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Rational points and derived equivalence

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Abstract

We give the first examples of derived equivalences between varieties defined over non-closed fields where one has a rational point and the other does not. We begin with torsors over Jacobians of curves over $\mathbb{Q}$ and $\mathbb{F}_q(t)$, and conclude with a pair of hyperkähler 4-folds over $\mathbb{Q}$. The latter is independently interesting as a new example of a transcendental Brauer–Manin obstruction to the Hasse principle. The source code for the various computations is supplied as supplementary material with the online version of this article.

1. Introduction

The following question was posed by Esnault and stated in [AB18, Question 2].

Question. For smooth projective varieties defined over a field k that is not algebraically closed, is the existence of a k-rational point preserved under derived equivalence?

Honigs *et al.* [Hon15, Hon18] showed that over a finite field, the answer is yes up to dimension 3, and for Abelian varieties of any dimension; in fact the *number* of rational points is preserved in these cases. A conjecture of Orlov [Orl05, Conjecture 1] would imply that this continues to hold in higher dimensions.

Antieau, Krashen and Ward [AKW17, Theorem 1.1] showed that the answer is yes for curves of genus 1 over arbitrary fields. (For curves of genus $g \geq 2$ or $g = 0$, the canonical bundle is ample or anti-ample, so there are no interesting derived equivalences.)

Hassett and Tschinkel [HT17] studied the question for K3 surfaces, giving a positive answer over $\mathbb{R}$ and proving a number of suggestive results over local fields. They showed that the *index* of a K3 surface – that is, the greatest common divisor of the degrees of its closed points – is invariant under derived equivalence. Thus if $D^b(X) \cong D^b(Y)$ and X has a rational point then Y has a zero-cycle of degree 1, so it seems very difficult to find a counterexample among K3 surfaces. On the other hand, they asked whether the existence of a rational point is preserved under *twisted* derived equivalence, and Ascher, Dasaratha, Perry and Zhou [ADPZ17] showed that it is not, producing pairs of K3 surfaces over $\mathbb{Q}$, $\mathbb{Q}_2$, and $\mathbb{R}$ such that $D^b(X, \alpha) \cong D^b(Y, \beta)$ for some $\alpha \in \mathrm{Br}(X)$ and $\beta \in \mathrm{Br}(Y)$, where X has a rational point (at which the restriction of α is trivial) and Y has no rational points.

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Auel and Bernardara studied geometrically rational surfaces in [AB18], showing (among other things) that a del Pezzo surface S of degree at least 5 has a rational point if and only if $D^b(S)$ admits a full exceptional collection.

We give two classes of examples to show that the answer to the question above is no: first, some Abelian varieties and torsors over them, and second, a pair of hyperkähler 4-folds.

Theorem 1. *For every $g \geq 2$, there is an Abelian g-fold X defined over $\mathbb{Q}$, an X-torsor Y with no rational points, and a $\mathbb{Q}$-linear exact equivalence*

$$D^b(X) \cong D^b(Y).$$

The same holds over $\mathbb{F}_q(t)$ for any odd q.

Recall that a torsor over an Abelian variety has a rational point if and only if it has a zero-cycle of degree 1, so this theorem contrasts with the behavior of K3 surfaces discussed above.

To prove Theorem 1, we mine the literature for curves C defined over $\mathbb{Q}$ and $\mathbb{F}_q(t)$ such that Pic^{g-1}_C has no rational points. If $k = \mathbb{R}$, $\mathbb{Q}_p$, or $\mathbb{F}_q((t))$ then Pic^{g-1}_C always has k-points by work of Lichtenbaum [Lic69] and a remark of Poonen and Stoll [PS99, Footnote 10], so for the examples we are interested in, Pic^{g-1}_C is a counterexample to the Hasse principle. Then Theorem 1 follows from the next result.

Theorem 2. *If C is a smooth, projective, geometrically connected curve of genus $g \geq 1$ over an arbitrary field k, then there is a k-linear exact equivalence*

$$D^b(\mathrm{Pic}^0_C) \cong D^b(\mathrm{Pic}^{g-1}_C).$$

The proof is a repackaging of Mukai's classic derived equivalence [Muk81] between an Abelian variety and its dual. We include a broader discussion of torsors over Abelian varieties in § 2.

The varieties X and Y in Theorem 1 become isomorphic after a finite field extension, but in § 4 we present a more sophisticated counterexample over $\mathbb{Q}$, in which the varieties remain different even over $\mathbb{C}$. These are our hyperkähler 4-folds.

Theorem 3. *There is an explicit K3 surface S, defined over $\mathbb{Q}$, and two smooth, projective, four-dimensional moduli spaces X and Y of sheaves on S, such that X has infinitely many rational points, Y has no zero-cycle of degree* 1, *and there is a $\mathbb{Q}$-linear exact equivalence*

$$D^b(X) \cong D^b(Y).$$

The spaces X and Y are not birational, even over $\mathbb{C}$.

In fact Y has points over $\mathbb{R}$ and over $\mathbb{Q}_p$ for every prime p, so it is a counterexample to the Hasse principle. We use the class $\alpha \in \mathrm{Br}(Y)$ that obstructs the existence of a universal sheaf on $S \times Y$ as a Brauer–Manin obstruction. It is a transcendental Brauer class: it remains non-trivial in $\mathrm{Br}(Y_{\bar{\mathbb{Q}}})$ or $\mathrm{Br}(Y_{\mathbb{C}})$.

This counterexample is related to the Abelian counterexamples above in that X is fibered over $\mathbb{P}^2$ in Jacobians of curves of genus 2, and Y is fibered in Pic^1 of the same curves. The derived equivalence is a version of our earlier equivalence $D^b(\mathrm{Pic}^0_C) \cong D^b(\mathrm{Pic}^1_C)$ in families; the extension

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to the singular fibers is due to Arinkin [Ari13], and was used in work of Addington, Donovan and Meachan [ADM16], but we give a simplified description.

