

# M-theory, Black Holes and Cosmology

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**ABSTRACT:** This paper is dedicated to M. Duff on the occasion of his 70th birthday. I discuss some issues of M-theory/string theory/supergravity closely related to Mike's interests. I describe a relation between STU black hole entropy, Cayley hyperdeterminant, Bhargava cube and a 3-qubit Alice, Bob, Charlie triality symmetry. I shortly describe my recent work with Gunaydin, Linde, Yamada on M-theory cosmology [1], inspired by the work of Duff with Ferrara and Borsten, Levay, Marrani et al. Here we have 7-qubits, a party including Alice, Bob, Charlie, Daisy, Emma, Fred, George. Octonions and Hamming error correcting codes are at the base of these models. They lead to 7 benchmark targets of future CMB missions looking for primordial gravitational wave from inflation. I also show puzzling relations between the fermion mass eigenvalues in these cosmological models, exceptional Jordan eigenvalue problem, and black hole entropy. The symmetry of our cosmological models is illustrated by beautiful pictures of a Coxeter projection of the root system of E7.

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### 1 Introduction

Inspiring ideas of Mike Duff have influenced my work over decades. Here I would like to present two aspects of it, both rooted in Mike's and in my work from long time ago, where there is also a very recent progress.

The first story is about the black holes attractors [2] in  $\mathcal{N} = 2$  4d supergravity, originating from M-theory/string theory/11d supergravity. These black holes have interesting properties which were initially understood in [3] and [4] where the STU black holes with string theory triality symmetry were described. These were followed by [5] and [6] where the STU black holes and their entropies were related to quantum information theory. In these papers the relation between quantum entanglement in a 3-qubit system, Alice, Bob and Charlie, and and 3-moduli STU black holes was discussed.

The recent stage of this story has to do with a renewed interest in mathematical aspects of black holes in string theory/supergravity as studied in [7–10]. The relation between STU black holes and Bhargava cube was observed and discussed earlier in [11, 12]. We will add to the recent advances in all these papers the analysis of the triality symmetry, which exists for these black holes in addition to the well known and well studied U-duality  $[SL(2, \mathbb{Z})]^3$  symmetry. Basically triality symmetry is a statement that Alice, Bob and Charlie are on equal footing. The aspects of Bhargava cube related to properties of the Cayley hyperdeterminant will be discussed here. We will clarify the concept of equivalence of black holes with the same entropy with U-duality symmetry  $[SL(2, \mathbb{Z})]^3 \ltimes S_3$ .

It was noticed in [6] that the black holes in  $\mathcal{N} = 8$  4d supergravity can be brought to a canonical basis. Their entropy formula defined in general by 56 charges in the quartic Cartan-Cremmer-Julia  $E_{7(7)}$  invariant, in the canonical basis depends only on 8 charges and coincides with the Cayley hyperdeterminant defining the STU black holes area of the horizon/entropy. In the Bhargava cube terminology this  $[SL(2, \mathbb{Z})]^3$  invariant is a discriminant of the associated binary quadratic forms.

The  $[SL(2, \mathbb{Z})]^3 \ltimes S_3$  symmetry of the Cayley hyperdeterminant/Bhargava cube is also a symmetry following from the Kähler potential which is given by

$$K_{3\text{mod}} = - \sum_{i=1}^3 \log \left( T^i + \bar{T}^i \right) . \quad (1.1)$$

STU black holes can be associated with M-theory first truncated to 7 moduli,  $T^i, i = 1, \dots, 7$  with  $[SL(2, \mathbb{Z})]^7$  and  $S_7$  symmetry and the Kähler potential given by

$$K_{7\text{mod}} = - \sum_{i=1}^7 \log \left( T^i + \bar{T}^i \right) . \quad (1.2)$$

When 4 of the 7 moduli are truncated we have the remaining  $[SL(2, \mathbb{Z})]^3$  duality as well as triality permutation symmetry  $S_3$ , and we recover the kinetic term of  $\mathcal{N} = 2$  supergravity STU model. A detailed derivation of the STU model from string theory/10d supergravity was performed in [3].

The second story of this paper is about the new ideas in cosmology based on 7-moduli model of M-theory compactified on a manifold with  $G_2$  holonomy and with  $[SL(2, \mathbb{Z})]^7$  symmetry and Kähler potential in eq. (1.2). M. Duff was the first to point out in [13] that the maximal supersymmetry of M-theory is spontaneously broken down to  $\mathcal{N} = 1$  supersymmetry in 4d when compactified on a manifold with  $G_2$  holonomy. More recently 11d M-theory/supergravity compactified on a twisted 7-tori with holonomy group  $G_2$  was investigated in [14].

During the last few years I studied the issues in cosmology initiated by discussions with S. Ferrara which resulted in our paper [15]. This work, in turn, originated from S. Ferrara's work with M. Duff and his collaborators [12, 16–19]. One of the central ideas in all these studies is based on the fact that  $E_{7(7)}(\mathbb{R})$  symmetry of  $\mathcal{N} = 8$  4d supergravity has a subgroup  $[SL(2, \mathbb{R})]^7$ . For the discrete subgroups this becomes a following relation

