear force Δw^2, typical for Boussinesq models. It is known that the Boussinesq plate alone, without restoring forces, can give rise to blowing up in finite-time solutions, see e.g. [32,36]. However, the presence of restoring forces will provide, as expected, some stabilizing effect for the low frequencies. On the other hand, it is precisely this term that introduces new challenges in the analysis of wellposedness of weak solutions due to its supercriticality and the fact that the term is not locally Lipschitz on the underlying phase space. The PDE model is described below.

Acoustic Medium. The following wave equation describes the temporal evolution of the acoustic dynamics in Ω

$$\begin{cases} z_{tt} - c^2 \Delta z + d(x) z_t = 0 & \text{in} \quad Q \equiv \Omega \times (0,\infty); \\ \partial_\nu z + l(x) z_t = \begin{cases} -l_0 z & \text{on} \quad \Sigma_1 \equiv \Gamma_1 \times (0,\infty); \\ w_t & \text{on} \quad \Sigma_0 \equiv \Gamma_0 \times (0,\infty); \end{cases} \\ z(0) = z_0; \quad z_t(0) = z_1 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $c > 0$, $l_0 \geq 0$. The function $l(x) \geq 0$ corresponds to a potential boundary dissipation and $d(x) \geq 0$ in Ω corresponds to a potential frictional damping in the acoustic environment. Both functions $l(x)$ and $d(x)$ are assumed sufficiently smooth.

Structural Wall. Let f be a continuous function. The following nonlinear 2D Kirchhoff-Boussinesq equation subjected to nonlinear restoring forces and a Boussinesq source describes the temporal dynamics on the wall Γ_0.

$$\begin{cases} w_{tt} + \Delta^2 w + k w_t + \rho z_t|_{\Gamma_0} = \mathsf{div}\{|\nabla w|^2 \nabla w\} + \sigma \Delta\{w^2\} - f(w) \text{ in } \Sigma_0; \\ \text{Boundary Conditions of type (C), (SS) or (F) on } \partial\Gamma_0 \times (0,\infty); \\ w(0) = w_0, \quad w_t(0) = w_1 \quad \text{on} \quad \Gamma_0; \end{cases} \tag{1.2}$$

where $\rho, \sigma, k \geq 0$, and the boundary conditions are given by

- *Clamped* (C): $w = 0$ and $\nabla w = 0$ on $\partial\Gamma_0 \times (0,\infty)$;
- *Simply Supported* (SS): $w = \Delta w = 0$ on $\partial\Gamma_0 \times (0,\infty)$;
- *Free* (F): $\begin{cases} \Delta w + (1-\mu) B_1 w = 0; \\ \partial_\nu \Delta w + (1-\mu) B_2 w = (|\nabla w|^2 \nabla w + \sigma \nabla [w^2]) \cdot \nu, \end{cases}$

on $\partial\Gamma_0 \times (0,\infty)$, where $\nu = (\nu_1, \nu_2)$ denotes the normal exterior vector to the boundary of Γ_0 and the boundary operators B_1 and B_2 are given by

$$B_1 u = 2\nu_1\nu_2 u_{x_1,x_2} - \nu_1^2 u_{x_2x_2} - \nu_2^2 u_{x_1x_1}; \tag{1.4}$$

$$B_2 u = \partial_\tau \left[(\nu_1^2 - \nu_2^2) u_{x_1x_2} + \nu_1\nu_2 (u_{x_2x_2} - u_{x_1x_1}) \right], \tag{1.5}$$

for every $u \in H^2(\Gamma_0)$, $\tau = (-\nu_2, \nu_1)$ is the unit tangential vector and $0 < \mu < 1/2$ is the Poisson's ratio.

Remark 1.1. The structural model governed by a nonlinear plate equation of Kirchhoff-Boussinesq type is a limit of Midlin Timoshenko system in 3 variables: two rotation angles of filaments and the transverse displacement, also known as bending component. Letting the parameters corresponding to shear and rotation tend to zero yields the nonlinear K-B plate for the variable representing transverse displacement. The rigorous asymptotic argument is provided in [19,29].
It is known that the presence of the "Boussinesq" forcing term ($\sigma > 0$) may lead to blow-up in finite-time of structural solutions. This effect is counteracted by a restoring internal force $\mathit{div}\{|\nabla w|^2 \nabla w\}$. In some sense, the competition between the two determines the global behavior of the model. At the same time, the restoring force gives rise to supercritical nonlinear terms making the analysis of wellposedness challenging. In order to handle these effects, strong damping was added to the plate equation in earlier works [32,36]. The latter has regularizing effect on the dynamics making it of parabolic nature. (Semigroup generated by the linear plate model becomes analytic.) Our goal is *to refrain from introducing these regularizing effects* by considering the hyperbolic-like character of the plate equation. This leads to a subtle analysis of the effects caused by the supercritical (not locally Lipschitz) nonlinear terms affecting the plate in conjunction with *boundary* dissipation affecting the acoustic medium.

Remark 1.2. Boundary conditions imposed on the structure include the three basic sets: clamped, hinged and free. However, from the mathematical point of view, free boundary conditions are the most challenging as they present several subtleties in the study. One of the reasons is that $\Delta w \in L_2(\Gamma_0)$ does not control the $H^2(\Gamma_0)$ topology. On the other hand, free boundary conditions have a lot of applications in structural theories. This is the reason for the emphasis on free boundary conditions in this work.

Our main goal is to study the evolution of the system (1.1)-(1.2), which describes the interaction between the waves propagating in the acoustic medium and the vibration of a portion of the structural wall that confines the medium. Since the ultimate goal of such studies is the long-time behavior of finite energy solutions, we will consider dissipative effects acting upon the interaction such as frictional $k > 0, d(x) \geq 0$ and boundary $l(x) \geq 0$ dissipation. The model is of interest not only because of the presence of a "supercritical" restoring term, given by the divergence, but also the semilinear energy building source ($\sigma > 0$) potentially causing a blow-up of the plate energy, and in general the so called "leak of the energy". The combination of the two produces interesting phenomenological effects for which some of the analytical tools have been developed in the past within the context of plate theory, cf. [19,20] and references therein. However, our main interest is in studying interaction between plate oscillations and propagation of acoustic waves in an acoustic medium. Such interaction is of hybrid type, where the spatial domain Γ_0 supporting plate oscillations becomes part of the boundary of the acoustic medium. It is precisely the interaction between the "leak of energy", supercriticality of restoring forces acting upon the structure and boundary forces acting upon the acoustic medium which brings new and interesting mathematical and phenomenological phenomena at the level of wellposedness (uniqueness, robustness and regularity) and stability of weak solutions to the entire hybrid interaction. It should also be noted that the structural dynamics does not account for any regularizing effects (like Kelvin-Voigt damping making the plate dynamics related to analytic semigroup), where the latter was widely considered in past literature [2,10,11,23].

Structural acoustic models with an interface between structural and acoustic medium have been of major interest due to an array of applications arising in engineering and life sciences. For more details on structural acoustic models we refer to [10,11,16,23,27] and references therein, mentioning suppression of noise in an helicopter, control of sound in an acoustic chamber as examples. On the other hand, mathematically, they provide an interesting problem due to an interface where the interaction and propagation of effects take place on the boundary, see [1,5,10,26]. It is known that boundary behavior of solutions in hyperbolic-like dynamics (without inherent smoothing) is challenging and requires a number of a priori estimates, often based on microlocal analysis which exhibits peculiar behavior of boundary hyperbolic traces. In fact, there has been a considerable activity in this area, also within a context of acoustic models, see e.g. [16,22,26,34]. One of the findings in the present work is that boundary behavior of the traces to an acoustic pressure plays dominant role in the analysis of supercritical nonlinear effects of the elastic medium and in propagating strong Hadamard wellposedness for the entire structure. Thus, the interaction between the two media and propagation of relevant effects is at the heart of the problem and a source of challenges.

In fact, the final result depends on the type of dissipation imposed on the acoustic environment. In case when the dissipation is internal $d(x) > 0, l = 0$, the resulting dynamical system provides a continuous flow with respect to the strong topology of the phase space. In case when the acoustic dissipation is localized on the boundary $l(x) > 0$, the flow is generally continuous (with respect to the initial data) in a *weak* topology only. The main reason is due to the fact that in the latter case energy *identity for weak solution* may not hold in general. The method based on time reversibility fails as the reverse acoustic dynamic is ill-posed while the method based on finite difference approximation (see [25]) appears to fail due to supercritical nonlinearity of the structure. Therefore, derivation of energy equality for weak solutions is a challenging issue in the boundary case. However, in situations where boundary dissipation has a "strategic" placement, the two methods referred to above cooperate leading to strong semigroup.

The rest of this paper is organized as follows: Section 2 presents the main result while Section 3 provides necessary auxiliary results and analysis of the linear associated problem along with several nonlinear estimates. Finally, Section 4 is dedicated to the proof of the main result stated in Section 2.

2. Main results

Our objective in this paper is to establish wellposedness of the dynamics represented by a coupled PDE system presented above, which should culminate with the statement of representation of the system by a dynamical system defined on a "natural" phase space. To this end, Hadamard wellposedness of the semi-flow at the finite energy level, is the main goal. The corresponding results are formulated in the present section while the proofs are relegated to Sections 3 and 4.

2.1. Notation

We begin by introducing some notation that will be used thorough the text. We consider the Hilbert space $\mathcal{H} \equiv H_z \times H_w$, also called *finite energy space*, where $H_z \equiv H^1(\Omega) \times L_2(\Omega)$ and $H_w \equiv H \times L_2(\Gamma_0)$, and H depends on the boundary conditions imposed on the structure:

$$H \equiv \begin{cases} H_0^2(\Gamma_0) & \text{if } (C); \\ H^2(\Gamma_0) \cap H_0^1(\Gamma_0) & \text{if } (SS); \\ H^2(\Gamma_0) & \text{if } (F). \end{cases}$$

The total energy functional associated with solutions of system (1.1)-(1.2) and induced by the topology of $\mathcal{H}$ is given by

$$\mathcal{E}(t) \equiv E(t) + \sigma \int_{\Gamma_0} w\,|\nabla w|^2 d\Gamma_0 + \int_{\Gamma_0} F(w) d\Gamma_0, \tag{2.1}$$

where F is the antiderivative of f, as specified in Assumption 3.1, and E denotes the positive part of the total energy, given by $E(t) \equiv E_z(t) + E_w(t)$, where E_z and E_w stand for the portions corresponding to the wave and plate equations, respectively, and are given by

$$E_z(t) \equiv \frac{1}{2} \int_{\Omega} |z_t|^2 + c^2 |\nabla z|^2 d\Omega + \frac{c^2 l_0}{2} \int_{\Gamma_1} \left| z|_{\Gamma_1} \right|^2 d\Gamma_1, \tag{2.2a}$$

$$E_w(t) \equiv \frac{1}{2} \int_{\Gamma_0} \left[|w_t|^2 + \frac{1}{2} |\nabla w|^4 \right] d\Gamma_0 + \frac{1}{2} a(w,w), \tag{2.2b}$$

where $a(\cdot,\cdot)$ is a bilinear form in $H^2(\Gamma_0)$ given by

$$a(u,v)=\begin{cases}\displaystyle\int_{\Gamma_0}\Delta u\Delta v d\Gamma_0, \quad \text{in the cases (C) and (SS);}\\ \displaystyle\int_{\Gamma_0}\left[u_{x_1x_1}v_{x_1x_1}+u_{x_2x_2}v_{x_2x_2}\right]d\Gamma_0+\int_{\Gamma_0}\left[\mu(u_{x_1x_1}v_{x_2x_2}+u_{x_2x_2}v_{x_1x_1})\right]d\Gamma_0\\ \quad +\displaystyle\int_{\Gamma_0}2(1-\mu)u_{x_1x_2}v_{x_1x_2}d\Gamma_0, \quad \text{in the case (F),}\end{cases} \tag{2.3}$$

for $u, v \in H^2(\Gamma_0)$.

Remark 2.1. Throughout the text, we will assume the following relation among the coefficients: $1 - c^{-2}\rho = 0$. This is without a loss of generality. Simple rescaling of the energy allows to eliminate this condition.

Notice that the functional E is equivalent to the usual topology of $\mathcal{H}$. On the other hand, the total energy $\mathcal{E}$ is not positive and does not exhibit dissipative aspect. Indeed, a straightforward *formal* computation of the energy identity shows that $\mathcal{E}(t)$ satisfies the following equation

$$\frac{d}{dt}\mathcal{E}(t)+k\int_{\Gamma_0}|w_t(t)|^2\,d\Gamma_0+c^2\int_{\Gamma}l(x)\,|z_t(t)|_\Gamma|^2\,d\Gamma+\int_{\Omega}d(x)\,|z_t(t)|^2\,d\Omega$$
$$=\sigma\int_{\Gamma_0}w_t(t)\,|\nabla w(t)|^2\,d\Gamma_0.$$

The above identity illustrates the so called "leak" of energy when $\sigma > 0$.

2.2. Formulation of the main result

In order to establish our wellposedness result, we start by providing the definition of *weak* and *strong* solutions of the system (1.1)-(1.2). We say a pair of functions (z, w) is a *weak solution* on the interval $[0, T]$, for $T > 0$, if $(z, z_t, w, w_t) \in L_\infty(0, T; \mathcal{H})$. Moreover, the following properties are satisfied:

i. the map $t \in [0, T] \mapsto (z(t), z_t(t), w(t), w_t(t)) \in \mathcal{H}$ is weakly continuous and, in addition, $l^{\frac{1}{2}}z_t|_\Gamma \in L_2(0, T; L_2(\Gamma))$;
ii. $z(0) = z_0$, $z_t(0) = z_1$, $w(0) = w_0$ and $w_t(0) = w_1$;
iii. the pair (z, w) is a distributional (in time) solution of the following equation

$$\begin{aligned}0=&\frac{d}{dt}\left[(z_t(t),\phi)_{L_2(\Omega)}+(w_t(t),\psi)_{L_2(\Gamma_0)}+\rho(z(t)|_{\Gamma_0},\psi)_{L_2(\Gamma_0)}\right]+c^2(\nabla z(t),\nabla\phi)_{L_2(\Omega)}\\&+a(w(t),\psi)+(d\cdot z_t(t),\phi)_{L_2(\Omega)}+c^2(l^{\frac{1}{2}}z_t(t)|_\Gamma,l^{\frac{1}{2}}\phi|_\Gamma)_{L_2(\Gamma)}+k(w_t(t),\psi)_{L_2(\Gamma_0)}\\&-c^2(w_t(t),\phi|_{\Gamma_0})_{L_2(\Gamma_0)}+c^2l_0(z(t)|_{\Gamma_1},\phi|_{\Gamma_1})_{L_2(\Gamma_1)}+G(w(t),\psi),\end{aligned} \tag{2.4}$$

for all $(\phi, \psi) \in H^1(\Omega) \times H$, where

$$G(u, v) \equiv (|\nabla u|^2 \nabla u, \nabla v)_{[L_2(\Gamma_0)]^2} + \sigma(\nabla\{u^2\}, \nabla v)_{[L_2(\Gamma_0)]^2} + (f(u), v)_{L_2(\Gamma_0)}, \tag{2.5}$$

for every $u, v \in H^2(\Gamma_0)$. Furthermore, we say that a weak solution (z, w) in the interval $[0, T]$ is *strong* (classical) if $(z, z_t, z_{tt}) \in C(0, T; H^2(\Omega) \times H_z)$ and $(w, w_t, w_{tt}) \in C(0, T; H^4(\Gamma_0) \times H_w)$.

Our main result reads.

Theorem 2.1 *(Existence, uniqueness, Hadamard wellposedness and regularity). Assume that $f \in C^1(\mathbb{R})$. With respect to the dynamics of system (1.1)-(1.2), subject to any of the boundary conditions (C), (SS) or (F), the following holds:*

1. **Existence of a strongly continuous semigroup in a weak topology of the phase space $\mathcal{H}$.** *For every initial data $U_0 \equiv (z_0, z_1, w_0, w_1) \in \mathcal{H}$, there exists a unique weak solution (z, w) in the class $U \equiv \{(z, z_t, w, w_t) \in C([0, T]; \mathcal{H})\}$, for some $T > 0$, which may depend on the initial data. This solution is global ($T > 0$ can be taken arbitrary) provided that f satisfies the non-explosion Assumption 3.1. Furthermore, the solutions generate a strongly continuous semigroup S_t with respect to **weak topology** in $\mathcal{H}$, given by the formula*

$$S_t U_0 \equiv U(t), \quad \text{for every } U_0 \in \mathcal{H}. \tag{2.6}$$

This is to say that for $U_{0n} \rightharpoonup U_0$ in $\mathcal{H}$ one has $U_n(t) \rightharpoonup U(t)$ in $\mathcal{H}$ uniformly in $t \in [0, T]$. Continuous dependence of solutions on the initial data is with respect to weak topology of $\mathcal{H}$. Corresponding solutions satisfy the following energy inequality

$$\begin{aligned}\mathcal{E}(t) + k\int_0^t \|w_t(s)\|^2_{L_2(\Gamma_0)} ds + \int_0^t \|d^{\frac{1}{2}} z_t(s)\|^2_{L_2(\Omega)} ds + \int_0^t \|l^{\frac{1}{2}} z_t(s)|_\Gamma\|^2_{L_2(\Gamma)} ds \\ \le \mathcal{E}(0) + \sigma\int_0^t (w_t(s), |\nabla w(s)|^2)_{L_2(\Gamma_0)} ds.\end{aligned} \tag{2.7}$$

2. **Energy identity and strong continuity with respect to the initial data**. *Let us impose the non-explosion condition (3.1). In case when $l(x) = 0$ on Γ or $\Gamma_0 \subset \operatorname{supp} l(x)$, the weak solution (z, w) must satisfy the following energy identity:*

$$\begin{aligned}\mathcal{E}(t) + k\int_0^t \|w_t(s)\|^2_{L_2(\Gamma_0)} ds + \int_0^t \|d^{\frac{1}{2}} z_t(s)\|^2_{L_2(\Omega)} ds + \int_0^t \|l^{\frac{1}{2}} z_t(s)|_\Gamma\|^2_{L_2(\Gamma)} ds \\ = \mathcal{E}(0) + \sigma\int_0^t (w_t(s), |\nabla w(s)|^2)_{L_2(\Gamma_0)} ds.\end{aligned} \tag{2.8}$$

In addition, the nonlinear semigroup in part 1 becomes ***continuous with respect to the*** *strong* ***topology of*** $\mathcal{H}$.

