OPTIMAL CONTROL FOR THE INFINITY OBSTACLE PROBLEM

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Abstract. In this note, we show that a natural optimal control problem for the ∞-obstacle problem admits an optimal control which is also an optimal state. Moreover, we show the convergence of the minimal value of an optimal control problem for the p-obstacle problem to the minimal value of our optimal control problem for the ∞-obstacle problem, as $p \to \infty$.

1. Introduction

The obstacle problem corresponding to an obstacle f in

$$W_g^{1,2}(\Omega) = \{u \in W^{1,2}(\Omega) : \quad u = g \quad \text{on} \quad \partial\Omega\} \tag{1.1}$$

consists of minimizing the Dirichlet energy

$$\int_\Omega |Du(x)|^2 \, dx$$

over the set

$$\mathbb{K}_{f,g}^2 = \{u \in W_g^{1,2}(\Omega) : \quad u(x) \geq f(x) \quad \text{in } \Omega\} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain, Du is the gradient of u, and $g \in tr(W^{1,2}(\Omega))$ with tr the trace operator. In (1.1), the equality $u = g$ on $\partial\Omega$ is in the sense of trace. This problem is used to model the equilibrium position of an elastic membrane whose boundary is held fixed at g and is forced to remain above a given obstacle f. It is known that the obstacle problem admits a unique solution $v \in \mathbb{K}_{f,g}^2$. That is, there is a unique $v \in \mathbb{K}_{f,g}^2$ such that

$$\int_\Omega |Dv(x)|^2 \, dx \leq \int_\Omega |Du(x)|^2 \, dx, \quad \forall u \in \mathbb{K}_{f,g}^2.$$

In [3] Adams, Lenhart and Yong introduced an optimal control problem for the obstacle problem by studying the minimizer of the functional

$$J_2(\psi) = \frac{1}{2} \int_\Omega (|T_2(\psi) - z|^2 + |D\psi|^2) \, dx.$$

In the above variational problem, following the terminology in control theory [16], ψ is called the control variable and $T_2(\psi)$ is the corresponding state. The control ψ lies in the space $W_0^{1,2}(\Omega)$, the state $T_2(\psi)$ is the unique solution for the obstacle problem corresponding to the obstacle ψ and the profile z is in $L^2(\Omega)$. The authors proved that there exists a unique minimizer $\bar{\psi} \in W_0^{1,2}(\Omega)$ of the functional J_2. Furthermore, they showed that $T_2(\bar{\psi}) = \bar{\psi}$.

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Following suit, for $1 < p < \infty$, and $z \in L^p(\Omega)$, Lou in [17] considered the variational problem of minimizing the functional

$$(P_p) \qquad \bar{J}_p(\psi) = \frac{1}{p}\int_\Omega |T_p(\psi) - z|^p + |D\psi|^p \, dx$$

for $\psi \in W_0^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega) : \quad u = 0 \quad \text{on} \quad \partial\Omega\}$ and established that the problem admits a minimizer $\bar{\psi}$. Here $T_p(\psi)$ is the unique solution for the $p-$obstacle problem with obstacle $\psi \in W_0^{1,p}(\Omega)$, see [6] and references therein for discussions about the p-obstacle problem. We remind the reader that the $p-$obstacle problem with obstacle $f \in W_g^{1,p}(\Omega)$ refers to the problem of minimizing the $p-$Dirichlet energy

$$\int_\Omega |Du(x)|^p \, dx$$

among all functions in the class

$$\mathbb{K}^p_{f,g} = \{u \in W^{1,p}(\Omega) : \quad u \geq f \quad \text{in} \quad \Omega \quad \text{and} \quad u = g \quad \text{on} \quad \partial\Omega\},$$

with $g \in tr(W^{1,p}(\Omega)$. It is further shown in [17] that, as in the case of $p = 2$, $T_p(\bar{\psi}) = \bar{\psi}$.

For the boundary data $g \in Lip(\partial\Omega)$, letting $p \to \infty$, one obtains a limiting variational problem of L^∞-type which is referred in the literature as the infinity obstacle problem or ∞-obstacle problem (see [20]) . That is, given an obstacle $f \in W_g^{1,\infty}(\Omega)$ one considers the minimization problem:

$$(1.3) \qquad \text{Finding} \quad u_\infty \in \mathbb{K}^\infty_{f,g} : \quad ||Du_\infty||_\infty = \inf_{u \in \mathbb{K}^\infty_{f,g}} ||Du||_\infty,$$

where

$$\mathbb{K}^\infty_{f,g} = \{u \in W^{1,\infty}(\Omega) : \quad v \geq f \quad \text{in} \quad \Omega \quad u = g \quad \text{on} \quad \partial\Omega\}, \quad \text{and} \quad ||\cdot||_\infty := ess\,sup\,|\cdot|.$$