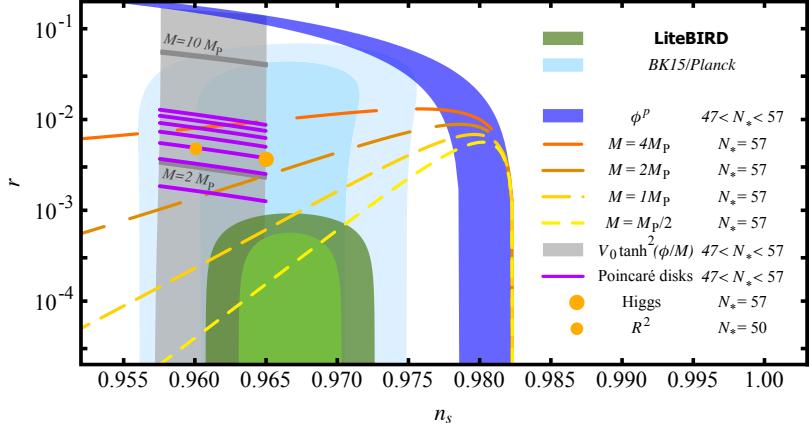
$$E_{7(7)}(\mathbb{Z}) \supset [SL(2, \mathbb{Z})]^7 . \quad (1.3)$$

When the relevant cosmological models were constructed in [15, 20–22], 7 targets for early universe future searches of gravitational waves from inflation were proposed. These are shown here in Fig. 1 by 7 purple lines.

The theoretical underpinning of the cosmological models in [15, 20–22] was very recently proposed in my paper [1] with M. Gunaydin, A. Linde, Y. Yamada. The entangled 7-qubit system corresponds to 7 parties: Alice, Bob, Charlie, Daisy, Emma, Fred and George, and it is related to 7 imaginary units of octonions.

M. Duff had a long and deep appreciation of the fact that there are four normed division algebras: the real numbers ( $\mathbb{R}$ ), complex numbers ( $\mathbb{C}$ ), quaternions ( $\mathbb{H}$ ), and octonions ( $\mathbb{O}$ ). He and his collaborators have developed many new aspects of the relations between octonions and physics, see for example [12]. I will show here how octonions, Fano planes

and error correcting Hamming (7,4) codes help to build cosmological models which will be tested by future cosmological observations.



**Figure 1:** This is a figure A.2 from the Astro2020 APC White Paper [LiteBIRD: an all-sky cosmic microwave background probe of inflation](#) with a forecast of Litebird constraints in the  $n_s$  -  $r$  plane [23]. The 7 purple lines in the figure, were derived most recently in [1] using M-theory compactified on  $G_2$ , octonions, Fano planes and error correcting codes.

I will also show that the mass eigenvalues of heavy scalars in cosmological models in [1] described by a pair of cubic equations  $x^3 - 7x - 7 = 0$ ,  $y^3 - 7x - 7 = 0$  have a particular relation to exceptional Jordan<sup>1</sup> eigenvalue problem [24–28]. There is an interesting connection between the product of the mass eigenvalues of fermions in cosmological models and the entropy of the STU black holes. Both correspond to a determinant of a certain relevant in each case Jordan matrix.

Another interesting feature of our cosmological models [1] is the symmetry of the fermion mass matrix at Minkowski vacua. It is invariant under the  $O(7)$  symmetry and its subgroups. The discrete subgroup of it is the Weyl group  $W(E_7)$ . We show the Coxeter plane of the root system of  $W(E_7)$  in Figs. 7, 8. When one imposes the invariance of the octonion algebra on the transformations one obtains a finite subgroup of  $G_2$ , the adjoint Chevalley group  $G_2(2)$  of order 12,096 as discussed in [29–32]. This is interesting since it is expected that neutrino physics will require an extension of the standard model. Some of these extensions might include discrete subgroups of  $G_2$ , see for example [33, 34].

Thus, both of these stories, STU black holes and M-theory cosmology 7-moduli models have interesting connection to  $E_7$  symmetry. I would like to notice here that the current status of 4d  $\mathcal{N} = 8$  supergravity and its perturbative UV behavior remain puzzling. Some heroic efforts were made by Z. Bern et al in amplitude loop computations, see the review [35]. They have shown that maximal supergravity behaves in UV much better than expected. It was suggested in [36–38], that  $E_{7(7)}$  symmetry together with maximal supersymmetry of

<sup>1</sup>Studies of octonions and Jordan algebras are based on the work of H. Freudenthal in ‘Oktaven, Ausnahmegruppen und Oktavengeometrie’, Mathematisch Instituut der Rijksuniversiteit te Utrecht, 1951.

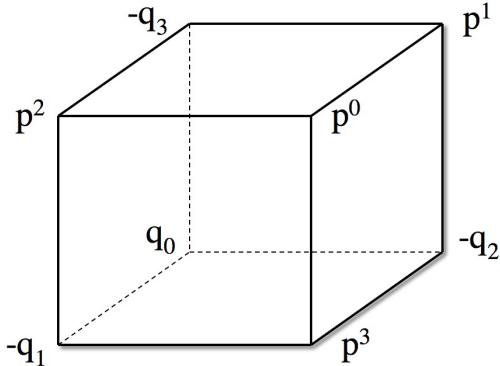
perturbative maximal supergravity in 4d might explain the cancellation of UV infinities observed in ‘theoretical experiments’ as described in [35]. It would be very interesting to learn more about these exceptional symmetries and their role in physics.

