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Bistability in Cylindrical Developable Mechanisms Through the Principle of Reflection

We present a resource for designing bistable developable mechanisms (BDMs) that reach their second stable positions while exterior or interior to a cylindrical surface. Analysis of the necessary conditions to create extramobile and intramobile cylindrical BDMs is conducted through a series of three tests. These tests contain elements of both existing and new mechanism design tools, including a novel graphical method for identifying stable positions of linkages using a single dominant torsional spring, called the principle of reflection. These tests are applied to all possible mechanism cases and configurations to identify why certain configurations will always, sometimes, or never be a BDM. Two tables summarize these results as a guide when designing extramobile and intramobile BDMs. The results are compared and demonstrated with a numerical simulation of 30,000+ mechanisms, including several example mechanisms that illustrate the concepts discussed in the work. Discussion is then provided on the implication of these results. [DOI: 10.1115/1.4049655]

Keywords: design of innovative devices design of machine elements, kinematics, linkages, mechanism synthesis

1 Introduction

Developable mechanisms are devices that can conform to or be embedded within a developable surface [1], such as a cone or cylinder. This class of mechanisms shows promise for use in applications that demand devices be highly compact when stored but capable of deploying and performing specific functions. Their terminology [2], as well as investigatory work on their implementation in cylindrical [3,4] and conical [5] surfaces, has been an area of recent interest. Developable mechanisms that conform to cylinders are of particular interest because of the prevalence of cylinders in many applications.

One area of interest in the analysis of developable mechanisms is their ability to move exterior to or interior to (extramobile and intramobile motion) a developable surface [3]. By remaining entirely outside or inside of a surface, the mechanism ensures that it does not interfere with other subsystems that may be within or without a specified region. This enables developable mechanisms to be implemented in many applications without adjusting geometry to accommodate for the motion path of the mechanism.

The potential impact of developable mechanisms could be increased by combining their unique geometry with other types of mechanisms that exhibit specialized behaviors. One such behavior is bistability, the capacity of a mechanism to utilize strain energy to reach multiple stable positions. The use of strain energy allows for more economical use of input or actuation power [6]. For example, an actuator can move a device from one stable state to another and remain inactive otherwise, reducing energy consumption, wear, and cost. Many methods of introducing and using strain energy into mechanisms have been proposed, including the use of architected materials [7,8], orthogonal compliant mechanisms [9], and micromechanisms [10]. Other methods for synthesizing these mechanisms have also been developed [11–15]. These

approaches and devices are all dependent upon the same basic principles of energy storage and stable equilibrium.

Theorems have been developed to accurately predict the conditions that must exist to make a planar mechanism bistable [16]. These theorems are based upon the Grashof-condition [17] and initial geometry of a mechanism. Because cylindrical developable mechanisms are a subset of planar mechanisms, these theorems may be applied without additional consideration to create bistable developable mechanisms (BDMs). Mechanisms created with this approach, called regular BDMs or simply BDMs, are created without consideration of their motion relative to the developable surface on which they conform.

Of particular interest are BDMs whose second stable position resides where all links are exterior or interior to a reference surface and can be reached through only extramobile or intramobile motion. The design of these mechanisms, referred to here as extramobile and intramobile BDMs, must consider the geometry of the developable surface to ensure their motion and stable positions reside outside or inside the surface.

This work identifies the conditions necessary to design extramobile and intramobile cylindrical BDMs. A series of tests are introduced and implemented to develop a design reference that identifies possible configurations of extramobile and intramobile BDMs. A novel graphical method for identifying stable positions of linkages using a single-dominant torsional spring, called the *Principle of Reflection*, is introduced and implemented in these tests. The test results are compared with a numerical simulation of several thousand mechanisms to identify any incongruities. Several example mechanisms demonstrate the approach. Two tables, located at the end of this work, summarize the test results as a guide for creating extramobile and intramobile BDMs. Discussion is then provided on the design reference.

2 Background

To understand the concepts and approaches used in this paper, we provide a brief review of relevant topics. This includes a discussion

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on developable mechanisms, extramobile and intramobile motion, bistability in planar mechanisms, and a geometric proof that is used frequently in this work.

2.1 Developable Mechanisms. Developable mechanisms are defined as mechanisms that are contained within or can conform to a developable surface (called a reference surface) when modeled with zero-thickness [2]. There are three major subsets of developable mechanism, which are correlated with the types of surfaces to which they conform. Planar, spherical, and spatial mechanisms can be designed to conform to cylindrical, conical, and tangent-developed surfaces, respectively. This is accomplished by aligning mechanism joints along the ruling lines of these developable surfaces. Mobility of these mechanisms is therefore only possible when a mechanism conforms to a developable surface, excluding other geometries such as dome-like or saddle-like surfaces.

Developable mechanisms that conform to cylinders are a subset of general planar mechanisms because the ruling lines of cylindrical surfaces are all parallel [2]. Therefore, cylindrical developable mechanisms are subject to the same rules that govern the behavior of traditional planar mechanisms. For example, the determination of Grashof criteria based on the distance between pins on a link [17] is still directly applicable to developable mechanisms. In a zero-thickness model, developable mechanisms have curved links. Because the kinematics of the links depend on the distance between pin joints, the rigid link may take on any shape, as in Fig. 1. For clarity, in this work, we will refer to the straight lines as r_i and the curved links as links.

Greenwood et al. showed how to determine if a mechanism is capable of extramobile or intramobile behavior based on its

straight-linkage representations at the conformed positions [3]. Six classes of four-bar mechanisms that achieve this behavior were identified. Three of these classes (1A, 2A, and 3A) are symmetric equivalents of the other three (1B, 2B, and 3B). As such, this paper will only look at Class A mechanisms (shown in Fig. 1).

2.2 Conditions for Intramobile and Extramobile Motion.

Intramobile (or extramobile) motion is the range of motion where a mechanism is interior (or exterior) to the reference surface. In previous work [18], we demonstrated that predicting intramobile and extramobile motion only requires analysis of grounded links; floating links do not affect the limits of extramobile or intramobile motion. Assurance of extramobile and intramobile motion can then be simplified into two conditions:

Condition 1: No grounded link may rotate from the conformed position far enough to again intersect the reference surface.

Condition 2: No grounded link may rotate interior to (exterior to) the reference surface for extramobile (intramobile) motion.

These conditions are comprehensive to all cylindrical developable mechanisms that are subject to the following assumptions. All links are modeled with zero thickness, have the same curvature as the reference surface, and have an arc length $\leq \pi R$. All grounded links only extend in one direction past their grounded pivot, while the coupler must not extend beyond either of the moving pivots.

Condition 1 can be visually illustrated and is shown in Figs. 2(a) and 2(b). The maximum possible rotations of any grounded link i for intramobile or extramobile motion are constrained by $\delta_{i\max}$ and are expressed mathematically as

$$\delta_{i\ ex,\max} = \pi \quad \text{for } (0 < S_i \leq \pi R) \quad (1)$$

$$\delta_{i\ in,\max} = \pi - \frac{S_i}{R} \quad \text{for } (0 < S_i \leq \pi R) \quad (2)$$

where S_i is the arc length of link i and R is the radius of the cylindrical reference surface. Condition 2 is violated when a grounded link moves exterior/interior to the surface, then reverses directions, returns to the initial position, and continues interior/exterior to the surface, as shown in Fig. 2(c). Conditions 1 and 2 will therefore serve as a means to determine if second stable positions are reached prior to a mechanism reaching its limit of intramobile or extramobile motion.

2.3 Bistability. Bistability in a mechanism exists when there are two separate stable equilibrium configurations within the range of motion of the mechanism, such as is seen in light switches, cabinet doors, and toothpaste lids. Bistability in four-bar mechanisms can be achieved by placing one or more torsional springs at the pin joints, each with a stiffness K_i . These torsional springs allow the mechanism to reach a variety of stable configurations, depending on the relative stiffnesses of each spring. The angle ψ indicates the angular deviation of any two links from their initial

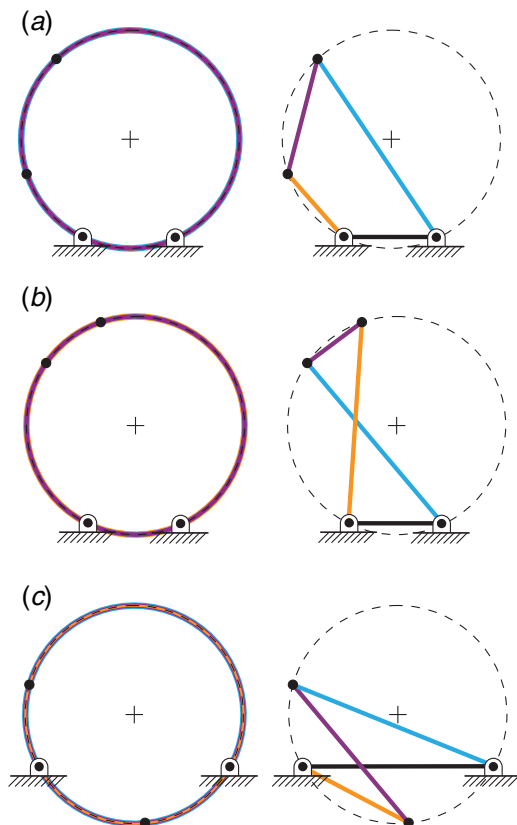


Fig. 1 Examples of the three classes of extramobile and intramobile mechanisms. While developable mechanisms are created using curved links (left), straight lines can be used to better visualize their positions (right): (a) Class 1A, (b) Class 2A, and (c) Class 3A.

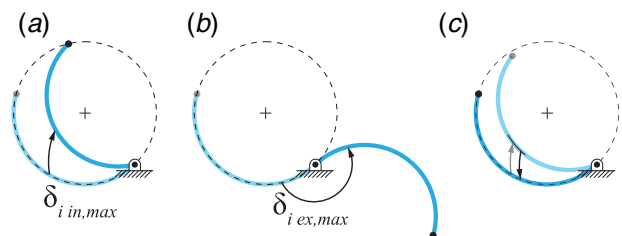


Fig. 2 Condition 1 provides $\delta_{i\max}$, the maximum (a) interior or (b) exterior rotation for any grounded link. Subfigure (c) shows Condition 2, where the link has returned to the surface. (a) Condition 1, (b) Condition 1, and (c) Condition 2.

relative angle and is given by [19]

$$\begin{aligned}\psi_1 &= \theta_2 - \theta_{2o} \\ \psi_2 &= \theta_2 - \theta_{2o} - (\theta_3 - \theta_{3o}) \\ \psi_3 &= \theta_4 - \theta_{4o} - (\theta_3 - \theta_{3o}) \\ \psi_4 &= \theta_4 - \theta_{4o}\end{aligned}\quad (3)$$

where θ_i and θ_{io} are the initial and rotated orientations of link i , respectively, relative to horizontal.