3. **Regularity**. *Assume that Assumption 3.1 is in place. In addition assume that* Ω *is sufficiently smooth. Weak solutions defined above become strong provided that* $U_0 \in H^2(\Omega) \times H^1(\Omega) \times H^4(\Gamma_0) \times H^2(\Gamma_0)$ *satisfies the following compatibility conditions:*

   For the acoustic medium:

$$\partial_\nu z_0 + l(x) z_1 = \begin{cases} -l_0 z_0 & on\ \Gamma_1 \\ w_1 & on\ \Gamma_0 \end{cases}; \tag{2.9a}$$

   For the structural wall:

   ii. *in the clamped (C) case:* $w_0 = w_1 = 0$ *and* $\nabla w_0 = \nabla w_1 = 0$ *on* $\partial\Gamma_0$*;*
   iii. *in the simply supported (SS) case:* $w_0 = w_1 = 0$ *and* $\Delta w_0 = 0$ *on* $\partial\Gamma_0$*;*
   iv. *in the free (F) case:*

$$\begin{cases} \Delta w_0 + (1-\mu) B_1 w_0 = 0, & on\ \partial\Gamma_0; \\ w_0 + (1-\mu) B_2 w_0 = (|\nabla w_0|^2 \nabla w_0, \nu)_{\mathbb{R}^2} + \sigma(\nabla\{w_0^2\}, \nu)_{\mathbb{R}^2}, & on\ \partial\Gamma_0, \end{cases} \tag{2.9b}$$

The above result shows that system (1.1)-(1.2) always defines a *weak* semigroup in the finite energy space $\mathcal{H}$. It is the effect of the boundary dissipation in the acoustic medium $l(x) \geq 0$ which may compromise continuity properties of this semigroup. However, a strategic placement of the dissipation allows to upgrade the weak continuity of the semiflow to a strong one. See part 2 of Theorem 2.1. The latter is due to the validity of the energy *identity* (2.8) an essential tool in this study.

Remark 2.2. As can be seen from the energy balance, the system is not dissipative. There is a "leak" of energy on Γ_0. In case when the boundary damping is active but *supp* $l(\cdot) \cap \Gamma_0$ is strictly contained in Γ_0. Part 1 of Theorem above leads to an existence of semigroup continuous in a *weak* topology only. Whether this weak continuity could be improved to a strong one is at present an open problem.

2.3. Comments

1. Structural acoustic models [12,30] have attracted considerable attention in both engineering and mathematical literature. We shall focus on the latter. A series of papers [9–11,23] studied control problems (piezoceramic, piezoelectric patches) formulated for linear models, often with Kelvin-Voigt damping imposed on the plate. These works were followed by [1,6,7,2,3,5,4,27], still within the context of linear models, with boundary/point, possibly nonlinear feedback controls. More recently, nonlinear plate/shell models, being more accurate from the physical point of view, have attracted much attention, particularly with respect to long-time behavior [16]. Typical models accounted for semilinear effects in acoustic waves along with large displacement models in plate theory such as Berger or Von Kármán [17,21]. Clearly, any theory of long-time behavior depends on a good understanding of the dynamical system associated with the flow [8,21]. While in case of Kármán-type models, the developments of past years in the area of existence, uniqueness and related compensated compactness [24] provide a good footing and background, this is not the case for K-B model under consideration in this paper. The supercriticality of the nonlinear internal force in the plate is a predicament for a construction of dynamical system, with respect to both weak and strong topology of the natural phase space. We note that in [13], supercritical terms in the plate model are mitigated by a supercritical structural damping. This is not the case in the present paper where the coupling via the boundary with the Neumann operator prevents standard estimates to be applied. The latter has to do with the fact that Lopatinski condition is not satisfied ($\dim \Omega > 1$), thus L_2 Neumann boundary data do not produce finite energy solutions. In the past, this predicament was circumvented by considering more regular plate models. In the present work, we shall rely on “hidden regularity” of solutions to acoustic wave equations [34] and compensated compactness methods associated with supercriticality of PDEs describing elastic structures. The key element of the analysis relies on exploiting boundary damping of an acoustic medium and its interaction via interface with the plate oscillations in order to establish full Hadamard wellposedness of the solutions. Interestingly enough, boundary damping alone leads to the problem of making the structure not time reversible, thus preventing well-established methods to show the energy *equality* to hold [8]. On the other hand, when the damping is properly placed, it does provide a mechanism for proving full Hadamard wellposedness of the entire structure, as ascertained by our main result. This is achieved by combining hidden regularity for the dynamic Neumann operator with the damping along with finite difference approximation developed in [20,25].

2. The model under consideration is the simplest one which exhibits the main feature/difficulties of the problem under study. These are (i) unbounded on the phase space nonlinearity along with (2) boundary damping on the acoustic medium. One could consider curved walls [22], extra semilinear terms in the acoustic wave or boundary damping in the plate, as a mechanism of stabilizing oscillations [1,19]. However, we opt for the simplest possible model where the features to be emphasized are the main focus.

3. Let us make a few comments on the strategy pursued for the proofs. Existence of weak solutions is proved rather standard Galerkin method supported by several critical estimates presented in Section 3. The key element in constructing *weak* semigroup is the uniqueness of weak solutions. This is accomplished (Section 4.1) by controlling the blow-up of $L_p(\Gamma_0)$ norms for finite-dimensional projections of $H^1(\Gamma_0)$ functions. In order to claim *strong* semigroup property, the energy *identity* satisfied by all *weak* solutions is an essential ingredient. It is here where the interaction between acoustic and structural media plays a dominant role, in particular, the “hidden regularity” of hyperbolic traces in the non-Lopatinski case. In fact, such identity is derived (Section 4.2) when the support of boundary dissipation contains Γ_0. Finally, the regularity of weak solutions is obtained in Section 4.3 by deriving an appropriate a priori bound satisfied by finite dimensional Galerkin approximations. This is possible due to the logarithmic control of Sobolev’s imbedding valid for $H^1(\Gamma_0) \cap H^2(\Gamma_0)$ functions. However, it should be noted that his task is particularly subtle in case of *free* boundary conditions which “spill over” unbounded and uncloseable trace operators in the variational formulation.

3. Preliminaries

In this section, we provide some preliminary estimates used in the proof of the main results.

3.1. Assumptions

In order to obtain global solutions and study the asymptotic behavior, the following assumption on the source term is imposed.

Assumption 3.1. Assume $f \in C^1(\mathbb{R})$ satisfies the following non-explosion condition

i. for the clamped (C) and simply supported (SS) cases:

$$F(s) \equiv \int_0^s f(\tau)d\tau \geq -\delta s^4 - \beta, \ \forall s \in \mathbb{R}, \tag{3.1a}$$

for some $\delta \geq 0$ sufficiently small and $\beta_\delta \in \mathbb{R}$;

ii. for the free (F) case:

$$F(s) \geq -\delta s^2 - \beta, \ \forall s \in \mathbb{R}, \tag{3.1b}$$

for some $\delta, \beta \in \mathbb{R}$.

Remark 3.1. Inequality (3.1a) holds for $\delta > 0$ arbitrarily small if

$$v(p) \equiv \liminf_{|s|\to\infty} \frac{f(s)}{s\,|s|^{p-1}} \geq 0, \tag{3.2}$$

for $p = 3$. Inequality (3.1b) will remain true when $v(1) > -\infty$.

3.2. A priori lower bound estimate

Let (z, w) be a weak solution to (1.1)-(1.2). Using the notation for the energy introduced in Section 1, we have

$$E_1(t) \equiv E(t) + \sigma(w(t), |\nabla w(t)|^2)_{L_2(\Gamma_0)}, \tag{3.3}$$

$$|(\sigma(w(t), |\nabla w(t)|^2)_{L_2(\Gamma_0)}| \leq \frac{\sigma}{2\alpha} \|\nabla w(t)\|^4_{L_4(\Gamma_0)} + \frac{\alpha\sigma}{2} \|w(t)\|^2_{L_2(\Gamma_0)}, \tag{3.4}$$

for any $\alpha > 0$. In the case of clamped (C) or simply supported (SS) boundary conditions, we can use Poincaré's inequality in order to estimate the L_2 norm as follows

$$\|w\|^2_{L_2(\Gamma_0)} \leq \frac{C^2\,|\Gamma_0|}{4\varepsilon} + \varepsilon\,\|\nabla w\|^4_{L_4(\Gamma_0)},$$

for any $\varepsilon > 0$. In this situation, we rewrite (3.4) as follows

$$|\sigma(w(t), |\nabla w(t)|^2)_{L_2(\Gamma_0)}| \leq \frac{\sigma\alpha C^2\,|\Gamma_0|}{8\varepsilon} + \left(\frac{\sigma\varepsilon\alpha}{2} + \frac{\sigma}{2\alpha}\right) \|\nabla w(t)\|^4_{L_4(\Gamma_0)}, \tag{3.5}$$

for any $\varepsilon, \alpha > 0$ and $w \in H^2(\Gamma_0) \cap H^1_0(\Gamma_0)$. Now, choosing $\alpha > 0$ and $\varepsilon > 0$ such that $\sigma\left(\frac{\varepsilon\alpha}{2} + \frac{1}{2\alpha}\right) = \frac{1}{8}$, we conclude

$$|\sigma(w(t), |\nabla w(t)|^2)_{L_2(\Gamma_0)}| \leq C(\Gamma_0) + \frac{1}{8}\|\nabla w(t)\|^4_{L_4(\Gamma_0)} \quad \text{if (C) or (SS).} \tag{3.6}$$

In the case of free (F) boundary conditions, Poincaré's inequality cannot be used. Therefore, choosing $\alpha = 4\sigma$ in (3.4), we obtain

$$|\sigma(w(t), |\nabla w(t)|^2)_{L_2(\Gamma_0)}| \leq \frac{1}{8} \|\nabla w(t)\|^4_{L_4(\Gamma_0)} + 2\sigma^2 \|w(t)\|^2_{L_2(\Gamma_0)} \quad \text{if (F)}. \tag{3.7}$$

Inequalities (3.6) and (3.7) allow us to conclude the following lower estimate for energy $E_1(t)$ given by (3.3):

$$\frac{1}{2}\left[E_z(t) + E_w(t)\right] - C(\Gamma_0) \leq E_1(t) \qquad \text{if (C) or (SS);} \tag{3.8a}$$

$$\frac{1}{2}\left[E_z(t) + E_w(t)\right] - 2\sigma^2 \|w(t)\|^2_{L_2(\Gamma_0)} \leq E_1(t) \qquad \text{if (F).} \tag{3.8b}$$

Combining (3.8a) and (3.8b) yields the following estimate valid for any of the boundary conditions under consideration.

$$\frac{1}{2} E(t) \leq E_1(t) + C(\Gamma_0) + \sigma^2 \|w(t)\|^2_{L_2(\Gamma_0)}. \tag{3.9}$$

Finally, it follows from (3.8a), (3.8b) and Assumption 3.1 that, for a proper choice of $\delta \in \mathbb{R}$, the total energy $\mathcal{E}(t)$ is bounded from below by its positive part, that is to say: there exist constants $C_1 > 0$ and $M_0 \in \mathbb{R}$ such that

$$C_1 E(t) + M_0 \leq \mathcal{E}(t), \quad \text{for } t \geq 0. \tag{3.10}$$

This inequality will be used in the next section for the proof of the main results.

3.3. Semigroup formulation of the linear problem

Let us consider in this section the following linear system associated with (1.1)-(1.2) and given by

Wave Problem:

$$\begin{cases} z_{tt} - c^2 \Delta z + z + d(x) z_t = 0 & \text{in} \quad Q; \\ \partial_\nu z + l(x) z_t = \begin{cases} -l_0 z & \text{in} \quad \Sigma_1; \\ w_t & \text{in} \quad \Sigma_0; \end{cases} \\ z(0) = z_0; \quad z_t(0) = z_1 & \text{in} \quad \Omega, \end{cases} \tag{3.11}$$

Plate Problem:

$$\begin{cases} w_{tt} + \Delta^2 w + k w_t + \rho z_t|_{\Gamma_0} = 0 \quad \text{in} \quad \Sigma_0; \\ \text{Boundary Conditions of type (C), (SS) or (F) on } \partial\Gamma_0 \times (0, \infty); \\ w(0) = w_0, \quad w_t(0) = w_1 \quad \text{on} \quad \Gamma_0; \end{cases} \tag{3.12}$$

In order to express the system in a semigroup framework, let A be an extension to $L_2(\Omega)$ of the Laplace operator with Robin-like boundary condition, given by

$$A \equiv -c^2\Delta + I$$
$$D(A) = \left\{ v \in H^1(\Omega) : \ \Delta v \in L_2(\Omega), \quad \partial_\nu v = 0 \text{ on } \Gamma_0 \quad \text{and} \quad \partial_\nu v + l_0 v = 0 \text{ on } \Gamma_1 \right\}.$$

The associate Neumann operators $N_i : L_2(\Gamma_i) \to L_2(\Omega)$, $i = 0, 1$ are defined by

$$v \in L_2(\Gamma) \mapsto N_i v = \phi \quad \text{iff } \phi \text{ is the solution of} \quad \begin{cases} (-c^2\Delta + I)\phi = 0, & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu}\phi + l_0\phi|_{\Gamma_1} = \begin{cases} v, & \text{in } \Gamma_i, \\ 0, & \text{in } \Gamma \backslash \Gamma_i. \end{cases} \end{cases}$$

Remark 3.2. It is well-known that A is a self-adjoint operator with a compact resolvent. Therefore, the fractional powers of A are well-defined, in particular, we have $D(A^{\frac{1}{2}}) = H^1(\Omega)$. Also, we are going to consider $H^1(\Omega)$ with the equivalent norm

$$\|u\|^2_{D(A^{\frac{1}{2}})} = \|A^{\frac{1}{2}}u\|^2_{L_2(\Omega)} = \int_\Omega c^2 |\nabla u|^2 + |u|^2 \, d\Omega + c^2 l_0 \int_{\Gamma_1} \left|u|_{\Gamma_1}\right|^2 d\Gamma_1,$$

for all $u \in D(A^{\frac{1}{2}})$.

Remark 3.3. We recall that, if N_i^* stands for the corresponding adjoint operator of N_i, then $N_i^* A u = c^2 u|_{\Gamma_i}$ for any $u \in H^1(\Omega)$ (see [28]) is the trace operator. In particular, if we consider $N_0 v = N\tilde{v}$ for every $v \in L_2(\Gamma_0)$, where $\tilde{v}$ stands for the extension of v to Γ by zero outside of Γ_0, then $N_0^* A u = c^2 u|_{\Gamma_0}$ for every $u \in H^1(\Omega)$.

In connection with the plate problem, we will consider the extension $\mathcal{A}$ to $L_2(\Gamma_0)$ of the biharmonic operator, given by

$$\mathcal{A} \equiv \Delta^2,$$
$$D(\mathcal{A}) = \begin{cases} H^4(\Gamma_0) \cap H_0^2(\Gamma_0) & \text{if (C);} \\ \left\{ v \in H^4(\Gamma_0) : \ v = 0 \text{ and } \Delta v = 0 \text{ in } \partial\Gamma_0 \right\} & \text{if (SS);} \\ \left\{ v \in H^4(\Gamma_0) \left| \begin{array}{l} \Delta u + (1-\mu)B_1 v = 0 \\ \partial_\nu \Delta v + (1-\mu) B_2 v = 0 \end{array} \right. \text{on } \partial\Gamma_0 \right\} & \text{if (F).} \end{cases}$$

Using the notation above, we will consider the following linear abstract problem associated with the system (3.11)-(3.12)

$$\begin{cases} z_{tt} + A(z + [N_0 l(x) N_0^* + N_1 l(x) N_1^*] A z_t - N_0 w_t) + dI z_t = 0, \\ w_{tt} + \mathcal{A} w + k\mathcal{I} w_t + c^{-2}\rho N_0^* A z_t = 0, \\ (z(0), z_t(0), w(0), w_t(0)) = (z_0, z_1, w_0, w_1) \in \mathcal{H}, \end{cases} \tag{3.13}$$

where I and $\mathcal{I}$ stand for identity operators on $L_2(\Omega)$ and $L_2(\Gamma_0)$, respectively.

Remark 3.4. Recall that (see [19]) for every $s \in [0, 1/2]$ we have

$$D(\mathcal{A}^s) = \begin{cases} H_0^{4s}(\Gamma_0), \ s \neq 1/8, 3/8 & \text{if } (C); \\ H^{4s}(\Gamma_0) \cap H_0^1(\Gamma_0), \ s \geq 1/4 & \text{if } (SS); \\ H_0^{4s}(\Gamma_0), \ s < 1/4, \ s \neq 1/8 & \text{if } (SS); \\ H^{4s}(\Gamma_0), & \text{if } (F). \end{cases}$$

Moreover, corresponding Sobolev norms are equivalent to the graph norm of respective fractional powers of $\mathcal{A}$, that is to say: there exist constants $c_1, c_2 > 0$ such that

$$c_1 \left\| \mathcal{A}^s u \right\|_{L_2(\Gamma_0)} \leq \|u\|_{H^{4s}(\Gamma_0)} \leq c_2 \left\| \mathcal{A}^s u \right\|_{L_2(\Gamma_0)},$$

for all admissible $s \in [0, \frac{1}{2}]$, and $u \in D(\mathcal{A}^s)$.

Denoting $U = (z, z_t, w, w_t)$ and $U_0 = (z_0, z_1, w_0, w_1)$, the system (3.13) can be written in the form

$$\frac{d}{dt}U(t) - \mathbf{A}U(t) = 0; \qquad U(0) = U_0,$$

where $\mathbf{A} : D(\mathbf{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathbf{A} \equiv \begin{pmatrix} 0 & I & 0 & 0 \\ -A & -A[N_0 l N_0^* - N_1 l N_1^*]A - dI & 0 & AN_0 \\ 0 & 0 & 0 & \mathcal{I} \\ 0 & -c^{-2}\rho N_0^* A & -\mathcal{A} & -k\mathcal{I} \end{pmatrix};$$

$$D(\mathbf{A}) = \left\{ (u_1, u_2, u_3, u_4) \in \mathcal{H} \,\middle|\, \begin{array}{r} u_1 \in H^2(\Omega); \quad u_2 \in D(A^{\frac{1}{2}}); \\ u_1 + [N_0 l N_0^* + N_1 l N_1^*]Au_2 - N_0 u_4 \in D(A); \\ u_3 \in D(\mathcal{A}); \quad u_4 \in D(\mathcal{A}^{\frac{1}{2}}). \end{array} \right\}.$$

Remark 3.5. Straightforward computations show that the adjoint $\mathbf{A}^*$ has a similar structure to $\mathbf{A}$ with same domain.

Lemma 3.1. *The operators* $\mathbf{A}$ *and* $\mathbf{A}^*$ *are dissipative.*

Proof. It is sufficient to prove that $\mathbf{A}$ is dissipative. Let $U = (u_1, u_2, u_3, u_4) \in D(\mathbf{A})$. We have

$$\begin{aligned} (\mathbf{A}U, U)_{\mathcal{H}} &= \left(\begin{pmatrix} u_2 \\ -A(u_1 + [N_0 l N_0^* + N_1 l N_1^*]Au_2 - N_0 u_4) - du_2 \\ u_4 \\ -c^{-2}\rho N_0^* A u_2 - \mathcal{A}u_3 - ku_4 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \right)_{\mathcal{H}} \\ &= (u_2, u_1)_{D(A^{1/2})} - \left(A(u_1 + [N_0 l N_0^* + N_1 l N_1^*]Au_2 - N_0 u_4), u_2\right)_{L_2(\Omega)} \\ &\quad - (du_2, u_2)_{L_2(\Omega)} \end{aligned}$$

$$+ (u_4, u_3)_{D(\mathcal{A}^{1/2})} - c^{-2}\rho\left(N_0^* A u_2, u_4\right)_{L_2(\Gamma_0)} - (\mathcal{A}u_3, u_4)_{L_2(\Gamma_0)} - k\,(u_4, u_4)_{L_2(\Gamma_0)}$$
$$= -k\,\|u_4\|^2_{L_2(\Gamma_0)} - \|d^{\frac{1}{2}} u_2\|^2_{L_2(\Omega)} - \|l^{\frac{1}{2}} u_2|_\Gamma\|^2_{L_2(\Gamma)} \le 0,$$

which proves the result. □

Lemma 3.2. *The operator* $\mathbf{A}$ *is maximally dissipative and, consequently, so is* $\mathbf{A}^*$.