## 2 STU black holes, triality and the Bhargava cube

A significant effort was dedicated over the years to understand the properties of black holes in M-theory/string theory /supergravity. The STU black holes are sufficiently simple, there are exact analytic solutions in classical  $\mathcal{N} = 2$  supergravity with the prepotential

$$F = \frac{d_{ijk} X^i X^j X^k}{X^0} = \frac{X^1 X^2 X^3}{X^0} \quad (2.1)$$

in the so-called double extreme approximation, when the values of 3 moduli  $z^i = \frac{X^i}{X^0}$  near the horizon are the same as the ones far away from the black hole,  $z^i|_{\text{inf}} = z^i|_{\text{hor}}$ . The solution depends on 8 charges, 4 electric and 4 magnetic. The area of the horizon/the entropy of these black holes was computed in [4] in terms of the 8 black hole charges  $(p^\Lambda, q_\Lambda)$ ,  $\Lambda = 0, 1, 2, 3$ , shown as corners of the 2x2x2 hypermatrix in Fig. 2. The entropy is the



**Figure 2:** The 2x2x2 hypermatrix corresponding to supergravity black holes given in Fig. 2 of [6]. It represents the STU black hole solution in [4] with 8 charges  $p^\Lambda = p^0, p^1, p^2, p^3$  and  $q_\Lambda = q_0, q_1, q_2, q_3$  where 3 moduli  $z^1, z^2, z^3$  at the black hole horizon are functions of these charges.

function of charges

$$\frac{S}{\pi} = (W(p^\Lambda, q_\Lambda))^{1/2}, \quad (2.2)$$

where

$$\begin{aligned} W(p^\Lambda, q_\Lambda) = & -(p \cdot q)^2 + 4((p^1 q_1)(p^2 q_2) + (p^1 q_1)(p^3 q_3) + (p^3 q_3)(p^2 q_2)) \\ & - 4p^0 q_1 q_2 q_3 + 4q_0 p^1 p^2 p^3 \end{aligned} \quad (2.3)$$

and

$$p \cdot q \equiv (p^0 q_0) + (p^1 q_1) + (p^2 q_2) + (p^3 q_3) . \quad (2.4)$$

The function  $W(p^\Lambda, q_\Lambda)$  is manifestly symmetric under transformations:

$$p^1 \leftrightarrow p^2 \leftrightarrow p^3, \quad q_1 \leftrightarrow q_2 \leftrightarrow q_3 . \quad (2.5)$$

Under  $[SL(2, \mathbb{Z})]^3$  transformations the charges and the moduli transform but the entropy is invariant.

The values of the 3 complex moduli near the horizon, for each  $i = 1, 2, 3$ , were computed in [4]

$$z^i = \frac{B^i}{2A_i} \mp i \frac{\sqrt{B^i - 4A_i C^i}}{2A_i} \quad (2.6)$$

where for each  $i = 1, 2, 3$

$$A_i = p^0 q_i - 3d_{ijk} p^j p^k \quad B^i = p \cdot q - 2p^i q_i \quad C^i = -(p^i q_0 + 3d^{ijk} q_j q_k) \quad (2.7)$$

and

$$W = -D = B^1 - 4A_1 C^1 = B^2 - 4A_2 C^2 = B^3 - 4A_3 C^3 . \quad (2.8)$$

It was pointed out in [5] that the classical expression for the entropy of the STU black holes  $W(p^\Lambda, q_\Lambda)$  (2.3) can be represented in a very beautiful form:

$$S^{\text{BPS}} = \pi \sqrt{W} = \frac{\pi}{2} \sqrt{-\text{Det } \psi} , \quad \text{Det } \psi < 0 , \quad (2.9)$$

where  $\text{Det } \psi$  is the Cayley's hyperdeterminant of the vector with components  $\psi_{ijk}$ , constructed in 1845. The dictionary between 8 charges  $p^\Lambda$  and  $q_\Lambda$  and components of  $\psi_{ijk}$  is the following:

$$\begin{array}{cccccccc} p^0 & p^1 & p^2 & p^3 & q_0 & q_1 & q_2 & q_3 \\ \psi_{000} & -\psi_{001} & -\psi_{010} & -\psi_{100} & \psi_{111} & \psi_{110} & \psi_{101} & \psi_{011} \end{array} \quad (2.10)$$

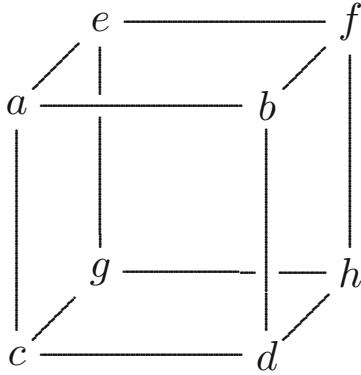
Cayley hyperdeterminant of the 2x2x2 hypermatrix  $\psi_{ijk}$  is defined as follows

$$\text{Det } \psi = -\frac{1}{2} \epsilon^{ii'} \epsilon^{jj'} \epsilon^{kk'} \epsilon^{mm'} \epsilon^{nn'} \epsilon^{pp'} \psi_{ijk} \psi_{i'j'm} \psi_{npk'} \psi_{n'p'm'} . \quad (2.11)$$

The new aspect of the STU black holes associated with Bhargava cube developed in [7–10] is the following. It is possible to attach a triple of quadratic forms

$$A_i x^2 + B_i x y + C_i y^2 \quad (2.12)$$

of the same discriminant  $D = B_i^2 - 4A_i C_i$  to a cube, with the corners given by an octuple  $a, b, c, d, e, f, g, h$ . We show this cube in Fig. 3. Even when only 2 of the forms are available, one can construct the third one as well as the Bhargava cube. The dictionary between 8



**Figure 3:** The Bhargava cube.

black hole charges in Fig. 2 and Bhargava octuple in Fig. 3 is

$$\begin{array}{cccccccc}
 p^2 & p^0 & -q_1 & p^3 & -q_3 & p^1 & q_0 & -q_2 \\
 & a & b & c & d & e & f & g & h
 \end{array} \tag{2.13}$$

The cube has a 3-way slicing: up-down, left-right, front-back, and many interesting properties. The discriminant of the cube is given by the following expression.

$$\begin{aligned}
 D_{Bha} = & a^2h^2 + b^2g^2 + c^2f^2 + d^2e^2 \\
 & -2(abgh + cdef + acfh + bdeg + aedh + bfch) \\
 & +4(adfg + bceh) .
 \end{aligned} \tag{2.14}$$