It is established in [20] that the minimization problem (1.3) has a solution

$$(1.4) \qquad u_\infty := u_\infty(f) \in \mathbb{K}^\infty_{f,g}$$

which verifies

$$(1.5) \qquad -\Delta_\infty u_\infty \geq 0 \quad \text{in} \quad \Omega \quad \text{in a weak sense .}$$

More importantly, the authors in [20] characterize u_∞ as the smallest infinity superharmonic function on Ω that is larger than the obstacle f and equals g on the boundary. Thus for a fixed $F \in Lip(\partial\Omega)$, this generates an obstacle to solution operator

$$T_\infty : W_F^{1,\infty}(\Omega) \longrightarrow W_F^{1,\infty}(\Omega)$$

defined by

$$(1.6) \qquad T_\infty(f) := u_\infty(f) \in W_F^{1,\infty}(\Omega), \quad f \in W_F^{1,\infty}(\Omega),$$

where

$$W_F^{1,\infty}(\Omega) := \{u \in W^{1,\infty}(\Omega) : u = F \quad \text{on} \quad \partial\Omega\}.$$

In this note, we consider a natural optimal control problem for the infinity obstacle problem. More precisely, for $F \in Lip(\partial\Omega)$ and for $z \in L^\infty(\Omega)$ fixed, we introduce the functional

$$J_\infty(\psi) = \max\{||T_\infty(\psi) - z||_\infty, \, ||D\psi||_\infty\}, \quad \psi \in W_F^{1,\infty}(\Omega)$$

and study the problem of existence of $\psi_\infty \in W_F^{1,\infty}(\Omega)$ such that:

$$(P_\infty) \qquad J_\infty(\psi_\infty) \leq J_\infty(\psi), \qquad \forall \quad \psi \in W_F^{1,\infty}(\Omega).$$

In deference to optimal control theory, a function ψ_∞ satisfying (P_∞) is called an *optimal control* and the state $T_\infty(\psi_\infty)$ is called an *optimal state.*

Several variants of control problems where the control variable is the obstacle have been studied by different authors since the first of such works appeared in [3]. The literature is vast, but to mention a few, in [2] the authors studied a generalization of [3] by adding a source term. In [1] a similar problem is studied when the state is a solution to a parabolic variational inequality. In [18] the author studied regularity of the optimal state obtained in [3]. When the state is governed by a bilateral variational inequality, results are obtained in [9], [10], [11] and [12]. Optimal control for higher order obstacle problems appears in [5] and [14]. Related works where the control variable is the obstacle are also studied in [13, 21] and the references therein.

In this note, we prove that the optimal control problem (P_∞) associated to J_∞ is solvable. Precisely we show the following result:

Theorem 1.1. *Assuming that $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain, $F \in Lip(\partial\Omega)$, and $z \in L^\infty(\Omega)$, J_∞ admits an optimal control $u_\infty \in W_F^{1,\infty}(\Omega)$ which is also an optimal state, i.e*

$$u_\infty = T_\infty(u_\infty).$$

Using also arguments similar to the ones used in the proof of Theorem 1.1, we show the convergence of the minimal value of an optimal control problem associated to $\bar{J}_p$ to the minimal value of the optimal control problem corresponding to J_∞ as p tends to infinity. Indeed we prove the following result:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain, $F \in Lip(\partial\Omega)$, and $z \in L^\infty(\Omega)$. Then setting*

$$J_p = (p\bar{J}_p)^{\frac{1}{p}}, \quad C_p = \min_{\psi \in W_F^{1,p}(\Omega)} J_p(\psi) \ \textit{for} \ \ 1 < p < \infty, \quad \textit{and} \qquad C_\infty = \min_{\psi \in W_F^{1,\infty}(\Omega)} J_\infty(\psi),$$

where $\bar{J}_p$ is as in (P_p), we have

$$\lim_{p \to \infty} C_p = C_\infty$$

In the proofs of the above results, we use the p-approximation technique as in the study of the ∞-obstacle problem combined with the classical methods of weak convergence in Calculus of Variations. As in the study of the ∞-obstacle problem, here also the key analytical ingredients are the L^q-characterization of L^∞ and Hölder's inequality. The difficulty arises from the the fact that the unicity question for the ∞-obstacle problem is still an open problem to the best of our knowledge. To overcome the latter issue, we make use of the characterization of the solution of the ∞-obstacle problem by Rossi-Teixeira-Urbano [20].

2. Preliminaries

One of the most popular way of approaching problems related to minimizing a functional of L^∞-type is to follow the idea first introduced by Aronsson in [7] and which involves interpreting an L^∞-type minimization problem as a limit when $p \to \infty$ of an L^p-type minimization problem. In this note, this p-approximation technique will be used to show existence of an optimal control for J_∞. In order to prepare for our use of the p-approximation technique, we are going to start this section by discussing some related L^p-type variational problems.

Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain and $g \in Lip(\partial\Omega)$. Moreover let $\psi \in W_g^{1,\infty}(\Omega)$ be fixed and $1 < p < \infty$. Then as described earlier the p-obstacle problem with obstacle ψ corresponds to finding a minimizer of the functional

$$I_p(v) = \int_\Omega |Dv(x)|^p dx \tag{2.1}$$

over the space $\mathbb{K}^p_{\psi,g} = \{v \in W^{1,p}(\Omega) : \quad v \geq \psi, \quad \text{and} \quad v = g \quad \text{on} \quad \partial\Omega\}$. The energy integral (2.1) admits a unique minimizer $u_p \in \mathbb{K}^p_{\psi,g}$. The minimizer u_p is not only p-superharmonic, i.e $\Delta_p u_p \leq 0$, but is also a weak solution to the following system

$$\begin{cases} -\Delta_p u \geq 0 & \text{in} \quad \Omega \\ -\Delta_p u\,(u-\psi) = 0 & \text{in} \quad \Omega \\ u \geq \psi & \text{in} \quad \Omega \end{cases} \tag{2.2}$$

where Δ_p is the p-Laplace operator given by

$$\Delta_p u := div(|Du|^{p-2}Du).$$

Moreover, it is known that the p-obstacle problem is equivalent to the system (2.2) (see [16] or [19]) and hence we will refer to (2.2) as the p-obstacle problem as well. On the other hand, by the equivalence of weak and viscosity solutions established in [19] (and [15]) u_p is also a viscosity solution of (2.2) according to the following definition.

Definition 2.1. *A function $u \in C(\Omega)$ is said to be a viscosity subsolution (supersolution) to*

$$\begin{aligned} F(x,u,Du,D^2u) &= 0 \quad in \quad \Omega \\ u &= 0 \quad in \quad \partial\Omega \end{aligned} \tag{2.3}$$

if for every $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ whenever $\phi - u$ has a minimum (resp. maximum) in a neighborhood of x_0 in Ω we have:

$$F(x,u,D\phi,D^2\phi) \leq 0 \quad (resp. \quad \geq 0).$$

The function u is called a viscosity solution of (2.3) in Ω if u is both viscosity subsolution and viscosity supersolution of (2.3) in Ω.

The asymptotic behavior of the sequence of minimizers $(u_p)_{p>1}$ as p tends to infinity has been investigated in [20]. In fact, in [20], it is established that for a fixed $\psi \in W_g^{1,\infty}(\Omega)$, there exists $u_\infty = u_\infty(\psi) \in \mathbb{K}^\infty_{\psi,g} = \{v \in W_g^{1,\infty}(\Omega) : v \geq \psi\}$ such that $u_p \to u_\infty$ locally uniformly in $\bar{\Omega}$, and that for every $q \geq 1$, u_p converges to u_∞ weakly in $W^{1,q}(\Omega)$. Furthermore, u_∞ is a solution to the ∞-obstacle problem

$$\min_{v \in \mathbb{K}^\infty_{\psi,g}} ||Dv||_\infty \tag{2.4}$$

For Ω convex (see [8]), the variational problem (2.4) is equivalent to the minimization problem

$$\min_{v \in \mathbb{K}^\infty_{\psi,g}} \mathcal{L}(v),$$

where

$$\mathcal{L}(v) = \inf_{(x,y)\in\Omega^2,\, x\neq y} \frac{|v(x) - v(y)|}{|x-y|}.$$

Moreover, in [20], it is show that u_∞ is a viscosity solution to the following system.

$$\begin{cases} -\Delta_\infty u \geq 0 & \text{in} \quad \Omega \\ -\Delta_\infty u\,(u-\psi) = 0 & \text{in} \quad \Omega \\ u \geq \psi & \text{in} \quad \Omega \end{cases}$$

where Δ_∞ is the ∞-Laplacian and is defined by

$$\Delta_\infty u = \langle D^2 u Du, Du\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j}.$$

Recalling that u is said to be *infinity superharmonic* or ∞- *superharmonic*, if $-\Delta_\infty u \geq 0$ in the viscosity sense, we have the following characterization of u_∞ in terms of infinity superharmonic functions and it is proven in [20]. We would like to emphasize that this will play an important role in our arguments.