By taking fibers of X and Y over $\mathbb{Q}$-points of $\mathbb{P}^2$ we can get more explicit examples of Theorem 1. By taking generic fibers we get an example of Theorem 1 over the function field $\mathbb{Q}(x,y)$, or indeed $\mathbb{C}(x,y)$. By taking the preimage of a general curve in $\mathbb{P}^2$ we get derived equivalent 3-folds of Kodaira dimension 1, where one has a rational point and one does not.

Theorem 3 stands in contrast to a result of Frei [Fre20, Theorem 1], who showed that, over a *finite* field, two smooth projective moduli spaces of sheaves on a given K3 surface have the same number of points as soon as they have the same dimension.

Conventions. For an arbitrary field k, we let k^s denote its separable closure, $\bar{k}$ its algebraic closure, and $G = \mathrm{Gal}(k^s/k)$ its absolute Galois group. For a variety X over k, we let $X^s = X \times_k k^s$ and $\bar{X} = X \times_k \bar{k}$. We let Pic_X denote the Picard scheme and $\mathrm{Pic}(X)$ the Picard group; the group of k-rational points of Pic_X is $\mathrm{Pic}(X^s)^G$, which may be strictly bigger than $\mathrm{Pic}(X)$, as we will discuss at length.

Because we consider sheaves of rank 0 and sheaves on reducible spaces, stability for us always means Gieseker stability, defined in terms of the reduced Hilbert polynomial, rather than slope stability.

2. Abelian counterexamples

2.1 Torsors over Abelian varieties

Let A be an Abelian variety over an arbitrary field k. As Poonen and Stoll discuss in [PS99, § 4], the short exact sequence of G-modules

$$0 \to \mathrm{Pic}^0(A^s) \to \mathrm{Pic}(A^s) \to \mathrm{NS}(A^s) \to 0$$

yields a long exact sequence

$$0 \to \mathrm{Pic}^0(A) \to \mathrm{Pic}(A) \to \mathrm{NS}(A^s)^G \to H^1(G, \mathrm{Pic}^0(A^s)) \to H^1(G, \mathrm{Pic}(A^s)) \cdots .$$

For any divisor class

$$\lambda \in \mathrm{NS}(A^s)^G,$$

we can consider the associated component of the Picard scheme

$$\mathrm{Pic}_A^\lambda \subset \mathrm{Pic}_A,$$

which is a torsor over the dual Abelian variety $\hat{A} = \mathrm{Pic}_A^0$, trivial if and only if λ is in the image of the map $\mathrm{Pic}(A) \to \mathrm{NS}(A^s)^G$ above: that is, in $\mathrm{NS}(A)$.

Theorem 4. *There is a k-linear exact equivalence*

$$D^b(A) \cong D^b(\mathrm{Pic}_A^\lambda).$$

It is interesting to contrast this statement with [AKW17, Corollary 5.2], which says that in characteristic 0, if $\mathrm{End}(A^s) = \mathbb{Z}$ and the inclusion $\mathrm{NS}(A) \subset \mathrm{NS}(A^s)^G = \mathbb{Z}$ is an equality, then two derived equivalent A-torsors necessarily generate the same subgroup of $H^1(G, \mathrm{Pic}^0(A^s))$.

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Proof of Theorem 4. Because A has a rational point, there is a Poincaré line bundle P on $A \times \mathrm{Pic}^{\lambda}_A$ by [Kle05, Exercise 9.4.3] or by Lemma 5 below. We claim that the functor

$$F\colon D^b(A) \to D^b(\mathrm{Pic}^{\lambda}_A)$$

induced by P is an equivalence. By [Orl02, Lemma 2.12], it is enough to prove it after base change to $\bar{k}$.

So assume that k is algebraically closed, choose a base point $[M] \in \mathrm{Pic}^{\lambda}(A)$, and use it to identify Pic^{λ}_A with $\hat{A}$. Then $P \otimes \pi_1^* M^{\vee}$ is a Poincaré bundle on $A \times \hat{A}$, hence induces an equivalence $D^b(A) \to D^b(\hat{A})$ by Mukai's theorem [Muk81]; see also Polishchuk's book [Pol03, Chapters 11 and 17], or Huybrechts' book [Huy06, Chapter 9] for another account in characteristic zero. Now the functor induced by $P \otimes \pi_1^* M^{\vee}$ is just tensoring with $M^{\vee}$ (which is an equivalence) followed by F, so it is an equivalence if and only if F is. □

2.2 Proof of Theorem 2 and discussion of Jacobians

We could deduce Theorem 2 from Theorem 4 by letting $A = \mathrm{Pic}^0_C$ and letting $\lambda \in \mathrm{NS}(A^s)^G$ be the class of the Θ-divisor; then $\mathrm{Pic}^{\lambda}_A \cong \mathrm{Pic}^{g-1}_C$ by [PS99, Corollary 4].[1]

But we can also describe the Poincaré bundle on $\mathrm{Pic}^0_C \times \mathrm{Pic}^{g-1}_C$ as the line bundle associated to a very explicit divisor D, and this will prove useful in § 4. Over $\bar{k}$, we can write

$$D = \left\{(L, M) : H^1(L \otimes M) \neq 0\right\} \subset \mathrm{Pic}^0_{\bar{C}} \times \mathrm{Pic}^{g-1}_{\bar{C}}.$$

We see that the fiber of D over a point of $\mathrm{Pic}^{g-1}_{\bar{C}}$ is a translate of the Θ-divisor on $\mathrm{Pic}^0_{\bar{C}}$, in such a way that $\mathcal{O}(D)$ is a universal bundle for $\mathrm{Pic}^{g-1}_{\bar{C}}$ as a moduli space of line bundles on $\mathrm{Pic}^0_{\bar{C}}$.