Using eqs. (2.3), (2.14) and the dictionary (2.10) we see that

$$D_{Bha} = -W. \tag{2.15}$$

The action of modular groups  $[SL(2, \mathbb{Z})]^3$  on the Bhargava cube was studied in detail in mathematical literature and recently applied in the context of STU black holes in [7–10]. However, the permutations symmetry of the discriminant of the Bhargava cube was not yet revealed in most of these studies<sup>2</sup>. Namely, the 3 permutation symmetries in black hole solutions which preserve the entropy and reflect the symmetry between 3 moduli  $z^i$  at the horizon and the relevant charges, are

$$\begin{aligned}
 z^1 \leftrightarrow z^2 & : & p^1 \leftrightarrow p^2 & & q_1 \leftrightarrow q_2 \\
 z^2 \leftrightarrow z^3 & : & p^2 \leftrightarrow p^3 & & q_2 \leftrightarrow q_3
 \end{aligned}$$

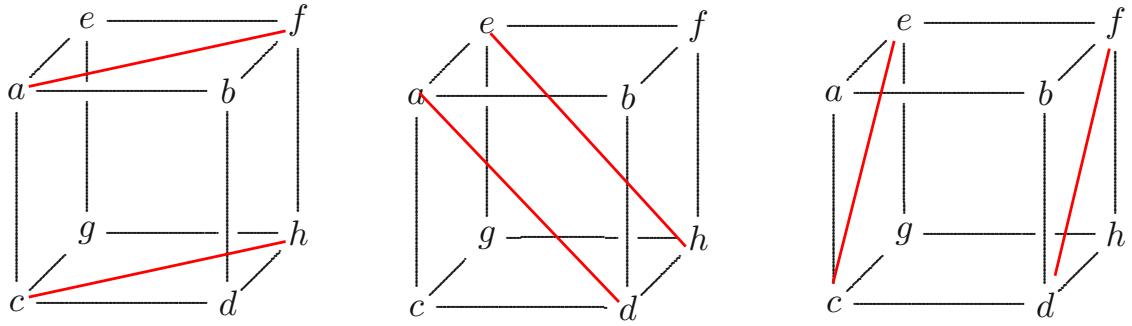
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<sup>2</sup>Examples with triality symmetry were given in [10], here we discuss a general case of triality symmetry in the context of Bhargava cube.

$$z^3 \leftrightarrow z^1 \quad : \quad p^3 \leftrightarrow p^1 \quad q_3 \leftrightarrow q_1 \quad (2.16)$$

Therefore the 3 symmetries of the discriminant of the Bhargava cube which reflect the corresponding black hole symmetries are

$$f \leftrightarrow a \quad c \leftrightarrow h$$



**Figure 4:** Three permutation symmetries of the discriminant of the Bhargava cube  $D_{Bha}$ , according to eq. (2.17)

## 2.1 The issue of black hole equivalence

Supersymmetric STU black holes are defined by their entropy as well as by the values of the 3 moduli near the horizon. In the basis where all 3 moduli  $z^i$  are on equal footing in the prepotential given in eq. (2.1), entropy is shown in eq. (2.3) and the values of the moduli  $z^i$  are given in eq. (2.6). The  $\mathcal{N} = 2$  supergravity in this basis and the black hole solution both have this symmetry. The symmetry of solutions is presented in eq. (2.16).

The permutation symmetry for black holes, a triality symmetry, is important when the physical question is asked: what kind of STU black holes are equivalent? It is known that the entropy might be the same for different set of 8 charges, for example  $(p^\Lambda, q_\Lambda)$  and  $((p^\Lambda)', (q_\Lambda)')$ . However, some of these 8 charges can be related to each other by a U-duality symmetry  $[SL(2, \mathbb{Z})]^3 \ltimes S_3$  transformation. In such case, these two sets of 8 charges belong to the same U-duality orbit. If however, they are not related to each other by an  $[SL(2, \mathbb{Z})]^3 \ltimes S_3$  transformation, they belong to different orbits.

There is a significant progress in understanding the discrete properties of Bhargava cube which may be useful in the context of string theory counting of states associated with supersymmetric black holes with integer charges. To use these properties it would be nice to take into account systematically also triality symmetry  $S_3$  of the discriminant of the Bhargava cube, in addition to modular  $[SL(2, \mathbb{Z})]^3$  symmetry which was already studied extensively.

In the basis  $a, b, c, d, e, f, g, h$  which is standard in Bhargava cube literature, this  $S_3$  symmetry (2.17) of the discriminant in eq. (2.14) is not obvious since it does not appear to be related to a supergravity  $S_3$  invariant prepotential (2.1). However, it is present there. The metric, and therefore the entropy of the STU black hole solution is U-duality invariant. The  $S_3$  symmetry is therefore manifest in eq. (2.3) since it follows from the triality invariant prepotential.

### 3 M-theory cosmology, octonions and error correcting codes

A short summary of the recent paper [1] suitable for this set up is the following. We have proposed an expression for the effective  $\mathcal{N} = 1$  4d supergravity following from M-theory/11d supergravity compactified on a manifold with  $G_2$  structure. Starting with general type  $G_2$ -structure manifolds one finds Minkowski vacua only in cases the twisted 7-tori are  $G_2$ -holonomy manifolds. Here again it was a crucial early insight of M. Duff that the maximal supersymmetry of M-theory is spontaneously broken by compactification to minimal  $\mathcal{N} = 1$  supersymmetry in 4d [13] when the compactification manifold has a  $G_2$  holonomy.

Our choice of the superpotential is based on a split of the 7-qubit system, Alice, Bob, Charlie, Daisy, Emma, Fred, George, into 3-qubits and 4-qubits. The 3-qubits codify the multiplication table of octonions, there are 7 associated triads there. The 7 complimentary 4-qubits define our superpotential. The automorphism group of octonions is  $G_2$ , therefore it is natural to define the superpotentials using octonions.