Lemma 2.2. *Setting*

$$\mathcal{F}^+ = \{v \in C(\Omega),\ \ -\Delta_\infty v \geq 0 \ \ \textit{in}\ \ \Omega \ \ \textit{in the viscosity sense}\}$$

and

$$\mathcal{F}^+_\psi = \{v \in \mathcal{F}^+,\ \ v \geq \psi \ \ \textit{in}\ \ \Omega, \ \ \textit{and}\ \ v = \psi \ \ \textit{on}\ \ \partial\Omega\},$$

we have

$$T_\infty(\psi) = u_\infty = \inf_{v \in \mathcal{F}^+_\psi} v, \tag{2.5}$$

with T_∞ *as defined earlier in* (1.6).

Lemma 2.2 implies the following characterization of infinity superharmonic functions as fixed points of T_∞. This charactreization plays a key role in our p-approximation scheme for existence.

Lemma 2.3. *Assuming that* $u \in W^{1,\infty}_g(\Omega)$, u *being infinity superharmonic is equivalent to* u *being a fixed point of* T_∞, *i.e*

$$T_\infty(u) = u.$$

Proof. Let $u \in W_g^{1,\infty}(\Omega)$ be an infinity superharmonic function and v be defined by $v = T_\infty(u)$. Then clearly the definition of v and lemma 2.2 imply $v \geq u$. On the other hand, since $u \in W_g^{1,\infty}(\Omega)$ and is an infinity superharmonic function, we deduce from lemma 2.2 that $u \geq T_\infty(u) = v$. Thus, we get $T_\infty(u) = u$. Now if $u = T_\infty(u)$, then using again lemma 2.2 or (1.4)-(1.6), we obtain u is an infinity superharmonic function. Hence the proof of the lemma is complete.

□

To run our p-approximation scheme for existence, another crucial ingredient that we will need is an appropriate characterization of the limit of sequence of solution w_p of the p-obstacle problem (2.2) with obstacle ψ_p under uniform convergence of both w_p and ψ_p. Precisely, we will need the following lemma.

Lemma 2.4. *If w_p is a solution to the p-obstacle problem (2.2) with obstacle ψ_p that is, w_p satisfies*

$$\begin{cases} -\Delta_p w_p \geq 0 & in \quad \Omega \\ -\Delta_p w_p \,(w_p - \psi_p) = 0 & in \quad \Omega \\ w_p \geq \psi_p & in \quad \Omega \end{cases} \tag{2.6}$$

in the viscosity sense and if also that $w_p \to u_\infty$ and $\psi_p \to \psi_\infty$ locally uniformly in $\overline{\Omega}$, then u_∞ is a solution in the viscosity sense of the following system

$$\begin{cases} -\Delta_\infty w_\infty \geq 0 & in \quad \Omega \\ -\Delta_\infty w_\infty \,(w_\infty - \psi_\infty) = 0 & in \quad \Omega \\ w_\infty \geq \psi_\infty & in \quad \Omega. \end{cases} \tag{2.7}$$

Proof. First of all, note that since $w_p \geq \psi_p$, $-\Delta_p w_p \geq 0$ in the viscosity sense in Ω for every p, $w_p \to u_\infty$, and $\psi_p \to \psi_\infty$ both locally uniformly in $\overline{\Omega}$, and $\overline{\Omega}$ is compact, we have $w_\infty \geq \psi_\infty$ and $-\Delta_\infty w_\infty \geq 0$ in the viscosity sense in Ω. It thus remains to prove that $-\Delta_\infty u_\infty \,(w_\infty - \psi_\infty) = 0$ in Ω which (because of $w_\infty \geq \psi_\infty$ in Ω) is equivalent to $-\Delta_\infty u_\infty = 0$ in $\{w_\infty > \psi_\infty\} := \{x \in \Omega : \; w_\infty(x) > \psi_\infty(x)\}$. Thus to conclude the proof, we are going to show $-\Delta_\infty w_\infty = 0$ in $\{w_\infty > \psi_\infty\}$. To that end, fix $y \in \{w_\infty > \psi_\infty\}$. Then, by continuity there exists an open neighborhood V of y in Ω such that $\overline{V}$ is a compact subset of Ω, and a small real number $\delta > 0$ such that $w_\infty > \delta > \phi_\infty$ in $\overline{V}$. Thus, from $w_p \to w_\infty$, $\psi_p \to \psi_\infty$ locally uniformly in $\overline{\Omega}$, and $\overline{V}$ compact subset of Ω, we infer that for sufficiently large p

$$w_p > \delta > \psi_p \quad \text{in} \quad \overline{V}. \tag{2.8}$$

On the other hand, since w_p is a solution to the p obstacle problem (2.2) with obstacle ψ_p, then clearly $-\Delta_p w_p = 0$ in $\{w_p > \psi_p\} := \{x \in \Omega : \; w_p(x) > \psi_p(x)\}$. Thus, (2.8) imply $-\Delta_p w_p = 0$ in the sense of viscosity in V. Hence, recalling that $w_p \to w_\infty$ locally uniformly in $\overline{\Omega}$ and letting $p \to \infty$, we obtain

$$-\Delta_\infty w_\infty = 0 \quad \text{in the sense of viscosity in} \quad V.$$

Thus, since $y \in V$ is arbitrary in $\{w_\infty > \psi_\infty\}$, then we arrive to

$$-\Delta_\infty w_\infty = 0 \quad \text{in the sense of viscosity in} \quad \{w_\infty > \psi_\infty\},$$

thereby ending the proof of the lemma. ☐

On the other hand, to show the convergence of the minimal values of J_p to that of J_∞, we will make use of the following elementary results.