To see that D is defined over k, we can describe it as the support of a certain twisted sheaf. Let $\alpha \in \mathrm{Br}(\mathrm{Pic}^0_C)$ be the obstruction to the existence of a universal line bundle on $C \times \mathrm{Pic}^0_C$, and let $\mathcal{L}$ be the $\pi_2^*\alpha$-twisted universal bundle. Similarly, let $\beta \in \mathrm{Br}(\mathrm{Pic}^{g-1}_C)$ and $\mathcal{M}$ on $C \times \mathrm{Pic}^{g-1}_C$. Then D is the support of the $(\alpha \boxtimes \beta)$-twisted sheaf

$$R^1\pi_{23,*}(\pi_{12}^*\mathcal{L} \otimes \pi_{13}^*\mathcal{M}), \tag{1}$$

where π_{ij} are the projections from $C \times \mathrm{Pic}^0_C \times \mathrm{Pic}^{g-1}_C$ onto any two factors.

For another argument that $\mathcal{O}(D)$ agrees with the Poincaré line bundle, we could use Deligne's description of the latter as the line bundle whose fiber at a geometric point (L, M) is

$$(\det H^*(L \otimes M))^{-1} \otimes \det H^*(L) \otimes \det H^*(M) \otimes (\det H^*(\mathcal{O}_C))^{-1}. \tag{2}$$

We learned this description from Arinkin's paper [Ari13]; see Polishchuk's book [Pol03, § 22.3] for another account. Globally, the first term of (2) is

$$(\det R^0\pi_{23,*}(\pi_{12}^*\mathcal{L} \otimes \pi_{13}^*\mathcal{M}))^{-1} \otimes (\det R^1\pi_{23,*}(\pi_{12}^*\mathcal{L} \otimes \pi_{13}^*\mathcal{M})).$$

Now $R^0\pi_{23,*}(\pi_{12}^*\mathcal{L} \otimes \pi_{13}^*\mathcal{M})$ vanishes, and $R^1\pi_{23,*}(\pi_{12}^*\mathcal{L} \otimes \pi_{13}^*\mathcal{M})$ is supported on D and has generic rank 1 there, so its determinant as a sheaf on $\mathrm{Pic}^0_C \times \mathrm{Pic}^{g-1}_C$ is the (untwisted) line bundle $\mathcal{O}(D)$. The other terms of (2) just modify $\mathcal{O}(D)$ by line bundles pulled back from Pic^0_C or Pic^{g-1}_C, so they do not change the fact that it induces a derived equivalence.

[1] Recall that the Θ-divisor lives most naturally in Pic^{g-1}_C, as the image of the Abel–Jacobi map $\mathrm{Sym}^{g-1}(C) \to \mathrm{Pic}^{g-1}_C$. If Pic^{g-1}_C has a k-point then we can identify $\mathrm{Pic}^0_C \cong \mathrm{Pic}^{g-1}_C$ and get a Θ-divisor on $A = \mathrm{Pic}^0_C$, unique up to translation. But if $\mathrm{Pic}^{g-1}_C(k) = \varnothing$ then we only get a class in $\mathrm{NS}(A^s)^G$, not $\mathrm{NS}(A)$.

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The interested reader may consult [ADM16, Proof of Proposition 2.1(a)] for more details on the generalization of Mukai's equivalence from $\mathrm{Pic}^0_C \times \mathrm{Pic}^0_C$ to $\mathrm{Pic}^m_C \times \mathrm{Pic}^n_C$, and [ADM16, Remark 2.3] for more (twisted) derived equivalences.

2.3 Proof of Theorem 1

To deduce Theorem 1 from Theorem 2, it is enough to find genus-g curves C such that Pic^{g-1}_C has no rational points. There is a subtlety, in that the inclusion

$$\mathrm{Pic}^d(C) \subset \mathrm{Pic}^d_C(k) = \mathrm{Pic}^d(C^s)^G \tag{3}$$

may be proper in general.

Coray and Manoil [CM96, Proposition 4.2] showed that, for any $g \geq 1$, the hyperelliptic curve C of genus g determined by

$$y^2 = 605 \cdot 10^6 x^{2g+2} + (18x^2 - 4400)(45x^2 - 8800)$$

has points over $\mathbb{R}$ and over $\mathbb{Q}_p$ for every prime p, so the inclusion (3) is an equality for all d by [CM96, Corollary 2.5]; but that $\mathrm{Pic}^1(C) = \varnothing$. Bhargava, Gross and Wang [BGW17, Theorem 2] later showed that a positive proportion of hyperelliptic curves over $\mathbb{Q}$ have this property, by studying pencils of quadrics. Note that if C is hyperelliptic and g is even then $\mathrm{Pic}^1_C \cong \mathrm{Pic}^{g-1}_C$.

Poonen and Stoll gave explicit hyperpelliptic curves of even genus over $\mathbb{Q}$ for which Pic^{g-1}_C has no rational points in [PS99, Propositions 26–28], and studied their density in [PS99, § 9]. They gave a non-hyperelliptic example of genus 3 over $\mathbb{Q}$, namely the plane quartic curve

$$x^4 + py^4 + p^2z^4 = 0$$

for any $p \equiv -1 \pmod{16}$, in [PS99, Proposition 29]. And they gave a genus-2 example over $\mathbb{F}_q(t)$ for q odd, namely

$$y^2 = tx^6 + x - at$$

for any a that is not a square in $\mathbb{F}_q$, in [PS99, Proposition 30].

While we expect that there are similar examples over $\mathbb{Q}$ in odd genus $g \geq 5$, and over $\mathbb{F}_q(t)$ in genus $g \geq 3$, we did not find them in the literature. But to finish the proof of Theorem 1, we can just take the examples above and cross with an elliptic curve. Let C be a genus-2 curve such that $Y = \mathrm{Pic}^1_C$ is a non-trivial torsor over the Abelian surface $X = \mathrm{Pic}^0_C$, and let E be an elliptic curve. Then for any $g > 2$, we see that $Y \times E^{g-2}$ is a non-trivial torsor over $X \times E^{g-2}$, and the two have equivalent derived categories [Huy06, Exercise 5.20].