The effective  $\mathcal{N} = 1$  4d supergravity following from M-theory/11d supergravity is defined as follows. The Kähler potential is given in eq. (1.2). The superpotential in general is given by a sum over 7 the 4-qubits  $\{ijkl\}$  of the form

$$\mathbb{W}\mathbb{O} = \sum_{\{ijkl\}} (T^i - T^j)(T^k - T^l) = \frac{1}{2} \mathbb{M}_{ij} T^i T^j. \quad (3.1)$$

It appears to have 28 terms of the form  $T^i T^j$ , however, half of them cancels and we are left with 14 terms. For example for Cartan-Shouten-Coxeter octonion conventions [39, 40]

$$\mathbb{W}\mathbb{O} = \sum_{r=0}^6 (T^{r+2} - T^{r+4})(T^{r+5} - T^{r+6}). \quad (3.2)$$

We can see these  $7 \times 4$  terms in right hand side of eq. (3.5). But actually, the formula simplifies to 14 terms

$$\mathbb{WO} = - \sum_{r=1}^7 T^r (T^{r+1} - T^{r+2}). \quad (3.3)$$

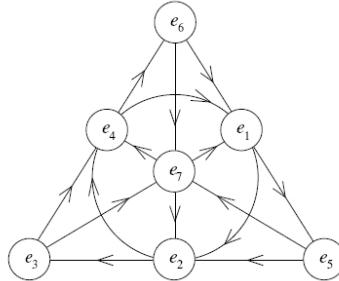
Explicitly the 14 terms are

$$\begin{aligned} \mathbb{WO} = & - \left( T^1(T^2 - T^3) + T^2(T^3 - T^4) + T^3(T^4 - T^5) + T^4(T^5 - T^6) \right. \\ & \left. + T^5(T^6 - T^7) + T^6(T^7 - T^1) + T^7(T^1 - T^2) \right). \end{aligned} \quad (3.4)$$

The set of 7 terms in the superpotential in the form (3.2) is easy to compare with 7 octonion associate triads, with 7 quadruples and with 7 codewords of the cyclic (7,4) Hamming error correcting code. We show this relation in eq. (3.5).

$$\mathbb{WO} = \begin{array}{c|c|c|c|c} \text{Triads} & \text{Codewords} & \text{Quadruples} & \Rightarrow & \mathbb{WO} \\ \hline (137) & 1 0 1 0 0 0 1 & (2456) & (T^2 - T^4)(T^5 - T^6) \\ (241) & 1 1 0 1 0 0 0 & (3567) & (T^3 - T^5)(T^6 - T^7) \\ (352) & 0 1 1 0 1 0 0 & (4671) & (T^4 - T^6)(T^7 - T^1) \\ (463) & 0 0 1 1 0 1 0 & (5712) & (T^5 - T^7)(T^1 - T^2) \\ (574) & 0 0 0 1 1 0 1 & (6123) & (T^6 - T^1)(T^2 - T^3) \\ (615) & 1 0 0 0 1 1 0 & (7234) & (T^7 - T^2)(T^3 - T^4) \\ (726) & 0 1 0 0 0 1 1 & (1345) & (T^1 - T^3)(T^4 - T^5) \end{array} \quad (3.5)$$

Let us show how the octonion triads are represented in the oriented Fano plane. Each of the 7 lines has 3 points, the arrows show the order, with possible cyclic permutations. For example the first one in eq. (3.5) is 137, we see it as the internal line going up and to the right, it shows 371. The next one is 241, it is a set of points on a circle. One more, 352 is the one at the bottom, going to the left, it shows as 523, etc.



**Figure 5:** An oriented Fano plane, Fig. 1 in [41]. On each of the 7 lines (including the circle) there are 3 points e.g. 1,2, and 4 on a circle. The octonion multiplication rule is build into the Fano table. For example, one can see from the oriented circle that  $e_1 \cdot e_2 = e_4$ .

We studied Minkowski vacua in 7-moduli models with octonionic superpotentials (3.1). We found that these models have a Minkowski minimum at

$$T^1 = T^2 = T^3 = T^4 = T^5 = T^6 = T^7 \equiv T \quad (3.6)$$

with one flat direction.

There are 480 different octonion conventions. We have presented a general formula of the superpotential for any octonion convention in [1]. In all these cases, the matrix  $\mathbb{M}_{ij}$  in eq. (3.1) can be computed either using the general formula or by performing a change of v  
o  
o

v

$$\left[ \begin{array}{ccccccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{ccccccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{ccccccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

**Figure 6:** Three sets of codewords with some terms excluded: they codify the superpotentials  $\mathbb{W}\mathbb{O}_{m,n}$  where the corresponding terms in eq. (3.2) are absent. These superpotentials lead to models with Minkowski vacua with two flat directions.

all these models with one and two flat directions are

$$K = -m \log (T_{(1)} + \bar{T}_{(1)}) - n \log (T_{(2)} + \bar{T}_{(2)}) \quad (3.7)$$

with  $m + n = 7$  and cases like  $m = 0, n = 7$ ;  $m = 1, n = 6$ ;  $m = 2, n = 5$ ;  $m = 3, n = 4$ . The superpotentials  $\mathbb{W}\mathbb{O}_{m,n}$  at the vacuum have the following properties:

$$\mathbb{W}\mathbb{O}_{m,n} = 0, \quad \partial_i \mathbb{W}\mathbb{O}_{m,n} = 0 \quad (3.8)$$

at

$$T^1 = \dots = T^m \equiv T_{(1)}, \quad T^{m+1} = \dots = T^n \equiv T_{(2)}. \quad (3.9)$$