Proof. Since $\mathbf{A}$ is dissipative, it suffices to show that $\mathbf{I} - \mathbf{A}$ is onto, that is $R(\mathbf{I} - \mathbf{A}) = \mathcal{H}$, see Theorem 4.6 in [31]. Therefore, letting $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$, consider the resolvent equation

$$F = (\mathbf{I} - \mathbf{A})U \Longleftrightarrow \begin{cases} f_1 = u_1 - u_2 \\ f_2 = u_2 + A(u_1 + [N_0 l N_0^* + N_1 l N_1^*]Au_2 - N_0 u_4) + du_2 \\ f_3 = u_3 - u_4 \\ f_4 = u_4 + c^{-2}\rho N_0^* A u_2 + \mathcal{A}u_3 + ku_4 \end{cases} \tag{3.14}$$

where $U = (u_1, u_2, u_3, u_4)$. Plugging $u_1 = f_1 + u_2$ and $u_3 = f_3 + u_4$ to Equations $(3.14)_1$ and $(3.14)_3$, respectively, the remaining equations reduce to the following system

$$\begin{cases} \phi = (I + d)u_2 + A(u_2 + [N_0 l N_0^* + N_1 l N_1^*]Au_2 - N_0 u_4); \\ \psi = -\dfrac{\rho}{c^2} N_0^* A u_2 + [(k+1)\mathcal{I} + \mathcal{A}]\, u_4, \end{cases} \tag{3.15}$$

where $\phi = f_2 - Af_1 \in D(A^{\frac{1}{2}})'$ and $\psi = f_4 - \mathcal{A}f_3 \in D(\mathcal{A}^{\frac{1}{2}})'$. In order to solve system (3.15), let us consider the following form on $\mathcal{V} \equiv D(A^{\frac{1}{2}}) \times D(\mathcal{A}^{\frac{1}{2}})$

$$\begin{aligned} b\left(\mathbf{u}, \tilde{\mathbf{u}}\right) =& (u + du, \tilde{u})_{L_2(\Omega)} + (u, \tilde{u})_{D(A^{\frac{1}{2}})} + (l^{\frac{1}{2}} N_0^* A u, l^{\frac{1}{2}} N_0^* A\tilde{u})_{L_2(\Gamma_0)} \\ &+ (l^{\frac{1}{2}} N_1^* A u, l^{\frac{1}{2}} N_1^* A\tilde{u})_{L_2(\Gamma_1)} \\ &- (v, N_0^* A\tilde{u})_{L_2(\Gamma_0)} - \frac{\rho}{c^2}\left(N_0^* A u, \tilde{v}\right)_{L_2(\Gamma_0)} + (k+1)\,(v, \tilde{v})_{L_2(\Gamma_0)} + (v, \tilde{v})_{D(\mathcal{A}^{\frac{1}{2}})}, \end{aligned}$$

for every $\mathbf{u} = (u, v)$, $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}) \in \mathcal{V}$. Note that $b(\cdot, \cdot)$ is a bilinear, continuous and coercive form in $\mathcal{V}$. Therefore, it follows from Lax-Milgram theorem that for $L = (\phi, \psi) \in \mathcal{V}'$ there exists a unique $\mathbf{u} = (u_2, u_4) \in \mathcal{V}$ such that

$$L(\tilde{\mathbf{u}}) = b(\mathbf{u}, \tilde{\mathbf{u}}), \quad \text{for every } \tilde{\mathbf{u}} \in \mathcal{V}. \tag{3.16}$$

In particular, for every $\tilde{u} \in D(A^{\frac{1}{2}})$ we have $\tilde{\mathbf{u}} = (\tilde{u}, 0)$ and equation (3.16) writes

$$\begin{aligned} \langle \phi, \tilde{u}\rangle_{D(A^{\frac{1}{2}})', D(A^{\frac{1}{2}})} &= (u_2 + du_2, \tilde{u})_{L_2(\Omega)} + (u_2 + [N_0 l N_0^* + N_1 l N_1^*]Au_2 - N_0 u_4, \tilde{u})_{D(A^{\frac{1}{2}})} \\ &= \langle (I + d)u_2 + A(u_2 + [N_0 l N_0^* + N_1 l N_1^*]Au_2 - N_0 u_4), \tilde{u}\rangle_{D(A^{\frac{1}{2}})', D(A^{\frac{1}{2}})} \end{aligned}$$

which implies Equation (3.15)$_1$. Moreover, since $u_1 = u_2 + f_1$ we have $A(u_1 + NlN^*Au_2 - N_0u_4) = f_2 - (I + d)u_2 \in L_2(\Omega)$, which implies $u_1 + [N_0lN_0^* + N_1lN_1^*]Au_2 - N_0u_4 \in D(A)$. Equation (3.15) (for ψ) can be obtained by setting $\tilde{\mathbf{u}} = (0, \tilde{v})$, for $\tilde{v} \in D(\mathcal{A}^{\frac{1}{2}})$. Therefore, we conclude that $U \in D(A)$ satisfies (3.14), which finishes the proof. □

As a consequence of Lemmas 3.1 and 3.2, it follows from Lumer-Phillips theorem (see Corollary 4.4 in [31]) we have the following.

Corollary 3.1. $\mathbf{A}$ *and* $\mathbf{A}^*$ *are generators of a strongly continuous semigroup of contractions on* $\mathcal{H}$.

It follows from the above results that the fractional powers of $\mathbf{A}$ and $\mathbf{A}^*$ are well-defined, see, for instance, [37] and references therein. Moreover, we have (see [35])

$$D(\mathbf{A}^\theta) = [D(\mathbf{A}), \mathcal{H}]_{1-\theta}, \quad \theta \in [0, 1],$$

where $[\cdot, \cdot]_\theta$ denotes the complex interpolation functor. Furthermore, $\mathbf{A}^{-\theta}$ is a bounded operator in $\mathcal{H}$, for $\theta \in [0, 1]$. Another useful property of $\mathbf{A}$ is the following

Lemma 3.3. *For any* $V = (v_1, v_2, v_3, v_4) \in \mathcal{H}$ *we have*

$$\|A^{\frac{1}{4}} v_1\|_{L_2(\Omega)} + \|\mathcal{A}^{\frac{1}{4}} v_3\|_{L_2(\Gamma_0)} \le C\|\mathbf{A}^{-\frac{1}{2}} V\|_{\mathcal{H}}. \tag{3.17}$$

Moreover, for any $v_4 \in L_2(\Gamma_0)$ *we have*

$$\|\mathbf{A}^{-\frac{1}{2}} V\|_{\mathcal{H}} \le C\left[\|A^{-\frac{1}{4}} v_2\|_{L_2(\Omega)} + \|\mathcal{A}^{-\frac{1}{4}} v_4\|_{L_2(\Gamma_0)}\right], \tag{3.18}$$

where $V = (0, v_2, 0, v_4)$.

Proof. We start by observing that

$$D(\mathbf{A}) \subset H^2(\Omega) \times H^1(\Omega) \times D(\mathcal{A}) \times D(\mathcal{A}^{\frac{1}{2}}).$$

Interpolating between $D(\mathbf{A})$ and $\mathcal{H}$ for $\theta = 1/2$ and taking into account the above inclusion, we obtain

$$D(\mathbf{A}^{\frac{1}{2}}) \subset H^{\frac{3}{2}}(\Omega) \times D(A^{\frac{1}{4}}) \times D(\mathcal{A}^{\frac{3}{4}}) \times D(\mathcal{A}^{\frac{1}{4}}).$$

Thus, for any $U = (u_1, u_2, u_3, u_4) \in D(\mathbf{A}^{\frac{1}{2}})$ we have in particular $u_2 \in D(A^{\frac{1}{4}})$ and $u_4 \in D(\mathcal{A}^{\frac{1}{4}})$, which implies

$$\|A^{\frac{1}{4}} u_2\|_{L_2(\Omega)} + \|\mathcal{A}^{\frac{1}{4}} u_4\|_{L_2(\Gamma_0)} \le C\|\mathbf{A}^{\frac{1}{2}} U\|_{\mathcal{H}}, \tag{3.19}$$

for some constant $C > 0$. Let $V = (v_1, v_2, v_3, v_4) \in \mathcal{H}$ be fixed. Since $\mathbf{A}$ is invertible, there exists a unique $U = (u_1, u_2, u_3, u_4) \in D(\mathbf{A})$ such that $V = \mathbf{A}U$. A straightforward computation provides

$$U = \mathbf{A}^{-1} V = \begin{pmatrix} N_0 v_3 - [N_0 l N_0^* + N_1 l N_1^*] A v_1 - A^{-1}(v_2 + d v_1) \\ v_1 \\ \mathcal{A}^{-1}[-c^{-2} \rho N_0^* A v_1 + k v_3 + v_4] \\ v_3 \end{pmatrix}. \tag{3.20}$$

Since $V \in D(\mathbf{A}^{-\frac{1}{2}})$, we have

$$\|\mathbf{A}^{-\frac{1}{2}} V\|_{\mathcal{H}} = \|\mathbf{A}^{\frac{1}{2}} \mathbf{A}^{-1} V\|_{\mathcal{H}} = \|\mathbf{A}^{\frac{1}{2}} U\|_{\mathcal{H}}. \tag{3.21}$$

Finally, for U in (3.20), inequality (3.17) follows by combining (3.19) and (3.21).

It remains to prove (3.18). To this end, first we note that for the particular case $V = (0, v_2, 0, v_4) \in \mathcal{H}$ we have

$$\|\mathbf{A}^{-\frac{1}{2}} V\|_{\mathcal{H}} = \|\mathbf{A}^{\frac{1}{2}} \mathbf{A}^{-1} V\|_{\mathcal{H}} = \left\| \mathbf{A}^{\frac{1}{2}} \begin{pmatrix} -A^{-1} v_2 \\ 0 \\ -\mathcal{A}^{-1} v_4 \\ 0 \end{pmatrix} \right\|_{\mathcal{H}}. \tag{3.22}$$

On the other hand, since $D(A) \times \{0\} \times D(\mathcal{A}) \times \{0\} \subset D(\mathbf{A})$ and $D(A^{\frac{1}{2}}) \times \{0\} \times D(\mathcal{A}^{\frac{1}{2}}) \times \{0\} \subset \mathcal{H}$ it follows by interpolation that $D(A^{1-\frac{\theta}{2}}) \times \{0\} \times D(\mathcal{A}^{1-\frac{\theta}{2}}) \times \{0\} \subset D(\mathbf{A}^{\theta})$ for any $\theta \in [0, 1]$. Hence, for $U = (u_1, 0, u_3, 0) \in D(\mathbf{A}^{\theta})$ we have

$$\|\mathbf{A}^{\theta} U\|_{\mathcal{H}} \le C \left[\|A^{1-\frac{\theta}{2}} u_1\|_{L_2(\Omega)} + \|\mathcal{A}^{1-\frac{\theta}{2}} u_3\|_{L_2(\Gamma_0)} \right], \quad \text{for } \theta \in [0, 1].$$

Applying the above inequality with $\theta = \frac{1}{2}$, $u_1 = -A^{-1} v_2$ and $u_3 = -\mathcal{A}^{-1} v_4$, also having in mind (3.22), we conclude (3.18) as desired. □

3.4. Nonlinear estimate

An interesting issue is the uniqueness of solutions of the nonlinear system (1.1)-(1.2), due to the presence of $\mathrm{div}\{|\nabla w|^2 \nabla w\}$, which is not bounded from $H^2(\Gamma_0)$ to $L_2(\Gamma_0)$. This prevents applicability of standard methods based on local Lipschitz regularity. In order to deal with this difficulty, one idea is to obtain the estimates on a negative scale of fractional powers $\mathcal{A}$ for solutions which are of finite energy. The differential of topology provides a chance for obtaining "uniqueness estimate", however, without continuous dependence on the data. We shall pursue this idea for the system under consideration. It should be noted that a related idea, tough applied to scalar single equation, was carried out in [19]. In order to obtain the estimate for the nonlinear term with respect to negative fractional powers of $\mathcal{A}$, we proceed as follows.

Let $\{\psi_N\}_{N \in \mathbb{N}}$ be the orthonormal basis consisting of eigenfunctions of $\mathcal{A}$ and consider $M : H^2(\Gamma_0) \to H^{-\epsilon}(\Gamma_0)$ to be the operator that describes the nonlinearity in the structural wall equation:

$$M(w) = \mathrm{div}\{|\nabla w|^2 \nabla w\} + \sigma \Delta\{w^2\} - f(w), \quad w \in H^2(\Gamma_0).$$

Note that with $w \in H^2(\Gamma_0)$, one obtains $\nabla w \in H^1(\Gamma_0) \subset L_p(\Gamma_0)$, for any $p \in [1, \infty)$, and by Sobolev's embeddings for every $\epsilon > 0$

$$||\mathrm{div}\{|\nabla w|^2 \nabla w\}||_{H^{-\epsilon}(\Gamma_0)} \leq C||w||^3_{H^2(\Gamma_0)}.$$

We will need the following sharp estimate:

Proposition 3.1. *Let $R > 0$ and $0 < s < 1$. Using the notation from previous section, if $\{\lambda_m : m \in \mathbb{N}\}$ is the set of eigenvalues of $\mathcal{A}$ then, for m large enough, there are constants $C_1, C_2 > 0$, which depend on R but not on m, such that*

$$\|\mathcal{A}^{-\frac{1}{4}}[M(w_1) - M(w_2)]\|_{L_2(\Gamma_0)} \leq C_1 \log(1 + \lambda_m)\|\mathcal{A}^{\frac{1}{4}}(w_1 - w_2)\|_{L_2(\Gamma_0)} + C_2\lambda_{m+1}^{-\frac{s}{4}}, \quad (3.23)$$

for any $w_1, w_2 \in H^2(\Gamma_0)$ such that $\|w_j\|_{H^2(\Gamma_0)} \leq R$, for $j = 1, 2$.

Proof. Let $w_1, w_2 \in H^2(\Gamma_0)$ such that $\|w_j\|_{H^2(\Gamma_0)} \leq R$, for $j = 1, 2$. We start with

$$\begin{aligned}
&\|\mathcal{A}^{-\frac{1}{4}}[\mathrm{div}\{|\nabla w_1|^2 \nabla w_1\} - \mathrm{div}\{|\nabla w_2|^2 \nabla w_2\}]\|_{L_2(\Gamma_0)} \\
&\leq C\||\nabla w_1|^2\nabla w_1 - |\nabla w_2|^2\nabla w_2\|_{L_2(\Gamma_0)} \\
&\leq C\|[|\nabla w_1|^2 + \nabla w_1 \cdot \nabla w_2 + |\nabla w_2|^2]\nabla(w_1 - w_2)\|_{L_2(\Gamma_0)} \\
&\leq C \textstyle\sum_{m,l=1}^2 \sum_{i,j,k=1}^2 \|\partial_{x_k} w_l \cdot \partial_{x_i} w_m \cdot \partial_{x_j} w\|_{L_2(\Gamma_0)},
\end{aligned} \quad (3.24)$$

where we denoted $w = w_1 - w_2$. Decomposing the cubic terms on the right-hand side of the above inequality in small and large frequencies as follows

$$\begin{aligned}
\partial_{x_k} w_l \cdot \partial_{x_i} w_m \cdot \partial_{x_j} w = {}& Q_N(\partial_{x_k} w_l) \cdot \partial_{x_i} w_m \cdot \partial_{x_j} w + P_N(\partial_{x_k} w_l) \cdot Q_N(\partial_{x_i} w_m) \cdot \partial_{x_j} w \\
&+ P_N(\partial_{x_k} w_l) \cdot P_N(\partial_{x_i} w_m) \cdot \partial_{x_j} w \equiv I_1 + I_2 + I_3,
\end{aligned}$$

for $i, j, k, m, l = 1, 2$, where P_N is the projector on $\mathrm{span}\{\psi_1, \ldots, \psi_N\}$ and $Q_N = I - P_N$.

We will first deal with I_1, for the same arguments hold for I_2 and yield to the same estimate. Let $0 < s < 1$. Using the embedding $H^r(\Gamma_0) \subset L_p(\Gamma_0)$ for $r = 1 - 2/p$ and $p \geq 2$ and Hölder's inequality, we have

$$\begin{aligned}
&\left\|Q_N(\partial_{x_k} w_l) \cdot \partial_{x_i} w_m \cdot \partial_{x_j} w\right\|_{L_2(\Gamma_0)} \\
&\leq \|Q_N(\partial_{x_k} w_l)\|_{L_{2/s}(\Gamma_0)}\|\partial_{x_i} w_m \cdot \partial_{x_j} w\|_{L_{2/(1-s)}(\Gamma_0)} \\
&\leq C\|Q_N(\partial_{x_k} w_l)\|_{H^{1-s}(\Gamma_0)}\|\partial_{x_i} w_m\|_{L_{4/(1-s)}(\Gamma_0)}\|\partial_{x_j} w\|_{L_{4/(1-s)}(\Gamma_0)} \\
&\leq C\left(\|\partial_{x_i} w_m\|_{H^1(\Gamma_0)}\|\partial_{x_j} w\|_{H^1(\Gamma_0)}\right)\|Q_N(\partial_{x_k} w_l)\|_{H^{1-s}(\Gamma_0)} \\
&\leq C_R\|Q_N(\partial_{x_k} w_l)\|_{H^{1-s}(\Gamma_0)}.
\end{aligned}$$

Since Q_N is the eigenprojector on $\mathrm{span}\{\psi_n : n \geq N + 1\}$, it follows from the characterization in Remark 3.4 that

$$\|Q_N(\partial_{x_k} w_l)\|_{H^{1-s}(\Gamma_0)} \leq C\|\mathcal{A}^{\frac{1-s}{4}} Q_N(\partial_{x_k} w_l)\|_{L_2(\Gamma_0)} \leq C\|w_l\|_{H^2(\Gamma_0)}\lambda_{N+1}^{-\frac{s}{4}},$$

from which we conclude

$$\|I_1\|_{L_2(\Gamma_0)}, \|I_2\|_{L_2(\Gamma_0)} \le C_R \lambda_{N+1}^{-\frac{s}{4}}, \quad \text{for every } i, j, k, l, m = 1, 2. \tag{3.25}$$

Finally, for I_3 we use Lemma 3.4 below in order to obtain

$$\begin{aligned}
&\|P_N(\partial_{x_k} w_l) \cdot P_N(\partial_{x_i} w_m) \cdot \partial_{x_j} w\|_{L_2(\Gamma_0)} \\
&\le \sup_{x\in\Gamma_0} \left|P_N(\partial_{x_k} w_l)\right| \cdot \sup_{x\in\Gamma_0} \left|P_N(\partial_{x_i} w_m)\right| \cdot \|\partial_{x_j} w\|_{L_2(\Gamma_0)} \\
&\le C \log(1+\lambda_{N+1}) \|\mathcal{A}^{\frac{1}{4}} \partial_{x_k} w_l\|_{L_2(\Gamma_0)} \|\mathcal{A}^{\frac{1}{4}} \partial_{x_i} w_m\|_{L_2(\Gamma_0)} \|\partial_{x_j} w\|_{L_2(\Gamma_0)} \\
&\le C \log(1+\lambda_{N+1}) \left[\|w_l\|_{H^2(\Gamma_0)} \|w_m\|_{H^2(\Gamma_0)}\right] \|w\|_{H^1(\Gamma_0)} \\
&\le C_R \log(1+\lambda_{N+1}) \|w\|_{H^1(\Gamma_0)}.
\end{aligned}$$