3. Brauer classes on compactified Picard schemes

Let X be a projective, geometrically integral, but not necessarily smooth variety over an arbitrary field k. For a divisor class $\lambda \in \mathrm{NS}(X^s)^G$, there is an inclusion

$$\mathrm{Pic}^\lambda(X) \ \subset \ \mathrm{Pic}^\lambda_X(k) = \mathrm{Pic}^\lambda(X^s)^G, \tag{4}$$

which we encountered for smooth curves in the proof of Theorem 1 but which we now consider more broadly in preparation for § 4.

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Coray and Manoil [CM96] have studied the inclusion (4) using the exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X^s)^G \to \mathrm{Br}(k) \to \mathrm{Br}(X) \to \cdots \tag{5}$$

coming from the Hochschild–Serre spectral sequence. We will study it using a class in $\mathrm{Br}(\mathrm{Pic}^\lambda_X)$ that arises from viewing the Picard scheme as a moduli space of stable sheaves on X.

For standard results on moduli spaces of sheaves, our reference is Huybrechts and Lehn's book [HL10], together with Langer's papers [Lan04b, Lan04a] which make the results available in positive and mixed characteristic. The key point for us is that a moduli space M of (S-equivalence classes of) semi-stable sheaves with given Hilbert polynomial is projective, and the open subspace M_{stab} parametrizing geometrically stable sheaves is in general only quasi-projective but carries a Brauer class that obstructs the existence of a universal sheaf on $X \times M_{\mathrm{stab}}$. This Brauer class is associated to a principal PGL_n-bundle coming from the construction of the moduli space via geometric invariant theory [HL10, Corollary 4.3.5]. It can be realized as an étale $\mathbb{P}^n$-bundle as follows. By boundedness, there is an $N \gg 0$, depending only on the Hilbert polynomial, such that for every semi-stable sheaf F, the twist $F(N)$ is globally generated and has no higher cohomology. Then we take the bundle over M_{stab} whose fiber at F is $\mathbb{P}H^0(F(N))$. For a stackier description, we could say that a geometrically stable sheaf is simple, so its automorphism group is $\mathbb{G}_m$, so the moduli stack of geometrically stable sheaves is a $\mathbb{G}_m$-gerbe over the moduli space; compare [Lie06, Corollary 4.3.3].

Because X is geometrically integral, every rank-1 torsion-free sheaf on X is geometrically stable with respect to any ample line bundle. If X is smooth, then Pic^λ_X is closed in the moduli space of rank-1 torsion-free sheaves on X, but if not then we can consider its closure

$$\overline{\mathrm{Pic}}^\lambda_X.$$

Let

$$\alpha_\lambda \in \mathrm{Br}(\overline{\mathrm{Pic}}^\lambda_X)$$

be the Brauer class that obstructs the existence of a universal sheaf on $X \times \overline{\mathrm{Pic}}^\lambda_X$.[2]

The inclusion (4) above can be understood as follows: for any extension field K/k, we have

$$\mathrm{Pic}^\lambda(X_K) = \{\ell \in \mathrm{Pic}^\lambda_X(K) : \alpha_\lambda|_\ell = 0 \in \mathrm{Br}(K)\}.$$

The following lemma is essentially well known for Pic^λ_X (see [CM96, Corollary 2.3] for an approach using the exact sequence (5)), but in the next section we need it for $\overline{\mathrm{Pic}}^\lambda_X$, which requires different methods.

LEMMA 5. *Let X be a projective, geometrically integral variety over an arbitrary field k, let $\lambda \in \mathrm{NS}(X^s)^G$, and let $\overline{\mathrm{Pic}}^\lambda_X$ and $\alpha_\lambda \in \mathrm{Br}(\overline{\mathrm{Pic}}^\lambda_X)$ be as above. If X has a smooth k-point, or more generally a zero-cycle of degree 1 supported in its smooth locus, then $\alpha_\lambda = 0$.*

Proof. Let c be the class in $K_{\mathrm{num}}(X^s)^G$ corresponding to $\lambda \in \mathrm{NS}(X^s)^G$. First we argue that for any vector bundle E on X, defined over k, the number $N_E := \chi(c \cdot [E])$ satisfies $N_E \cdot \alpha_\lambda = 0$. A similar claim is valid for any moduli space of geometrically stable sheaves, not only $\overline{\mathrm{Pic}}^\lambda_X$. Let π_1 and π_2 be the projections from $X \times \overline{\mathrm{Pic}}^\lambda_X$, and let U_λ be the $\pi_2^*\alpha_\lambda$-twisted universal sheaf on

[2] Note that α_λ is the restriction of a natural Brauer class that lives on the whole moduli space of geometrically stable sheaves, or indeed of simple sheaves, so there is no need to worry about ramification at the boundary components of $\overline{\mathrm{Pic}}^\lambda_X$.

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$X \times \overline{\mathrm{Pic}}^{\lambda}_X$. Then $R\pi_{2,*}(U_\lambda \otimes \pi_1^* E)$ is an α_λ-twisted perfect[3] complex of rank N_E, so $N_E \cdot \alpha_\lambda = 0$. (See [HL10, Theorem 4.6.5] for a similar argument.)

Now if $\xi \subset X_{\mathrm{sm}}$ is a subscheme of length d, then $\mathcal{O}_\xi$ admits a finite resolution by vector bundles and satisfies $\chi(c \cdot [\mathcal{O}_\xi]) = d$. Thus a zero-cycle of degree 1 forces α_λ to vanish, by a greatest common divisor argument. Note that if the support of ξ meets the singular locus of X then $\mathcal{O}_\xi$ might not admit a finite resolution by vector bundles. □

Lemma 5 will be used in conjunction with the following result.