As a mechanism moves, the magnitude of ψ will increase. If a single torsional spring is located at pin joint i , this deflection will cause an increase of strain energy as given by

$$V_i = \frac{1}{2} K_i \psi_i^2 \quad (4)$$

To reach a stable position using a single spring of stiffness K_i , the angle ψ_i must equal 0. This can be extended to mechanisms with multiple torsional springs so long as there is one dominant spring K_i (a spring with much greater stiffness than any other in the mechanism).

Jensen and Howell proposed three fundamental theorems that predict how compliant design affects bistable behavior in planar four-bar mechanisms [16]. These theorems are as follows:

THEOREM 1. *A compliant mechanism whose pseudo-rigid-body model behaves like a Grashof four-link mechanism with a torsional spring placed at one joint will be bistable if and only if the torsional spring is located opposite the shortest link and the spring's undeflected state does not correspond to a mechanism position in which the shortest link and the other link opposite the spring are collinear.*

THEOREM 2. *A compliant mechanism whose pseudo-rigid-body model behaves like a non-Grashof four-link mechanism with a torsional spring at any one joint will be bistable if and only if the spring's undeflected state does not correspond to a mechanism position in which the two links opposite the spring are collinear.*

THEOREM 3. *A compliant mechanism whose pseudo-rigid-body model behaves like a change-point four-link mechanism with a torsional spring placed at any one joint will be bistable if and only if the spring's undeflected state does not correspond to a mechanism position in which the two links opposite the spring are collinear.*

These theorems accurately predict the placement of torsional springs on four-bar mechanisms to obtain bistability. They may therefore be applied to help identify possible configurations of developable mechanisms that lead to bistable positions outside and inside a cylinder.

2.4 Motion Limits. The motion of a mechanism is often limited by its geometry. Specifically, the extreme values for the θ_2 and θ_4 occur at toggle positions, or when two links prohibit further motion of another link. If these limits exist for a given mechanism, the extreme values of θ_2 occur when r_3 and r_4 align and the extreme values for θ_4 occur when r_2 and r_3 align.

The extreme angle values can affect the intramobile and extramobile motion of developable mechanisms. These values are given by Ref. [18] for triple-rockers, double-rockers, and crank-rockers. These equations are also included below because they are frequently used in the analysis.

For triple-rockers (RRR4), the limits are as follows:

$$\theta_{2\min,RRR4} = \arccos\left(\frac{r_1^2 + r_2^2 - (r_4 - r_3)^2}{2r_1r_2}\right) \quad (5)$$

$$\theta_{2\max,RRR4} = 2\pi - \arccos\left(\frac{r_1^2 + r_2^2 - (r_4 - r_3)^2}{2r_1r_2}\right) \quad (6)$$

$$\theta_{4\min,RRR4} = \pi - \arccos\left(\frac{r_1^2 + r_4^2 - (r_2 + r_3)^2}{2r_1r_4}\right) \quad (7)$$

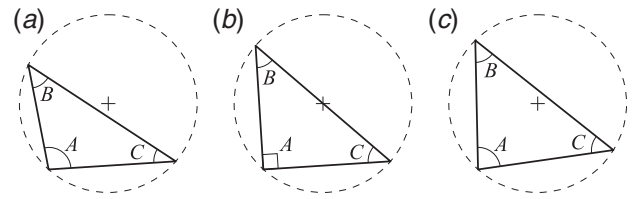


Fig. 3 Thale's theorem can be used to determine if an interior angle of a triangle is $>$, $<$, or $= \pi/2$. This is based on the relative location of the center of the triangle's circumcircle. (a) $A > \pi/2$, (b) $A = \pi/2$, and (c) $A, B, C < \pi/2$.

$$\theta_{4\max,RRR4} = \pi + \arccos\left(\frac{r_1^2 + r_4^2 - (r_2 + r_3)^2}{2r_1r_4}\right) \quad (8)$$

For double-rockers (GRCR), the limits are as follows:

$$\theta_{2\min,GRCR} = \theta_{2\min,RRR4} \quad (9)$$

$$\theta_{2\max,GRCR} = \arccos\left(\frac{r_1^2 + r_2^2 - (r_3 + r_4)^2}{2r_1r_2}\right) \quad (10)$$

$$\theta_{4\min,GRCR} = \theta_{4\min,RRR4} \quad (11)$$

$$\theta_{4\max,GRCR} = \pi - \arccos\left(\frac{r_1^2 + r_4^2 - (r_2 - r_3)^2}{2r_1r_4}\right) \quad (12)$$

For crank-rockers (GCRR), link 2 is fully revolute and thus θ_2 has no extreme value. The limits for θ_4 are as follows:

$$\theta_{4\min,GCRR} = \theta_{4\min,RRR4} \quad (13)$$

$$\theta_{4\max,GCRR} = \theta_{4\max,GRCR} \quad (14)$$

2.5 Thale's Theorem. Thale's Theorem [20] states that a triangle that is circumscribed by a circle and intersects the center of the circle must have the angle opposite the longest side equal to $\pi/2$. If the triangle does not contain the center of the circle, the angle opposite to the largest side must be greater than $\pi/2$. It follows that the other interior angles are less than $\pi/2$. If the triangle contains the center of the circle, all interior angles are less than $\pi/2$ (see Fig. 3).

3 Principle of Reflection: Finding Stable Positions of Bistable Linkages

Past work has developed analytical methods to identify the stable positions of bistable mechanisms through analysis of strain energy in deflected springs [21]. In general, analytical methods can provide robust numerical solutions to complex problems, while graphical methods can harness the designer's intuition through visual representation. This is especially true when mechanism design is constrained by the geometry of the device or its environment. The geometry-based nature of developable mechanisms makes graphical methods a logical approach for their design, particularly when the design utilizes CAD systems or geometry-based analysis programs.

We now introduce the *Principle of Reflection*, a graphical method for finding the stable positions of four-bar linkages with one dominant torsional spring. This method will later be used to analyze BDMs. Note that the dominant spring must be placed according to the theorems in Sec. 2.3. The Principle of Reflection is discussed in context of the dominant spring (K1, K2, K3, or K4, as shown in Fig. 4).

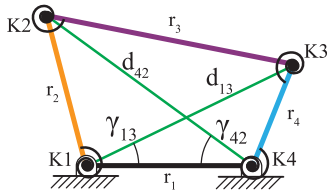


Fig. 4 The four possible torsional spring locations for a four-bar mechanism. Nomenclature for the Principle of Reflection is also shown.

We define the first stable position for K1 and K3 to be at the initial position, when $\angle P142 = \gamma_{42}$ and $\overline{P24} = d_{42}$, as shown in Fig. 4. Likewise, we define the first stable position for K2 and K4 at the initial position when $\angle P314 = \gamma_{13}$ and $\overline{P13} = d_{13}$. The values of γ can be found through the laws of sines and cosines. Since the arcsin function is a one-to-many relation, care must be taken to identify which quadrant γ lies in to ensure arcsin returns the correct value.

$$\gamma_{13} = \arcsin\left(\frac{r_4 \sin(\pi - \theta_{4o})}{\sqrt{r_1^2 + r_4^2 - 2r_1 r_4 \cos(\pi - \theta_{4o})}}\right) \quad (15)$$

$$\gamma_{42} = \arcsin\left(\frac{r_2 \sin(\theta_{2o})}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_{2o})}}\right) \quad (16)$$

Below and throughout this paper, the first stable position (the initial position) is noted by θ_o , and the second stable position is noted by θ_b . Without loss of generality we will let $\theta_1 = 0$.

K1 or K4. K1 has zero potential energy when $\theta_2 = \theta_{2o}$ ($\psi_1 = 0$), so the second stable position is reached when r_2 has returned to its original position (assuming it has not undergone a full rotation). Therefore, $\theta_{2b} = \theta_{2o}$. In the second stable position, r_3 and r_4 are reflected across d_{42} , as shown in Figs. 5(a) and 5(b).

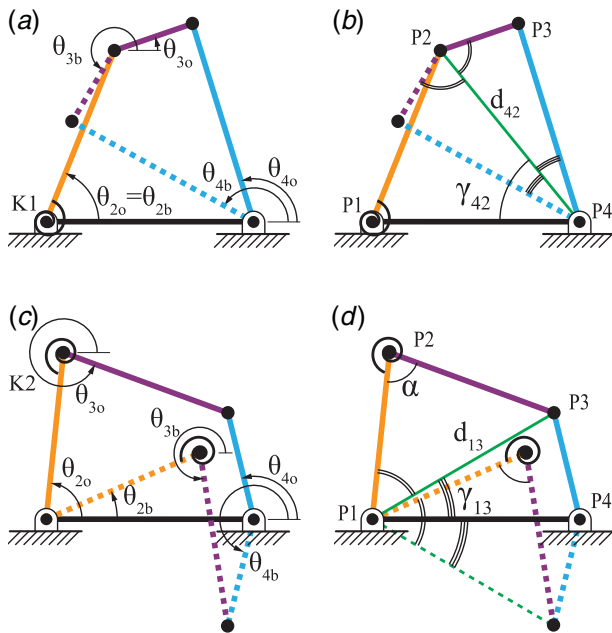


Fig. 5 (a, c) Stable states (solid and dashed) for a mechanism with one dominant torsional spring. (b, d) Principle of Reflection used to find the second stable position. (a) With spring K1 (b), reflecting r_3, r_4 across d_{42} , (c) with spring K2, and (d) reflecting r_2, r_4 across d_{13} .

Similarly, K4 has zero potential energy when $\theta_4 = \theta_{4o}$ ($\psi_4 = 0$). In the second stable position, $\theta_{4b} = \theta_{4o}$ where both r_2 and r_3 are reflected across d_{13} .

K2 or K3. For springs K2 or K3, the second stable position of the mechanism (when $\psi_2 = 0$ or $\psi_3 = 0$) can be found using reflection of the opposite grounded link, as described below and shown in Fig. 5.