Lemma 2.5. *Suppose* $\{a_p\}$ *and* $\{b_p\}$ *are nonnegative sequences with*

$$\liminf_{p\to\infty} a_p = a \quad \text{and} \quad \liminf_{p\to\infty} b_p = b.$$

Then

$$\liminf_{p\to\infty} \max\{a_p, b_p\} = \max\{a, b\}.$$

Proof. Let $\{b_{p_k}\}$ be a subsequence converging to $b = \liminf_{p\to\infty} b_p$. Then

$$\lim_{k\to\infty} \max\{a_{p_k}, b_{p_k}\} = \max\{a, b\}.$$

Since the lim inf is the smallest limit point we have

$$\liminf_{p\to\infty} \max\{a_p, b_p\} \le \max\{a, b\}. \tag{2.9}$$

On the other hand

$$a_p,\ b_p \le \max\{a_p, b_p\}, \quad \text{for all} \quad p.$$

Thus

$$b = \liminf_{p\to\infty} b_p \le \liminf_{p\to\infty} \max\{a_p, b_p\},$$

and likewise

$$a \le \liminf_{p\to\infty} \max\{a_p, b_p\}.$$

Consequently

$$\liminf_{p\to\infty} \max\{a_p, b_p\} \ge \max\{a, b\}. \tag{2.10}$$

Finally (2.9) and (2.10) conclude the proof of the lemma . ☐

Lemma 2.6. *Suppose* $\{a_p\}$ *and* $\{b_p\}$ *are nonnegative sequences with*

$$\liminf_{p\to\infty} a_p = a \quad \text{and} \quad \liminf_{p\to\infty} b_p = b.$$

Then

$$\liminf_{p\to\infty} (a_p^p + b_p^p)^{1/p} = \max\{a, b\}.$$

Proof. It follows directly from the trivial inequality

$$2^{\frac{1}{p}} \max\{a_p, b_p\} \ge (a_p^p + b_p^p)^{1/p} \ge \max\{a_p, b_p\}, \ \ \forall p \ge 1,$$

lemma 2.5 and the fact that $\liminf_n(a_n b_n) = (\lim_n a_n)(\liminf_n b_n)$ if $\lim_n a_n > 0$. ☐

3. Existence of optimal control for J_∞ and limit of C_p

In this section, we show the existence of an optimal control for J_∞ and show that C_p converges to C_∞ as $p \to \infty$. We divide it in two subsections. In the first one we show existence of an optimal control for J_∞ via the p-approximation technique, and in the second one we show that C_p converges to C_∞ as p tends to infinity.

3.1. **Existence of optimal control.** In this subsection, we show the existence of a minimizer of J_∞ via the p-approximation technique using solutions of the optimal control for J_p. For this end, we start by recalling some optimality facts about J_p inherited from $\bar{J}_p$ (see (P_p) for its definition) and mentioned in the introduction. For $\Omega \subset \mathbb{R}^n$ a bounded and smooth domain, $z \in L^\infty(\Omega)$, $F \in Lip(\partial\Omega)$, and $1 < p < \infty$, we recall that the functional J_p is defined by the formula

$$J_p(\psi) = \left[\int_\Omega |T_p(\psi) - z|^p + |D\psi|^p dx\right]^{1/p}, \quad \psi \in W_F^{1,p}(\Omega) \tag{3.1}$$

and that the optimal control problem for J_p is the variational problem of minimizing J_p, namely

$$\inf_{\psi \in W_F^{1,p}(\Omega)} J_p(\psi) \tag{3.2}$$

over $W_F^{1,p}(\Omega)$, where

$$W_F^{1,p}(\Omega) = \{\psi \in W^{1,p}(\Omega) : \ \psi = F \quad \text{on} \ \ \partial\Omega\},$$

and $T_p(\psi)$ is the solution to the p-obstacle problem with obstacle ψ. Moreover, as for the functional $\bar{J}_p$, J_p also admits a minimizer $\psi_p \in W_F^{1,p}(\Omega)$ verifying

$$T_p(\psi_p) = \psi_p. \tag{3.3}$$

As mentioned in the introduction, for more details about the latter results, see [3] for $p = 2$ and see [17] for $p > 2$.