Lemma 6. *Let X be a projective variety over a non-Archimedean local field $k = k_v$ with ring of integers $\mathcal{O}_v$ and residue field κ_v. Suppose that X has a model $\mathcal{X} \to \mathrm{Spec}\mathcal{O}_v$ whose special fiber is geometrically integral, or more generally has a component Y of multiplicity 1 that is geometrically integral. Then X has a zero-cycle of degree 1 supported in its smooth locus.*

Proof. By the Lang–Weil bounds, Y has a smooth point defined over a degree-r extension of κ_v for all $r \gg 0$ [Poo17, Theorem 7.7.1]. By Hensel's lemma, this lifts to a smooth point of X defined over a degree-r extension of k_v, so there is a zero-cycle of degree r defined over k_v, supported in the smooth locus of X. By considering r and $r+1$ we get a zero-cycle of degree 1. □

4. A hyperkähler counterexample

Recall that a K3 surface S of degree 2 can be obtained as a double cover of $\mathbb{P}^2$ branched over a smooth sextic curve. In § 4.1 we consider two moduli spaces X and Y of sheaves on such a K3 surface S and show that if the sextic satisfies a laundry list of conditions then the conclusions of Theorem 3 hold. In § 4.2 we exhibit a sextic satisfying those conditions. In § 4.3 we discuss computational issues that we faced in finding this sextic.

4.1 Steps of the proof

Let $f \in \mathbb{Z}[x,y,z]$ be a homogeneous polynomial of degree 6 that cuts out a smooth curve $B \subset \mathbb{P}^2_{\mathbb{Q}}$. Let S be the K3 surface over $\mathbb{Q}$ defined by

$$w^2 = f(x,y,z),$$

either in weighted projective space $\mathbb{WP}^3(3,1,1,1)$ or in the total space of $\mathcal{O}_{\mathbb{P}^2}(3)$. Let $\pi\colon S \to \mathbb{P}^2$ be the map that forgets w, which is a double cover branched over B.

The class $h := \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \in \mathrm{NS}(S)$ is ample and satisfies $h^2 = 2$. A curve in the linear system $|h|$ is the preimage of a line in $\mathbb{P}^2$, and we may identify $|h|$ with the space of lines in $\mathbb{P}^2$. A general member of $|h|$ is a smooth curve of genus 2.

Definition-Proposition 7. A line $L \subset \mathbb{P}^2$ (defined over any perfect field k) is a *tritangent line* to the sextic curve B if it satisfies one of the following conditions, which are equivalent:

(a) L is tangent to B at three points, in the scheme-theoretic sense;
(b) $f|_L = c \cdot g^2$, for some scalar c and some cubic $g \in H^0(\mathcal{O}_L(3))$;
(c) the curve $C = \pi^{-1}(L)$ is not geometrically integral.

[3] Because U is flat over $\overline{\mathrm{Pic}}^{\lambda}_X$ and X is proper; see [Sta18, Tag 0A1E].

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Proof. It is clear that (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c). To see that (c) $\Rightarrow$ (b), suppose for simplicity that the line is given by $z = 0$. Then the curve $w^2 = f(x, y)$ in $\mathbb{WP}^3(3, 1, 1)$ is not integral over $\bar{k}$, so $w^2 - f(x, y)$ factors, so $f(x, y)$ is a square in $\bar{k}[x, y]$. Because k is perfect, this implies that $f = c \cdot g^2$ for some $c \in k$ and some cubic $g \in k[x, y]$. □

Consider the moduli spaces of semi-stable sheaves on S of rank 0, first Chern class h, and Euler characteristic -1 or 0:

$$X := M_h(0, h, -1),$$
$$Y := M_h(0, h, 0).$$

Sheaves with these invariants include all line bundles of degree 0 or 1 supported on curves in the linear system $|h|$, together with some rank-1 torsion-free sheaves supported on singular curves. The moduli spaces are projective varieties of dimension 4. They map to $|h|$ by sending a sheaf to its support. Thus if $\mathcal{C} \to |h|$ is the tautological family of curves in the linear system $|h|$, that is,

$$\mathcal{C} = \{(x, C) : x \in C\} \subset S \times |h|,$$

then X is a compactification of the relative Picard scheme $\mathrm{Pic}^0_{\mathcal{C}/|h|}$, and Y is a compactification of $\mathrm{Pic}^1_{\mathcal{C}/|h|}$.

PROPOSITION 8. *The space X parametrizes only geometrically stable sheaves, hence is smooth. The same is true of Y if there are no tritangent lines to B defined over $\bar{\mathbb{Q}}$.*

Proof. Mukai showed that a moduli space of geometrically stable sheaves on a K3 surface is smooth [Muk84, Theorem 0.1].

Let F be a pure sheaf on $S_{\bar{\mathbb{Q}}}$ with $\mathrm{rank}(F) = 0$ and $c_1(F) = h$. Write $c_1(F) = m_1[C_1] + \cdots + m_k[C_k]$, where C_i are the irreducible components of the reduced support of F and $m_i > 0$. Since $c_1(F).h = h^2 = 2$ and h is ample, we see that either F is supported on an irreducible curve $C \in |h|$ and has generic rank 1 there, or it is supported on a reducible curve $C_1 \cup C_2$ with $C_1.h = C_2.h = 1$ and has generic rank 1 on each component.

In the first case (irreducible support), any saturated[4] subsheaf $G \subset F$ is either 0 or all of F, so F is necessarily stable. In the second case, which can only occur if there is a tritangent line defined over $\bar{\mathbb{Q}}$, a saturated subsheaf $G \subset F$ might be supported on C_1 or C_2 alone, with generic rank 1 there. In this case, the Hilbert polynomial $P_G(t)$ is $t + \chi(G)$, and the reduced Hilbert polynomial $p_G(t)$ is the same. If $\chi(F) = -1$ then $P_F(t) = 2t - 1$, so $p_F(t) = t - \frac{1}{2}$ so $p_G(t) \le p_F(t)$ implies $p_G(t) < p_F(t)$, so F is again stable. □

PROPOSITION 9. *Suppose there are no tritangent lines to B defined over $\bar{\mathbb{Q}}$. Then there is a $\mathbb{Q}$-linear exact equivalence $D^b(X) \cong D^b(Y)$.*

Proof. To emulate the construction of § 2.2 for the family of curves $\mathcal{C} \to |h|$, we might want to define a divisor

$$\{(L, M) : H^1(L \otimes M) \neq 0\} \subset X \times_{|h|} Y.$$

[4] See [HL10, Proposition 1.2.6] for why it is enough to consider saturated subsheaves.