Consider the four-bar linkage with torsional spring K2 in Fig. 5(c). The second stable position occurs when spring K2 returns to an undeflected state, $\psi_2 = 0$ (note that the links will be in different positions than the initial position). In both stable positions, $\overline{P13} = d_{13}$. This forms two triangles, $\Delta P123$ and $\Delta P134$. It follows that in the second stable position, triangle $\Delta P134$ is reflected across r_1 . Because triangle $\Delta P123$ also has side d , the positions of r_2 and r_3 are both located an angular displacement of $2\gamma_{13}$ about P1. Therefore, for a mechanism with torsional spring K2, the second stable position occurs when r_4 is reflected across r_1 and when r_2 and r_3 are rotated magnitude $2\gamma_{13}$ about P1.

A similar derivation can be done for K3. The result is that, for a mechanism with torsional spring K3, the second stable position occurs when r_2 is reflected across r_1 and r_3 and r_4 are located at an angular displacement of magnitude $2\gamma_{42}$ about P4.

4 Methods

The Theorems of Bistability, Principle of Reflection, and Conditions for Intramobile and Extramobile Motion provide us with the necessary tools to identify intramobile and extramobile BDMs. We will use these together to form three tests that filter which mechanism geometries always, sometimes, or never create an extramobile or intramobile BDM. If a mechanism fails a test, it cannot be an intramobile (or extramobile) BDM and will be noted by 7_j , where the subscript refers to the failed test. For convenience, these tests are generally performed sequentially. Mechanisms must pass all three tests to be an extramobile or intramobile BDM.

The first stable position is defined as the conformed and initial position of the mechanism, where all the links are conformed to the cylindrical reference surface. At this point, all joints are coincident with the reference surface. Without loss of generality, we will let $\theta_1 = 0$.

4.1 Test 1: Theorems of Bistability. For a developable mechanism to be a BDM, it must adhere to the three theorems of bistability for general planar four-bar linkages identified by Jensen and Howell [16] (see Sec. 2.3). Mechanisms that fail Test 1 (noted by χ_1) are no longer candidates for being extramobile or intramobile BDMs.

4.2 Test 2: Principle of Reflection. As cylindrical developable mechanisms are a subset of general planar mechanisms, the Principle of Reflection discussed in Sec. 3 may be used to identify the second stable position of a developable mechanism. (This principle assumes that the mechanism passes Test 1.) If either link 2 or 4 is exterior/interior to the surface in the second stable position, the mechanism fails Test 2 for intramobile/extramobile motion (noted by χ_2).

For K1, only the second stable position of link 4 needs to be checked (link 2 has returned to the conformed position). Similarly, for K4, only the second stable position of link 2 needs to be checked (link 4 has returned to the conformed position). If the second stable position of either link lies at a reflection across d toward the center of the circle, it cannot be an extramobile BDM. Likewise, a reflection away from the center of the circle prevents it from being an intramobile BDM.

4.3 Test 3: Motion Between Stable Positions. Test 3 is used to identify if the motion between the initial position and the second

Table 1 Possible locations of torsional springs to create bistable extramobile and intramobile cylindrical Class A developable mechanisms

			Extramobile			Intramobile				
Type			K1	K2	K3	K4	K1	K2	K3	K4
Class 1A	Grashof	GCCC	$\mathbf{X}_1$	✓	✓	$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_2$	$\mathbf{X}_1$
		GCRR	$\mathbf{X}_1$	$\mathbf{X}_1$	✓	$\mathbf{X}_2$	$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_2$	✓*
		GRCR	✓	$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_2$	$\mathbf{X}_1$	$\mathbf{X}_1$	✓*
	Non-Grashof Change-Point	RRR4	✓*	✓*	✓	$\mathbf{X}_2$	$\mathbf{X}_2$	$\mathbf{X}_2$	$\mathbf{X}_2$	✓*
		CPCCC	$\mathbf{X}_3$	✓	✓	$\mathbf{X}_2$	$\mathbf{X}_2$	$\mathbf{X}_2$	$\mathbf{X}_2$	$\mathbf{X}_3$
		CPCRR	$\mathbf{X}_3$	$\mathbf{X}_3$	✓	$\mathbf{X}_2$	$\mathbf{X}_2$	$\mathbf{X}_2$	$\mathbf{X}_2$	✓*
		CPRCR	✓	✓	✓*	$\mathbf{X}_2$	$\mathbf{X}_2$	$\mathbf{X}_2$	$\mathbf{X}_2$	✓*
Class 2A	Grashof	GCCC	$\mathbf{X}_1$	✓	✓*	$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_2$	$\mathbf{X}_1$
		GCRR	$\mathbf{X}_2$	$\mathbf{X}_1$	$\mathbf{X}_1$	✓	✓*	$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_2$
	Change-Point	CP2X-RCR	$\mathbf{X}_2$	✓	✓*	✓	$\mathbf{X}_3$	$\mathbf{X}_3$	$\mathbf{X}_3$	$\mathbf{X}_3$
		CP2X-CCR/RRC	$\mathbf{X}_2$	$\mathbf{X}_1$	✓*	$\mathbf{X}_1$	✓*	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_1$
		CP3X	$\mathbf{X}_2$	$\mathbf{X}_1$	✓*	$\mathbf{X}_1$	$\mathbf{X}_3$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_1$
Class 3A	Grashof	GCRR	$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_2$	✓	$\mathbf{X}_1$	$\mathbf{X}_1$	✓*	$\mathbf{X}_2$
	Change-Point	CP2X-CCR	✓	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_1$	✓*	$\mathbf{X}_1$
		CP3X	$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_1$

Note: Cases are identified as being always (✓), sometimes (✓*), and never ($\mathbf{X}_i$) possible, with subscripts denoting an associated failed test. Additional guidance for ✓* scenarios is given in Table 2.

stable position ever violates Condition 1. If it does, it fails Test 3 (noted by $\mathbf{X}_3$). This test assumes that a mechanism has already passed Tests 1 and 2 (the spring will provide bistability and its second stable position does not intersect the reference surface). With this assumption, there are two cases. First, if neither grounded link (links 2 and 4) changes direction between the stable states, the second stable position will be reached without either link violating Condition 1, and therefore, Tests 1 and 2 are sufficient.

In the second case, if a grounded link does change direction between the stable states, it is possible that the link may have crossed the reference surface prior to the mechanism reaching the second stable position (again, assuming the second stable position is valid, Test 2 was passed). The maximum displacement of that link must then be checked to see if that link violated Condition 1 en route to the second stable position. Links 2 and 4 will not change direction unless they reach an extreme value of θ_2 or θ_4 . These extreme values were discussed and provided in Sec. 2.4.

4.3.1 Test 3 for Grashof Mechanisms and Extramobile Motion. Using the equations in Sec. 2.4, we will show that a grounded link for a Grashof mechanism (excluding change-points) will either have no extremes in its motion (and therefore will not change direction) or it will never rotate past $\delta_{ex,max} = \pi$. The three possible types of Grashof mechanisms for Class A mechanisms are double-rockers (GRCR), crank-rockers (GCRR), and double-crankers (GCCC) [3].

GRCR: Both links 2 and 4 can reach extreme positions in their motion. The equations for the extremes of link 2 (Equations 9 and 10) directly use the $\arccos$ function, which is bounded between 0 and π , so the motion of link 2 cannot rotate more than π . The equations for link 4 (Eqs. 11 and 12) use π minus the $\arccos$ function, which still bounds the output to be between 0 and π so the motion of link 4 also cannot rotate more than π .

GCRR: Link 2 is fully revolute and therefore will not change direction between the first and second stable states. For link 4, the extremes of θ_4 are found by Eqs. (13) and (14), which are both π minus the $\arccos$ function. As discussed above, the output is bounded by 0 and π so the motion of link 4 also cannot rotate more than π .

GCCC: Both links 2 and 4 are fully revolute and therefore will not change direction between the first and second stable states.

Therefore, grounded links of Grashof mechanisms either do not change direction or will never rotate past $\delta_{ex,max} = \pi$. This means

that if a Grashof mechanism passes Test 1 and 2 for extramobile motion, it cannot fail Test 3 for extramobile motion. However, a Grashof mechanism that passed Tests 1 and 2 intramobile motion may fail Test 3 for intramobile motion because $\delta_{in,max}$ depends on the arc length of the link.

5 Analysis

The three tests are now applied to determine the geometries that always (noted by ✓), sometimes (noted by ✓*), or never (noted by $\mathbf{X}$) create an extramobile or intramobile BDM. A final summary is provided at the end of the paper in Table 1 and supplemented by a summary of ✓* cases in Table 2.

Each subsection analyzes one of the three Class A cylindrical developable mechanisms that are capable of extramobile and intramobile motion. Typically, the torsional springs will be discussed in order K1, K4, K2, and K3 since the bistable positions of K1 and K4 are found in a similar manner (as are K2 and K3). The analysis will further be broken down by Grashof criteria and link configuration, with a general notation given as “mechanism type, associated springs” (e.g., GCCC K1,K4). The relative lengths of links are also referred to throughout the work using s , l , p , and q , where s is the shortest link, l is the longest link, and p and q are the remaining links.

5.1 Class 1A. By definition, conformed Class 1A mechanisms (Fig. 1(a)) are always in an open configuration and the loop formed by the mechanism may not contain the center of the circle [3]. This constrains link 4 to be the longest link l as it subtends all other links and may not cross the center of the circle. It follows that for Class 1A mechanisms,

$$\begin{aligned} \pi/2 &\leq \theta_{2o} \leq \pi \\ \pi/2 &\leq \theta_{4o} \leq \pi \end{aligned} \quad (17)$$

5.1.1 Test 2 for All Class 1A BDMs. For conciseness, this section details Test 2 for all Class 1A BDMs. As such, it assumes that Test 1 has been passed. The second stable position for each of these cases are found by the Principle of Reflection. Figure 6 shows Class 1A mechanisms that use one dominant spring to obtain bistability.