Using the characterization in Remark 3.4, I_3 is estimated by

$$\|I_3\|_{L_2(\Gamma_0)} \le C_R \|\mathcal{A}^{\frac{1}{4}} w\|_{L_2(\Gamma_0)} \log(1+\lambda_{N+1}) \quad \text{for every } i, j, k, l, m = 1, 2. \tag{3.26}$$

Plugging estimates (3.25) and (3.26) into (3.24), we conclude

$$\begin{aligned}
&\|\mathcal{A}^{-\frac{1}{4}}[\operatorname{div}\{|\nabla w_1|^2 \nabla w_1\} - \operatorname{div}\{|\nabla w_2|^2 \nabla w_2\}]\|_{L_2(\Gamma_0)} \\
&\le C_{1,R} \log(1+\lambda_{N+1}) \|\mathcal{A}^{\frac{1}{4}}(w_1 - w_2)\|_{L_2(\Gamma_0)} + C_{2,R} \lambda_{N+1}^{-\frac{s}{4}}.
\end{aligned} \tag{3.27}$$

The estimates for the remaining two terms in the definition of operator M are more direct after exploiting local Lipschitz condition:

$$\begin{aligned}
&\sigma \|\mathcal{A}^{-\frac{1}{4}} \operatorname{div}\{\nabla(w_1^2 - w_2^2)\}\|_{L_2(\Gamma_0)} \\
&\le C \left\| \left|\nabla \left(w_1^2 - w_2^2\right)\right| \right\|_{L_2(\Gamma_0)} \\
&\le C \textstyle\sum_{i=1}^2 \|\partial_{x_i}(w_1^2 - w_2^2)\|_{L_2(\Gamma_0)} \\
&\le C \textstyle\sum_{i=1}^2 \left[\|w \cdot \partial_{x_i}(w_1 + w_2)\|_{L_2(\Gamma_0)} + \|(w_1 + w_2) \cdot \partial_{x_i} w\|_{L_2(\Gamma_0)}\right].
\end{aligned}$$

Using the embedding $H^2(\Gamma_0) \subset C(\overline{\Gamma_0})$, the second term on the right-hand side of the previous inequality reads

$$\begin{aligned}
\left\|(w_1 + w_2) \cdot \partial_{x_i} w\right\|_{L_2(\Gamma_0)} &\le \|w_1 + w_2\|_{L_\infty(\Gamma_0)} \|\partial_{x_i} w\|_{L_2(\Gamma_0)} \\
&\le C_R \|\mathcal{A}^{\frac{1}{4}} w\|_{L_2(\Gamma_0)}, \quad \text{for } i = 1, 2.
\end{aligned}$$

The first term is estimated using Hölder's inequality and the embedding $H^r(\Gamma_0) \subset L_p(\Gamma_0)$ for $r = 1 - 2/p$ and $p \ge 2$ as before

$$\|w \cdot \partial_{x_i}(w_1 + w_2)\|_{L_2(\Gamma_0)} = \left\{ \int_{\Gamma_0} \left|w \cdot \partial_{x_i}(w_1 - w_2) \cdot 1\right|^2 d\Gamma_0 \right\}^{\frac{1}{2}}$$

$$
\begin{aligned}
&\leq |\Gamma_0|^{\frac{1-s}{2}} \|\partial_{x_i}(w_1+w_2)\|_{L_{4/(1-s)}} \|w\|_{L_{4/(1-s)}} \\
&\leq C\|\partial_{x_i}(w_1+w_2)\|_{H^1(\Gamma_0)} \|w\|_{H^1(\Gamma_0)} \\
&\leq C_R \|\mathcal{A}^{\frac{1}{4}} w\|_{L_2(\Gamma_0)}, \quad \text{for } i=1,2 \text{ and } 0<s<1.
\end{aligned}
$$

Therefore, we conclude

$$
\sigma \|\mathcal{A}^{-\frac{1}{4}} \operatorname{div}\{\nabla(w_1^2 - w_2^2)\}\|_{L_2(\Gamma_0)} \leq C_R \|\mathcal{A}^{\frac{1}{4}}(w_1 - w_2)\|_{L_2(\Gamma_0)}. \tag{3.28}
$$

Finally, using Assumption 3.1, the last term in the operator M is estimated as follows

$$
\begin{aligned}
\|\mathcal{A}^{-\frac{1}{4}}[f(w_1) - f(w_2)]\|_{L_2(\Gamma_0)} &= \|\mathcal{A}^{-\frac{1}{4}}\|_{\mathcal{L}(L_2(\Gamma_0))} \|f(w_1) - f(w_2)\|_{L_2(\Gamma_0)} \\
&\leq \|\mathcal{A}^{-\frac{1}{4}}\|_{\mathcal{L}(L_2(\Gamma_0))} \sup_{|s|\leq R} \left|f'(s)\right| \|w_1 - w_2\|_{L_2(\Gamma_0)} \\
&\leq C_R \|\mathcal{A}^{\frac{1}{4}}(w_1 - w_2)\|_{L_2(\Gamma_0)}.
\end{aligned} \tag{3.29}
$$

Combining (3.27), (3.28) and (3.29) and choosing N sufficiently large, we obtain estimate (3.23) □

Lemma 3.4 *(see [14,19]). Let $\{\psi_i\}_{i\in\mathbb{N}}$ be the orthonormal basis in $L_2(\Gamma_0)$ of eigenvectors of $\mathcal{A}$, P_n be the projector in $L_2(\Gamma_0)$ onto the space spanned by $\{\psi_1, \psi_2, \dots, \psi_n\}$ and $u \in D(\mathcal{A}^{\frac{1}{4}})$. Then, there exists $n_0 > 0$ such that for $n \geq n_0$ we have*

$$
\max_{x\in\Gamma_0} |(P_n u)(x)| \leq C \left[\log(1+\lambda_n)\right]^{\frac{1}{2}} \|\mathcal{A}^{\frac{1}{4}} u\|_{L_2(\Gamma_0)}
$$

where λ_n is the corresponding eigenvalue, and the constant $C > 0$ does not depend on n.

Remark 3.6. The inequality stated above provides a "rate" of blowing-up estimates for projections of solutions under Sobolev's embedding at the critical level $H^1(\Gamma_0)$, where $\Gamma_0 \subset \mathbb{R}^2$.

4. Proof of the main results

In this section we present the proof of Theorem 2.1, announced in Section 2.

4.1. Weak Hadamard wellposedness

We start by proving the existence and uniqueness of weak solutions.

Proposition 4.1 ***(Existence of weak solutions)****. Let $f \in C^1(\mathbb{R})$ be given. For every $R > 0$ and $U_0 \equiv (z_0, z_1, w_0, w_1) \in \mathcal{H}$ such that $\|U_0\|_{\mathcal{H}} \leq R$ there exists $T_0 \equiv T_0(R) > 0$ and a pair of functions (z, w) which is a (local in time) weak solution of (1.1)-(1.2). In addition, we have the boundary regularity $l^{1/2} z_t|_\Gamma \in L_2(0, T; L_2(\Gamma))$. Moreover, the solution is global provided that f satisfies the non-explosion Assumption 3.1.*

Proof. Step 1. (*Existence of Local Solutions*) Since the nonlinearity in the structural wall is not locally Lipschitz, the proof of the above result relies on the Faedo-Galerkin method. Let $T > 0$ be fixed and consider $\{\phi_n\}_{n\in\mathbb{N}} \subset H^1(\Omega)$ and $\{\psi_n\}_{n\in\mathbb{N}} \subset H$ orthonormal basis in $L_2(\Omega)$ and $L_2(\Gamma_0)$, respectively. For each $n \in \mathbb{N}$, we define the approximate subspace

$$\mathcal{V}_n = \operatorname{span}\{\phi_i,\ i = 1,\dots,n\} \times \operatorname{span}\{\psi_i,\ i = 1,\dots,n\}.$$

The approximate problem is formulated as follows:

Find $(z^n, w^n) : [0, T] \to \mathcal{V}_n$ such that, for every $(\phi, \psi) \in \mathcal{V}_n$ and any case (C), (SS) or (F):

$$\begin{aligned} 0 = {} & \left(z^n_{tt}(t), \phi\right)_{L_2(\Omega)} + \left(w^n_{tt}(t), \psi\right)_{L_2(\Gamma_0)} + c^2 \left(\nabla z^n(t), \nabla\phi\right)_{[L_2(\Omega)]^2} + a(w^n(t), \psi) \\ & + \left(d \cdot z^n_t(t), \phi\right)_{L_2(\Omega)} + k \left(w^n_t(t), \psi\right)_{L_2(\Gamma_0)} + c^2 (l^{\frac{1}{2}} z^n_t(t)|_\Gamma, l^{\frac{1}{2}}\phi|_\Gamma)_{L_2(\Gamma)} \\ & - c^2 \left(w^n_t(t), \phi|_{\Gamma_0}\right)_{L_2(\Gamma_0)} + c^2 l_0 \left(z^n(t)|_{\Gamma_1}, \phi|_{\Gamma_1}\right)_{L_2(\Gamma_1)} + \rho \left(z^n_t(t)\big|_{\Gamma_0}, \psi\right)_{L_2(\Gamma_0)} \\ & + G(w^n(t), \psi); \end{aligned} \tag{4.1}$$

and initial conditions

$$\begin{aligned} &\left(z^n(0), \phi\right)_{L_2(\Omega)} = (z_0, \phi)_{L_2(\Omega)}\,; \quad \left(z^n_t(0), \phi\right)_{L_2(\Omega)} = (z_1, \phi)_{L_2(\Omega)}\,; \\ &\left(w^n(0), \psi\right)_{L_2(\Gamma_0)} = (w_0, \psi)_{L_2(\Gamma_0)}\,; \quad \left(w^n_t(0), \psi\right)_{L_2(\Gamma_0)} = (w_1, \psi)_{L_2(\Gamma_0)}\,, \end{aligned}$$

where $a(\cdot,\cdot)$ is given in (2.3) and G is given in (2.5). Denoting $z^n(t) = \sum_{i=1}^n \xi_{ni}(t)\phi_i$ and $w^n(t) = \sum_{i=1}^n \vartheta_{ni}(t)\psi_i$ where $\xi_{ni}(t)$ and $\vartheta_{ni}(t)$ are real functions, one can rewrite both problems above as the following ODE system

$$\begin{aligned} &\zeta''_n(t) + \mathbb{D}\zeta'_n(t) + \mathbb{A}\zeta_n(t) + \mathbb{F}(\zeta_n(t)) = 0, \quad \text{for } 0 < t < T; \\ &\zeta_n(0) = \zeta_0 \equiv (\xi^n_0, \vartheta^n_0); \quad \zeta'_n(0) = \zeta_1 \equiv (\xi^n_1, \vartheta^n_1); \end{aligned}$$

where $\zeta_n(t) \equiv (\xi_n(t), \vartheta_n(t))$ and $\xi_n(t) \equiv (\xi_{ni}(t))_{i=1,\dots,n}$ and $\vartheta_n(t) \equiv (\vartheta_{ni}(t))_{i=1,\dots,n}$. The initial data is given by

$$\xi^n_j \equiv \left(\left(z_j, \phi_i\right)_{L_2(\Omega)}\right)_{i=1,\dots,n}; \quad \vartheta^n_j \equiv \left(\left(w_j, \psi_i\right)_{L_2(\Gamma_0)}\right)_{i=1,\dots,n}, \quad \text{for } j = 0, 1,$$

and the matrices $\mathbb{A}$ and $\mathbb{D}$ depend on the basis, while the nonlinear vector $\mathbb{F}$ depends on G applied $w^n(t)$ and the basis of H, as we can see below:

$$\mathbb{A} \equiv \begin{pmatrix} c^2 \left[\left(\nabla\phi_i, \nabla\phi_j\right)_{L_2(\Omega)} + l_0 \left(\phi_i|_{\Gamma_1}, \phi_j|_{\Gamma_1}\right)_{L_2(\Gamma_1)}\right]_{j,i=1,\dots,n} & 0 \\ 0 & \left[a(\phi_j, \psi_i)\right]_{j,i=1,\dots,n} \end{pmatrix};$$

$$\mathbb{D} \equiv \begin{pmatrix} \left[\left(d \cdot \phi_i, \phi_j\right)_{L_2(\Omega)} + c^2 \left(l^{\frac{1}{2}}\phi_i|_\Gamma, l^{\frac{1}{2}}\phi_j|_\Gamma\right)_{L_2(\Gamma)}\right]_{j,i=1,\dots,n} & 0 \\ 0 & 0 \end{pmatrix}$$

$$+\begin{pmatrix} \rho\left(\phi_i|_{\Gamma_0},\psi_j\right)_{L_2(\Gamma_0)} & 0 \\ 0 & -c^2\left[\left(\psi_i,\phi_j|_{\Gamma_0}\right)_{L_2(\Gamma_0)}\right]_{j,i=1,\dots,n}\end{pmatrix};$$

$$\mathbb{F}(\zeta_n(t)) \equiv \begin{pmatrix} 0 \\ \left[G\left(\sum_{i=1}^n \vartheta_{ni}(t)\psi_i,\psi_j\right)\right]_{j=1,\dots,n}\end{pmatrix}.$$

The above system admits a unique solution $\zeta_n(t)$ on a maximal interval $[0,T^n_{max})$, via Carathéodory's theorem (see e.g. [18]). Consequently, there is a unique solution $(z^n(t),w^n(t))$ of the approximate variational problems (4.1).

In order to pass to the limit on the above approximate variational problems, we must establish some a priori estimates. Since $1-c^{-2}\rho=0$, using the energy functional introduced in Section 3.2 applied to the approximate solutions, the corresponding (approximate) energy identity reads as

$$\begin{aligned} E_1^n(t)+k\int_0^t \|w_t^n(s)\|^2_{L_2(\Gamma_0)}ds+\int_0^t \|d^{\frac12}z_t^n(s)\|^2_{L_2(\Omega)}ds+c^2\int_0^t \|l^{\frac12}z_t^n(s)|_\Gamma\|^2_{L_2(\Gamma)}ds \\ = E_1^n(0)+\int_0^t \sigma(w_t^n(s),|\nabla w^n(s)|^2)_{L_2(\Gamma_0)}ds-\int_0^t (f(w^n(s)),w_t^n(s))_{L_2(\Gamma_0)}ds,\end{aligned} \tag{4.2}$$

where $E_1^n(t)\equiv E^n(t)+\sigma(w^n(t),|\nabla w^n(t)|^2)_{L_2(\Gamma_0)}$. Using the embeddings $H^1(\Gamma_0)\to L_4(\Gamma_0)$ and $H^2(\Gamma_0)\to C(\overline{\Gamma}_0)$ as well as the continuity of f, the right-hand side integrals of (4.2) are estimated as follows

$$\left|\int_0^t \sigma(w_t^n(s),|\nabla w^n(s)|^2)_{L_2(\Gamma_0)}ds\right|\le 2\sigma\int_0^t E_w^n(s)ds;$$

$$\left|\int_0^t \left(f(w^n(s)),w_t^n(s)\right)_{L_2(\Gamma_0)}ds\right|\le \int_0^t E_w^n(s)ds+\frac{t}{2}\cdot\Psi\left(\tilde{E}_w^n(t)\right),$$

where $\Psi:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous and increasing function, and $\tilde{E}_w^n(t)\equiv \max_{s\in[0,t]}E_w^n(s)$.

Let $R>0$ such that $\|(z_0,z_1,w_0,w_1)\|_{\mathcal{H}}\le R$. Thus, it follows from the convergence of the initial data of the approximate problem that there exists n_0 sufficiently large such that $E^n(0)\le C(R)$, for every $n\ge n_0$. Using the embedding $H^2(\Gamma_0)\to C(\overline{\Gamma}_0)$ and the expression of $E_1^n(t)$ we also have $E_1^n(0)\le C(\sigma,R)$, for $n\ge n_0$. Therefore, identity (4.2) implies

$$E_1^n(t)\le\left[C(\sigma,R)+\frac{t}{2}\cdot\Psi\left(\tilde{E}_w^n(t)\right)\right]+C_\sigma\int_0^t E^n(s)ds, \tag{4.3}$$

for $n\ge n_0$ and $t\in[0,T^n_{max})$. On the other hand, using the lower estimate obtained in Section 3.2, keeping in mind that E_w^n does not control the $L_2(\Gamma_0)$ norm of plate solutions in the case of free (F) boundary condition, we consider the functional $\Phi_n(t)=E^n(t)+\|w^n(t)\|^2_{L_2(\Gamma_0)}$. Rewriting (3.9) using $\Phi_n(t)$ we have

$$\frac{1}{2}\Phi_n(t) \le E_1^n(t) + C(\Gamma_0) + \left(\frac{1}{2}+\sigma^2\right)\left\|w^n(t)\right\|_{L_2(\Gamma_0)}^2. \tag{4.4}$$

Finally, using the following estimate

$$\left\|w^n(t)\right\|_{L_2(\Gamma_0)}^2 \le C(R) + 2\int_0^t \Phi_n(s)ds$$

and inequalities (4.3), (4.4) we conclude that

$$\Phi_n(t) \le \left[C(\Gamma_0,\sigma,R) + t\cdot\Psi\left(\tilde{\Phi}_n(t)\right)\right] + C_\sigma \int_0^t \Phi_n(s)ds,$$

where the constant $C_\sigma > 1$ and $\tilde{\Phi}_n(t) \equiv \max\{\Phi_n(s) : s \in [0,t]\}$. Previous estimate along with Gronwall's inequality implies

$$\tilde{\Phi}_n(t) \le \left[C(\Gamma_0,\sigma,R) + t\cdot\Psi\left(\tilde{\Phi}_n(t)\right)\right] e^{C_\sigma\cdot t}, \quad \text{for } 0 \le t < T_{max}^0. \tag{4.5}$$

Let $0 < T_1^n \le \min\{1, T_{max}^n\}$. Since $\Phi_n(t)$ is continuous and T_1^n is finite, there exists $C_{1n} > 0$ such that $\tilde{\Phi}_n(t) \le C_{1n}$ for all $t \le T_1^n$. Using (4.5) and the fact that $\Psi(C_{1n}) \ne 0$, we arrive at

$$\tilde{\Phi}_n(t) \le 2C_R e^{C_\sigma} \tag{4.6}$$

for $0 \le t \le \min\{T_1^n, C_R\Psi(C_{1n})^{-1}\}$, where $C_R \equiv C(\Gamma_0,\sigma,R)$. Let T_0^n be the maximum value for which inequality (4.6) holds for $t \le T_0^n$. Thus, we have $T_1^n \le T_0^n \le T_{max}^n$, for every $n \ge n_0$. Also, either $T_0^n = \infty$ or $T_0^n < \infty$, and in the last case we have $\tilde{\Phi}_n(T_0^n) = 2C_R e^{C_\sigma}$.