To continue, let us pick $\eta \in W_F^{1,\infty}(\Omega)$. Since η competes in the minimization problem (3.2), we have

$$\int_\Omega |D\psi_p|^p dx \le J_p(\eta) = \int_\Omega |T_p(\eta) - z|^p + |D\eta|^p dx.$$

Since $\overline{\Omega}$ is compact and $T_p(\eta) \to T_\infty(\eta)$ as $p \to \infty$ locally uniformly on $\overline{\Omega}$ (which follows from the definition of $T_\infty(\eta)$), we deduce that for p very large

$$\int_\Omega |D\psi_p|^p dx \le M^p |\Omega| \tag{3.4}$$

for some M which depends only on $||\eta||_{W^{1,\infty}}$, $||T_\infty(\eta)||_{C^0}$ and $||z||_\infty$. Furthermore, let us fix $1 < q < p$. Then by using Holder's inequality, we can write

$$\int_\Omega |D\psi_p|^q dx \le \left\{\int_\Omega (|D\psi_p|^q)^{p/q} dx\right\}^{q/p} |\Omega|^{\frac{p-q}{p}} \tag{3.5}$$

and we obtain by using (3.4) that for p very large

$$\int_\Omega |D\psi_p|^q dx \le M^q |\Omega|^{\frac{q}{p}} |\Omega|^{\frac{p-q}{p}}$$

and raising both sides to $1/q$, we derive that for p very large, there holds

$$||D\psi_p||_{L^q} \le M|\Omega|^{1/q},$$

with $||\cdot||_{L^q}$ denoting the classical $L^q(\Omega)$-norm. This shows, that the sequence $\{\psi_p\}$ is bounded in $W_F^{1,q}(\Omega)$ in the gradient norm for every q with a bound independent of q, and by Poincare's inequality, that for every $1 < q < \infty$, the sequence $\{\psi_p\}$ is bounded in $W^{1,q}_F(\Omega)$ in the standard $W^{1,q}(\Omega)$-norm. Therefore , by classical weak compactness arguments, we have that, up to a subsequence,

(3.6)
$$\psi_p \longrightarrow \psi_\infty, \text{ as } p\to\infty \text{ locally uniformly in } \overline{\Omega} \text{ and weakly in } W^{1,q}(\Omega)\ \forall\ 1<q<\infty.$$

Notice that consequently $||D\psi_\infty||_{L^q} \le M|\Omega|^{1/q}$ for all $1 < q < \infty$. Thus, we deduce once again by Poincare's inequality that

$$\psi_\infty \in W^{1,\infty}_F(\Omega). \tag{3.7}$$

We want now to show that ψ_∞ is a minimizer of J_∞. To that end, we make the following observation which is a consequence of lemma 2.4.

Lemma 3.1. *The function ψ_∞ is a fixed point of T_∞, namely*

$$T_\infty(\psi_\infty) = \psi_\infty,$$

and the solutions $T_p(\psi_p)$ of the p-obstacle problem with obstacle ψ_p verify: as $p\to\infty$,

$$T_p(\psi_p) \longrightarrow T_\infty(\psi_\infty) \text{ locally uniformly in } \overline{\Omega} \text{ and weakly in } W^{1,q}(\Omega)\ \forall\ 1<q<\infty.$$

Proof. We know that $T_p(\psi_p) = \psi_p$ (see (3.3)) Thus using (3.6) and Lemma 2.4 with $\phi_p = \psi_p$ and $w_p = T_p(\psi_p) = \psi_p$, we have $T_p(\psi_p) \to \psi_\infty$ locally uniformly in $\overline{\Omega}$, weakly in $W^{1,q}(\Omega)$ for every $1 < q < \infty$, and ψ_∞ is a infinity superharmonic. Thus, recalling (3.7), we have lemma 2.3 implies $T_\infty(\psi_\infty) = \psi_\infty$. Hence the proof of the lemma is complete. □

Now, with all the ingredients at hand, we are ready to show that ψ_∞ is a minimizer of J_∞. Indeed, we are going to show the following proposition:

Proposition 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and smooth domain, $F \in Lip(\partial\Omega)$ and $z \in L^\infty(\Omega)$. Then ψ_∞ is a minimizer of J_∞ on $W^{1,\infty}_F(\Omega)$ That is:*

$$J_\infty(\psi_\infty) = \min_{\eta\in W^{1,\infty}_F(\Omega)} J_\infty(\eta)$$

Proof. We first introduce for $n < p < \infty$ and $\psi \in W^{1,p}_F(\Omega)$

$$H_p(\psi) = \max\{||T_p(\psi) - z||_\infty,\ ||D\psi||_\infty\},$$

which is well defined by Sobolev Embedding Theorem. Then for any $\eta \in W^{1,\infty}_F(\Omega)$

$$\int_\Omega |D\psi_p|^p dx \le J_p^p(\eta) = \int_\Omega \left(|T_p(\eta) - z|^p + |D\eta|^p\right) dx.$$