Table 2 This table provides a summary of all $\checkmark^*$ scenarios in Table 1

Class	Mechanism	Spring	If True, $\checkmark$
1A	GCRR	K4 (intramobile)	$\theta_{4o} - \theta_{4\min,GCRR} \leq \delta_{4in, \max}$
		GRCR	$\theta_{4o} - \theta_{4\min,GRCR} \leq \delta_{4in, \max}$
		RRR4	$\theta_{2\max,RRR4} - \theta_{2o} \leq \delta_{2ex, \max}$
	CPCRR CPRCR		AND
			$\theta_{4\max,RRR4} - \theta_{4o} \leq \delta_{4ex, \max}$
		K2 (extramobile)	$\angle P132 \leq \angle P134$
			OR
			$\angle P132 > \angle P134, \theta_{2\max,RRR4} - \theta_{2o} \leq \delta_{2ex, \max}$
		K4 (intramobile)	$\theta_{4o} - \theta_{4\min,RRR4} \leq \delta_{4in, \max}$
		K4 (intramobile)	$\theta_{4o} - \theta_{4\min,GCRR} \leq \delta_{4in, \max}$
		K3 (extramobile)	$\angle P132 \leq \angle P134$
			OR
			$\angle P132 > \angle P134, \theta_{2\max,RRR4} - \theta_{2o} \leq \delta_{2ex, \max}$
2A	GCCC	K3 (extramobile)	$\theta_{2o} \geq \pi/2$
		GRCR	$\theta_{4o} - \theta_{4\min,GRCR} \leq \delta_{4in, \max}$
		CP2X-RCR	$\theta_{2o} \geq \pi/2$
		CP2X-CCR/RRC	$r_2 = r_3 = s$
			AND
	CP3X		$\theta_{2o} \geq \pi/2$
			AND
			$\theta_{4o} - \theta_{4\min,RRR4} \leq \delta_{4in, \max}$
		K3 (extramobile)	$\theta_{2o} \geq \pi/2$
		K3 (extramobile)	$\theta_{2o} \geq \pi/2$
3A	GCRR	K3 (intramobile)	$2\gamma_{13} \leq \delta_{4in, \max}$
	CP2X-CCR	K3 (intramobile)	$2\gamma_{42} \leq \delta_{4in, \max}$

K1 (Figure 6(a)): In the second stable position link 4 is exterior to the reference surface (intramobile K1: $\mathbf{X}_2$).

The second stable position of link 4 lies at an angular displacement of $2\angle P243$ (rotated counter-clockwise). Thale's Theorem shows that $\angle P243 < \pi/2$, so link 4 cannot be past $\delta_{4ex, \max}$. Hence, it passes Test 2 for extramobile motion.

K4 (Fig. 6(b)): In the second stable position link 2 is rotated toward the center of the circle (extramobile K4: $\mathbf{X}_2$).

For the second stable position of link 2 to be located further than $\delta_{2in, \max}$, d_{13} must cross through the center of the circle, but this cannot happen due to Eq. (17) (link 4 may not cross the center of the circle). So, K2 passes Test 2 for intramobile motion.

K2 (Fig. 6(c)): In the second stable position, links 2 and 4 are exterior to the reference surface (intramobile K2: $\mathbf{X}_2$). Link 2's second stable position is located at a rotation of $2\gamma_{13}$ (rotated counter-clockwise). By Thale's Theorem, $\gamma_{13} \geq \pi/2$; so, it is not past $\delta_{2ex, \max}$. The second stable position of link 4 is found by a reflection of r_4 across r_1 , so $\theta_{4b} = -\theta_{4o}$ (link 4 rotated counter-clockwise). Because $\theta_{4o} \geq \pi/2$ (Eq. (17)), the second stable position for link 4 will not be past $\delta_{4ex, \max}$. So, it passes Test 2 for extramobile motion.

K3 (Fig. 6(d)): In the second stable position links 2 and 4 are exterior to the reference surface (intramobile K3: $\mathbf{X}_2$). The second stable position of link 2 is found by a reflection of r_2 across r_1 , so $\theta_{2b} = -\theta_{2o}$ (link 2 rotated counter-clockwise). Because $\theta_{2o} \geq \pi/2$ (Eq. (17)), the position of link 2 cannot exceed $\delta_{2ex, \max}$. The second stable position of link 4 is located at a rotation of $2\gamma_{42}$ (counter-clockwise). By Thale's Theorem, $\gamma_{42} < \pi/2$; so, the location of link 4 cannot exceed $\delta_{4ex, \max}$. Hence, it passes Test 2 for extramobile motion.

5.1.2 Grashof. There exist three possible mechanism configurations for a Grashof Class 1A mechanism: a double-crank ($r_1 = s$), a crank-rocker ($r_2 = s$), and a double-rocker ($r_3 = s$). Theorem 1 states that a torsional spring must be placed opposite s for bistability to be possible. We can therefore use Test 1 to eliminate all springs attached to s (GCCC K1, K4: $\mathbf{X}_1$) (GCRR K1, K2: $\mathbf{X}_1$) (GRCR K2, K3: $\mathbf{X}_1$).

Each mechanism configuration for Class 1A Grashof mechanisms will now be analyzed to identify when intramobile and extramobile BDMs are possible. Note that Test 2 was already completed in Sec. 5.1.1. Grashof mechanisms cannot fail Test 3 for extramobile motion, as discussed in Sec. 4.3.1. Therefore, all the remaining extramobile candidates (those that passed Tests 1-3) are always

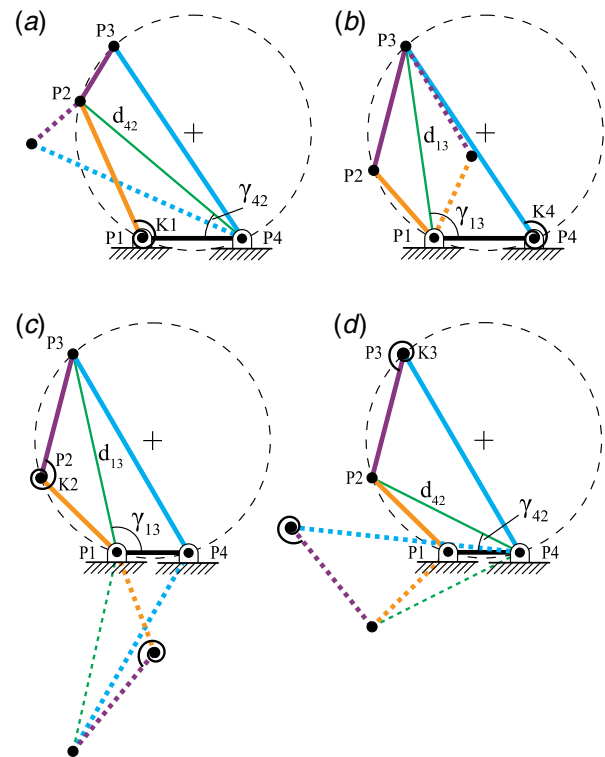


Fig. 6 Class 1A mechanisms shown in their conformed (solid) and second stable (dashed) positions: (a) with spring K1, (b) with spring K4, (c) with spring K2, and (d) with spring K3

extramobile BDMs (extramobile GCCC K2,K3: ✓) (extramobile GCRR K3: ✓) (extramobile GRRCR K1: ✓).

The two remaining intramobile candidates are now discussed.

Test 3 for intramobile GCRR K4. After Tests 1 and 2 for the crank-rocker, K4 is still a candidate for an intramobile BDM. Test 3 shows that link 2 has no extreme positions; so, it will not change directions. r_2 and r_3 must align to reach the second stable state, causing θ_4 to reach its minimum value θ_{4min} (see Eq. 13). Therefore, Test 3 requires that any Class 1A crank-rockers using K4 to be checked to ensure $\theta_{4o} - \theta_{4min,GCRR} \leq \delta_{4in,max}$ to guarantee an intramobile BDM (intramobile GCRR K4: ✓*).

Test 3 for intramobile GRRCR K4. After Tests 1 and 2 for the double-rocker, K4 is still a candidate for an intramobile BDM. The results for K4 are identical to the previous section (GCRR), meaning that $\theta_{4o} - \theta_{4min,GRRCR}$ must be less than or equal to $\delta_{4in,max}$ to guarantee an intramobile BDM (intramobile GRRCR K4: ✓*).

5.1.3 Non-Grashof. Since link 4 is constrained to be l for all Class 1A mechanisms, only one type of non-Grashof mechanism (RRR4) exists in Class 1A. According to Theorem 2 of Test 1, non-Grashof mechanisms will always be bistable as long as the links opposite the torsional spring are not collinear when the spring is undeflected (when $\psi=0$). Class 1A mechanisms are unable to have any links collinear in the conformed position, so all Class 1A non-Grashof BDMs pass Test 1.

The Test 2 analysis from Sec. 5.1.1 applies to non-Grashof mechanisms, thereby eliminating K4 extramobile BDMs and K1 - K3 intramobile BDMs. We will now check Test 3.

Test 3 for intramobile RRR4 K4. After Tests 1 and 2, K4 is still a candidate for an intramobile RRR4 BDM. To reach the second stable position for K4, the mechanism must reach a toggle position and move back to the initial positions of link 4. When link 2 is at its minimum angular value ($\theta_{2min,RRR4}$), P2 is in contact with link 4. This means that the most extreme displaced position of link 2 can also be reached by link 4. We therefore only must check the displacement of link 4. Using Eq. (7), $\theta_{4o} - \theta_{4min,RRR4}$ must be less than or equal to $\delta_{4in,max}$ to guarantee the second stable position may be reached while interior to the reference surface (intramobile RRR4 K4: ✓*).

Test 3 for extramobile RRR4 K1, K2, K3. After Tests 1 and 2, K1, K2, and K3 are all still candidates for extramobile RRR4 BDMs.

K1: The rotation of both links 2 and 4 must be checked to ensure neither $\theta_{2max,RRR4} - \theta_{2o}$ or $\theta_{4max,RRR4} - \theta_{4o}$ (using Eqs. (6) and (8)) exceeds $\delta_{ex,max}$ (extramobile RRR4 K1: ✓*).

K2: We will first show that link 4 never changes direction prior to reaching the stable position. $\angle P123$ (the angle between r_2 and r_3) decreases as the mechanism moves exterior to the reference surface and reaches a minimum value when r_4 and r_1 are collinear. It then increases in value until link 4 reaches its maximum angular displacement $\theta_{4max,RRR4}$, when r_2 and r_3 are collinear. Between these two positions, $\angle P123$ is guaranteed to have reached its initial magnitude. Therefore, the mechanism will always reach the second stable position for K2 prior to reaching $\theta_{4max,RRR4}$.

Link 2 is capable of reaching its maximum displacement $\theta_{2max,RRR4}$ prior to the second stable position. If this does not happen, the mechanism is guaranteed to be an extramobile BDM (because it has already passed Test 2). However, if $\theta_{2max,RRR4}$ is reached, we must check to ensure $\theta_{2max,RRR4} - \theta_{2o} \leq \delta_{ex}$.