Finally, let $T_0 \equiv \inf\{T_0^n : n \ge 0\}$. We claim that $T_0 > 0$. Indeed, if it were not the case, there would exist $T_0^{n_k} \to 0+$ as $n_k \to \infty$. For such sequence, using (4.5), we have

$$2C_R e^{C_\sigma} = \tilde{\Phi}_n(T_0^{n_k}) \le \left[C_R + T_0^{n_k}\Psi\left(2C_R e^{C_\sigma}\right)\right] e^{C_\sigma T_0^{n_k}}.$$

Letting $n_k \to \infty$ in the previous inequality, we conclude $2C_R e^{C_\sigma} \le C_R$ which leads to contradiction due to the fact $C_\sigma > 1$. Since $T_0 < \infty$, we conclude

$$\Phi_n(t) \le 2C_R e^{C_\sigma}, \quad \text{for } t \in [0,T_0] \text{ and } n \ge n_0.$$

It follows from the above estimate and (4.2) that

$$\begin{aligned} &\left\{(z^n, z_t^n, w^n, w_t^n)\right\} \text{ is bounded in } L_\infty(0,T_0;\mathcal{H}), \\ &\text{and } \{l^{\frac{1}{2}} z_t^n|_\Gamma\} \text{ is bounded in } L_2((0,T_0)\times\Gamma). \end{aligned} \tag{4.7}$$

In order to pass to the limit in (4.1), we need an estimate for the second-order time derivatives. Considering these approximate variational problems and performing straightforward computations, one can prove that for every $(\phi,\psi) \in \mathcal{V}_n$,

$$\left|\left((z_{tt}^n(t), w_{tt}^n(t)), (\phi, \psi)\right)_{L_2(\Omega)\times L_2(\Gamma_0)}\right| \le C(\sigma, R)\,\|(\phi, \psi)\|^2_{H^1(\Omega)\times H^2(\Gamma_0)},$$

where $C(\sigma, R) > 0$ depends on the initial data. The above inequality implies

$$\left\{(z_{tt}^n, w_{tt}^n)\right\} \text{ is bounded in } L_\infty(0, T_0; (H^1(\Omega)\times H^2(\Gamma_0))'). \tag{4.8}$$

It follows from (4.7) and (4.8) that there exists a pair of functions (z, w) and a subsequence such that

$$\begin{aligned}&(z^n, z_t^n, z_{tt}^n) \to (z, z_t, z_{tt}), \text{ weak star in } L_\infty(0, T_0; H_z \times (H^1(\Omega))');\\ &(w^n, w_t^n, w_{tt}^n) \to (w, w_t, w_{tt}), \text{ weak star in } L_\infty(0, T_0; H_w \times H');\end{aligned}$$

The above weak-star convergences and compactness results in [33] imply that $(z, z_t) : [0, T_0] \to H_z$ and $(w, w_t) : [0, T_0] \to H_w$ are weakly continuous. This weak continuity will allow us to prove the validity of the initial condition. In order to pass to the limit in the variational problems, we observe that the above weak convergences also imply that (see again [33])

$$\begin{aligned}&(z^n, z_t^n) \to (z, z_t) \text{ strong in } C(0, T_0; H^{1-\varepsilon}(\Omega)\times H^{-\varepsilon}(\Omega));\\ &(w^n, w_t^n) \to (w, w_t) \text{ strong in } C(0, T_0; H^{2-\varepsilon}(\Gamma_0)\times H^{-\varepsilon}(\Gamma_0)),\end{aligned}$$

for any $\varepsilon > 0$. Another ingredient is the following embedding $W^{s,p}(\Gamma_0) \to L_r(\Gamma_0)$, for $0 < s < 1$ and $p < r < np/(n - sp)$. This embedding implies that the following maps $w \mapsto |\nabla w|^2 \nabla w$, $w \mapsto |\nabla w|^2$ and $w \mapsto f(w)$ are well defined from H to $L_2(\Gamma_0)$ and are continuous. This continuity together with the above weak and strong convergences allows us to pass to the limit in the approximate variational problems obtaining a local solution.

Step 2. (*Global Solutions*) Our next step is to prove that the solution obtained in the previous step is global, provided that f satisfies Assumption 3.1. To this end, we are going to consider the total energy (2.1) applied to the approximate solutions. Using the approximate variational problem and assumption $1 - c^2\rho = 0$, we have

$$\begin{aligned}\mathcal{E}^n(t) + k\int_0^t \|w_t^n(s)\|^2_{L_2(\Gamma_0)}ds + \int_0^t \|d^{\frac12} z_t^n(s)\|^2_{L_2(\Omega)}ds + \rho\int_0^t \|l^{\frac12} z_t^n(s)|_\Gamma\|^2_{L_2(\Gamma)}ds\\ = \mathcal{E}^n(0) + \sigma\int_0^t (w_t^n(s), \left|\nabla w^n(s)\right|^2)_{L_2(\Gamma_0)}ds.\end{aligned}$$

The right-hand side of the previous identity can be estimated in terms of the initial data and the energy functional $\Phi_n(t)$ in order to conclude

$$\begin{aligned}\mathcal{E}^n(t) + k\int_0^t \|w_t^n(s)\|^2_{L_2(\Gamma_0)}ds + \int_0^t \|d^{\frac12} z_t^n(s)\|^2_{L_2(\Omega)}ds + \rho\int_0^t \|l^{\frac12} z_t^n(s)|_\Gamma\|^2_{L_2(\Gamma)}ds\\ \le C(\sigma, R) + C_\sigma\int_0^t \Phi_n(s)ds.\end{aligned} \tag{4.9}$$

Further, using the expression of $\mathcal{E}^n(t)$ as well as inequalities (4.4), (4.9) and Assumption 3.1, for an appropriate choice of δ, we have

$$\Phi_n(t) \le C_R + C_\sigma \int_0^t \Phi_n(s)ds, \quad \text{for } t \in [0, T_0] \text{ and } n \ge n_0,$$

where C_R is a positive constant which depends on the initial data. Finally, using the lower semi-continuity of the energy functional and by passing to the limit in the previous inequality, one has

$$\Phi(t) \le C_R + C_\sigma \int_0^t \Phi(s)ds, \quad \text{for } t \in [0, T_0],$$

where $\Phi(t) = E(t) + \|w(t)\|^2_{L_2(\Gamma_0)}$. Gronwall's inequality implies that the solution is global with the desired regularity. □

In the previous result, we have proved that system (1.1)-(1.2) admits a (local in time) finite energy solution, which is global provided f satisfies Assumption 3.1. Next, we will show that the solution is unique. The argument is based on an adaptation of Sedenko's method, as presented in [19], in what follows, we use the notion established in Section 3.3.

Proposition 4.2 *(Uniqueness of weak solutions). Under the assumptions of Proposition 4.1, for every initial data in $\mathcal{H}$ the corresponding weak solution is unique.*

Proof. Let $T > 0$, $U_0 \equiv (z_0, z_1, w_0, w_1) \in \mathcal{H}$ and suppose that (z^1, w^1) and (z^2, w^2) are two weak solutions of (1.1)-(1.2) for the same initial data U_0. Since $(z^i, w^i) \in L_\infty(0, T; D(\mathcal{A}^{\frac{1}{2}}) \times D(A^{\frac{1}{2}}))$ and $l^{1/2} z^i_t|_\Gamma \in L_2(0, T; L_2(\Gamma))$, for $i = 1, 2$, there exists $R > 0$ such that

$$\sup_{t\in[0,T]} \left[\|\mathcal{A}^{\frac{1}{2}} w^i(t)\|_{L_2(\Gamma_0)} + \|A^{\frac{1}{2}} z^i(t)\|_{L_2(\Omega)} \right] + \int_0^T ||l^{1/2} z_t(s)|_\Gamma||^2_{L_2(\Gamma)} ds < R. \tag{4.10}$$

Define $z \equiv z^1 - z^2$ and $w \equiv w^1 - w^2$. Using the notation introduced in Section 3.3, we can conclude that (z, w) is the weak solution of the following first order variational formulation

$$\begin{cases} \dfrac{d}{dt}\left[(z_t(t), \phi)_{L_2(\Omega)} + (w_t(t), \psi)_{L_2(\Gamma_0)} + \rho \left(N_0^* A z(t), \psi\right)_{L_2(\Gamma_0)}\right] \\ \quad + (A^{\frac{1}{2}} z(t), A^{\frac{1}{2}}\phi)_{L_2(\Omega)} + (l^{\frac{1}{2}} N^* A z_t(t), l^{\frac{1}{2}} N^* A\phi)_{L_2(\Gamma)} - (w_t(t), N_0^* A\phi)_{L_2(\Gamma_0)} \\ \quad + (\mathcal{A}^{\frac{1}{2}} w(t), \mathcal{A}^{\frac{1}{2}}\psi)_{L_2(\Gamma_0)} + (d^{\frac{1}{2}} z_t(t), d^{\frac{1}{2}}\phi)_{L_2(\Omega)} + k\,(w_t(t), \psi)_{L_2(\Gamma_0)} \\ = (Z(t), \phi)_{L_2(\Omega)} + (M(t), \psi)_{L_2(\Gamma_0)}; \\ z(0) = z_t(0) = 0; \quad w(0) = w_t(0) = 0, \end{cases} \tag{4.11}$$

where the time derivative is understood in the sense of distributions, $Z(t) = z^1(t) - z^2(t)$ and the non homogeneous term $M(t)$ is given by

$$M(t) \equiv \mathsf{div}\{|\nabla w_1|^2 \nabla w_1 - |\nabla w_2|^2 \nabla w_2\} + \sigma \Delta\{w_1^2 - w_2^2\} - [f(w_1) - f(w_2)].$$

It follows from Proposition 3.1 that there are positive constants C_1 and C_2, which depend on R, such that $M(t)$ must satisfy the following estimate

$$\|\mathcal{A}^{-\frac{1}{4}} M(t)\|_{L_2(\Gamma_0)} \leq C_1 \log(1+\lambda_m)\|\mathcal{A}^{\frac{1}{4}} w(t)\|_{L_2(\Gamma_0)} + C_2 \lambda_{m+1}^{-s/4}, \tag{4.12}$$

for $0 < s < 1$ and $t \in [0, T]$, where λ_m is an eigenvalue of $\mathcal{A}$ large enough. It follows from the previous inequality and (3.18) in Lemma (3.3) that $\mathbf{A}^{-\frac{1}{2}} \tilde{M} \in L_\infty(0, T; \mathcal{H})$, where $\tilde{M}(t) = (0, Z(t), 0, M(t))^T$. Moreover, if we denote $U(t) \equiv (z(t), z_t(t), w(t), w_t(t))$ then $U(t)$ is the mild solution of the abstract inhomogeneous problem

$$\frac{d}{dt}U(t) - \mathbf{A}U(t) = \tilde{M}(t); \quad U(0) = 0.$$

On the other hand, this solution must satisfy

$$\frac{d}{dt}(U(t), V)_{\mathcal{H}} - (U(t), \mathbf{A}^* V)_{\mathcal{H}} = (\tilde{M}(t), V)_{\mathcal{H}}, \quad \forall\, V \in D(\mathbf{A}^*),$$

in the sense of distributions. Thus, $\mathbf{A}^{-\frac{1}{2}} U$ must satisfy

$$\frac{d}{dt}(\mathbf{A}^{-\frac{1}{2}} U(t), V)_{\mathcal{H}} - (\mathbf{A}^{-\frac{1}{2}} U(t), \mathbf{A}^* V)_{\mathcal{H}} = (\mathbf{A}^{-\frac{1}{2}} \tilde{M}(t), V)_{\mathcal{H}},$$

for every $V \in D(\mathbf{A}^*)$, in the sense of distributions. In this case, the solution must be given by

$$\mathbf{A}^{-\frac{1}{2}} U(t) = \int_0^t e^{\mathbf{A}(t-s)} \mathbf{A}^{-\frac{1}{2}} \tilde{M}(s) ds \quad \text{in } \mathcal{H},$$

where $\{e^{\mathbf{A}(t-s)}\}_{t \geq 0}$ stands for the semigroup generated by $\mathbf{A}$. The later identity and (3.17) in Lemma 3.3 imply that

$$\begin{aligned} \psi(t) &\equiv \|A^{\frac{1}{4}} z(t)\|_{L_2(\Omega)} + \|\mathcal{A}^{\frac{1}{4}} w(t)\|_{L_2(\Gamma_0)} \\ &\leq C \int_0^t \Big[\|A^{-\frac{1}{4}} z(t)\|_{L_2(\Omega)} + \|\mathcal{A}^{-\frac{1}{4}} M(s)\|_{L_2(\Gamma_0)} \Big] ds. \end{aligned} \tag{4.13}$$

Inequalities (4.12) and (4.13) imply

$$\psi(t) \leq C_1(1 + \log(1+\lambda_m)) \int_0^t \psi(s) ds + C_2 T \lambda_{m+1}^{-s/4}, \quad t \in [0, T],$$

for some $0 < s < 1$. Using Gronwall's inequality we conclude

$$\psi(t) \le C_2 T \lambda_{m+1}^{-s/4} (1+\lambda_m)^{C_2 \cdot t}, \quad t \in [0,T], \quad 0 < s < 1.$$

Letting $N \to \infty$ and for $0 \le t < t_0 \equiv s \cdot (4C_1)^{-1}$ we obtain $\psi(t) = 0$ for $0 \le t < t_0$, which implies that $z^1 = z_2$ and $w^1 = w^2$ for in the interval $[0, t_0)$. Repeating the process, one can conclude the equalities in the whole interval $[0, T]$, which concludes the proof. □

Existence and uniqueness of weak solutions leads to the continuity of the flow in the weak topology of $\mathcal{H}$ (see [19]). Our next challenge is to show that the said continuity also holds with respect to the strong topology. This property depends on the validity of the energy identity.

4.2. Strong Hadamard wellposedness

In order to establish our next result, i.e., part 2 of Theorem 2.1- we will appeal to an approximation argument used for the purpose of proving energy *equality*. To proceed, recall the following finite difference setting and result, as presented in [25]. Let $h > 0$ a parameter that goes to 0. If X denotes a Hilbert space and $g \in B([0,T]; X)$, we extend $g(t)$ to $\mathbb{R}$ by setting: $g(t) = g(0)$ if $t \le 0$ and $g(t) = g(T)$ for $t \ge T$. With this notation, we define the operation

$$D_h g(t) \equiv \frac{1}{2h}\left[g_h^+(t) + g_h^-(t)\right], \quad \text{for every } g \in B([0,T]; X),$$
$$\text{where} \quad g_h^+(t) \equiv g(t+h) - g(t) \quad \text{and} \quad g_h^-(t) \equiv g(t) - g(t-h).$$

With the above notation, we have the following result.

Lemma 4.1 *(Proposition 4.3 in [25]). Assume that g is weakly continuous with values in X. Then*

(1) $\displaystyle\lim_{h\to 0}\int_0^T (g(t), D_h g(t))_X \, dt = \frac{1}{2}\left[\|g(T)\|_X^2 - \|g(0)\|_X^2\right]$;

(2) *If $g \in H^1(0,T;X)$, then the following limits are well defined in $L_2(0,T;X)$:*

$$\lim_{h\to 0} D_h g = g_t; \quad \lim_{h\to 0} \frac{1}{h} g_h^+ = g_t; \quad \lim_{h\to 0} \frac{1}{h} g_h^- = g_t;$$

Moreover, if g_t is weakly continuous with values in X, then for every $t \in (0,T)$, $D_h g(t) \to g_t(t)$ weakly in X, and

$$\frac{1}{h} g_h^-(T) \to g_t(T); \quad \frac{1}{h} g_h^+(0) \to g_t(0); \quad \text{weakly in } X;$$

(3) *In addition to previous assumptions, let $V \subset X \subset V'$, $g_{tt} \in L_2(0,T;V')$, $g \in L_2(0,T;V)$. Then*

$$\lim_{h\to 0}\int_0^T (g_{tt}(t), D_h g(t))_X \, dt = \frac{1}{2}\left[\|g_t(T)\|_X^2 - \|g_t(0)\|_X^2\right].$$

With the above setting, we obtain our next result in which weak solutions established above also satisfy energy *equality*. This step is critical to establish Hadamard wellposedness and continuity of nonlinear semigroup. We recall that energy equality is satisfied for the Galerkin approximations which lead to the construction of weak solution. On the other hand, we have also shown that weak solutions are unique and enjoy additional boundary regularity on the support of $l(x)$ inside Γ.

Proposition 4.3 *(Energy identity). Let $U = (z, z_t, w, w_t)$ be a weak solution in the interval* $[0, T]$. *In addition we assume that either $l(x) \equiv 0$ or* $\operatorname{supp} l(x) \supset \Gamma_0$. *Then the following energy identity holds*

$$\mathcal{E}(t) + k\int_0^t \|w_t(s)\|^2_{L_2(\Gamma_0)}ds + \rho\int_0^t \|d^{\frac{1}{2}} z_t(s)\|^2_{L_2(\Omega)}ds + \int_0^t \|l^{\frac{1}{2}} z_t(s)|_\Gamma\|^2_{L_2(\Gamma)}ds$$
$$= \mathcal{E}(0) + \sigma\int_0^t (w_t(s), |\nabla w(s)|^2)_{L_2(\Gamma_0)}ds, \quad \textit{for } t > 0. \tag{4.14}$$

Proof. We note that the assumption on the support of $l(x)$ implies the additional boundary regularity

$$\int_0^T \big\|z_t(t)|_{\Gamma_0}\big\|^2_{L_2(\Gamma_0)}\,dt \le C(\|U\|_{L_\infty(0,T;\mathcal{H})}) \tag{4.15}$$

Using equality (4.2) and the weak-star convergence $w_t^n \to w_t$ in $L_\infty(0, T; L_2(\Gamma_0))$ as well as the lower semicontinuity of the energy functional $E_1^n(t)$, we arrive at

$$E_1(t) + k\int_0^t \|w_t(s)\|^2_{L_2(\Gamma_0)}ds + \rho\int_0^t \|d^{\frac{1}{2}} z_t(s)\|^2_{L_2(\Omega)}ds + \int_0^t \|l^{\frac{1}{2}} z_t(s)|_\Gamma\|^2_{L_2(\Gamma)}ds$$
$$\le E_1(0) + \int_0^t \sigma(w_t(s), |\nabla w(s)|^2)_{L_2(\Gamma_0)}ds - \int_0^t (f(w(s)), w_t(s))_{L_2(\Gamma_0)}ds,$$

which can be rewritten as follows

$$\int_0^t \frac{d}{dt}\left\{E_1(s) + \int_{\Gamma_0} F(w(s))d\Gamma_0\right\}ds + k\int_0^t \|w_t(s)\|^2_{L_2(\Gamma_0)}ds + \rho\int_0^t \|d^{\frac{1}{2}} z_t(s)\|^2_{L_2(\Omega)}ds$$
$$+\int_0^t \|l^{\frac{1}{2}} z_t(s)|_\Gamma\|^2_{L_2(\Gamma)}ds \le \sigma\int_0^t (w_t(s), |\nabla w(s)|^2)_{L_2(\Gamma_0)}ds.$$

Using the expression of $\mathcal{E}(t)$ and the last inequality, we conclude that

$$\begin{aligned}\mathcal{E}(t)+k\int_0^t \|w_t(s)\|^2_{L_2(\Gamma_0)}ds+\rho\int_0^t \|d^{\frac{1}{2}}z_t(s)\|^2_{L_2(\Omega)}ds+\int_0^t \|l^{\frac{1}{2}}z_t(s)|_\Gamma\|^2_{L_2(\Gamma)}ds\\ \le \mathcal{E}(0)+\sigma\int_0^t (w_t(s),|\nabla w(s)|^2)_{L_2(\Gamma_0)}ds, \quad \text{for } t\ge 0.\end{aligned} \tag{4.16}$$