Based on these M-theory Minkowski vacua we have build  $\mathcal{N} = 1$  supergravity phenomenological models with the potential

$$V = F(T, \bar{T}) \left( 1 + \frac{|\mathcal{W}^{\text{oct}}|^2}{W_0^2} \right) + \sum_{i=1}^7 (T^i + \bar{T}^i)^2 |\partial_i \mathcal{W}^{\text{oct}}|^2 \quad (3.10)$$

where

$$\mathcal{W}^{\text{oct}} \equiv \frac{1}{\sqrt{\prod_{i=1}^7 (2T^i)}} \mathbb{W}\mathbb{O}. \quad (3.11)$$

Along the supersymmetric Minkowski flat directions we have  $\mathcal{W}^{\text{oct}} = \partial_i \mathcal{W}^{\text{oct}} = 0$ . Therefore the full expression for the inflaton potential, for example in the simplest T-models, is given by an inflationary potential for the  $\alpha$ -attractor models and a cosmological constant

$$V = \Lambda + m^2 \tanh^2 \sqrt{\frac{1}{6\alpha}} \phi. \quad (3.12)$$

Inflation along various flat directions with these kinetic terms leads to  $3\alpha = 7, 6, 5, 4, 3, 2, 1$  and therefore to 7 possible values of the tensor to scalar ratio  $r = 12\alpha/N_e^2$  in the range  $10^{-2} \gtrsim r \gtrsim 10^{-3}$ , which should be accessible to future cosmological observations. They are shown by 7 purple lines in Fig. 1 here, taken from the LiteBIRD satellite mission forecast.

#### 4 Properties of the mass matrix in octonion cosmological models

The octonion superpotentials for models in [1] with  $G_2$  holonomy and 7 moduli have 14 terms in the form

$$\mathbb{W}\mathbb{O} = \frac{1}{2} \mathbb{M}_{ij} T^i T^j. \quad (4.1)$$

The matrix  $\mathbb{M}_{ij}$  for the simplest case  $\mathbb{W}\mathbb{O}$  for Cartan-Shouten-Coxeter octonion notations is

$$\mathbb{M}_{\mathbb{W}\mathbb{O}} = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 \end{pmatrix} \quad (4.2)$$

One can see that it has the property  $M_{ii} = \sum_j M_{ij} = 0, \forall i$ . In Minkowski vacuum with  $\mathbb{W}\mathbb{O} = \mathbb{W}\mathbb{O}_{,i} = 0$  the fermion mass matrix is

$$\frac{1}{2} e^{\frac{K}{2}} \chi^i \mathbb{M}_{ij} \chi^j. \quad (4.3)$$

The non-vanishing 6 eigenvalues of the  $\mathbb{M}$  matrix, defining the fermion mass eigenstates in Minkowski vacua solve a double set of cubic equations

$$x^3 - 7x - 7 = 0, \quad y^3 - 7y - 7 = 0. \quad (4.4)$$

The eigenvalues of the fermion mass matrix are

$$\mathbb{M}_{\text{WO}}^{\text{EV}} = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.5)$$

where  $x_a = y_a$  with  $a = 1, 2, 3$  are solutions of the cubic eqs. (4.4). Numerically this gives for a set of  $x_1, y_1$ ;  $x_2, y_2$ ;  $x_3, y_3$  and a massless one, the following values

$$3.0489, 3.0489; -1.6920, -1.6920; -1.3569, -1.3569; 0 \quad (4.6)$$

as shown in [1]. It looks like the numerical sum of all 3 eigenvalues vanishes

$$3.0489 - 1.6920 - 1.3569 \approx 0. \quad (4.7)$$

Meanwhile, can also solve eqs. (4.4) analytically. With  $z_a = x_a = y_a$  and  $\theta = \text{Arctan} 3^{-3/2}$  the solutions are

$$\begin{aligned} z_1 &= 2\sqrt{\frac{7}{3}} \text{Re } e^{i\frac{\theta}{3}} \\ z_2 &= 2\sqrt{\frac{7}{3}} \text{Re } e^{i\frac{\theta+2\pi}{3}} \\ z_3 &= 2\sqrt{\frac{7}{3}} \text{Re } e^{i\frac{\theta+4\pi}{3}} \end{aligned} \quad (4.8)$$

Also one finds that

$$z_1 + z_2 + z_3 = 0, \quad (4.9)$$

which means that indeed the sum of the 3 eigenvalues of the fermions mass matrix vanishes exactly. A number of other relations can be seen in the exact solution:

$$\begin{aligned} z_1 z_2 + z_2 z_3 + z_1 z_3 &= -7 \\ z_1 z_2 z_3 &= 7 \\ z_1^2 + z_2^2 + z_3^2 &= 2 \cdot 7 \\ z_1^3 + z_2^3 + z_3^3 &= 3 \cdot 7 \\ z_1^4 + z_2^4 + z_3^4 &= 2 \cdot 7^2 \\ z_1^5 + z_2^5 + z_3^5 &= 5 \cdot 7^2. \end{aligned} \quad (4.10)$$

We can compare the eigenvalues of the 3x3 part of the fermion mass matrix with the eigenvalues of the 3x3 octonionic Hermitian matrix studied in [26, 27] which defines the supersymmetric black hole entropy in 5d. This entropy was shown in [42] to be equal to a square root of the cubic invariant  $I_3$  of  $E_{6(6)}$ . In [26, 27] it was shown how this cubic invariant is related to the Jordan algebra  $J_3^{\mathcal{O}}$  of the 3x3 hermitian matrices over the composition algebra of octonions  $\mathcal{O}$ .