Figure 7 shows two RRR4 mechanisms in their conformed and second stable positions. The toggle position for link 2 occurs when r_2 and r_3 become collinear. It can then be seen that the mechanism in Fig. 7(a) has not yet reached the toggle position of link 2 and therefore is guaranteed to be an extramobile BDM. The mechanism in Fig. 7(b) must have had r_3 cross r_4 to reach its second stable position; so, we must check if $\theta_{2max,RRR4} - \theta_{2o} \leq \delta_{ex}$.

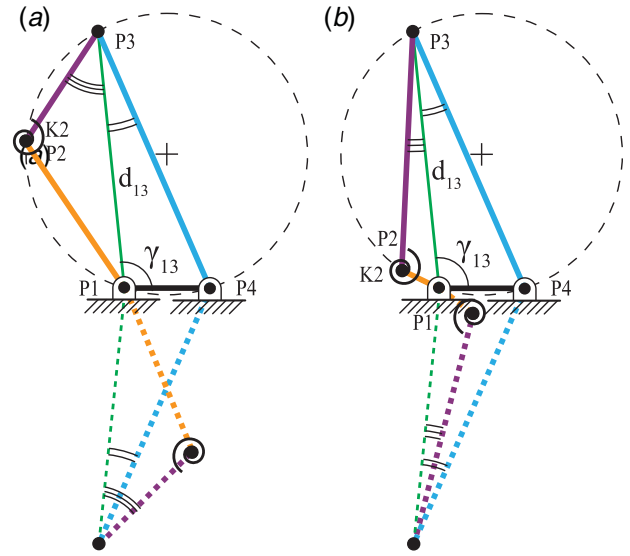


Fig. 7 Two possible bistable positions for RRR4 mechanisms using K2 are shown. In (a), the mechanism reaches its second stable position without link 2 changing direction. In (b), link 2 has reached a maximum and changed direction before reaching the second stable position. (a) With spring K2 and (b) with spring K2.

A quick check can be made of a K2 RRR4 mechanism in its conformed position to see which of the two cases (Fig. 7(a) or 7(b)) the mechanism is. In case (a), $\angle P132 \geq \angle P134$ while in case (b), $\angle P132 < \angle P134$. Because these angles are the same in the second stable position, it can be seen that their relative magnitude demonstrates if r_3 and r_4 have passed their collinear position.

In summary, if $\angle P132 \geq \angle P134$, K2 RRR4 mechanisms are guaranteed to be an extramobile BDM. If $\angle P132 < \angle P134$, $\theta_{2max,RRR4} - \theta_{2o}$ must be less than or equal to $\delta_{ex,max}$ to ensure a viable extramobile BDM (extramobile RRR4 K2: ✓*).

K3: $\angle P234$ (the angle between r_3 and r_4 , see Fig. 6(d)) increases as the mechanism moves exterior to the reference surface and reaches a maximum value when r_2 and r_1 are collinear. It then decreases in value until link 2 reaches its maximum angular displacement $\theta_{2max,RRR4}$, when r_3 and r_4 are collinear. Between these positions, $\angle P234$ is guaranteed to have reached its initial magnitude. Therefore, the mechanism will always reach the second stable position for K2 prior to reaching $\theta_{2max,RRR4}$.

Because $l = r_4$, r_2 reaches its maximum prior to r_4 , meaning that $\theta_{4max,RRR4}$ is never reached prior to the second stable position. Therefore, the K3 RRR4 mechanism is guaranteed to be an extramobile BDM (extramobile RRR4 K3: ✓).

5.1.4 Change Point. Change-point mechanisms are unique due to their ability to move between their open and crossed circuits without disassembly, and their capacity to be fabricated monolithically with a single surface [22]. To reach a change-point position (position where all pin joints lie in a line), a mechanism must satisfy

$$s + l = p + q \quad (18)$$

Because $\theta_{1o} = 0$ and $r_4 = l$ (requirement for Class 1A), there are three possible Class 1A change-point mechanisms: CPCCC, CPCRR, and CPRCR. Each of these have only one change-point position, and since no links are collinear in the initial position, they all pass Test 1. Test 2 was already completed for all Class 1A mechanisms (Sec. 5.1.1). Hence, after Tests 1 and 2, mechanisms with springs K1–K3 are still candidates for extramobile BDMs, while mechanisms with spring K4 are still candidates for intramobile BDMs. Test 3 will now be discussed for these remaining mechanisms, organized by change-point type.

CPCCC. For a CCCCC, the link lengths and the positions of these links at the change-point position are

$$\begin{aligned} r_1 &= s; \quad r_2 = p; \quad r_3 = q; \quad r_4 = l \\ \text{CP Position 1: } \theta_1 &= 0; \quad \theta_2 = 0; \quad \theta_3 = 0; \quad \theta_4 = 0 \end{aligned} \quad (19)$$

The mechanism cannot reach this change-point position without either link 2 or 4 exceeding $\delta_{ex,max}$ or $\delta_{in,max}$. Therefore, any additional stable positions are not accessible without violating Condition 1, and the possible extramobile and intramobile BDMs are the same as for a double-crank (extramobile CCCCC K1,K4: $\mathbf{X}_3$, extramobile K2,K3: $\checkmark$, extramobile K4: $\mathbf{X}_3$).

CPCRR. The change-point position for a CPCRR mechanism occurs when the position of each link is

$$\begin{aligned} r_1 &= p; \quad r_2 = s; \quad r_3 = q; \quad r_4 = l \\ \text{CP Position 1: } \theta_1 &= 0; \quad \theta_2 = 0; \quad \theta_3 = \pi; \quad \theta_4 = \pi \end{aligned} \quad (20)$$

To reach this position, link 2 must move interior while link 4 must move exterior to the reference surface. Additional stable positions can therefore not be reached without violating either Condition 1 or 2. Consequentially, CPCRR mechanisms do not reach any additional stable positions than the crank-rocker (extramobile CPCRR K1,K2: $\mathbf{X}_3$, K3: $\checkmark$, intramobile K4: $\checkmark^*$).

CPRCR. The positions of each link in the CPRCR mechanism are given by

$$\begin{aligned} r_1 &= p; \quad r_2 = q; \quad r_3 = s; \quad r_4 = l \\ \text{CP Position 1: } \theta_1 &= 0; \quad \theta_2 = \pi; \quad \theta_3 = 0; \quad \theta_4 = \pi \end{aligned} \quad (21)$$

This change-point position may only be reached while exterior to the reference surface, meaning any changes from the GRCR case do not apply to intramobile mechanisms (intramobile K4: $\checkmark^*$). We will therefore only look at the extramobile case.

At the change-point position, the mechanism may move upwards and downwards in either its open and crossed configurations. If the mechanism moves up toward the reference surface, it follows the same logic presented for the GRCR mechanism, allowing K1 to pass Test 3 (extramobile CPRCR K1: $\checkmark$).

If the mechanism moves downward in its crossed configuration, the mechanism follows the same logic as was presented for the extramobile triple-rocker. The mechanism always passes Test 2 for both moving springs and always passes Test 3 for K3 (extramobile CPRCR K3: $\checkmark$). It therefore fails Test 2 for the intramobile K2 and K3. For K2, $\angle P132$ must be greater than $\angle P134$ to guarantee an extramobile BDM. If $\angle P132 < \angle P134$, $\theta_{2max,RRR4}$ must be less than $\delta_{ex,max}$ to guarantee an extramobile BDM (extramobile CPRCR K2: $\checkmark^*$).

5.2 Class 2A. Class 2A mechanisms are linkages whose two grounded links (r_2 and r_3) are crossed in the conformed position (Fig. 1(b)). Hyatt et al. demonstrated that any crossed cyclic quadrilateral will result in a Grashof mechanism (including change-points) [23]. Therefore, the discussion on Class 2A mechanisms need not discuss non-Grashof mechanisms.

To remain a Class 2A mechanism, the area enclosed between the four links may not contain the center of the circle [3]. Because $\theta_{1o} = 0$, both grounded links must remain on the left side of the center of the circle. The initial angular position of each grounded link must then reside with the ranges

$$\begin{aligned} 0 &\leq \theta_{2o} \leq \pi \\ \pi/2 &\leq \theta_{4o} \leq \pi \end{aligned} \quad (22)$$

5.2.1 Grashof Mechanisms. There are two possible configurations of Grashof Class 2A mechanisms. Because link 4 always subtends link 1, and link 2 always subtends link 3, either link 1 or link 3

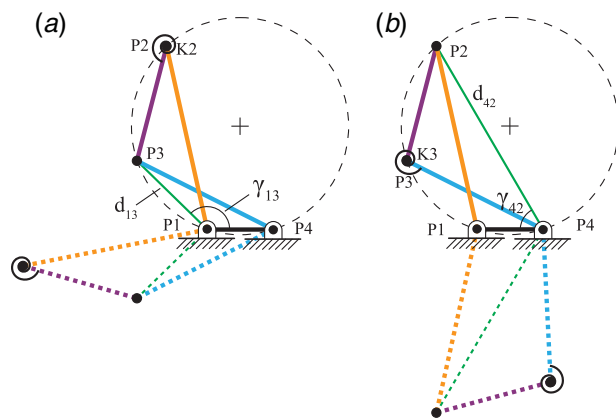


Fig. 8 Class 2A GCCC mechanisms shown in their conformed (solid) and second stable (dashed) positions: (a) with spring K2 and (b) with spring K3

will always be s . These configurations lead to a double-crank and a double-rocker.

Double-Crank (GCCC). By Test 1, the double-crank ($r_1 = s$) cannot use spring K1 or K4 (GCCC K1,K4: $\mathbf{X}_1$). We will therefore analyze the use of K2 and K3, as shown in Fig. 8. The second stable positions are found by the Principle of Reflection.

K2 (Fig. 8(a)): In the second stable position, links 2 and 4 are exterior to the reference surface (intramobile GCCC K2: $\mathbf{X}_2$). At this position, link 2 has undergone a net rotation (counter-clockwise) of $2\gamma_{13}$. By Thale's Theorem, $\gamma_{13} > \pi/2$, so link 2 cannot be past $\delta_{2,ex,max}$. The second stable position of link 4 is when $\theta_{4b} = -\theta_{4o}$ (net rotation counter-clockwise). Equation (22) shows that $\theta_{4o} \leq \pi$, so link 4 cannot be past $\delta_{4,ex,max}$. Hence, K2 passes Test 2 for extramobile motion. Grashof mechanisms cannot fail Test 3 in extramobile motion (extramobile GCCC K2: $\checkmark$).