For the reverse inequality, let us consider first the case $l\equiv 0$. As presented in [20], the argument to prove the reverse inequality relies on time reversibility property. For this purpose, let $0\le t\le T$ and consider the problem (1.1)-(1.2) with reversing time. In this case, functions $\tilde{z}(t)=z(T-t)$ and $\tilde{w}(t)=w(T-t)$ constitute a weak solution on $[0,T]$ of the (backwards) system

wave equation:

$$\begin{aligned}&\tilde{z}_{tt}-c^2\Delta\tilde{z}=dz_t(T-t), \quad \text{in } Q;\\ &\partial_\nu\tilde{z}=\begin{cases}-l_0\tilde{z} & \text{on } \Sigma_1;\\ -\tilde{w}_t & \text{on } \Sigma_0;\end{cases}\\ &\tilde{z}(0)=z(T); \quad \tilde{z}_t(0)=z_t(T) \quad \text{in } \Omega;\end{aligned}$$

plate equation:

$$\begin{aligned}&\tilde{w}_{tt}+\Delta^2\tilde{w}-\rho\tilde{z}_t|_{\Gamma_0}=\mathrm{div}\{|\nabla\tilde{w}|^2\nabla\tilde{w}\}+W(T-t) \quad \text{on } \Sigma_0;\\ &\text{Boundary Conditions on } \partial\Gamma_0\times(0,\infty);\\ &\tilde{w}(0)=w(T); \quad \tilde{w}_t(0)=w_t(T) \quad \text{in } \Gamma_0,\end{aligned}$$

where $W=-kw_t+\sigma\Delta\{w^2\}-f(w)$. Since $W\in L_\infty(0,T;L_2(\Gamma_0))$ and $z_t\in L_\infty(0,T;L_2(\Omega))$, we have that $W(T-t)\in L_\infty(0,T;L_2(\Gamma_0))$, as well as $z_t(T-t)\in L_\infty(0,T;L_2(\Omega))$. Note also the change of the sign on the interface. In view of the above, we apply the same Galerkin-argument for existence of solutions as before, however, applied to the $\tilde{z}$, $\tilde{w}$ problem running over negative times in $[0,T]$. In this case, the energy *inequality* valid for the new variables on the interval $[\tilde{s},\tilde{t}]\subset[0,T]$ is given by

$$E_{\tilde{z},\tilde{w}}(\tilde{t})\le E_{\tilde{z},\tilde{w}}(\tilde{s})+\int_{\tilde{s}}^{\tilde{t}}(\tilde{z}(\tau),d\cdot z_t(T-\tau))_{L_2(\Omega)}\,d\tau+\int_{\tilde{s}}^{\tilde{t}}(\tilde{w}(\tau),W(T-\tau))_{L_2(\Gamma_0)}\,d\tau$$

where $E_{\tilde{z},\tilde{w}}$ stands for the linear energy functional (see (2.2a)-(2.2b)) applied to the solution $(\tilde{z},\tilde{w})$ of the reverse-in-time system. It follows from the *uniqueness* of weak solutions that $(\tilde{z}(T-t),\tilde{w}(T-t))$ must coincide with $(z(t),w(t))$ in $[0,T]$. Therefore, if $[s,t]\subset[0,T]$ then we choose $\tilde{s}=T-t$ and $\tilde{t}=T-s$ in the above inequality which implies, after a change of variable

$$E(s)\le E(t)-\int_s^t(z(\tau),d\cdot z_t(\tau))_{L_2(\Omega)}\,d\tau-\int_s^t(w(\tau),W(\tau))_{L_2(\Gamma_0)}\,d\tau.$$

Choosing $s=0$ and keeping in mind that $W \in L_\infty(0,T;L_2(\Gamma_0))$ and $z_t \in L_\infty(0,T;L_2(\Omega))$ we directly evaluate the last (supercritical) integral on the right-hand side of the above inequality by using classical Sobolev's embeddings, obtaining

$$\begin{aligned}\mathcal{E}(t) + k\int_0^t \|w_t(\tau)\|^2_{L_2(\Gamma_0)} d\tau + \int_0^t \|d^{\frac{1}{2}} z_t(\tau)\|^2_{L_2(\Omega)} d\tau \\ \geq \mathcal{E}(0) + \sigma \int_0^t (|\nabla w(\tau)|^2, w_t(\tau))_{L_2(\Gamma_0)} d\tau.\end{aligned} \tag{4.17}$$

Inequalities (4.16) and (4.17) imply identity (4.14), for this first case ($l=0$), as desired.

Let us now consider the case $l(x) \geq 0$ and *supp* $l(x) \supset \Gamma_0$. This means that the $l(x) \geq l_0 > 0$ on Γ_0. We first observe that $z_t|_{\Gamma_0} \in L_2(0,T;L_2(\Gamma_0))$ and $\partial_\nu z \in L_2(0,T;L_2(\Gamma))$ imply, by hidden regularity, that the map $L_2(0,T;L_2(\Gamma_0)) \ni g \mapsto z$, where z is the solution of the problem $\Box z = 0$ with boundary conditions $\frac{\partial}{\partial\nu} z + l(x) z_t = g$ in Σ_0 and $\frac{\partial}{\partial\nu} z + l_0 z = 0$ in Σ_1, has the property z is in $C(0,T;H^1(\Omega)) \cap C^1(0,T;L_2(\Omega))$. Hence, the strategy used here is to obtain both (wave and plate) energy separately, by using different methods. For the structural problem, we apply the same reversibility-in-time argument as before with $W = -kw_t + \sigma\Delta\{w^2\} - f(w) + \rho z_t|_{\Gamma_0}$. This will provide the identity

$$\begin{aligned}E_w(t) + \int_{\Gamma_0} F(w(t)) d\Gamma_0 + k\int_0^t \|w_t(s)\|^2_{L_2(\Gamma_0)} ds + \rho\int_0^t (z_t(s)|_{\Gamma_0}, w_t(s))_{L_2(\Gamma_0)} ds \\ = E_w(0) + \sigma\int_0^t (|\nabla w(s)|^2, w_t(s))_{L_2(\Gamma_0)} ds, \quad \text{for } t>0.\end{aligned} \tag{4.18}$$

Finally, for the acoustic problem, we start from the variational problem (2.4) (with $\psi = 0$) and write

$$\begin{aligned}0 = &(z_t(t), \phi(t))_{L_2(\Omega)} - (z_t(0), \phi(0))_{L_2(\Omega)} + c^2\int_0^t (\nabla z(s), \nabla\phi(s))_{L_2(\Omega)} ds \\ &+ \int_0^t (d\cdot z_t(t), \phi(s))_{L_2(\Omega)} ds \\ &+ \int_0^t (l^{\frac{1}{2}} z_t(s)|_\Gamma, l^{\frac{1}{2}}\phi(s)|_\Gamma)_{L_2(\Gamma)} ds + c^2 l_0 \int_0^t (z(s)|_\Gamma, \phi(s)|_\Gamma)_{L_2(\Gamma)} ds \\ &- c^2\int_0^t (w_t(s), \phi(s)|_{\Gamma_0})_{L_2(\Gamma_0)} ds\end{aligned}$$

for any $\phi \in H^1(0,T;L_2(\Omega)) \cap L_2(0,T;D(A^{\frac{1}{2}}))$. Choosing $\phi = D_h z$ and using the above notation, we rewrite the previous identity as follows

$$
\begin{aligned}
0 = \frac{1}{2}\left[(z_t(t), \frac{1}{h}z_h^-(t))_{L_2(\Omega)} - (z(0), \frac{1}{h}z_h^+(0))_{L_2(\Omega)}\right] - \int_0^t (z_t(s), [D_h z(s)]_t)_{L_2(\Omega)} ds \\
+c^2 \int_0^t (\nabla z(s), D_h(\nabla z(s)))_{L_2(\Omega)} ds + c^2 l_0 \int_0^t (z(s)|_\Gamma, D_h(z(s)|_\Gamma))_{L_2(\Gamma)} ds \\
+ \int_0^t ([d^{\frac{1}{2}} z(s)]_t, D_h(d^{\frac{1}{2}} z_t(s)))_{L_2(\Omega)} ds + \int_0^t ([l^{\frac{1}{2}} z(s)|_\Gamma]_t, D_h(l^{\frac{1}{2}} z(s)|_\Gamma))_{L_2(\Gamma)} ds \\
-c^2 \int_0^t (w_t(s), D_h z(s)|_{\Gamma_0})_{L_2(\Gamma_0)} ds.
\end{aligned}
$$

Note that by the virtue of Lemma 4.1, see also [25], we have

$$
\int_0^t (z_t(s), [D_h z(s)]_t)_{L_2(\Omega)} \, ds = 0, \qquad \text{for every } h > 0.
$$

Indeed, using the definition of D_h and performing straightforward computations with change of variables, we have

$$
\begin{aligned}
&\int_0^t (z_t(s), [D_h z(s)]_t)_{L_2(\Omega)} \, ds = \frac{1}{2h} \int_0^t (z_t(s), z_t(s+h) - z_t(s-h))_{L_2(\Omega)} \, ds \\
&= \frac{1}{2h} \int_0^h (z_t(s), z_t(t+h))_{L_2(\Omega)} \, ds + \frac{1}{2h} \int_h^{t-h} (z_t(s), z_t(s+h) - z_t(s-h))_{L_2(\Omega)} \, ds \\
&\quad - \frac{1}{2h} \int_{t-h}^t (z_t(s), z_t(s-h))_{L_2(\Omega)} \, ds \\
&= \frac{1}{2h} \int_0^h (z_t(s), z_t(t+h))_{L_2(\Omega)} \, ds + \frac{1}{2h} \int_h^{t-h} (z_t(s), z_t(s+h))_{L_2(\Omega)} \, ds \\
&\quad - \frac{1}{2h} \int_0^{t-2h} (z_t(s), z_t(s+h))_{L_2(\Omega)} \, ds - \frac{1}{2h} \int_{t-h}^t (z_t(s), z_t(s-h))_{L_2(\Omega)} \, ds,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_0^t (z_t(s), [D_h z(s)]_t)_{L_2(\Omega)}\, ds \\
&= \frac{1}{2h}\Bigg[\int_0^h (z_t(s), z_t(t+h))_{L_2(\Omega)}\, ds + \int_{t-2h}^{t-h} (z_t(s), z_t(s+h))_{L_2(\Omega)}\, ds \\
&\quad + \int_h^{t-2h} (z_t(s), z_t(s+h))_{L_2(\Omega)}\, ds - \int_0^h (z_t(s), z_t(s+h))_{L_2(\Omega)}\, ds \\
&\quad - \int_h^{t-2h} (z_t(s), z_t(s+h))_{L_2(\Omega)}\, ds - \int_{t-2h}^{t-h} (z_t(s), z_t(s+h))_{L_2(\Omega)}\, ds\Bigg] \\
&= 0,
\end{aligned}
$$

as desired.

Our assumption on the support of $l(x)$ allows to deduce

$$
\int_0^t ||z_t(s)||^2_{\Gamma_0} ds \le C[E_z(0) + \int_0^t ||w_t(s)||^2_{\Gamma_0} ds]. \tag{4.19}
$$

In fact, the above inequality results from the so called "hidden regularity", which in this case can be simply deduced from the following Lemma.

Lemma 4.2. *Let $g \in L_2(0, T; L_2(\Gamma_0))$ and z be a solution of $\Box z = 0$, subject to zero initial data and the boundary conditions*

$$
\frac{\partial}{\partial \nu} z + l(x) z_t = \begin{cases} g & on\ \Sigma_0 \\ -l_0 z & on\ \Sigma_1 \end{cases}
$$

where supp $l(x) \supset \Gamma_0$. *Then, the following inequality holds:*

$$
||z_t(t)||^2 + ||\nabla z(t)||^2 + ||z(t)||^2_{L_2(\Sigma_1)} + \int_0^t ||l^{1/2} z_t(s)||^2_{L_2(\Gamma)} ds \le C \int_0^t ||g(s)||^2_{L_2(\Gamma_0)} ds.
$$

Proof. Since the solutions are smooth for $z_t(t) \in H^1(\Omega)$, $z_{tt}(t) \in L_2(\Omega)$ and compatible boundary data g, it suffices to prove the inequality for smooth solutions only. Multiplying the D'Alambertian by z_t and integrating by parts gives

$$
||z_t(t)||^2 + ||\nabla z(t)||^2 + ||z(t)||^2_{L_2(\Sigma_1)} + 2\int_0^t ||l^{1/2} z_t||^2_{L_2(\Gamma)} \le 2\int_0^t \int_{\Gamma_0} g(t) z_t\, dx dt.
$$

Exploiting the condition on the support of $l(x)$ this yields

$$||z_t(t)||^2 + ||\nabla z(t)||^2 + ||z(t)||^2_{L_2(\Sigma_1)} + \int_0^t ||z_t||^2_{L_2(\Gamma_0)} \leq C \int_0^t \int_{\Gamma_0} |g(t)|^2 dx dt. \quad \square$$

Continuing with the proof, inequality (4.19) and Lemma 4.1 allows taking the limit as $h \to 0$ in the expression

$$\lim_{h\to 0} \int_0^t ([l^{\frac{1}{2}} z(s)|_\Gamma]_t, D_h(l^{\frac{1}{2}} z(s)|_\Gamma))_{L_2(\Gamma)} ds = \int_0^t ([l^{\frac{1}{2}} z(s)|_\Gamma]_t, (l^{\frac{1}{2}} z_t(s)|_\Gamma))_{L_2(\Gamma)} ds.$$

For the same reason, and having in mind that $w_t \in C(0, T; L_2(\Gamma_0))$, we obtain

$$\lim_{h\to 0} \int_0^t (w_t(s), D_h z(s)|_{\Gamma_0})_{L_2(\Gamma_0)} ds = \int_0^t (w_t(s), z_t(s)|_{\Gamma_0})_{L_2(\Gamma_0)} ds.$$

Taking the limit when $h \to 0$ and using Lemma 4.1,

$$\begin{aligned} &E_z(t) + \int_0^t \|d^{\frac{1}{2}} z_t(s)\|^2_{L_2(\Omega)} ds + \int_0^t \|l^{\frac{1}{2}} z_t(s)|_\Gamma\|^2_{L_2(\Gamma)} ds \\ &= E_z(0) + c^2 \int_0^t (w_t(s), z_t(s)|_{\Gamma_0})_{L_2(\Gamma_0)} ds, \quad \text{for } t > 0. \end{aligned} \tag{4.20}$$

Adding up equations (4.18) and (4.20), having in mind relation $1 - c^{-2}\rho = 0$, we obtain (4.14) as desired. $\square$

Our next result shows that the weak solutions depend continuously on the initial data with respect to the strong topology of $\mathcal{H}$. The proof adopts some ideas in [25].

Proposition 4.4 *(Strong continuous dependence). Under the assumptions of Proposition 4.1, the corresponding weak solutions of* (1.1)-(1.2) *depend continuously on the initial data with respect to the strong topology of* $\mathcal{H}$.

Proof. Let $T > 0$ and $\{U_0^n \equiv (z_0^n, z_1^n, w_0^1, w_1^n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $U_0^n \to U_0 \equiv (z_0, z_1, w_0, w_1)$ in $\mathcal{H}$. If $U^n, U : [0, T] \to \mathcal{H}$ are the corresponding weak solutions of (1.1)-(1.2), then using the energy identity (2.8) and Assumption 3.1, we have

$$\begin{aligned} &\Phi^n(t) \equiv E_z^n(t) + E_w^n(t) + \left\| w^n(t) \right\|^2_{L_2(\Gamma_0)} \\ &\leq C(\left\| U_0^n \right\|_{\mathcal{H}}) + C(\sigma) \int_0^t \Phi^n(s) ds, \quad \text{for } t \in [0, T], \quad n \in \mathbb{N}, \end{aligned}$$

which implies that $\{U^n\}$ is bounded in $L_\infty(0,T;\mathcal{H})$. Reducing to a subsequence if necessary, we conclude that

$$U^n \to U \text{ weak star in } L_\infty(0,T;\mathcal{H}). \tag{4.21}$$

In order to conclude the proof, it suffices to prove that

$$\left\|U^n(t)\right\|_{\mathcal{H}} \to \|U(t)\|_{\mathcal{H}} \text{ in } C[0,T]. \tag{4.22}$$

Since the functional $\Phi(t) \equiv E(t) + \|w(t)\|^2_{L_2(\Gamma_0)}$ is equivalent to the topology of $\mathcal{H}$, the convergence (4.22) will follow from $\lim_{n\to\infty}\Phi^n(t) = \Phi(t)$. Using the energy identity (2.8) and the continuity of $E^n(t) = E^n_z(t) + E^n_w(t)$, we have

$$\begin{aligned}\mathcal{E}(0) &= \lim_{n\to\infty}\mathcal{E}^n(0)\\ &= \lim_{n\to\infty}\left[\mathcal{E}^n(t) + k\int_0^t \|w^n_t(s)\|^2_{L_2(\Gamma_0)}ds - \sigma\int_0^t (w^n_t(s), |\nabla w^n(s)|^2)_{L_2(\Gamma_0)}ds\right].\end{aligned}$$

Using the energy identity once more and the uniqueness of weak solutions, it follows from the previous identity that

$$\begin{aligned}&\lim_{n\to\infty}\left[\mathcal{E}^n(t) + k\int_0^t \|w^n_t(s)\|^2_{L_2(\Gamma_0)}ds - \sigma\int_0^t (w^n_t(s), |\nabla w^n(s)|^2)_{L_2(\Gamma_0)}ds\right]\\ &\quad = \mathcal{E}(t) + k\int_0^t \|w_t(s)\|^2_{L_2(\Gamma_0)}ds - \sigma\int_0^t (w_t(s), |\nabla w(s)|^2)_{L_2(\Gamma_0)}ds.\end{aligned}$$

Moreover, using the weak-star convergence (4.21) and the compactness $H^1(\Gamma_0) \to L_4(\Gamma_0)$, we obtain from the previous identity that

$$\lim_{n\to\infty}\mathcal{E}^n(t) \le \mathcal{E}(t).$$

Now, using the expression of $\mathcal{E}^n$ and $\mathcal{E}$, the compactness of $u \in H^2(\Gamma_0) \mapsto u\,|\nabla u|^2 \in L_1(\Gamma_0)$, $u \in H^2(\Omega) \mapsto F(u) \in L_1(\Omega)$ and the lower semicontinuity of $E^n(t) = E^n_z(t) + E^n_w(t)$, we conclude

$$\lim_{n\to\infty}\left[E^n_z(t) + E^n_w(t)\right] = E_z(t) + E_w(t). \tag{4.23}$$

Finally, it follows from the weak convergence (4.21) that (see [33]) $w_n \to w$ in $C([0,T]; H^{2-\varepsilon}(\Gamma_0)$ for any $\varepsilon > 0$. This strong convergence implies that

$$\lim_{n\to\infty}\left\|w^n(t)\right\|_{L_2(\Gamma_0)} = \|w(t)\|_{L_2(\Gamma_0)} \tag{4.24}$$

Limits (4.23) and (4.24) imply $\lim_{n\to\infty} \Phi^n(t) = \Phi(t)$ as desired and, therefore, convergence (4.22) follows, which concludes the proof. $\square$

Propositions 4.1, 4.2 and 4.4 prove items 1 and 2 in Theorem 2.1. It remains to prove the regularity result, which will be done next.