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But on a singular curve $C \in |h|$, if L and M both fail to be locally free at some singular point, then the underived tensor product $L \otimes M$ is the wrong thing to write, and $\mathrm{Tor}_i^C(L, M)$ may be non-zero for infinitely many $i > 0$.

So following Arinkin [Ari13], we consider the open subschemes

$$X^\circ = \mathrm{Pic}^0_{\mathcal{C}/|h|} \subset X \quad \text{and} \quad Y^\circ = \mathrm{Pic}^1_{\mathcal{C}/|h|} \subset Y$$

and the divisor

$$D = \{(L, M) : H^1(L \otimes M) \neq 0\} \subset X^\circ \times_{|h|} Y \cup X \times_{|h|} Y^\circ,$$

which avoids the issue just discussed because one of L and M is always locally free. To see that D is defined over $\mathbb{Q}$, we can describe it as the support of the analogue of (1). Next, consider the inclusion

$$j \colon X^\circ \times_{|h|} Y \cup X \times_{|h|} Y^\circ \hookrightarrow X \times_{|h|} Y.$$

Our equivalence will be induced by the sheaf $j_*\mathcal{O}(D)$. Presumably this coincides with $\mathcal{O}(\bar{D})$, or more precisely, the dual of the ideal sheaf of $\bar{D}$, where $\bar{D}$ is the closure of D in $X \times_{|h|} Y$, but we will not need this.

To prove that the functor $F \colon D^b(X) \to D^b(Y)$ induced by $j_*\mathcal{O}(D)$ is an equivalence, we can again base-change to $\bar{\mathbb{Q}}$ by [Orl02, Lemma 2.12]. Then for closed points $x_1, x_2 \in X$, we consider the natural map

$$\mathrm{Ext}^*_X(\mathcal{O}_{x_1}, \mathcal{O}_{x_2}) \longrightarrow \mathrm{Ext}^*_Y(F(\mathcal{O}_{x_1}), F(\mathcal{O}_{x_2})). \tag{6}$$

The skyscraper sheaves $\mathcal{O}_{x_i}$ are a spanning class [Huy06, Proposition 3.17], so if (6) is an isomorphism for all x_1 and x_2 then F is fully faithful [Huy06, Proposition 1.49], and hence is an equivalence because ω_X and ω_Y are trivial [Huy06, Proposition 7.6].

Let $p \colon X \to |h|$ and $q \colon Y \to |h|$ be the Abelian fibrations discussed earlier, which map a sheaf to its support. Because our kernel $j_*\mathcal{O}(D)$ is supported on the fiber product $X \times_{|h|} Y$, we see that $F(\mathcal{O}_{x_1})$ is supported on the fiber $q^{-1}(p(x_1)) \subset Y$, and similarly with x_2. Thus if x_1 and x_2 lie in different fibers of p, then $F(x_1)$ and $F(x_2)$ have disjoint support, so both sides of (6) are zero.

If x_1 and x_2 lie in the same fiber, choose an étale neighborhood $U \to |h|$ of $p(x_1) = p(x_2)$ over which the family of cures $\mathcal{C} \to |h|$ admits a section, and use the section to identify $Y|_U$ with $X|_U$. Now we use [Ari13, Theorem C], which applies to $\overline{\mathrm{Pic}}^0$ of any integral curve with planar singularities, and indeed to any family of such curves by remark (2) after [Ari13, Theorem C]. Our curves have planar singularities because they are contained in a smooth surface S, and they are integral thanks to our hypothesis on tritangent lines. As at the end of § 2.2, our $\mathcal{O}(D)$ now coincides with Arinkin's Poincaré line bundle P up to tensoring with a line bundle pulled back from $X|_U$ on either side, so our $j_*\mathcal{O}(D)$ coincides with his Poincaré sheaf $\bar{P} = j_*P$ [Ari13, Lemma 6.1(2)], again up to line bundles on either side. Thus $F(\mathcal{O}_{x_1})$ and $F(\mathcal{O}_{x_2})$ agree, up to a line bundle, with the images of $\mathcal{O}_{x_1}$ and $\mathcal{O}_{x_2}$ under the functor induced by $\bar{P}$, which is an equivalence; thus (6) is an isomorphism. □

Proposition 10. *The space X contains a copy of $\mathbb{P}^2$, hence has infinitely many $\mathbb{Q}$-points.*

Proof. The map $X \to |h|$ that sends a sheaf to its support has a section, given by mapping a curve $C \in |h|$ to the trivial line bundle $\mathcal{O}_C$. □

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PROPOSITION 11. *The space Y has points over $\mathbb{R}$ and over $\mathbb{Q}_p$ for every prime p.*

Proof. Let $C \in |h|$ be any smooth curve. Then $\mathrm{Pic}^1_C \subset Y$ has both $\mathbb{R}$- and $\mathbb{Q}_p$-points by Lichtenbaum's result [Lic69] as discussed in the introduction. □

PROPOSITION 12. *Suppose f is chosen so that*

(a) $S(\mathbb{R}) = \varnothing$,
(b) *there are no tritangent lines to B defined over $\bar{\mathbb{F}}_2$, and*
(c) *for every tritangent line L to B defined over $\mathbb{F}_q$ with q odd, the curve $C = \pi^{-1}(L) \subset S_{\mathbb{F}_q}$ consists of two reduced rational curves defined over $\mathbb{F}_q$ rather than $\mathbb{F}_{q^2}$.*[5]

Then $Y(k) = \varnothing$ for every number field k of odd degree over $\mathbb{Q}$.

Proof. Because there are no tritangent lines over $\bar{\mathbb{F}}_2$, there are no tritangent lines defined over $\bar{\mathbb{Q}}$, so Y parametrizes only geometrically stable sheaves by Proposition 8. Thus there is a Brauer class $\alpha \in \mathrm{Br}(Y)$ that obstructs the existence of a universal sheaf on $S \times Y$. We will use α as a Brauer–Manin obstruction to the existence of k-points on Y.