K3 (Fig. 8(c)): In the second stable position, links 2 and 4 are exterior to the reference surface (intramobile GCCC K3: $\mathbf{X}_2$). Here, $\theta_{2b} = -\theta_{2o}$ (net rotation counter-clockwise). Since θ_{2o} can be less than $\pi/2$ (Eq. (22)), θ_{2o} must be checked to ensure link 2 will be exterior to the surface to pass Test 2. The second stable position of link 4 is a net rotation (counter-clockwise) of $2\gamma_{42}$. By Thale's Theorem, $\gamma_{42} \leq \pi$ (r_2 cannot cross the center of the circle), so link 4 cannot be past $\delta_{4,ex,max}$. As above, Grashof mechanisms in extramobile motion cannot fail Test 3. Therefore (extramobile GCCC K3: $\checkmark^*$), depending on the result from Test 2.

Double-Rocker (GRCR). Using Test 1 for double-rockers ($r_3 = s$), we can eliminate springs K2 and K3 (GRCR K2,K3: $\mathbf{X}_1$). We will therefore analyze the use of K1 and K4, as shown in Fig. 9. The second stable positions are found by the Principle of Reflection.

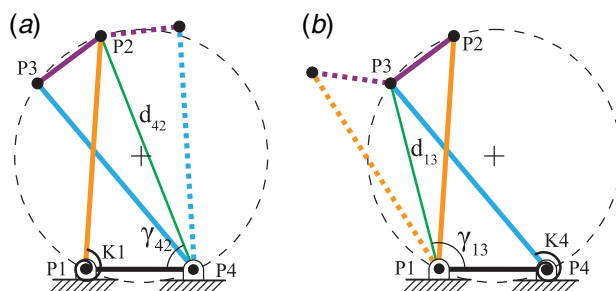


Fig. 9 Class 2A GRGR mechanisms shown in their conformed (solid) and second stable (dashed) positions: (a) with spring K1 and (b) with spring K4

K1 (Fig. 9(a)): The second stable position exists when r_4 is rotated toward the center of the circle (extramobile GRCR K1: $\mathbf{X}_2$). At this position, link 2 has undergone a net rotation (clockwise) of $2\angle P243$. To pass Test 2 for intramobile motion, $\angle P243$ must be checked to see if it is less than $\delta_{4in, \max}/2$. Test 3 shows that r_3 and r_4 must align prior to the second stable position. If $\theta_{4o} - \theta_{4min, GRCR} \leq \delta_{4in, \max}$, it will pass Test 3 and be an intramobile BDM. Note that the corresponding $\theta_{4min, GRCR}$ always supersedes the checking of $\angle P243$. Using the same logic as the Class 1A GRCR mechanism, the extreme positions of θ_2 do not need to be checked. Hence, (intramobile GRCR K1: $\checkmark^*$) depending on Test 3 for link 4.

K4 (Fig. 9(b)): In the second stable position, link 2 is exterior to the reference surface (intramobile GRCR K4: $\mathbf{X}_2$). Here, link 2 is located at a net rotation (counter-clockwise) of $2\angle P312$. By Thale's Theorem, $\angle P312 \leq \pi/2$, so link 2 cannot be past $\delta_{2, \max}$, and this mechanism passes Test 2. Grashof mechanisms cannot fail Test 3 for extramobile motion (extramobile GRCR K4: $\checkmark$).

5.2.2 Change-Point Mechanisms. The three different configurations of Class 2A change-point mechanisms (CP2X-RCR, CP2X-CCR/RRC, and CP3X) are shown in Fig. 10. All of these mechanisms have at least two pairs of links that have equal length, allowing them to have more than one change-point position.

CP2X-RCR. The CP2X-RCR mechanism (Fig. 10(a)) has no links collinear in the conformed position and therefore passes Test 1. This mechanism has all links collinear in the following two positions:

$$\begin{aligned} r_1 = r_3 = s; \quad r_2 = r_4 = l \\ \text{CP Position 1: } \theta_1 = 0; \quad \theta_2 = \pi; \quad \theta_3 = 0; \quad \theta_4 = \pi \\ \text{CP Position 2: } \theta_1 = 0; \quad \theta_2 = 0; \quad \theta_3 = 0; \quad \theta_4 = 0 \end{aligned} \quad (23)$$

CP Position 1 can be achieved during extramobile motion because link 2 and link 4 are exterior to the reference surface. In CP Position 2, link 2 has moved inside the reference surface and has then violated Condition 1 (since $r_2 > r_1$). Tests 2 and 3 will now be completed for each dominant torsional spring.

K1: For Test 2, the same logic as the Class 2A GRCR applies (extramobile CP2X-RCR K1: $\mathbf{X}_2$). To achieve bistability using K1, r_3 and r_4 must align. Since neither CP position 1 or 2 can occur during intramobile motion, this mechanism fails Test 3 (intramobile CP2X-RCR K1: $\mathbf{X}_3$).

K4: For Test 2, the same logic as the Class 2A GRCR applies (intramobile CP2X-RCR K4: $\mathbf{X}_2$). Test 3 shows that r_2 and r_3 must align prior to the second stable position for K4. As shown above, CP Position 1 can be achieved in extramobile motion; so, this mechanism passes Test 3 (extramobile CP2X-RCR K4: $\checkmark$).

K2: For Test 2, the same logic as the Class 2A GRCR applies (intramobile CP2X-RCR K2: $\mathbf{X}_2$). The mechanism must pass through one of the two CP Positions prior to the second stable position for K2. As shown above, CP Position 1 can be achieved in extramobile motion; so, this mechanism passes Test 3 (extramobile CP2X-RCR K2: $\checkmark$).

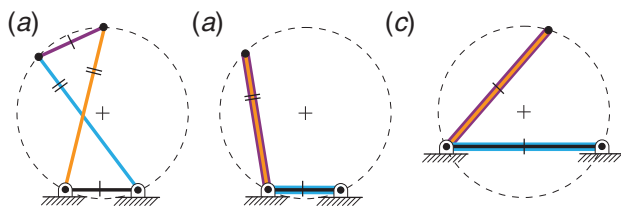


Fig. 10 Class 2A change-point mechanisms in their conformed positions. Hatch marks indicate equal lengths: (a) CP2X-RCR, (b) CP2X-CCR/RRC, and (c) CP3X.

K3: For Test 2, the same logic as the Class 2A GRCR applies (intramobile CP2X-RCR K3: $\mathbf{X}_2$), including the fact θ_{2o} must be checked to ensure link 2 will be exterior to the surface. For the same reason as K2, this mechanism passes Test 3. Hence, (extramobile CP2X-RCR K3: $\checkmark^*$) depending on the result from Test 2.

CP2X-CCR/RRC and CP3X. The analysis for CP2X-CCR/RRC and CP3X is similar and will be discussed together, with differences noted. Both of these types of mechanisms require links to be collinear in the conformed position. Because Theorem 3 states that bistability cannot be achieved if links are collinear when opposite an undeflected torsional spring, K2 and K4 fail Test 1 (CP2X-CCR/RRC K2, K4: $\mathbf{X}_1$; CP3X K2, K4: $\mathbf{X}_1$).

These mechanisms reach change-point positions when

$$\begin{aligned} \text{CP2X - CCR/RRC: } r_1 = r_4; \quad r_2 = r_3 \\ \text{CP3X: } r_1 = r_2 = r_3 = r_4 \\ \text{CP Position 1: } \theta_1 = 0; \quad \theta_2 = \pi; \quad \theta_3 = 0; \quad \theta_4 = \pi \\ \text{CP Position 2: } \theta_1 = 0; \quad \theta_2 = 0; \quad \theta_3 = \pi; \quad \theta_4 = \pi \\ \text{CP Position 3 (CP3X): } \theta_1 = 0; \quad \theta_2 = 0; \quad \theta_3 = 0; \quad \theta_4 = 0 \end{aligned} \quad (24)$$

where CP Position 3 only applies to the CP3X mechanism. CP position 1 lies outside the surface and can be reached by extramobile motion. CP position 2 lies interior to the surface and can be reached by intramobile motion only if links 2 and 3 are s and links 1 and 4 are $l > s$ (which is by definition the CP2X-CCR mechanism). For example, the mechanism in Fig. 11(a) passes Test 2 but may never reach this position as r_2 and r_3 cannot become collinear with r_1 and r_4 without exceeding $\delta_{4in, \max}$. CP position 3 (only applicable to CP3X) also lies interior to the surface, but the mechanism cannot unfold into a kite directly from this position (due to sequential folding).

Because links of equal length are adjacent, the second stable positions using K1 and K3 are always in a kite shape (see Fig. 11).

K1 (Fig. 11(a)): To pass Test 2, K1's kite must fit within the reference surface (extramobile CP2X-CCR/RRC K1: $\mathbf{X}_2$; extramobile CP3X K1: $\mathbf{X}_2$). This happens if $\theta_{2o} \geq \pi/2$, which is possible, but not required (by Thale's Theorem). Test 3 shows that the mechanism must reach a CP Position prior to reaching the second stable (kite) position. Only CP Position 2 can be reached during intramobile motion and unfold to a kite (and when $r_2 = r_3 = s$ and $r_1 = r_4 = l > s$, as discussed above). All links in the CP3X mechanism are the same length; so, it does not meet this criteria and fails Test 3 (CP3X intramobile K1: $\mathbf{X}_3$). The CP2X-CCR/RRC mechanism can meet this criteria.

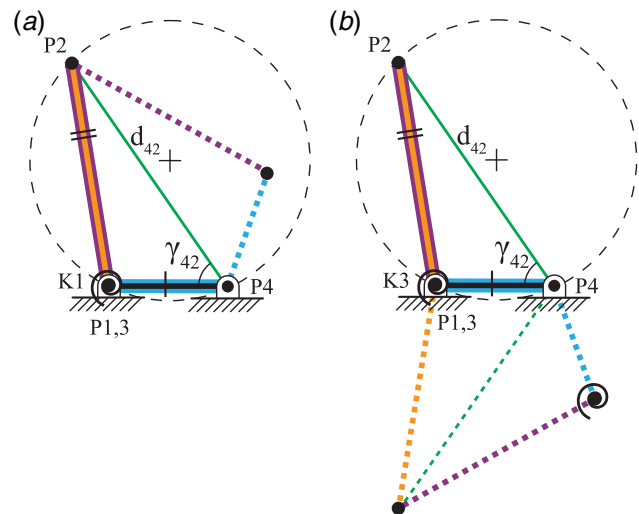


Fig. 11 Class 2A CP2X-RRC mechanism in its initial and second stable positions: (a) with spring K1 and (b) with spring K3

Test 3 also shows that r_2 and r_3 must align (extended) prior to the second stable position, so the minimum position of θ_4 (which is equivalent to $\theta_{4min,RRR4}$) must be checked to ensure it does not cross the reference surface: $\theta_{4o} - \theta_{4min} < \delta_{4in, max}$. Hence, it will be an intramobile BDM, depending on the results from Test 2 and Test 3 (CP2X-CCR/RRC intramobile K1: ✓*).