A generic element  $J$  of  $J_3^{\mathcal{O}}$  has the form

$$J = \begin{pmatrix} \alpha_1 & o_3 & o_2^* \\ o_3^* & \alpha_2 & o_1 \\ o_2 & o_1^* & \alpha_3 \end{pmatrix} \quad (4.11)$$

where  $\alpha_a$  are real numbers and  $o_a$  with  $a = 1, 2, 3$  are elements of  $\mathcal{O}$ . The automorphism of the split exceptional Jordan algebra is the non-compact  $F_{4(4)}$  group. In case of the non-split octonions the automorphism group is  $F_4$ . An element of  $J_3^{\mathcal{O}}$  can be brought to a diagonal form by an  $F_4$  rotation [24–26, 28]. In case of the black holes the generic element of  $J$  has eigenvalues  $\lambda_a$ ,  $a = 1, 2, 3$  and the cubic norm of  $J_3^{\mathcal{O}}$  is, as shown in [26].

In case of non-split octonions, one also start with the element (4.11) and diagonalize it using  $F_4$  transformation [24]. The 3 eigenvalues in this case were shown to satisfy certain characteristic cubic equation [28]

$$-\det(J - \lambda I) = \lambda^3 - (\text{Tr} J) \lambda^2 + \text{Tr}(J \times J) \lambda - (\det J) I = 0 \quad (4.12)$$

where in notation of [25]

$$J \times J = J^{-1} \det J. \quad (4.13)$$

In our cosmological model the analogous cubic equation  $x^3 - 7x - 7 = 0$  corresponds to the choice

$$\text{Tr} J = 0, \quad \text{Tr} J^{-1} = -I, \quad \det J = 7 \quad (4.14)$$

The choice  $\text{Tr} J = 0$  according to [25] means that our matrix (4.11) depends only on 26 parameters and therefore it is a 26-dimensional representation of  $F_4$ . It is also explained there that the invariants of  $F_4$  are

$$\text{Tr} J, \quad \text{Tr}(J \times J), \quad \det J. \quad (4.15)$$

Thus we find that our fermion mass matrix eigenvalues are defined by a cubic equation  $x^3 - 7x - 7 = 0$  of the kind which defines the exceptional Jordan matrix eigenvalues [28] with special  $F_4$  invariant properties.

$$\text{Tr} J = 0, \quad (J \times J) = -7, \quad \det J = 7. \quad (4.16)$$

Meanwhile, the relation between the  $\det$  of the fermion mass matrix  $\mathbb{M}_{\mathbb{W}\mathbb{O}}$  and black hole entropy in the diagonal basis  $\sqrt{I}_3$  is

$$\det \mathbb{M}_{\mathbb{W}\mathbb{O}} = x_1 x_2 x_3 \quad I_3 = \lambda_1 \lambda_2 \lambda_3 = p^1 p^2 p^3. \quad (4.17)$$

In 5d black holes the values of magnetic charges,  $p^1, p^2, p^3$  are less restricted, they do not satisfy a cubic equation of the kind  $x^3 - 7x - 7 = 0$ . In fact the entropy of 4d STU black holes we started with in eq. (2.3) is the same as the one in 5d under condition that  $p^0 = q_1 = q_2 = q_3 = 0$ .

Thus, in addition to numerous relations between various BPS and non-BPS black holes, we have observed here an interesting relation to octonion based cosmological models and fermion mass matrix.

## 5 Discrete symmetry of fermions in cosmological models

The fermion mass matrix in eq. (4.3) at Minkowski vacua in cosmological models [1] can be brought to a diagonal form as shown in eqs. (4.5), (4.6). Since it is a 7x7 matrix, its eigenvalues are invariant under the  $O(7)$  symmetry and its subgroups. The discrete subgroup of it is the Weyl group  $W(E_7)$ . It is isomorphic to a finite subgroup of  $O(7)$  which is the direct product  $Z_2 \times SO_7(2)$ . The group  $SO_7(2)$  is the adjoint Chevalley group of order 1,451, 520. The Weyl group  $W(E_7)$  has 2,903,040 symmetries. The root system of the Weyl group  $W(E_7)$  cannot be visualized since it is an object in 7 dimensions, but the 2-dimensional projections of them, the Coxeter planes, are well known. We present them in Figs. 7, 8.

However, the Weyl group  $W(E_7)$  does not preserve the octonion algebra. When one imposes the invariance of the octonion algebra on the transformations of the  $E_7$  roots one obtains a finite subgroup of  $G_2$ , as expected, the adjoint Chevalley group  $G_2(2)$  of order 12,096. We now review the analysis of  $E_7$  roots and its  $G_2(2)$  symmetry following [29–32] and show that it applies to the fermions in cosmological models of [1]. First we notice that  $E_8$  roots can be defined by the integral octonions of the following form. We take Cartan-Shoten-Coxeter octonion conventions, which we used in eq. (3.5). The triples are 124,235,346,457,561,672,713 and the quadruples are 3567, 4671, 5712, 6123, 7234, 1345, 2456. The set of 240 integer octonions is