Therefore, using the trivial identity $(|a|^p + |b|^p)^{\frac{1}{p}} \le 2^{\frac{1}{p}} \max\{|a|,\ |b|\}$, we get

$$\left(\int_\Omega |D\psi_p|^p dx\right)^{1/p} \le 2^{1/p}|\Omega|^{1/p} H_p(\eta).$$

If we now set

$$I_p = \inf_{\eta\in W^{1,\infty}_F(\Omega)} H_p(\eta), \tag{3.8}$$

we deduce that

$$\left(\int_\Omega |D\psi_p|^p dx\right)^{1/p} \leq 2^{1/p}|\Omega|^{1/p} I_p.$$

Let us fix q such that $n < q < \infty$. Then for $q < p < \infty$, by proceeding as in (3.5), we obtain

$$||D\psi_p||_{L^q} \leq 2^{1/p} I_p |\Omega|^{1/q}.$$

Similarly,

$$||T_p(\psi_p) - z||_{L^q} \leq 2^{1/p} I_p |\Omega|^{1/q}.$$

Thus

$$\max\{||T_p(\psi_p) - z||_{L^q}, ||D\psi_p||_{L^q}\} \leq 2^{1/p} I_p |\Omega|^{1/q}. \tag{3.9}$$

For any $\eta \in W_F^{1,\infty}(\Omega)$ we also have $I_p \leq H_p(\eta)$ and $\liminf_{p\to\infty} I_p \leq \liminf_{p\to\infty} H_p(\eta)$. Thus, since ψ_p converges weakly in $W^{1,q}(\Omega)$ to ψ_∞ as $p \to \infty$ and (3.9) holds, then by weak lower semicontinuity, we conclude that

$$||D\psi_\infty||_{L^q} \leq \liminf_{p\to\infty} ||D\psi_p||_{L^q} \leq |\Omega|^{1/q} \liminf_{p\to\infty} H_p(\eta).$$

Moreover, since $T_p(\eta)$ converges locally uniformly on $\overline{\Omega}$ to $T_\infty(\eta)$ as $p \to \infty$ and $\overline{\Omega}$ is compact, then clearly

$$\lim_{p\to\infty} H_p(\eta) = J_\infty(\eta),$$

and hence

$$||D\psi_\infty||_{L^q} \leq J_\infty(\eta)|\Omega|^{1/q}.$$

Since this holds for any element η of $W_F^{1,\infty}(\Omega)$, we conclude that by taking the infimum over $W_F^{1,\infty}(\Omega)$ and letting $q \to \infty$

$$||D\psi_\infty||_\infty \leq \inf_{\eta\in W_F^{1,\infty}(\Omega)} J_\infty(\eta) \leq J_\infty(\psi_\infty). \tag{3.10}$$

Using lemma 3.1 and equation (3.9) combined with Rellich compactness Theorem or the continuous embedding of L^∞ into L^q, we conclude that

$$||T_\infty(\psi_\infty) - z||_{L^q} = \lim_{p\to\infty} ||T_p(\psi_p) - z||_{L^q} \leq |\Omega|^{1/q} \liminf_{p\to\infty} H_p(\eta).$$

Thus, as above letting q goes to infinity and taking infimum in η over $W_F^{1,\infty}(\Omega)$, we also have

$$||T_\infty(\psi_\infty) - z||_\infty \leq \inf_{\eta\in W_F^{1,\infty}(\Omega)} J_\infty(\eta) \leq J_\infty(\psi_\infty). \tag{3.11}$$

Finally, from (3.7), (3.10) and (3.11) we deduce

$$J_\infty(\psi_\infty) = \min_{\eta\in W_F^{1,\infty}(\Omega)} J_\infty(\eta),$$

as desired. □

3.2. **Convergence of Minimum Values.** In this subsection, we show the convergence of the minimal value of the optimal control problem of J_p to the one of J_∞ as $p \to \infty$, namely Theorem 1.2 via the following proposition:

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain, $F \in Lip(\partial\Omega)$ and $1 < p < \infty$. Then recalling that*

$$C_p = \min_{\psi \in W_F^{1,p}(\Omega)} J_p(\psi) \quad and \quad C_\infty = \min_{\psi \in W_F^{1,\infty}(\Omega)} J_\infty(\psi),$$

we have

$$\lim_{p\to\infty} C_p = C_\infty.$$

Proof. Let $\psi_p \in W_F^{1,p}(\Omega)$ and $\psi_\infty \in W_F^{1,\infty}(\Omega)$ be as in subsection 3.1. Then they satisfy $J_p(\psi_p) = C_p$ and $J_\infty(\psi_\infty) = C_\infty$. Moreover, up to a subsequence, we have ψ_p and ψ_∞ verify (3.6) and the conclusions of lemma 3.1. On the other hand, by minimality and Hölder's inequality, we have

$$J_p(\psi_p) \le J_p(\psi_\infty) \le 2^{1/p}|\Omega|^{1/p} \max\{||T_p(\psi_\infty) - z||_\infty, ||D\psi_\infty||_\infty\}.$$