K3 (Fig. 11(b)): It can be seen that the second stable position occurs when links 2 and 4 are exterior to the surface (intramobile CP2X-CCR/RRC K3: ✗₂; intramobile CP3X K3: ✗₂). To pass Test 2, K3's kite must be entirely exterior the surface: θ_{2o} must be greater than or equal to $\pi/2$.

For Test 3, CP position 3 lies outside the surface and can be reached by extramobile motion, as discussed earlier. Other than at this position, neither r_2 and r_3 nor r_3 and r_4 will align prior to the second stable configuration; so, this mechanism passes Test 3 assuming it has passed Test 2 (extramobile CP2X-CCR/RRC and CP3X K3: ✓*).

5.3 Class 3A. Class 3A mechanisms are linkages whose coupler (r_3) crosses ground (r_1) in the conformed position (Fig. 1(c)). In order to remain a Class 3A mechanism, the entire mechanism must be on one side of the circle. Hence,

$$\begin{aligned} 3\pi/2 \leq \theta_{2o} \leq 2\pi \\ \pi/2 \leq \theta_{4o} \leq \pi \end{aligned} \quad (25)$$

Because Class 3A mechanisms are always crossed, they must always be Grashof mechanisms [23]. It can also be seen that r_2 is subtended by all other links, making it s for all non-change-point mechanisms. These constraints therefore require all Class 3A mechanisms to be either a crank-rocker or change-point mechanism.

5.3.1 Grashof Mechanisms

Crank-Rocker (GCRR). Test 1 eliminates the possibility of using K1 or K2 for bistability because they are always attached to s (GCRR K1,K2: ✗₁). We will therefore analyze the use of K4 and K3, as shown in Fig. 12. The second stable positions are found by the Principle of Reflection.

K4 (Fig. 12(a)): In the second stable position, link 2 is exterior to the surface (intramobile GCRR K4: ✗₂). Here, link 2 is located at a net rotation (clockwise) of $2\angle P213$. By Thale's Theorem, $\angle P213 \geq \pi/2$; so, link 2 cannot be past $\delta_{2 ex, max}$ and this mechanism passes Test 2. Grashof mechanisms cannot fail Test 3 for extramobile motion (extramobile GCRR K4: ✓). **K3** (Fig. 12(b)): Link 2 must be rotated toward the center of the circle in the second stable position (extramobile GCRR K3: ✗₂). The second stable position for link 2 is a reflection of r_2 across r_1 . Since r_1 must be below the center of the circle, link 2 for this mechanism always passes Test 2. For link 4 to pass Test 2 for intramobile motion, its position in the second stable position (a net rotation of $2\gamma_{42}$) must be checked to ensure its displacement is less than $\delta_{4 in, max}$.

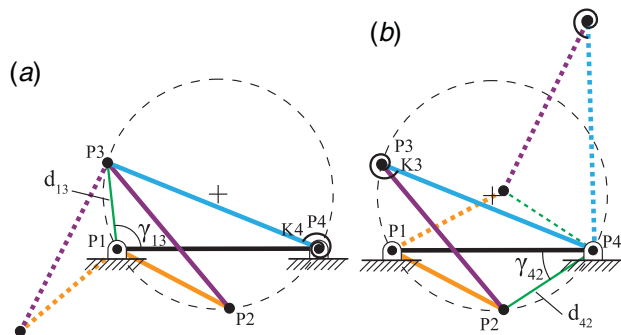


Fig. 12 Class 3A mechanisms shown in their conformed (solid) and second stable (dashed) positions: (a) with spring K4 and (b) with spring K3

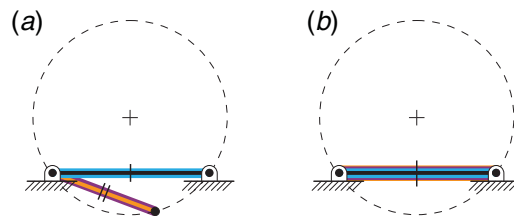


Fig. 13 Class 3A change-point mechanisms in their conformed positions. Hatch marks indicate equal lengths. (a) CP2X-CCR and (b) CP3X.

For Test 3, link 2 is fully revolute and does not change directions. Furthermore, Test 3 shows that θ_4 will not reach an extreme minimum value prior to the second state. This is because, in both stable states, the angle between links 2 and 3 remains less than π (r_2 and r_3 are never collinear prior to reaching the second stable position). By Thale's Theorem, $\angle P124 \geq \pi/2$, $\angle P324 \geq \pi/2$, and $\angle P123$ (the angle between links 2 and 3) is less than π in the first stable position. By the Principle of Reflection, in the second stable position, the angle between links 2 and 3 is equal to $2\pi - (\angle P124 + \angle P324)$, which must also be less than π . Therefore, r_2 and r_3 cannot align (and θ_4 cannot reach its maximum) prior to the second stable position using K3. Hence, this mechanism may be an intramobile BDM, depending on the result for Test 2 (intramobile GCRR K3: ✓*).

5.3.2 Change-Point Mechanisms. The two possible change-point mechanisms in Class 3A are a CP2X-CCR and a CP3X mechanism, as shown in Fig. 13. Their change-point positions are the same as those for the CP2X-CCR/RRC and CP3X in Class 2A (Eq. (24)).

CP2X-CCR. The CP2X-CCR mechanism is unable to use K2 and K4 as they are opposite collinear links (CP2X-CCR K2,K4: ✗₁).

The second stable positions for each spring can be found with the Principle of Reflection and are shown in Fig. 14. As was the case in Class 2A, these mechanisms form a kite.

K1 (Fig. 14(a)): Link 2 is exterior to the reference surface in the second stable position (intramobile CP2X-CCR K1: ✗₂). In this second stable position, it has undergone a net rotation of $2\gamma_{42}$ (counter-clockwise). By Thale's Theorem, $\gamma_{42} < \pi/2$; so, it cannot be past $\delta_{2 ex, max}$ and therefore passes Test 2 for extramobile motion. CP position 1 can be reached through extramobile motion because at this point link 2 will never have rotated more than π radians (Eq. (25)). From CP position 1, the mechanism can unfold into the kite configuration, but must pass the toggle position when r_2 and r_3 align, causing θ_{4max} . However, the equation for $\theta_{4max, CP2X-CCR} = \theta_{4max, RRR4}$ and is therefore bounded by 0 and π .

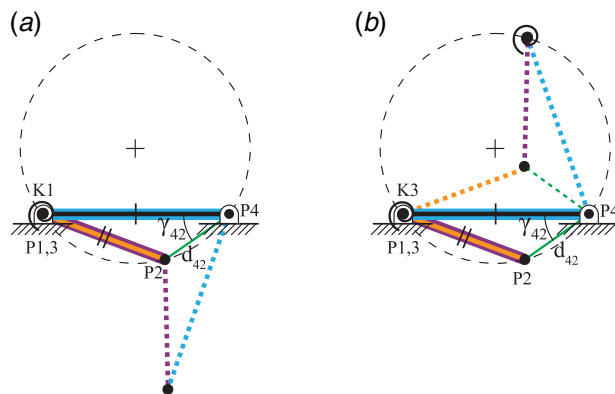


Fig. 14 Class 3A CP2X-CCR mechanism in its second stable positions: (a) with spring K1 and (b) with spring K3

Hence, this mechanism passes Test 3 (extramobile CP2X-CCR K1: ✓).

K3 (Fig. 14(b)): Link 4 is rotated toward the center of the circle in the second stable position (extramobile CP2X-CCR K3: ✗₂). This mechanism will pass Test 2 for intramobile motion if $\angle P243$ (which equals γ_{42}) is less than $\delta_{in}/2$. For Test 3, the mechanism must use change-point position 2 to remain interior to the reference surface. From this position, no link will change direction until it reaches the second stable position. Therefore, assuming the mechanism has passed Test 2, it will pass Test 3 (intramobile CP2X-CCR K3: ✓*).

CP3X. Because the CP3X mechanism requires all four links to be collinear in the conformed position (which is the first stable position), it fails Test 1 (CP3X K1,K2,K3,K4: ✗₁).

6 Discrete Sampling and Demonstration

A sampling of cylindrical developable mechanisms throughout the design space were numerically simulated and analyzed. This section is not intended to be a proof, but instead is used as a demonstration of the principles explored in the preceding sections. The simulation also allows for useful visualization of BDMs, their motion, and how their strain energy changes as they move. Accordingly, this section also includes two of the simulated mechanisms as examples that succinctly illustrate the approach and outcomes of this paper.

6.1 Generating Mechanism Samples. The design space for Class 1A, 2A, and 3A mechanisms can be represented by the non-dimensional arc lengths of links 1, 2, and 3 [3]. This design space was systematically sampled in 7.5 deg increments to create 8096 mechanisms capable of intramobile and extramobile motion (2024 in Class 1A, 4048 in Class 2A, 2024 in Class 3A). Note that change-point mechanisms were not included due to the bifurcations in their motion path.

The motion of each mechanism was simulated using a simple MATLAB script based on four-bar linkages. This script mapped the position of the curved links as the device moved through its

motion (both intramobile and extramobile). It was able to determine when the mechanism violated Condition 1 or 2 by using Eqs. (1) and (2). The motion was also correlated with the expected strain energy due to the deflection of dominant torsional spring K1 (Equation 4). By tracking the changes in strain energy, it would identify if/when the mechanism had reached a minimum in the strain energy function (the second stable position). If this occurred after the mechanism violated Condition 1 or 2, the mechanism configuration is not a valid BDM. Otherwise, it is. This was then repeated for springs K2, K3, and K4.

In all, 32,384 mechanism configurations (four each for 8096 mechanisms) were simulated. Each binary result from the simulation (i.e., valid or invalid BDM) was compared with the expected result found from Tables 1 and 2. All mechanism configurations returned results in agreement with those provided in Tables 1 and 2.

6.2 Examples. We now provide two example mechanisms to provide a holistic demonstration of how different mechanism configurations may or may not create an extramobile or intramobile BDM. The mechanisms discussed in this section were arbitrarily selected from the simulated mechanisms in Sec. 6.1. For both examples, the motion of each mechanism is represented along the x -axis, where both ends of the axis represent the conformed position.