4.3. Regularity

Proposition 4.5 *(Regularity). In addition to the assumptions of Proposition 4.1, we assume that Assumption 3.1 holds along with the compatibility conditions* (2.9a)-(2.9b)*. If* $(z_0, z_1, w_0, w_1) \in H^2(\Omega) \times H^1(\Omega) \times H^4(\Gamma_0) \times H^2(\Gamma_0)$ *and* Ω *is sufficiently smooth, then the corresponding solution* (z, w) *is strong.*

Remark 4.1. So far we have proved existence and uniqueness of weak solutions, which satisfy the variational form. One of the issues is to be able to show that in the case of smooth and compatible initial data, these solutions satisfy an appropriate form of PDE. Technical difficulties appear, particularly, in the case of free boundary conditions. These produce boundary terms which "spill over" in the variational form and are not controlled by the energy. To handle the obstacle, we shall work with variational forms satisfied by *finite dimensional* approximations. In what follows below, we provide a brief synopsis of steps to be followed.

- We prove that time derivatives display finite energy regularity. This step requires:
  - Differentiation in time of variational equality produces new terms on the boundary which are not controlled by the energy. To handle these, Green's maps and fractional powers of the biharmonic operator are critically used.
  - In addition, time differentiation produces interior nonlinear term which is also supercritical. To handle the latter, Brezis-Gallouët inequality is used. However, this requires control of H^3 norms.
  - The next step is to obtain the enhanced H^3 space regularity for the plate. This is obtained from the variational finite-dimensional formulation with critical use of compatibility conditions and, again, control of "blow-up" the Sobolev's embeddings.
- Regularity of time derivatives is obtained from logarithmic control of Gronwall's inequality, and the H^4 regularity for the plate and H^2 regularity for the wave is obtained by duality.

Proof. The argument is based on establishing *uniform estimates* for Faedo-Galerkin approximations which then need to be reconstructed as strong solutions. Let us consider $\{\phi_n\}_{n\in\mathbb{N}} \subset H^2(\Omega)$ and $\{\psi_n\}_{n\in\mathbb{N}} \subset H^4(\Gamma_0)$ basis of eigenvectors of the Laplacian operator with Neumann boundary conditions and the biharmonic operator with either (C), (SS) or (F) boundary conditions (see Section 3.3). For each $n \in \mathbb{N}$, let $\mathcal{V}_n$ be the approximation subspace as defined in Proposition 4.1, and consider corresponding solution (z^n, w^n) of the following variational problem

$$
\begin{aligned}
&\text{Find } (z^n(t), w^n(t)) \in \mathcal{V}_n \text{ such that:}\\
&0 = \left(z^n_{tt}(t), \phi\right)_{L_2(\Omega)} + \left(w^n_{tt}(t), \psi\right)_{L_2(\Gamma_0)} + c^2\left(\nabla z^n(t), \nabla\phi\right)_{L_2(\Omega)} + a(w^n(t), \psi)\\
&\quad + \left(d\cdot z^n_t(t), \phi\right)_{L_2(\Omega)} + k\left(w^n_t(t), \psi\right)_{L_2(\Gamma_0)} + c^2\left(l\cdot z^n_t(t)|_\Gamma, \phi|_\Gamma\right)_{L_2(\Gamma)}\\
&\quad - c^2\left(w^n_t(t), \phi|_{\Gamma_0}\right)_{L_2(\Gamma_0)} + c^2 l_0\left(z^n(t)|_{\Gamma_1}, \phi|_{\Gamma_1}\right)_{L_2(\Gamma_1)} + \rho\left(z^n_t(t)\big|_{\Gamma_0}, \psi\right)_{L_2(\Gamma_0)}
\end{aligned}
\tag{4.25}
$$

$$
\begin{aligned}
&+G(w^n(t),\psi), \qquad \text{for any } (\phi,\psi)\in\mathcal{V}_n;\\
&(z^n(0), z^n_t(0), w^n(0), w^n_t(0)) \equiv (z^n_0, z^n_1, w^n_0, w^n_1) \to (z_0, z_1, w_0, w_1) \text{ in } \mathcal{H},
\end{aligned}
$$

where $a(\cdot,\cdot)$ is given in (2.3). It follows from Propositions 4.1 and 4.4 that $\{(z^n, z^n_t, w^n, w^n_t)\}_{n\in\mathbb{N}}$ converges to (z, z_t, w, w_t) weakly star in $L_\infty(0,T;\mathcal{H})$, for every $T>0$. Let us consider $R>0$ such that $\|(z_0, z_1, w_0, w_1)\|_{\mathcal{H}} \le R$.

Differentiating (4.25) in time and setting $(u^n, v^n) \equiv (z^n_t, w^n_t)$, we obtain

$$
\begin{aligned}
0 = \frac{d}{dt}&\left[\left(u^n_t(t),\phi\right)_{L_2(\Omega)} + \left(v^n_t(t),\psi\right)_{L_2(\Gamma_0)} + \rho\left(u^n(t)|_{\Gamma_0},\psi\right)_{L_2(\Gamma_0)}\right]\\
&+c^2\left(\nabla u^n(t),\nabla\phi\right)_{L_2(\Omega)} + a(v^n(t),\psi) + \left(d\cdot u^n_t(t),\phi\right)_{L_2(\Omega)}\\
&+c^2\left(l\cdot u^n_t(t)|_\Gamma,\phi|_\Gamma\right)_{L_2(\Gamma)} + k\left(v^n_t(t),\psi\right)_{L_2(\Gamma_0)} + c^2 l_0\left(u^n(t)|_{\Gamma_1},\phi|_{\Gamma_1}\right)_{L_2(\Gamma_1)}\\
&-c^2\left(v^n_t(t),\phi|_{\Gamma_0}\right)_{L_2(\Gamma_0)} + \frac{d}{dt}G(w^n(t),\psi), \qquad \forall(\phi,\psi)\in\mathcal{V}_n,
\end{aligned}
\tag{4.26}
$$

where, we recall, $G(w,\psi)$ is given by (2.5).

Note that

$$
\begin{aligned}
\frac{d}{dt}G(w^n(t),\psi) &= (|\nabla w^n(t)|^2\nabla v^n(t) + 2(\nabla w^n(t),\nabla v^n(t))_{\mathbb{R}^2}\nabla w^n(t),\nabla\psi)_{L_2(\Gamma_0)}\\
&\quad + 2\sigma\left(\nabla\{w^n(t)v^n(t)\},\nabla\psi\right)_{L_2(\Gamma_0)} + \left(f'(w^n(t))v^n(t),\psi\right)_{L_2(\Gamma_0)},
\end{aligned}
$$

for every $\psi\in\operatorname{span}\{\psi_i : i=1,\dots,n\}$.

Choosing $(\phi,\psi) = (u^n_t(t), v^n_t(t))$ and recalling that $1-c^{-2}\rho=0$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{W}_n(t) + k\|v^n_t(t)\|^2_{L_2(\Gamma_0)} + \|d^{1/2}u^n_t(t)\|^2_{L_2(\Omega)} + c^2\|l^{1/2}u^n_t(t)|_\Gamma\|^2_{L_2(\Gamma)}\\
&= 3(\nabla w^n(t), |\nabla v^n(t)|^2\nabla v^n(t))_{L_2(\Gamma_0)} - 2\sigma\left(\nabla\{w^n(t)v^n(t)\},\nabla v^n_t(t)\right)_{L_2(\Gamma_0)}\\
&\quad - \left(f'(w^n(t))v^n(t), v^n_t(t)\right)_{L_2(\Gamma_0)}\\
&\equiv 3I_1(w^n,v^n) + 2\sigma I_2(w^n,v^n) + I_3(w^n,v^n),
\end{aligned}
\tag{4.27}
$$

where $\mathcal{W}_n(t) \equiv \mathcal{U}_n(u^n(t), u^n_t(t)) + \mathcal{V}_n(v^n(t), v^n_t(t))$ and

$$
\mathcal{U}_n(u^n, u^n_t) \equiv \frac{1}{2}\left[\|u^n_t\|^2_{L_2(\Omega)} + c^2\|\nabla u^n\|^2_{L_2(\Omega)} + c^2 l_0\|u^n|_{\Gamma_1}\|^2_{L_2(\Gamma_1)}\right];
$$

$$
\mathcal{V}_n(v^n, v^n_t) \equiv \frac{1}{2}\left[\|v^n_t\|^2_{L_2(\Gamma_0)} + a(v^n,v^n) + \int_{\Gamma_0}|\nabla w^n|^2|\nabla v^n|^2 + 2\left|(\nabla w^n,\nabla v^n)_{\mathbb{R}^2}\right|^2 d\Gamma_0\right].
$$

To estimate integrals I_i, $i=1,2,3$, we shall follow the arguments presented in [20].

Computing I_1. This integral is the most critical one due to the superlinearity. To handle it, we shall use logarithmic estimates that follow from Brezis-Gallouët inequality (see [15])

$$\|u\|_{L_\infty} \le C\|u\|_{H^1} \ln^{1/2}(1+K) + C\|u\|_{H^2}(1+K)^{-1}, \tag{4.28}$$

for some constant C depending on the domain and $K > 0$ is arbitrary. Using the above estimate for ∇w^n and $K = ||w^n||_{H^3(\Gamma_0)}$

$$\|\nabla w^n\|_{L_\infty(\Gamma_0)} \le C\|w^n\|_{H^2(\Gamma_0)}[1 + \ln^{1/2}(1 + \|w^n\|_{H^3(\Gamma_0)})] + C \tag{4.29}$$

By (4.29) and classical Sobolev embedding, we arrive at

$$\begin{aligned} |I_1| \le \|\nabla w^n\|_{L_\infty(\Gamma_0)} \|v^n\|^3_{W^{1,3}(\Gamma_0)} &\le C\|\nabla w^n\|_{L_\infty(\Gamma_0)}\|v^n\|_{L_2(\Gamma_0)}\|v^n\|^2_{H^2(\Gamma_0)} \\ &\le C_{T,R}\|v^n\|^2_{H^2(\Gamma_0)}[1 + \ln^{1/2}(1 + \|w^n\|_{H^3(\Gamma_0)})], \end{aligned}$$

where $C_{T,R}$ is a constant that depends on the domain obtained by using the uniform bounds from Section 4.1.

Our next step is to obtain the estimate for the H^3 norm of w^n. While this is relatively straightforward for clamped or simply supported boundary conditions, the treatment of *free* boundary conditions requires more involved and delicate arguments. To proceed, we go back to the approximated variational problem (4.25) setting $\phi = 0$ and using Green's formula on the biharmonic operator

$$\begin{aligned} &(\Delta^2 w^n, \psi)_{L_2(\Gamma_0)} + < BC, \psi|_{\partial\Gamma_0} >_{\partial\Gamma_0} \\ &= -\left(w^n_{tt}, \psi\right)_{L_2(\Gamma_0)} - k\left(w^n_t, \psi\right)_{L_2(\Gamma_0)} - \rho\left(z^n_t|_{\Gamma_0}, \psi\right)_{L_2(\Gamma_0)} \\ &+(\mathrm{div}\{|\nabla w^n|^2\nabla w^n\}, \psi)_{L_2(\Gamma_0)} + \sigma(\Delta\{w^{n2}\}, \psi)_{L_2(\Gamma_0)} - (f(w^n), \psi)_{L_2(\Gamma_0)}, \end{aligned} \tag{4.30}$$

where $BC \equiv [|\nabla w^n|^2\nabla w^n + \sigma\nabla\{w^{n2}\}] \cdot \nu$. Note that, in the clamped (C) or simply supported (SS) cases we have $BC \equiv 0$. In order to deal with free boundary conditions (F), we will consider the following Green's map $\mathcal{G} : L_2(\partial\Gamma_0) \to L_2(\Gamma_0)$ given by

$$\mathcal{G}g = v \quad \text{iff } v \text{ is the solution of } \begin{cases} \Delta^2 v = 0, & \text{in } \Gamma_0; \\ \Delta v + (1-\mu)B_1 v = 0, \quad \frac{\partial}{\partial\nu}\Delta v + (1-\mu)B_2 v = g, & \text{on } \partial\Gamma_0. \end{cases}$$

It is known that if $\mathcal{G}^*$ is its adjoint in L_2 then $\mathcal{G}^*\mathcal{A}\psi = -\psi|_{\partial\Gamma_0}$ for every $\psi \in H^2(\Gamma_0)$. Thus, we rewrite (4.30) as follows

$$< \mathcal{A}(w^n - \mathcal{G}BC), \psi >= (R(w^n) - \rho z^n_t, \psi)_{L_2(\Gamma_0)}, \tag{4.31}$$

for every $\psi \in \mathrm{span}\{\psi_1, \ldots, \psi_n\}$, where

$$R(w^n) = -w^n_{tt} - kw^n_t + \mathrm{div}\{|\nabla w^n|^2\nabla w^n\} + \sigma\Delta\{w^{n2}\} - f(w^n).$$

Note that $w^n - \mathcal{G}BC$ does not belong to $D(\mathcal{A})$ due to the boundary effects caused by the Green's map. Thus, identity (4.31) is understood in the dual sense, i.e., with respect to $[D(\mathcal{A}^{1/2})]'$ topology (or $[H^2(\Omega_0)]'$), Hence, we conclude

$$\mathcal{P}_N\mathcal{A}(w^n - \mathcal{G}BC) = \mathcal{P}_N[R(w_n) - \rho z^n_t]$$

where $\mathcal{P}_N$ denotes the orthonormal projector in span$\{\psi_1, \dots, \psi_N\}$. Since $\mathcal{P}_N$ commutes with $\mathcal{A}$ (and its fractional powers) we have

$$\mathcal{A}\mathcal{P}_N w^n = \mathcal{P}_N \mathcal{A}\mathcal{G}BC + \mathcal{P}_N R(w^n) - \rho \mathcal{P}_N z_t^n, \quad \text{for } n \geq N. \tag{4.32}$$

By elliptic regularity and characterization of fractional powers of $\mathcal{A}$ where $\mathcal{A}^\theta \sim H^{4\theta}(\Gamma_0)$ for $\theta < \frac{7}{8}$ [21].

$$\mathcal{G} \in L(H^{-1/2}(\partial\Gamma_0) \to H^3(\Gamma_0) \subset D(\mathcal{A}^{3/4})) \tag{4.33}$$

and rescaling equation (4.32) by $\mathcal{A}^{-1/4}$ yields:

$$\mathcal{A}^{3/4}\mathcal{P}_N w^n = \mathcal{P}_N \mathcal{A}^{3/4}\mathcal{G}BC + \mathcal{A}^{-1/4}\mathcal{P}_N R(w^n) - \rho \mathcal{A}^{-1/4}\mathcal{P}_N z_t^n,$$

for $n \geq N$. This yields the estimate

$$\begin{aligned} \|\mathcal{A}^{3/4}\mathcal{P}_N w^n\|_{L_2(\Gamma_0)} \leq \|\mathcal{P}_N \mathcal{A}^{3/4}\mathcal{G}BC\|_{L_2(\Gamma_0)} + \|\mathcal{A}^{-1/4}\mathcal{P}_N R(w^n)\|_{L_2(\Gamma_0)} \\ +\rho \|\mathcal{A}^{-1/4}\mathcal{P}_N z_t^n\|_{L_2(\Gamma_0)} \equiv J_1 + J_2 + J_3. \end{aligned} \tag{4.34}$$

Estimates for J_1. By elliptic regularity in (4.33) combined with the embedding $H^{1/2}(\partial\Gamma_0) \subset L_p(\partial\Gamma_0)$ for $1 \leq p < \infty$, we have

$$\begin{aligned} \|\mathcal{P}_N \mathcal{A}^{3/4}\mathcal{G}BC\|_{L_2(\Gamma_0)} &\leq C\|BC\|_{H^{-1/2}(\partial\Gamma_0)} = C\|[|\nabla w^n|^2 \nabla w^n + \sigma \nabla\{{w^n}^2\}] \cdot \nu\|_{H^{-1/2}(\partial\Gamma_0)} \\ &\leq C(\|\nabla w^n\|^3_{H^{1/2}(\partial\Gamma_0)} + \|w^n\|_{H^{1/2}(\partial\Gamma_0)}\|\nabla w^n\|_{L_2(\partial\Gamma_0)}) \leq C(1 + \|w^n\|^3_{H^2(\Gamma_0)}). \end{aligned}$$

Estimates for J_2.

$$\begin{aligned} &\|\mathcal{A}^{-1/4}\mathcal{P}_N R(w^n)\|_{L_2(\Gamma_0)} \leq \|R(w^n)\|_{H^{-1/2+\varepsilon}(\Gamma_0)} \\ &\leq C(1 + \|w^n\|^3_{H^2(\Gamma_0)} + \|w^n_{tt}\|_{L_2(\Gamma_0)} + \|w^n_t\|_{L_2(\Gamma_0)}) + C\Psi(\|w^n\|_{H^2(\Gamma_0)}), \end{aligned}$$

where $\Psi(r) = \max_{|s|\leq r} |f(s)|$ and C stands for a constant that does not depend on n or N.

Estimates for J_3. Using trace and interpolation theorems, we have

$$\|\mathcal{A}^{-1/4}\mathcal{P}_N z_t^n\|_{L_2(\Gamma_0)} \leq C\|z_t^n\|_{L_2(\Gamma_0)} \leq C\left(\varepsilon\|z_t^n\|_{H^1(\Omega)} + C_\varepsilon\|z_t^n\|_{L_2(\Omega)}\right).$$

Plugging these estimates into (4.34), having in mind that $D(\mathcal{A}^\theta) \subset H^{4\theta}(\Gamma_0)$ for $\theta \in [0, 1]$ and the substitution $w_t^n = v^n$ and $z_t^n = u^n$, we obtain

$$\|w^n\|_{H^3(\Gamma_0)} \leq C_1\left[\|v_t^n\|_{L_2(\Gamma_0)} + \|v^n\|_{L_2(\Gamma_0)} + \|u^n\|_{H^1(\Omega)}\right] + C_2\tilde{\Psi}(\|w^n\|_{H^2(\Gamma_0)}) \tag{4.35}$$

where $\tilde{\Psi}(r) = 1 + r^3 + \Psi(r)$ is an increasing function. The above leads to:

Lemma 4.3. *Under any of the boundary conditions, one has the following estimate for Galerkin approximation* w^n

$$\|w^n\|_{H^3(\Gamma_0)} \le C_1 \left[\mathcal{W}_n + \|v^n\|^2_{L_2(\Gamma_0)}\right] + C_{T,R}$$

where $\mathcal{W}_n(t)$ *is defined below in (4.27).*

Finally, the estimate in Lemma 4.3 together with uniform bounds obtained for approximated solutions in Section 4.1 implies the following

$$|I_1| \le C_{T,R}\|v^n\|^2_{H^2(\Gamma_0)} \left[1 + \ln^{1/2}(1 + \mathcal{W}_n + \|v^n\|^2_{L_2(\Gamma_0)})\right]. \tag{4.36}$$

Computing I_2. If the boundary conditions are of type clamped (C) or simply supported (SS), then I_2 can be rewritten as

$$\begin{aligned} I_2 &= -(\nabla\{w^n v^n\}, \nabla\{v_t^n\})_{L_2(\Gamma_0)} \\ &= -< \frac{\partial}{\partial\nu}\{w^n v^n\}, v_t^n >_{\partial\Gamma_0} + (\Delta\{w^n v^n\}, v_t^n)_{L_2(\Gamma_0)} = (\Delta\{w^n v^n\}, v_t^n)_{L_2(\Gamma_0)} \\ &= (v^n\Delta\{w^n\}, v_t^n)_{L_2(\Gamma_0)} + (w^n\Delta\{v^n\}, v_t^n)_{L_2(\Gamma_0)} + 2(\nabla\{w^n\}\cdot\nabla\{v^n\}, v_t^n)_{L_2(\Gamma_0)}. \end{aligned}$$

Using the inclusion $H^2(\Gamma_0) \subset L_\infty(\Gamma_0)$ and the uniform boundedness for the approximated solutions, we obtain

$$|I_2| \le C_{T,R}\|v^n\|_{H^2(\Gamma_0)}\|v_t^n\|_{L_2(\Gamma_0)}.$$

In the case of free boundary conditions, we use a different approach, although it might be used also for the other boundary conditions. We return to the variational problem (4.25) and rewrite I_2 as follows

$$\begin{aligned} I_2 &= -\frac{d}{dt}\left\{(v_n\nabla w^n, \nabla v^n)_{L_2(\Gamma_0)} + \frac{1}{2}(w^n, |\nabla v^n|^2)_{L_2(\Gamma_0)}\right\} \\ &\quad + \frac{3}{2}(v^n, |\nabla v^n|^2)_{L_2(\Gamma_0)} + (v_t^n\nabla w^n, \nabla v^n)_{L_2(\Gamma_0)} \\ &\equiv -\frac{d}{dt}I_{21}(t) + I_{22}(t). \end{aligned} \tag{4.37}$$

In order to the obtain estimates for $I_{21}(t)$ and $I_{22}(t)$, we are going to use inclusion $H^s(\Gamma_0) \subset L_\infty(\Gamma_0)$ for $s > 1$, compactness imbedding and uniform boundedness for the approximate solutions.