$$\pm 1, \pm e_i \quad (5.1)$$

$$\frac{1}{2}(\pm 1 \pm e_i \pm e_j \pm e_k) \quad (5.2)$$

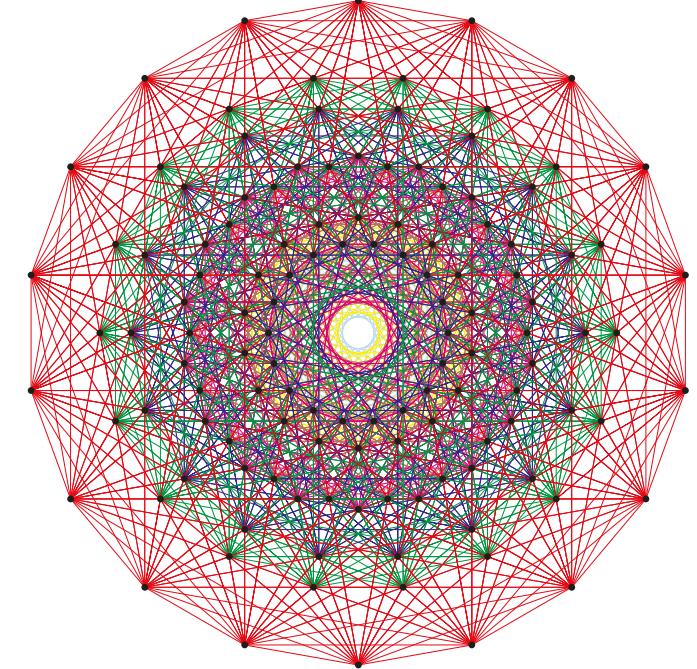
$$\frac{1}{2}(\pm e_i \pm e_j \pm e_k \pm e_l). \quad (5.3)$$

Here in (5.2)  $ijk$  belong to triples, so we have  $7 \times 16 = 112$  and in (5.3)  $ijkl$  belong to complementary quadruples, so we have again  $7 \times 16 = 112$ . To this we add 16 from eq. (5.1). This gives the total of 240 integral octonions, which make the  $E_8$  roots, also called Cayley integers or octavians. From the set of integral octonions above we keep only the ones in

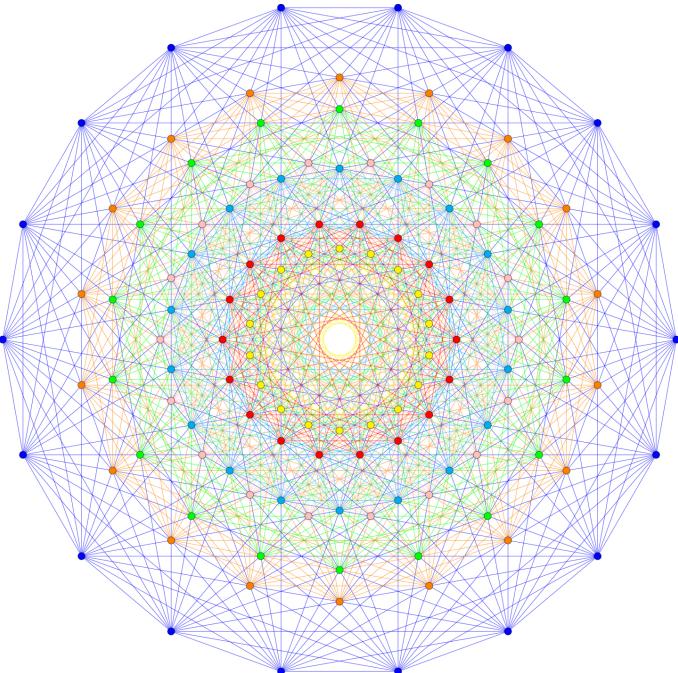
$$\pm e_i, \quad \frac{1}{2}(\pm e_i \pm e_j \pm e_k \pm e_l). \quad (5.4)$$

There are  $14 + 7 \times 16 = 126$  integral octonions. It was shown in [31] that the set of transformations which preserve the octonion algebra of the root system of  $E_7$  in (5.4) is the adjoint Chevalley group  $G_2(2)$ . It is possible to decompose these 126 imaginary octonions into 18 sets of 7 imaginary octonionic units that can be transformed to each other by the finite subgroup of matrices. These lead to 18 sets of 7 which we see in Figs. 7, 8.

Thus it appears that the cosmological models in [1] derived from compactification of 11d supergravity on a manifold with  $G_2$  holonomy, have some hidden  $E_7$  symmetry. It would be nice to understand a relation of all this to the maximal 4d supergravity with  $E_{7(7)}$  symmetry.



**Figure 7:** This is the root system of the Weyl group of  $E_7$  projected into the Coxeter plane as given by [John Stembridge](#). The Lie group  $E_7$  has a root system of 126 points in 7-dimensional space. One can see these 126 points as 7 groups of 18 points. These 126 points are tightly packed together and this configuration has a total of 2,903,040 symmetries.



**Figure 8:** The Coxeter projections of all exceptional root systems are given by [Tomás Görbe](#), including the  $E_7$  case shown here. As in Fig. 7 we can see 7 circles with 18 points each, a total of 126 points representing a root system of  $E_7$ .

## 6 Discussion

The entanglement of 7 qubits (Alice, Bob, Charlie, Daisy, Emma, Fred and George) was important in M. Duff's studies of black holes in M-theory. In the context of M-theory cosmology, it is not surprising that the concept of 7 qubits and the related tools like octonions,  $G_2$  symmetry, Fano Planes, (7,4) Hamming correcting codes also are playing an important role. It would be interesting to develop more understanding of both black holes and cosmological models in M-theory and of the role of octonions in physics.

Neutrino physics may also require some new ideas to satisfy the current and future data. It was advocated in [33, 34] that some discrete subgroups of  $G_2$ , like  $\mathcal{PSL}_2(13)$  might be useful for this purpose.

A nice feature of our cosmological models in [1] is that they describe a case of maximal supersymmetry spontaneously broken down to a minimal supersymmetry. These models will be tested by the future cosmological observations as we show in Figure. 1. The most recent forecast of the CMB-S4 in [43] suggests that the ground based Stage-4 experiments will achieve the science goals of detecting primordial gravitational waves for  $r > 0.003$  at greater than  $5\sigma$ , or, in the absence of a detection, of reaching an upper limit of  $r < 0.001$  at 95% CL. Therefore the benchmark targets of cosmological models in [1] will be tested during the next decade or two.

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