6.2.1 Class 2A GCCC. Our first example is that of a Class 2A GCCC mechanism. From Table 1, we find that Test 1 has eliminated springs K1 and K4 from consideration, and Test 2 has eliminated K2 and K3 for intramobile BDMs. Spring K2 should always create an extramobile BDM, and K3 may create an extramobile BDM as long as $\theta_{2o} \geq \pi/2$.

Figure 15 shows the analysis for the simulated Class 2A GCCC mechanism. Subfigures (a) and (b) show the analysis of Conditions 1 and 2. While traveling exterior to the surface (left to right), link 2 violates the Conditions (in this case, Condition 1 because it passed $\delta_{ex,max}$) before link 4. While traveling interior to the surface (right to left), link 2 again violates the Conditions before link 4. These become the limits of intramobile and extramobile motion, respectively, in Subfigure (c), which shows the strain energy curves for springs K1–K4.

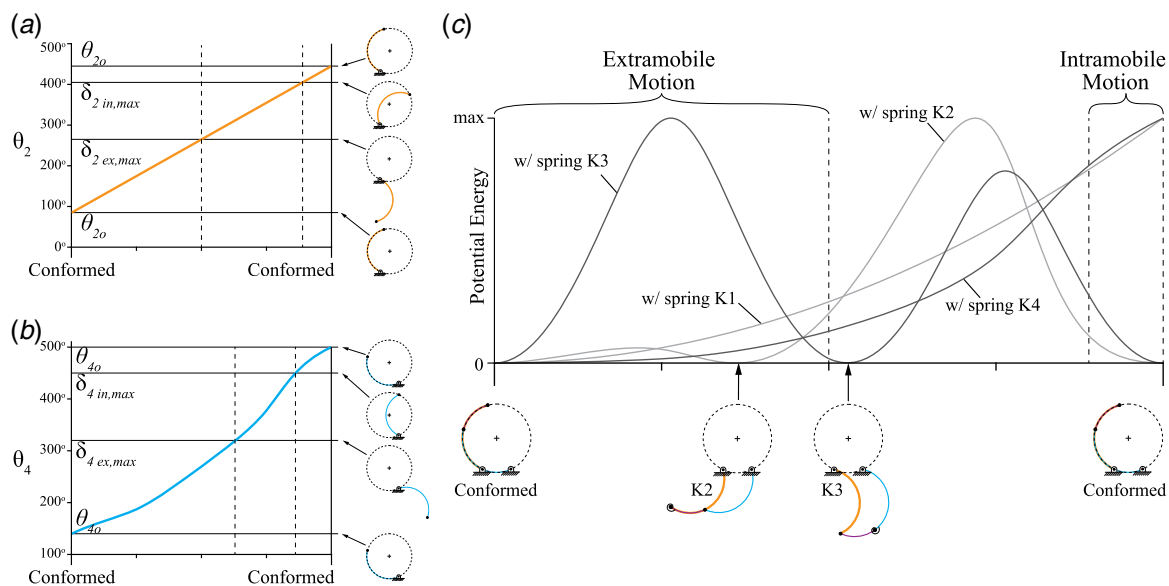


Fig. 15 Subfigures (a) and (b) show θ_2 and θ_4 , respectively, and their corresponding limits for extramobile and intramobile motion for a simulated Class 2A GCCC mechanism. From these, we find where link 2 or 4 violates Conditions 1 or 2 (marked with dashed lines). These translate to the limits of extramobile and intramobile motion in subfigure (c). K1 and K4 cannot produce a BDM because they only have one minimum. K2 produces a valid extramobile BDM, but K3 does not (its stable position is outside the range of extramobile motion). (a) Motion of link 2, (b) motion of link 4, (c) energy curves along the full range of motion of the mechanism, where each curve corresponds to a different dominant torsional spring. Stable states are also noted and shown.

There is only one minimum in the strain energy using K1 and K4 (and therefore no second stable position, as predicted by Test 1). For K2, the second stable position occurs while the mechanism is in extramobile motion (therefore precluding it from creating an intramobile BDM, as was predicted by Test 2). While K3 also has a second stable position, this occurs outside both intramobile and extramobile motions. The conformed position of link 2, seen in both Subfigures (a) and (c), has an orientation less than $\pi/2$ (in agreement with Table 2). Therefore, this mechanism with K2 is an extramobile BDM. It is not an intramobile BDM with any spring.

6.2.2 Class 1A RRR4. Our second example is a Class 1A RRR4 mechanism. Table 1 states that only extramobile K3 should always work, while extramobile K1–K2 and intramobile K4 are conditional cases that depend on the displacements of links 2 and 4.

Figure 16 shows the analysis for a Class 1A RRR4 mechanism. While traveling exterior to the surface (left to right), link 2 violates the Conditions (in this case, Condition 2 because it moved inside the surface, then crossed back outside the surface) before link 4 does. While traveling interior to the surface (right to left), link 4 violates the Condition 2 before link 2 does. These become the limits of intramobile and extramobile motion, respectively, in subfigure (c), which shows the strain energy curves for springs K1–K4.

Each spring produces a minimum, as expected by Test 1. K1, K2, and K3 produce extramobile BDMs because the stable positions are within the extramobile motion range. K4 produces an intramobile BDM because its stable position is within the intramobile motion range. All of these stable positions are therefore in agreement with Table 1.

7 Discussion and Conclusion

By highlighting all possible cases of when mechanisms will always, sometimes, and never be an extramobile or intramobile BDM, Tables 1 and 2 make the design of these bistable mechanisms

approachable and straightforward. A survey of the tables provides a few points for discussion.

The Intramobile section of Table 1 indicates that there are no cases of intramobile BDMs that will always work. This requires that the design of developable mechanisms that will reach a second stable position interior to a cylinder must always take into consideration the geometry and/or motion of the mechanism during the design process to ensure a valid intramobile BDM. It should also be noted that there are few cases of possible intramobile mechanism configurations that result in a viable intramobile BDM. Furthermore, there are no cases where two or more different springs on a given mechanism will create an intramobile BDM.

Class 1 mechanisms provide more variety in mechanism types than the other classes, including the only possible non-Grashof mechanisms. In contrast, Class 3 provides the least variety but is the only class that can create an intramobile BDM that uses a moving spring.

Lastly, as can be seen with the sample mechanisms in Figs. 15(c) and 16(c), the second stable positions can be visually identified when two links joined by a torsional spring form an arc. This behavior is unique to developable mechanisms and could be utilized as a design feature.

Bistable developable mechanisms that remain interior or exterior to a cylinder show promise for implementation in many applications. Their ability to reach stable positions with minimal power consumption and minimal interference with existing systems make them a great candidate in settings where power may be limited and volume is at a premium, including space and medical applications. To fully realize their potential, additional steps could be taken, such as developing methods to incorporate mechanical compliance to induce strain. The implementation of compliance could be combined with recent work to model the deflection of compliant parts [24], enable out-of-plane motion [25] while maintaining bistability or plausibly adapt geometry to integrate constant-force behaviors [26] with extramobile or intramobile bistability. Additionally, understanding bistable behaviors can form a basis for considerations of actuation in BDMs.

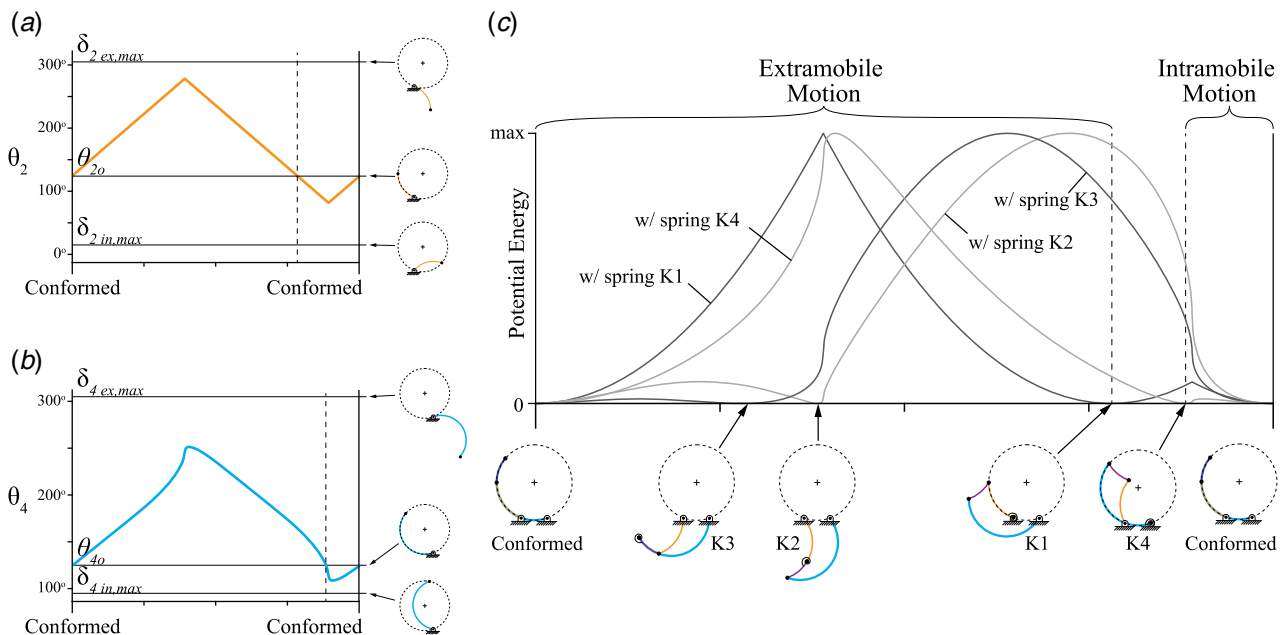


Fig. 16 Subfigures (a) and (b) show θ_2 and θ_4 , respectively, and their corresponding limits for extramobile and intramobile motion for a simulated Class 1A RRR4 mechanism. From these, we find where link 2 or 4 violates Conditions 1 or 2 (marked with dashed lines). These translate to the limits of extramobile and intramobile motion in subfigure (c). K4 produces a valid intramobile BDM, while K1, K2, and K3 produce valid extramobile BDMs. (a) Motion of link 2, (b) motion of link 4, and (c) energy curves along the full range of motion of the mechanism, where each curve corresponds to a different dominant torsional spring. Stable states are also noted and shown.

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Conflict of Interest

There are no conflicts of interest.

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