Thus

$$\limsup_{p\to\infty} J_p(\psi_p) \le J_\infty(u_\infty). \tag{3.12}$$

Now we are going to show the following

$$J_\infty(\psi_\infty) \le \liminf_{p\to\infty} J_p(\psi_p). \tag{3.13}$$

To that end observe that by definition of J_∞, we have

$$J_\infty(\psi_\infty) = \max\{||T_\infty(\psi_\infty) - z||_\infty,\ ||D\psi_\infty||_\infty\}. \tag{3.14}$$

Thus, using the L^q-characterization of L^∞, we have that (3.14) imply

$$J_\infty(\psi_\infty) = \max\{\lim_{q\to\infty} ||T_\infty(\psi_\infty) - z||_{L^q},\ \lim_{q\to\infty} ||D\psi_\infty||_{L^q}\}, \tag{3.15}$$

and by using lemma 2.5, we get

$$J_\infty(\psi_\infty) = \lim_{q\to\infty} \max\{||T_\infty(\psi_\infty) - z||_{L^q},\ ||D\psi_\infty||_{L^q}\}. \tag{3.16}$$

On the other hand, by weak lower semicontinuity, and corollary 3.1, we have

$$||D\psi_\infty||_{L^q} \le \liminf_{p\to\infty} ||D\psi_p||_{L^q}. \tag{3.17}$$

Now, combining (3.16) and (3.17), we obtain

$$J_\infty(\psi_\infty) \le \liminf_{q\to\infty} \max\{||T_\infty(\psi_\infty) - z||_{L^q}, \liminf_{p\to\infty} ||D\psi_p||_{L^q}\}. \tag{3.18}$$

Next, using lemma 2.6, corollary 3.1, and (3.18), we get

$$J_\infty(\psi_\infty) \le \liminf_{q\to\infty} \liminf_{p\to\infty} \left\{(||T_p(\psi_p) - z||_{L^q})^p + (||D\psi_p||_{L^q})^p\right\}^{1/p}. \tag{3.19}$$

To continue, we are going to estimate the right hand side of (3.19). Indeed, using Hölder's inequality, we have

$$\begin{aligned}
(||T_p(\psi_p) - z||_{L^q})^p &= \left\{ \int_\Omega |T_p(\psi_p) - z|^q dx \right\}^{p/q} \\
&\leq \left\{ \int_\Omega |T_p(\psi_p) - z|^p dx \right\} |\Omega|^{(1-q/p)p/q} \\
&= \left\{ \int_\Omega |T_p(\psi_p) - z|^p dx \right\} |\Omega|^{(1-q/p)p/q}.
\end{aligned}$$

Similarly, we obtain

$$(||D\psi_p||_{L^q})^p \leq \left\{ \int_\Omega |D\psi_p|^p \, dx \right\} |\Omega|^{(1-q/p)p/q}.$$

By using the latter two estimates in (3.19), we get

$$\begin{aligned}
J_\infty(\psi_\infty) &\leq \liminf_{q\to\infty} \liminf_{p\to\infty} \left[\left\{ \int_\Omega (|T_p(\psi_p) - z|^p + |D\psi_p|^p) \, dx \right\}^{1/p} |\Omega|^{(1-q/p)p/q(1/p)} \right] \\
&= \liminf_{q\to\infty} \liminf_{p\to\infty} \left[\left\{ \int_\Omega (|T_p(\psi_p) - z|^p + |D\psi_p|^p) \, dx \right\}^{1/p} |\Omega|^{\frac{1}{q} - \frac{1}{p}} \right] \\
&= \liminf_{q\to\infty} \left[|\Omega|^{\frac{1}{q}} \liminf_{p\to\infty} J_p(\psi_p) \right] = \liminf_{p\to\infty} J_p(\psi_p) \qquad (3.20)
\end{aligned}$$

proving claim (3.13). Combining (3.12) with (3.20) we obtain

$$\lim_{p\to\infty} J_p(\psi_p) = J_\infty(u_\infty),$$

and recalling that we were working with a possible subsequence, then we have that up to a subsequence

$$\lim_{p\to\infty} C_p = C_\infty.$$

Hence, since the limit is independent of the subsequence, we have

$$\lim_{p\to\infty} C_p = C_\infty$$

as required. □

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