First we claim that for all $y \in Y(\mathbb{R})$ we have $\alpha|_y \neq 0$. If on the contrary $\alpha|_y = 0$, then y represents a sheaf F on $S_{\mathbb{R}}$. Because $\chi(F) = 0$ and $c_1(F)^2 = h^2 = 2$, the degree of $c_2(F) \in \mathrm{CH}_0(S_{\mathbb{R}})$ is 1 by Riemann–Roch. But a zero-cycle defined over $\mathbb{R}$ is a linear combination of $\mathbb{R}$-points and $\mathbb{C}$-points, and we have $S(\mathbb{R}) = \varnothing$, so there can be no zero-cycle of odd degree.

Now fix a number field k of odd degree over $\mathbb{Q}$. We claim that for all non-Archimedean places v of k and all $y \in Y(k_v)$ we have $\alpha|_y = 0$. Even stronger, we will show that, for all curves $C \in |h|$ defined over k_v, the restriction of α to the fiber $\overline{\mathrm{Pic}}^1_C \subset Y_{k_v}$ is zero. Let $\mathcal{O}_v$ be the ring of integers of k_v and $\kappa_v \cong \mathbb{F}_q$ the residue field. Because the sextic f was defined over $\mathbb{Z}$, we get a model of C over $\mathrm{Spec}\mathcal{O}_v$. By hypothesis, the reduction C_{κ_v} is either geometrically integral or a reduced union of two rational curves; in either case, Lemmas 6 and 5 imply that the restriction of α to $\overline{\mathrm{Pic}}^1_C$ is zero.

To conclude, suppose that $y \in Y(k)$. Because $[k : \mathbb{Q}]$ is odd, k has an odd number of real places, so we see that $\sum_v \mathrm{inv}_v(\alpha|_y)$ is an odd multiple of $1/2$, which is impossible; see, for example, [Poo17, Proposition 8.2.2]. □

PROPOSITION 13. *Suppose that $\mathrm{Pic}(S_{\bar{\mathbb{Q}}}) = \mathbb{Z}h$.*

(a) *The schemes X and Y are not birational, even over $\mathbb{C}$.*
(b) *If the Brauer class $\alpha \in \mathrm{Br}(Y)$ that obstructs the existence of a universal sheaf on $S \times Y$ obstructs the Hasse principle (for example, if the hypotheses of Proposition 12 are satisfied) then α is transcendental, that is, its image in $\mathrm{Br}(Y_{\bar{\mathbb{Q}}})$ is non-zero.*

Proof. Part (a) is due to Sawon [Saw08, Proposition 15]: the discriminant of the Beauville–Bogomolov form on $\mathrm{Pic}(X_{\mathbb{C}})$ is -4, while on $\mathrm{Pic}(Y_{\mathbb{C}})$ it is -1.

For part (b), recall the filtration of the Brauer group

$$\mathrm{Br}_0(Y) \subset \mathrm{Br}_1(Y) \subset \mathrm{Br}(Y),$$

where

$$\mathrm{Br}_0(Y) := \mathrm{im}(\mathrm{Br}(\mathbb{Q}) \to \mathrm{Br}(Y)) \quad \text{and} \quad \mathrm{Br}_1(Y) := \ker(\mathrm{Br}(Y) \to \mathrm{Br}(Y_{\bar{\mathbb{Q}}})).$$

[5] That is to say, in Definition–Proposition 7(b) we can take $c = 1$ and $g \neq 0$.

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Classes in $\mathrm{Br}_0(Y)$ are called constant, those in $\mathrm{Br}_1(Y)$ are called algebraic, and the rest are called transcendental. The Hochschild–Serre spectral sequence gives an isomorphism

$$\mathrm{Br}_1(Y)/\mathrm{Br}_0(Y) \cong H^1(G, \mathrm{Pic}(Y_{\bar{\mathbb{Q}}}));$$

see, for example, [Poo17, Corollary 6.7.8 and Remark 6.7.10].

The usual identification of $\mathrm{Pic}(Y_{\bar{\mathbb{Q}}})$ with the orthogonal to the Mukai vector $(0, h, 0)$ in the Mukai lattice $\mathbb{Z} \oplus \mathrm{Pic}(S_{\bar{\mathbb{Q}}}) \oplus \mathbb{Z}$ is valid as G-modules; to see this, note that all the maps appearing in [Cha16, Theorem 2.4(vi)] are G-equivariant, and see [Fre20, § 2] for the techniques needed to relax condition (C) of [Cha16, Definition 2.3].

Because $\mathrm{Pic}(S_{\bar{\mathbb{Q}}}) = \mathbb{Z}h$, we see that $\mathrm{Pic}(Y_{\bar{\mathbb{Q}}}) \cong \mathbb{Z}^2$ with trivial G-action, so $H^1(G, \mathrm{Pic}(Y_{\bar{\mathbb{Q}}})) = 0$. Thus if α were algebraic then it would be constant, but a constant class cannot obstruct the Hasse principle. □

4.2 The explicit example

The sextic polynomial

$$\begin{aligned} f(x,y,z) = &-x^6 - x^5z - x^4y^2 - x^4z^2 - x^3yz^2 - x^2y^2z^2 \\ &- xy^5 - xy^4z - xz^5 - y^6 - y^3z^3 - y^2z^4 - yz^5 - z^6 \end{aligned}$$

satisfies all the hypotheses laid out in the previous section. Magma code to verify the claims below is included in the supplementary file `verify.magma`, available with the online version of this article. We also provide `generate.magma` for readers who want to search for more examples, or adapt the code for their own purposes.

We find that $f(x, y, z) < 0$ for all $x, y, z \in \mathbb{R}^3 \setminus 0$, so $S(\mathbb{R}) = \varnothing$.