Estimates for I_{21}.

$$\begin{aligned} |I_{21}(t)| &\le |(v_n\nabla w^n, \nabla v^n)_{L_2(\Gamma_0)} + \frac{1}{2}(w^n, |\nabla v^n|^2)_{L_2(\Gamma_0)}| \\ &\le C[\|v^n\|_{L_\infty(\Gamma_0)}\|v^n\|_{H^1(\Gamma_0)}\|w^n\|_{H^1(\Gamma_0)} + \|w^n\|_{L_\infty(\Gamma_0)}\|v^n\|^2_{H^1(\Gamma_0)}] \end{aligned} \tag{4.38}$$

$$\leq C_{T,R}[\|v^n\|_{L_\infty(\Gamma_0)}\|v^n\|_{H^1(\Gamma_0)} + \|v^n\|^2_{H^1(\Gamma_0)}] \leq C_{T,R}\|v^n\|^2_{H^{2-\varepsilon}(\Gamma_0)}$$
$$\leq \eta\|v^n\|^2_{H^2(\Gamma_0)} + C_{\eta,T,R},$$

for $\eta > 0$ can be taken arbitrarily small.

Estimates for I_{22}.

$$\begin{aligned}
|I_{22}(t)| &\leq \frac{3}{2}|(v^n, |\nabla v^n|^2)_{L_2(\Gamma_0)}| + |(v_t^n \nabla w^n, \nabla v^n)_{L_2(\Gamma_0)}| \\
&\leq \frac{3}{2}\|w_t^n\|_{L_2(\Gamma_0)}\|\nabla v^n\|^2_{L_4(\Gamma_0)} + \|v_t^n\|_{L_2(\Gamma_0)}\|\nabla w^n\|_{L_4(\Gamma_0)}\|\nabla v^n\|_{L_4(\Gamma_0)} \\
&\leq C_{T,R}[\|\nabla v^n\|^2_{L_4(\Gamma_0)} + \|v_t^n\|_{L_2(\Gamma_0)}\|\nabla v^n\|_{L_4(\Gamma_0)}] \leq C_{\eta,T,R}\|\nabla v^n\|^2_{L_4(\Gamma_0)} + \eta\|v_t^n\|^2_{L_2(\Gamma_0)} \\
&\leq C_{\eta,T,R}\|v^n\|^2_{H^{2-\varepsilon}(\Gamma_0)} + \eta\|v_t^n\|^2_{L_2(\Gamma_0)} \leq C_{\eta,T,R}\|v^n\|^2_{H^2(\Gamma_0)} + \eta\|v_t^n\|^2_{L_2(\Gamma_0)},
\end{aligned} \tag{4.39}$$

for any $\eta > 0$ arbitrarily small.

Computing I_3. Since f is of class C^1, it follows from the embedding $H^2(\Gamma_0) \subset L_\infty(\Gamma_0)$ and uniform bounds for the approximated solutions,

$$|I_3| \leq C_{T,R}\|v^n\|_{H^2(\Gamma_0)}\|v_t^n\|_{L_2(\Gamma_0)} \leq \eta\|v_t^n\|^2_{L_2(\Gamma_0)} + C_{\eta,T,R}\|v^n\|^2_{H^2(\Gamma_0)}, \tag{4.40}$$

for $\eta > 0$ arbitrarily small.

Plugging (4.37) into (4.27) and using estimates (4.36), (4.39) and (4.40), also choosing $\eta < k/2(2\sigma + 1)$, we conclude

$$\begin{aligned}
&\frac{d}{dt}[\mathcal{W}_n + 2\sigma I_{21}] + \frac{k}{2}\|v_t^n\|^2_{L_2(\Gamma_0)} + \|d^{1/2}u_t^n\|^2_{L_2(\Omega)} + c^2\|l^{1/2}u_t^n|_\Gamma\|^2_{L_2(\Gamma)} \\
&\leq C_{T,R}\|v^n\|^2_{H^2(\Gamma_0)}[1 + \ln^{1/2}(1 + \mathcal{W}_n + \|v^n\|^2_{L_2(\Gamma_0)})] + 2\sigma C_{\eta,T,R}\|v^n\|^2_{H^2(\Gamma_0)}.
\end{aligned} \tag{4.41}$$

Let us define

$$\tilde{\mathcal{W}}_n \equiv \mathcal{W}_n + 2\sigma I_{21} + \|v^n\|^2_{L_2(\Gamma_0)} + c_0, \tag{4.42}$$

where $c_0 > 0$ is an appropriate constant. Note that, since $\|v\|^2_{H^2}$ is equivalent to $a(v, v) + \|v\|^2_{L_2}$, it follows from inequality (4.38) that, making η sufficiently small and taking $c_0 > 0$ appropriate, it follows that

$$\tilde{\mathcal{W}}_n \geq C_{1,\eta}[\mathcal{W}_n + \|v_n\|^2_{L_2(\Gamma_0)}] \geq C_{2,\eta}\|v^n\|^2_{H^2(\Gamma_0)}.$$

Also, taking the derivative of $\tilde{\mathcal{W}}_n$ we obtain

$$\frac{d}{dt}\tilde{\mathcal{W}}_n = \frac{d}{dt}[\mathcal{W} + 2\sigma I_{21}] + 2(v^n, v_t^n)_{L_2(\Gamma_0)},$$

which implies

$$\frac{d}{dt}\tilde{\mathcal{W}}_n + (k/2-\varepsilon)\|v_t^n\|^2_{L_2(\Gamma_0)} - C_{\eta,\varepsilon}\|v_n\|^2_{H^2(\Gamma_0)} \le \frac{d}{dt}[\mathcal{W}+2\sigma I_{21}] + \frac{k}{2}\|v_t^n\|^2_{L_2(\Gamma_0)}.$$

Choosing $\varepsilon < k/2$, and returning to (4.41) and with the obtained bounds we conclude

$$\frac{d}{dt}\tilde{\mathcal{W}}_n \le C_1\tilde{\mathcal{W}}_n \ln^{1/2}(1+\tilde{\mathcal{W}}_n) + C_2\tilde{\mathcal{W}}_n. \tag{4.43}$$

Solving the differential inequality (4.43), above we obtain

$$\mathcal{W}_n(t) + \|v^n\|^2_{L_2(\Gamma_0)} \le C\tilde{\mathcal{W}}_n(t) \le C_{T,R}(1+\tilde{\mathcal{W}}_n(0))^{\beta_{T,R}}, \quad \text{for } t \ge 0,$$

where $C_{T,R}$ and $\beta_{T,R}$ are positive constants.

Using the expression of $\tilde{\mathcal{W}}_n$ and inequality (4.38), the above estimate implies

$$\sup_{t\in[0,T]}\{\|z_{tt}^n\|^2_{L_2(\Omega)} + \|z_t^n\|^2_{H^1(\Omega)} + \|w_{tt}^n\|^2_{L_2(\Gamma_0)} + \|w_t^n\|^2_{H^2(\Gamma_0)}\} \le C_n^*,$$

for $n = 1, 2, \ldots,$ where

$$C_n^* \equiv C_{T,R}\Big[1 + \|z_{tt}^n(0)\|^2_{L_2(\Omega)} + \|z_1^n\|^2_{H^1(\Omega)} + \|w_{tt}^n(0)\|^2_{L_2(\Gamma_0)} + \|w_1^n\|^2_{H^2(\Gamma_0)} + \|w_1^n\|^4_{H^2(\Gamma_0)} + \|w_0^n\|^4_{H^2(\Gamma_0)}\Big].$$

Therefore, in order to obtain uniform estimates for the derivatives in higher-energy spaces, we must find a bound for the initial data in the above expression. If we denote $z_2^n \equiv z_{tt}^n(0)$ and $w_2^n \equiv w_{tt}^n(0)$, then these elements are given by the system

$$\begin{aligned}
\left(z_2^n,\phi\right)_{L_2(\Omega)} + \left(w_2^n,\psi\right)_{L_2(\Gamma_0)} &= -c^2\left(\nabla z_0^n,\nabla\phi\right)_{L_2(\Omega)} - a(w_0^n,\psi) + c^2\left(w_1^n,\phi|_{\Gamma_0}\right)_{L_2(\Gamma_0)}\\
&- \left(d\cdot z_1^n,\phi\right)_{L_2(\Omega)} - k\left(w_1^n,\psi\right)_{L_2(\Gamma_0)} - c^2\left(l\cdot z_1^n|_\Gamma,\phi|_\Gamma\right)_{L_2(\Gamma)} - c^2 l_0\left(z_0^n|_{\Gamma_1},\phi|_{\Gamma_1}\right)_{L_2(\Gamma_1)}\\
&-\rho\left(z_1^n|_{\Gamma_0},\psi\right)_{L_2(\Omega)} - G(w_0^n,\psi),
\end{aligned} \tag{4.44}$$

for $(\phi,\psi) \in \mathcal{V}_n$.

In order to obtain an additional estimate for the approximate solutions, we need to show that the inner products above are uniformly bounded for every $(\phi,\psi) \in L_2(\Omega)\times L_2(\Gamma_0)$ by choosing $(z_0^n, z_1^n, w_0^n, w_1^n) \in \mathcal{V}_n$ sufficiently close to (z_0, z_1, w_0, w_1) such that $\sup_{n\in\mathbb{N}} C_n^* < \infty$. Let $(z_0, z_1, w_0, w_1) \in H^2(\Omega)\times H^1(\Omega)\times H^4(\Gamma_0)\times H^2(\Gamma_0)$ satisfy the compatibility conditions described in Theorem 2.1. Define $z_1^n = \hat{P}_n z_1$ and $w_1^n = P_n w_1$, where $\hat{P}_n$ and P_n are the orthoprojectors over span$\{\phi_i : i = 1,\ldots,n\}$ and span$\{\psi_i : i = 1,\ldots,n\}$, respectively. Note that, in this case we have

$$\left\|(z_t^n(0), w_t^n(0))\right\|_{H^1(\Omega)\times H^2(\Gamma_0)} = \left\|(z_1^n, w_1^n)\right\|_{H^1(\Omega)\times H^2(\Gamma_0)} \le C\,\|(z_1,w_1)\|_{H^1(\Omega)\times H^2(\Gamma_0)}, \forall\, n\in\mathbb{N};$$

$$(z_1^n, w_1^n) \to (z_1, w_1) \text{ in } H^1(\Omega)\times H^2(\Gamma_0) \text{ as } n\to\infty.$$

The above inequality provides a uniformly bounded approximation for the elements z_1 and w_1. Let us now construct the remaining approximated initial data. To this end, let us consider the following functional $\Pi : H^1(\Omega) \times H^2(\Gamma_0) \to \mathbb{R}$ given by

$$\begin{aligned}\Pi(\phi, \psi) \equiv & c^2(\nabla z_0, \nabla\phi)_{L_2(\Omega)} + c^2 l_0(z_0|_{\Gamma_1}, \phi|_{\Gamma_1})_{L_2(\Gamma_1)} + c^2(l \cdot z_1|_{\Gamma}, \phi|_{\Gamma})_{L_2(\Gamma)} \\ & - c^2(w_1, \phi|_{\Gamma_0})_{L_2(\Gamma_0)} \\ & + a(w_0, \psi) + (|\nabla w_0|^2 \nabla w_0, \nabla\psi)_{L_2(\Gamma_0)} + (\nabla\{w_0^2\}, \nabla\psi)_{L_2(\Gamma_0)} + M(w_0, \psi)_{L_2(\Gamma_0)},\end{aligned}$$

where $M > 0$ is a constant that will be specified later. Using Green's formulas and the compatibility conditions between the initial data, we obtain

$$\begin{aligned}|\Pi(\phi, \psi)| \le & |c^2(\Delta z_0, \phi)_{L_2(\Omega)}| + |(\Delta^2 w_0 - \mathrm{div}\{|\nabla w_0|^2 \nabla w_0\} - \sigma\Delta\{w_0^2\} + M w_0, \psi)_{L_2(\Gamma_0)}| \\ & \le C\left(\|(z_0, w_0)\|_{H^2(\Omega)\times H^4(\Gamma_0)}\right) \|(\phi, \psi)\|_{L_2(\Omega)\times L_2(\Gamma_0)},\end{aligned}$$

for every $(\phi, \psi) \in H^1(\Omega) \times H^2(\Gamma_0)$, which shows that Π is a continuous functional with respect to L_2-topology. We use the same notation Π for its extension to $L_2(\Omega) \times L_2(\Gamma_0)$. Hence, for every $n \in \mathbb{N}$, we consider $(z_0^n, w_0^n) \in \mathcal{V}_n$ satisfying

$$\begin{aligned}& c^2(\nabla z_0^n, \nabla\phi)_{L_2(\Omega)} + c^2 l_0(z_0^n|_{\Gamma_1}, \phi|_{\Gamma_1})_{L_2(\Gamma_1)} + a(w_0^n, \psi) + (\left|\nabla w_0^n\right|^2 \nabla w_0^n, \nabla\psi)_{L_2(\Gamma_0)} \\ & \quad + \sigma(\nabla\{(w_0^n)^2\}, \nabla\psi)_{L_2(\Gamma_0)} + M(w_0^n, \psi)_{L_2(\Gamma_0)} = \Pi(\phi, \psi), \quad \text{for } (\phi, \psi) \in \mathcal{V}_n.\end{aligned} \tag{4.45}$$

The sequence $\{(z_0^n, w_0^n)\}_{n\in\mathbb{N}}$ is a Galerkin approximation sequence for the following (nonlinear) elliptic variational problem

$$\begin{aligned}& c^2(\nabla z, \nabla\phi)_{L_2(\Omega)} + c^2 l_0(z|_{\Gamma_1}, \phi|_{\Gamma_1})_{L_2(\Gamma_1)} + a(w, \psi) + (|\nabla w|^2 \nabla w, \nabla\psi)_{L_2(\Gamma_0)} \\ & \quad + \sigma(\nabla\{w^2\}, \nabla\psi)_{L_2(\Gamma_0)} \\ & \quad + M(w, \psi)_{L_2(\Gamma_0)} = \Pi(\phi, \psi), \quad \text{for } (\phi, \psi) \in H^1(\Omega) \times H^2(\Gamma_0).\end{aligned}$$

Since the elliptic operator associated with the previous variational problem is a locally Lipschitz perturbation of a monotone operator and, in addition, is coercive for suitable large M, it follows that $\{w_0^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $H^2(\Gamma_0)$ and $w_0^n \to w_0$ strongly in $H^2(\Gamma_0)$.

Finally, identities (4.44) and (4.45) imply

$$\begin{aligned}& \left(z_2^n, \phi\right)_{L_2(\Omega)} + \left(w_2^n, \psi\right)_{L_2(\Gamma_0)} \\ & = -[\Pi(\phi, \psi) + \left(d \cdot z_1^n, \phi\right)_{L_2(\Omega)} + \left(k w_1^n + \rho z_1^n|_{\Gamma_0} - M w_0^n + f(w_0^n), \psi\right)_{L_2(\Gamma_0)}]\end{aligned}$$

for every $(\phi, \psi) \in \mathcal{V}_n$. Since $\Pi(\cdot, \cdot)$ is continuous in $L_2(\Omega) \times L_2(\Gamma_0)$, $\{(z_1^n, w_1^n)\}$ is uniformly bounded in $H^1(\Omega) \times H^2(\Gamma_0)$ and $\{w_0^n\}$ is uniformly bounded in $H^2(\Gamma_0)$, we conclude from the previous identity that $\{(z_2^n, w_2^n)\}$ is uniformly bounded in $L_2(\Omega) \times L_2(\Gamma_0)$, which provides an additional a priori estimate for the first and second derivatives of the solutions, namely,

$$\{(z_t^n, z_{tt}^n, w_t^n, w_{tt}^n)\}_{n\in\mathbb{N}} \text{ is bounded in } L_\infty(0, T; H^1(\Omega) \times L_2(\Omega) \times H^2(\Gamma_0) \times L_2(\Gamma_0)).$$

Feeding this time regularity back into variational form allows us to boost the regularity in space by improving $H^3(\Gamma_0)$ to $D(\mathcal{A})$ for the plate component. The above boundedness together with the weak-star convergence of $\{(z^n, z^n_t, w^n, w^n_t)\}$ to (z, z_t, w, w_t) in $L_\infty(0, T; \mathcal{H})$ allows us to pass to the limit in (4.25) and conclude that the solution (z, w) belongs to the class

$$(z, z_t, z_{tt}) \in C([0, T]; D(A) \times H^1(\Omega) \times L_2(\Omega));$$
$$(w, w_t, w_{tt}) \in C([0, T]; H^4(\Gamma_0) \times H^2(\Gamma_0) \times L_2(\Gamma_0)),$$

which concludes the proof. □

Remark 4.2. Notice that the estimates (uniform with respect to the discretization parameter "n") for the integrals I_1 I_2 and I_3 are more delicate. This is particularly true in the case of free (F) boundary conditions. The estimate for I_1 has two hurdles. The above critical unboundedness of the restoring forces of the plate equation and the inhomogeneity in the boundary conditions which are not controlled by the finite energy topology. To address the difficulties, logarithmic control of critical Sobolev embedding is employed. However, the Galerkin approximations do not yield $H^4(\Gamma_0)$ regularity. (This is because the operator $\mathcal{A}G$ has its range only in $H^3(\Gamma_0)$ and this is the effect of free boundary conditions.) Therefore, the plate solution is shown to have $H^3(\Gamma_0)$ space regularity. The additional boost to $H^4(\Gamma_0)$ requires a subtle approximation procedure for initial data, as presented above.

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