There are tritangent lines to B defined over $\bar{\mathbb{F}}_p$ for five primes. For $p = 5$, the line $z = 4x + y$ is tritangent. For $p = 31$, the line $y = 24x + 23$ is tritangent. For $p = 7517$, 8471603739813611030879 9, and

$$\begin{aligned} &4424904772196959344085200612883251617292465803437757948 \\ &5992572698404066491363246248977477562371729031497984350 \\ &0902180031058767256453958545754450340721124283977338015 \\ &3664612642260759001523868554216076825404419681, \end{aligned}$$

there are tritangent lines whose equations we omit. In each case, there is a single tritangent line defined over $\mathbb{F}_p$, and its preimage in S consists of two reduced rational curves defined over $\mathbb{F}_p$ rather than $\mathbb{F}_{p^2}$.

The curve B is smooth over $\mathbb{F}_{31}$, hence over $\mathbb{Q}$. Using the Magma routine `WeilPolynomialOfDegree2K3Surface`, due to Elsenhans, we find the characteristic polynomial of Frobenius acting on $H^2_{\text{ét}}(S_{\bar{\mathbb{F}}_{31}}, \mathbb{Q}_\ell(1))$:

$$\begin{aligned} (t-1)^2\big(&t^{20} - \tfrac{12}{31}t^{19} + \tfrac{15}{31}t^{18} - \tfrac{6}{31}t^{17} - \tfrac{3}{31}t^{16} - \tfrac{6}{31}t^{15} - \tfrac{5}{31}t^{14} \\ &+ \tfrac{5}{31}t^{13} - \tfrac{13}{31}t^{12} + \tfrac{15}{31}t^{11} - \tfrac{22}{31}t^{10} + \tfrac{15}{31}t^9 - \tfrac{13}{31}t^8 + \tfrac{5}{31}t^7 \\ &- \tfrac{5}{31}t^6 - \tfrac{6}{31}t^5 - \tfrac{3}{31}t^4 - \tfrac{6}{31}t^3 + \tfrac{15}{31}t^2 - \tfrac{12}{31}t + 1\big). \end{aligned}$$

The degree-20 factor is irreducible and is not cyclotomic, so S has geometric Picard rank 2 over $\mathbb{F}_{31}$, and hence geometric Picard rank 1 over $\mathbb{Q}$ by a result of Hassett and Várilly-Alvarado

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[HV13, Proposition 5.3]. (Note that this reference requires the geometric Picard group of $S_{\mathbb{F}_{31}}$ to have rank 2 *and* be generated by the curves in the preimage of the tritangent line, but the latter is automatic: the intersection pairing between the two curves is

$$\begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix},$$

whose discriminant -5 is squarefree, whereas if they generated an index-N sublattice then the discriminant would be divisible by N^2.)

4.3 Computational discussion

The computational difficulty is in finding all primes p such that there is a tritangent line to the sextic curve defined over $\bar{\mathbb{F}}_p$. For any given p, we can find tritangent lines defined over $\bar{\mathbb{F}}_p$ in a fraction of a second using Elsenhans and Jahnel's algorithm [EJ08, Algorithm 8]: we write equations in the coefficients of a general line that say it is tritangent to the sextic, and compute a Gröbner basis of the ideal they generate. But if we want to find tritangent lines for all primes at once, we must compute a Gröbner basis of the same ideal over $\mathbb{Z}$, which typically takes about half an hour in our implementation. Moreover, the running time is very sensitive to the details of the implementation: if we make a seemingly trivial change, like switching the order of two variables, it might take hours or days. In the supplementary file `verify.magma`, we carry out the Gröbner basis computation over $\mathbb{Z}$, but in our initial search for f we needed something much faster.

Following advice from Elsenhans, we computed a Gröbner basis over $\mathbb{Q}$ in Magma with the `ReturnDenominators` option enabled; this returns a list of all the denominators used in the division steps of the Gröbner basis algorithm, so if there are no tritangent lines over $\bar{\mathbb{Q}}$ then the primes at which tritangent lines occur must divide one of those denominators. In our case the list was very long, and many of the denominators were more than 1000 digits, which is too big to factor. So we ran the computation twice, with slight variations in the details, and then took common factors between the two lists. This yielded a list of small numbers and one 300-digit number. This is still too big to factor in general, but we tested many sextics and occasionally found candidates for which the big number had a few small prime factors and one big prime factor, as in the example above.

We also modified [EJ08, Algorithm 8] as follows. The main step in the algorithm is to take the ideal in

$$R[a, b, c_0, \dots, c_3],$$

where $R = \mathbb{F}_p$ or $\mathbb{Q}$ or $\mathbb{Z}$, generated by equating coefficients in

$$f(1, t, a + bt) = (c_0 + c_1 t + c_2 t^2 + c_3 t^3)^2,$$

and eliminate the variables $c_0, \dots, c_3$ to get a Gröbner basis of an ideal in $R[a, b]$ that says 'the line $z = ax + by$ is a tritangent line'. Over $\mathbb{F}_p$ and $\mathbb{Q}$, this is very efficient. But over $\mathbb{Z}$ and with `ReturnDenominators` over $\mathbb{Q}$ it was profitable to split it into two steps. First, take the ideal in

$$\mathbb{Z}[c_0, \dots, c_3, d_0, \dots, d_6]$$

generated by equating coefficients in

$$d_0 + d_1 t + \cdots + d_6 t^6 = (c_0 + \cdots + c_3 t^3)^2,$$

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and eliminate $c_0, \ldots, c_3$ to give an ideal in $\mathbb{Z}[d_0, \ldots, d_6]$ with 51 generators. This takes a fraction of a second. Then substitute the coefficients of $f(1, t, a + bt)$ into the latter ideal and compute a Gröbner basis. Our modified algorithm ran at least an order of magnitude faster than the original when using ReturnDenominators over $\mathbb{Q}$. Over $\mathbb{Z}$, it ran in about half an hour, whereas the original ran out of memory before returning an answer. Our modification produces a much bigger set of generators for the ideal, with elements of much higher degree, so we were surprised that it performed better.

Source code

Magma scripts are available as supplementary material with the online version of this article available at https://doi.org/10.1112/S0010437X21007089.

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